Fundamental Groups of Topological Groups

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1 Introduction

This thesis is about fundamental groups of topological groups, a notion of topological space and group that occurs naturally in, for example, the Euclidean topology on the real numbers. As known from basic analysis, the operations of addition, subtraction, multiplication and division are continuous operations on the real numbers, so that the additive and multiplicative groups of the real numbers become topological groups. Like other structures, topological groups come with naturally defined subs and morphisms.

The notion of fundamental group requires a base point. In the case of topological groups, there is an obvious choice for the base point, namely the identity element of the underlying group. This is not just an 'obvious choice' but actually gives rise to correspondence between the group operation on the fundamental group and the group operation of the underlying group. The identity element as base point is a necessary condition for this observation.

One of the main results is that the fundamental group of a topological group, as defined above, is abelian. Thus, any topological space with non-abelian fundamental group cannot be equipped with a topological group structure. Put more precisely, if X is a topological space, and there does not exist an $x \in X$ such that the fundamental group of X with base point x is abelian, then X cannot be a topological group. In this thesis, several proofs of this statement will be discussed, each from a different angle, offering different insights.

The notion of covering map is viable in the context of topological groups. In such a case, a covering group is the result. If a topological group is 'nice', then any cover of it can be given a covering group structure. As known from topology and group theory, the kernel of a covering group is a discrete normal subgroup. This means that the kernel of a covering group is a central subgroup, as will be discussed.

Besides the real numbers, matrix groups are other important examples of topological groups. This thesis focuses on the real matrix groups. The real invertible $n \times n$ matrices form a topological group when given the usual Euclidean topology, for any $n \in \mathbb{Z}_{>0}$. This matrix group has interesting subgroups, such as the set of matrices with positive determinant and the orthogonal matrices, which are sub-topological groups when given the subspace topology.

A main result is that the Gram-Schmidt algorithm defines a so-called 'deformation retraction' from the real invertible $n \times n$ matrices to the orthogonal $n \times n$ matrices. This allows us to relate their fundamental groups. The Spin Cover is such an application, which will be discussed in the final section.

2 Topological Groups

A topological group is a topological space and a group such that the two structures are compatible.

2.1 Definitions

Definition. Let G be a set, T a topology on G and m a group operation on G. Then (G, T, m) is a paratopological group if m is continuous with respect to the product topology of T on $G \times G$.

Definition. A paratopological group (G, T, m) is a topological group if the inversion map $i : G \to G, g \mapsto g^{-1}$ is continuous.

The structure of topological group is 'nice' in the sense that it has naturally defined subs, as follows.

Lemma 2.1. *Let (G, T, m) be a topological group and $H \subseteq G$ a subgroup. Then H equipped with the subspace topology of T is a topological group.*

Proof. By definition of subspace topology, the restriction to $H \times H$ of the continuous map m, which is the group operation on H by definition of subgroup, is continuous. The restriction of the inversion map $i : G \to G$ to H is a bijection $H \to H$ by definition of subgroup. By the same argument, this restriction of i is continuous. So H equipped with the subspace topology of T is a topological group. □

This defines a sub:

Definition. Let (G, T, m) be a topological group. A sub-topological group of (G, T, m) is a subgroup H equipped with the subspace topology of T.

2.2 Examples

1. Let T_E be the Euclidean topology on $\mathbb{R}$. Then $(\mathbb{R}, T_E, +)$ is a topological group, since addition and taking negatives are continuous.

2. $(\mathbb{R}^\times, T_E, \cdot)$ is also a topological group, since multiplication and division are continuous operations.

3. Let (G, m) be a group. Equip G with the discrete topology D_G. Then (G, D_G, m) is a topological group. To see this, note that $G \times G$ is also discrete, and a map from any discrete topological space is continuous.

4. Let (G, m) be a group. Equip G with the chaotic topology C_G. Then (G, C_G, m) is a topological group, since any map to a chaotic topological space is continuous.

5. Let $\mathbb{R}^{n\times n}$ be the set of $n \times n$-matrices over $\mathbb{R}$. Equip this with the Euclidean topology on $\mathbb{R}^{n^2}$. Then $GL_n(\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det(A) \in \mathbb{R}^\times\}$ with the subspace topology and the matrix multiplication is a topological group. This is an example of a *matrix group*. Matrix groups will be discussed in Section 7.

6. Proposition: *Let (G, T, m) be a paratopological group, and suppose T contains a singleton. Then (G, T) is discrete.*
   Proof. Let $A = \{a\} \in T$. As m is continuous,
   $$m^{-1}(A) = m^{-1}\{a\} = \bigcup_{g\in G} \{(g, g^{-1}a)\} = \bigcup_{g\in G} \{g\} \times \{g^{-1}a\} \tag{1}$$
   is open in $G \times G$. An open in $G \times G$ is a union of the form $\bigcup_{i\in I} U_i \times V_i$ where $U_i, V_i \in T$. Since for all $g \neq h \in G$ we have $g^{-1}a \neq h^{-1}a$, $m^{-1}(A)$ is a union of pairs (a, b) such that for every a there is a unique b and vice versa. So for all $g \in G$ we have $\{g\} \in T$. We conclude that T is the discrete topology on G. □

7. Consider $V_4 = \{e, a, b, c\}$ the Kleinian group. Then $(V_4, \{V_4, \emptyset, \langle a\rangle, \{b, c\}\}, \cdot)$ is a topological group. Note that there is no open singleton in the topology.
   Proof. We have
   $$\cdot^{-1}\langle a\rangle = \{(e, a), (a, e), (b, c), (c, b), (a, a), (b, b), (c, c), (e, e)\} = \langle a\rangle^2 \cup \{b, c\}^2 \tag{2}$$
   and
   $$\cdot^{-1}\{b, c\} = \{(e, b), (e, c), (a, b), (a, c), (b, e), (c, e), (b, a), (c, a)\} = \langle a\rangle \times \{b, c\} \cup \{b, c\} \times \langle a\rangle \tag{3}$$
   and these are both open in $V_4 \times V_4$. Moreover, the inversion i equals id_{V_4}, which is continuous as well. □

8. This is an example of a paratopological group that is not a topological group. This example was suggested to me by H.W. Lenstra.
   Consider $\mathbb{Q}^\times$ with the following topology: a subset $X \subseteq \mathbb{Q}^\times$ is open if and only if
   $$\forall x \in X : \exists m \in \mathbb{Z}_{>0} : (x + m\mathbb{Z})\backslash\{0\} = \{x + mz : z \in \mathbb{Z}, x + mz \neq 0\} \subseteq X \tag{4}$$
   Then $\mathbb{Q}^\times$ with this topology and the standard multiplication is a paratopological group and not a topological group.
   Proof.

   - First we need to show that (4) actually defines a topology on $\mathbb{Q}^\times$. So let I be a set, and $(X_i)_{i\in I}$ a collection of open sets in this context. Then
     $$\forall i \in I, x \in X_i : \exists m_x \in \mathbb{Z}_{>0} : (x + m_x\mathbb{Z})\backslash\{0\} \subseteq X_i \tag{5}$$
     Let $x \in \bigcup_{i\in I} X_i$. Then there exists $j \in I$ such that $x \in X_j$. X_j is open, so there exists $m_x \in \mathbb{Z}_{>0}$ such that $(x + m_x\mathbb{Z})\backslash\{0\} \subseteq X_j \subseteq \bigcup_{i\in I} X_i$. So $\bigcup_{i\in I} X_i$ is open.

Now suppose I is finite. Let $y \in \bigcap_{i\in I} X_i$. For all $i \in I$, let $m_i \in \mathbb{Z}_{>0}$ such that $(y+m_i\mathbb{Z})\backslash\{0\} \subseteq X_i$. Let m be a common multiple of the (finitely many!) m_i. So for all $i \in I$ there exists $a_i \in \mathbb{Z}$ such that $m = a_i m_i$. Since for all $i \in I$, $m_i > 0$, also $a_i > 0$ for all $i \in I$. Now for all $i \in I$ we have that $(y + a_i m_i\mathbb{Z})\backslash\{0\} = (y + m\mathbb{Z})\backslash\{0\} \subseteq (y + m_i\mathbb{Z})\backslash\{0\} \subseteq X_i$, so $(y + m\mathbb{Z})\backslash\{0\} \subseteq \bigcap_{i\in I} X_i$. So $\bigcap_{i\in I} X_i$ is open. Also $\emptyset$ and $\mathbb{Q}^\times$ clearly satisfy (4). So (4) defines a topology on $\mathbb{Q}^\times$.

- Next, we need to show that the multiplication $\cdot$ is continuous with respect to (4). To do this, note that for an open $X \subseteq \mathbb{Q}^\times$ we can write

$$X = \bigcup_{x\in X} (x + m_x\mathbb{Z})\backslash\{0\} \tag{6}$$

by (4). So

$$\{(x + m\mathbb{Z})\backslash\{0\} | x \in \mathbb{Q}^\times, m \in \mathbb{Z}\} \tag{7}$$

is a base of the topology. The continuity of a map only needs to be checked on a base by Lemma A.1 . So, let $a \in \mathbb{Z}_{>0}$ and $b = \frac{r}{s} \in \mathbb{Q}^\times$, where $r, s \in \mathbb{Z}$ (and nonzero). We need to show that

$$\cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\}) = \{(c, c^{-1}(ax + b) : c \in \mathbb{Q}^\times, x \in \mathbb{Z}, ax + b \neq 0\} \tag{8}$$

is open in $\mathbb{Q}^\times \times \mathbb{Q}^\times$.
Let $(c, d) \in \cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\})$. Then there exists $x \in \mathbb{Z}$ such that $d = c^{-1}(ax + b)$. Write $c = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{\neq 0}$. Then we have

$$(c, d) = \left(\frac{p}{q}, \frac{q}{p}\left(ax + \frac{r}{s}\right)\right) = \left(\frac{p}{q}, \frac{axq}{p} + \frac{qr}{ps}\right) = \left(\frac{p}{q}, \frac{axqs + qr}{ps}\right) \tag{9}$$

Let $z \in \mathbb{Z}$. Then

$$\left(\frac{p}{q} + apsz\right)\left(\frac{axqs + qr}{ps}\right) = ax + b + a^2xqsz + aqrz = a(x + axqsz + qrz) + b \in (a\mathbb{Z} + b)\backslash\{0\} \tag{10}$$

Moreover, we have for all $y \in \mathbb{Z}$

$$\left(\frac{p}{q} + apsz\right)\left(\frac{axqs + qr}{ps} + aqy\right) = ax + b + a^2xqsz + aqrz + apy + a^2pszqy \tag{11}$$

$$= a(x + axqsz + qrz + py + apszqy) + b \in (a\mathbb{Z} + b)\backslash\{0\} \tag{12}$$

meaning that there exist $\alpha, \beta \in \mathbb{Z}$ such that for all $y, z \in \mathbb{Z}$, $(c + \alpha z, d + \beta y) \in \cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\})$, or

$$((c + \alpha\mathbb{Z}) \times (d + \beta\mathbb{Z}))\backslash\{(0, 0)\} \subseteq \cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\}) \tag{13}$$

Since (c, d) was arbitrary, we conclude

$$\bigcup_{(c,d)\in\cdot^{-1}((a\mathbb{Z}+b)\backslash\{0\}} ((c + \alpha_c\mathbb{Z}) \times (d + \beta_d\mathbb{Z}))\backslash\{(0, 0)\} \subseteq \cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\}) \tag{14}$$

where the α_c and β_d are chosen suitably, as shown in (13). Since the other inclusion is obviously true, we conclude equality in (14), meaning $\cdot^{-1}((a\mathbb{Z} + b)\backslash\{0\})$ is open in $\mathbb{Q}^\times \times \mathbb{Q}^\times$. So $\cdot$ is continuous.

- Finally, we need to show that the inversion $i : \mathbb{Q}^\times \to \mathbb{Q}^\times$ mapping $\frac{p}{q}$ to $\frac{q}{p}$ is not continuous. Obviously, $\mathbb{Z}\backslash\{0\}$ is open with respect to (4), we can take $m = 1$. However, $i^{-1}(\mathbb{Z}\backslash\{0\})$ is bounded by 1, since for any $c \in \mathbb{Z}\backslash\{0\}$ we have $|c| \geq 1$, so $|\frac{1}{c}| \leq 1$. So $i^{-1}(\mathbb{Z}\backslash\{0\})$ is not open with respect to (4).

□

3 Properties of Topological Groups

Let's look at some properties of topological groups. From a group theory point of view, for any group G and $g \in G$, we have that the left multiplication sending $h \in G$ to gh, the right multiplication sending $h \in G$ to hg and the inversion map sending $h \in G$ to h^{-1} are bijections whose inverses are $h \mapsto g^{-1}h$, $h \mapsto hg^{-1}$ respectively the inversion map itself. If G has a structure of a topological group, these maps are homeomorphisms.

Lemma 3.1. *Let (G, T, m) be a topological group.*

1. *The inversion map*

$$\begin{array}{rccc} i: & G & \to & G \\ & h & \mapsto & h^{-1} \end{array} \tag{15}$$

is a homeomorphism.

2. *For any $g \in G$, the left multiplication*

$$\begin{array}{rccc} {}_g m: & G & \to & G \\ & h & \mapsto & m(g,h) = gh \end{array} \tag{16}$$

is a homeomorphism.

3. *For any $g \in G$, the right multiplication*

$$\begin{array}{rccc} m_g: & G & \to & G \\ & h & \mapsto & m(h,g) = hg \end{array} \tag{17}$$

is a homeomorphism.

Proof. 1. Since $i \circ i = \mathrm{id}_G$, i is the continuous inverse of i.

2. Since g is any element of G, if ${}_g m$ is continuous, then its inverse ${}_{g^{-1}}m$ is also continuous. So it suffices to show that ${}_g m$ is continuous. Let ${}_g i : G \to G \times G$ be defined as ${}_g i(h) = (g, h)$ for all $h \in G$. Then ${}_g m = m \circ {}_g i$ since

$$m({}_g i(h)) = m(g,h) = gh = {}_g m(h) \tag{18}$$

for all $h \in G$. Since m is continuous, it suffices to show that ${}_g i$ is continuous. Let c_g be the constant, hence continuous, map $h \mapsto g$, then

$${}_g i = (c_g, \mathrm{id}_G) \tag{19}$$

in the notation of Lemma A.2, and since both the constant map c_g and id_g are continuous, so is ${}_g i$.

3. Same argument as (2).

□

Hence, given a subset A of a topological group, the translations gA and Ag and the set A^{-1} of its inverses are homeomorphic to the original set:

Corollary 3.2. *Let (G, T, m) be a topological group, $A \subseteq G$ a subset, and $g \in G$ an element.*

1. *Let $Ag = \{ag : a \in A\}$. Then the right multiplication m_g maps A to Ag homeomorphically.*
2. *Let $gA = \{ga : a \in A\}$. Then the left multiplication ${}_g m$ maps A to gA homeomorphically.*
3. *Let $A^{-1} = \{a^{-1} : a \in A\}$. Then the inversion map $i : G \to G$ maps A to A^{-1} homeomorphically.*

Proof. The maps ${}_g m$, m_g and i are homeomorphisms, so their restrictions to A are homeomorphisms onto their image. □

This means that to show that a set A is open, closed, compact, Hausdorff, connected, path connected or any property that is invariant under homeomorphism, it suffices to show that the translation gA or Ag has the particular property, for *any* $g \in G$, or even the set of its inverses A^{-1}.

Next, we can note that the repeated multiplication in a topological group is continuous.

Lemma 3.3. *Let (G, T, m) be a topological group. Then the map*

$$\begin{array}{rccl} m_3: & G \times G \times G & \to & G \\ & (a, b, c) & \mapsto & abc = m(a, m(b, c)) = m(m(a, b), c) \end{array} \tag{20}$$

is continuous.

Proof. By Lemma A.2,

$$\begin{array}{rccl} (m, \mathrm{id}_G): & G \times G \times G & \to & G \times G \\ & (a, b, c) & \mapsto & (m(a, b), c) \end{array} \tag{21}$$

is continuous, so $m_3 = m \circ (m, \mathrm{id}_G)$ is continuous. □

Repeating this argument, we get that repeated multiplication is continuous:

Corollary 3.4. *Let (G, T, m) be a topological group. Then for any $n \in \mathbb{Z}_{\geq 2}$ the n-fold multiplication*

$$\begin{array}{rccl} m_n: & G \times G \times G \times \ldots \times G & \to & G \\ & (g_1, g_2, \ldots, g_n) & \mapsto & g_1 g_2 \ldots g_n \end{array} \tag{22}$$

is continuous.

Proof. Induction on n. For $n = 2$ we have m which is continuous. Induction hypothesis: m_{n-1} is continuous. We can now repeat the proof of Lemma 3.3 to see that m_n is continuous. □

In the case of a connected topological group, we get that any discrete normal subgroup is central, so in particular abelian.

Proposition 3.5. *Let (G, T, m) be a connected topological group, and $N \subseteq G$ a discrete normal subgroup. Then N is central.*

Proof. Since N is normal, any $n \in N$ gives rise to a well defined map

$$\begin{array}{rccl} f_n: & G & \to & N \\ & g & \mapsto & gng^{-1} \end{array} \tag{23}$$

The map

$$\begin{array}{rccl} \tilde{\sigma}_n: & G & \to & G \times G \times G \\ & g & \mapsto & (g, n, g^{-1}) \end{array} \tag{24}$$

which can be written as $\tilde{\sigma}_n = (\mathrm{id}_G, c_n, i)$ where c_n is the constant map $g \mapsto n$, is continuous by Lemma A.2, so by Lemma 3.3, $f_n = m_3 \circ \tilde{\sigma}_n$ is continuous. Since G is connected, $\mathrm{im}(f_n)$ is connected. Because N is discrete, this means that f_n is constant with image $\{n\}$. Hence, $gng^{-1} = n$ for all $n \in N$ and $g \in G$, so N is central. □

In particular, discrete normal subgroups of connected topological groups are abelian:

Corollary 3.6. *Let (G, T, m) be a connected topological group, and $N \subseteq G$ a discrete normal subgroup. Then N is abelian.* □

The following example will be useful to study the Spin Cover, which will be discussed in Section 7.

Example 3.7. Consider S^3 as the set of unit quaternions:

$$\begin{array}{rccl} \beta: & S^3 & \to & U = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\} \\ & (p,q,r,s) & \mapsto & p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k} \end{array} \tag{25}$$

Now, since U closed under multiplication and taking inverses by [Schwartz, Lemma 1.2], U is a group. Hence, through β, S^3 can be given a group structure. Moreover, S^3 has a natural topology defined by taking the subspace topology of the Euclidean topology of $\mathbb{R}^4$. We can transfer this topology to U through β. Since the multiplication on U is defined by multiplications and additions in $\mathbb{R}$ on each coordinate the multiplication on U is continuous. A similar argument shows that division in U is contunuous, so taking inverses in U is continuous. This means that $S^3 \cong U$ is a topological group in this way. The subgroup

$$Q_8 = \{1, -1, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\} \tag{26}$$

is not abelian, so by Corollary 3.6 Q_8 is not a normal subgroup of $S^3 \cong U$.

3.1 (Path) Connected Component of the Identity Element is a Normal Subgroup

It turns out that the connected and path connected components of the identity element of a topological group are normal subgroups. We will prove this in the following steps.

Proposition 3.8. *Let (G, T, m) be a topological group.*

1. *Let G_0 be the path connected component of the identity element $e \in G$. Then G_0 is a subgroup.*
2. *Let H_0 be the connected component of $e \in G$. Then H_0 is a subgroup.*

Proof. (1) Consider

$$m|_{G_0 \times G_0} : G_0 \times G_0 \to G \tag{27}$$

G_0 is path connected, so $G_0 \times G_0$ is path connected by Lemma A.3 in the appendix. As m is continuous, $m(G_0 \times G_0)$ is path connected. Furthermore $m(e, e) = e \in G_0$, so $m(G_0 \times G_0) \subseteq G_0$ since G_0 is a path connected component. Also, the inversion i is continuous and $i(e) = e$, so by the same argument $i(G_0) \subseteq G_0$. So G_0 is a subgroup.
(2): same argument, replacing 'path connected' by 'connected', again using Lemma A.3. □

Proposition 3.9. *Let (G, T, m) be a topological group, let $g \in G$ and let ${}_gm$ and m_g be the left respectively right multiplications by g, as in* (16) *and* (17)*.*

1. *Let G_0 be the path connected component of the identity element $e \in G$. Then ${}_gm$ en m_g map G_0 to the path connected component of g homeomorphically.*
2. *Let H_0 be the connected component of $e \in G$. Then ${}_gm$ and m_g map H_0 to the connected component of g homeomorphically.*

Proof. (1) By Lemma 3.1, m_g and ${}_gm$ are homeomorphisms so map path connected components to path connected components. Furthermore

$$m_g(e) = eg = g = ge = {}_gm(e)$$

so $g \in m_g(G_0) = G_0 g$ and $g \in {}_gm(G_0) = gG_0$.
(2): same argument. □

Now we can conclude these subgroups are normal:

Corollary 3.10. *The subgroups G_0 en H_0 as in Propositions 3.8 and 3.9 are normal.*

Proof. Distinct (path)connected components are disjoint, and for all $g \in G$ we have $g \in gG_0 \cap G_0g$ and $g \in H_0g \cap gH_0$, so we have $gG_0 = G_0g$ and $gH_0 = H_0g$ for all $g \in G$. This is exactly to say that G_0 and H_0 are normal. □

Remark 3.11. By Lemma 2.1, G_0 and H_0 are normal sub-topological groups of G.

Corollary 3.12. *The connected component of the identity element of a topological group is a closed normal subgroup.*

Proof. Connected components are closed. By Corollary 3.10 the connected component of the identity element is a normal subgroup. □

4 The Morphisms

A morphism of topological groups is a continuous group homomorphism.

Definition. Let G, H be topological groups. A map $f : G \to H$ is a topological group homomorphism if f is a group homomorphism and f is continuous.

Definition. Let G, H be topological groups. A map $f : G \to H$ is a topological group isomorphism if f is a topological group homomorphism and there exists a topological group homomorphism $g : H \to G$ such that $g \circ f = id_G$ and $f \circ g = id_H$. If G, H are topological groups such that there exists a topological group isomorphism between them, then G, H are isomorphic.

Remark 4.1. A topological group isomorphism is thus a homeomorphism and a group isomorphism.

Remark 4.2. In order to show that a map $f : G \to H$ between topological groups is a topological group isomorphism, it is not necessary to show both homeomorphism and group isomorphism, since one of the two already gives bijectivity. Hence, if you have homeomorphism, then group homomorphism is enough. Conversely, if you have group isomorphism, continuity and openness (or closedness) is enough.

Proposition 4.3. *Let G, H, K be topological groups and $f : G \to H$,$g : H \to K$ topological group homomorphisms. Then $g \circ f : G \to K$ is a topological group homomorphism.*

Proof. Since f and g are continuous, so is $g \circ f$. Also, since f and g are group homomorphisms, so is $g \circ f$. □

So, as with continuous maps and homomorphisms, compositions of topological group homomorphisms are topological group homomorphisms.

Lemma 4.4. *Let G be an abelian topological group. Then the inversion map $i : G \to G$ is a topological group isomorphism.*

Proof. By Lemma 3.1 i is a homeomorphism. Moreover,

$$i(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = i(g)i(h) \tag{28}$$

since G is abelian. Finally, i is a group isomorphism by Remark 4.2. □

Proposition 4.5. *Let G be a topological group. Then any conjugation map $\sigma_h : G \to G, g \mapsto hgh^{-1}$ is a topological group isomorphism.*

Proof. A conjugation map σ_h is an inner automorphism of the group G, so in particular a group isomorphism. Moreover, σ_h is the composition of ${}_hm$ and $m_{h^{-1}}$:

$$g \overset{{}_hm}{\longmapsto} hg \overset{m_{h^{-1}}}{\longmapsto} hgh^{-1} \tag{29}$$

so a homeomorphism by Lemma 3.1. □

Let's look at kernels:

Definition. Let G, H be topological groups and $f : G \to H$ a topological group homomorphism. The kernel of f, denoted $\ker(f)$, is the kernel of the group homomorphism f.

Remark 4.6. By Lemma 2.1, $\ker(f)$ and $\mathrm{im}(f)$ are sub-topological groups of G respectively H. Moreover, $\ker(f)$ is a normal sub-topological group of G.

5 The Fundamental Group of a Topological Group

The notion of fundamental group of a topological space requires a choice of base point. In the case of a topological group (G, T, m), there is a natural base point, namely the identity element e of the group (G, m). This leads to the following definition.

Definition. Let (G, T, m) be a topological group. The fundamental group of (G, T, m) is $\pi_1((G, T), e)$ where e is the identity element of (G, m).

Notation. When T and m are clear, we usually say the fundamental group of G, denoted $\pi_1(G)$.

Notation. For a loop γ with base point e, we write $[\gamma]$ for the class of γ in $\pi_1(G)$.

It is not immediately clear that the identity element e is an interesting choice for the base point for the fundamental group. The following lemma will make this more clear.

Lemma 5.1. *Let (G, T, m) be a topological group with identity element e. Let $\alpha, \beta, \gamma, \delta : [0, 1] \to G$ be loops with base point e, such that α is path homotopic to β and γ is path homotopic to δ. Then the pointwise multiplications $\alpha \cdot \gamma = m \circ (\alpha, \gamma)$ mapping $s \in [0, 1]$ to $\alpha(s)\gamma(s) = m(\alpha(s), \gamma(s))$ and $\beta \cdot \delta$ mapping $s \in [0, 1]$ to $\beta(s)\delta(s)$ are path homotopic.*

Proof. First note that since m is continuous and also (α, β) and (γ, δ) are continuous by Lemma A.2, $\alpha \cdot \gamma = m \circ (\alpha, \gamma)$ and $\beta \cdot \delta = m \circ (\beta, \delta)$ are actually paths (meaning they are continuous) with base point e, so path homotopy makes sense here. Let $A : [0, 1]^2 \to G$ be a path homotopy from α to β and $B : [0, 1]^2 \to G$ a path homotopy from γ to δ. Then the pointwise multiplication $C : [0, 1]^2 \to G$ mapping (s, t) to $m(A(s, t), B(s, t)) = A(s, t)B(s, t)$ is a path homotopy from $\alpha \cdot \gamma$ to $\beta \cdot \delta$, since

$$C(s, 0) = A(s, 0)B(s, 0) = \alpha(s)\gamma(s) \tag{30}$$
$$C(s, 1) = A(s, 1)B(s, 1) = \beta(s)\delta(s) \tag{31}$$
$$C(0, t) = A(0, t)B(0, t) = ee = e \tag{32}$$
$$C(1, t) = A(1, t)B(1, t) = ee = e \tag{33}$$

and C is continuous by the same argument as in the beginning of this proof, so C is a path homotopy from $\alpha \cdot \gamma$ to $\beta \cdot \delta$. □

Alternative proof using fundamental groups. We need to show that if $[\alpha] = [\beta]$ and $[\gamma] = [\delta]$ then $[\alpha \cdot \gamma] = [\beta \cdot \delta]$. Note

$$[\alpha \cdot \gamma] = [m \circ (\alpha, \gamma)] = m_*[(\alpha, \gamma)] \tag{34}$$

and by the same argument $[\beta \cdot \delta] = m_*[(\beta, \delta)]$. Since $[\alpha] = [\beta]$ and $[\gamma] = [\delta]$ we get $[(\alpha, \gamma)] = [(\beta, \delta)]$. Also $m_* : \pi_1(G \times G) \to \pi_1(G)$ is well defined, so

$$[\alpha \cdot \gamma] = m_*[(\alpha, \gamma)] = m_*[(\beta, \delta)] = [\beta \cdot \delta] \tag{35}$$

completing the proof. □

Remark 5.2. In the first (direct) proof, the base point being e is very important, as (32) and (33) would not necessarily hold for a different base point. However, e is not used in the second proof. This is because this proof generalizes to the case $m_* : \pi_1(G \times G, (g, g)) \to \pi_1(G, g^2)$ for any $g \in G$. But this only defines an operation if $g = e$, hence the choice of e for the base point.

Let (G, T, m) be a topological group. Consider $m_* : \pi_1(G \times G) \to \pi_1(G)$. By [Fulton, Exercise 12.9] we have

$$\pi_1(G \times G) \cong \pi_1(G) \times \pi_1(G) \tag{36}$$

canonically, where $[(\alpha, \beta)]$ corresponds to $([\alpha], [\beta])$. This gives

$$m_* : \pi_1(G) \times \pi_1(G) \to \pi_1(G) \tag{37}$$

defined by $m_*([\alpha],[\beta]) = m_*[(\alpha,\beta)] = [\alpha\cdot\beta]$. On the other hand, denote by

$$\odot : \pi_1(G)\times\pi_1(G)\to\pi_1(G) \tag{38}$$

the group operation on $\pi_1(G)$, defined by concatenation. These turn out to be the same.

Proposition 5.3. *Let (G,T,m) be a topological group with inversion $i : G\to G$ and identity element e.*

1. *Let $\odot$ be the group operation of $\pi_1(G)$. Then $m_* = \odot$.*
2. *Let $\iota : \pi_1(G)\to\pi_1(G)$ be the inversion map in $\pi_1(G)$. Then $i_* = \iota$.*

Proof. 1. We need to show that if γ,δ are two loops with base point e, then there exists a path homotopy from $\gamma\cdot\delta$ (with the notation of Lemma 5.1) to the concatenation $\gamma\odot\delta$.
Let E be the constant loop with image $\{e\}$. Then $[E]$ is the identity element of $\pi_1(G)$, so $[\gamma\odot E] = [\gamma]$ and $[E\odot\delta] = [\delta]$. By Lemma 5.1 we get

$$[(\gamma\odot E)\cdot(E\odot\delta)] = [\gamma\cdot\delta] \tag{39}$$

On the other hand,

$$(\gamma\odot E)\cdot(E\odot\delta)(s) = \begin{cases}\gamma(2s)e = \gamma(2s) & s\le\frac{1}{2}\\ e\delta(2s-1) = \delta(2s-1) & s\ge\frac{1}{2}\end{cases} \tag{40}$$

$$= (\gamma\odot\delta)(s) \tag{41}$$

so $[\gamma\cdot\delta] = [(\gamma\odot E)\cdot(E\odot\delta)] = [\gamma\odot\delta]\ (= [\gamma]\odot[\delta])$, which proves the result.

2. Let γ be a loop with base point e. On the one hand, we have

$$\iota[\gamma]\odot[\gamma] = [E] \tag{42}$$

On the other hand

$$(i\circ\gamma)\cdot\gamma = m\circ(i\circ\gamma,\gamma) = E \tag{43}$$

so $\iota[\gamma]\odot[\gamma] = [E] = m_*(i_*([\gamma]),[\gamma]) = i_*([\gamma])\cdot[\gamma] \overset{(1)}{=} i_*([\gamma])\odot[\gamma]$ by part 1. So i_* is the inversion map in $\pi_1(G)$, and therefore equal to ι.

□

Remark 5.4. This is quite interesting: it relates the group operation on a topological group to the group operation on its fundamental group. This has the following consequence:

5.1 The Fundamental Group is Abelian

The fundamental group of a topological group turns out to be abelian. This is a fundamental result, we will discuss various proofs.

Theorem 5.5. *Let (G,T,m) be a topological group with identity element e. Then its fundamental group $\pi_1(G)$ is an abelian group.*

5.1.1 Direct Proof

Let γ,δ be loops with base point e and let E be the constant loop with base point e. By (40) in the proof of Proposition 5.3 we have $(\gamma\odot E)\cdot(E\odot\delta) = \gamma\odot\delta$. Similarly,

$$(E\odot\gamma)\cdot(\delta\odot E)(s) = \begin{cases}e\delta(2s) = \delta(2s) & s\le\frac{1}{2}\\ \gamma(2s-1)e = \gamma(2s-1) & s\ge\frac{1}{2}\end{cases} \tag{44}$$

$$= (\delta\odot\gamma)(s) \tag{45}$$

so $(E\odot\gamma)\cdot(\delta\odot E) = \delta\odot\gamma$. Hence,

$$[\gamma\odot\delta] = [\gamma]\odot[\delta] = ([E]\odot[\gamma])\odot([\delta]\odot[E]) = [E\odot\gamma]\odot[\delta\odot E] = [(E\odot\gamma)\cdot(\delta\odot E)] = [\delta\odot\gamma] \tag{46}$$

5.1.2 Proof using Hilton-Eckmann Argument

This proof uses the Hilton-Eckmann Argument [Eckmann/Hilton]:

Proposition 5.6 (Hilton-Eckmann Argument)**.** *Let X be a set equipped with two binary operations m_1, m_2, such that*

1. *m_1 and m_2 both have identity elements e_1 and e_2 respectively.*
2. *For all $a, b, c, d \in X$ we have*

$$m_2(m_1(a,b), m_1(c,d)) = m_1(m_2(a,c), m_2(b,d)) \tag{47}$$

Then $m_1 = m_2$ and they are commutative and associative.

Proof. First note that $e_1 = e_2$:

$$e_1 = m_1(e_1,e_1) = m_1(m_2(e_2,e_1), m_2(e_1,e_2)) = m_2(m_1(e_2,e_1), m_1(e_1,e_2)) = m_2(e_2,e_2) = e_2 \tag{48}$$

So we can write $e = e_1 = e_2$. Let $a, b \in X$:

$$m_1(a,b) = m_1(m_2(a,e), m_2(e,b)) = m_2(m_1(a,e), m_1(e,b)) = m_2(a,b) \tag{49}$$

so $m_1 = m_2$. Write $m = m_1 = m_2$. For commutativity,

$$m(a,b) = m(m(e,a), m(b,e)) = m(m(e,b), m(a,e)) = m(b,a) \tag{50}$$

and for associativity, let $c \in X$,

$$m(m(a,b),c) = m(m(a,b), m(e,c)) = m(m(a,e), m(b,c)) = m(a, m(b,c)) \tag{51}$$

which finishes the proof. □

Now we need to show that the two Hilton-Eckmann conditions hold for m_* and $\odot$.

1. Identities: $\odot$ has an identity, since it is a group operation. Moreover, if E is the constant loop with image e, then $[E]$ is the identity for m_*, since for any loop γ with base point e we have $(\gamma \cdot E)(s) = \gamma(s)E(s) = \gamma(s)e = \gamma(s)$ and $(E \cdot \gamma) = E(s)\gamma(s) = e\gamma(s) = \gamma(s)$.
2. Interchange: let $[\alpha], [\beta], [\gamma], [\delta] \in \pi_1(G)$. Then

$$((\alpha \cdot \beta) \odot (\gamma \cdot \delta))(s) = \begin{cases} \alpha(2s)\beta(2s) & s \leq \frac{1}{2} \\ \gamma(2s-1)\delta(2s-1) & s \geq \frac{1}{2} \end{cases} = (\alpha \odot \gamma) \cdot (\beta \odot \delta)(s) \tag{52}$$

   so not only $([\alpha] \cdot [\beta]) \odot ([\gamma] \cdot [\delta]) = ([\alpha] \odot [\gamma]) \cdot ([\beta] \odot [\delta])$, but this is even an equality on the level of loops.

So the pair $(m_*, \odot)$ satisfies the Hilton-Eckmann conditions, hence are the same, commutative and associative. In particular, $\odot$ is commutative, which was to be shown.

Remark 5.7. This proof also proves the first part of Proposition 5.3.

5.1.3 Proof Sketch using Category Theory

This will be a sketch of the proof. In this section, denote by **Grp** the category of groups, **Top** the category of topological spaces and **Set** the category of sets.
Let $\mathcal{C}$ be a category, and X an object of $\mathcal{C}$. Then we obtain two functors $h_X, h^X : \mathcal{C} \to \textbf{Set}$ sending an object T to $\mathrm{Hom}(X,T)$ respectively $\mathrm{Hom}(T,X)$ in **Set**, and sending a morphism $f : T \to U$ to the map that composes with f from the appropriate side.

Definition. Let X be an object of a category $\mathcal{C}$. Then X is a group object if one of the following conditions is satisfied:

1. There exists a functor $\tilde{h}_X : \mathcal{C} \to \mathbf{Grp}$ such that the diagram

$$\begin{array}{ccc} & & \mathbf{Grp} \\ & \tilde{h}_X \nearrow & \downarrow \\ \mathcal{C} & \xrightarrow{h_X} & \mathbf{Set} \end{array} \tag{53}$$

commutes, where $\mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor.

2. There exists a functor $\tilde{h}^X : \mathcal{C} \to \mathbf{Grp}$ such that the diagram

$$\begin{array}{ccc} & & \mathbf{Grp} \\ & \tilde{h}^X \nearrow & \downarrow \\ \mathcal{C} & \xrightarrow{h^X} & \mathbf{Set} \end{array} \tag{54}$$

commutes, where $\mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor.

Proposition 5.8. *A group object in* ***Top*** *is precisely a topological group.*

Remark 5.9. This is the reason we do not discuss paratopological groups.

Sketch proof. Let (G, T, m) be a topological group with identity element e and let X be a topological space. Let $f, g : X \to G$ be continuous maps. By Lemma A.2, $(f, g) : X \to G \times G$ is continuous, so $m \circ (f, g)$ is continuous. This gives a map

$$\begin{array}{rccc} m_X : & \mathrm{Hom}_{\mathbf{Top}}(X, G) \times \mathrm{Hom}_{\mathbf{Top}}(X, G) & \to & \mathrm{Hom}_{\mathbf{Top}}(X, G) \\ & (f, g) & \mapsto & m \circ (f, g) \end{array} \tag{55}$$

The verification that $(\mathrm{Hom}_{\mathbf{Top}}(X, G), m_X)$ is a group is left to the reader. It now easily follows that for any topological space S and continuous map $c : S \to X$, the implied map c^* defined by composition from the right by c is a group homomorphism.
For the other implication, we will use Yoneda's Lemma:

Lemma 5.10 (Yoneda)**.** *Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and $h^X : \mathcal{C} \to$* ***Set*** *the functor as in* (54)*. Then for any functor $\mathcal{F} : \mathcal{C} \to$* ***Set*** *there exists a functorial bijection*

$$\mathrm{Hom}_{Fun}(h^X, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X) \tag{56}$$

Proof: see [Etingof, p183-184].

Example 5.11. For $\mathcal{F} = h^Y$ for some object Y of $\mathcal{C}$, we get

$$\mathrm{Hom}_{Fun}(h^X, h^Y) \xrightarrow{\sim} h^Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \tag{57}$$

Now let G be a group object in **Top**. Then for every topological space X there exists a group operation $m_X : \mathrm{Hom}_{\mathbf{Top}}(X, G) \times \mathrm{Hom}_{\mathbf{Top}}(X, G) \to \mathrm{Hom}_{\mathbf{Top}}(X, G)$ and for every continuous map $c : S \to X$ the induced map $c^* : \mathrm{Hom}_{\mathbf{Top}}(X, G) \to \mathrm{Hom}_{\mathbf{Top}}(S, G)$ sending ϕ to $\phi \circ c$ is a group homomorphism.
We apply (57) to $\mathrm{Hom}_{Fun}(h^{G \times G}, h^G)$ to obtain a continuous map $G \times G \to G$. For any topological spaces X, Y and any continuous map $f : Y \to X$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Top}}(X, G \times G) = \mathrm{Hom}_{\mathbf{Top}}(X, G) \times \mathrm{Hom}_{\mathbf{Top}}(X, G) = h^{G \times G}(X) & \xrightarrow{m_X} & h^G(X) = \mathrm{Hom}_{\mathbf{Top}}(X, G) \\ (f, f)_* \big\downarrow & & \big\downarrow f_* \\ \mathrm{Hom}_{\mathbf{Top}}(Y, G \times G) = \mathrm{Hom}_{\mathbf{Top}}(Y, G) \times \mathrm{Hom}_{\mathbf{Top}}(Y, G) = h^{G \times G}(Y) & \xrightarrow{m_Y} & h^G(Y) = \mathrm{Hom}_{\mathbf{Top}}(Y, G) \end{array} \tag{58}$$

commutes since $f_* : h^G(X) \to h^G(Y)$ is a group homomorphism. So this indeed gives a functorial morphism $h^{G \times G} \to h^G$, which by Yoneda's Lemma can be identified with a continuous map $m : G \times G \to G$.
Again, the map m is indeed a group operation, with continuous inversion $i : G \to G$. □

The next statement can be proved using the Hilton Eckmann Argument:

Proposition 5.12. *The group objects in **Grp** are exactly the abelian groups.*

Sketch proof. Let G be a group object in **Grp**. Then the multiplication on G and the induced multiplication on G through Yoneda's Lemma satisfy the Hilton-Eckmann conditions, so are the same and commutative. On the other hand, if G is an abelian group, G is a group so any $\mathrm{Hom}(H, G)$ is a group. □

Finally, the functor π_1 sending a based topological group to its fundamental group sends group objects to group objects, so topological groups to abelian groups.

6 Covering Groups

The notion of covering maps in topology can be transferred to the case of topological groups in a compatible way, given some conditions. The main result that shows this uses the Existence and Uniqueness of Lifts theorems, a powerful tool in algebraic topology.

6.1 The Existence and Uniqueness of Lifts Theorems

The Existence and Uniqueness of Lifts theorems [Fulton, Lemma 11.5 and Proposition 13.5] provide a useful way to create continuous maps between topological spaces and determine them uniquely up to a choice in a fiber.

Theorem 6.1 (Uniqueness of Lifts)**.** *Let X, Y, Z be topological spaces, with Z connected. Let $p : Y \to X$ be a covering map and $q : Z \to X$ a continuous map.*

$$\begin{array}{ccc} & & Y \\ & f \nearrow g & \Big\downarrow p \\ Z & \xrightarrow{q} & X \end{array} \tag{59}$$

Suppose $f, g : Z \to Y$ are continuous and $q = p \circ g = p \circ f$, and suppose there exists $z \in Z$ such that $f(z) = g(z)$. Then $f = g$.

Proof. See [Fulton, Lemma 11.5]. □

Theorem 6.2 (Existence of Lifts)**.** *Let X, Y, Z be topological spaces, with Z connected and locally path connected. Let $x \in X$, $y \in Y$, $z \in Z$ and let $p : Y \to X$ be a covering map and $q : Z \to X$ a continuous map, such that $p(y) = q(z) = x$. Suppose $q_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(Y, y))$ in $\pi_1(X, x)$. Then there exists a continuous map $f : Z \to Y$ such that $f(z) = y$ and $p \circ f = q$.*

Proof. See [Fulton, Proposition 13.5]. □

6.2 The Theorem of the Covering Group

The Theorem of the Covering Group states that if we have a covering map from a 'nice' topological space into a topological group then we can transfer the topological group structure. The precise statement of the theorem is as follows.

Theorem 6.3 (Covering Group)**.** *Let (G, T, m) be a topological group. Let $p : (G', T') \to (G, T)$ be a covering map where (G', T') is connected and locally path connected. Let $e \in G$ be the identity element and $e' \in G'$ such that $p(e') = e$. Then there exists a unique map $m' : G' \times G' \to G'$ such that (G', T', m') is a topological group with identity element e' and p a group homomorphism.*

Proof. Consider the diagram

$$\begin{array}{ccc} G' \times G' & \dashrightarrow & G' \\ {\scriptstyle (p,p)}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{m} & G \end{array} \tag{60}$$

where (p,p) is the map that sends (x,y) to $(p(x),p(y))$. This map is continuous by Lemma A.2 in the appendix. So the map $m \circ (p,p)$ is now a continuous map from a connected and locally path connected (Lemma A.3) space into G. On the level of fundamental groups this leads to the following diagram:

$$\begin{array}{ccc} & & \pi_1(G',e') \\ & & \downarrow{\scriptstyle p_*} \\ \pi_1(G' \times G', (e',e')) & \xrightarrow{(m\circ(p,p))_*} & \pi_1(G,e) \end{array} \tag{61}$$

Now the Existence of Lifts Theorem states that if

$$(m \circ (p,p))_*(\pi_1(G' \times G', (e',e'))) \subseteq p_*(\pi_1(G',e')) \tag{62}$$

then there exists a continuous map $m' : G' \times G' \to G'$ such that $p \circ m' = m \circ (p,p)$. The Uniqueness of Lifts Theorem states that in this case there exists a unique continuous map $m' : G' \times G' \to G'$ such that $p \circ m' = m \circ (p,p)$ and $m'(e',e') = e'$. So we need to show (62), we do this in the following steps:

1. We have $(m \circ (p,p))_* = m_* \circ (p,p)_*$.
2. By (36), $\pi_1(G' \times G', (e',e')) \cong \pi_1(G',e') \times \pi_1(G',e')$.
3. This gives

$$(m \circ (p,p))_*(\pi_1(G' \times G', (e',e'))) = m_*((p,p)_*(\pi_1(G' \times G', (e',e')))) \tag{63}$$
$$= m_*(p_*(\pi_1(G',e')) \times p_*(\pi_1(G',e'))) \tag{64}$$

4. Since p_* is a group homomorphism, $p_*(\pi_1(G',e'))$ is a subgroup of $\pi_1(G,e)$.
5. Denote by $\odot$ the group operation of $\pi_1(G,e)$. By Proposition 5.3, we have $m_* = \odot$. Since $\odot$ is a group operation, the statement follows.

By Existence and Uniqueness of Lifts, it follows that there is a unique continuous map $m' : G' \times G' \to G'$ such that $m \circ (p,p) = p \circ m'$ and $m'(e',e') = e'$.
From now on, denote $m'(a,b) = ab$. The map p has a 'homomorphism property' (note that we can't say that p is a homomorphism yet, because we did not yet show that (G',m') is a group):

$$p(a)p(b) = m(p(a),p(b)) = p(m'(a,b)) = p(ab) \tag{65}$$

for all $a,b \in G'$.
We now need to show that (G',m') is a group, in other words m' is associative, e' is the identity element with respect to m' and there exist inverses with respect to m'.

- Associative: we need to show that the maps

$$G' \times G' \times G' \to G', (a,b,c) \mapsto a(bc) \tag{66}$$
$$G' \times G' \times G' \to G', (a,b,c) \mapsto (ab)c \tag{67}$$

are the same. Since (e',e',e') is mapped to e' by both maps, the statement again follows from the Uniqueness of Lifts Theorem, applied to the diagram

$$\begin{array}{ccc} G' \times G' \times G' & \overset{(a,b,c)\mapsto a(bc)}{\underset{}{\overset{(a,b,c)\mapsto(ab)c}{\dashrightarrow}}} & G' \\ {\scriptstyle (p,p,p)}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G \times G & \xrightarrow{(g,h,k)\mapsto ghk} & G \end{array} \tag{68}$$

because indeed $p(a)p(b)p(c) = p(a(bc)) = p((ab)c)$ by homomorphism property of p. Here, $G' \times G' \times G'$ is connected and locally path connected by Lemma A.3.

- e' is the identity element: follows from Uniqueness of Lifts applied to the diagram

$$\begin{array}{ccc} & & G' \\ \mathrm{id}_{G'}, m'_{e'}, {}_{e'}m' & \nearrow & \Big\downarrow p \\ G' & \xrightarrow{p} & G \end{array} \tag{69}$$

where we compare the identity on G' with the maps $m'_{e'} = a \mapsto m'(a, e')$ and ${}_{e'}m' = a \mapsto m'(e', a)$. These are continuous since (for example)

$$a \overset{i_{e'}}{\longmapsto} (a, e') \overset{m'}{\longmapsto} ae' \tag{70}$$

and $i_{e'}$ is continuous by the proof of Lemma 3.1. By the same argument, also $a \mapsto m'(e', a)$ is continuous.
All three maps map e' to e', so by Uniqueness of Lifts they are the same, so e' is indeed the identity element of (G', m').

- Inverses: the inversion map $i : G \to G$ mapping $g \in G$ to its inverse g^{-1} is continuous, so we can consider the following diagram of continuous maps:

$$\begin{array}{ccc} G' & \overset{i'}{-\to} & G' \\ p\Big\downarrow & & \Big\downarrow p \\ G & \xrightarrow{i} & G \end{array} \tag{71}$$

By Proposition 5.3, we have

$$i_*(p_*(\pi_1(G', e'))) \subseteq p_*(\pi_1(G', e')) \tag{72}$$

so by Existence and Uniqueness of Lifts there exists a unique continuous map $i' : G' \to G'$ making this diagram commutative and sending e' to e'.
To show that this map i' is indeed the inversion map with respect to m', we must show that the maps $a \mapsto m'(a, i'(a))$ and $a \mapsto m'(i'(a), a)$ equal the 'null-map' $a \mapsto e'$. To show that these are continuous, let $d : G' \to G' \times G'$ be the 'diagonal' map that sends g to (g, g). Then d is continuous by Lemma A.2, first part with $f = g = id_{G'}$. Now we have

$$a \overset{d}{\longmapsto} (a, a) \overset{(id, i')}{\longmapsto} (a, i'(a)) \overset{m'}{\longmapsto} ai'(a) \tag{73}$$

so $a \mapsto m'(a, i'(a))$ is continuous, and by the same argument so is $a \mapsto m'(i'(a), a)$. We can now apply Uniqueness of Lifts to

$$\begin{array}{ccc} & a \mapsto e' & \\ & a \mapsto m'(i'(a), a) & \\ G' & \xrightarrow{a \mapsto m'(a, i'(a))} & G' \\ p\Big\downarrow & & \Big\downarrow p \\ G & \xrightarrow{g \mapsto e} & G \end{array} \tag{74}$$

Noting that $m'(e', i'(e')) = m'(e', e') = e' = m'(i'(e'), e'))$ gives the result.

So m' is a group operation, making (G', m') into a group. In particular, the 'homomorphism property' (65) of p now means p is a group homomorphism, which finishes the proof. $\square$

6.3 Properties of Covering Groups

The Theorem of the Covering Group gives rise to a notion of 'covering group'.

Definition. Let (G, T, m) be a topological group. A covering group of (G, T, m) is a topological group (G', T', m') together with a covering map $p : G' \to G$ that is a group homomorphism as well.

Covering groups have discrete kernels:

Lemma 6.4. *Let $p : G' \to G$ be a covering group. Then $\ker(p)$ is a discrete normal subgroup of G'.*

Proof. Since p is a group homomorphism, its kernel is a normal subgroup. Also, by definition of covering space, the fibers of a covering space are discrete, so in particular $\ker(p)$ is discrete. □

Is the converse also true? That is, if we have a topological group homomorphism $p : G' \to G$, surjective and $\ker(p)$ is discrete, is p then a covering group? The answer is no:

Example 6.5. Let $V_4 = \{e, a, b, c\}$ be the Kleinian group with identity element e and $C = \{1, \sigma\}$ the order 2 group, where σ has order 2. Equip V_4 with the discrete topology and C with the chaotic topology. Define

$$\begin{array}{rccc} f: & V_4 & \to & C \\ & e, b & \mapsto & 1 \\ & a, c & \mapsto & \sigma \end{array} \tag{75}$$

f is exactly the homomorphism with kernel $\langle b \rangle$ and f is continuous as a map to a chaotic topological space. So, f is a topological group homomorphism. Since V_4 is discrete, $\ker(f) = \langle b \rangle$ is discrete. f is also surjective. Now we will show that f is not a covering map. Since C is chaotic, there is only one open neighborhood of any point in C, namely C itself. Now

$$f^{-1}(C) = \langle a \rangle \cup \{b, c\} = \langle b \rangle \cup \{a, c\} = \langle c \rangle \cup \{a, b\} \tag{76}$$

$f|_{\langle b \rangle}$ and $f|_{\{a,c\}}$ are not injective so cannot be homeomorphisms. On the other hand, the inverse of the restriction of f to any of the four other size 2 sets is not constant, so not continuous. Hence, f is not a covering map.

If G' is connected, then $\ker(p)$ is central:

Corollary 6.6. *Let $p : G' \to G$ be a covering group with G' connected. Then $\ker(p)$ is central.*

Proof. Follows directly from Proposition 3.5 and Lemma 6.4. □

Central subgroups are abelian:

Corollary 6.7. *Let $p : G' \to G$ be a covering group with G' connected. Then $\ker(p)$ is abelian.* □

Moreover, this is yet another proof for the fact that the fundamental group of G is abelian, where this time G is assumed to be nice, which means that there exists a universal cover $u : \tilde{G} \to G$. This already follows from Theorem 5.5, but the proofs given below offer some extra insight.

Corollary 6.8. *Let G be a nice topological group. Then $\pi_1(G)$ is abelian.*

Proof. Let $u : \tilde{G} \to G$ be a universal cover. By the Theorem of the Covering Group, $\tilde{G}$ can be naturally equipped with a topological group structure such that u is a group homomorphism. Let d be the identity element of $\tilde{G}$ in this setting. By [Bruin, Theorem 14.1], the monodromy action of $\pi_1(G)$ on $\ker(u)$ is free and transitive. Hence by Lemma B.1 the map

$$\begin{array}{rccc} a: & \pi_1(G) & \to & \ker(u) \\ & [\gamma] & \mapsto & d * [\gamma] \end{array} \tag{77}$$

is a bijection, where $*$ is the monodromy action. It is now enough to show that a is a group homomorphism, because then $\pi_1(G)$ is isomorphic to the abelian group $\ker(u)$. So we need to show that

$$d * (\gamma \odot \delta) = (d * \gamma)(d * \delta) \tag{78}$$

for any two loops γ, δ in G with base point the identity e of G. By Lemma 5.3, $\gamma \odot \delta$ is path homotopic to the pointwise multiplication $\gamma \cdot \delta$, so

$$d * (\gamma \odot \delta) = d * (\gamma \cdot \delta) \tag{79}$$

Let $\tilde{\gamma}_d$ be the path lift of γ with begin point d and let $\tilde{\delta}_d$ be the path lift of δ with begin point d. Since u is a group homomorphism, we have for all $t \in [0, 1]$

$$u \circ (\tilde{\gamma}_d \cdot \tilde{\delta}_d)(t) = u(\tilde{\gamma}_d(t)\tilde{\delta}_d(t)) = u(\tilde{\gamma}_d(t))u(\tilde{\delta}_d(t)) \tag{80}$$

so

$$u \circ (\tilde{\gamma}_d \cdot \tilde{\delta}_d) = (u \circ \tilde{\gamma}_d) \cdot (u \circ \tilde{\delta}_d) = \gamma \cdot \delta \tag{81}$$

so $\tilde{\gamma}_d \cdot \tilde{\delta}_d$ is the path lift of $\gamma \cdot \delta$ with begin point d. Therefore,

$$(d * \gamma)(d * \delta) = (\tilde{\gamma}_d)(1)(\tilde{\delta}_d)(1) = (\tilde{\gamma}_d \cdot \tilde{\delta}_d)(1) = d * (\gamma \cdot \delta) \tag{82}$$

□

Alternative proof. As in the previous proof, let $u : \tilde{G} \to G$ be the universal covering group with identity element d. Let

$$\mathrm{Aut}(u) = \{\text{homeomorphisms } f : \tilde{G} \to \tilde{G} : u \circ f = u\} \tag{83}$$

be the automorphism group of u. By [Fulton, Corollary 13.15] there is a canonical group isomorphism

$$\rho : \pi_1(G) \xrightarrow{\sim} \mathrm{Aut}(u) \tag{84}$$

so it is enough to show that $\mathrm{Aut}(u)$ is isomorphic to $\ker(u)$. Define

$$\begin{array}{rccc} r: & \ker(u) & \to & \mathrm{Aut}(u) \\ & x & \mapsto & {}_x m \end{array} \tag{85}$$

where ${}_x m$ is the left multiplication as in (16). This is indeed a u-homeomorphism: by Lemma 3.1 ${}_x m$ is a homeomorphism and

$$u(xy) = u(x)u(y) = u(y) \tag{86}$$

since $x \in \ker(u)$. Now we need to show that r is a group isomorphism.

- Homomorphism: let $x, y \in \ker(u)$, then

  $$r(x)r(y) = {}_x m \circ {}_y m = (z \overset{{}_y m}{\longmapsto} yz \overset{{}_x m}{\longmapsto} xyz) = {}_{xy} m = r(xy) \tag{87}$$

  so r is a group homomorphism.
- Injective: Suppose ${}_x m = {}_y m$. Then for all $z \in \tilde{G}$ we have $xz = {}_x m(z) = {}_y m(z) = yz$, so $x = y$.
- Surjective: let $\phi \in \mathrm{Aut}(u)$. Define $x := \phi(d)$. Now also $x = xd = {}_x m(d)$, so by Uniqueness of Lifts $\phi = {}_x m$.

Therefore r is a group isomorphism, giving that $\rho^{-1} \circ r$ is a group isomorphism $\ker(u) \to \pi_1(G)$. Since $\ker(u)$ is abelian, so is $\pi_1(G)$. □

7 Matrix Groups

Important examples of topological groups are matrix groups, such as $GL_n(\mathbb{R})$, $SO_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$ and $GL_n(\mathbb{C})$. In this section, we will consider the real matrix groups. Let's look at some properties of these matrix groups. First of all, since any matrix group is considered as being equipped with the subspace topology of the Euclidean topology as explained below, any matrix group is Hausdorff.

7.1 The Real Square Matrices with Nonzero Determinant

We first consider

$$GL_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det(A) \in \mathbb{R}^\times\} \tag{88}$$

We define a topology on $GL_n(\mathbb{R})$ as follows. We take the Euclidean topology on $\mathbb{R}^{n^2}$, take a 'coordinate bijection' $\phi : \mathbb{R}^{n^2} \to \mathbb{R}^{n\times n}$ and transfer the topology accordingly, and finally we take the subspace topology on $GL_n(\mathbb{R})$. It is easy to see that this definition does not depend on the choice of the coordinate bijection ϕ, since the Euclidean topology on $\mathbb{R}^{n^2}$ comes from the Euclidean distance on $\mathbb{R}^{n^2}$ and this distance is invariant under the action of S_{n^2} on the set of standard basis vectors of $\mathbb{R}^{n^2}$. We claim that $GL_n(\mathbb{R})$ together with this topology and the operation of matrix multiplication yields a topological group.

Theorem 7.1. *$GL_n(\mathbb{R})$ with the above defined topology and matrix multiplication is a topological group.*

Proof. The matrix multiplication consists of taking standard scalar products of rows and columns of the two matrices. Since multiplication and addition in $\mathbb{R}$ are continuous, so is matrix multiplication. The inversion can be computed using the cofactor matrix:

$$A^{-1} = \frac{\mathrm{cof}(A)}{\det(A)} \tag{89}$$

By definition of cofactor, any entry (i, j) in $\mathrm{cof}(A)$ is determined by taking the determinant of A with some entries replaced by zeros and the (i, j)-th entry by 1, so by Lemma 7.2 taking cofactor is continuous, so taking inverse is continuous, again by Lemma 7.2, since then also $1/\det$ is continuous since it is nonzero on $GL_n(\mathbb{R})$. ☐

Lemma 7.2. *The determinant $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ is a continuous map.*

Proof. By definition of determinant, if $A = (a_{i,j})_{i,j=1}^n$:

$$\det(A) = \sum_{\sigma\in S_n} sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} ... a_{n,\sigma(n)} \tag{90}$$

and this is continuous. ☐

This immediately yields

Proposition 7.3. *The determinant $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ is a topological group homomorphism.*

Proof. By Lemma 7.2 det is continuous. Moreover, det is multiplicative hence a group homomorphism. ☐

Since det is continuous, surjective and $\mathbb{R}^\times$ is neither compact nor path connected, we obtain:

Proposition 7.4. *$GL_n(\mathbb{R})$ is neither compact nor path connected.* ☐

7.2 The Real Square Matrices with Positive Determinant

Next, let's look at the sub-topological group of $GL_n(\mathbb{R})$ of real square matrices with positive determinant:

$$GL_n(\mathbb{R})^+ := \{A \in \mathbb{R}^{n\times n} : \det(A) \in \mathbb{R}_{>0}\} \tag{91}$$

This is indeed a sub-topological group of $GL_n(\mathbb{R})$: for $A, B \in \mathbb{R}^{n\times n}$ with $\det(A), \det(B) > 0$ we have $\det(AB) = \det(A)\det(B) > 0$, and $\det(A^{-1}) = \det(A)^{-1} > 0$, so $GL_n(\mathbb{R})^+$ is closed under multiplication and taking inverse, so by Lemma 2.1 $GL_n(\mathbb{R})^+$ is a sub-topological group of $GL_n(\mathbb{R})$. Again, since $\mathbb{R}_{>0}$ is not compact, we obtain that $GL_n(\mathbb{R})^+$ is not compact. However, since $\mathbb{R}_{>0}$ is path connected, $GL_n(\mathbb{R})^+$ might be path connected, and this is indeed the case.

Theorem 7.5. *$GL_n(\mathbb{R})^+$ is path connected.*

Proof. Let $A \in GL_n(\mathbb{R})^+$. Since A is invertible, the reduced row echelon form of A is the identity matrix I. Hence, by Gauss elimination there exist elementary matrices $E_1, E_2, ..., E_k$ of the form $L_i(\lambda) = I + (\lambda - 1)E_{ii}$, $\lambda \in \mathbb{R}^\times$, or $M_{ij}(\lambda) = I + \lambda E_{ij}$, $\lambda \in \mathbb{R}^\times, i \neq j$, where E_{ij} is the matrix with zeros except a 1 on the (i, j)-th entry, such that

$$I = E_1 E_2 ... E_k A \tag{92}$$

For any $h \in \{1, ..., k\}$ and $t \in [0, 1]$, define

$$E_h(t) := \begin{cases} I + t(\lambda - 1)E_{ii} & E_h = L_i(\lambda) \\ I + t\lambda E_{ij} & E_h = M_{ij}(\lambda) \end{cases} \tag{93}$$

Both $t \mapsto I + t(\lambda - 1)E_{ii}$ and $t \mapsto I + t\lambda E_{ij}$ are continuous, giving us a path $\gamma_h : [0, 1] \to GL_n(\mathbb{R})$. Now, by Lemma A.2 the map $\gamma_{h,j} : [0, 1] \to GL_n(\mathbb{R})$ sending t to $E_h(t)E_j(t)$ is continuous. Continuing this way, we get that the map $\gamma : [0, 1] \to GL_n(\mathbb{R})$ sending t to $E_1(t)E_2(t)...E_k(t)A$ is continuous, so a path with $\gamma(0) = A$ and $\gamma(1) = I$.
We need to show that the image of γ lies in $GL_n(\mathbb{R})^+$, that is to say, the determinant of $E_1(t)E_2(t)...E_k(t)A$ is positive for all $t \in [0, 1]$.
First, note that $\det(L_i(\lambda)) = \lambda$ since the (i, i)-th coordinate is λ and the other diagonal entries are 1 and the rest is 0. Also $\det(M_{ij}(\lambda)) = 1$ since it is a lower or upper triangular matrix with diagonal entries all 1. Now suppose one of the E_h is $L_i(\lambda)$ with $\lambda < 0$. Since the product of the determinants of the E_i equals $\frac{1}{\det A} > 0$, there is another $L_j(\mu)$ with $\mu < 0$. Now we can perform elementary row operations only of the form $M_{ij}(\alpha)$ on the two corresponding rows

$$\begin{pmatrix} A \\ B \end{pmatrix} \rightsquigarrow \begin{pmatrix} A + B \\ B \end{pmatrix} \rightsquigarrow \begin{pmatrix} A + B \\ -2A - B \end{pmatrix} \rightsquigarrow \begin{pmatrix} -A \\ -2A - B \end{pmatrix} \rightsquigarrow \begin{pmatrix} -A \\ -B \end{pmatrix} \tag{94}$$

to change the sign of both of them. Therefore, we can assume that all E_h of the form $L_i(\lambda)$ have $\lambda > 0$. Now we can check the determinant:

$$\det(E_h(t)) = \begin{cases} \det(I + t(\lambda - 1)E_{ii}) & E_h = L_i(\lambda) \\ \det(I + t\lambda E_{ij}) & E_h = M_{ij}(\lambda) \end{cases} \tag{95}$$

$$= \begin{cases} \det(I + (1 - t + t\lambda - 1)E_{ii}) & E_h = L_i(\lambda) \\ 1 & E_h = M_{ij}(\lambda) \end{cases} \tag{96}$$

$$= \begin{cases} t(\lambda - 1) + 1 & E_h = L_i(\lambda) \\ 1 & E_h = M_{ij}(\lambda) \end{cases} \tag{97}$$

this is always positive if $\lambda > 0$. Combining everything, we obtain a path from A to I in $GL_n(\mathbb{R})^+$. So if $B \in GL_n(\mathbb{R})^+$ is another matrix, we have a path from A to I and a path from B to I, so a path from I to B, and concatenating them yields a path from A to B. □

Remark 7.6. $GL_n(\mathbb{R})^+$ is path connected and contains the identity matrix. We claim that $GL_n(\mathbb{R})^+$ is the path connected component of $I \in GL_n(\mathbb{R})$. To prove the claim, let $A \in GL_n(\mathbb{R})^+, B \in GL_n(\mathbb{R}) \backslash GL_n(\mathbb{R})^+$. Then $\det(A) > 0$ and $\det(B) < 0$. Let γ be a path from A to B in $GL_n(\mathbb{R})$. Since γ is continuous, the composite map

$$[0, 1] \xrightarrow{\gamma} GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times \tag{98}$$

is continuous. We have $(\det \circ \gamma)(0) > 0$ and $(\det \circ \gamma)(1) < 0$. By the Intermediate Value Theorem, there exists $t \in [0, 1]$ such that $(\det \circ \gamma)(t) = 0$, which contradicts the fact that γ is a path in $GL_n(\mathbb{R})$. Hence, γ could not have been continuous, so there does not exist a path from A to B, so $GL_n(\mathbb{R})^+$ is the path connected component of $I \in GL_n(\mathbb{R})$.

Corollary 7.7. *The fundamental groups of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{R})^+$ coincide.*

Proof. Any loop with base point I in $GL_n(\mathbb{R})$ does not leave $GL_n(\mathbb{R})^+$ by Remark 7.6, so is a loop with base point I in $GL_n(\mathbb{R})^+$. □

7.3 The Kernel of the Determinant

Consider

$$SL_n(\mathbb{R}) := \ker(\det) \tag{99}$$

This definition makes $SL_n(\mathbb{R})$ into a normal sub-topological group of $GL_n(\mathbb{R})$. Since $SL_n(\mathbb{R})$ is not discrete, we obtain that det is not a covering map. Now, since $\{1\}$ is compact and path connected, $SL_n(\mathbb{R})$ might be compact or path connected. In fact, $SL_n(\mathbb{R})$ is path connected, but not compact for $n \geq 2$.

Proposition 7.8. *$SL_n(\mathbb{R})$ is path connected.*

Proof. Let $f : GL_n(\mathbb{R})^+ \to SL_n(\mathbb{R})$ be the map that, for any $A \in GL_n(\mathbb{R})^+$, replaces the first column A_1 of A by $\frac{A_1}{\det(A)}$ and fixes the other columns of A. This map is well defined by linearity of det in the columns, so continuous by Lemma 7.2. f is also surjective, since any $A \in SL_n(\mathbb{R})$ is sent to itself by f. Now, $GL_n(\mathbb{R})^+$ is path connected, so $\mathrm{im}(f) = SL_n(\mathbb{R})$ is path connected. □

Proposition 7.9. *$SL_1(\mathbb{R})$ is compact, and $SL_n(\mathbb{R})$ is not compact for $n \geq 2$.*

Proof. $SL_1(\mathbb{R}) = \{1\}$ is finite, so compact. For $n \geq 2$, we do induction on n. For $n = 2$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1 \tag{100}$$

so $ad = bc + 1$ and this does not bound a, b, c, d, so $SL_2(\mathbb{R})$ is not bounded as a Euclidean metric subspace of $GL_2(\mathbb{R})$, so not compact. For the induction step, we consider the inclusion

$$\begin{array}{rccc} j: & SL_n(\mathbb{R}) & \to & SL_{n+1}(\mathbb{R}) \\ & A & \mapsto & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) \end{array} \tag{101}$$

where the zeros are the zero column and zero row of size n. Now for any $A \in SL_n(\mathbb{R})$, the formula for the determinant of $j(A)$ is the same as the formula of the determinant of A, so $\mathrm{im}(j) \subseteq SL_{n+1}(\mathbb{R})$ is not bounded by the induction hypothesis, so $SL_{n+1}(\mathbb{R})$ is not bounded, so not compact by Heine-Borel. □

7.4 The Orthogonal Matrices and the Gram-Schmidt Process

Let

$$O_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) : A^\top A = I\} \tag{102}$$

be the set of orthogonal n by n matrices. As for $A, B \in O_n(\mathbb{R})$ we have

$$(AB)^\top AB = B^\top A^\top AB = B^\top IB = B^\top B = I \tag{103}$$

and

$$(A^{-1})^\top A^{-1} = (A^\top)^{-1} A^{-1} = (AA^\top)^{-1} = ((A^\top A)^\top)^{-1} = (I^\top)^{-1} = I \tag{104}$$

we see that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ and therefore a sub-topological group of $GL_n(\mathbb{R})$. Since for any $A \in O_n(\mathbb{R})$ its columns all lie on S^{n-1} and $O_n(\mathbb{R})$ is closed in $GL_n(\mathbb{R})$, we obtain that $O_n(\mathbb{R})$ is compact.

Next, we will show that the well-known Gram-Schmidt algorithm [Stoll, Theorem 9.7] defines a deformation retraction $GS : GL_n(\mathbb{R}) \to O_n(\mathbb{R})$. This is interesting, because if we have a deformation retraction between topological spaces X and Y, then their fundamental groups are isomorphic: let $r : X \to Y$ be the deformation retraction, and $i : Y \to X$ the inclusion map. Since $i \circ r$ is homotopic to id_X, we have that $i_* : \pi_1(Y, y) \to \pi_1(X, i(y))$ and $r_* : \pi_1(X, i(y)) \to \pi_1(Y, y)$, for some $y \in Y$, are each others inverses.

Theorem 7.10. *The Gram-Schmidt algorithm determines a deformation retraction $GS : GL_n(\mathbb{R}) \to O_n(\mathbb{R})$. Here, we view a basis of $\mathbb{R}^n$ as an element of $GL_n(\mathbb{R})$ by placing the basis vectors as column vectors next to each other in the matrix.*

Proof. For any $A \in GL_n(\mathbb{R})$ denote by A_i its ith column, for all $i \in \{1, ..., n\}$. For all $i \in \{1, ..., n\}$, let $f_i : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the map that fixes A_j for $j \neq i$ and replaces A_i by

$$A_i \mapsto A_i - \sum_{j=1}^{i-1} \frac{\langle A_j, A_i \rangle}{\langle A_j, A_j \rangle} A_j \tag{105}$$

All the f_i are well-defined and hence continuous: the f_i act as performing one or more column operations of the form $M_{ij}(\lambda)$, which as we have seen in the proof of Theorem 7.5, do not change the determinant. Let $g : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be the map that replaces A_i by $\frac{1}{\sqrt{\langle A_i, A_i \rangle}} A_i$ for all $i \in \{1, ..., n\}$. Since $\langle A_i, A_i \rangle \neq 0$ for all $A \in GL_n(\mathbb{R})$ and all $i \in \{1, ..., n\}$ and $\langle A_i, A_i \rangle$ depends continuously on A, g is well-defined and continuous. The Gram-Schmidt Theorem states that the image of

$$GS = g \circ f_n \circ f_{n-1} \circ ... \circ f_2 \circ f_1 \tag{106}$$

which is continuous as a composition of continuous maps, is contained in $O_n(\mathbb{R})$. And indeed, $GS|_{O_n(\mathbb{R})} = \mathrm{id}_{O_n(\mathbb{R})}$ so $GS \circ i = \mathrm{id}_{O_n(\mathbb{R})}$ where $i : O_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is the inclusion map. So we need to show that $i \circ GS : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is homotopic to $id_{GL_n(\mathbb{R})}$. For any $i \in \{1, ..., n\}$, define

$$\begin{array}{rccl} H_i : & [0,1] \times GL_n(\mathbb{R}) & \to & GL_n(\mathbb{R}) \\ & (t, (A_1, ..., A_n)) & \mapsto & (A_1, ..., A_{i-1}, A_i - t\sum_{j=1}^{i-1} \frac{\langle A_j, A_i \rangle}{\langle A_j, A_j \rangle} A_j, A_{i+1}, ..., A_n) \end{array} \tag{107}$$

H_i is continuous and well-defined by the same argument the f_i are well-defined, so H_i is a homotopy from $\mathrm{id}_{GL_n(\mathbb{R})}$ to f_i. Moreover,

$$\begin{array}{rccl} H_g : & [0,1] \times GL_n(\mathbb{R}) & \to & GL_n(\mathbb{R}) \\ & (t, (A_1, ..., A_n)) & \mapsto & (\frac{1}{t\sqrt{\langle A_i, A_i \rangle} + 1 - t} A_i)_{i=1}^n \end{array} \tag{108}$$

is a homotopy from $\mathrm{id}_{GL_n(\mathbb{R})}$ to g. Now we can define a homotopy from $\mathrm{id}_{GL_n(\mathbb{R})}$ to GS as follows: denote $f_i(t) := H_i(t, -)$ and $g(t) := H_g(t, -)$. Then the map

$$\begin{array}{rccl} H : & [0,1] \times GL_n(\mathbb{R}) & \to & GL_n(\mathbb{R}) \\ & (t, A) & \mapsto & (g(t) \circ f_n(t) \circ ... \circ f_2(t) \circ f_1(t))(A) \end{array} \tag{109}$$

is a homotopy from $\mathrm{id}_{GL_n(\mathbb{R})}$ to GS. □

Let

$$SO_n(\mathbb{R}) := \{A \in O_n(\mathbb{R}) : \det(A) = 1\} \tag{110}$$

be the orientation preserving orthogonal matrices ('the special orthogonal matrices'). This is a sub-topological group of $GL_n(\mathbb{R})$: by (103) and (104) and the fact that det is multiplicative, $SO_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

Corollary 7.11. *The restricted Gram-Schmidt algorithm $GS|_{GL_n(\mathbb{R})^+} : GL_n(\mathbb{R})^+ \to SO_n(\mathbb{R})$ is well-defined and a deformation retraction.*

Proof. It is enough to show well-definedness: deformation retraction follows from the same construction as in Theorem 7.10. Again, by the proof of Theorem 7.5 the f_i which act as column operations of the form $M_{ij}(\lambda)$, do not change the determinant. Therefore, for any $A \in GL_n(\mathbb{R})^+$, we have

$$\det((f_n \circ f_{n-1} \circ ... \circ f_1)(A)) = \det(A) \tag{111}$$

so in particular $(f_n \circ f_{n-1} \circ ... \circ f_1)(A) \in GL_n(\mathbb{R})^+$. Moreover, g multiplies every column A_i of A by $\frac{1}{\sqrt{\langle A_i, A_i \rangle}}$ and since det is linear in the columns, the determinant of $g(A)$ is the determinant of A multiplied by n positive numbers, so remains positive. By the Gram-Schmidt Theorem, $GS(A) \in O_n(\mathbb{R})$. Hence, $GS(A) \in SO_n(\mathbb{R})$. □

Corollary 7.12. *The fundamental groups of $GL_n(\mathbb{R})$, $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ coincide.*

Proof. This was discussed in the beginning of this section. □

7.5 Example: the Spin Cover

As an application of the material in this section we will consider the Spin Cover, as it allows us to calculate the fundamental groups of $GL_3(\mathbb{R})$, $GL_3(\mathbb{R})^+$, $O_3(\mathbb{R})$ and $SO_3(\mathbb{R})$.

As in Example 3.7, consider S^3 as the set of unit quaternions U. As in [Schwartz, Lemma 2.1], let P be the set of pure quaternions, i.e. quaternions with real component 0, and identify P with $\mathbb{R}^3$ via the canonical bijection

$$\begin{array}{rccc} \theta: & P & \to & \mathbb{R}^3 \\ & b\mathbf{i}+c\mathbf{j}+d\mathbf{k} & \mapsto & (b,c,d) \end{array} \tag{112}$$

By [Schwartz, Lemma 2.1], for any $q \in S^3$ the map

$$\begin{array}{rccc} T_q: & P & \to & P \\ & p & \mapsto & qpq^{-1} \end{array} \tag{113}$$

is well-defined. If we consider $SO_3(\mathbb{R})$ as the orientation preserving isometries of P which fix the origin through θ, then by [Schwartz, Lemma 3.1] $T_q \in SO_3(\mathbb{R})$. So the map

$$\begin{array}{rccc} \Psi: & U & \to & SO_3(\mathbb{R}) \\ & q & \mapsto & T_q \end{array} \tag{114}$$

is well-defined. Moreover, by [Schwartz, Lemmas 3.2, 3.3, 3.4], Ψ is a surjective group homomorphism with kernel $\{-1, 1\}$.

Now we will show that Ψ is a covering map.

Lemma 7.13. *The map Ψ defined above is a covering map.*

Proof. Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in U$.

- Continuous: the map T_q sends $\mathbf{i} \in P$ to

$$T_q(\mathbf{i}) = (a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k})\mathbf{i}(a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}) = (a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k})(a\mathbf{i}+b-c\mathbf{k}+d\mathbf{j}) \tag{115}$$

$$= (a^2+b^2-c^2-d^2)\mathbf{i} + (ad+bc+bc+ad)\mathbf{j} + (-ac+bd-ac+bd)\mathbf{k} \tag{116}$$

  Similarly,

$$T_q(\mathbf{j}) = ... = (-ad+bc+bc-ad)\mathbf{i} + (a^2-b^2+c^2-d^2)\mathbf{j} + (ab+ab+cd+cd)\mathbf{k} \tag{117}$$

  and

$$T_q(\mathbf{k}) = ... = (ac+bd+ac+bd)\mathbf{i} + (-ab-ab+cd+cd)\mathbf{j} + (a^2-b^2-c^2+d^2)\mathbf{k} \tag{118}$$

  now since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ generate P and T_q is linear, these images determine T_q, giving

$$T_q(t\mathbf{i}+u\mathbf{j}+v\mathbf{k}) = \theta^{-1}\left(\begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix}\right) \tag{119}$$

  This is a combination of addition and multiplication on each coordinate of the matrix and therefore depends continuously on a, b, c, d.

- Covering map: First, Ψ is surjective. Next, $\{\pm 1\}$ acts evenly on S^3: this is intuitively clear, since for any $q \in S^3$ we can pick for example the open neighborhood

$$W_q = \{r \in S^3 : |q - r| < 0.1\} \tag{120}$$

  Now since $|q-(-q)| = |q+q| = |2q| = 2 > 2 \cdot 0.1 = 0.2$, the neighborhood $(-1)W_q$ of $-q$ has empty intersection with W_q, so $\{\pm 1\}$ acts evenly on S^3. By the alternative proof of Corollary 6.8, we have $\{\pm 1\} = \ker(\Psi) \cong \mathrm{Aut}(\Psi)$. Now by [Fulton, Lemma 11.17] this means that Ψ is a covering map.

□

Now since S^3 is simply connected, Ψ is a universal covering group. By [Fulton, Corollary 13.15] and Corollary 6.8, we have $\pi_1(SO_3(\mathbb{R})) \cong \ker(\Psi) = \{\pm 1\}$. Hence by Corollaries 7.7 and 7.12, we have

$$\pi_1(GL_3(\mathbb{R})) \cong \pi_1(GL_3(\mathbb{R})^+) \cong \pi_1(O_3(\mathbb{R})) \cong \pi_1(SO_3(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z} \tag{121}$$

A Some Basic Lemmas in Topology

Lemma A.1. *Let X, Y be topological spaces, and B a base of Y. Then a map $f : X \to Y$ is continuous if and only if for every $U \in B$, $f^{-1}(U)$ is open in X.*

Proof. The 'only if' part is clear, since B is a subset of the topology T_Y of Y. For the 'if' part, let $U \subseteq Y$ be open, and write

$$U = \bigcup_{i \in I} U_i \tag{122}$$

with $U_i \in B$ for all $i \in I$. Then

$$\begin{aligned} f^{-1}(U) &= f^{-1}\left(\bigcup_{i \in I} U_i\right) && (123)\\ &= \{x \in X : f(x) \in \bigcup_{i \in I} U_i\} && (124)\\ &= \bigcup_{i \in I}\{x \in X : f(x) \in U_i\} && (125)\\ &= \bigcup_{i \in I} f^{-1}(U_i) && (126)\end{aligned}$$

is a union of open sets, so open. (See also [Bruin, Exercise 7.4].) □

Lemma A.2. *Let W, X, Y, Z be topological spaces.*

1. *Let $f : X \to Y, g : X \to Z$ be continuous. Then the map*

$$\begin{array}{rccc} (f, g) : & X & \longrightarrow & Y \times Z \\ & x & \longmapsto & (f(x), g(x)) \end{array} \tag{127}$$

   is continuous.

2. *Let $f : W \to Y, g : X \to Z$ be continuous. Then the map*

$$\begin{array}{rccc} (f, g) : & W \times X & \longrightarrow & Y \times Z \\ & (w, x) & \longmapsto & (f(w), g(x)) \end{array} \tag{128}$$

   is continuous.

Proof. (1): let $U \subseteq Y, V \subseteq Z$ be open. Then

$$(f, g)^{-1}(U \times V) = \{x \in X : (f(x), g(x)) \in U \times V\} = \{x \in X : f(x) \in U \text{ and } g(x) \in V\} = f^{-1}(U) \cap g^{-1}(V) \tag{129}$$

is an intersection of two open sets in X so open.
(2): let $U \subseteq Y, V \subseteq Z$ be open. Then

$$(f, g)^{-1}(U \times V) = \{(w, x) \in W \times X : f(w) \in U \text{ and } g(x) \in V\} = f^{-1}(U) \times g^{-1}(V) \tag{130}$$

is open in $W \times X$. □

Lemma A.3.

1. *Let X, Y be path connected topological spaces. Then $X \times Y$ is path connected.*
2. *Let X, Y be connected topological spaces. Then $X \times Y$ is connected.*
3. *Let X, Y be locally path connected spaces. Then $X \times Y$ is locally path connected.*

Proof.

1. Let $(a,b),(c,d)$ be two points in $X \times Y$. Since X is path connected, there exists a path $\gamma : [0,1] \to X$ from a to c and since Y is path connected there exists a path $\delta : [0,1] \to Y$ from b to d. Then

$$\begin{array}{rccc} (\gamma,\delta): & [0,1] & \to & X \times Y \\ & s & \mapsto & (\gamma(s),\delta(s)) \end{array} \tag{131}$$

is a path from (a,b) to (c,d) by Lemma A.2.

2. Let $g : X \times Y \to \{0,1\}$ be a continuous map. Let $y \in Y$ and $f_y : X \to X \times Y$ be the map sending $x \in X$ to (x,y). Then f_y is continuous since for all $U \subseteq X, V \subseteq Y$ open $f_y^{-1}(U \times V)$ equals either U if $y \in V$ or $\emptyset$ if $y \notin V$. Since X is connected, the composition $g \circ f_y : X \to \{0,1\}$ is a constant map for all $y \in Y$. So for all $y \in Y$, $X \times \{y\} \subseteq g^{-1}\{0\}$ or $g^{-1}\{1\}$.
We could have played this game the other way around using the map ${}_y f$ sending $y \in Y$ to (x,y) which is continuous by the same argument. This leads to $\{x\} \times Y \subseteq g^{-1}\{0\}$ or $g^{-1}\{1\}$.
Suppose there exist $y,z \in Y$ such that $g(x,y) = 0$ and $g(x,z) = 1$ for all $x \in X$. Contradiction, since we then have that g is not constant on $\{x\} \times Y$ for any $x \in X$. Since $\bigcup_{y \in Y} X \times \{y\} = X \times Y$, we conclude that g must be constant, so $X \times Y$ is connected.

3. Let $(x,y) \in X \times Y$ and $\bigcup_{i \in I} U_i \times V_i$ be an open neighborhood of (x,y), where all the U_i are open in X and V_i are open in Y. Then there exists $j \in I$ such that $x \in U_j$ and $y \in V_j$. Since X is locally path connected, there exist an open $U \subseteq U_j$ such that $x \in U$ and U is path connected. Since Y is locally path connected, there exists an open $V \subseteq V_j$ such that $y \in V$ and V is path connected. By definition of product topology $U \times V$ is open in $X \times Y$ and by part (1), $U \times V$ is path connected. Furthermore $(x,y) \in U \times V \subseteq \bigcup_{i \in I} U_i \times V_i$, so $X \times Y$ is locally path connected.

□

B Some Basic Lemmas in Group Theory

Lemma B.1. *1. Let $\phi : G \times X \to X$ be a left-G-action on a set X that is free and transitive. Then for all $x \in X$ the map*

$$\begin{array}{rccc} \phi_x: & G & \to & X \\ & g & \mapsto & gx \end{array} \tag{132}$$

is a bijection.

2. Let $\phi : X \times G \to X$ be a right-G-action on a set X that is free and transitive. Then for all $x \in X$ the map

$$\begin{array}{rccc} \phi_x: & G & \to & X \\ & g & \mapsto & xg \end{array} \tag{133}$$

is a bijection.

Proof. Let e be the identity element of G.

1.
   - Injective: suppose $gx = hx$ for some $g,h \in G$. Then

   $$(g^{-1}h)x = g^{-1}(hx) = g^{-1}(gx) = (g^{-1}g)x = ex = x \tag{134}$$

   Since ϕ is free, we have $g^{-1}h = e$, so $g = h$.
   - Surjective: by transitivity of ϕ, for any $y \in X$ there exists $g \in G$ such that $gx = y$. This is exactly to say that ϕ_x is surjective.

2. Exactly the same proof, except that we have to consider $x(gh^{-1}) = (xg)h^{-1} = xh(h^{-1}) = xe = x$.

□

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