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Constructions of Lüroth normal numbers

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Constructions of Lüroth normal numbers

Master's Thesis Mathematics

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1 Introduction: A needle in a haystack

This is a thesis on *normal numbers*. These are numbers such that their digits faithfully represent a specific distribution. The most famous example of such a number was given by Champernowne in 1933 [Cha33]:

$$c := 0.1234567891011121314\ldots$$

Practically, Champernowne proved that if one selects a digit of c at random, its value is uniformly distributed over the integers from 0 to 9. So if we pick any integer, for example 1, a random digit x of c has a 10% chance of being equal to 1. Furthermore, any combination of two digits, say 11 or 86, yields exactly a 1% chance of success. Consequently, the probability of selecting any triple is 0.1%, and so on. In other words, the digits of normal numbers represent "total randomness" [Bok09]. This yields applications in areas such as cryptography [BC02].
The concept of normality was introduced earlier by Borel in 1909 [Bor09] in the setting of base b expansions, where $b \geq 2$ is an integer. These are expressions for numbers $x \in [0, 1]$ of the form

$$x = \sum_{i \geq 1} \frac{a_i}{b^i},$$

with digits $a_i \in \{0, 1, \ldots, b-1\}$. A number x is said to be normal in base b if for any $k \geq 1$ a block of digits $d_1 d_2 \ldots d_k$ occurs with frequency b^{-k} in its base b expansion:

$$\lim_{n \to \infty} \frac{\#\{0 \leq i \leq n : a_{i+1} = d_1, \ldots, a_{i+k} = d_k\}}{n} = b^{-k}.$$

According to this definition, Champernowne's number c is normal in base 10. Another celebrated example of a number with this property is the Copeland-Erdős constant, obtained by concatenating all prime numbers [CE46]:

$$p := 0.23571113171923\ldots$$

It is important to keep in mind that the definition of normality always depends on the choice for the underlying distribution or measure. This is usually the Lebesgue measure, but different measures can also be used [DLR20].
Normality may be easy to understand, yet it is unclear whether the digits of well-known numbers such as e or π yield a normal number in any base. Although especially the number p is an impressive example, the perhaps unsatisfying truth is that already in 1909, Borel showed that Lebesgue almost every real number is *absolutely normal*, i.e. normal in all integer bases $b \geq 2$. The proof was however not constructive, thus not providing an answer for the normality of $e - 2$ or $\pi - 3$. So as much as the search for normal numbers seems to be like one of a needle in a haystack, the actual haystack was always known to be more of a "needlestack".
This intriguing fact may have been the fuel for the ongoing search for normal numbers. Over the years, many generalizations and extensions to other areas have been made. Constructions were for example made for β-expansions [IS75]. For $\beta \in (1, 2)$ these arise by writing $x \in [0, \frac{1}{\beta - 1}]$ as

$$x = \sum_{i \geq 1} \frac{b_i}{\beta^i},$$

where $b_i \in \{0, 1\}$ for all $i \geq 1$.

Another example is the area of continued fraction expansions, where $x \in (0,1)$ is written as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, a_i \in \mathbb{N}.$$

See [AKS81, DLR20] for results on normal numbers in this context.

In 2009, a first attempt was made for *Lüroth expansions* in the bachelor thesis of Boks [Bok09]. These expansions were introduced in 1883 by Lüroth [Lür83], representing real numbers $x \in (0,1]$ by writing them as

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + ... + \frac{1}{a_1(a_1-1)...a_{n-1}(a_{n-1}-1)a_n} + \dots,$$

with $a_k \in \mathbb{N}_{\geq 2}$ for each $k \geq 1$. The expansion is based on partitioning the unit interval with subintervals $(\frac{1}{n+1}, \frac{1}{n}], n \geq 1$. Lüroth expansions were later generalized to *Generalized Lüroth Series* [BBDK96]. This generalization incorporates different partitions of the unit interval as well as negative terms in the expression of x.
Boks made a promising construction for the classical setting, but the proof of normality was unfortunately not completed. Successful constructions for Generalized Lüroth Series were made later by Madritch and Mance [MM16], Vandehey [Van14] and Dajani, De Lepper and Robinson [DLR20]. Another implicit construction was presented by Aehle and Paulsen [AP15]. All of this research uses the same "natural" definition of normality with respect to the Lebesgue measure on $(0,1]$.
Most of these constructions have some limitations. The research of [AP15] shows that it is sufficient to construct a so called *equidistributed sequence*. The problem is that finding such sequences seems to be as difficult as finding normal numbers. The method from [DLR20] yields explicit numbers, but they are only proven normal for one specific partition of the unit interval. The construction in [Van14] has a similar drawback as it works only for finite partitions.
The approach of [MM16] is very effective as it produces Lüroth normal numbers in full generality. Moreover, it also works in other settings like continued fractions or β-expansions. The major drawback is that the construction is very inefficient in the sense that it uses many repetitions.

This thesis presents a new family of Lüroth normal numbers with several advantages. The construction is simple and intuitive, avoids the use of many repetitions and works for any Generalized Lüroth Series.
The outline of this thesis is as follows. Lüroth expansions will be introduced in Chapter 2 among other fundamental definitions and classical results. In Chapter 3, we construct a family of Lüroth expansions and discuss some of its properties. We will show that this construction leads to a family of Lüroth normal numbers in Chapter 4. Chapter 5 will exhibit similar results for Generalized Lüroth Series. Some reflections, including a more detailed comparison with other constructions will be discussed in Chapter 6.

2 Lüroth expansions

We study Lüroth expansions by means of a dynamical system and discuss and prove the important properties of this system such as ergodicity. We define Lüroth normality and show that as in the case of b-adic numbers, Lebesgue almost all numbers are Lüroth normal.

2.1 The Lüroth map

Lüroth expansions are generated by the *Lüroth map* $T_L : (0,1] \to (0,1]$:

$$T_L x := n(n+1)x - n,$$

where $n \in \mathbb{N}$ is the integer such that $x \in (\frac{1}{n+1}, \frac{1}{n}]$.

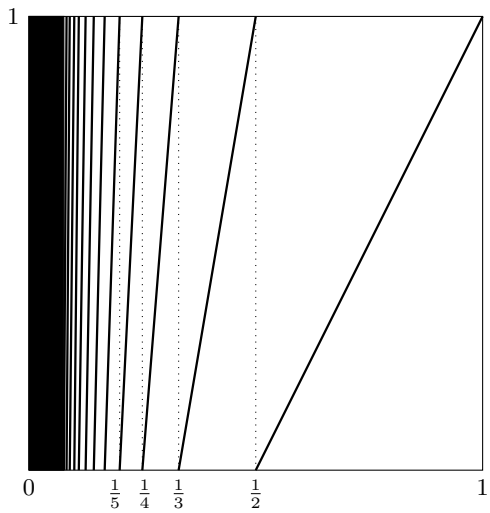


Figure 1: The Lüroth map.

The digits in this expansion come from the *Lüroth alphabet* $\mathcal{A} = \mathbb{N}_{\geq 2}$. The set $\Sigma = \mathcal{A}^{\mathbb{N}}$ denotes the set of all infinite sequences $(a_i)_{i=1}^{\infty}$, $a_i \in \mathcal{A}$.
For $x \in (0,1]$, we define its Lüroth expansion to be the sequence $A(x) = (a_i)_{i=1}^{\infty}$ such that $a_i := n+1$ if $T_L^{i-1}x \in (\frac{1}{n+1}, \frac{1}{n}]$, $i \geq 1$. We will often use the term sequence instead of expansion. The definition allows us to write

$$T_L x = a_1(a_1 - 1)x - (a_1 - 1),$$

and we can continue as follows:

$$T_L^2 x = a_2(a_2 - 1)T_L x - (a_2 - 1),$$

so that

$$\begin{aligned} x &= \frac{1}{a_1} + \frac{T_L x}{a_1(a_1-1)} \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)}\left(\frac{1}{a_2} + \frac{T_L^2 x}{a_2(a_2-1)}\right) \\ &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \frac{T_L^2 x}{a_1(a_1-1)a_2(a_2-1)}. \end{aligned}$$

Proceeding inductively, we obtain after i steps that

$$\begin{aligned} x &= \frac{1}{a_1} + ... + \frac{1}{a_1(a_1-1)...a_{i-1}(a_{i-1}-1)a_i} + \frac{T_L^i x}{a_1(a_1-1)...a_i(a_i-1)} \\ &= \sum_{k=1}^{i}(a_i-1)\prod_{j=1}^{k}\frac{1}{a_j(a_j-1)} + \frac{T_L^i x}{a_1(a_1-1)...a_i(a_i-1)}. \end{aligned} \tag{2.1}$$

Observe that $a_i \geq 2$ and $T_L^i x \in (0,1]$ for all $i \geq 1$, so that the last term can be estimated as

$$0 \leq \frac{T_L^i x}{a_1(a_1-1)....a_i(a_i-1)} \leq \frac{1}{2^i},$$

and this yields

$$|x - \sum_{k=1}^{i}(a_k-1)\prod_{j=1}^{k}\frac{1}{a_j(a_j-1)}| \to 0$$

as $i \to \infty$. Hence,

$$x = \sum_{k=1}^{\infty}(a_k-1)\prod_{j=1}^{k}\frac{1}{a_j(a_j-1)}.$$

This shows that all Lüroth expansions uniquely represent a real number $x \in (0,1]$.

Observe that in our definition, Lüroth expansions are always infinite. We have deliberately chosen to do so because a normal expansion can never be finite.
The Lüroth map is often defined slightly differently, allowing also finite expansions. For example in [JdV68, DKa21], the open and closed end of the intervals $(\frac{1}{n+1}, \frac{1}{n}]$ are reversed, which gives that $T_L x = 0$ if $x = \frac{1}{n}$. The rest of the definition is the same, so that a finite expansion $(a_1, ..., a_m)$ coincides with the infinite expansion $(a_i, ..., a_m + 1, 2, 2, ...)$. In [KM22], the finite expansion is written as $(a_1, ..., a_m, \infty, \infty, ...)$, so that both expansions are *ultimately periodic*: there exist $n \geq 0$ and $r \geq 0$ such that $a_{n+i} = d_{n+r+i}$ for all $i \geq 1$. Already in 1883 [Lür83], Lüroth showed that an expansion $A(x)$ is ultimately periodic if and only if $x \in \mathbb{Q}$.

2.2 Fundamental intervals and cylinder sets

The previous section shows that the map

$$\begin{aligned} L : \Sigma &\to (0,1], \\ (a_i)_{i=1}^{\infty} &\mapsto \sum_{i\geq 1}^{\infty}(a_i - 1)\prod_{j=1}^{i}\frac{1}{a_j(a_j-1)} \end{aligned}$$

is one to one and well defined.
We introduce some more notation. We define $\mathcal{A}^l$ as the set of sequences of finite length l, and $\mathcal{A}^* = \cup_{l\geq 1}\mathcal{A}^l$ as the set of all finite sequences. These elements are referred to as *words*, often written as $\alpha = \alpha_1...\alpha_k$ or $\mathbf{u} = u_1...u_k$. We write $|\alpha| = k$ for the length of these sequences. The empty word ε of length 0 is excluded from $\mathcal{A}^*$. Finally, two words $\alpha = \alpha_1...\alpha_k, \beta = \beta_1...\beta_l \in \mathcal{A}^*$ can be *concatenated* to a word $\alpha\beta := \alpha_1...\alpha_k\beta_1...\beta_l \in \mathcal{A}^*$.

As in [DKa21], we write

$$\Delta(i) := \left(\frac{1}{i}, \frac{1}{i-1}\right] = \{x \in [0,1) : a_1(x) = i\}.$$

For $n \geq 1$, this leads to the notion of the *fundamental intervals* of rank n. For a word $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{A}^*$, let

$$\begin{aligned} \Delta_\alpha &:= \Delta(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &:= \Delta(\alpha_1) \cap T_L^{-1}\Delta(\alpha_2) \cap \dots \cap T_L^{-(n-1)}\Delta(\alpha_n) \end{aligned}$$

be the set containing all real numbers of which the first k digits of their Lüroth expansion are equal to those of α.
The length of fundamental intervals can be calculated:

$$\begin{aligned} \Delta_\alpha &= \Delta(\alpha_1) \cap T_L^{-1}\Delta(\alpha_2) \cap \dots \cap T_L^{-(n-1)}\Delta(\alpha_n) \\ &= \{x \in [0,1) : a_1(x) = \alpha_1, a_2(x) = \alpha_2, \dots, a_n(x) = \alpha_n\} \\ &= \left\{x \in [0,1) : \frac{P_n}{Q_n} < x \leq \frac{P_n}{Q_n} + \prod_{m=1}^{n}\frac{1}{\alpha_m(\alpha_m - 1)}\right\}, \end{aligned}$$

where

$$\frac{P_n}{Q_n} = \frac{1}{\alpha_1} + \frac{1}{\alpha_1(\alpha_1-1)\alpha_2} + \dots + \frac{1}{\alpha_1(\alpha_1-1)\dots\alpha_{n-1}(\alpha_{n-1}-1)\alpha_n}.$$

This shows that

$$\lambda(\Delta_\alpha) = \prod_{m=1}^{n}\frac{1}{\alpha_m(\alpha_m-1)},$$

where λ denotes the Lebesgue measure on $[0,1]$.

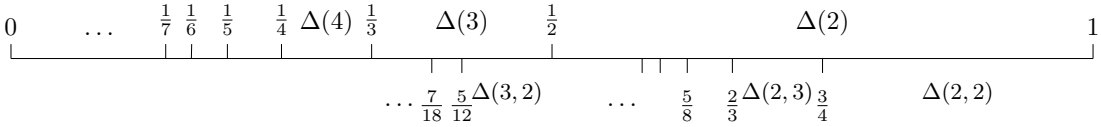


Figure 2: Illustration of a few fundamental intervals of rank 1 and 2.

The corresponding notion of fundamental intervals in Σ are the *cylinder sets*:

$$\langle \alpha_i \dots \alpha_j \rangle := \{A \in \Sigma : a_i = \alpha_i, ..., a_j = \alpha_j\}.$$

For a word $\alpha = \alpha_1 ... \alpha_k \in \mathcal{A}^*$ we write $\langle \alpha \rangle = \langle \alpha_1, \dots \alpha_k \rangle$, and observe that $L\langle \alpha \rangle = \Delta_\alpha$.
We write $\mathcal{C}$ for the σ-algebra on Σ generated by the cylinder sets, which we equip with the *Bernoulli measure* μ:

$$\mu(\langle \alpha_i \dots \alpha_j \rangle) := \prod_{l=i}^{j} \frac{1}{\alpha_l(\alpha_l - 1)}. \tag{2.2}$$

The transformation $T : \Sigma \to \Sigma$ given by $(a_i)_{i=1}^\infty \mapsto (a_{i+1})_{i=1}^\infty$ is the *left shift*. The left shift is closely related to the Lüroth map T_L as for any sequence $A \in \Sigma$, it holds that

$$L(T(A)) = T_L(L(A)).$$

Note that $\mu(\langle \alpha \rangle) = \lambda(L(\langle \alpha \rangle))$. With abuse of notation, we will write $\mu(\alpha)$ instead of $\mu(\langle \alpha \rangle)$.

2.3 Measure preservingness and normality

Next, we show that the Lüroth map T_L is *measure preserving* with respect to λ. That is, $\lambda(T_L^{-1}B) = \lambda(B)$ for all $B \in \mathcal{B}([0,1])$. It suffices to show this for all open intervals in $[0,1]$ by Theorem 1.2.1 of [DKa21].
Observe that for an interval $(a,b) \subset (0,1)$, we have

$$\begin{aligned} T_L^{-1}(a,b) &= \left(\frac{1}{2} + \frac{1}{2}a, \frac{1}{2} + \frac{1}{2}b\right) \cup \left(\frac{1}{3} + \frac{1}{6}a, \frac{1}{3} + \frac{1}{6}b\right) \cup \left(\frac{1}{4} + \frac{1}{12}a, \frac{1}{4} + \frac{1}{12}b\right) \cup ... \\ &= \bigcup_{n=2}^{\infty} \left(\frac{1}{n} + \frac{1}{n(n-1)}a, \frac{1}{n} + \frac{1}{n(n-1)}b\right), \end{aligned}$$

so that

$$\begin{aligned} \lambda(T_L^{-1}(a,b)) &= \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n} + \frac{1}{n(n-1)}a, \frac{1}{n} + \frac{1}{n(n-1)}b\right) \\ &= \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n(n-1)}a, \frac{1}{n(n-1)}b\right) \\ &= \sum_{n=2}^{\infty} \frac{1}{n(n-1)}\lambda(a,b) \\ &= \lambda(a,b). \end{aligned}$$

The left shift T is also measure preserving with respect to μ as

$$T^{-1}\{A \in \Sigma : a_i = \alpha_i, ..., a_j = \alpha_j\}. = \{A \in \Sigma : a_{i+1} = \alpha_i, ..., a_{j+1} = \alpha_j\}.$$

This shows that $(\Sigma, \mathcal{C}, \mu, T)$ is *measure preservingly isomorphic* to $((0,1], \mathcal{B}((0,1]), \lambda, T_L)$ via L. This allows the following definition.

2.1 Definition (Lüroth normality)**.** A sequence $A = (\alpha_i)_{i=1}^{\infty} \in \Sigma$ is μ-normal if for every $\alpha = \alpha_1 ... \alpha_k \in \mathcal{A}^*, k \geq 1$, it holds that

$$\lim_{n\to\infty} \frac{\#\{i \leq n : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{n} = \mu(\alpha).$$

Consequently, a number $x \in (0,1]$ is *Lüroth normal* if there exists a μ-normal sequence $A_x \in \Sigma$ such that $L(A_x) = x$.

As mentioned in the introduction, normality is defined with respect to a certain measure. In our case this is μ, the isomorphic image of the Lebesgue measure. This is in a sense the most "natural" choice, as most examples like Champernowne's constant are also normal with respect to this measure. It is however perfectly possible to define normality with respect to different invariant measures as is done in [DLR20]. We will often drop the μ in the notation, as this thesis will only refer to Lüroth normality with respect to μ.

2.4 Ergodicity

We now show how the famous Pointwise Ergodic Theorem implies that Lebesgue almost all numbers are Lüroth normal. Apart from measure preservingness it also requires *ergodicity.*

The Lüroth map is indeed ergodic. That is, for every measurable set B satisfying $T_L^{-1}B = B$, we have $\lambda(B) \in \{0,1\}$. We prove this using the following lemma: (Lemma 2.3.1 in [DKa21])

2.2 Lemma (Knopp's Lemma)**.** *If $E \in \mathcal{B}((0,1])$ is a Lebesgue set and $\mathcal{S}$ is a class of subintervals of $(0,1]$, satisfying*

a) every open subinterval of $(0,1]$ is at most a countable union of disjoint elements from $\mathcal{S}$,

b) $\forall B \in \mathcal{S}, \lambda(B \cap E) \geq \gamma\lambda(B)$, where $\gamma > 0$ is independent of B,

then $\lambda(E) = 1$.

The collection of all fundamental intervals of all ranks is our candidate for $\mathcal{S}$.
For any choice of digits $\alpha_1, \alpha_2, \ldots,$ it holds that $\lim_{n\to\infty} \lambda(\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)) = 0$. Since $\cup_{\alpha_1 \in \mathbb{N}_{\geq 2}} \Delta(\alpha_1) = (0,1]$ and $\cup_{\alpha_n \in \mathbb{N}_{\geq 2}} \Delta(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Delta(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, the endpoints of $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are dense in $(0,1]$. This shows that $\mathcal{S}$ generates the Lebesgue σ-algebra of $(0,1]$, thus satisfying part a) of Knopp's Lemma. Also observe that $\lambda(\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)) = \prod_{m=1}^{n} \frac{1}{\alpha_m(\alpha_m - 1)}$, and that $T_L^n(\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)) = (0,1]$, so T_L^n maps a fundamental interval of rank n linearly to $(0,1]$. If E is a T_L-invariant set such that $\lambda(E) > 0$ and $B \in \mathcal{S}$, this yields

$$\lambda(B \cap E) = \lambda(B \cap T_L^{-n}E) = \lambda(B) \cdot \lambda(T_L^n B \cap E) = \lambda(B)\lambda(E).$$

This addresses part b) : we can take $\gamma = \lambda(E)$. Hence T_L is ergodic. We are now in a position to apply the following theorem.

2.3 Theorem (Pointwise Ergodic Theorem)**.** *Let $(X, \mathcal{F}, \nu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then, for any $f \in L^1(X, \mathcal{F}, \nu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = f^*(x)$$

exists ν-a.e., is T-invariant and $\int_X f d\nu = f_X f^* d\nu$. If moreover T is ergodic, then f^* is a constant ν-a.e. and $f^* = \int_X f d\nu$.

Let $x \in (0, 1]$, $A(x) = (a_i)_{i=1}^{\infty}$. By definition, $T_L^i(x) \in \Delta_\alpha$ if and only if $a_{i+1}(x) = \alpha_1, ..., a_{i+k}(x) = \alpha_k$. So taking $f = \mathbb{1}_{\Delta_\alpha}$, we immediately obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{\#\{i \le n : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T_L^i(x)) \\
&= \int_{(0,1]} \mathbb{1}_{\Delta_\alpha} d\lambda, \ \lambda\text{-a.e.} \\
&= \lambda(\Delta_\alpha) \\
&= \mu(\alpha).
\end{aligned}$$

Observe that this result holds for $x \in (0, 1]$ λ-a.e. In other words, $x \in (0, 1]$ is μ-normal with probability 1. In this thesis we set out to find such x.

2.5 Generalized Lüroth Series

In [BBDK96], the generalization of Lüroth expansions is twofold.
Firstly, any countable partition of the unit interval is allowed. Instead of the digits $n \ge 2$ we have an arbitrary *digit set* $\mathcal{D} \subset \mathbb{N}$. Let $I_n = (l_n, r_n], n \in \mathcal{D}$, be a countable set of disjoint intervals such that their lengths

$$L_n := r_n - l_n$$

satisfy

$$\sum_{n\in\mathcal{D}} L_n = 1$$

and

$$0 < L_i \le L_j < 1 \text{ for all } i, j \in \mathcal{D}, i > j. \tag{2.3}$$

We define

$$I_\infty = [0, 1] \setminus \cup_{n\in\mathcal{D}} I_n, \ L_\infty = 0,$$

and use this to define the map

$$Tx := \begin{cases} \frac{x-l_n}{r_n-l_n}, & x \in I_n, \\ 0, & x \in I_\infty. \end{cases}$$

For $x \in \cup_{n\in\mathcal{D}} I_n$, we define

$$s(x) := \frac{1}{r_n - l_n},\ h(x) := \frac{l_n}{r_n - l_n},\ x \in I_n,$$

so that for all $x \in [0,1]$, we define

$$s_n = s_n(x) = \begin{cases} s(T^{n-1}x) & T^{n-1}x \notin I_\infty, \\ \infty, & T^{n-1}x \in I_\infty, \end{cases}$$

and

$$h_n = h_n(x) = \begin{cases} h(T^{n-1}x) & T^{n-1}x \notin I_\infty, \\ 1, & T^{n-1}x \in I_\infty. \end{cases}$$

For $x \in (0,1)$ such that $T^{n-1}x \notin I_\infty$, we have

$$Tx = s(x)x - h(x) = s_1 x - h_1,$$

and just like before, we can continue as follows:

$$T^2 x = s(Tx) \cdot Tx - h(Tx) = s_2 Tx - h_2,$$

so that

$$\begin{aligned} x &= \frac{h_1}{s_1} + \frac{1}{s_1} Tx \\ &= \frac{h_1}{s_1} + \frac{1}{s_1}\left(\frac{h_2}{s_2} + \frac{1}{s_2} T^2 x\right) \\ &= \frac{h_1}{s_1} + \frac{h_2}{s_1 s_2} + \cdots + \frac{h_n}{s_1 \dots s_n} + \frac{1}{s_1 \dots s_n} T^n x. \end{aligned}$$

As $s_n > 1$ and $T^n x \in [0,1]$ for all $n \geq 1$, the infinite series

$$\sum_{n=1}^{\infty} \frac{h_n}{s_1 \dots s_n} \tag{2.4}$$

converges to x.

The second part of the generalization allows changes of the transformation T. It is also possible to use the following transformation S:

$$Sx := 1 - Tx = \begin{cases} \frac{t_n - x}{r_n - l_n}, & x \in I_n, \\ 0, & x \in I_\infty. \end{cases}$$

Let $\varepsilon = (\varepsilon(n))_{n\in\mathcal{D}}$ be an arbitrary, fixed sequence of zeroes and ones. We can now define a map $T_\varepsilon : [0,1] \to [0,1]$ that "mixes" T and S:

$$T_\varepsilon x := \varepsilon(x)Sx + (1-\varepsilon(x))Tx, x \in [0,1],$$

where

$$\varepsilon(x) = \begin{cases} \frac{t_n - x}{r_n - l_n}, & x \in I_n, \\ 0, & x \in I_\infty. \end{cases}$$

We obtain new definions of s_n and h_n by replacing T by T_ε and like in (2.4), this yields the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{\varepsilon_0 + \cdots + \varepsilon_{n-1}} \frac{h_n + \varepsilon_n}{s_1 \dots s_n}. \tag{2.5}$$

The expansion (2.5) is called the (I,ε)-generalized Lüroth series (GLS) of x. As $\varepsilon_n = \varepsilon(a_n)$, we can again denote the series by $A(x) = (a_i)_{i=1}^{\infty}$. All definitions from Section 2.2 now apply with $\mathcal{D}$ instead of $\mathcal{A}$. Thus for a cylinder set $\langle \alpha_1, \dots, \alpha_k \rangle$, we have

$$\mu(\langle \alpha \rangle) := \prod_{i=1}^{k} L_{\alpha_i} \tag{2.6}$$

yielding the same definition of normality.

As shown in [BBDK96], the same ergodic properties hold for generalized Lüroth series, so that again Lebesgue almost all numbers are μ-normal. It is important to note that the second part of the generalization is not relevant for the normality of numbers, as Definition 2.1 is only concerned with the Lebesgue measure of the fundamental intervals. Alternating the Lüroth map by changing signs of course changes the value of $L(A)$, but it does not change the length of these intervals. Therefore the μ-normality of a Lüroth expansion holds under any such alteration. Thus in our search for normal numbers for any GLS we only have to worry about changes to the partition.

3 The Lüroth Tree

We construct a specific graph, establish a couple of important properties and construct a Lüroth sequence based on this graph.

3.1 Construction

For the construction of our Lüroth normal sequence A we use some elementary graph theory. A *graph* $G := (V, E)$ is a pair with a set of *vertices* $V := V(G)$ and a set of *edges* $E := E(G)$. An edge e is said to *join* the vertices x and y and is denoted by (x, y). A *finite path* in G is a finite sequence of edges $(x_1, x_2), (x_2, x_3), ..., (x_n, x_{n+1})$ such that all $x_i, 1 \leq i \leq n$, are distinct. A *tree* is a graph in which any two vertices are connected by exactly one path.

We gratefully make use of the so called *Lüroth Tree* that was described in [Bok09]. The tree starts with the word 2, and then uses the rule

$$\begin{tikzcd}[column sep=small] & u_1u_2...u_k \arrow[dl, "\frac{1}{u_k}"'] \arrow[dr, "\frac{u_k-1}{u_k}"] & \\ u_1u_2...u_k2 & & u_1u_2...u_k+1. \end{tikzcd}$$

This leads to the following formal definition.

3.1 Definition (Lüroth Tree)**.** Let $\mathbf{u} = u_1...u_k \in \mathcal{A}^*$. Write $\mathbf{u}2 := u_1...u_k2$ and $\mathbf{u}^+ =: u_1...u_{k-1}(u_k + 1)$. Then we define the Lüroth Tree as the graph G_L such that

$$\begin{aligned} V(G_L) &= \mathcal{A}^*, \\ E(G_L) &= \{(\mathbf{u}, \mathbf{u}2), (\mathbf{u}, \mathbf{u}^+) : \mathbf{u} \in \mathcal{A}^*\}. \end{aligned}$$

We further define a map f that assigns a *label* to each edge.:

$$\begin{aligned} f :& E(G_L) \to \mathbb{R}_+, \\ e \mapsto& \begin{cases} \frac{1}{u_k}, & \text{if } e = (\mathbf{u}, \mathbf{u}2) \text{ for some } \mathbf{u} = u_1...u_k \in \mathcal{A}^*, \\ \frac{u_k-1}{u_k}, & \text{if } e = (\mathbf{u}, \mathbf{u}^+) \text{ for some } \mathbf{u} = u_1...u_k \in \mathcal{A}^*. \end{cases} \end{aligned} \tag{3.1}$$

If we start the construction at $\mathbf{u} = 2$, we obtain the following figure.

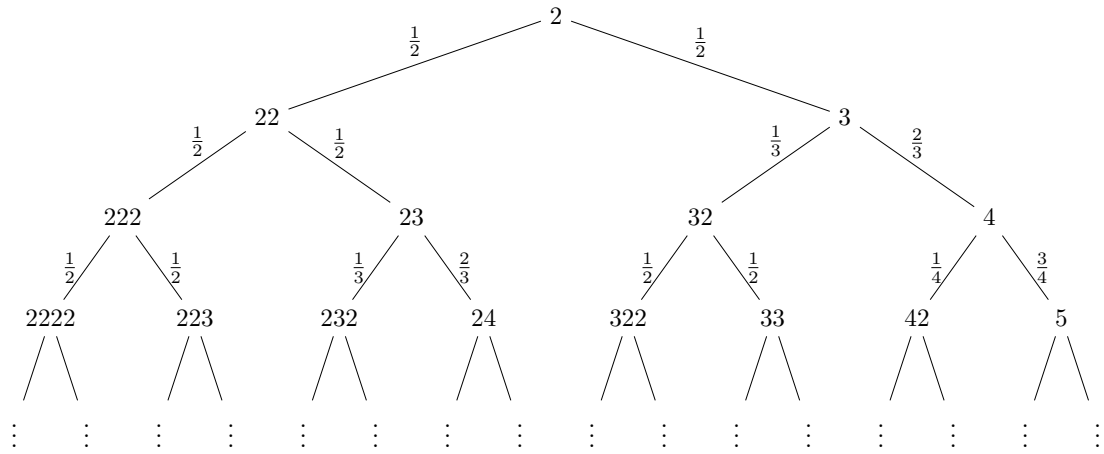


Figure 3: The Lüroth Tree.

Before we move on to the construction of our normal number, we define a graph homomorphism from G_L to itself.

3.2 Definition. For each $\mathbf{u} \in \mathcal{A}^*$ we define

$$\begin{aligned} \varphi_{\mathbf{u}} : G_L(V,E) &\to G_L(V,E), \\ V(G_L) \ni \mathbf{v} &\mapsto \mathbf{uv} \in V(G_L), \\ E(G_L) \ni (\mathbf{v}_1, \mathbf{v}_2) &\mapsto (\mathbf{uv}_1, \mathbf{uv}_2) \in E(G_L). \end{aligned}$$

Observe that this graph homomorphism is well defined as $(\mathbf{uv})2 = \mathbf{u}(\mathbf{v}2)$, and similarly $(\mathbf{uv})^+ = \mathbf{u}(\mathbf{v}^+)$.

3.3 Lemma (Recursion)**.** *Let* $\mathbf{u} \in \mathcal{A}^* \cup \{\varepsilon\}$. *The label mapping* $f : E(G_L) \to \mathbb{R}_{\geq 0}$ *is* $\varphi_{\mathbf{u}}$ *invariant:*

$$f(\varphi_{\mathbf{u}}(e)) = f(e) \text{ for all } e \in E(G_L).$$

Proof. Let $e \in E(G_L)$ and let $\mathbf{v} \in \mathcal{A}^*$ be its initial vertex, so $e = (\mathbf{v}, \mathbf{v}2)$ or $e = (\mathbf{v}, \mathbf{v}^+)$. The label assigned to e and $\varphi_{\mathbf{u}}(e)$ only depends on the last digit of $\mathbf{v}$. We then immediately obtain $f(\varphi_{\mathbf{u}}(e)) = f(e)$ for all $e \in E(G_L)$, as the last digit of $\mathbf{v}$ equals the last digit of $\mathbf{uv}$. $\square$

3.2 Depth and weight

We introduce two key concepts: *depth* and *weight*.

3.4 Definition. Let $\mathbf{u} = u_1...u_k \in \mathcal{A}^* \cup \{\varepsilon\}$. We define

$$D(\mathbf{u}) := \sum_{i=1}^{k} (u_i - 1)$$

as the *depth* of $\mathbf{u}$, where the empty sum has value 0. We also write

$$G(n) = \{\mathbf{u} \in V(G_L) : D(\mathbf{u}) = n\}$$

as the collection of all words of depth n.

Observe that the depth $D(\mathbf{u})$ represents the number of vertices in the path connecting the word 2 and the word $\mathbf{u}$ in the Lüroth Tree.
Immediately from the definition we obtain the useful identity

$$D(\mathbf{uv}) = D(\mathbf{u}) + D(\mathbf{v}). \tag{3.2}$$

For each $\mathbf{u} = u_1...u_k \in \mathcal{A}^*$ we write $P(\mathbf{u})$ for the (unique) path that connects the top of the Lüroth tree with $\mathbf{u}$. Formally this means that

$$P(\mathbf{u}) := (\mathbf{u}_1, \mathbf{u}_2), ..., (\mathbf{u}_{D(\mathbf{u})-1}, \mathbf{u}_{D(\mathbf{u})})$$

such that

$$\begin{aligned} \mathbf{u}_1 &= 2, \\ \mathbf{u}_{D(u)} &= \mathbf{u}, \text{ and} \\ (\mathbf{u}_j, \mathbf{u}_{j+1}) &\in E(G_L). \end{aligned}$$

This leads to the following notion.

3.5 Definition. Let $\mathbf{u} = u_1...u_k \in \mathcal{A}^* \cup \{\varepsilon\}$. We define

$$W(\mathbf{u}) := \prod_{j=1}^{D(\mathbf{u})-1} f(\mathbf{u}_j, \mathbf{u}_{j+1})$$

as the *weight* of $\mathbf{u}$. It is the product of all the labels along the path $P(\mathbf{u})$ from 2 to $\mathbf{u}$, where the empty product has value 1.

For example, the word 32 has depth 3. It's path is $P(32) = (2,3),(3,32)$, yielding weight $W(32) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. Observe since $\frac{1}{u_k} + \frac{u_k - 1}{u_k} = 1$, it holds that

$$W(\mathbf{u}) = W(\mathbf{u}2) + W(\mathbf{u}^+),$$

so that for all $n \geq 2$,

$$\sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) = \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}2) + \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}^+) = \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}).$$

As $W(2) = 1$, we obtain for all $n \in \mathbb{N}$ the useful identity

$$\sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) = 1. \tag{3.3}$$

3.6 Lemma. *Let* $\mathbf{uv}$ *be the concatenation of* $\mathbf{u}, \mathbf{v} \in \mathcal{A}^*$*. The following holds:*

$$W(\mathbf{uv}) = W(\mathbf{u}2) \cdot W(\mathbf{v}). \tag{3.4}$$

Proof. Write

$$P(\mathbf{u}) = (\mathbf{u}_1, \mathbf{u}_2), ..., (\mathbf{u}_{D(\mathbf{u})-1}, \mathbf{u}_{D(\mathbf{u})}), P(\mathbf{v}) = (\mathbf{v}_1, \mathbf{v}_2), ..., (\mathbf{v}_{D(\mathbf{v})-1}, \mathbf{v}_{D(\mathbf{v})}),$$

and observe that

$$\begin{aligned} P(\mathbf{uv}) &= ((\mathbf{uv})_1, (\mathbf{uv})_2), ..., ((\mathbf{uv})_{D(\mathbf{uv})-1}, (\mathbf{uv})_{D(\mathbf{uv})}) \\ &= (\mathbf{u}_1, \mathbf{u}_2), ..., (\mathbf{u}_{D(\mathbf{u})-1}, \mathbf{u}_{D(\mathbf{u})}), (\mathbf{u}_{D(\mathbf{u})}, \mathbf{u}_{D(\mathbf{u})}\mathbf{v}_1), ..., (\mathbf{u}_{D(\mathbf{u})}\mathbf{v}_{D(\mathbf{v})-1}, \mathbf{u}_{D(\mathbf{u})}\mathbf{v}_{D(\mathbf{v})}), \end{aligned}$$

where $\mathbf{u}_{D(\mathbf{u})}\mathbf{v}_1 = \mathbf{u}2$. Using Lemma 3.3, we obtain

$$\begin{aligned} W(\mathbf{uv}) &= \prod_{j=1}^{D((\mathbf{uv}))-1} f((\mathbf{uv})_j, (\mathbf{uv})_{j+1}) \\ &= \left(\prod_{j=1}^{D(\mathbf{u})-1} f(\mathbf{u}_j, \mathbf{u}_{j+1}) \right) \cdot f(\mathbf{u}, \mathbf{u}2) \cdot \prod_{j=D(\mathbf{u})+1}^{D(\mathbf{uv})-1} f((\mathbf{uv})_j, (\mathbf{uv})_{j+1}) \\ &= \prod_{j=1}^{D(\mathbf{u}2)-1} f((\mathbf{u}2)_j, (\mathbf{u}2)_{j+1}) \cdot \prod_{j=1}^{D(\mathbf{v})-1} f(\varphi_{\mathbf{u}}(\mathbf{v}_j, \mathbf{v}_{j+1})) \\ &= W(\mathbf{u}2) \cdot W(\mathbf{v}). \end{aligned}$$

□

3.7 Corollary. Let $\mathbf{u} = u_1 ... u_k \in \mathcal{A}^*$. Then

$$W(\mathbf{u}2) = \mu(\mathbf{u}).$$

Proof. We start with the assumption that $k = 1$, so that $\mathbf{u} = u_1$. Then

$$W(\mathbf{u}) = \prod_{j=2}^{u_1} \frac{j-1}{j} = \frac{1}{u_1 - 1},$$

so that also $W(\mathbf{u}2) = \frac{1}{u_1(u_1-1)}$. We now let $k > 1$ and view $\mathbf{u}$ as the concatenation of single-digit words $\mathbf{u}_i = u_i$. Repeated application of Lemma 3.6 then yields

$$W(\mathbf{u}2) = \prod_{i=1}^{k} W(\mathbf{u}_i 2) = \prod_{i=1}^{k} \frac{1}{u_i(u_i - 1)} = \mu(\mathbf{u}). \tag{3.5}$$

□

3.3 Lüroth Tree sequences

In a sense, a sequence $A \in \Sigma$ can be viewed as an infinite set of data. A normal number then has the property that if we take larger and larger samples, these samples will ultimately be "correctly distributed". (What one considers to be correct of course depends on the underlying measure.) The Lüroth Tree itself can also be considered as a data structure. And as Corollary 3.7 suggests, "sampling" from this tree (whatever that may mean) might yield good results. The goal is thus to find a sequence that effectively reflects these characteristics of the Lüroth Tree.

The constructed sequence will be an infinite concatenation of words from the Lüroth Tree. In order to properly analyze such a sequence, we not only view it as a sequence of digits, but also as a sequence of concatenated words. The following makes this precise.

3.8 Definition (Concatenation)**.** Let $A = (a_i)_{i=1}^{\infty} \in \Sigma$ be a sequence of digits and let $c : \mathbb{N} \to \mathbb{N}, i \mapsto c(i)$ be a map. Then c is a *concatenation map* if it satisfies the following:

$$\begin{aligned} &c(1) = 1, \\ &c(i) - c(i-1) \in \{0, 1\} \text{ for all } j \in \mathbb{N}, \\ &\#\{i : c(i) = j\} < \infty \text{ for all } j \in \mathbb{N}. \end{aligned}$$

The idea is that c assigns to each digit a_i a number $j = c(i)$ to indicate the word that it belongs to. After a finite amount of digits, this $c(i)$ increases by 1 and thus marks the beginning of a new word.
We further define

$$\begin{aligned} m(j) &= \min\{i \in \mathbb{N} : c(i) = j\}, \\ M(j) &= \max\{i \in \mathbb{N} : c(i) = j\}. \end{aligned}$$

This allows us to construct an auxiliary sequence

$$A_c := (\mathbf{u}_j)_{j=1}^{\infty}$$

of words $\mathbf{u}_j \in \mathcal{A}^*$ such that

$$\mathbf{u}_j = a_{m(j)} ... a_{M(j)}.$$

In this way, one could say that the sequence A of digits splits into a sequence A_c of words via the map c.
For each $\alpha \in \mathcal{A}^*$, we write

$$A_c(\alpha) := \{j \in \mathbb{N} : \mathbf{u}_j = \alpha\},$$

as the indices j of words of A_c that are equal to α.
As we will see, the most important aim of the construction is ensuring that the value $\#A_c(\alpha)$ is proportional to $W(\alpha)$. This means that we have to add some words multiple times, since $W(\alpha)$ is a fractional value in (0,1) for all $\alpha \neq 2$.

For a word $\alpha \in \mathcal{A}^*$,

$$W(\alpha) = \prod_{j=1}^{D(\alpha)-1} \frac{n_j}{d_j}, \; n_j, d_j \in \mathbb{N},$$

is a product of fractions such that $d_j \leq j+1$, so that

$$W(\alpha) \cdot \prod_{j=1}^{D(\alpha)-1} d_j \in \mathbb{N}.$$

Now for $n \in \mathbb{N}$, the *superfactorial* $\mathrm{sf}(n)$ is defined as

$$\mathrm{sf}(n) := \prod_{k=1}^{n} k!.$$

Observe that for all $\alpha \in \mathcal{A}^*$, it holds that $\mathrm{sf}(D(\alpha))$ is a multiple of $\prod_{j=1}^{D(\alpha)-1} d_j$ so that

$$W(\alpha) \cdot \mathrm{sf}(D(\alpha)) \in \mathbb{N}.$$

This leads to the following.

3.9 Definition (Lüroth Tree sequence)**.** A sequence $A = (a_i)_{i=1}^{\infty} \in \Sigma$ is a *Lüroth Tree sequence* if there exists a concatenation map $c : \mathbb{N} \to \mathbb{N}$ such that for all $\alpha \in \mathcal{A}^*$ it holds that

$$\#A_c(\alpha) = W(\alpha) \cdot \mathrm{sf}(D(\alpha)). \tag{3.6}$$

Observe that by (3.3), the following holds:

$$\sum_{\mathbf{u} \in G(n)} \#A_c(\mathbf{u}) = \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) \cdot \mathrm{sf}(D(\mathbf{u})) = \mathrm{sf}(n). \tag{3.7}$$

This means that in a Lüroth Tree sequence exactly $\mathrm{sf}(n)$ words are element of $G(n)$.

3.10 Example. In this example, we concatenate the words of the Lüroth Tree from left to right, starting at the top. That is, we add the words of $G(n)$ in the order $n = 1, 2, 3...$, and write them as a sequence A. This is very similar to the approach found in [Bok09].
We start at $n = 1$, so (3.7) tells us that we only have to add $\mathrm{sf}(1) = 1$ word. Indeed, there is only one word to add as $G(1) = \{2\}$. This is all in accordance with the fact that $W(2) = 1$ so that $W(2) \cdot \mathrm{sf}(D(2)) = 1$. The concatenation map c so far maps $1 \mapsto 1$. Thus we start like this:

$$A = (2, ...),$$
$$A_c = (2, ...).$$

At $n = 2$, we have $\mathrm{sf}(2) = 2$ words to add from the collection $G(2) = \{22, 3\}$. It is all still very managable, as $W(22) = W(3) = \frac{1}{2}$ so that both words are also added once.

This yields

$$A = (2, 2, 2, 3, ...),$$
$$A_c = (2, 22, 3, ...),$$

and the updated version of c maps $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2$, and $4 \mapsto 3$.
Moving on to $n = 3$, we are going to add $\mathrm{sf}(3) = 12$ words selected from $G(3) = \{222, 23, 32, 4\}$. The respective weights are $W(222) = \frac{3}{12}, W(23) = \frac{3}{12}, W(32) = \frac{2}{12}$ and $W(4) = \frac{4}{12}$. This yields $\#A_c(222) = 3, \#A_c(23) = 3, \#A_c(32) = 2, \#A_c(4) = 4$, so that

$$A = (2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 3, 2, 3, 3, 2, 3, 2, 4, 4, 4, 4, ...),$$
$$A_c = (2, 22, 3, 222, 222, 222, 23, 23, 23, 32, 32, 4, 4, 4, 4, ...),$$

and $c(5) = c(6) = c(7) = 4, c(8) = c(9) = c(10) = 5$, etc.
We stop the example here as (3.7) forces us to add $\mathrm{sf}(4) = 288$ words at depth 4!

We take another look at the example. Observe that $\mu(2) = \frac{1}{2}$ of the words of $G(2)$ and $G(3)$ start with a 2. If we look more closely at the words of $G(3)$, $\mu(3) = \frac{1}{6}$ of them start with a 3, while $\mu(22) = \frac{1}{4}$ of them starts with 22. This is no coincidence, as we will show next.
Let $\alpha \in \mathcal{A}^*$, and write

$$A_c(\alpha, n) := \{j \in \mathbb{N} : \mathbf{u}_j \in G(n), \mathbf{u}_j = \alpha\gamma, \gamma \in \mathcal{A}^*\},$$

as the set of indices of words of depth n starting with α. Note that this set is empty if $D(\alpha) \geq n$, as we exclude the case $\gamma = \varepsilon$ in the definition.

3.11 Lemma. *Fix $\alpha \in \mathcal{A}^*$ and a Lüroth Tree sequence $A = (a_i)_{i=1}^{\infty}$ with concatenation map c. Then for $n \in \mathbb{N}$ such that $D(\alpha) < n$, it holds that*

$$\frac{\#A_c(\alpha, n)}{\mathrm{sf}(n)} = \mu(\alpha).$$

Proof. For any $\gamma \in \mathcal{A}^*$ such that $\mathbf{u}_j = \alpha\gamma \in G(n)$, it holds that

$$D(\gamma) = n - D(\alpha) > 0$$

by (3.2). This yields

$$\begin{aligned}
\#A_c(\alpha, n) &= \sum_{\gamma \in G(n-D(\alpha))} \#A_c(\alpha\gamma) \\
&= \sum_{\gamma \in G(n-D(\alpha))} W(\alpha\gamma) \cdot \mathrm{sf}(n) \\
&= \mathrm{sf}(n) \cdot \sum_{\gamma \in G(n-D(\alpha))} W(\alpha 2) \cdot W(\gamma) \\
&= \mathrm{sf}(n) \cdot W(\alpha 2) \cdot \sum_{\gamma \in G(n-D(\alpha))} W(\gamma) \\
&= \mathrm{sf}(n) \cdot \mu(\alpha).
\end{aligned}$$

using Definition 3.9, Lemma 3.6, Corollary 3.7 and (3.3). □

3.4 Length of words

Another important element of a Lüroth Tree sequence is the average length of words. We write

$$d(n) := \sum_{j:\mathbf{u}_j \in G(n)} |\mathbf{u}_j|$$

for the number of digits of depth n. Since each word $\mathbf{u} \in G(n)$ is added $\mathrm{sf}(n)W(\mathbf{u})$ times by (3.6), we can denote the weighted average number of digits per word of depth n as

$$\frac{d(n)}{\mathrm{sf}(n)} = \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) \cdot |\mathbf{u}|.$$

All words $\mathbf{u} \in G(n+1)$ can be written as $\mathbf{v}2$ or $\mathbf{v}^+, \mathbf{v} \in G(n)$, and $|\mathbf{u}2| = |\mathbf{u}| + 1 = |\mathbf{u}^+| + 1$. This yields the following calculation:

$$\begin{aligned}
\frac{d(n+1)}{\mathrm{sf}(n+1)} &= \sum_{\mathbf{u} \in G(n+1)} W(\mathbf{u}) \cdot |\mathbf{u}| \\
&= \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}2) \cdot |\mathbf{u}2| + \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}^+) \cdot |\mathbf{u}^+| \\
&= \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) \cdot W(\mathbf{u}) \cdot |\mathbf{u}2| + \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}^+) \cdot W(\mathbf{u}) \cdot |\mathbf{u}^+| \\
&= \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) \cdot W(\mathbf{u}) \cdot (|\mathbf{u}| + 1) + \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}^+) \cdot W(\mathbf{u}) \cdot |\mathbf{u}| \\
&= \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) \cdot W(\mathbf{u}) + \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) \cdot W(\mathbf{u}) \cdot |\mathbf{u}| + \sum_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}^+) \cdot W(\mathbf{u}) \cdot |\mathbf{u}| \\
&\geq \inf_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) + \sum_{\mathbf{u} \in G(n)} [f(\mathbf{u}, \mathbf{u}2) + f(\mathbf{u}, \mathbf{u}^+)] \cdot W(\mathbf{u}) \cdot |\mathbf{u}| \\
&= \inf_{\mathbf{u} \in G(n)} f(\mathbf{u}, \mathbf{u}2) + \frac{d(n)}{\mathrm{sf}(n)}.
\end{aligned}$$

Clearly $\frac{d(1)}{\mathrm{sf}(1)} = 1$. By (3.1), for $\mathbf{u}_j \in G(n)$ it holds that $f(\mathbf{u}_j, \mathbf{u}_j^+) \geq \frac{1}{n+1}$, and therefore

$$\frac{d(n+1)}{\mathrm{sf}(n+1)} \geq \frac{d(n)}{\mathrm{sf}(n)} + \frac{1}{n+1}, \tag{3.8}$$

hence

$$\frac{d(n)}{\mathrm{sf}(n)} \geq \sum_{m=1}^{n} \frac{1}{m}. \tag{3.9}$$

We also define

$$\mathrm{sf}^*(n) := \sum_{m=1}^{n} \mathrm{sf}(m),\ d^*(n) := \sum_{m=1}^{n} d(m),$$

as the total number of words and the total number of digits added up to and including those of $G(n)$ respectively. Observe that

$$\mathrm{sf}(n) = \prod_{k=1}^{n} k! \geq n \cdot \prod_{k=1}^{n-1} k! \geq \sum_{m=1}^{n-1} \prod_{k=1}^{m} k! = \sum_{m=1}^{n-1} \mathrm{sf}(m),$$

so that

$$\mathrm{sf}^*(n) = \sum_{m=1}^{n} \mathrm{sf}(m) \leq 2\mathrm{sf}(n). \tag{3.10}$$

The weighted average of digits per word over all words of depth at most n is given by

$$\frac{d^*(n)}{\mathrm{sf}^*(n)}.$$

We then have the following result.

3.12 Proposition. *The cumulative weighted average number of digits per word up until depth n is unbounded in n, i.e.*

$$\lim_{n\to\infty} \frac{d^*(n)}{\mathrm{sf}^*(n)} = \infty.$$

Proof. We calculate

$$\frac{d^*(n)}{\mathrm{sf}^*(n)} \geq \frac{d(n)}{\mathrm{sf}^*(n)} \geq \frac{d(n)}{2\mathrm{sf}(n)} \geq \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i}.$$

This yields the result, as the harmonic series diverges.

□

3.13 Remark. This result will enable us to ignore certain digits in the count for normality. As long as this is a fixed amount per word, Proposition 3.12 shows that these digits become negligible for n large enough. In for example [Bok09, Theorem1] and [DLR20, Lemma 4.3], this argument is used to get rid of the so called *divided occurrences*. Informally, this is the case when a word α overlaps with multiple words $\mathbf{u}_j$. In Example 3.10, we see a divided occurence of 32 at $i = 4$ as it contains a digit from $\mathbf{u}_3$ and $\mathbf{u}_4$. An undivided occurence would be the "full" word $\mathbf{u}_7 = 32$. We do not explicitly need this particular notion, although our use of Proposition 3.12 is very similar, as we will see in the next chapter.

4 Normality

In this chapter we prove the Lüroth normality of a specific type of Lüroth Tree sequences. We first examine general sequences by classifying the digits as either "good" or "bad". We argue that the latter are negligible in the limit, but also show that this does not imply normality. We therefore define two extra conditions, and prove that such sequences are normal. We provide many examples along the way.

4.1 Classification of digits

Let $A = (a_i)$ be a Lüroth Tree sequence with concatenation map c, and let $i \in \mathbb{N}$. We introduce some more terminology.
Similar to the previous chapter, we denote m_i and M_i as the least and greatest integer for which $c(i) = c(m_i) = c(M_i)$ respectively. Observe that they now depend on the index i of digits, not the index j of words. We then define the *prefix* $p_c(i)$ and *postfix* $s_c(i)$ as the words

$$p_c(i) := a_{m_i}...a_{i-1} \in \mathcal{A}^*,$$
$$s_c(i) := a_i...a_{M_i} \in \mathcal{A}^*.$$

We furthermore refer to $D_c(i) := D(\mathbf{u}_{c(i)})$ as the depth of a digit a_i, and observe that as $\mathbf{u}_{c(i)} = p_c(i)s_c(i)$, (3.2) implies that

$$D_c(i) = D(p_c(i)) + D(s_c(i)). \tag{4.1}$$

Now let $\beta \in \mathcal{A}^* \cup \{\varepsilon\}$ be a word. We write

$$A(\beta, n) = \{i \in \mathbb{N} : p_c(i) = \beta, D_c(i) = n\}$$

for the set of indices i of depth n with prefix $p_c(i) = \beta$. In this way, we divide the digits into separate sets depending on their prefix and depth. We will sometimes refer to these sets as *classes* of digits. Note that this time, we include the possibility $\beta = \varepsilon$, as the first digits of words have empty prefix. Further observe that the indices in $A(\beta, n)$ are equal to the indices of the first digits of γ in the definition of $A_c(\beta, n)$. Formally, we have

$$i \in A(\beta, n) \iff c(i) \in A_c(\beta, n), i = m_i + |\beta| - 1,$$

so there is a bijection

$$A(\beta, n) \longleftrightarrow A_c(\beta, n) \tag{4.2}$$

mapping $i \mapsto c(i)$. Note that if $D(\beta) \geq n$, both sets are empty.

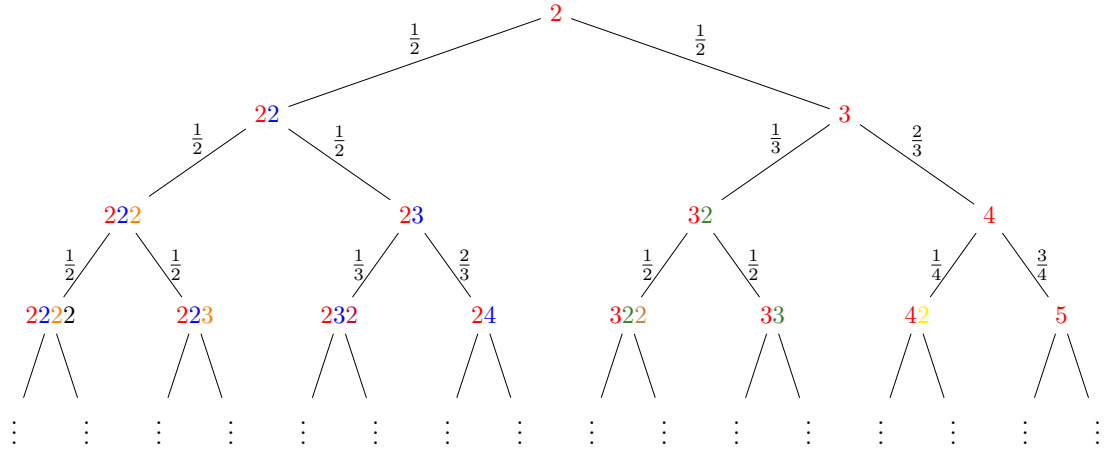


Figure 4: The classes $A(\beta, n)$ for $1 \leq n \leq 4$. Digits of classes with the same β share colors.

4.1 Example. We will work out some classes in detail in low depths n for Example 3.10. We use the following overview to keep track of the relationship between the digits and indices for $n \leq 3$:

a_i	2	2	2	3	2	2	2	2	2	2	2	2	2	2	3	2	3	2	3	3	2	3	2	4	4	4	4
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$c(i)$	1	2	2	3	4	4	4	5	5	5	6	6	6	7	7	8	8	9	9	10	10	11	11	12	13	14	15

At depth $n = 1$, there is only one class $A(\varepsilon, 1) = \{1\}$.

At depth $n = 2$, we have two classes: $A(\varepsilon, 2) = \{2, 4\}$ and $A(2, 2) = \{3\}$.

At depth 3, we have more classes and they already become much larger as we need to add words multiple times. The full overview:

- $A(\varepsilon, 3) = \{5, 8, 11, 14, 16, 18, 20, 22, 24, 25, 26, 27\}$, (red)
- $A(2, 3) = \{6, 9, 12, 15, 17, 19\}$, (blue)
- $A(22, 3) = \{7, 10, 13\}$, (orange)
- $A(3, 3) = \{21, 23\}$. (green)

Similar to Example 3.10, we can make the following observation.
Let $\alpha = \alpha_1 \ldots \alpha_k \in \mathcal{A}^*$. We define

$$\mathcal{A}^*(\alpha) = \{\mathbf{u} \in \mathcal{A}^* : \mathbf{u} = \alpha\gamma, \gamma \in \mathcal{A}^* \cup \{\varepsilon\}\},$$

as the set of words that share their first digits with α.

If we look at $A(\varepsilon, 3)$ (the red digits of depth 3), we see that $\mu(2) = \frac{1}{2}$ of the digits satisfy $s_c(i) \in \mathcal{A}^*(2)$. Another $\mu(3) = \frac{1}{6}$ of these digits have the property that $s_c(i) \in \mathcal{A}^*(3)$, and for $\mu(22) = \frac{1}{4}$ digits it holds that $s_c(i) \in \mathcal{A}^*(22)$. This is not surprising in light of Lemma 3.11: the postfix of a digit i with prefix ε is of course equal to the complete word $\mathbf{u}_{c(i)}$ itself.
However, observe that half of the blue digits of $A(2, 3)$ *also* satisfy $s_c(i) \in \mathcal{A}^*(2)$, just like the red digits of $A(\varepsilon, 2)$. A closer look at Figure 4 reveals a recursive pattern: the blue digits of depth $n + 1$ have exactly the same postfixes as the red digits of level n. Similarly, the postfixes of the orange and green digits with prefix 22 and 3 are also equal to those of the red digits, but with a respective delay of two depths instead of one. Important to note is that Lemma 3.3 ensures that the respective weights also show this pattern. If we look at $A(2, 3)$ for example, observe that $\mu(2) = \frac{1}{2}$ of the digits starts with a 2.
In order to make this precise, fix $\alpha, \beta \in \mathcal{A}^*$. Then we define

$$A(\beta, \alpha, n) = \{i \in \mathbb{N} : p_c(i) = \beta, s_c(i) \in \mathcal{A}^*(\alpha), D_c(i) = n\}$$

as the set of digits i with prefix β and depth n such that their postfix shares the first digits with those of α. Observe that for every such digit, we have exactly one word $\mathbf{u}_j = \beta\alpha\gamma, \gamma \in \mathcal{A}^* \cup \{\varepsilon\}$. In the event that $D(\beta) < n - D(\alpha)$, we have $\gamma \in \mathcal{A}^*$, yielding a bijection

$$A(\beta, \alpha, n) \longleftrightarrow A_c(\beta\alpha, n) \tag{4.3}$$

mapping $i \mapsto c(i)$.

4.2 Proposition. *Let $A = (a_i)_{i=1}^{\infty}$ be a Lüroth Tree sequence with concatenation map c. Fix $\alpha \in \mathcal{A}^*$. For $n \in \mathbb{N}$ and $\beta \in \mathcal{A}^*$ such that $D(\beta) < n - D(\alpha)$, it holds that*

$$\frac{\#A(\beta, \alpha, n)}{\#A(\beta, n)} = \mu(\alpha).$$

Proof. By (4.2) and (4.3), we may write

$$\begin{aligned}
\frac{\#A(\beta, \alpha, n)}{\#A(\beta, n)} &= \frac{\#A_c(\beta\alpha, n)}{\#A_c(\beta, n)} \\
&= \frac{\mu(\beta\alpha)}{\mu(\beta)} \\
&= \frac{W(\beta\alpha 2)}{W(\beta 2)} \\
&= \frac{W(\beta 2) \cdot W(\alpha 2)}{W(\beta 2)} \\
&= W(\alpha 2) \\
&= \mu(\alpha),
\end{aligned}$$

using Lemma 3.11 and Lemma 3.6 along the way. Note that it is also possible to calculate the ratio $\frac{\mu(\beta\alpha)}{\mu(\beta)}$ directly from (2.2). □

4.3 Example. If we take $\alpha = \beta = 2, n = 3$, then $D(\alpha) = D(\beta) = 1$, so the condition of Proposition 4.2 is satisfied as $1 < 3-1$. In the case of Example 4.1, we have $A(2,2,3) = \{6,9,12\}$ and $A(2,3) = \{6,9,12,15,17,19\}$, yielding the correct value: $\frac{\#A(2,2,3)}{\#A(2,3)} = \frac{3}{6} = \mu(2)$.

4.4 Definition. Let A be a Lüroth Tree sequence, and let $l \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we define the union

$$\begin{aligned} U_A(n,l) &:= \bigcup_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-l}} A(\beta, n) \\ &= \{i \in \mathbb{N} : D_c(i) = n, D(p_c(i)) < n-l\} \end{aligned}$$

as the collection of indices i of depth n such that their prefix has depth less than $n-l$. By (4.1), we have

$$U_A(n,l) = \{i \in \mathbb{N} : D_c(i) = n, D(s_c(i)) > l\},$$

the indices i of depth n such that their postfix has depth greater than l.
For $\alpha \in \mathcal{A}^*$, we further define a subset of $U_A(n, D(\alpha))$ as

$$U_{A,\alpha}(n) := \bigcup_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} A(\beta, \alpha, n) = \{i \in U_A(n, D(\alpha)) : s_c(i) \in \mathcal{A}^*(\alpha)\},$$

containing the indices of digits of depth n such that $s_c(i) \in \mathcal{A}^*(\alpha)$.

4.5 Corollary. Fix $\alpha \in \mathcal{A}^*, n \in \mathbb{N}$. Then

$$\frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} = \mu(\alpha).$$

Proof. A digit can not have more than one prefix, so both unions consist of pairwise disjoint sets. This yields, using the same arguments as in Proposition 4.2,

$$\begin{aligned} \frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} &= \frac{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} \#A(\beta, \alpha, n)}{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} \#A(\beta, n)} \\ &= \frac{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} W(\beta 2) \cdot W(\alpha 2)}{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} W(\beta 2)} \\ &= W(\alpha 2) \cdot \frac{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} W(\beta 2)}{\sum_{\substack{\beta \in \mathcal{A}^*: \\ D(\beta) < n-D(\alpha)}} W(\beta 2)} \\ &= W(\alpha 2) \\ &= \mu(\alpha). \end{aligned}$$

□

For $\alpha \in \mathcal{A}^*, |\alpha| = k$, we take a closer look at the set $U_A(n, D(\alpha))$. For $i \in U_A(n, D(\alpha))$, assume that $a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k$. Then as $D(s_c(i)) > D(\alpha)$, we must have $s_c(i) = \alpha\gamma$ for some $\gamma \in \mathcal{A}^*$, so that $s_c(i) \in \mathcal{A}^*(\alpha)$.
Conversely, suppose that $s_c(i) \in \mathcal{A}^*(\alpha)$. Then clearly have $a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k$.
So for $i \in U_A(n, D(\alpha))$ it holds that

$$s_c(i) \in \mathcal{A}^*(\alpha) \iff a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k.$$

Now let $T \subset \mathbb{N}$. Such a T will often be a set of consecutive integers, that we denote as

$$[k, m] := [k+1, m] \cap \mathbb{N},$$

and in the case that $k = 0$ we abbreviate $[m] := [1, m] \cap \mathbb{N}$.

If we define

$$N_A(\alpha, T) := \frac{\#\{i \in T : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{\#T},$$

the above shows that we can write

$$\begin{aligned} N_A(\alpha, U_A(n, D(\alpha))) &= \frac{\#\{i \in U_A(n, D(\alpha)) : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{\#U_A(n, D(\alpha))} \\ &= \frac{\#\{i \in U_A(n, D(\alpha)) : s_c(i) \in \mathcal{A}^*(\alpha)\}}{\#U_A(n, D(\alpha))} \\ &= \frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} \\ &= \mu(\alpha). \end{aligned}$$

Writing

$$U_A^*(n, D(\alpha)) = \bigcup_{m=1}^{n} U_A(m, D(\alpha)),$$

we obtain

$$\begin{aligned} N_A(\alpha, U_A^*(n, D(\alpha))) &= \frac{\#\{i \in U_A^*(n, D(\alpha)) : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{\#U_A^*(n, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} \#\{i \in U_A(m, D(\alpha)) : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} \#\{i \in U_A(m, D(\alpha)) : s_c(i) \in \mathcal{A}(\alpha)\}}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} N_A(\alpha, U_A(m, D(\alpha))) \cdot \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} \mu(\alpha) \cdot \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &= \mu(\alpha), \end{aligned} \tag{4.4}$$

as the sets $U_A(m, D(\alpha))$ are clearly pairwise disjoint.

This result is very promising, as it shows that if we only look at the digits of $U_A^*(n, D(\alpha))$, we have exactly the result we want. Informally, one can think of $U_A^*(n, D(\alpha))$ as the "good" digits: digits such that their postfix "allows enough room for α to occur". So what about the "bad" digits with indices not in $U_A^*(n, D(\alpha))$?

Observe that for a word $\mathbf{u} \in \mathcal{A}^*$ it always holds that

$$|\mathbf{u}| \leq D(\mathbf{u}).$$

(Note we have $|\mathbf{u}| = D(\mathbf{u})$ if and only if $\mathbf{u} = \underbrace{222...222}_{D(\mathbf{u}) \text{ times}}$.) For a postfix $s_c(i)$, this yields

$$|s_c(i)| \leq D_c(i) - D(p_c(i)).$$

If we fix $\alpha \in \mathcal{A}^*$, and the postfix $s_c(i)$ has length at least $D(\alpha) + 1$, then

$$D(\alpha) < D_c(i) - D(p_c(i)),$$

so that $i \in U_A(n, D(\alpha))$. This means that at most $D(\alpha)$ digits of each word do not belong to $U_A^*(n, D(\alpha))$. This yields the following estimate:

$$\#U_A^*(n, D(\alpha)) \geq d^*(n) - \mathrm{sf}^*(n)D(\alpha). \tag{4.5}$$

The next proposition shows that in the limit, almost all digits belong to $U_A^*(n, D(\alpha))$.

4.6 Proposition. *Fix $\alpha \in \mathcal{A}^*$, and let A be a Lüroth Tree sequence with concatenation map c. Then*

$$\lim_{n\to\infty} \frac{\#U_A^*(n, D(\alpha))}{d^*(n)} = 1.$$

Proof. Let $n \in \mathbb{N}$. Then by (4.5) and Proposition 3.12,

$$\begin{aligned} 1 \geq \frac{\#U_A^*(n, D(\alpha))}{d^*(n)} &\geq \frac{d^*(n) - \mathrm{sf}^*(n)D(\alpha)}{d^*(n)} \\ &= 1 - \frac{\mathrm{sf}^*(n)}{d^*(n)} D(\alpha) \to 1. \end{aligned}$$

□

Later on, we will also use the following result for a single depth n only.

4.7 Corollary. Fix $\alpha \in \mathcal{A}^*$, and let A be a Lüroth Tree sequence with concatenation map c. Then

$$\lim_{n\to\infty} \frac{\#U_A(n, D(\alpha))}{d(n)} = 1.$$

Proof. By (3.9), we also have that $\frac{d(n)}{\mathrm{sf}(n)} \to \infty$. The result follows in exactly the same way. □

Recall that a sequence $A = (a_i)_{i=1}^{\infty}$ is normal if for every $\alpha = \alpha_1 ... \alpha_k \in \mathcal{A}^*$ it holds that

$$\lim_{k\to\infty} N_A(\alpha, [k]) = \lim_{n\to\infty} \frac{\#\{i \leq k : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{k} = \mu(\alpha).$$

The limit is naturally calculated "from left to right", causing trouble for certain orderings of words.

4.8 Example. Suppose that c is such that before each word $\mathbf{u}_i$ that has at least one digit that is not a 2, we would have $\mathbf{u}_{i-1} = 222\dots222$ such that $|\mathbf{u}_{i-1}|$ is at least $2|\mathbf{u}_i|$. The resulting digit sequence cannot be normal, as this ordering yields $N_A(2, [k]) \geq \frac{2}{3} > \frac{1}{2} = \mu(2)$. For example, we could take c such that

$$A_c = (2, 22, 3, 222, 4, 222, 4, 222, 4, 2222, 4, 2222, 23, 2222, 23, 2222, 23, 2222, 32, 2222, 32, \dots).$$

Note that in this way, we're adding words of depth 4 such as 2222 "too early", i.e. we assign them a smaller index j than for example the words 32 or 4 of depth 3. Such c does not violate the conditions of Definition 3.9, as the set of words consisting of only 2's and its complement in $\mathcal{A}^*$ are both countable sets.

4.2 A first condition

The example suggests that forcing c to respect the order of depths is a step in the right direction. We can choose c to be such that for all $n \geq 1$, it holds that

$$\{j \in \mathbb{N} : D(\mathbf{u}_j) \leq n\} = [\mathrm{sf}^*(n)]. \tag{4.6}$$

Note that the c of Example 3.10 satisfies this criterion. Also observe that (4.6) implies that

$$\{i \in \mathbb{N} : D_c(i) \leq n\} = [d^*(n)].$$

for all $n \in \mathbb{N}$. This yields the following.

4.9 Lemma. *Fix $\alpha \in \mathcal{A}^*$, and let A be a Lüroth Tree sequence with concatenation map c satisfying* (4.6)*. Then*

$$\lim_{n\to\infty} N_A(\alpha, [d^*(n)]) = \mu(\alpha).$$

Proof. Write $|\alpha| = k$. We then calculate

$$\begin{aligned} N_A(\alpha, [d^*(n)]) &= \frac{\#\{i \leq d^*(n) : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{d^*(n)} \\ &= \frac{\#\{i \in \mathbb{N} : D_c(i) \leq n, a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{d^*(n)} \\ &= \frac{\#\{i \in U_A^*(n, D(\alpha)) : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{d^*(n)} \\ &\quad + \frac{\#\{i \notin U_A^*(n, D(\alpha)) : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{d^*(n)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{N_A(\alpha, U_A^*(n, D(\alpha))) \cdot \#U_A^*(n, D(\alpha))}{d^*(n)} \\
&\quad + \frac{N_A(\alpha, [d^*(n)] \backslash U_A^*(n, D(\alpha))) \cdot (d^*(n) - \#U_A^*(n, D(\alpha)))}{d^*(n)} \\
&= N_A(\alpha, U_A^*(n, D(\alpha))) \cdot \frac{\#U_A^*(n, D(\alpha))}{d^*(n)} \\
&\quad + N_A(\alpha, [d^*(n)] \backslash U_A^*(n, D(\alpha))) \cdot \frac{(d^*(n) - \#U_A^*(n, D(\alpha)))}{d^*(n)} \\
&\to \mu(\alpha)
\end{aligned}
$$

as $n \to \infty$ by Proposition 4.6 and (4.4). □

This is not yet enough for normality, as the values of $N_A(\alpha, [k])$ can significantly move away from $\mu(\alpha)$ in between the "milestones" $d^*(n)$, as the following example shows.

4.10 Example. We return once more to Example 3.10, where we started each new depth with the word $222\ldots222$. We again make use of the table that was displayed in Example 4.1 for the first three depths.

a_i	2	2	2	3	2	2	2	2	2	2	2	2	2	2	3	2	3	2	3	3	2	3	2	4	4	4	4
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Suppose that we are interested in the value $N_A(3, [k])$. We know by Lemma 4.9 that for $k \in \{1, 4, 27\ldots\}$ these values should converge to $\mu(3) = \frac{1}{6}$, and indeed, we see that

$$
\begin{aligned}
N_A(3, [1]) &= 0, \\
N_A(3, [4]) &= \tfrac{1}{4}, \\
N_A(3, [27]) &= \tfrac{6}{27} = \tfrac{2}{9}.
\end{aligned}
$$

But what happens if we measure at different "milestones"? If we take the subsequence $(k_n)_{n\geq 1} = (d^*(n) + \frac{\mathrm{sf}(n)}{2^{n-1}} \cdot n)_{n\geq 1}$, we have $k_1 = 1, k_2 = 3, k_3 = 13$, and this yields

$$
\begin{aligned}
N_A(3, [1]) &= 0, \\
N_A(3, [3]) &= 0, \\
N_A(3, [13]) &= \tfrac{1}{13}.
\end{aligned}
$$

These values of k_n do not mark the point where we have finished adding all words of depth n, but include the first batch of 2's that stem from all added words $\beta = 222\ldots222$ of the next depth $n+1$. This word β of depth $n+1$ has weight $\frac{1}{2^n}$, so it is added $2^{-n} \cdot \mathrm{sf}(n+1)$ times. If we compare this to all $\mathrm{sf}^*(n)$ previous words we obtain using (3.10) the following estimate on the ratio:

$$
\begin{aligned}
\frac{2^{-n} \cdot \mathrm{sf}(n+1)}{\mathrm{sf}^*(n)} &\geq \frac{2^{-n} \cdot \mathrm{sf}(n+1)}{2\mathrm{sf}(n)} \\
&= \frac{(n+1)!}{2^{n+1}}.
\end{aligned}
$$

In other words, we add the word $222\ldots222$ at least $\frac{(n+1)!}{2^{n+1}}$ times more than all previous words.

Also note that this word has more digits than all previous words, so that we also add at least this many digits. As we clearly have

$$\frac{n!}{2^n} \to \infty$$

as $n \to \infty$, this implies that

$$\lim_{n\to\infty} N_A(3, [k_n]) = 0.$$

So unfortunately, we have found an α and a subsequence (k_n) for which $N_A(3, [k_n])$ does not converge to the right value. In fact, note that for *any* $\alpha \neq 222\ldots222$ we have

$$\lim_{n\to\infty} N_A(\alpha, [k_n]) = 0,$$

while for any $\beta = 222\ldots222$ it holds that

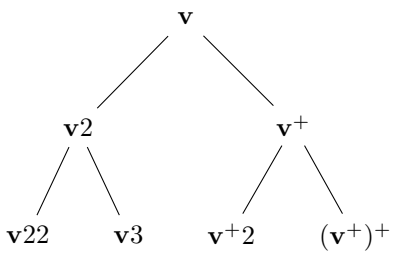
$$\lim_{n\to\infty} N_A(\beta, [k_n]) = 1.$$

This shows that a continuation of the procedure that we used in Example 3.10 will not yield a normal Lüroth expansion. As nice as the limit behaves for the "milestones" $d^*(n)$, this example also revealed the existence of subsequences with contrasting properties. "Stumbling blocks" like $d^*(n) + \frac{\mathrm{sf}(n)}{2^{n-1}} \cdot n$ in our example may prevent a good convergence result, as $N_A(\alpha, [k])$ infinitely often fluctuates between values arbitrarily close to $\mu(\alpha)$ and arbitrarily close to either 0 or 1. The construction that we describe in the next section is designed to tackle exactly this problem.

4.3 A second condition

The aim in our construction is to not just add all words of the same depth together, but also order the words within one depth. This should prevent the massive addition of just one word as exhibited in Example 4.10.
In order to make this precise, we introduce some more terminology.

4.11 Definition. Let $\mathbf{u} \in \mathcal{A}^*, D(\mathbf{u}) \geq 3$. Then we define the *grandparent* $\mathbf{u}^{-2}$ of $\mathbf{u}$ as the word $\mathbf{v}$ such that $\mathbf{u} \in \{\mathbf{v}22, \mathbf{v}3, \mathbf{v}^+2, (\mathbf{v}^+)^+\}$.



So if for example $\mathbf{v} = 3$, then it is the grandparent of the words 322, 33, 42 and 5.

4.12 Lemma. *Fix* $\mathbf{v} \in \mathcal{A}^*$. *The following holds:*

$$\sum_{\mathbf{u}:\mathbf{u}^{-2}=\mathbf{v}} W(\mathbf{u}) = W(\mathbf{v}).$$

Proof. Using (3.1), we have

$$\sum_{\mathbf{u}:\mathbf{u}^{-2}=\mathbf{v}} W(\mathbf{u}) = W(\mathbf{v}22) + W(\mathbf{v}3) + W(\mathbf{v}^+2) + W((\mathbf{v}^+)^+)$$
$$= W(\mathbf{v}2) + W(\mathbf{v}^+) = W(\mathbf{v}).$$

□

The idea is that we add the words of depth n in such a way that their grandparents are a faithful representation of the words of depth $n-2$. This is formalized in the following definition.

4.13 Definition (Strongly ordered Lüroth Tree sequence)**.** A sequence $A = (a_i)_{i=1}^{\infty} \in \Sigma$ is a *strongly ordered Lüroth Tree sequence* if there exists a concatenation map $c : \mathbb{N} \to \mathbb{N}$ that satisfies the following two conditions.

1. For all $n \geq 1$ and $\beta \in G(n)$, it must hold that

$$\#\{j \leq \mathrm{sf}^*(n) : \mathbf{u}_j = \beta\} = W(\beta) \cdot \mathrm{sf}(n). \tag{4.7}$$

2. Furthermore, for all $n > 3, k \in [n!(n-1)!]$, and $\beta \in G(n-2)$, it must hold that

$$\#\{j \in [\mathrm{sf}^*(n-1) + (k-1) \cdot \mathrm{sf}(n-2), \mathrm{sf}^*(n-1) + k \cdot \mathrm{sf}(n-2)] : \mathbf{u}_j^{-2} = \beta\} = W(\beta) \cdot \mathrm{sf}(n-2). \tag{4.8}$$

4.14 Remark. Observe that the first condition inductively implies (4.6) and (3.6). This is because (4.7) holds for all n, so it holds for $n = 1$ in particular. Thus the first word must be 2. It also holds for $n = 2$, forcing the second and third word to be 22 and 3, and so on. This shows that a strongly ordered Lüroth Tree sequence is also a Lüroth Tree sequence that satisfies condition (4.6) on the depth of words.
Past depth $n = 3$, the second condition divides the $\mathrm{sf}(n)$ words of depth n into $n!(n-1)!$ blocks of $\mathrm{sf}(n-2)$ words. Viewing this block as a collection of $\mathrm{sf}(n-2)$ grandparents, the frequencies must be in accordance with the sequence of words of depth $n-2$ that satisfied (4.7).

4.15 Example. Clearly Example 3.10 satisfies (4.7). As all words of depth 3 have the single grandparent 2, the first three depths also did not violate (4.8). But what happens at depth $n = 4$?
All words of depth 4 have grandparent either 22 or 3. According to (4.8), every $\mathrm{sf}(4-2) = 2$ consecutive words need to be such that $W(22) \cdot \mathrm{sf}(2) = 1$ of them has grandparent 22, and the other $W(33) \cdot \mathrm{sf}(2) = 1$ has grandparent 3. So if we start adding the words from the Lüroth Tree at depth 4, we have to alternate between words $\mathbf{u} \in \{2222, 223, 232, 24\}$ and $\mathbf{v} \in \{322, 33, 42, 5\}$, so A_c will look for example like this:

$$A_c = (\dots, 2222, 322, 223, 33, 232, 42, 24, 5, \dots).$$

We can confidently continue this process for depth 4 as we can easily see that exactly half of the words we have to add of depth 4 has grandparent 2 and the other half has grandparent 3.

This principle is in fact guaranteed to work for all higher depths by Lemma 4.12: for every $\beta \in G(n-2)$ we have

$$\sum_{\mathbf{u}:\mathbf{u}^{-2}=\beta} \#A_c(\mathbf{u}) = \sum_{\mathbf{u}:\mathbf{u}^{-2}=\beta} W(\mathbf{u}) \cdot \mathrm{sf}(n) = W(\beta) \cdot \mathrm{sf}(n)$$

words of depth n that have β as their grandparent. This shows that we can choose exactly $W(\beta) \cdot \mathrm{sf}(n-2)$ of these words to be part of one of the $n!(n-1)!$ batches of $\mathrm{sf}(n-2)$ words.

Example 4.15 explicitly shows that strongly ordered Lüroth Tree sequences exist, and we will show that any such sequence is Lüroth normal.

4.16 Remark. The construction of [Bok09] is also an example of a strongly ordered Lüroth Tree sequence. This construction is ordered even stronger because it not only respects the weights of words one could call "parents", but all "higher-order parents" simultaneously. In fact, the condition (4.8) does not have to be on the grandparents, it can be altered to any fixed "higher-order parent". Whether this yields a larger or smaller degree of freedom seems to be a matter of taste. See the last chapter for more details.

4.4 Proof of normality

Let A be a strongly ordered Lüroth Tree sequence. We write

$$I_A(n) := \{i \in \mathbb{N} : D_c(i) = n\}$$

for the digits of depth n, and observe that by (4.7), it holds that

$$I_A(n) = [d^*(n-1) \dots d^*(n)].$$

For this set we have the following result.

4.17 Lemma. *Fix $\alpha \in \mathcal{A}^*$, and let A be a strongly ordered Lüroth Tree sequence. Then*

$$\lim_{n\to\infty} N_A(\alpha, I_A(n)) = \mu(\alpha).$$

Proof. We follow the proof of Lemma 4.9, using Corollary 5.18 instead of Proposition 4.6. □

Let $A = (a_i)_{i=1}^{\infty}, a_i' \in \mathcal{A}^*$, be any sequence, and $x, y \in \mathbb{N}$ integers such that $x < y$. We take a closer look at $N_A(\alpha, [x, y])$. Suppose that we change the value of a single digit $a_{i^*}, i^* \in [x, y]$. We write this altered sequence as $A' = (a_i')_{i=1}^{\infty}\Sigma$ where $a_i' = a_i$ if and only if $i \neq i^*$. Observe that $N_{A'}(\alpha, \{i\}) = N_A(\alpha, \{i\})$ if $i \notin [i^* - (|\alpha| - 1), i^*]$, so the value of $N_A(\alpha, \{i\})$ changes for at most $|\alpha|$ digits. This leads to the following bound:

$$\big|(y-x) \cdot N_A(\alpha, [x, y]) - (y-x) \cdot N_{A'}(\alpha, [x, y])\big| \leq |\alpha|.$$

Now suppose that we *add* a digit at some index $i^* \in [x, y]$. We can again create a new sequence A' such that $a_i = a_i'$ for $i < i^*$ and $a_{i+1}' = a_i$ for $i > i^*$. The digit a_{i^*}' can be arbitrarily chosen from $\mathcal{A}^*$. We now have $N_{A'}(\alpha, \{i\}) = N_A(\alpha, \{i\})$ for $i \leq i^* - |\alpha|$ and $N_A'(\alpha, i+1) = N_A'(\alpha, i)$ for $i > i^*$. This shows that we have at most $|\alpha| + 1$ changes:

$$\big|(y-x) \cdot N_A(\alpha, [x, y]) - (y+1-x) \cdot N_{A'}(\alpha, [x, y+1])\big| \leq |\alpha| + 1.$$

Instead of considering any subsequent string of digits with index $i \in [x, y]$, we take a look at the digits of $I_A(n)$ of a strongly ordered Lüroth Tree sequence A. We assume $n > 3$ and write

$$I_A(n) = \{i \in \mathbb{N} : \mathrm{sf}^*(n-1) < c(i) \leq \mathrm{sf}^*(n)\}.$$

Now suppose that for $j \in [\mathrm{sf}^*(n-1), \mathrm{sf}^*(n)]$, we replace a word $\mathbf{u}_j \in A_c$ by $(\mathbf{u}_j^+)^+$, thus creating a new sequence A' with concatenation map c' and sequence of words $A'_{c'}$. Observe that this comes down to the change of exactly one digit of index $i \in I_A(n)$. Writing

$$I_{A'}(n) = \{i \in \mathbb{N} : \mathrm{sf}^*(n-1) < c'(i) \leq \mathrm{sf}^*(n)\},$$

we obtain again

$$\left|\#I_A(n) \cdot N_A(\alpha, I_A(n)) - \#I_{A'}(n) \cdot N_{A'}(\alpha, I_{A'}(n))\right| \leq |\alpha|.$$

If we replace $\mathbf{u}_j$ by $\mathbf{u}_j 3$ instead, this amounts to adding exactly one digit, thus yielding exactly the same statement with $|\alpha| + 1$.
If we instead replace $\mathbf{u}_j$ by $\mathbf{u}_j 22$ or $\mathbf{u}_j^+ 2$, we change or add two digits. Writing A' for any new sequence such that one word $\mathbf{u}_j$ is replaced by either $\mathbf{u}_j 22, \mathbf{u}_j 3, \mathbf{u}_j^+ 2$ or $(\mathbf{u}_j^+)^+$ we have the following bound:

$$\left|\#I_A(n) \cdot N_A(\alpha, I_A(n)) - \#I_{A'}(n) \cdot N_{A'}(\alpha, I_{A'}(n))\right| \leq 2|\alpha| + 2.$$

Now fix $n > 3, k \in [n!(n-1)!]$. We define

$$I_A^k(n) := \{i \in \mathbb{N} : c(i) \in [\mathrm{sf}^*(n-1) + (k-1) \cdot \mathrm{sf}(n-2), \mathrm{sf}^*(n-1) + k \cdot \mathrm{sf}(n-2)]\}$$

as the set of indices of digits belonging to the k-th batch of $\mathrm{sf}(n-2)$ words of depth n.

4.18 Example. If we look at the sequence A in Example 4.15, we can make the following table for the first words of depth 4:

a_i	2	2	2	2	3	2	2	2	2	3	3	3	2	3	2	4	2	2	4	5
i	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47

We have $I_A^1(4) = [27, 34], I_A^2(4) = [34, 39], I_A^3(4) = [39, 44], I_A^4(4) = [44, 47]$, and so on.

As we saw in Definition 4.11, the words $\{\mathbf{u}_j : j \in [\mathrm{sf}^*(n-1) + (k-1) \cdot \mathrm{sf}(n-2), \mathrm{sf}^*(n-1) + k \cdot \mathrm{sf}(n-2)]\}$ to which the digits of $I_A^k(n)$ belong can all be written as $\mathbf{v}22, \mathbf{v}3, \mathbf{v}^+2$ or $(\mathbf{v}^+)^+$ for some $\mathbf{v} \in G(n-2)$.

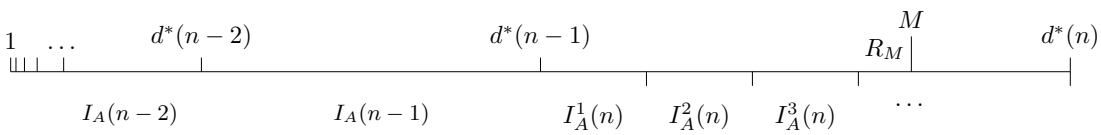


Figure 5: Illustration (not on scale) of the digit sets for a strongly ordered Lüroth sequence A.

The second condition (4.8) now assures that the distribution of words $\mathbf{u}_j$ is exactly such that the words $\mathbf{u}_j^{-2} \in G(n-2)$ are distributed in accordance with (3.6). In other words, (4.8) guarantees a one-to-one correspondence (including multiple instances) between the words of depth $n-2$ and the grandparents of words that supply the digits of $I_A^k(n)$.
This shows that for every $k \in [n!(n-1)!]$, there exists a strongly ordered Lüroth Tree sequence A' with concatenation map c' such that for every $l \in [\mathrm{sf}(n-2)]$,

$$A'_{c'} \ni \mathbf{v}_{\mathrm{sf}^*(n-3)+l} = \mathbf{u}^{-2}_{\mathrm{sf}^*(n)+(k-1)\cdot \mathrm{sf}(n-2)+l} \in A_c. \tag{4.9}$$

The words on the left side of this equation represent the digits of A' indexed by $I_{A'}(n-2)$, while the words on the rights side exactly represent the digits of A indexed by $I_A^k(n)$. Thus in order to transform $I_{A'}(n-2)$ into $I_A^k(n)$, we change or add at most 2 digits per word. This shows that for any n and k, there exists an A' such that

$$\left|\#I_{A'}(n-2)\cdot N_{A'}(\alpha, I_{A'}(n-2)) - \#I_A^k(n)\cdot N_A(\alpha, I_A^k(n))\right| \leq (2|\alpha|+2)\cdot \mathrm{sf}(n-2). \tag{4.10}$$

If we look at Example 4.18, we can compare all examples $I_A^k(4)$ to $I_A(2) = [1,4]$ of A itself, here printed again:

a_i	2	2	3
i	2	3	4

According to (4.10), at most $2\cdot \mathrm{sf}(4-2) = 4$ digits are added or changed in the transformation of $I_A(2)$ into $I_A^k(4)$.
Indeed, if we look at $I_A^1(4)$, we see that we have to add a 2 a total of 4 times. For $I_A^2(4)$, we add a 3 twice. The set $I_A^3(4)$ is a little more complicated: we change a 2 into a 3, add a 2, change a 3 into a 4, and add a 2, altogether a total of 4 modifications. The last set $I_A^4(4)$ has only 2 changes: we turn a 2 into a 4 and a 3 into a 5.

We are now in a position to prove the following proposition.

4.19 Proposition. *Fix $\alpha \in \mathcal{A}^*$, and let A be a strongly ordered Lüroth Tree sequence. Then for all $k \in [n!(n-1)!]$,*

$$\lim_{n\to\infty} N_A(\alpha, I_A^k(n)) = \mu(\alpha).$$

Proof. Take $n > 3, k \in [n!(n-1)!]$, and let A' be a sequence satisfying (4.9). Observe that $\#I_A^k(n) \geq d(n-2) = \#I_{A'}(n-2)$. This gives the following bound from above:

$$\begin{aligned} N_A(\alpha, I_A^k(n)) &\leq \frac{N_{A'}(\alpha, I_{A'}(n-2))\cdot d(n-2) + (2|\alpha|+2)\cdot \mathrm{sf}(n-2)}{\#I_A^k(n)} \\ &\leq \frac{N_{A'}(\alpha, I_{A'}(n-2))\cdot d(n-2) + (2|\alpha|+2)\cdot \mathrm{sf}(n-2)}{d(n-2)} \\ &= N_{A'}(\alpha, I_{A'}(n-2)) + (2|\alpha|+2)\cdot \frac{\mathrm{sf}(n-2)}{d(n-2)}. \end{aligned}$$

Also observe that $\#I_A^k(n) \leq d(n-2)+(2|\alpha|+2)\cdot \mathrm{sf}(n-2)$ by (4.10). This yields a similar bound from below:

$$\begin{aligned}
N_A(\alpha, I_A^k(n)) &\geq \frac{N_{A'}(\alpha, I_{A'}(n-2))\cdot d(n-2) - (2|\alpha|+2)\cdot \mathrm{sf}(n-2)}{d(n-2)+(2|\alpha|+2)\cdot \mathrm{sf}(n-2)} \\
&= N_{A'}(\alpha, I_{A'}(n-2))\cdot \frac{d(n-2)}{d(n-2)+(2|\alpha|+2)\cdot \mathrm{sf}(n-2)} \\
&\quad - \frac{(2|\alpha|+2)\cdot \mathrm{sf}(n-2)}{d(n-2)+(2|\alpha|+2)\cdot \mathrm{sf}(n-2)} \\
&= N_{A'}(\alpha, I_{A'}(n-2))\cdot \frac{1}{1+(2|\alpha|+2)\cdot \frac{\mathrm{sf}(n-2)}{d(n-2)}} - \frac{(2|\alpha|+2)\cdot \frac{\mathrm{sf}(n-2)}{d(n-2)}}{1+(2|\alpha|+2)\cdot \frac{\mathrm{sf}(n-2)}{d(n-2)}}.
\end{aligned}$$

Applying (3.9), we see that for every $\varepsilon \in \mathbb{R}_{>0}$, there exists an $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, we have

$$\left|N_A(\alpha, I_A^k(n)) - N_{A'}(\alpha, I_{A'}(n-2))\right| < \frac{1}{2}\varepsilon$$

for any $k \in [n!(n-1)!]$. By Lemma 4.17, there also exists an $N_2 \in \mathbb{N}$ such that all $n \geq N_2$, the identity

$$|\mu(\alpha) - N_{A'}(\alpha, I_{A'}(n-2))| < \frac{1}{2}\varepsilon$$

holds. Thus for all $n \geq \max\{N_1, N_2\}$,

$$\left|\mu(\alpha) - N_A(\alpha, I_A^k(n))\right| < \varepsilon.$$

□

4.20 Remark. Instead of using auxiliary strongly ordered Lüroth Tree sequences A', we could also obtain this result by examining permutations of words. Observe that in our construction, we demand no particular ordering of words as long as they satisfy the conditions (4.7) and (4.8). Consequently, results such as Lemma 4.17 hold for any permutation of words within the batches $I_A^k(n)$ marked by (4.8).
In order to compare the words of $I_A^k(n)$ to those of $I_A(n-2)$, we had to deal with the fact that the ordering of $I_A^k(n)$ might not equal to the ordering of grandparents in $I_A(n-2)$. Instead of what we just did, it is also possible to compute directly that permuting words has no significant effects in the limit.
In order to do so, note that Proposition 4.6 allows us to leave the last $|\alpha|$ digits of every word out of consideration, thus effectively discarding all divided occurrences mentioned earlier in Remark 3.13. This means that permuting words has no effect on $N_A(\alpha, I_A(n))$ in the limit, so that we can indeed compare the words of $I_A^k(n)$ to those of $I_A(n-2)$ without having to worry about the specific order.
Note that this fact was always silently implied by the nature of our construction.

Either way, Proposition 4.19 has proven that the sets $I_A^k(n)$ are similar enough to $I_A(n-2)$ in the limit giving us a lot more "milestones". As the next theorem shows, these milestones will eliminate the possibility of subsequences $\{k_n\}$ as in Example 4.10.

4.21 Theorem (Lüroth normality)**.** *Let A be a strongly ordered Lüroth Tree sequence. Then for any $\alpha \in \mathcal{A}^*$,*

$$\lim_{m\to\infty} \frac{\#\{i \le m : a_i = \alpha_1, \dots, a_{i+k-1} = \alpha_k\}}{m} = \mu(\alpha).$$

Proof. Let $M \in \mathbb{N}$. Then there are unique integers $n = n_M \in \mathbb{N}, k = k_M \in [n_M!(n_M-1)!]$ such that $M \in I_A^k(n)$. Write

$$R_M := M - d^*(n-1) - \sum_{l=1}^{k-1} \#I_A^l(n) \in \mathbb{N},$$

and observe that $R_M \in [d(n-2) + 2\mathrm{sf}(n-2)]$. See also Figure 5.
By (3.8), we have that

$$\frac{\mathrm{sf}(n-2)}{d(n-2)} \ge \frac{\mathrm{sf}(n-1)}{d(n-1)},$$

so that

$$\frac{d(n-2)}{d(n-1)} \le \frac{\mathrm{sf}(n-2)}{\mathrm{sf}(n-1)}.$$

We use this to show that the last R_M digits are negligible compared to M:

$$\begin{aligned}
\frac{R_M}{M} &\le \frac{d(n-2) + 2\mathrm{sf}(n-2)}{d^*(n-1)} \\
&\le \frac{d(n-2)}{d(n-1)} + 2\frac{\mathrm{sf}(n-2)}{d(n-2)} \\
&\le \frac{\mathrm{sf}(n-2)}{\mathrm{sf}(n-1)} + 2\frac{\mathrm{sf}(n-2)}{d(n-2)} \\
&\le \frac{1}{(n-1)!} + 2\frac{\mathrm{sf}(n-2)}{d(n-2)} \to 0,
\end{aligned}$$

as $n \to \infty$ by (3.9).
Clearly $\lim_{M\to\infty} n = \infty$. Hence for any sequence A, we have

$$\begin{aligned}
N_A(\alpha, [M]) &= \frac{N_A(\alpha, [M - R_M]) \cdot (M - R_M) + N_A(\alpha, [M, R_M]) \cdot R_M}{M} \\
&= N_A(\alpha, [M - R_M]) \cdot \frac{(M - R_M)}{M} + N_A(\alpha, [M, R_M]) \cdot \frac{R_M}{M} \\
&= N_A(\alpha, [M - R_M]) \cdot (1 - \frac{R_M}{M}) + N_A(\alpha, [M, R_M]) \cdot \frac{R_M}{M},
\end{aligned}$$

so that

$$N_A(\alpha, [M]) \to N_A(\alpha, [M - R_M])$$

as $M \to \infty$.

Combining this with the other convergence results of Lemma 4.9 and Proposition 4.19, we fix $\varepsilon \in \mathbb{R}_{>0}$. Then there exist N_1, N_2, N_3 such that for all $M \geq \max\{N_1, N_2, N_3\}$, we have

$$\begin{aligned} |N_A(\alpha, [d^*(n)]) - \mu(\alpha)| &< \frac{1}{2}\varepsilon, \\ \left|N_A(\alpha, I_A^l(n)) - \mu(\alpha)\right| &< \frac{1}{2}\varepsilon, \text{ (for all } l \in [n!(n-1)!]) \\ |N_A(\alpha, [M]) - N_A(\alpha, [M - R_M])| &< \frac{1}{2}\varepsilon, \end{aligned}$$

so that

$$\begin{aligned} & |N_A(\alpha, [M]) - \mu(\alpha)| \\ < \; & |N_A(\alpha, [M - R_M]) - \mu(\alpha)]| + \frac{1}{2}\varepsilon \\ = \; & \left| \frac{N_A(\alpha, [d^*(n-1)]) \cdot d^*(n-1) + \sum_{l=1}^{k-1} N(\alpha, I_A^l(n)) \cdot \#I_A^l(n)}{M - R_M} - \mu(\alpha) \right| + \frac{1}{2}\varepsilon \\ = \; & \left| \frac{N_A(\alpha, [d^*(n-1)]) \cdot d^*(n-1) + \sum_{l=1}^{k-1} N(\alpha, I_A^l(n)) \cdot \#I_A^l(n)}{M - R_M} \right. \\ & \left. - \frac{(d^*(n-1) + \sum_{l=1}^{k-1} \#I_A^l(n)) \cdot \mu(\alpha)}{M - R_M} \right| + \frac{1}{2}\varepsilon \\ \leq \; & \frac{|N_A(\alpha, [d^*(n-1)]) - \mu(\alpha)| \cdot d^*(n-1) + \sum_{l=1}^{k-1} |N(\alpha, I_A^l(n)) - \mu(\alpha)| \cdot \#I_A^l(n)}{M - R_M} + \frac{1}{2}\varepsilon \\ < \; & \frac{\frac{1}{2}\varepsilon \cdot d^*(n-1) + \sum_{l=1}^{k-1} \frac{1}{2}\varepsilon \cdot \#I_A^l(n)}{M - R_M} + \frac{1}{2}\varepsilon \\ = \; & \frac{M - R_M}{M - R_M} \cdot \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ = \; & \varepsilon. \end{aligned}$$

□

4.22 Example. Theorem 4.21 shows that a number $x \in (0, 1]$ is Lüroth normal if its Lüroth expansion A_x is strongly ordered. If we take the sequence of Example 4.15, we get the following sequence of words:

$$A_c = (2, 22, 3, 222, 222, 222, 23, 23, 23, 32, 32, 4, 4, 4, 4, 2222, 322, 223, 33, 232, 42, 24, 5, \dots).$$

Using (2.1), we have

$$x = L(A_x) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{24} + \frac{1}{96} + \frac{1}{192} + \cdots = 0.936197\dots$$

as an example of a Lüroth normal number.

Of course the value of $L(A_x)$ is completely determined by the transformation T. In the next chapter, we will generalize the material of Chapters 3 and 4 so that we also obtain Lüroth normal numbers for the family of transformations that we described in Section 2.5.

5 Generalization

We generalize Chapters 3 and 4 to an arbitrary infinite partition of the unit interval. It turns out that we can significantly reduce the amount of words added per depth. Thus the generalization can be also seen as an improvement if applied to the "classical" setting. As most of this chapter is very similar to earlier material, we omit many proofs and examples.

5.1 Construction

Recall the definition of Generalized Lüroth Series as discussed in Chapter 2. Without loss of generality, we take $\mathcal{A} = \mathcal{D} := \mathbb{Z}_{\geq 1}, \Sigma = \mathcal{A}^{\mathbb{N}}$. This leads to the following generalization of the Lüroth Tree.

5.1 Definition (Generalized Lüroth Tree)**.** Let $\mathbf{u} = u_1 \dots u_k \in \mathcal{A}^*$..Write $\mathbf{u}1 := u_1 \dots u_k 1$ and $\mathbf{u}^+ = u_1 \dots u_{k-1}(u_k + 1)$, . Then we define the Lüroth Tree as the graph G_L such that

$$\begin{aligned} V(G_L) &= \mathcal{A}^*, \\ E(G_L) &= \{(\mathbf{u}, \mathbf{u}1), (\mathbf{u}, \mathbf{u}^+) : \mathbf{u} \in \mathcal{A}^*\}. \end{aligned}$$

For the generalized labels, we write

$$p_k := \frac{L_k}{1 - \sum_{i=1}^{k-1} L_i}, \; q_k := 1 - \frac{L_k}{1 - \sum_{i=1}^{k-1} L_i}.$$

Then the following map f assigns these labels to each edge:

$$\begin{aligned} f :& E(G_L) \to \mathbb{R}_+, \\ e \mapsto& \begin{cases} p_{u_k}, & \text{if } e = (\mathbf{u}, \mathbf{u}1) \text{ for some } \mathbf{u} = u_1 \dots u_k \in \mathcal{A}^*, \\ q_{u_k}, & \text{if } e = (\mathbf{u}, \mathbf{u}^+) \text{ for some } \mathbf{u} = u_1 \dots u_k \in \mathcal{A}^*. \end{cases} \end{aligned} \tag{5.1}$$

So for $u_k = k$, the generalized tree uses the rule

$$\begin{tikzcd} & u_1 u_2 \dots u_k \arrow[dl, dash, "p_k"'] \arrow[dr, dash, "q_k"] & \\ u_1 u_2 \dots u_k 1 & & u_1 u_2 \dots u_k + 1. \end{tikzcd}$$

If we start the construction at $\mathbf{u} = 1$, we can draw the following figure:

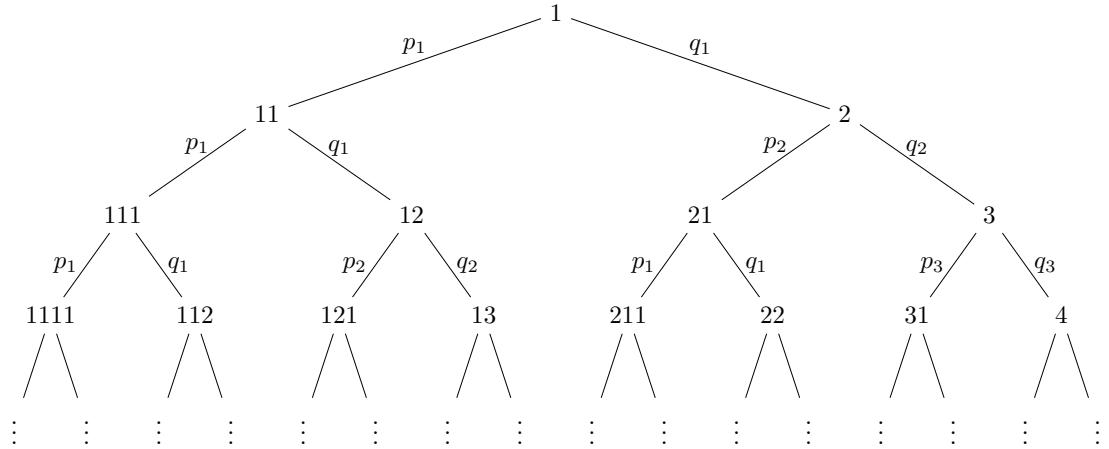


Figure 6: The Generalized Lüroth Tree.

Using the same definition of $\phi_{\mathbf{u}}$ as in Definition 3.2, we obtain an exact analogy of Lemma 3.3.

5.2 Lemma. *Let* $\mathbf{u} \in \mathcal{A}^* \cup \{\varepsilon\}$. *The label mapping* $f : E(G_L) \to \mathbb{R}_{\geq 0}$ *is* $\varphi_{\mathbf{u}}$ *invariant:*

$$f(\varphi_{\mathbf{u}}(e)) = f(e) \text{ for all } e \in E(G_L).$$

□

Zooming in on the labels, we have the following useful result.

5.3 Lemma. *Let* $n \in \mathbb{N}$. *Then*

$$\prod_{i=1}^{n} q_i = 1 - \sum_{i=1}^{n} L_i.$$

Proof. Clearly $q_1 = 1 - L_1$. Suppose the statement holds for $n-1$. Then

$$\begin{aligned}
\prod_{i=1}^{n} q_i &= \prod_{i=1}^{n-1} q_i \cdot q_n \\
&= \left(1 - \sum_{i=1}^{n-1} L_i\right) \cdot \left(1 - \frac{L_n}{1 - \sum_{i=1}^{n-1} L_i}\right) \\
&= 1 - \sum_{i=1}^{n-1} L_i - L_n \\
&= 1 - \sum_{i=1}^{n} L_i.
\end{aligned}$$

□

5.4 Corollary. Let $n \in \mathbb{N}$. Then

$$\prod_{i=1}^{n-1} q_i \cdot p_n = L_n.$$

We continue with a generalization of *depth* and *weight*.

5.5 Definition. For each $i \in \mathcal{D}$ we define the depth of a digit as $D(i) = i$. For a word $\mathbf{u} = u_1 \dots u_k \in \mathcal{A}^* \cup \{\varepsilon\}$ we define

$$D(\mathbf{u}) := \sum_{i=1}^{k} u_i$$

as the *depth* of $\mathbf{u}$, where the empty sum has value 0. We also write

$$G(n) = \{\mathbf{u} \in V(G_L) : D(\mathbf{u}) = n\}$$

as the collection of all words of depth n.

We again have the identity

$$D(\mathbf{uv}) = D(\mathbf{u}) + D(\mathbf{v}), \tag{5.2}$$

and continue with the same definition of weight.

5.6 Definition. Let $\mathbf{u} = u_1 \dots u_k \in \mathcal{A}^* \cup \{\varepsilon\}$. We define

$$W(\mathbf{u}) := \prod_{j=1}^{D(\mathbf{u})-1} f(\mathbf{u}_j, \mathbf{u}_{j+1})$$

as the *weight* of $\mathbf{u}$. It is again the product of all the labels along the path $P(\mathbf{u})$ from 1 to $\mathbf{u}$, where the empty product has value 1.

As $p_{u_k} + q_{u_k} = 1$, it holds that

$$W(\mathbf{u}) = W(\mathbf{u}1) + W(\mathbf{u}^+),$$

so that for all $n \geq 2$,

$$\sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) = \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}1) + \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}^+) = \sum_{\mathbf{u} \in G(n-1)} W(\mathbf{u}).$$

As $W(1) = 1$, we again obtain the useful identity

$$\sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) = 1 \tag{5.3}$$

for all $n \in \mathbb{N}$.

5.7 Lemma. *Let* $\mathbf{uv}$ *be the concatenation of* $\mathbf{u}, \mathbf{v} \in \mathcal{A}^*$. *The following holds:*

$$W(\mathbf{uv}) = W(\mathbf{u}1) \cdot W(\mathbf{v}). \tag{5.4}$$

Proof. The proof is completely analogous to the proof of Lemma 3.6, using Lemma 5.2 instead of Lemma 3.3. □

5.8 Corollary. Let $\mathbf{u} = u_1 \dots u_k \in \mathcal{A}^*$. Then

$$W(\mathbf{u}1) = \mu(\mathbf{u}).$$

Proof. We start with the assumption that $k = 1$, so that $\mathbf{u} = u_1$. Then

$$W(\mathbf{u}1) = \prod_{j=1}^{u_1-1} q_j \cdot p_{u_1} = L_{u_1} \tag{5.5}$$

by Corollary 5.4. Just like in the proof of Corollary 3.7, we let $k > 1$ and view $\mathbf{u}$ as the concatenation of single-digit words $\mathbf{u}_i = u_i$. Repeated application of Lemma 5.7 then yields

$$W(\mathbf{u}1) = \prod_{i=1}^{k} W(\mathbf{u}_i 1) = \prod_{i=1}^{k} L_{u_i} = \mu(\mathbf{u}), \tag{5.6}$$

where the last equation follows from (2.6). □

5.2 Generalized Lüroth Tree sequences

Let $A = (a_i)_{i=1}^{\infty} \in \Sigma$ again be a sequence of digits. Concatenation maps are defined exactly as in Definition 3.8. As in Section 3.3, we define

$$m(j) = \min\{i \in \mathbb{N} : c(i) = j\},$$
$$M(j) = \max\{i \in \mathbb{N} : c(i) = j\},$$

yielding the auxiliary sequence

$$A_c := (\mathbf{u}_j)_{j=1}^{\infty}$$

of words $\mathbf{u}_j \in \mathcal{A}^*$ such that

$$\mathbf{u}_j = a_{m(j)} ... a_{M(j)}.$$

For each $\alpha \in \mathcal{A}^*$, we write again

$$A_c(\alpha) := \{j \in \mathbb{N} : \mathbf{u}_j = \alpha\},$$

as the indices j of words of A_c that are equal to α.

As the lengths L_i can be irrational for any $\beta \in \mathcal{A}^*$, an integer m such that

$$W(\beta) \cdot m \in \mathbb{N}$$

may however not exist. As before, for each $\alpha \in \mathcal{A}^*$, we write

$$A_c(\alpha) := \{j \in \mathbb{N} : \mathbf{u}_j = \alpha\},$$

as the indices j of words of A_c that are equal to α. This leads to an altered definition of a Generalized Lüroth Tree sequence, where we aim for a value close to $W(\beta)$ instead of exactly achieving it.

5.9 Definition (Generalized Lüroth Tree sequence)**.** A sequence $A = (a_i)_{i=1}^{\infty} \in \Sigma$ is a *Generalized Lüroth Tree sequence* if there exists a concatenation map $c : \mathbb{N} \to \mathbb{N}$ with the following properties.

1. For all $\beta \in \mathcal{A}^*$, there exists an *error* $e_\beta \in (-1, 1)$ for which it holds that

$$\#A_c(\beta) = W(\beta) \cdot D(\beta)! + e_\beta. \tag{5.7}$$

2. Also for every $n \in N$, the following holds:

$$\sum_{\mathbf{u} \in G(n)} \#A_c(\mathbf{u}) = n!.$$

Thus in a Generalized Lüroth Tree sequence, exactly $n!$ words are element of $G(n)$.

Although perhaps less clear from the definition, Generalized Lüroth Tree sequences certainly exist and are not difficult to construct. For all $\beta \in G(n)$, one can calculate $W(\beta) \cdot n!$ and add it $\lfloor W(\beta) \cdot n! \rfloor$ times. Let $g_n \in [2^n]$ be the number of words such that $W(\beta) \cdot n! \in \mathbb{N}$. Observe that

$$t_n := n! - \sum_{\beta \in G(n)} \lfloor W(\beta) \cdot n! \rfloor < 2^n - g_n,$$

so that we always have less then $2^n - g_n$ "open spots" left. If we now add any collection of t_n distinct words of

$$E_n := \{\beta \in G(n) : W(\beta) \cdot n! \notin \mathbb{N}\}$$

once, then these t_n words will have error $e_\beta \in (0, 1)$. The other words of E_n that were not chosen will consequently have error $e_\beta \in (-1, 0)$. Of course the words outside E_n have error exactly 0.

5.10 Example. The consequences of this new definition are perhaps best exhibited by taking the "classical" partition $L_i = \frac{1}{i(i+1)}, i \geq 1$, yielding the same labels on the edges of the corresponding Lüroth Tree. The only difference is the digit set, as we essentially map any integer d of the classical setting to $d - 1$.
Once more following the approach of Example 3.10, we start with $n = 1$. We only have a 1, it thus has weight $W(1) = 1$ and we add it once as $1! = 1$.
At $n = 2$, we have two words: 11 and 2. Both have weight $\frac{1}{2}$, so we add both once as we can add $2! = 2$ words.
At $n = 3$, things start to look more different as we add $3! = 6$ words instead of 12. The weights are as follows:

Word	111	12	21	3
Weight	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$

This yields

$$W(111) \cdot 6 = 1.5, W(12) \cdot 6 = 1.5, W(21) \cdot 6 = 1, \text{ and } W(3) \cdot 6 = 2.$$

The last two words are clear, we add 21 once and 3 twice. For 111 and 12 we have two choices: we can 111 once and 12 twice or vice versa. Picking the first choice, we have $e_{111} = -\frac{1}{2}$ and $e_{12} = \frac{1}{2}$. So far our word looks like this:

$$\begin{aligned} A &= (1,1,1,2,1,1,1,1,2,1,2,2,1,3,3,\dots), \\ A_c &= (1,11,2,111,12,12,21,3,3,\dots). \end{aligned}$$

The good news is that we can easily look at $n = 4$, as we only have to add $4! = 24$ words instead of 288. We first calculate some weights:

Word	1111	112	121	13	211	22	31	4
Weight	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$

Multiplying all these weights with 24, we obtain the integer values 3,3,2,4,2,2,2 and 6. This yields the following:

$$\begin{aligned} A = (&1,1,1,2,1,1,1,1,2,1,2,2,1,3,3, \\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,1,1,2,1,1,2,1,2,1,1,2,1,1,3,1,3,1,3,1,3, \\ &2,1,1,2,1,1,2,2,2,2,3,1,3,1,4,4,4,4,4,4,\dots), \\ A_c = (&1,11,2,111,12,12,21,3,3,1111,1111,1111,112,112,112,121,121,13,13,13,13, \\ &211,211,22,22,31,31,4,4,4,4,4,4,\dots). \end{aligned}$$

We stop here, because at depth 5 we have to add $5! = 120$ words. That is quite a lot, but $\mathrm{sf}(5) = 34560$ is a lot more!

Now let $\alpha \in \mathcal{A}^*$. We again write

$$A_c(\alpha, n) := \{j \in \mathbb{N} : \mathbf{u}_j \in G(n), \mathbf{u}_j = \alpha\gamma, \gamma \in \mathcal{A}^*\},$$

as the set of words of depth n starting with α.

5.11 Lemma. *Fix $\alpha \in \mathcal{A}^*$ and a Generalized Lüroth Tree sequence $A = (a_i)_{i=1}^{\infty}$ with concatenation map c. Then it holds that*

$$\lim_{n\to\infty} \frac{\#A_c(\alpha, n)}{n!} = \mu(\alpha).$$

Proof. Fix $n \in \mathbb{N}$ such that $n > D(\alpha)$. Then for $\gamma \in \mathcal{A}^*$ such that $\mathbf{u}_j = \alpha\gamma \in G(n)$, it holds that

$$D(\gamma) = n - D(\alpha) > 0$$

by (5.2).

This yields

$$\begin{aligned}
\#A_c(\alpha, n) &= \sum_{\gamma \in G(n-D(\alpha))} \#A_c(\alpha\gamma) \\
&= \sum_{\gamma \in G(n-D(\alpha))} (W(\alpha\gamma) \cdot n! + e_{\alpha\gamma}) \\
&= n! \cdot \sum_{\gamma \in G(n-D(\alpha))} W(\alpha 1) \cdot W(\gamma) + \sum_{\gamma \in G(n-D(\alpha))} e_{\alpha\gamma} \\
&= n! \cdot W(\alpha 1) \cdot \sum_{\gamma \in G(n-D(\alpha))} W(\gamma) + \sum_{\gamma \in G(n-D(\alpha))} e_{\alpha\gamma} \\
&= n! \cdot \mu(\alpha) + \sum_{\gamma \in G(n-D(\alpha))} e_{\alpha\gamma}.
\end{aligned}$$

using Definition 5.9, Lemma 5.7, Corollary 5.8 and (5.3). Observe that

$$\sum_{\gamma \in G(n-D(\alpha))} e_{\alpha\gamma} \in (-2^n, 2^n), \tag{5.8}$$

and that

$$\lim_{n \to \infty} \frac{2^n}{n!} \to 0.$$

This yields the result. □

5.3 Length of words

For Generalized Lüroth Tree sequences, we need to come up with a more general argument to prove that the weighted average length of words is unbounded. We write

$$\begin{aligned}
d(n) &:= \sum_{j : \mathbf{u}_j \in G(n)} |\mathbf{u}_j| \\
&= \sum_{\mathbf{u} \in G(n)} (W(\mathbf{u}) n! + e_{\mathbf{u}}) |\mathbf{u}| \\
&= n! \cdot \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) |\mathbf{u}| + \sum_{\mathbf{u} \in G(n)} e_{\mathbf{u}} |\mathbf{u}|
\end{aligned}$$

for the number of digits of depth n. Using (5.7), we can denote the "abstract" weighted average number of digits per word of depth n as

$$\frac{d(n) + e_n}{n!} = \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) \cdot |\mathbf{u}|, \tag{5.9}$$

where $e_n \in (-n2^n, n2^n)$ as $|\mathbf{u}| \leq n$ and $\#G(n) = 2^n$.
For convenience, we write

$$\Delta_n := \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) \cdot |\mathbf{u}|.$$

In order to prove that Δ_n is unbounded, we use the following inductive argument.

Let $n \in \mathbb{N}$, and suppose that all values of Δ_i are known for $i \in [n-1]$.
For $d \in [n-1]$, observe that the following holds:

$$\{\mathbf{v} \in \mathcal{A}^* : d\mathbf{v} \in G(n)\} = G(n-d).$$

This means that if we take the set of words of depth n that start with digit d and remove this first digit, we obtain exactly the set of words of depth $n-d$. For example, taking $d = 1, n = 4$, the words of depth 4 that start with a 1 are 1111,112,121 and 13, and removing every first digit yields 111,12,21 and 3: exactly the words of depth $4 - 1 = 3$.

By Lemma 5.7 and (5.5), for $\mathbf{u} = d\mathbf{v} \in \mathcal{A}^*$ we obtain

$$W(\mathbf{u}) = W(d1) \cdot W(\mathbf{v}) = L_d \cdot W(\mathbf{v}).$$

If we now split $G(n)$ up as a union of the word n itself with the words that have starting digit $d \in [n]$, we obtain the following, where we use that $W(n) = 1 - \sum_{d=1}^{n-1} L_d$ by Lemma 5.3:

$$\begin{aligned}
\Delta_n &= \sum_{\mathbf{u} \in G(n)} W(\mathbf{u}) \cdot |\mathbf{u}| \\
&= W(n) + \sum_{d=1}^{n-1} \sum_{\mathbf{u} \in G(n): \mathbf{u} = d\mathbf{v}} W(\mathbf{u}) \cdot |\mathbf{u}| \\
&= W(n) + \sum_{d=1}^{n-1} \sum_{\mathbf{v} \in G(n-d)} L_d W(\mathbf{v}) \cdot (|\mathbf{v}| + 1) \\
&= W(n) + \sum_{d=1}^{n-1} L_d \cdot \left(\sum_{\mathbf{v} \in G(n-d)} W(\mathbf{v}) \cdot (|\mathbf{v}| + 1) \right) \\
&= W(n) + \sum_{d=1}^{n-1} L_d \cdot (\Delta_{n-d} + 1) \\
&= 1 - \sum_{d=1}^{n-1} L_d + \sum_{d=1}^{n-1} L_d \cdot \Delta_{n-d} + \sum_{d=1}^{n-1} L_d \\
&= 1 + \sum_{d=1}^{n-1} L_d \cdot \Delta_{n-d}
\end{aligned}$$

As $\Delta_1 = 1$, we have $\Delta_2 > \Delta_1$. Now let $N \in \mathbb{N}_{\geq 2}$, and suppose that $\Delta_n > \Delta_{n-1}$ for all $n \in [1, N]$. Then by (2.3), we see that $\Delta_{N+1} > \Delta_N$. This shows that Δ_n is strictly increasing in n. In particular, this shows that $\Delta_i \geq 1 + \delta_1$ for $i \in \mathbb{N}_{>1}$ and some $\delta_1 \in (0, 1)$.

Observe that as $\sum_{d=1}^{\infty} L_d = 1$, there exists an $N_1 \in \mathbb{N}$ such that

$$\sum_{d=1}^{N_1} L_d + \frac{1}{2}\delta_1 \geq 1.$$

This leads to the following estimate:

$$\begin{aligned}
\Delta_{N_1+2} &= 1 + \sum_{d=1}^{N_1+1} L_d \cdot \Delta_{N_1+2-d} \\
&\geq 1 + \sum_{d=1}^{N_1} L_d \cdot \Delta_{N_1+2-d} \\
&\geq 1 + (1+\delta_1) \cdot \sum_{d=1}^{N_1} \Delta_{N_1+2-d} \\
&\geq 1 + (1+\delta_1) \cdot (1 - \frac{1}{2}\delta_1) \\
&= 2 + \frac{1}{2}(\delta_1 - \delta_1^2).
\end{aligned}$$

This shows that $\Delta_n \geq 2 + \frac{1}{2}(\delta_1 - \delta_1^2)$ for $n > N_1 + 1$. We continue by taking

$$\delta_m = \frac{1}{m}(\delta_{m-1} - \delta_{m-1}^2),\ m \in \mathbb{N}_{\geq 2}.$$

Let $M \in \mathbb{N}$. Arguing inductively, we assume that for all $m < M$, there exists N_m such that $\Delta_n \geq m + \delta_m$ for $n > N_m + 1$. As $\delta_m \in (0,1)$ for $m \geq 1$, there exists an N_M such that

$$\sum_{d=1}^{N_M} L_d + \frac{1}{M+1}\delta_M \geq 1.$$

This yields

$$\begin{aligned}
\Delta_{N_M+N_{M-1}+2} &= 1 + \sum_{d=1}^{N_M+N_{M-1}+1} L_d \cdot \Delta_{N_M+N_{M-1}+2-d} \\
&\geq 1 + \sum_{d=1}^{N_M} L_d \cdot \Delta_{N_M+N_{M-1}+2-d} \\
&\geq 1 + \sum_{d=1}^{N_M} L_d \cdot (M - 1 + \delta_{M-1}) \\
&\geq 1 + (M - 1 + \delta_{M-1}) \cdot (1 - \frac{1}{M}\delta_{M-1}) \\
&= M + \frac{1}{M}(\delta_{M-1} - \delta_{M-1}^2). \\
&= M + \delta_M
\end{aligned}$$

This shows that for all $m \in \mathbb{N}$, there exists an N_m such that $\Delta_{N_m+N_{m-1}+2} > m$. Thus $\Delta_n \to \infty$ as $n \to \infty$.

As

$$\frac{e_n}{n!} \to 0,$$

we conclude from (5.9) that

$$\frac{d(n)}{n!} \to \infty. \tag{5.10}$$

Observe that

$$n! = n \cdot (n-1)! \geq \sum_{m=1}^{n-1} m!,$$

thus for the total number $w^*(n)$ of words up to depth n it holds that

$$w^*(n) := \sum_{m=1}^{n} m! \leq 2n!. \tag{5.11}$$

We write

$$d^*(n) := \sum_{m=1}^{n} d(m), \tag{5.12}$$

so that

$$\frac{d^*(n)}{w^*(n)}$$

represents the weighted average of digits per word over all words of depth at most n. We then have the following result.

5.12 Proposition. *The cumulative weighted average number of digits per word up until depth n is unbounded in n, i.e.*

$$\lim_{n\to\infty} \frac{d^*(n)}{w^*(n)} = \infty.$$

Proof. We calculate

$$\frac{d^*(n)}{w^*(n)} \geq \frac{d(n)}{w^*(n)} \geq \frac{d(n)}{2n!} \geq \frac{1}{2}\frac{d(n)}{n!},$$

which diverges as $n \to \infty$ by (5.10). □

5.4 Classification of digits

We can classify the digits of a Generalized Lüroth Tree sequence A with concatenation map c as we did in Chapter 4. Using the same definitions, we again denote m_i and M_i as the least and greatest integer for which $c(i) = c(m_i) = c(M_i)$ respectively. We then have

$$p_c(i) := a_{m_i} ... a_{i-1} \in \mathcal{A}^*,$$
$$s_c(i) := a_i ... a_{M_i} \in \mathcal{A}^*.$$

We again write $D_c(i) = D(\mathbf{u}_{c(i)})$, yielding the definitions

$$A(\beta, n) = \{i \in \mathbb{N} : p_c(i) = \beta, D_c(i) = n\},$$
$$A(\beta, \alpha, n) = \{i \in \mathbb{N} : p_c(i) = \beta, s_c(i) \in \mathcal{A}^*(\alpha), D_c(i) = n\},$$

where again

$$\mathcal{A}^*(\alpha) = \{\mathbf{u} \in \mathcal{A}^* : \mathbf{u} = \alpha\gamma, \gamma \in \mathcal{A}^* \cup \{\varepsilon\}\}.$$

By exactly the same arguments, we have the bijections

$$A(\beta, n) \longleftrightarrow A_c(\beta, n), \quad A(\beta, \alpha, n) \longleftrightarrow A_c(\beta\alpha, n), \tag{5.13}$$

mapping $i \mapsto c(i)$. This leads to a generalization of Proposition 4.2.

5.13 Proposition. *Let $A = (a_i)_{i=1}^{\infty}$ be a Generalized Lüroth Tree sequence with concatenation map c. Fix $\alpha, \beta \in \mathcal{A}^*$. Then it holds that*

$$\lim_{n\to\infty} \frac{\#A(\beta, \alpha, n)}{\#A(\beta, n)} = \mu(\alpha).$$

Proof. By (5.13) and Lemma 5.11, we may write

$$\begin{aligned}\lim_{n\to\infty} \frac{\#A(\beta, \alpha, n)}{\#A(\beta, n)} &= \lim_{n\to\infty} \frac{\#A_c(\beta\alpha, n)}{\#A_c(\beta, n)} \\ &= \frac{\mu(\beta\alpha)}{\mu(\beta)} \\ &= \mu(\alpha),\end{aligned}$$

exactly as in Proposition 4.2. □

5.14 Definition. Let A be a Lüroth Tree sequence, and let $l \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we define the unions

$$\begin{aligned}U_A(n, l) &:= \bigcup_{\substack{\beta\in\mathcal{A}^*: \\ D(\beta)<n-l}} A(\beta, n) \\ &= \{i \in \mathbb{N} : D_c(i) = n, D(p_c(i)) < n - l\},\end{aligned}$$

$$U_{A,\alpha}(n) := \bigcup_{\substack{\beta\in\mathcal{A}^*: \\ D(\beta)<n-D(\alpha)}} A(\beta, \alpha, n) = \{i \in U_A(n, D(\alpha)) : s_c(i) \in \mathcal{A}^*(\alpha)\},$$

as in Definition 4.4.

This leads to a generalized version of Corollary 4.5.

5.15 Proposition. *Fix $\alpha \in \mathcal{A}^*$. Then*

$$\lim_{n\to\infty} \frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} = \mu(\alpha).$$

Proof. We argue a bit more carefully, as the sums become infinite sums in the limit. We fix $n \in \mathbb{N}$. Then as in the proof of Lemma 5.11,

$$\begin{aligned}\frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} &= \frac{\sum_{\substack{\beta\in\mathcal{A}^*:\\ D(\beta)<n-D(\alpha)}} \#A(\beta,\alpha,n)}{\sum_{\substack{\beta\in\mathcal{A}^*:\\ D(\beta)<n-D(\alpha)}} \#A(\beta,n)} \\ &= \frac{\sum_{\beta\in G(n-D(\alpha))}(n!\cdot\mu(\beta\alpha)+e_{\beta\alpha})}{\sum_{\beta\in G(n-D(\alpha))}(n!\cdot\mu(\beta)+e_{\beta})},\end{aligned}$$

where

$$\begin{aligned}e_{\beta\alpha} &= \sum_{\gamma\in G(n-D(\beta\alpha))} e_{\beta\alpha\gamma} \in (-2^n, 2^n), \\ e_{\beta} &= \sum_{\gamma\in G(n-D(\beta))} e_{\beta\gamma} \in (-2^n, 2^n),\end{aligned}$$

as in (5.8).
Consequently,

$$\sum_{\beta\in G(n-D(\alpha))} e_{\beta\alpha}, \sum_{\beta\in G(n-D(\alpha))} e_{\beta} \in (-4^n, 4^n).$$

Clearly it also holds that $\frac{4^n}{n!} \to 0$ as $n \to \infty$, so that both errors converge to zero. This shows that

$$\frac{\sum_{\beta\in G(n-D(\alpha))}(\mu(\beta)+\frac{e_{\beta\alpha}}{\mu(\alpha)\cdot n!})}{\sum_{\beta\in G(n-D(\alpha))}(\mu(\beta)+\frac{e_{\beta}}{\mu(\alpha)\cdot n!})} \to 1$$

as $n \to \infty$,.
Writing

$$\frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))} = \mu(\alpha)\cdot\frac{\sum_{\beta\in G(n-D(\alpha))}(\mu(\beta)+\frac{e_{\beta\alpha}}{\mu(\alpha)\cdot n!})}{\sum_{\beta\in G(n-D(\alpha))}(\mu(\beta)+\frac{e_{\beta}}{\mu(\alpha)\cdot n!})},$$

we obtain the result. □

5.16 Remark. Observe that in our construction we now use $n!$ instead of $\mathrm{sf}(n)$, thus decreasing the amount of words added. There are slower growing sequences available such as $k_n = 5^n$, since we only need $\frac{4^n}{k_n}$ to go to zero in the limit, as the above shows. We chose $n!$ because it is so similar to $\mathrm{sf}(n)$, so that some new definitions resemble the old approach as much as possible.

We again write

$$N_A(\alpha, T) := \frac{\#\{i \in T : a_i = \alpha_1, ..., a_{i+k-1} = \alpha_k\}}{\#T},$$

so that

$$N_A(\alpha, U_A(n, D(\alpha))) = \frac{\#U_{A,\alpha}(n)}{\#U_A(n, D(\alpha))}.$$

Applying Proposition 5.15, we find

$$\lim_{n\to\infty} N_A(\alpha, U_A(n, D(\alpha))) = \mu(\alpha). \tag{5.14}$$

Fixing $n \in \mathbb{N}$ and writing

$$U_A^*(n, D(\alpha)) = \bigcup_{m=1}^{n} U_A(m, D(\alpha)),$$

we calculate

$$\begin{aligned} N_A(\alpha, U_A^*(n, D(\alpha))) &= \frac{\#\{i \in U_A^*(n, D(\alpha)) : a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k\}}{\#U_A^*(n, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} \#\{i \in U_A(m, D(\alpha)) : a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k\}}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &= \frac{\sum_{m=1}^{n} N_A(\alpha, U_A(m, D(\alpha)) \cdot \#U_A(m, D(\alpha)))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))}. \end{aligned}$$

Fix $\varepsilon \in (0, \mu(\alpha)), \alpha \in \mathcal{A}^*$. We know by (5.14) that there exists an $N_1 \in \mathbb{N}$ such that for all $m \geq N_1$, we have

$$|\mu(\alpha) - N_A(\alpha, U_A(m, D(\alpha)))| < \frac{1}{2}\varepsilon.$$

As $\#U_A(m, D(\alpha)) \geq 1$ for $m > D(\alpha)$, there also exists an $N_2 > N_1$ such that for all $n \geq N_2$

$$0 < \frac{\sum_{m=1}^{N_1-1} \#U_A(m, D(\alpha))}{\sum_{m=N_1}^{n} \#U_A(m, D(\alpha))} < \frac{1}{2}\varepsilon.$$

As $N_A(\alpha, U_A(m, D(\alpha))) \in [0, 1]$ for all $m \in \mathbb{N}$, we have the following estimate for $n \geq N_2$:

$$\begin{aligned} N_A(\alpha, U_A^*(n, D(\alpha))) &= \frac{\sum_{m=1}^{n} N_A(\alpha, U_A(m, D(\alpha))) \cdot \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &\leq \frac{\sum_{m=1}^{N_1-1} \#U_A(m, D(\alpha)) + \sum_{m=N_1}^{n} (\mu(\alpha) + \frac{1}{2}\varepsilon) \cdot \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\ &\leq \frac{\sum_{m=1}^{N_1-1} \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} + \frac{\sum_{m=N_1}^{n} (\mu(\alpha) + \frac{1}{2}\varepsilon) \cdot \#U_A(m, D(\alpha))}{\sum_{m=N_1}^{n} \#U_A(m, D(\alpha))} \\ &\leq \frac{1}{2}\varepsilon + \mu(\alpha) + \frac{1}{2}\varepsilon \\ &= \mu(\alpha) + \varepsilon. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
N_A(\alpha, U_A^*(n, D(\alpha))) &\geq \frac{\sum_{m=N_1}^{n} (\mu(\alpha) - \frac{1}{2}\varepsilon) \cdot \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\
&= \frac{\sum_{m=N_1}^{n} (\mu(\alpha) - \frac{1}{2}\varepsilon) \cdot \#U_A(m, D(\alpha))}{\sum_{m=N_1}^{n} \#U_A(m, D(\alpha))} \cdot \frac{\sum_{m=N_1}^{n} \#U_A(m, D(\alpha))}{\sum_{m=1}^{n} \#U_A(m, D(\alpha))} \\
&= (\mu(\alpha) - \frac{1}{2}\varepsilon) \cdot \left(1 - \frac{\sum_{m=1}^{N_1-1} \#U_A(m, D(\alpha))}{\sum_{m=N_1}^{n} \#U_A(m, D(\alpha))}\right) \\
&\geq (\mu(\alpha) - \frac{1}{2}\varepsilon) \cdot (1 - \frac{1}{2}\varepsilon) \\
&\geq \mu(\alpha) - (1 + \mu(\alpha))\frac{1}{2}\varepsilon \\
&\geq \mu(\alpha) - \varepsilon.
\end{aligned}$$

This shows that

$$\lim_{n\to\infty} N_A(\alpha, U_A^*(n, D(\alpha))) = \mu(\alpha). \tag{5.15}$$

This result is as promising as before: in the limit, the "good" digits $U_A^*(n, D(\alpha))$ still behave exactly like we want them to. As in (4.5), it also still holds that

$$\#U_A^*(n, D(\alpha)) \geq d^*(n) - w^*(n)D(\alpha), \tag{5.16}$$

leading to the following generalization of Proposition 4.6.

5.17 Proposition. *Fix $\alpha \in \mathcal{A}^*$, and let A be a Lüroth Tree sequence with concatenation map c. Then*

$$\lim_{n\to\infty} \frac{\#U_A^*(n, D(\alpha))}{d^*(n)} = 1.$$

Proof. Let $n \in \mathbb{N}$. Then by (5.16) and Proposition 5.12,

$$\begin{aligned}
1 \geq \frac{\#U_A^*(n, D(\alpha))}{d^*(n)} &\geq \frac{d^*(n) - w^*(n)D(\alpha)}{d^*(n)} \\
&= 1 - \frac{w^*(n)}{d^*(n)} \to 1.
\end{aligned}$$

□

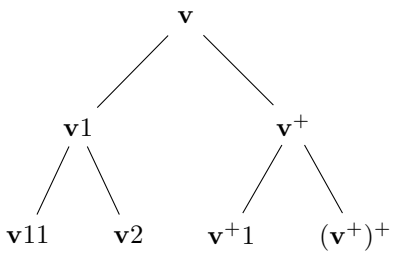
5.18 Corollary. Fix $\alpha \in \mathcal{A}^*$, and let A be a Lüroth Tree sequence with concatenation map c. Then

$$\lim_{n\to\infty} \frac{\#U_A(n, D(\alpha))}{d(n)} = 1.$$

□

So far we have established a generalized convergence result on the milestones $w^*(n)$. Unfortunately, we are not able to present an example similar to Example 4.10. An extra condition analogous to Definition 4.13 is sufficient as we will show next, but we're uncertain about the necessity. It might be able to prove that the stronger ordering is not needed, perhaps by choosing a different value for $w(n)$ in light of Remark 5.16.

5.19 Definition. Let $\mathbf{u} \in \mathcal{A}^*, D(\mathbf{u}) \geq 3$. Then we define the *grandparent* $\mathbf{u}^{-2}$ of $\mathbf{u}$ as the word $\mathbf{v}$ such that $\mathbf{u} \in \{\mathbf{v}11, \mathbf{v}2, \mathbf{v}^+1, (\mathbf{v}^+)^+\}$.



5.20 Lemma. *Fix* $\mathbf{v} \in \mathcal{A}^*$. *The following holds:*

$$\sum_{\mathbf{u}:\mathbf{u}^{-2}=\mathbf{v}} W(\mathbf{u}) = W(\mathbf{v}).$$

Proof. Using (5.1), we have

$$\begin{aligned}\sum_{\mathbf{u}:\mathbf{u}^{-2}=\mathbf{v}} W(\mathbf{u}) &= W(\mathbf{v}11) + W(\mathbf{v}2) + W(\mathbf{v}^+1) + W((\mathbf{v}^+)^+) \\ &= W(\mathbf{v}1) + W(\mathbf{v}^+) = W(\mathbf{v}).\end{aligned}$$

□

5.21 Definition (Strongly ordered Generalized Lüroth Tree sequence)**.** A sequence $A = (a_i)_{i=1}^{\infty} \in \Sigma$ is a *strongly ordered Generalized Lüroth Tree sequence* if there exists a concatenation map $c : \mathbb{N} \to \mathbb{N}$ that satisfies the following three conditions.

1. For all $n \geq 1$, it holds that
$$\sum_{\mathbf{u} \in G(n)} \#A_c(\mathbf{u}) = n!.$$

2. For all $n \geq 1$ and $\beta \in G(n)$, we have there is an $e_\beta \in (-1, 1)$ such that
$$\#\{j \leq w^*(n) : \mathbf{u}_j = \beta\} = W(\beta) \cdot n! + e_\beta. \tag{5.17}$$

3. Furthermore, for all $n > 3, k \in [n(n-1)]$ and $\beta \in G(n-2)$, there is an $e_\beta^k \in (-4, 4)$ such that
$$\#\{j \in [w^*(n-1)+(k-1)\cdot(n-2)!, w^*(n-1)+k\cdot(n-2)!] : \mathbf{u}_j^{-2} = \beta\} = W(\beta)\cdot(n-2)!+e_\beta^k. \tag{5.18}$$

Note that the error e_β^k can be four times larger since any grandparent has four children. It does not affect any argument, as $\lim_{n\to\infty} 4 \cdot \frac{4^n}{n!} = 0$.

Using (5.17), we obtain the analog of Lemma 4.9.

5.22 Lemma. *Fix $\alpha \in \mathcal{A}^*$, and let A be a Generalized Lüroth Tree sequence.*

$$\lim_{n\to\infty} N_A(\alpha, [d^*(n)]) = \mu(\alpha).$$

Proof. Fix $\varepsilon \in \mathbb{R}_{>0}$. Using Proposition 5.17, we can perform the calculation of Lemma 4.9 to assure the existence of an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$|N_A(\alpha, [d^*(n)]) - N_A(\alpha, U_A^*(n, D(\alpha)))| < \frac{1}{2}\varepsilon.$$

By (5.15), we also have an $N_2 \in \mathbb{N}$ such that for all $N \geq N_2$,

$$|N_A(\alpha, U_A^*(n, D(\alpha))) - \mu(\alpha)| < \frac{1}{2}\varepsilon,$$

yielding the result. □

Let A be a strongly ordered Generalized Lüroth Tree sequence. We write

$$I_A(n) := \{i \in \mathbb{N} : D_c(i) = n\},$$

for the digits of depth n, and observe that by (5.17), it holds that

$$I_A(n) = [d^*(n-1), d^*(n)].$$

For this set we have the following analog of Lemma 4.17. We omit the proof, as it is analogous to the proof of Lemma 5.22.

5.23 Lemma. *Fix $\alpha \in \mathcal{A}^*$, and let A be a strongly ordered Generalized Lüroth Tree sequence. Then*

$$\lim_{n\to\infty} N_A(\alpha, I_A(n)) = \mu(\alpha).$$

We write again

$$I_A(n) = \{i \in \mathbb{N} : w^*(n-1) < c(i) \leq w^*(n)\}.$$

Now fix $n > 3, k \in [n(n-1)]$. We define

$$I_A^k(n) := \{i \in \mathbb{N} : c(i) \in [w^*(n-1) + (k-1)\cdot(n-2)!, w^*(n-1) + k\cdot(n-2)!]\}$$

as the set of indices of digits belonging to the k-th batch of $(n-2)!$ words of depth n. As before, Definition 5.19 implies that the words $\{\mathbf{u}_j : j \in [w^*(n)+(k-1)\cdot(n-2)!, w^*(n)+k\cdot(n-2)!]\}$ that belong to the digits of $I_A^k(n)$ can all be written as $\mathbf{v}11, \mathbf{v}2, \mathbf{v}^+1$ or $(\mathbf{v}^+)^+$ for some $\mathbf{v} \in G(n-2)$. Observe that (5.18) no longer guarantees a one-to-one correspondence between the words of depth $n-2$ and the grandparents of words that supply the digits of $I_A^k(n)$. This is because we do not put particular restrictions on the distribution of errors e_β^k.

However, for every $k \in [n(n-1)]$ there still exists a strongly ordered Generalized Lüroth Tree sequence A' with concatenation map c' such that for every $l \in [(n-2)!]$,

$$A'_{c'} \ni \mathbf{v}_{w^*(n-3)+l} = \mathbf{u}^{-2}_{w^*(n)+(k-1)\cdot(n-2)!+l} \in A_c,$$

as the grandparents of the words in $I^k_A(n)$ satisfy (5.7) for $n-2$. For this A', it again holds that

$$\left|\#I_{A'}(n-2) \cdot N_{A'}(\alpha, I_{A'}(n-2)) - \#I^k_A(n) \cdot N_A(\alpha, I^k_A(n))\right| \leq 2|\alpha| \cdot (n-2)!. \tag{5.19}$$

This leads to the analog of Proposition 4.19.

5.24 Proposition. *Fix $\alpha \in \mathcal{A}^*$, and let A be a strongly ordered Generalized Lüroth Tree sequence. Then for all $k \in [n(n-1)]$,*

$$\lim_{n\to\infty} N_A(\alpha, I^k_A(n)) = \mu(\alpha).$$

Proof. Fix $\varepsilon \in \mathbb{R}_{>0}$. By exactly the same arguments as in Proposition 4.19, we find an $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, it holds that

$$\left|N_A(\alpha, I^k_A(n)) - N_{A'}(\alpha, I_{A'}(n-2))\right| < \frac{1}{2}\varepsilon$$

for any $k \in [n(n-1)]$. By Lemma 5.23, there also exists an $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$|\mu(\alpha) - N_{A'}(\alpha, I_{A'}(n-2))| < \frac{1}{2}\varepsilon,$$

so for all $n \geq \max\{N_1, N_2\}$,

$$\left|\mu(\alpha) - N_A(\alpha, I^k_A(n))\right| < \varepsilon.$$

□

5.25 Theorem. *Let A be a strongly ordered Generalized Lüroth Tree sequence. Then for any $\alpha \in \mathcal{A}^*$,*

$$\lim_{m\to\infty} \frac{\#\{i \leq m : a_i = \alpha_1, \ldots, a_{i+k-1} = \alpha_k\}}{m} = \mu(\alpha).$$

Proof. Let $M \in \mathbb{N}, M \geq d^*(6)$. Then there are unique integers $n = n_M \in \mathbb{N}, k = k_M \in [n_M(n_M-1)]$ such that $M \in I^k_A(n)$. Write

$$R_M := M - d^*(n_M - 1) - \sum_{l=1}^{k-1} \#I^l_A(n_M) \in \mathbb{N},$$

with $R_M \in [d(n_M-2) + 2(n_M-2)!]$.

We estimate

$$\begin{aligned}
\frac{d(n-2)}{d^*(n-1)} &\leq \frac{d(n-2)}{d(n-1)} \\
&\leq \frac{\Delta_{n-2}\cdot(n-2)! + (n-2)2^{n-2}}{\Delta_{n-1}\cdot(n-1)! - (n-1)2^{n-1}} \\
&\leq \frac{\frac{\Delta_{n-2}}{n-1} + \frac{2^{n-2}}{(n-2)!}}{\Delta_{n-1} - \frac{2^{n-1}}{(n-2)!}} \\
&\leq \frac{\frac{1}{n-1} + \frac{2^{n-2}}{\Delta_{n-2}(n-2)!}}{1 - \frac{2^{n-1}}{\Delta_{n-2}(n-2)!}}.
\end{aligned}$$

Thus by (5.10), the last R_M digits are negligible compared to M:

$$\begin{aligned}
\frac{R_M}{M} &\leq \frac{d(n_M-2) + 2(n_M-2)!}{d^*(n_M-1)} \\
&\leq \frac{d(n_M-2)}{d(n_M-1)} + 2\frac{(n_M-2)!}{d(n_M-2)} \to 0,
\end{aligned}$$

as $n_M \to \infty$.
Clearly $\lim_{M\to\infty} n_M = \infty$, so that for any sequence A we have

$$N_A(\alpha, [M]) \to N_A(\alpha, [M - R_M]) \tag{5.20}$$

as $M \to \infty$.
The convergence results Lemma 5.22 and Proposition 5.24 now yield the desired result. □

6 Reflection

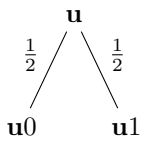
This chapter aims to sketch a little bit of context. We will talk about the relationship to Champernowne's approach, discuss some other constructions of Lüroth numbers and reflect on some ideas and intuition behind our construction and proofs.

6.1 Champernowne

Most constructions of normal numbers found in literature are said to be an extension or "in the spirit" of the Champernowne construction. Our construction very much qualifies as such. Champernowne's famous constant c we mentioned in the introduction is normal in base 10. The same idea works in any base. If we do this for base $b = 2$, we obtain the *dyadic Champernowne number*

$$\mathcal{C} = 0.0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ \ldots$$

that appears in for example [DLR20]. If we now draw a "Champernowne Tree", using a similar but simpler rule



note that we obtain this constant by starting with the empty word ε and concatenating the numbers on the vertices from left to right.

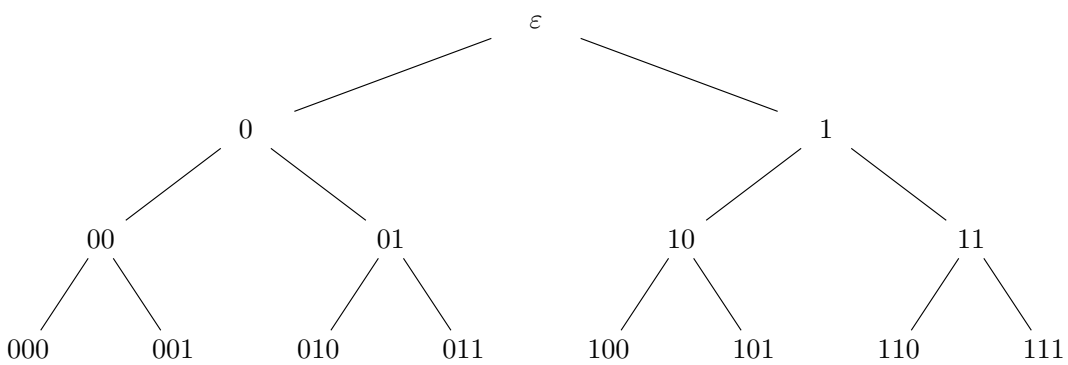


Figure 7: The dyadic Champernowne Tree.

The construction is in this case simple: just add 2^n words of every depth n, adding all words exactly once. This shows how nicely the construction of Boks dealt with the fact that Lüroth numbers have an infinite digit set: adding a new digit at every depth, and simultaneously design the labels so that all older digits but this last digit have exactly the right frequency.

6.2 Other constructions

In 2012, Madritsch and Mance published the first preprint of [MM16], describing a very general method for finding normal numbers. The method can be applied to any Generalized Lüroth Series. The approach is also similar to Champernowne, but much more abstract. They construct a sequence of measures ν_i converging to μ, but such that only finitely many words have nonzero measure ν_i. In this way, one can make a collection of words that exactly represents ν_i. This ν_i measure is very similar to our weight function W viewed as a measure on the space $G(n)$. They then concatenate these finite collections of words yielding a μ-normal number, as the repeated additions of the improved collection "swamps the behavior of what came before them" [Van14].

This last quote is from an article of Vandehey that appeared around the same time. As mentioned in the introduction, it only deals with Generalized Lüroth Series such that the alphabet $\mathcal{D}$ is finite.
As the construction of Aehle and Paulsen [AP15] is not explicit, it makes the most sense to compare our construction to the one found in [MM16]. Their procedure adds words of the same length rather then of the same depth. Working out their method for the classical partition, the construction starts with 16 concatenations of words of length 2 containing the digits 2 and 3:

$$M := 22\ 22\ 22\ 22\ 22\ 22\ 22\ 22\ 22\ 23\ 23\ 23\ 32\ 32\ 32\ 33\ldots,$$

after which words of length 3 are added that contain only the digits 2, 3 and 4, and then all combinations of length 4 consisting of the digits up until 5, etc. Not surprisingly, the resulting number is different from ours. This can be seen immediately as we encounter higher digits much earlier.

The most recent example of a Lüroth normal number is provided by Dajani, De Lepper and Robinson [DLR20]. This article introduces Minkowski normality and provides a construction. They also show how their method can be extended to produce an expansion that is normal with respect to the Lebesgue measure. It however only applies to a specific generalized Lüroth series with intervals $I_n = (\frac{1}{2^n}, \frac{2}{2^n}]$.

6.3 Some additional remarks

The arguments and proofs of this thesis are all about convergence. In fact: we often had multiple quantities simultaneously converging. A very concise outline of our proof could be sketched as follows:

$$N_A(\alpha, [n]) \to N_A(\alpha, [d^*(n)]) \to N_A(\alpha, U_A(n, D(\alpha)) \to \mu(\alpha).$$

(Note that the last arrow was actually an equality in Chapter 4.) One might perhaps intuitively expect some kind of uniform convergence to be necessary to deal with subsequent convergence statements, but this never turned out to be necessary. Moreover, the saving grace in our proofs is always the fact that we are allowed to first fix a word $\alpha \in \mathcal{A}^*$ before having to do anything else. This was very useful, as it also meant that quantities like $D(\alpha)$ were fixed. In a broader sense, one realizes that a number can actually never be something like "uniformly normal": for any n, there will always be words α that can not even be seen once as their length exceeds the position of the digit we're looking at.

Another remark is the conditions we had to put on the concatenation map c. As mentioned in Remark 4.16, the definition of "strongly ordered" is mostly a matter of taste. Instead of splitting the $\mathrm{sf}(n)$ words up in to $n!(n-1)!$ batches, it is also possible to split it up in $n!(n-1)!(n-2)!$ batches. The $\mathrm{sf}(n-3)$ words of each batch will then have to be a faithful representation of their "greatgrandparents" of depth $n-3$. This gives more milestones, and thus in a sense more restrictions, but allows more freedom within the batch itself: there are twice as many children to choose from. One can pick any number m and make $n!(n-1)\dots(n-m)!$ batches for $n > m$. As said, it's a matter of taste if a higher m is more free, but at least for the examples the lowest option $m = 1$ was convenient. The construction in [Bok09] is very particular, but went (viewed in this regard) even further. Their method ensures that the words alternate between descendants of 22 and 3, which boils down to changing m every construction step by taking $m = n-1$.
A small side note is that these extra conditions are not necessary when discussing the weighted average length of words. We only had to use this mechanism in the context of ignoring finite amounts of digits per word, always viewed within the complete batch of words within or up to a certain depth.

We'd like to conclude with the realization $x = L(A)$ of the example A that was constructed in Example 5.10. This number starts as follows:

$$x = 0.9157128458323956189986282578875171467764060356652949245541838134430727023 31\dots$$

$$\dots \text{ would } x \text{ be normal in base } 10 \dots \text{ ?}$$

(Any bookmaker ought to say yes.)

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