Group Completions of Commutative Monoids in Orthogonal Spaces
Jubitana, J.

L.J. Jubitana

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Thesis supervisors:
Dr. S. Sagave (RU Nijmegen)
Dr. R.S. de Jong

Leiden University
Mathematical Institute

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1. Introduction

The construction of integers out of natural numbers is one of the best-known examples of group completion. While addition on the natural numbers is unital, associative and commutative, it does not have inverse elements. In contrast, the set of integers with addition is an abelian group. This construction is just a particular case of the group completion of a commutative monoid. We can also construct group completions of topological monoids. The set of path-connected components of a topological monoid is a monoid. If it is a group, we say that the topological monoid is grouplike. If a map of topological monoids is a group completion, it induces an ordinary group completion on the sets of path-connected components. The orthogonal group $O(\mathbb{R}^n)$, consisting of isometries on $\mathbb{R}^n$, is a prominent example of a grouplike topological monoid. Its set of path-connected components is a group with two elements $\{1, -1\}$.

Constructing group completions of topological monoids will be one of our main goals. Group completion is usually dependent on the commutativity of a monoid. However, note that the orthogonal group $O(\mathbb{R}^n)$ is not commutative while $\{1, -1\}$ is. This suggests that one could impose a condition weaker than commutativity on topological monoids to construct group completions. The commutativity of topological monoids is indeed often too restrictive. A common and better type of topological space is an E_∞ space, which has a multiplication that is commutative up to homotopy. See [May72] and [May77] for a good exposition of these spaces. Group completions of monoids have been studied extensively, for example, in [BP72], [May74], [Qui94] and [BM05]. Instead of changing the notion of commutativity, it will be more convenient to replace the category of topological spaces with a type of functor category and consider commutative monoids here. The category we use is that of orthogonal spaces $\mathbf{Top}^{\mathcal{V}}$. This approach will be similar to that of Steffen Sagave and Christian Schlichtkrull in [SS13].

An orthogonal space is a continuous functor from the category $\mathcal{V}$ to the category of (compactly generated weak Hausdorff) topological spaces **Top**, where $\mathcal{V}$ consists of all finite-dimensional standard inner product spaces $\mathbb{R}^n$ as objects and all isometric embeddings as morphisms. An orthogonal space is also called a $\mathcal{V}$-space. Many useful results about $\mathcal{V}$-spaces are found in [Lin13], [SS19] and [Sch18]. One of the motivating examples is the $\mathcal{V}$-space defined by orthogonal groups $O : \mathbb{R}^n \mapsto O(\mathbb{R}^n)$. Some other examples of $\mathcal{V}$-spaces are $V_k : \mathbb{R}^n \mapsto V_k(\mathbb{R}^n)$, where $V_k(\mathbb{R}^n)$ is the Stiefel manifold consisting of orthonormal k-frames in $\mathbb{R}^n$, and $\mathrm{Gr}_k : \mathbb{R}^n \mapsto \mathrm{Gr}_k(\mathbb{R}^n)$, where $\mathrm{Gr}_k(\mathbb{R}^n)$ is the Grassmannian consisting of k-dimensional linear subspaces of $\mathbb{R}^n$. The notion of (commutative) monoids in sets or topological spaces can be generalized to $\mathcal{V}$-spaces. These (commutative) monoids are called (commutative) $\mathcal{V}$-space monoids. The $\mathcal{V}$-space O has a multiplication that makes it a commutative $\mathcal{V}$-space monoid.

We can turn a $\mathcal{V}$-space into a topological space using a functor known as the homotopy colimit. This is a 'fattened up' colimit that preserves more information about a $\mathcal{V}$-space. The homotopy colimit of a commutative $\mathcal{V}$-space monoid is an E_∞-space. There are various definitions of homotopy colimits, all weakly equivalent to each other. We give examples of the $\mathcal{V}$-spaces mentioned above. The topological spaces $V_k(\mathbb{R}^\infty)$, consisting of orthonormal k-frames in $\mathbb{R}^\infty$, and $\mathrm{Gr}_k(\mathbb{R}^\infty)$, consisting of k-dimensional linear subspaces of $\mathbb{R}^\infty$, are homotopy colimits of V_k and Gr_k. The infinite orthogonal group $O(\mathbb{R}^\infty)$, consisting of isometries that act non-trivially on a finite subspace of $\mathbb{R}^\infty$ and are the identity everywhere else, is a homotopy colimit of O. Another example is the additive Grassmannian. It is a commutative $\mathcal{V}$-space monoid $\mathrm{Gr} : n \mapsto \mathrm{Gr}(\mathbb{R}^n)$, where $\mathrm{Gr}(\mathbb{R}^n)$ consists of all linear subspaces of $\mathbb{R}^n$. Its homotopy colimit is the topological monoid $\mathrm{Gr}(\mathbb{R}^\infty)$ consisting of all finite-dimensional linear subspaces of $\mathbb{R}^\infty$. Multiplication is defined using the direct sum, and this multiplication is not commutative. The corresponding set of path-connected components is the monoid $\mathbb{N}_{\geq 0}$ of natural numbers, which is commutative. The strict commutativity of topological monoids is indeed not necessary. We can construct a morphism from Gr to some other commutative $\mathcal{V}$-space monoid that will then realize the group completion $\mathbb{N}_{\geq 0} \to \mathbb{Z}$.

To construct group completions for all commutative $\mathcal{V}$-space monoids, we use model structures on categories. A category with a model structure is a model category. Daniel Quillen introduced these in [Qui67, Chapter 1]. In a model category, every object X has what is known as a fibrant replacement $X \to \widehat{X}$. We will construct a model structure called the group completion model structure on the category of commutative $\mathcal{V}$-space monoids $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, where the fibrant replacement is the group completion of every commutative $\mathcal{V}$-space monoid. We express this as the following theorem, where the subscripts of $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ denote the model structure and $B^{\mathbf{F}}$ is a variation of the classifying space of a topological monoid. This theorem is the analogue of [SS13, Theorem 1.3].

Theorem 1.1. *A morphism of commutative $\mathcal{V}$-space monoids $M \to N$ is a weak equivalence in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\mathrm{gp}}$ if and only if the induced map*

$$B^{\mathbf{F}}(\mathrm{hocolim}_{\mathcal{V}} M) \to B^{\mathbf{F}}(\mathrm{hocolim}_{\mathcal{V}} N)$$

is a weak homotopy equivalence. The fibrant objects in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\mathrm{gp}}$ are those objects that are fibrant in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\mathrm{pos}}$ and grouplike. A fibrant replacement $M \to \widehat{M}$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\mathrm{gp}}$ is a group completion.

Conventions

We will assume that all categories are locally small, that is, all Hom-classes $\mathscr{C}(X, Y)$, with X and Y objects in a category $\mathscr{C}$, are sets. Therefore a category $\mathscr{C}$ is small if and only if its class of objects $\mathrm{Ob}\,\mathscr{C}$ is a set. The category of all topological spaces is denoted as $\mathbf{S}$. The more convenient category of compactly generated weak Hausdorff topological spaces is denoted as **Top**. Unless specified, the term space will refer to an object in **Top**. Other notable categories are the category **Set** of sets, **sSet** of simplicial sets and **sTop** of simplicial spaces. Morphisms in these categories will be referred to as maps. When necessary, we will specify if a map is continuous or simplicial.

There are definitions where a given morphism or natural transformation comes with the condition that it must fit in a certain commutative diagram. We will not write out these diagrams but provide a reference where they can be found. These conditions imply that every sensible diagram that one expects to be commutative is commutative. Some examples are the 'coherence axioms' mentioned in Definition 2.2 that are given in [Mac78].

2. The category of $\mathcal{K}$-spaces

While $\mathcal{V}$-spaces will be our primary focus, we also need functor categories over other small categories like $\mathcal{N}$, with $\mathrm{Ob}\,\mathcal{N} = \mathbb{Z}_{\geq 0}$, and the simplex category Δ. Therefore this chapter will be dedicated to constructing $\mathcal{K}$-spaces, which are continuous functors from a small category $\mathcal{K}$ to the category of topological spaces. While topological spaces play an essential role, it turns out that it is not very convenient to work with the category of all topological space $\mathbf{S}$. The category $\mathbf{S}$ is, in particular, not a closed symmetric monoidal category. Therefore we will start by defining what symmetric monoidal categories are and when such a category is closed. This is followed by an introduction to the category **Top**, which is a closed subcategory of $\mathbf{S}$. With this convenient category, we can rigorously define $\mathcal{K}$-spaces.

2.1. Monoidal categories

Given categories $\mathscr{C}$ and $\mathscr{D}$, let $[\mathscr{D}, \mathscr{C}]$ denote the functor category. A functor $\mathscr{D} \to \mathscr{C}$ is called a *diagram* if $\mathscr{D}$ is small. A category $\mathscr{C}$ is *(co)complete* if for all small categories $\mathscr{D}$ and all diagrams $F : \mathscr{D} \to \mathscr{C}$ the (co)limit of F exists. It is *bicomplete* if it is both complete and cocomplete. There exists a functor

$$c : \mathscr{C} \to [\mathscr{D}, \mathscr{C}]$$

that sends an object $X \in \mathrm{Ob}\,\mathscr{C}$ to the constant functor $cX : d \mapsto X$. If $\mathscr{C}$ is (co)complete, then the (co)limit is right (left) adjoint to c.

$$[\mathscr{D}, \mathscr{C}] \underset{c}{\overset{\mathrm{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{C} \underset{\lim}{\overset{c}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathscr{D}, \mathscr{C}]$$

Proposition 2.1 ([Mac78, Section V.3]). *If $\mathscr{C}$ is a (co)complete category and $\mathscr{D}$ a category, then $[\mathscr{D}, \mathscr{C}]$ is (co)complete.*

If $\mathscr{C}$ is (co)complete, then (co)limits are defined level-wise: Let $G : \mathscr{E} \to [\mathscr{D}, \mathscr{C}]$ be a diagram and let the *evaluation functor* $\mathrm{ev}_d : [\mathscr{D}, \mathscr{C}] \to \mathscr{C}$ be defined by $\mathrm{ev}_d(F) = F(d)$, then we have $\lim G : d \mapsto \lim(\mathrm{ev}_d \circ G)$ and $\mathrm{colim}\, G : d \mapsto \mathrm{colim}(\mathrm{ev}_d \circ G)$.

Definition 2.2 ([Mac78]). A *monoidal category* $(\mathscr{C}, \otimes, 1)$ is a category $\mathscr{C}$ together with a functor $(-) \otimes (-) : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, called the *tensor product*, an object $1 \in \mathrm{Ob}\,\mathscr{C}$, called the *identity object*, and natural isomorphisms a, l and r, called the *associator*, *left unitor* and *right unitor* respectively, with components of the form

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$
$$l_X : 1 \otimes X \to X$$
$$r_X : X \otimes 1 \to X$$

satisfying ‘coherence axioms’ given in [Mac78, Section VII.1].

A *symmetric monoidal category* is a monoidal category with an additional natural isomorphism b, called the *symmetry isomorphism*, with components of the form

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying ‘coherence axioms’ given in [Mac78, Section VII.7].

A *permutative category* is a symmetric monoidal category where a, l and r are identities. In this case, the coherence axioms in [Mac78, Section VII.1] are automatically satisfied.

Definition 2.3. A symmetric monoidal category $(\mathscr{C}, \otimes, 1)$ is *closed* if for every $X \in \operatorname{Ob}\mathscr{C}$ the functor $(-) \otimes X : \mathscr{C} \to \mathscr{C}$ has a right adjoint $[X, -] : \mathscr{C} \to \mathscr{C}$. The object $[X, Y]$ is the *internal hom* of the objects X and Y.

Definition 2.4 ([Mac78])**.** A *monoid* in a monoidal category $(\mathscr{C}, \otimes, 1)$ is an object $M \in \operatorname{Ob}\mathscr{C}$ together with morphisms $u_M : 1 \to M$ and $\mu_M : M \otimes M \to M$, called the *unit* and *multiplication* respectively, satisfying conditions given in [Mac78, Section VII.3]. If $\mathscr{C}$ is symmetric then a monoid is *commutative* if $\mu_M \circ b_{M,M} = \mu_M$.

A *morphism of monoids* is a morphism $f \in \mathscr{C}(M, N)$, with M and N monoids, such that $f \circ \mu_M = \mu_N \circ (f \oplus f)$ and $f \circ u_M = u_N$. Let $\operatorname{Mon}(\mathscr{C})$ denote the category of monoids and morphisms of monoids in $\mathscr{C}$ and let $\mathcal{C}\mathscr{C}$ denote the full subcategory of commutative monoids in $\operatorname{Mon}(\mathscr{C})$.

A typical example of a tensor product is the (cartesian) product of two objects, given that such products exist. In a complete category, these certainly exist. Many examples of symmetric monoidal categories are of this form.

Proposition 2.5. *Let $\mathscr{C}$ be a complete category, with product $\times$ and terminal object $*$, then $(\mathscr{C}, \times, *)$ is a symmetric monoidal category.*

Proof. The required natural isomorphisms follow directly from the universal property of finite products. ☐

Example 2.6. The category **Set** of sets is bicomplete. Let $\mathscr{D}$ be a small category and $F : \mathscr{D} \to \mathbf{Set}$ a diagram. Its limit is

$$\lim F = \left\{ (x_d)_d \in \prod_{d \in \operatorname{Ob}\mathscr{D}} F(d) \;\middle|\; \forall f \in \mathscr{D}(d_0, d_1) : x_{d_1} = F(f)(x_{d_0}) \right\},$$

which is a subset of the cartesian product $\prod_{d \in \operatorname{Ob}\mathscr{D}} F(d)$. Its colimit is

$$\operatorname{colim} F = \left(\coprod_{d \in \operatorname{Ob}\mathscr{D}} F(d) \right) / \sim,$$

which is a quotient of the disjoint union $\coprod_{d \in \operatorname{Ob}\mathscr{D}} F(d)$, where the equivalence relation $\sim$ is generated by the relation, $x \sim y$ if there exists morphism $f \in \mathscr{D}(d_0, d_1)$ such that $y = F(f)(x)$, for $x \in F(d_0), y \in F(d_1)$. Thus $(\mathbf{Set}, \times, *)$, with $\times$ the cartesian product and $*$ the singleton set, is a symmetric monoidal category. Let X, Y and Z be sets and let $[X, Y] = \mathbf{Set}(X, Y)$, then the bijection $\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, [Y, Z])$ makes **Set** closed. Monoids in **Set** are ordinary monoids, and morphisms of monoids in **Set** are monoid homomorphisms.

Example 2.7. Let **Cat** be the category of small categories and functors. Then $(\mathbf{Cat}, \times, \mathbf{1})$, with $\mathbf{1}$ the trivial category containing a single object and morphism, is a closed symmetric monoidal category as mentioned in [Mac78, Section VII.7]. The internal-hom of small categories $\mathscr{D}$ and $\mathscr{C}$ is the functor category $[\mathscr{D}, \mathscr{C}]$, justifying this notation.

Example 2.8. The category **S** of all topological spaces is bicomplete. A limit is constructed by taking the limit in **Set** and giving it the subspace topology via the inclusion $\lim F \subseteq \prod_d F(d)$. Similarly, a colimit is constructed by taking the colimit in **Set** and giving it the quotient topology via the quotient map $\coprod_d F(d) \to \operatorname{colim} F$. Thus $(\mathbf{S}, \times, *)$ is a symmetric monoidal category. However, it is not closed by [Bor94, Proposition 7.1.2].

Example 2.9. Let Δ be the simplex category, with objects finite ordered sets $[p] = \{0, \ldots, p\}$ and morphisms order-preserving maps. For a category $\mathscr{C}$ we let $\mathbf{s}\mathscr{C} = [\Delta^{\mathrm{op}}, \mathscr{C}]$ denote the category of simplicial objects in $\mathscr{C}$. The category $\mathbf{sSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets is bicomplete by Proposition 2.1. Thus it is also a symmetric monoidal category. It is also closed: Given simplicial sets K and L the internal-hom is the simplicial set $[K, L]$ with set of p-simplices $[K, L]_p = \mathbf{sSet}(K \times \Delta[p], L)$, as shown in [Hov99, Section 3.1] after Remark 3.1.7. The internal-hom $[K, L]$ is usually called the mapping space and denoted as $\mathrm{Map}(K, L)$.

Example 2.10. Let $\mathcal{V}$ be the category with the standard inner product spaces $\mathbb{R}^n$ as objects, for all $n \in \mathbb{Z}_{\geq 0}$, and the isometric embeddings as morphisms. The canonical isomorphism $\mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$ makes the *direct sum* a functor $(-) \oplus (-) : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. The triple $(\mathcal{V}, \oplus, \mathbb{R}^0)$ is a permutative category.

We introduce monoidal functors between monoidal categories. They prove to be useful since they preserve monoids.

Definition 2.11. A *monoidal functor* $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}}) \to (\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$, of monoidal categories, is a functor $F : \mathscr{D} \to \mathscr{C}$, with a natural transformation with components of the form

$$F(X) \otimes_{\mathscr{C}} F(Y) \to F(X \otimes_{\mathscr{D}} Y)$$

and a morphism $1_{\mathscr{C}} \to F(1_{\mathscr{D}})$, satisfying conditions given in [Mac78, Section XI.2]. If these are isomorphisms, F is *strong*, and if they are identities, F is *strict*.

2.2. Topological spaces

The category **S** not being closed is a problem, and we wish to replace it with a suitable subcategory of spaces that is a bicomplete closed symmetric monoidal category. Before moving on to this subcategory we mention the following definition. An *inclusion* is an injective continuous map $i : A \hookrightarrow B$ such that for every open $U \subset A$ there exists some open $V \subset B$ with $U = i^{-1}(V)$. It is *closed* if $i(A)$ is closed in B and it is a *closed T_1 inclusion* if every point in $B \backslash i(A)$ is closed in B.

Definition 2.12. Let $A \in \mathrm{Ob}\, \mathbf{S}$ be a topological space. A subset S of A is *compactly closed* if for every compact Hausdorff space C and every map $f : C \to A$, the preimage $f^{-1}(S)$ is closed in C. The topological space A is *compactly generated* if every compactly closed subset is closed.

Definition 2.13. Let $A \in \mathrm{Ob}\, \mathbf{S}$ be a topological space. It is *weak Hausdorff* if for every compact Hausdorff space C and every map $f : C \to A$ the image $f(C)$ is closed in A.

The category **Top** is the full subcategory of **S** with objects the compactly generated weak Hausdorff spaces. In weak Hausdorff spaces, all points are closed. Hence in **Top**, every closed inclusion is a closed T_1 inclusion. the category **Top** is, like **S**, bicomplete. A proof of this is given in Appendix A [Sch18]. We provide a short explanation.

Consider the full subcategory $\mathbf{S}_0$ of **S**, with objects the compactly generated spaces. Let $A \in \operatorname{Ob}\mathbf{S}$. Keeping the underlying set of A the same but changing its topology to include all compactly closed subsets as closed subsets defines a functor $\mathfrak{cg} : \mathbf{S} \to \mathbf{S}_0$ (discussion after [Sch18, Proposition A.2]). If A is compactly generated, then there exists a suitable equivalence relation $\sim$ on A that defines a functor $\mathfrak{h} : \mathbf{S}_0 \to \mathbf{Top}$, that sends A to a quotient space $A/\sim$ that is compactly generated weak Hausdorff ([Sch18, Proposition A.10]). If A is compactly generated, then $\mathfrak{cg}(A) = A$. If A is weak Hausdorff then $\mathfrak{cg}(A)$ is as well ([Sch18, Proposition A.4(viii)]). If A is compactly generated weak Hausdorff, then $\mathfrak{h}(A) \cong A$ (discussion after [Sch18, Proposition A.10]).

Let $F : \mathscr{D} \to \mathbf{Top}$ be a diagram in **Top**. Then F is a diagram in **S**. In **S**, we can take the limit and colimit, which we denote by $\lim^{\mathbf{S}}(F)$ and $\operatorname{colim}^{\mathbf{S}}(F)$. It turns out that $\mathfrak{cg}(\lim^{\mathbf{S}}(F))$ and $\mathfrak{h}(\operatorname{colim}^{\mathbf{S}}(F))$ are the limit and colimit of F in **Top** (discussions after [Sch18, Proposition A.2, Proposition A.10]). For the limit and colimit in **Top**, we write

$$\lim F = \mathfrak{cg}(\lim^{\mathbf{S}}(F)), \tag{2.1}$$

$$\operatorname{colim} F = \mathfrak{h}(\operatorname{colim}^{\mathbf{S}}(F)). \tag{2.2}$$

We remark that $\lim^{\mathbf{S}}(F)$ is weak Hausdorff and $\operatorname{colim}^{\mathbf{S}}(F)$ is compactly generated. Hence these (co)limits are indeed spaces in **Top**.

We focus our attention on **Top**: A space will always be a compactly generated weak Hausdorff space, and (co)limits of diagrams in spaces will always be taken in **Top** as in (2.1) and (2.2) unless stated differently. In particular, the product of spaces X and Y is $X \times Y = \mathfrak{cg}(X \times_{\mathbf{S}} Y)$, where $\times_{\mathbf{S}}$ denotes the cartesian product with the typical product topology, which is in general not compactly generated. Coproducts are the same in **Top** and **S** since coproducts of weak Hausdorff spaces are weak Hausdorff in **S** unlike general colimits ([Sch18, Propositions A.4(vi), A.5(iv)]).

Because **Top** is bicomplete, it is also a symmetric monoidal category. We can endow the Hom-sets $\mathbf{Top}(A, B)$ with the *compact-open topology* generated by a subbasis consisting of the subsets $W(C, U) = \{f \in \mathbf{Top}(A, B) \mid f(C) \subseteq U\}$, with C a compact Hausdorff subset of A and U an open subset of B. Let $\mathrm{C}(A, B)$ denote $\mathbf{Top}(A, B)$ with the compact-open topology. This space is, in general, not compactly generated. The mapping space is defined as $B^A = \mathfrak{cg}(\mathrm{C}(A, B))$. By [Sch18, Theorem A.23(i)] the spaces $\mathrm{C}(A, B)$ and B^A are weak Hausdorff and by [Sch18, Theorem A.23(ii)] we get a natural bijection

$$\mathbf{Top}(A \times B, C) \cong \mathbf{Top}(A, C^B), \tag{2.3}$$

for all spaces $A, B, C \in \operatorname{Ob}\mathbf{Top}$. Thus $(\mathbf{Top}, \times, *)$ is a closed symmetric monoidal category, with the internal-hom $[A, B] = B^A$. We usually implicitly assume that $\mathbf{Top}(A, B)$ is equipped with the topology of the mapping space B^A and refer to it as a Hom-space. Then [Sch18, Theorem A.23(ii)] even implies that (2.3) is a homeomorphism. Finally, note that by Proposition 2.1, the category $\mathbf{sTop} = [\Delta^{\mathrm{op}}, \mathbf{Top}]$ of simplicial spaces is bicomplete and, therefore, a symmetric monoidal category.

2.3. Enriched categories

To construct $\mathcal{K}$-spaces, we must define categories enriched over **Top**. The Hom-sets and composition maps in a category are objects and morphisms in the category $(\mathbf{Set}, \times, *)$. The previous chapter shows that category **Top** is equipped with Hom-spaces and continuous composition maps. Such a category is enriched over **Top**. Later on in Section 7.1, we generalize this to categories enriched over some closed symmetric monoidal category $\mathscr{E}$, equipped with Hom-objects and composition morphisms in $\mathscr{E}$ satisfying axioms similar to those of an ordinary category. From this point of view, an ordinary category is a category enriched over $(\mathbf{Set}, \times, *)$.

In a bicomplete category $\mathscr{C}$ taking the (co)product of copies of an object $X \in \operatorname{Ob}\mathscr{C}$ over some set S as

$$X^S = \prod_{s \in S} X, \qquad\qquad X \times S = \coprod_{s \in S} X.$$

defines functors $\mathscr{C} \times \mathbf{Set}^{\mathrm{op}} \to \mathscr{C}$ and $\mathscr{C} \times \mathbf{Set} \to \mathscr{C}$ known as the *cotensor* and *tensor* over **Set** respectively. It is not hard to see that we have the following natural bijections

$$\mathscr{C}(X \times S, Y) \cong \mathbf{Set}(S, \mathscr{C}(X, Y)) \cong \mathscr{C}(X, Y^S),$$

In Section 7.1 these cotensor and tensor will also be generalized over more general closed symmetric monoidal categories. However, this chapter is restricted to categories enriched, tensored and cotensored over **Top**.

Definition 2.14. A category $\mathscr{C}$ is a **Top**-*category* if for all $X, Y, Z \in \operatorname{Ob}\mathscr{C}$ the Hom-sets $\mathscr{C}(X, Y)$ have a topology such that the composition map

$$\mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$$

is continuous. A functor $F : \mathscr{D} \to \mathscr{C}$, of **Top**-categories $\mathscr{D}$ and $\mathscr{C}$ is *continuous* if the maps $\mathscr{D}(d_0, d_1) \to \mathscr{C}(F(d_0), F(d_1))$ are continuous, for all $d_0, d_1 \in \operatorname{Ob}\mathscr{D}$.

Definition 2.15. A category $\mathscr{C}$ is *enriched, tensored and cotensored* over **Top** if there exist functors

$$\mathrm{Map} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Top}, \quad (-) \times (-) : \mathscr{C} \times \mathbf{Top} \to \mathscr{C}, \quad (-)^{(-)} : \mathscr{C} \times \mathbf{Top}^{\mathrm{op}} \to \mathscr{C},$$

called the *enrichment*, *tensor* and *cotensor* over **Top** respectively, such that we have natural bijections

$$\mathscr{C}(X \times A, Y) \cong \mathbf{Top}(A, \mathrm{Map}(X, Y)) \cong \mathscr{C}(X, Y^A), \tag{2.4}$$

for $X, Y \in \operatorname{Ob}\mathscr{C}, A \in \operatorname{Ob}\mathbf{Top}$, and natural isomorphisms

$$(X \times A) \times B \cong X \times (A \times B), \tag{2.5}$$

$$X \times * \cong X, \tag{2.6}$$

for $X \in \operatorname{Ob}\mathscr{C}$ and $*, A, B \in \operatorname{Ob}\mathbf{Top}$, that satisfy 'coherence axioms' given in [Hov99, Definition 4.1.6].

If $\mathscr{C}$ is enriched, tensored and cotensored over **Top** then $\mathscr{C}$ is a **Top**-category. The underling set of $\mathrm{Map}(X, Y)$ is $\mathbf{Top}(*, \mathrm{Map}(X, Y)) \cong \mathscr{C}(X \times *, Y) \cong \mathscr{C}(X, Y)$. The identity map $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ corresponds by (2.4) to a map $X \times \mathrm{Map}(X, Y) \to Y$. Together with the map $Y \times \mathrm{Map}(Y, Z) \to Z$ we obtain $X \times (\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z)) \to Z$ after applying (2.5). After applying the symmetry isomorphism in **Top**, this corresponds by (2.4) to the composition map $\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$, which must now be continuous. We usually write $\mathscr{C}(X, Y)$ instead of $\mathrm{Map}(X, Y)$ to denote the Hom-spaces, leaving the topology implicit.

Example 2.16. In $(\mathbf{Top}, \times, *)$, let the tensor be the tensor product and the enrichment and cotensor be the internal-hom. Then **Top** is enriched, tensored and cotensored over itself. The bijections (2.4) follow from (2.3) by one application of the symmetry isomorphism in $(\mathbf{Top}, \times, *)$. The natural isomorphisms (2.5) and (2.6) are the associator and right unitor in $(\mathbf{Top}, \times, *)$

Example 2.17. The category $\mathcal{V}$ is a **Top**-category. The Hom-set $\mathcal{V}(\mathbb{R}^n, \mathbb{R}^m)$ inherits the subspace topology from $\mathrm{Mat}_{m,n}(\mathbb{R}) \cong \mathbb{R}^{m\times n}$. The continuity of matrix multiplication makes the composition in $\mathcal{V}$ continuous. The direct sum $\oplus$ in $\mathcal{V}$ is continuous. For this we need to check that all maps $\mathcal{V}(\mathbb{R}^n, \mathbb{R}^m) \times \mathcal{V}(\mathbb{R}^k, \mathbb{R}^l) \to \mathcal{V}(\mathbb{R}^{n+k}, \mathbb{R}^{m+l})$ are continuous. This is the case because the assignment

$$(A, B) \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

is continuous for all matrices A and B.

2.4. The category of $\mathcal{K}$-spaces

We can generalize **Top** valued diagrams using topologically enriched category theory. These constructions are $\mathcal{K}$-spaces.

Definition 2.18. An *index category* is a small **Top**-category $\mathcal{K}$. A *$\mathcal{K}$-space* is a continuous functor $\mathcal{K} \to \mathbf{Top}$. Let $\mathbf{Top}^{\mathcal{K}}$ denote the full subcategory of $\mathcal{K}$-spaces in $[\mathcal{K}, \mathbf{Top}]$.

An index category $\mathcal{K}$ is *discrete* if all its Hom-spaces are discrete. In this case $\mathbf{Top}^{\mathcal{K}} = [\mathcal{K}, \mathbf{Top}]$. Equipping the Hom-sets of a category with discrete topologies makes it a **Top**-category. Hence any diagram $\mathscr{D} \to \mathbf{Top}$ can be interpreted as a $\mathscr{D}$-space with $\mathscr{D}$ discrete. If $\mathcal{K}$ and $\mathcal{L}$ are index categories then so are $\mathcal{K}^{\mathrm{op}}$ and $\mathcal{K} \times \mathcal{L}$. Let X and Y be $\mathcal{K}$- and $\mathcal{K}^{\mathrm{op}}$-spaces respectively. Given $k \in \mathrm{Ob}\,\mathcal{K}$ we often write $X_k = X(k)$. Given $\phi \in \mathcal{K}(k, l)$, $x \in X_k$ and $y \in Y_l$ we write $\phi_* x = X(\phi)(x)$ and $\phi^* y = Y(\phi)(y)$. By the continuity of a $\mathcal{K}$-space X we have continuous maps $\mathcal{K}(k, l) \to \mathbf{Top}(X_k, X_l)$ and by (2.3) we then get continuous evaluation maps $\mathcal{K}(k, l) \times X_k \to X_l$. Therefore a diagram $X : \mathcal{K} \to \mathbf{Top}$ is a $\mathcal{K}$-space if and only if the evaluation maps are continuous. We list some examples.

Example 2.19. The continuity of the composition $\mathcal{K}(k, l) \times \mathcal{K}(n, k) \to \mathcal{K}(n, l)$ of an index category $\mathcal{K}$ implies that $\mathcal{K}(n, -) : k \to \mathcal{K}(n, k)$ defines a $\mathcal{K}$ space. Similarly $\mathcal{K}(-, n) : k \to \mathcal{K}(k, n)$ defines a $\mathcal{K}^{\mathrm{op}}$-space and $\mathcal{K}(-, -) : (k, l) \mapsto \mathcal{K}(k, l)$ is a $\mathcal{K}^{\mathrm{op}} \times \mathcal{K}$-space. We sometimes write $\mathcal{K}$ to indicate the $\mathcal{K}^{\mathrm{op}} \times \mathcal{K}$-space $\mathcal{K}(-, -)$ instead of the index category.

Example 2.20. Let A be a space and X a $\mathcal{K}$-space. The evaluation map $\mathcal{K}(k, l) \times X_k \to X_l$ immediately shows that the functors $A \times X$ and $X \times A$ defined level-wise by $(A \times X)_k = A \times X_k$ and $(X \times A)_k = X_k \times A$ are $\mathcal{K}$-spaces. By (2.3) we obtain a map $(X_k)^A \times A \to X_k$. By taking the tensor product with $\mathcal{K}(k, l)$ and composing it with the evaluation map, we obtain a map $\mathcal{K}(k, l) \times (X_k)^A \times A \to X_l$. Applying (2.3) once more shows that the functor X^A defined level-wise by $(X^A)_k = (X_k)^A$ is a $\mathcal{K}$-space.

Example 2.21. Consider a $\mathcal{K}$-space X and an $\mathcal{L}$-space Y. Then the functor (X, Y) defined level-wise by $(X \times Y)_{(k,l)} = X_k \times Y_l$ is a $\mathcal{K} \times \mathcal{L}$-space. Let $\mathcal{L} = \mathcal{K}$, then the diagonal functor $\mathrm{d} : \mathcal{K} \to \mathcal{K} \times \mathcal{K}, k \mapsto (k, k)$ is a continuous functor. Thus $X \times Y = (X, Y) \circ \mathrm{d}$ is a $\mathcal{K}$-space with $(X \times Y)_k = X_k \times Y_k$.

Example 2.22. The category $\mathcal{N}$ with $\operatorname{Ob}\mathcal{N} = \mathbb{Z}_{\geq 0}$ equipped with a single morphism $n \to m$ for all $n \leq m$ is a discrete index category. Note that an $\mathcal{N}$-space X is precisely a sequence $X_0 \to X_1 \to X_2 \to \dots$ of spaces.

Example 2.23. The category $\mathcal{I}$ with $\operatorname{Ob}\mathcal{I} = \{\mathbf{n} = \{1, \dots, n\} \mid n \in \mathbb{Z}_{\geq 0}\}$, where $\mathbf{0}$ denotes the empty set, equipped with the injections is a discrete index category. There exists a continuous functor $\mathcal{N} \to \mathcal{I}$ sending n to $\mathbf{n}$ and $n \to m$ to $\mathbf{n} \subseteq \mathbf{m}$. This functor can be viewed as an inclusion. For an $\mathcal{I}$-space X we write X_n instead of $X(\mathbf{n})$ or $X_{\mathbf{n}}$.

Example 2.24. As shown in Example 2.17, the small category $\mathcal{V}$ is an index category. There exists a continuous functor $\mathcal{N} \to \mathcal{V}$ sending n to $\mathbb{R}^n$ and $n \to m$ to

$$\mathbb{R}^n \to \mathbb{R}^m, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0 \dots, 0).$$

This functor can be viewed as an inclusion. Hence we sometimes write n instead of $\mathbb{R}^n$. We write $\mathcal{V}(n, m)$ instead of $\mathcal{V}(\mathbb{R}^n, \mathbb{R}^m)$ and, for a $\mathcal{V}$-space X, we write X_n instead of $X(\mathbb{R}^n)$ for example. Via this inclusion, we can interpret a $\mathcal{V}$-space as an $\mathcal{N}$-space.

Example 2.25. The simplex category Δ, given in Example 2.9, is a discrete index category. Thus Δ^{op}-spaces are precisely simplicial spaces and $\mathbf{Top}^{\Delta^{\mathrm{op}}} = \mathbf{sTop}$. The category $\mathbf{Top}^{\Delta} = [\Delta, \mathbf{Top}]$ is the category of cosimplicial spaces. An important cosimplicial space is $\Delta^\bullet : [p] \mapsto \Delta^p$ where Δ^p is the standard topological p-simplex. [Dug08, Chapter 3] gives a good introduction to simplicial spaces and $\Delta^\bullet$.

Example 2.26. The trivial category $\mathbf{1}$ with a single object and morphism is a discrete index category. We have $\mathbf{Top}^{\mathbf{1}} \cong [\mathbf{1}, \mathbf{Top}] \cong \mathbf{Top}$. Therefore any space can be viewed as a $\mathbf{1}$-space and any $\mathcal{K}$-space can be viewed as a $\mathcal{K} \times \mathbf{1}$- or $\mathbf{1}^{\mathrm{op}} \times \mathcal{K}$-space, where $\mathbf{1}^{\mathrm{op}}$ is of course equal to $\mathbf{1}$. Let X be a $\mathcal{K}$-space and A be a space. Via the unique continuous functor $\mathcal{K} \to \mathbf{1}$, the space A defines a constant $\mathcal{K}$-space. Then the definitions for $X \times A$ in Example 2.20 and Example 2.21 coincide.

Since $\mathbf{Top}$ is cocomplete and constant diagrams in $[\mathcal{K}, \mathbf{Top}]$ are certainly $\mathcal{K}$-spaces we get the adjunction

$$\operatorname{colim} : \mathbf{Top}^{\mathcal{K}} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top} : c. \tag{2.7}$$

We finish this chapter by showing that the category $\mathbf{Top}^{\mathcal{K}}$ is bicomplete and enriched, tensored and cotensored. This will imply that $(\mathbf{Top}^{\mathcal{K}}, \times, *)$ is a symmetric monoidal category. However, a different tensor product, called the boxproduct, will be of greater interest. This will be discussed in the next chapter.

Theorem 2.27. *Let $\mathcal{K}$ be an index category. The category $\mathbf{Top}^{\mathcal{K}}$ is bicomplete.*

Proof. We will show that (co)limits can be taken in the underlying functor category $[\mathcal{K}, \mathbf{Top}]$. Let $\mathscr{D}$ be a small category and $F : \mathscr{D} \to \mathbf{Top}^{\mathcal{K}}$ a diagram. Then $\operatorname{colim} F$ is a functor in $[\mathcal{K}, \mathbf{Top}]$ defined by

$$\operatorname{colim} F : k \mapsto \operatorname{colim}(\mathrm{ev}_k \circ F).$$

Fix $k, l \in \operatorname{Ob}\mathcal{K}$. For every $d \in \operatorname{Ob}\mathscr{D}$ we have a continuous map $\mathcal{K}(k, l) \times F(d)_k \to F(d)_l$. These maps are components of a natural transformation $\mathcal{K}(k, l) \times F(-)_k \to F(-)_l$ that can also be written as $\mathcal{K}(k, l) \times (\mathrm{ev}_k \circ F) \to (\mathrm{ev}_k \circ F)$. For a diagram $G : \mathscr{D} \to \mathbf{Top}$ and a space A we have $\operatorname{colim}(A \times G) \cong A \times \operatorname{colim} G$, since $\mathbf{Top}$ is closed by (2.3). Therefore taking the colimit of the natural transformation induces a continuous map

$$\mathcal{K}(k, l) \times (\operatorname{colim} F)_k \to (\operatorname{colim} F)_l.$$

Thus $\operatorname{colim} F$ is a $\mathcal{K}$-space. The isomorphism $\lim(A \times G) \cong A \times \lim G$ is immediate for limits. The rest of the argument is the same. $\square$

Theorem 2.28 ([Bor94, Propositions 6.3.1, 6.5.7]). *Let $\mathcal{K}$ be an index category. The category $\mathbf{Top}^{\mathcal{K}}$ is enriched, tensored and cotensored over $\mathbf{Top}$.*

3. Constructions on $\mathcal{K}$-spaces

Having defined $\mathcal{K}$-spaces, we need to construct a variety of tools. First is the homotopy colimit. It is one of our main functors sending $\mathcal{V}$-spaces to spaces. It is more favourable than the colimit since it preserves weak equivalences. Next is the box product. It gives the category $\mathbf{Top}^{\mathcal{K}}$ a structure of a closed symmetric monoidal category. Later on, we will see that this product relates nicely to the symmetric monoidal category $(\mathbf{Top}, \times, *)$ via the homotopy colimit. Constructing these tools requires us to build up more enriched category theory. Using this, we also obtain the geometric realization. We finish by returning to $\mathcal{V}$ and listing interesting examples of $\mathcal{V}$-spaces.

3.1. Enriched coends

In this section, we consider continuous diagrams of the form $\mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ for index categories $\mathcal{K}$ and $\mathcal{L}$. If such a diagram is continuous it is equivalent to a $\mathcal{K} \times \mathcal{L}$-space, which is just a $\mathcal{K}$-space if $\mathcal{L}$ is the terminal category. The colimit of a diagram of parallel morphisms $X \rightrightarrows Y$ is called a *coequalizer*. Coends are a type of coequalizer dependent on a diagram of the form $\mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Top}^{\mathcal{L}}$. There are many important constructions that can be made with this.

Definition 3.1. Let $F : \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Top}^{\mathcal{L}}$ be a diagram. A morphism $f \in \mathscr{D}(d_0, d_1)$ induces two morphisms $F(\mathrm{id}_{d_1}, f) : F(d_1, d_0) \to F(d_1, d_1)$ and $F(f, \mathrm{id}_{d_0}) : F(d_1, d_0) \to F(d_0, d_0)$. These induce parallel morphisms

$$\coprod_{d_0, d_1 \in \mathrm{Ob}\,\mathscr{D}} \mathscr{D}(d_0, d_1) \times F(d_1, d_0) \rightrightarrows \coprod_{d \in \mathrm{Ob}\,\mathscr{D}} F(d, d).$$

Its coequalizer is the *coend* of F and is denoted as

$$\int^{d \in \mathscr{D}} F(d, d).$$

This definition uses the enrichment and tensor over **Set**, motivating us to generalize to **Top**. Let $\mathcal{K}$ and $\mathcal{L}$ be index categories and note that $\mathbf{Top}^{\mathcal{L}}$ is bicomplete, enriched, tensored and cotensored over **Top**. A diagram $X : \mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ is continuous if and only if the evaluation morphisms $\mathcal{K}(k, l) \times X_k \to X_l$ are in $\mathrm{Mor}(\mathbf{Top}^{\mathcal{L}})$. Let $Z : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ be a continuous diagram. For $k_1 \in \mathrm{Ob}\,\mathcal{K}$ the map $k \mapsto (k_1, k)$ defines a continuous functor $\mathcal{K} \to \mathcal{K}^{\mathrm{op}} \times \mathcal{K}$. Then $Z(k_1, -)$ is continuous and $\mathcal{K}(k_0, k_1) \times Z(k_1, k_0) \to Z(k_1, k_1)$ is in $\mathrm{Mor}(\mathbf{Top}^{\mathcal{L}})$. Similarly $\mathcal{K}(k_0, k_1) \times Z(k_1, k_0) \to Z(k_0, k_0)$ is in $\mathrm{Mor}(\mathbf{Top}^{\mathcal{L}})$.

Definition 3.2. Let $\mathcal{K}$ and $\mathcal{L}$ be index categories and $Z : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ be a continuous diagram. Consider the parallel morphisms

$$\coprod_{k_0, k_1 \in \mathrm{Ob}\,\mathcal{K}} \mathcal{K}(k_0, k_1) \times Z(k_1, k_0) \rightrightarrows \coprod_{k \in \mathrm{Ob}\,\mathcal{K}} Z(k, k). \tag{3.1}$$

Its coequalizer is the *enriched coend* of Z and is denoted as

$$\int^{k\in\mathcal{K}} Z(k,k).$$

Given a set S and a space A, the space $\coprod_{s\in S} A$ is homeomorphic to $S \times A$ when S is given the discrete topology. Thus Definition 3.1 is a particular case of Definition 3.2, where $\mathscr{D}$ is viewed as a discrete index category.

Definition 3.3. Let $\mathcal{K}$ and $\mathcal{L}$ be index categories and $X : \mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ and $Y : \mathcal{K}^{\mathrm{op}} \to \mathbf{Top}^{\mathcal{L}}$ be continuous diagrams. The *tensor product* of Y and X over $\mathcal{K}$ is the enriched coend

$$Y \otimes_{\mathcal{K}} X = \int^{k\in\mathcal{K}} Y_k \times X_k.$$

When $\mathcal{L} = \mathbf{1}$, this tensor product is equivalent to [HV92, Definition 2.3] after taking the functor $\mathfrak{h}$. In [HV92], a $\mathcal{K}$-space X is called a 'left $\mathcal{K}$-module' since morphisms in $\mathcal{K}$ have a type of action on the spaces X_k. This motivates the notation and the use of the term tensor product over $\mathcal{K}$. It should be noted that this product is not a tensor product in the sense of Definition 2.2. The subscript makes sure to differentiate $\otimes_{\mathcal{K}}$ from a monoidal tensor product.

Proposition 3.4. *Let X and Y be $\mathcal{K}$- and $\mathcal{K}^{\mathrm{op}}$-spaces respectively. Let $\overline{\mathcal{K}}$ denote the discrete index category with the same underlying category as $\mathcal{K}$. We have*

$$\operatorname{colim} X \cong * \otimes_{\overline{\mathcal{K}}} X, \tag{3.2}$$

$$Y \otimes_{\mathcal{K}} X \cong X \otimes_{\mathcal{K}^{\mathrm{op}}} Y. \tag{3.3}$$

Proof. Equation (3.2) is the dual of [Mac78, Theorem V.2.2]. Equation (3.3) is a consequence of using the symmetry isomorphism. □

A continuous functor $\mathcal{K} \to \mathbf{Top}^{\mathcal{L}}$ is equivalent to an $\mathcal{L} \times \mathcal{K}$-space. The tensor product over $\mathcal{K}$ of an $\mathcal{L} \times \mathcal{K}^{\mathrm{op}}$-space Y and an $\mathcal{L} \times \mathcal{K}$-space X is the $\mathcal{L}$-space defined as

$$Y \otimes_{\mathcal{K}} X : l \mapsto Y_l \otimes_{\mathcal{K}} X_l.$$

We can even construct the tensor product of an $\mathcal{L} \times \mathcal{K}^{\mathrm{op}}$-space Y and an $\mathcal{M} \times \mathcal{K}$-space X over $\mathcal{K}$. The projections $\mathrm{pr}_{\mathcal{L}} : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ and $\mathrm{pr}_{\mathcal{M}} : \mathcal{L} \times \mathcal{M} \to \mathcal{M}$ are continuous functors. Let

$$Y \otimes_{\mathcal{K}} X = (Y \circ (\mathrm{pr}_{\mathcal{L}} \times \mathcal{K}^{\mathrm{op}})) \otimes_{\mathcal{K}} (Y \circ (\mathrm{pr}_{\mathcal{M}} \times \mathcal{K})).$$

Then this is the $\mathcal{L} \times \mathcal{M}$-space with

$$Y \otimes_{\mathcal{K}} X : (l,m) \mapsto Y_l \otimes_{\mathcal{K}} X_m.$$

By setting $\mathcal{L} = \mathcal{M} = \mathbf{1}$, we return to the case in [HV92].

3.2. The realization

The geometric realization is a functor sending either simplicial sets or simplicial spaces to spaces and is denoted as $|-|$. It is a vital tool to relate the homotopy theory of these categories. For convenience, we call it the realization. There are many ways to construct this functor. One way is to use a tensor product over the simplex category Δ.

Definition 3.5. Recall the cosimplicial space $\Delta^\bullet : [p] \mapsto \Delta^p$, where Δ^p is the standard topological p-simplex. The *realization* of a simplicial space K is the space

$$|K| = K \otimes_\Delta \Delta^\bullet.$$

To take the realization of a simplicial set K, we first endow every set of p-simplices K_p with the discrete topology. The *realization* of a $\mathcal{K} \times \Delta^{\mathrm{op}}$-space (or simplicial $\mathcal{K}$-space) X is the $\mathcal{K}$-space

$$|X| = X \otimes_\Delta \Delta^\bullet.$$

Note that $|X|_k = |X_k|$. These realizations define functors $\mathbf{sTop} \to \mathbf{Top}$, $\mathbf{sSet} \to \mathbf{Top}$ and $[\Delta^{\mathrm{op}}, \mathbf{Top}^{\mathcal{K}}] \to \mathbf{Top}^{\mathcal{K}}$, all denoted by $|-|$.

The realization $|-| : \mathbf{sSet} \to \mathbf{Top}$ has a right adjoint functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ called the *singular complex functor* which is defined by $\mathrm{Sing}_p(A) = \mathbf{Top}(\Delta^p, A)$ as shown in [Hov99, Section 3.1] after Remark 3.1.7. While the realization of simplicial spaces is useful, it has a disadvantage: It does not preserve weak equivalences. For this reason, we introduce the fat realization.

Definition 3.6. Consider the subcategory $\Delta^{\mathbf{F}} \subset \Delta$ having the same objects, but only the injective maps, and inheriting the discrete topology of Δ. The *fat realization* of a simplicial space K is the space

$$||K|| = K \otimes_{\Delta^{\mathbf{F}}} \Delta^\bullet,$$

where we interpret $\Delta^\bullet$ as a $\Delta^{\mathbf{F}}$-space and K as a $(\Delta^{\mathbf{F}})^{\mathrm{op}}$-space via the inclusion $\Delta^{\mathbf{F}} \subset \Delta$.

Later, we will see that the fat realization does preserve weak equivalences.

3.3. The homotopy colimit

A homotopy $H : S^n \times [0,1] \to A$, usually written as $H_t(x) = H(x,t)$, corresponds by (2.3) to a path in $\mathbf{Top}(S^n, A)$. Let $\mathbf{Top}_*$ be the category of based spaces and let (A, a) be a based space. A based homotopy $H : S^n \times [0,1] \to A$ corresponds to a path in $\mathbf{Top}_*(S^n, A)$. Therefore the n-th homotopy group $\pi_n(A, a)$ can be defined as the set of path-connected components $\pi_0(\mathbf{Top}_*(S^n, A))$.

Definition 3.7. A continuous map $f : A \to B$ is a *weak homotopy equivalence* if the induced maps $\pi_n(A, a) \to \pi_n(B, f(a))$ are bijections for all $n \geq 0$ and $a \in A$.

Definition 3.8. A morphism of $\mathcal{K}$-spaces $f : X \to Y$ is a *level equivalence* if all components f_n are weak homotopy equivalences.

In the adjunction (2.7), the functor c sends weak homotopy equivalences to level equivalences. The colimit does not do the converse in general. [Dug08, Example 2.1] shows this in the case $\mathcal{K} = \{1 \leftarrow 0 \rightarrow 2\}$. For this reason, we introduce the homotopy colimit that does send level equivalences to weak homotopy equivalences. We discuss this in Section 5.2. [Dug08, Section 2] gives some examples of homotopy colimits. The homotopy colimit can be defined in several ways up to weak homotopy equivalences. We follow the approach of [SS19, Section 2.1], which is derived from the definition given in [HV92, Section 3]. For this, we need the bar construction.

Definition 3.9. Let X and Y be $\mathcal{K}$- and $\mathcal{K}^{\text{op}}$-spaces respectively. The *bar construction* $B(Y, \mathcal{K}, X)$ is the realization of the simplicial space $B_\bullet(Y, \mathcal{K}, X)$ whose space of p-simplices is

$$B_p(Y, \mathcal{K}, X) = \coprod_{k_0, \dots k_p \in \text{Ob}\,\mathcal{K}} Y_{k_p} \times \mathcal{K}(k_{p-1}, k_p) \times \cdots \times \mathcal{K}(k_0, k_1) \times X_{k_0}.$$

The boundaries are induced by the composition in $\mathcal{K}$, and the evaluation maps

$$Y_{k_p} \times \mathcal{K}(k_{p-1}, k_p) \to Y_{k_{p-1}} \quad \text{and} \quad \mathcal{K}(k_0, k_1) \times X_{k_0} \to X_{k_1}.$$

The degeneracies are induced by the maps $* \to \mathcal{K}(k_i, k_i), i = 0, \dots, p$.

Peeking under the hood shows us that the underlying sets are

$$B_p(Y, \mathcal{K}, X) = \{(y; \phi_p, \dots, \phi_1; x) \mid \phi_n \in \text{Mor}\,\mathcal{K}, \mathfrak{t}(\phi_n) = \mathfrak{s}(\phi_{n+1}), x \in X_{\mathfrak{s}(\phi_1)}, y \in Y_{\mathfrak{t}(\phi_p)}\}.$$

Here $\mathfrak{t}$ and $\mathfrak{s}$ are the functions that send a morphism to its domain and codomain, respectively. The boundaries and degeneracies can be expressed as

$$d^i(y; \phi_p, \dots, \phi_1; x) = \begin{cases} (y; \phi_p, \dots, \phi_2; \phi_{1*}(x)) & , i = 0 \\ (y; \phi_p, \dots, \phi_{i+1}\phi_i, \dots, \phi_1; x) & , 0 < i < p \\ (\phi_p^*(y); \phi_{p-1}, \dots, \phi_1; x) & , i = p \end{cases}$$
$$s^i(y; \phi_p, \dots, \phi_1; x) = (y; \phi_p, \dots, \text{id}, \phi_i, \dots, \phi_1; x).$$

Definition 3.9 can be generalized to the case where X and Y are $\mathcal{M} \times \mathcal{K}$- and $\mathcal{L} \times \mathcal{K}^{\text{op}}$-spaces respectively. In that case $B_p(Y, \mathcal{K}, X)$ and $B(Y, \mathcal{K}, X)$ are $\mathcal{L} \times \mathcal{M}$-spaces and

$$B(Y, \mathcal{K}, X) : (l, m) \mapsto B(Y_l, \mathcal{K}, X_m).$$

Let $F : \mathcal{K} \to \mathcal{L}$ be a continuous functor of index categories. Given $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{K}^{\text{op}}$-, $\mathcal{L}^{\text{op}}$-spaces X, X', Y, Y' respectively, and natural transformations $\alpha : X \to X' \circ F$ and $\beta : Y \to Y' \circ F$ we get an induced map

$$B(Y, \mathcal{K}, X) \to B(Y', \mathcal{L}, X'). \tag{3.4}$$

On the space of p-simplices, this map is $(y; \phi_p, \dots, \phi_1; x) \mapsto (\beta(y); F(\phi_p), \dots, F(\phi_1); \alpha(x))$ which commutes with boundaries and degeneracies.

Definition 3.10. The *homotopy colimit* of a $\mathcal{K}$-space X is the space

$$\text{hocolim}_{\mathcal{K}} X = B(*, \mathcal{K}, X)$$

also denoted as $X_{h\mathcal{K}}$. A morphism of $\mathcal{K}$-spaces $f : X \to Y$ induces a map $f_* : X_{h\mathcal{K}} \to Y_{h\mathcal{K}}$, as a special case of the map (3.4). This defines a functor $\text{hocolim}_{\mathcal{K}} : \mathbf{Top}^{\mathcal{K}} \to \mathbf{Top}$. The *classifying space* of an index category $\mathcal{K}$ is the space $B\mathcal{K} = B(*, \mathcal{K}, *) = *_{h\mathcal{K}}$.

We mention some useful properties of the bar construction and homotopy colimit.

Proposition 3.11 ([HV92, Proposition 3.1(1)])**.** *Let W be an $\mathcal{L}^{\text{op}}$-space, Y be an $\mathcal{L} \times \mathcal{K}^{\text{op}}$-space, X be a $\mathcal{K} \times \mathcal{M}^{\text{op}}$-space and Z be an $\mathcal{M}$-space. We have*

$$B(Y, \mathcal{K}, X) \otimes_{\mathcal{M}} Z \cong B(Y, \mathcal{K}, X \otimes_{\mathcal{M}} Z),$$
$$W \otimes_{\mathcal{L}} B(Y, \mathcal{K}, X) \cong B(W \otimes_{\mathcal{L}} Y, \mathcal{K}, X).$$

Corollary 3.12. *The homotopy colimit preserves colimits.*

Proof. Let $F : \mathscr{D} \to \mathbf{Top}^{\mathcal{K}}$ be a diagram. We can interpret it as an $\mathcal{K} \times \mathscr{D}$-space by equipping $\mathscr{D}$ with the discrete topology. Then

$$\begin{aligned}(\operatorname{colim} F)_{h\mathcal{K}} &\cong B(*, \mathcal{K}, F \otimes_{\mathscr{D}^{\mathrm{op}}} *) \\ &\cong B(*, \mathcal{K}, F) \otimes_{\mathscr{D}^{\mathrm{op}}} * \\ &\cong \operatorname{colim}(F_{h\mathcal{K}})\end{aligned}$$

by Proposition 3.4 and Proposition 3.11. □

Lemma 3.13. *The homotopy colimit preserves tensors.*

Proof. The realization preserves tensors since it preserves finite products by [Hov99, Lemma 3.1.8]. Since tensors of simplicial spaces are constructed level-wise, the result follows. □

Lemma 3.14. *If $\mathcal{K}$ has an initial or terminal object, then $B\mathcal{K}$ is contractible.*

Proof. If $\mathcal{K}$ has an initial object there exists an adjunction $\mathbf{1} \rightleftarrows \mathcal{K}$. As a consequence of [Seg68, Proposition 2.1], this induces a homotopy equivalence $* \simeq B\mathcal{K}$. For a terminal object, the proof is dual. □

Given a $\mathcal{K}$-space X, the *bar resolution* is the $\mathcal{K}$-space $\overline{X} := B(\mathcal{K}, \mathcal{K}, X)$. It is used to construct a natural transformation from the homotopy colimit to the colimit. For every $k \in \operatorname{Ob} \mathcal{K}$ and $p \geq 0$ there exists a map

$$\begin{aligned}\epsilon_{k,p} : B_p(\mathcal{K}(-,k), \mathcal{K}, X) &\to X_k \\ (f, \phi_p, \ldots, \phi_1, x) &\mapsto (f\phi_p \cdots \phi_1)_*(x)\end{aligned}$$

which is continuous since it is an iterated evaluation map. Let cX be the simplicial $\mathcal{K}$-space defined by $(cX)_{k,p} = X_k$. Note that the realization of cX is just X. The maps $\epsilon_{k,p}$ are natural in both k and p and therefore define a morphism of simplicial $\mathcal{K}$-spaces. Its realization

$$\epsilon : \overline{X} \to X$$

is called the *evaluation*. It is natural in X.

Proposition 3.15. *Let X be a $\mathcal{K}$-space. There exists an isomorphism* $\operatorname{hocolim}_{\mathcal{K}} X \cong \operatorname{colim} \overline{X}$ *that is natural in X.*

Proof. Consider the following chain of isomorphisms

$$\begin{aligned}\operatorname{colim} \overline{X} &\cong * \otimes_{\overline{\mathcal{K}}} B(\mathcal{K}, \mathcal{K}, X) \\ &\cong B(* \otimes_{\overline{\mathcal{K}}} \mathcal{K}, \mathcal{K}, X) \\ &\cong B(\operatorname{colim}_k \mathcal{K}(-,k), \mathcal{K}, X) \\ &\cong B(*, \mathcal{K}, X) \\ &= \operatorname{hocolim}_{\mathcal{K}} X.\end{aligned}$$

The first and third isomorphisms come from equation (3.2). The second uses [HV92, Proposition 3.1(1)], which is natural in X. Since colimits are constructed level-wise we have $(\operatorname{colim}_k \mathcal{K}(-,k))_l = \operatorname{colim}_k \mathcal{K}(l,k)$. By the adjunction (2.7) and the Yoneda lemma ([Kel05, Section 1.9]) we have bijections $\mathbf{Top}(\operatorname{colim} \mathcal{K}(k,-), A) \cong \mathbf{Top}^{\mathcal{K}}(\mathcal{K}(k,-), cA) \cong \mathbf{Top}(*, A)$ natural in A. Thus $\operatorname{colim} \mathcal{K}(k,-) \cong *$ by the corollary in [Mac78, Section III.2]. Since equations (3.2) is natural in X, the isomorphism $\operatorname{hocolim}_{\mathcal{K}} X \cong \operatorname{colim} \overline{X}$ is natural in X. □

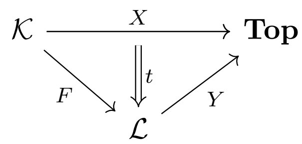
Taking the colimit of the evaluation ϵ and composing with the isomorphism in Proposition 3.15, we get a map

$$\pi : \operatorname{hocolim}_{\mathcal{K}} X \to \operatorname{colim} X,$$

called the *projection*, that is natural in X.

3.4. The box product

The boxproduct is defined as a left Kan extension. Let $F : \mathcal{K} \to \mathcal{L}$ be a continuous functor of index categories. This induces a functor $F^* : \mathbf{Top}^{\mathcal{L}} \to \mathbf{Top}^{\mathcal{K}}$. The *left Kan extension* Lan_F is the left adjoint of this functor if it exists. Let X be an $\mathcal{K}$-space. A pair (Y, t) consisting of an $\mathcal{L}$-space and a morphism of K-spaces $t : X \to F^*Y$ fits in the diagram:



(3.5)

This diagram is, in general, not commutative. A universal pair (Y, t) is also known as the left Kan extension.

Definition 3.16. Let $F : \mathcal{K} \to \mathcal{L}$ be a continuous functor of index categories and X a $\mathcal{K}$-space. The *left Kan extension* of X along F, if it exists, is a pair $(\mathrm{Lan}_F X, t)$ that fits in (3.5) such that for any other pair (Y, t') that fits in (3.5) there exists a unique morphism $s : Y \to \mathrm{Lan}_F X$ such that $F^*(s) \circ t = t'$. The morphism t is often left implicit.

By the universal property of a left Kan extension, it is unique up to unique isomorphism if it exists.

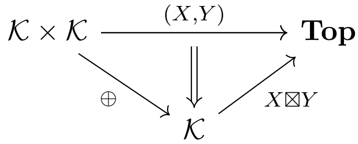
Proposition 3.17 ([Kel05, Proposition 4.33, (4.25)])**.** *Let $F : \mathcal{K} \to \mathcal{L}$ and X be given as in Definition 3.16. Then the left Kan extension of X along F exists and is an $\mathcal{L}$-space defined by* $(\mathrm{Lan}_F X)(l) = \mathcal{L}(F(-), l) \otimes_{\mathcal{K}} X$.

For the remainder of this section, we fix a symmetric monoidal index category $(\mathcal{K}, \oplus, 0)$ whose tensor product $(-) \oplus (-) : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is continuous.

Definition 3.18. Let X and Y be $\mathcal{K}$-spaces and let (X, Y) denote the $\mathcal{K}\times\mathcal{K}$-space defined by $(X, Y)_{k,l} = X_k \times Y_l$. The *box product* $X \boxtimes Y$ is the left Kan extension of (X, Y) along $\oplus$. This $\mathcal{K}$-space is unique up to unique isomorphism. Proposition 3.17 shows that it exists and that it can be written out as

$$X \boxtimes Y(k) = \mathcal{K}(- \oplus -, k) \otimes_{\mathcal{K}\times\mathcal{K}} (X, Y).$$

The box product fits in the following diagram:



(3.6)

By the universal property of the left Kan extension, there is a one-to-one correspondence between morphisms of $\mathcal{K}$-spaces $X \boxtimes Y \to Z$ and morphisms of $\mathcal{K} \times \mathcal{K}$-spaces $(X, Y) \to Z(- \oplus -)$. Hence the former morphism is entirely determined by the maps

$$X(k) \times Y(l) \to Z(k \oplus l).$$

The identity on $X \boxtimes Y$ induces maps

$$X(k) \times Y(l) \to (X \boxtimes Y)(k \oplus l)$$

that are precisely the components of the morphism of $\mathcal{K}$-space in (3.6).

Theorem 3.19 ([Day70, Theorem 3.3], [Sch18, Theorem C.10, Remark C.12])**.** *The triple* $(\mathbf{Top}^{\mathcal{K}}, \boxtimes, \mathcal{K}(0, -))$ *is a closed symmetric monoidal category.*

A monoid in the category of $\mathcal{K}$-spaces with respect to $\boxtimes$ is called a *$\mathcal{K}$-space monoid* and $\mathcal{C}\mathbf{Top}^{\mathcal{K}}$ denotes the category of *commutative $\mathcal{K}$-space monoids.*

Example 3.20. The category $\mathcal{V}$ is an index category by Example 2.24 and $(\mathcal{V}, \oplus, \mathbb{R}^0)$ is a symmetric monoidal category with $(-) \oplus (-)$ continuous by Example 2.17. The object $\mathbb{R}^0$ is initial object in $\mathcal{V}$, therefore $\mathcal{V}(0, -) = *$ is the terminal object in $\mathbf{Top}^{\mathcal{V}}$. It follows that $(\mathbf{Top}^{\mathcal{V}}, \boxtimes, *)$ is a closed symmetric monoidal category that is enriched, tensored and cotensored over $\mathbf{Top}$. Let M be a $\mathcal{V}$-space monoid. The unit $u_M : * \to M$ is entirely determined by a basepoint $* \to M_0$. The multiplication $\mu_M : M \boxtimes M \to M$ is entirely determined by maps $\mu_{n,m} : M_n \times M_m \to M_{n \oplus m}$. In this context, M is commutative if and only if the diagram

$$\begin{array}{ccc} M_n \times M_m & \xrightarrow{\mu_{n,m}} & M_{n\oplus m} \\ \downarrow & & \downarrow \\ M_m \times M_n & \xrightarrow{\mu_{m,n}} & M_{m\oplus n} \end{array} \tag{3.7}$$

commutes, where the symmetry isomorphisms in $\mathbf{Top}$ and $\mathcal{V}$ induce the vertical maps.

3.5. Examples of $\mathcal{V}$-space monoids

To better understand the category $\mathbf{Top}^{\mathcal{V}}$, we will discuss examples of $\mathcal{V}$-spaces and commutative $\mathcal{V}$-space monoids.

Example 3.21. Fixing some $\mathbb{R}^k$ we obtain the $\mathcal{V}$-space $\mathcal{V}(k, -)$, as seen in Example 2.19. The space $\mathcal{V}(k, n)$ is isomorphic to the space $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$ known as the *real Stiefel manifold*. The space $\mathcal{V}(k, k) = O(k)$, in particular, is the *orthogonal group*.

Example 3.22. The *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$ is the set consisting of all k-dimensional linear subspaces of $\mathbb{R}^n$. Consider the map $\mathcal{V}(k, n) \to \mathrm{Gr}_k(\mathbb{R}^n), \psi \to \operatorname{im} \psi$. The right action from $O(k)$ on $\mathcal{V}(k, n)$ by precomposition induces a bijection $\mathcal{V}(k, n)/O(k) \leftrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$. Thus we can endow $\mathrm{Gr}_k(\mathbb{R}^n)$ with a topology making this bijection a homeomorphism. Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be an isometry and $V \subseteq \mathbb{R}^n$ be a k-dimensional linear subspace. Then $\phi(V)$ is a k-dimensional linear subspace of $\mathbb{R}^m$. Thus ϕ induces a map

$$\mathrm{Gr}_k(\phi) : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^m)$$

for every $k \geq 0$. The continuity of $\phi_* : \mathcal{V}(k, n) \to \mathcal{V}(k, m)$ makes $\mathrm{Gr}_k(\phi)$ continuous. Thus we obtain a functor

$$\mathrm{Gr}_k : \mathcal{V} \to \mathbf{Top}$$

for every $k \geq 0$. The continuity of $\mathcal{V}(n, m) \times \mathcal{V}(k, n) \to \mathcal{V}(k, m)$ makes

$$\mathcal{V}(n, m) \times \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^m)$$

continuous. Therefore Gr_k is a $\mathcal{V}$-space for every $k \geq 0$.

Example 3.23 ([Sch18, Example 2.3.6]). The assignment $\mathbb{R}^n \mapsto O(n)$ induced by the orthogonal group defines a $\mathcal{V}$-space. Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be an isometry. We can write $\mathbb{R}^m = \operatorname{im}\phi \oplus V$, where V is the orthogonal complement of $\operatorname{im}\phi$. Let $\varphi : \mathbb{R}^n \to \operatorname{im}\phi$ be the isomorphic isometry with $\varphi(x) = \phi(x)$. For $f \in O(n)$ we then set

$$O(\phi)(f) = \varphi f \varphi^{-1} \oplus \mathrm{id}_V$$

which is an isometry. Thinking of ϕ as a matrix, we can consider its transpose $\phi^\top : \mathbb{R}^m \to \mathbb{R}^n$, which is a left inverse of ϕ since it is an isometry. Then $\phi^\top = \varphi^{-1} \circ \mathrm{pr}_{\operatorname{im}\phi}$, where $\mathrm{pr}_{\operatorname{im}\phi}$ is the projection. We can then write $O(\phi)(f) = \phi f \phi^\top + \mathrm{pr}_V$, where $+$ denotes pointwise addition. Then $O(\phi)$ is a composite of continuous maps. Namely the composition of linear maps and pointwise addition of linear maps. Thus we obtain a functor

$$O : \mathcal{V} \to \mathbf{Top}\,.$$

The maps $(\phi, f) \mapsto O(\phi)(f)$ are continuous for the same reason making O a continuous functor. Thus O is a $\mathcal{V}$-space.

The $\mathcal{V}$-space O is even a commutative $\mathcal{V}$-space monoid. We automatically obtain a basepoint since $O(0)$ is a one-point space. Next, consider the composite map

$$\mu^O_{n,m} : O(n) \times O(m) \to O(n \oplus m) \times O(n \oplus m) \to O(n \oplus m)$$

where the left map is induced by the inclusions $\mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^m$ and $\mathbb{R}^m \to \mathbb{R}^n \oplus \mathbb{R}^m$ on their respective summands, and the right map is the composition of linear maps. The maps $\mu^O_{n,m}$ form a natural transformation, which induces a multiplication $\mu_O : O \boxtimes O \to O$. Together with the basepoint, this makes O a $\mathcal{V}$-space monoid. By writing $\mu^O_{n,m}$ out explicitly we get

$$\mu^O_{n,m}(f, g) = (f \oplus \mathrm{id}_m) \circ (\mathrm{id}_n \oplus g) = f \oplus g.$$

Thus O is a commutative $\mathcal{V}$-space monoid since the maps $\mu^O_{n,m}$ make the diagram (3.7) commute.

Note that since $O(n)$ is a group for all $\mathbb{R}^n$, the $\mathcal{V}$-space O is also a monoid with respect to the naive product $\times$. In this case, however, it is not commutative since the groups $O(n)$ are, in general, not abelian.

Example 3.24 ([Sch18, Example 2.3.12]). The *additive Grassmannian* is the coproduct of all Grassmannians

$$\mathrm{Gr} = \coprod_{k \geq 0} \mathrm{Gr}_k\,.$$

The space $\mathrm{Gr}(\mathbb{R}^n) = \coprod_{k \geq 0} \mathrm{Gr}_k(\mathbb{R}^n)$ has all linear subspaces of $\mathbb{R}^n$ as elements.

The maps

$$\mathrm{Gr}_k(\mathbb{R}^n) \times \mathrm{Gr}_l(\mathbb{R}^m) \to \mathrm{Gr}_{k+l}(\mathbb{R}^n \oplus \mathbb{R}^m), (V, W) \mapsto V \oplus W$$

induce maps $\mu^{\mathrm{Gr}}_{n,m} : \mathrm{Gr}(\mathbb{R}^n) \times \mathrm{Gr}(\mathbb{R}^n) \to \mathrm{Gr}(\mathbb{R}^n \oplus \mathbb{R}^m)$ that form a natural transformation, which induces a multiplication $\mu_{\mathrm{Gr}} : \mathrm{Gr} \boxtimes \mathrm{Gr} \to \mathrm{Gr}$. Together with the basepoint given by the one-point space $\mathrm{Gr}(0)$, this makes Gr a $\mathcal{V}$-space monoid. It is commutative since the maps $\mu^{\mathrm{Gr}}_{n,m}$ make the diagram (3.7) commute.

Example 3.25 ([Sch18, Example 2.4.1]). The *periodic Grassmannian* is a $\mathcal{V}$-space BOP with values

$$\mathrm{BOP}(n) = \coprod_{k \geq 0} \mathrm{Gr}_k(\mathbb{R}^n \oplus \mathbb{R}^n).$$

Let $\phi \in \mathcal{V}(n,m)$, $\phi^2 = \phi \oplus \phi$ and $V \in \mathrm{Gr}_k(\mathbb{R}^n \oplus \mathbb{R}^n)$, for some $k \geq 0$, then

$$\mathrm{BOP}(\phi)(V) = \phi^2(V) + (\mathbb{R}^m - \mathrm{im}\,\phi) \oplus 0.$$

Since $\phi^2(V) \subseteq \mathrm{im}\,\phi^2$ and $(\mathbb{R}^m - \mathrm{im}\,\phi) \oplus 0 \subseteq (\mathbb{R}^m \oplus \mathbb{R}^m - \mathrm{im}\,\phi^2)$ the summands are disjoint. Thus $\dim(\mathrm{BOP}(\phi)(V)) = \dim(V) + m - n$ and $\mathrm{BOP}(\phi)(V) \in \mathrm{Gr}_{k+m-n}(\mathbb{R}^m \oplus \mathbb{R}^m)$. The continuity of the maps $(\phi, V) \mapsto \mathrm{BOP}(\phi)(V)$ follows from the continuity of the $\mathcal{V}$-space $\mathrm{Gr}((-) \oplus (-))$, where $(-) \oplus (-)$ is the composite of the diagonal $\mathrm{d} : \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ and the direct sum $\oplus$, which are all continuous. Thus BOP is also continuous and, therefore, a $\mathcal{V}$-space.

The isomorphism

$$k_{n,m} : \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m \cong \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^n \oplus \mathbb{R}^m,$$
$$(x, x', y, y') \mapsto (x, y, x', y')$$

induces the map

$$\mu^{\mathrm{BOP}}_{n,m} : \mathrm{BOP}(n) \times \mathrm{BOP}(m) \to \mathrm{BOP}(n \oplus m),$$
$$(V, W) \mapsto k_{n,m}(V \oplus W).$$

The maps $\mu^{\mathrm{BOP}}_{n,m}$ form a natural transformation. Together with the basepoint given by the one-point space $\mathrm{BOP}(0)$, this makes BOP a commutative $\mathcal{V}$-space monoid.

For any $d \in \mathbb{Z}$ the subspaces $\mathrm{BOP}_d(n) = \mathrm{Gr}_{d+n}(\mathbb{R}^n \oplus \mathbb{R}^n) \subset \mathrm{BOP}(n)$ form a $\mathcal{V}$-space BOP_d such that $\mathrm{BOP} = \coprod_{d \in \mathbb{Z}} \mathrm{BOP}_d$. Indeed $\mathrm{BOP}(\phi)$ restricts to $\mathrm{BOP}_d(n) \to \mathrm{BOP}_d(m)$. We say that BOP is $\mathbb{Z}$-graded. We can construct a morphism of $\mathcal{V}$-spaces $\Phi : \mathrm{Gr} \to \mathrm{BOP}$ defined by

$$\Phi_n : \mathrm{Gr}(n) \to \mathrm{BOP}(n), V \mapsto \mathbb{R}^n \oplus V.$$

This morphism is well defined since it includes V in the second summand of $\mathbb{R}^n \oplus \mathbb{R}^n$. Since $k_{n,m}(\mathbb{R}^n \oplus V \oplus \mathbb{R}^m \oplus W) = \mathbb{R}^{n+m} \oplus V \oplus W$ the morphism f is a morphism of commutative $\mathcal{V}$-space monoids. For any $d \in \mathbb{Z}_{\geq 0}$ the morphism Φ restricts to

$$\Phi|_d : \mathrm{Gr}_d \to \mathrm{BOP}_d\,.$$

We say that Φ is graded.

4. Model categories

A *model structure* on a category consists of three classes of morphisms called *weak equivalences*, *fibrations* and *cofibrations* satisfying certain axioms. A category with a model structure is called a *model category*. Model categories are used to generalize homotopy theory of topological spaces. Indeed important examples are model structures on **Top**. The classical model structure on **Top** has weak homotopy equivalences as weak equivalences and Serre fibrations as fibrations. Daniel Quillen introduced model categories in [Qui67, Chapter 1]. A good introduction to model categories is [DS95], which also proves that the classical model structure on **Top** is indeed a model structure. Other notable examples include the Strøm model structure on **Top** and the classical model structure on **sSet**. Proofs that these structures are indeed model structures are given by Arne Strøm in [Str72] and Mark Hovey in [Hov99, Chapter 3], respectively. Proving that the axioms are satisfied can be quite a challenge. We introduce Kan's recognition theorem to aid us in this process.

4.1. Model categories

Let $\mathscr{C}$ be a category and $A, B \in \operatorname{Ob}\mathscr{C}$. The object A is a *retract* of B if there exist morphisms $s : A \to B$ and $r : B \to A$ such that $rs = \mathrm{id}_A$. Let $\operatorname{Arr}(\mathscr{C})$ be the arrow category with objects the morphisms in $\mathscr{C}$ and morphisms pairs of morphisms $(s_1, s_2) : f \to g$ such that $gs_1 = s_2 f$. Let $f, g \in \operatorname{Mor}\mathscr{C}$. The morphism f is a *retract* of g if it is a retract of g as objects in $\operatorname{Arr}(\mathscr{C})$. In this case, we have the following commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \longrightarrow & Y & \longrightarrow & X \end{array}$$

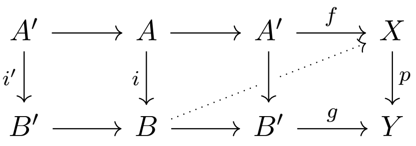
where the composites of the horizontal maps are the identities on A and X.

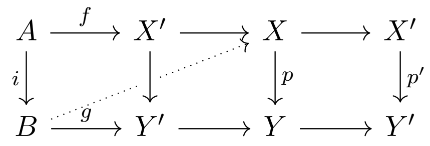
In a commutative diagram in $\mathscr{C}$

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \tag{4.1}$$

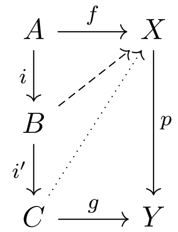
the dotted arrow is called a *lift*, if it exists. If such a lift exists for any pair (f, g) that makes the diagram commute, we say that i has the *left lifting property (LLP)* with respect to p and p has the *right lifting property (RLP)* with respect to i.

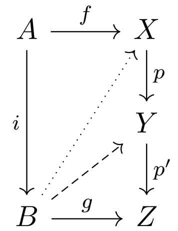
If i has the LLP with respect to p as in diagram (4.1), a retract i' of i has the LLP with respect to p, and a retract p' of p has the RLP with respect to i. To see this note that the dotted arrows induce lifts in the outer squares:





A lift always exists if i or p is an identity morphism. If i and i' are composable morphisms that have the LLP with respect to p, then so does $i'i$. Similarly, if p and p' are composable morphisms that have the RLP with respect to i, then so does $p'p$. To see the former case, note that in the left diagram, the LLP of i induces the dashed arrow, allowing the LLP of i' to induce the dotted arrow. A similar argument holds for the latter case.





Definition 4.1. A *model category* is a category $\mathscr{C}$ with three classes of morphisms

($\xrightarrow{\sim}$) *Weak equivalences*
($\twoheadrightarrow$) *Fibrations*
($\rightarrowtail$) *Cofibrations*

each of which contains all identity morphisms and is closed under composition. A morphism that is both a (co)fibration and a weak equivalence is an *acyclic (co)fibration*. The following axioms must be satisfied:

MC1 The category $\mathscr{C}$ is bicomplete.

MC2 If f and g are morphisms in $\mathscr{C}$ such that gf is defined and if two of the three morphisms f, g and gf are weak equivalences, then so is the third.

MC3 If f is a retract of g and g is a weak equivalence, a fibration or a cofibration, then so is f.

MC4 In the commutative diagram (4.1) a lift exists if either

1. i is a cofibration and p an acyclic fibration,
2. i is an acyclic cofibration and p a fibration.

MC5 Any morphism f can be factored as $f = pi$ such that either

1. i is a cofibration and p an acyclic fibration,
2. i is an acyclic cofibration and p a fibration.

We also say that the given classes form a *model structure* on $\mathscr{C}$.

A class of morphisms that satisfies MC2 has the *two-out-of-three* property. A class of morphisms that satisfies MC3 is *closed under retracts*. Since a model category is bicomplete, it has an initial and terminal object. We say that an object is *fibrant* (*cofibrant*) if the unique morphism to the terminal object (from the initial object) is a fibration (cofibration). For any object X, we can use MC5.1 to factor the unique morphism to the terminal object $X \to *$ and obtain an acyclic cofibration to a fibrant object $X \xrightarrow{\sim} \widehat{X}$ called the *fibrant replacement*. We can take the fibrant replacement of a morphism and choose it such that it is a fibration. Dually there exists a *cofibrant replacement* $\widetilde{X} \xrightarrow{\sim} X$. The remainder of this section will consist of useful tools about model categories.

Proposition 4.2 ([DS95, Proposition 3.13]). *Let $\mathscr{C}$ be a model category.*

1. *The cofibrations are precisely those morphisms which have the LLP with respect to the acyclic fibrations.*
2. *The acyclic cofibrations are precisely those morphisms which have the LLP with respect to the fibrations.*
3. *The fibrations are precisely those morphisms which have the RLP with respect to the acyclic cofibrations.*
4. *The acyclic fibrations are precisely those morphisms which have the RLP with respect to the cofibrations.*

What Proposition 4.2 shows is that the model structure is entirely pinned down by the class of weak equivalences and either the class of fibrations or cofibrations. Because of this, we often take these lifting properties to be the defining properties of either the fibrations or cofibrations. We have already seen that identity morphisms have both lifting properties and that compositions and retracts preserve lifting properties.

If we have the following pushout and pullback diagrams

$$\begin{CD} A @>{j}>> C \\ @V{i}VV @VV{i'}V \\ B @>>> D \end{CD} \qquad\qquad \begin{CD} Z @>>> X \\ @V{p'}VV @VV{p}V \\ W @>{q}>> Y \end{CD}$$

then i' is called the *pushout* of i along j and p' is called the *pullback* of p along q. If i has the LLP with respect to p, then by the universal property, so does i'. If p has the RLP with respect to i, then by the universal property, so does p'.

Together with Proposition 4.2, we obtain the following corollary.

Corollary 4.3 ([DS95, Proposition 3.14]). *Pushouts of (acyclic) cofibrations are (acyclic) cofibrations. Pullbacks of (acyclic) fibrations are (acyclic) fibrations.*

Definition 4.4. A model category $\mathscr{C}$ is *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence.

There is an obvious dual notion of a right proper model category that we will not need.

Definition 4.5 ([Hov99, Definition 1.3.1], [Hir03, Definition 8.5.2]). A *Quillen adjunction* is an adjunction

$$L : \mathscr{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{C} : R.$$

between model categories $\mathscr{D}$ and $\mathscr{C}$, such that

1. the functor L preserves cofibrations and acyclic cofibrations,
2. the functor R preserves fibrations and acyclic fibrations.

Proposition 4.6 ([Hir03, Proposition 8.5.3]). *Let $(L \dashv R)$ be an adjunction between model categories. The following are equivalent:*

1. *The adjunction $(L \dashv R)$ is a Quillen adjunction.*
2. *The functor L preserves cofibrations and acyclic cofibrations.*
3. *The functor R preserves fibrations and acyclic fibrations.*
4. *The functor L preserves cofibrations and the functor R preserves fibrations.*
5. *The functor L preserves acyclic cofibrations and the functor R preserves acyclic fibrations.*

Proposition 4.7 (Ken Brown's Lemma, [Hov99, Lemma 1.1.12]). *Let $(L \dashv R)$ be a Quillen adjunction. The functor L preserves weak equivalences between cofibrant objects. The functor R preserves weak equivalences between fibrant objects.*

Let $\mathscr{C}$ be a category and $C \in \operatorname{Ob}\mathscr{C}$. An object in the *overcategory* $\mathscr{C}/C$ is a morphism $X \to C$ in $\mathscr{C}$. A morphism in $\mathscr{C}/C$ from $f : X \to C$ to $g : Y \to C$ is a morphism $h : X \to Y$ in $\mathscr{C}$ such that $f = gh$. An object in the *undercategory* $C/\mathscr{C}$ is a morphism $C \to X$ in $\mathscr{C}$. A morphism in $C/\mathscr{C}$ from $f : C \to X$ to $g : C \to Y$ is a morphism $h : X \to Y$ in $\mathscr{C}$ such that $g = hf$.

Proposition 4.8 ([Hir03, Theorem 7.6.5]). *Let $\mathscr{C}$ be a model category and $C \in \operatorname{Ob}\mathscr{C}$. The overcategory $\mathscr{C}/C$ is a model category in which a morphism is a weak equivalence, fibration or cofibration if the underlying morphism in $\mathscr{C}$ is. Similarly, the undercategory $C/\mathscr{C}$ is a model category in which a morphism is a weak equivalence, fibration or cofibration if the underlying morphism in $\mathscr{C}$ is.*

The forgetful functor $\mathscr{C}/C \to \mathscr{C}$ sends $X \to C$ to its domain X. It is left adjoint to the functor $\mathscr{C} \to \mathscr{C}/C$ that sends $X \in \operatorname{Ob}\mathscr{C}$ to the projection $X \times C \to C$. By Proposition 4.8, the forgetful functor preserves weak equivalences and cofibrations, so the adjunction is a Quillen adjunction by Proposition 4.6. Dually the forgetful functor $C/\mathscr{C} \to \mathscr{C}$ is right adjoint to the functor sending $X \in \operatorname{Ob}\mathscr{C}$ to $C \to X \coprod C$. This adjunction is also a Quillen adjunction.

Definition 4.9. Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories, with $\mathscr{E}$ cocomplete and let $(-) \times (-) : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ be a functor. Given $f \in \mathscr{C}(X, Y)$ and $g \in \mathscr{D}(A, B)$ the *pushout product* is the morphism

$$f \square g : X \times B \bigcup_{X \times A} Y \times A \to Y \times B.$$

Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be model categories. The functor $\times$ satisfies the *pushout product property* if, given cofibrations f in $\mathscr{C}$ and g in $\mathscr{D}$, the pushout product $f \square g$ is a cofibration in $\mathscr{E}$ and f or g being acyclic implies that $f \square g$ is acyclic.

Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category that is also a model category. We say $\mathscr{C}$ satisfies the pushout product property if the tensor product $\otimes$ satisfies it.

4.2. Cofibrantly generated model categories

Proving that the axioms in Definition 4.1 are satisfied for three given classes of morphisms can be very cumbersome. However, Proposition 4.2 has shown that we only need to consider the weak equivalences and cofibrations. The model structure is, in a sense, generated by the weak equivalences and the cofibrations. It turns out that many model structures are generated by weak equivalences and two sets of cofibrations. These model categories are called *cofibrantly generated* model categories. Instead of directly checking the axioms in Definition 4.1, we can check conditions on these sets of cofibrations. While this is usually easier, one disadvantage is that not every model category is cofibrantly generated. A consequence of working with sets is that we need some set theory to continue.

Definition 4.10. A *partially ordered set* or *poset* is a set W together with a relation $\leq$ that is

1. reflexive: $\forall x \in W : x \leq x$,
2. transitive: $\forall x, y, z \in W : [x \leq y \wedge y \leq z \Rightarrow x \leq z]$,
3. antisymmetric: $\forall x, y \in W : [x \leq y \wedge y \leq x \Rightarrow x = y]$.

We can view a poset W as a small category with objects the elements of W and a unique morphism $x \to y$ if $x \leq y$.

Definition 4.11. A *well-ordered set* or *woset* is a poset $(W, \leq)$ such that for all non-empty subsets $S \subseteq W$ there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. An isomorphism $f : (W, \leq) \to (X, \leq')$ of wosets is a bijection $f : W \leftrightarrow X$ such that $x \leq y$ implies $f(x) \leq' f(y)$.

The isomorphism classes of wosets are instrumental. Since every set can be well-ordered, these isomorphism classes can be used to 'count' elements. An *ordinal* is a unique representative of an isomorphism class of wosets. We will not state the definition in full detail, but it is vital to know that an ordinal λ is the woset of all lesser ordinals. See [Dug66, Definition II.6.1] for a rigorous definition. The first ordinals are constructed as

$$\emptyset, \qquad \{\emptyset\}, \qquad \{\emptyset, \{\emptyset\}\}, \qquad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.$$

Given an ordinal λ its *successor ordinal* is $\lambda + 1 = \lambda \cup \{\lambda\}$. An ordinal that is not a successor ordinal is a *limit ordinal.* For example, the woset of all finite ordinals is a limit ordinal. For a subset $S \subseteq \lambda$ of an ordinal, let $\sup(S)$ be the smallest ordinal that contains S. For ordinals λ and β we write $\beta \leq \lambda$ and $\beta < \lambda$ instead of $\beta \subseteq \lambda$ and $\beta \subsetneq \lambda$ respectively.

We can also consider the 'size' or *cardinality* of a set, that is, its isomorphism class in **Set** (An isomorphism in **Set** is just a bijection). A *cardinal* is a unique representative of an isomorphism class of sets. It is an ordinal of greater cardinality than all its lesser ordinals. See [Dug66, Definition II.7.5] for a rigorous definition. For a set S, let $|S|$ be the unique cardinal isomorphic to S.

Definition 4.12. Let $\mathscr{C}$ be a cocomplete category, λ an ordinal and $\mathfrak{I}$ a class of morphisms. A λ-*sequence* is a functor $E : \lambda \to \mathscr{C}$ often written as

$$E_0 \to E_1 \to \cdots \to E_\beta \to \ldots \qquad , \beta < \lambda,$$

such that for all limit ordinals $\gamma < \lambda$ the morphism $\operatorname{colim}_{\beta<\gamma} E_\beta \to E_\gamma$ is an isomorphism. The morphism $E_0 \to \operatorname{colim}_{\beta<\lambda} E_\beta$ is the *composition* of the λ-sequence. If every morphism $E_\beta \to E_{\beta+1}, \beta + 1 < \lambda$, is in $\mathfrak{I}$ then the composition is a *transfinite composition of morphisms in* $\mathfrak{I}$.

Definition 4.13. Let $\mathscr{C}$ be a cocomplete category, I a set of morphisms and $\emptyset$ the initial object.

1. An *I-injective* is a morphism that has the RLP with respect to all morphisms in I. The class of I-injectives is denoted as I-inj.
2. An *I-cofibration* is a morphism that has the LLP with respect to all morphisms in I-inj. The class of I-cofibrations is denoted as I-cof.
3. A *relative I-cell complex* is a transfinite composition of pushouts of morphisms in I. The class of relative I-cell complexes is denoted as I-cell. An object $X \in \operatorname{Ob}\mathscr{C}$ is an *I-cell complex* if the morphism $\emptyset \to X$ is a relative I-cell complex

The lifting properties used in this definition hint at how we want to construct a model structure. We will seek a set I such that the I-injectives become the acyclic fibrations. This way, the I-cofibrations must be the cofibrations. A second set is used for the fibrations and acyclic cofibrations. Note that I-inj and I-cof contain all identity morphisms and are closed under composition and retracts. The relative I-cell complexes generalize the concept of a CW-complex in the category **Top**. By [Hir03, Proposition 10.5.11], every retract of a relative I-cell complex is an I-cofibration. These relative I-cell complexes will then be used to factorize morphisms as in axiom MC5 in Definition 4.1. We will sketch out how to do this. The following definition is fairly abstract. Example 4.16 shows a case where it is necessary.

Definition 4.14. Let κ be a cardinal. An ordinal λ is *κ-filtered* if it is a limit ordinal and, if $S \subseteq \lambda$ and $|S| \leq \kappa$, then $\sup(S) < \lambda$.

If κ is finite, every limit ordinal is κ-filtered since the supremum of a finite subset of a limit ordinal is finite, and a limit ordinal is infinite. A cardinal κ is itself a κ-filtered ordinal.

Definition 4.15. Let $\mathscr{C}$ be a cocomplete category, $\mathfrak{I}$ a class of morphisms and κ a cardinal. An object $X \in \operatorname{Ob}\mathscr{C}$ is *κ-small relative to* $\mathfrak{I}$ if for all κ-filtered ordinals λ and all λ-sequences

$$E_0 \to E_1 \to \cdots \to E_\beta \to \dots \qquad , \beta < \lambda,$$

where $E_\beta \to E_{\beta+1}, \beta + 1 < \lambda$, is in $\mathfrak{I}$, the map

$$\operatorname{colim}_{\beta<\lambda} \mathscr{C}(X, E_\beta) \to \mathscr{C}(X, \operatorname{colim}_{\beta<\lambda} E_\beta) \tag{4.2}$$

is a bijection. An object in $\mathscr{C}$ is *small relative to* $\mathfrak{I}$ if it is κ-small relative to $\mathfrak{I}$ for some cardinal κ. It is *finite relative to* $\mathfrak{I}$ if κ is finite.

The critical takeaway of this definition is to specify cases where, given a transfinite composition $E_0 \to \operatorname{colim} E$, we can factor a morphism $X \to \operatorname{colim} E$ as $X \to E_\beta \to \operatorname{colim} E$ for some ordinal β.

Example 4.16 ([Hov99, Example 2.1.5])**.** A set $X \in \operatorname{Ob}\mathbf{Set}$ is small relative to $\operatorname{Mor}\mathbf{Set}$. Let λ be an $|X|$-filtered ordinal, E a λ sequence and consider a map $f : X \to \operatorname{colim} E$. For every $x \in X$ there exists an ordinal $\beta(x)$ such that $f(x) \in E_{\beta(x)}$. Let $S = \{\beta(x)\}_{x\in X}$. Then $S \subseteq \lambda$ and $|S| \leq |X|$ and therefore $\sup(S) < \lambda$. Now f factors as $X \to E_{\sup(S)} \to \operatorname{colim} E$. Thus (4.2) is surjective. A similar result shows injectivity.

Lemma 4.17. *Spaces are small relative to closed inclusions. Compact spaces are finite relative to closed inclusions.*

Proof. Let λ be a limit ordinal and let $E : \lambda \to \mathbf{Top}$ be a functor. For a limit ordinal $\gamma < \lambda$ we have the composite $\operatorname{colim}^{\mathbf{S}}_{\beta<\gamma} E_\beta \to \operatorname{colim}_{\beta<\gamma} E_\beta \to E_\gamma$ in $\mathbf{S}$. If E is a λ-sequence in $\mathbf{S}$ then it is a λ-sequence in $\mathbf{Top}$. The converse holds if $\operatorname{colim}^{\mathbf{S}}_{\beta<\gamma} E_\beta$ is weak Hausdorff for every limit ordinal γ. This is the case if every map $E_\beta \hookrightarrow E_{\beta+1}, \beta + 1 < \lambda$, is a closed inclusion. We prove this using transfinite induction. Note that $\operatorname{colim}^{\mathbf{S}}_{\beta<\gamma} E_\beta = \cup_{\beta<\gamma} E_\beta$. Let $\cup_{\beta<\gamma'} E_\beta$ be weak Hausdorff for ever limit ordinal $\gamma' < \gamma$. Then $E' : \gamma \to \mathbf{Top}$, with $E'_\beta = E_\beta, \beta < \gamma$ and $E'_\gamma = \cup_{\beta<\gamma} E_\beta$, is a γ-sequence in $\mathbf{S}$ of closed inclusions. By [Hov99, Proposition 2.4.2] any map $h : K \to E'_\gamma$, with K compact Hausdorff, factors through some weak Hausdorff space $E'_\beta, \beta < \gamma$. Thus $\operatorname{im} h$ is closed in E'_β and therefore closed in E'_γ making it weak Hausdorff.

By [Hov99, Lemma 2.4.1], every space in $\mathbf{S}$ is small relative to inclusions, and by [Hov99, Proposition 2.4.2], every compact space in $\mathbf{S}$ is finite relative to closed T_1 inclusions. Any λ sequence in $\mathbf{Top}$ of closed inclusions is a λ-sequence in $\mathbf{S}$ of closed T_1 inclusions. Hence the result follows. □

The following definition and proposition give the final two ingredients that allow us to factorize as in axiom MC5 in Definition 4.1. We present a sketch of the proof of Proposition 4.19 to show why small objects and the following small object argument are used. Full proofs are given in [Hov99] and [Hir03]. Both are generalizations of a gluing construction given in [DS95, Chapter 7].

Definition 4.18. Let $\mathscr{C}$ be a cocomplete category and I a set of morphisms. The set I *permits the small object argument* if all domains of morphisms in I are small relative to I-cell.

Proposition 4.19 ([Hov99, Theorem 2.1.14],[Hir03, Proposition 10.5.16])**.** *Let $\mathscr{C}$ be a cocomplete category and I a set of morphisms that permits the small object argument. Every morphism $f \in \operatorname{Mor}\mathscr{C}$ factorizes as $f = pi$ such that $i \in I$-cell and $p \in I$-inj.*

Proof sketch. We can choose a cardinal λ such that every domain of morphisms in I is λ-small relative to I-cell. Let S be the set of commutative squares

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & Y \end{array}$$

where $j \in I$ and consider the commutative square:

$$\begin{array}{ccc} \coprod_{s\in S} Z_s & \longrightarrow & X \\ {\scriptstyle \coprod_{s\in S} j_s}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_{s\in S} W_s & \longrightarrow & Y \end{array}$$

Then we can factor $f = f_0$ as $X_0 \to X_1 \xrightarrow{f_1} Y$, with $X_0 = X$ and $X_1 = W' \coprod_{Z'} X$, where W' and Z' are the domain and codomain of $\coprod_{s\in S} j_s$, respectively. By repeating this process a transfinite amount of times up to λ, we factor f as $X \xrightarrow{i} X_\lambda \xrightarrow{p} Y$ where i is a transfinite composition of pushouts of coproducts of morphisms in I, which is in I-cell. Consider the left commutative square

$$\begin{array}{ccc} A & \longrightarrow & X_\lambda \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccccccc} A & \longrightarrow & X_\beta & \longrightarrow & X_{\beta+1} & \longrightarrow & X_\lambda \\ {\scriptstyle j}\downarrow & & & \nearrow & & & \downarrow{\scriptstyle p} \\ B & & \longrightarrow & & \longrightarrow & & Y \end{array}$$

with $j \in I$. The top morphism factors through some X_β because of the small object argument, as shown in the right diagram. Then by the construction of $X_\beta \to X_{\beta+1}$, the dotted lift exists. □

Let $\mathscr{C}$ be a cocomplete category and I a set of morphisms that permits the small object argument. Factorize an I-cofibration as $f = pi$ using Proposition 4.19. Then p is an I-injective, so there exists a lift in the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & E \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Y & = & Y \end{array}$$

showing that f is a retract of the relative I-cell complex i. Thus I-cof is precisely the class of retracts of morphisms in I-cell.

We can finally define cofibrantly generated model categories and state the Recognition Theorem.

Definition 4.20. A model category $\mathscr{C}$ is *cofibrantly generated* if there exist sets of morphisms I and J such that

1. the sets I and J permit the small object argument,
2. the class of fibrations is J-inj,
3. the class of acyclic fibrations is I-inj.

The sets I and J are called *generating cofibrations* and *generating acyclic cofibrations*, respectively. The classes of cofibrations and acyclic cofibrations are necessarily I-cof and J-cof, respectively.

Theorem 4.21 (Recognition theorem, [Hov99, Theorem 2.1.19], [Hir03, Theorem 11.3.1])**.** *Let $\mathscr{C}$ be a bicomplete category. Let W be a class of morphisms that contains all identity morphisms and is closed under composition, and let I and J be sets of morphisms such that*

RT1 the class W has the two-out-of-three property and is closed under retracts,
RT2 the sets I and J permit the small object argument,
RT3 J-cell $\subset W \cap I$-cof,
RT4 I-inj $\subset W \cap J$-inj,
RT5 either $W \cap I$-cof $\subset J$-cof or $W \cap J$-inj $\subset I$-inj.

Then $\mathscr{C}$ is a cofibrantly generated model category with W the weak equivalences, I the generating cofibrations and J the generating acyclic cofibrations.

4.3. Model structures on spaces and simplicial sets

Two of the best-known model structures are the classical model structures on **Top** and **sSet**, from this point onward, referred to as the model structures on **Top** or **sSet**. The model structure on **Top** is essential since we will use it to construct various other model structures. All these model structures will be cofibrantly generated, including those on **Top** and **sSet**. The relevant proofs for **Top** and **sSet**, given in [Hov99], use Theorem 4.21. Nonetheless, these proofs are still very long and complicated, and we will not state them here. Instead, we focus on building some more machinery to help us down the road.

Definition 4.22. A continuous map $f : A \to B$ is a *Serre fibrations* if it has the RLP with respect to all inclusions $D^n \times \{0\} \hookrightarrow D^n \times [0,1], n \geq 0$.

Let $J = \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] \mid n \geq 0\}$, then the class of Serre fibrations is J-cof. Since all the inclusions in J are weak homotopy equivalences, this suggests that the Serre fibrations are the fibrations in a cofibrantly generated model category. This is the model structure on **Top**. Let $I = \{S^n \hookrightarrow D^{n+1} \mid n \geq -1\}$.

Theorem 4.23 (Model structure on **Top**, [Hov99, Theorem 2.4.19, 2.4.23, 2.4.25])**.** *The category* **Top** *is a cofibrantly generated model category with weak equivalences, the weak homotopy equivalences, fibrations the Serre fibrations, cofibrations those maps that have the LLP with respect to all Serre fibrations that are weak homotopy equivalences and the sets I and J the generating cofibrations and acyclic cofibrations respectively.*

Cofibrations are retracts of relative I-cell complexes. Since the maps in I are of the form $S^n \hookrightarrow D^{n+1}$, we find that relative CW-complexes (see [Spa66, Section 7.6]) are cofibrations. Since all maps in J are strong deformation retracts, they admit retractions. Therefore every space is fibrant. Most model categories we work with are enriched, tensored and cotensored over **Top**. We want the tensor, here denoted as $\times$, to satisfy the pushout product property. We will give a criterion from Schwede that allows us to check this.

Definition 4.24. Let $\mathscr{C}$ be a model category that is enriched, tensored and cotensored over **Top**. Then $\mathscr{C}$ is a *topological model category* if the tensor satisfies the pushout product property in Definition 4.9.

Proposition 4.25 ([Sch18, Proposition B.5])**.** *Let $\mathscr{C}$ be a model category that is enriched, tensored and cotensored over* **Top**. *Let $\mathcal{G}$ be a set of cofibrant objects and $\mathcal{Z}$ a set of acyclic cofibrations of $\mathscr{C}$ such that*

1. *the acyclic fibrations are precisely those morphisms which have the RLP with respect to the morphisms $X \times i$, for all $X \in \mathcal{G}$ and $i \in I = \{S^n \hookrightarrow D^{n+1} \mid n \geq -1\}$,*
2. *the fibrations are precisely those morphisms which have the RLP with respect to the morphisms $X \times j$, for all $X \in \mathcal{G}$ and $j \in J = \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] \mid n \geq 0\}$, and the morphisms $c \square i$, for all $c \in \mathcal{Z}$ and $i \in I = \{S^n \hookrightarrow D^{n+1} \mid n \geq -1\}$.*

Then $\mathscr{C}$ is a topological model category.

Of course, the category **Top** must be a topological model category. We check if **Top** satisfies the pushout product property as a monoidal category.

Proposition 4.26. *The symmetric monoidal model category* $(\mathbf{Top}, \times, *)$ *satisfies the pushout product property. Therefore it is a topological model category.*

Proof. This is a consequence of Proposition 4.25 when we set $\mathcal{G} = \{*\}$ and $\mathcal{Z} = \emptyset$, because I and J are the generating cofibrations and acyclic cofibrations respectively. Also note that $\emptyset \to *$ is an element of I, making $*$ cofibrant. □

Let $\mathscr{C}$ be a topological model category. If g is a cofibration in **Top**, then $(-)\square g : \mathscr{C} \to \mathscr{C}$ is a functor that preserves (acyclic) cofibrations. By (2.4), the object $\emptyset_{\mathscr{C}} \cong X \times \emptyset$ is the initial object since $\emptyset$ is the initial space. So if $f : X \to Y$ is a morphism in $\mathscr{C}$ and $g : \emptyset \to A$ is a map, then $f \square g = f \times A$, since $\emptyset_{\mathscr{C}} \cup_{\emptyset_{\mathscr{C}}} (X \times A) = X \times A$. Let $(-) \times A : \mathscr{C} \to \mathscr{C}$ denote the tensor with some space A. If A is cofibrant, this functor preserves (acyclic) cofibrations. By definition, it must also be left adjoint to the cotensor $(-)^A : \mathscr{C} \to \mathscr{C}$.

Thus if A is cofibrant, this adjunction is a Quillen adjunction. In a completely similar fashion, the functor $X \times (-) : \mathbf{Top} \to \mathscr{C}$ is left adjoint to $\mathrm{Map}(X, -) : \mathscr{C} \to \mathbf{Top}$ and this adjunction is a Quillen adjunction if X is a cofibrant object in $\mathscr{C}$. In **Top** in particular, taking the product with a cofibrant space preserves (acyclic) cofibrations.

Definition 4.27. Let $\mathscr{C}$ be a cocomplete category enriched, tensored and cotensored over **Top**. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and let $\mathrm{incl}_t : X \to X \times [0, 1]$ be the morphism induced by the tensor of X and the closed inclusion $\{t\} \hookrightarrow [0, 1]$ and the isomorphism $X \cong X \times \{t\}$. The pushout $C(f) = X \times [0, 1] \cup_X Y$ of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{incl}_1}\downarrow & & \downarrow{\scriptstyle q(f)} \\ X \times [0,1] & \xrightarrow{f'} & C(f) \end{array}$$

is called the *mapping cylinder*.

The constant map $[0, 1] \to \{1\}$ induces a retraction for all morphisms incl_t. By the universal property of the pushout, this gives rise to a retraction of $q(f)$, which allows us to factor f through $f' \mathrm{incl}_t$, for all $t \in [0, 1]$. The morphism $c(f) = f' \mathrm{incl}_0$ is called the *front* of the mapping cylinder.

Lemma 4.28. *Let $\mathscr{C}$ be a topological model category and $f : X \to Y$ be a morphism between cofibrant objects. Then the front of $C(f)$ is a cofibration.*

Proof. The morphism

$$\mathrm{incl}_0 \coprod \mathrm{incl}_1 : X \coprod X \cong X \times \{0, 1\} \rightarrowtail X \times [0, 1]$$

is a cofibration since $\{0, 1\} \rightarrowtail [0, 1]$ is a generating cofibration and X is cofibrant. The mapping cylinder fits in the following pushout diagram:

$$\begin{array}{ccc} X \amalg X & \xrightarrow{\mathrm{id}_X \amalg f} & X \amalg Y \\ {\scriptstyle \mathrm{incl}_0 \amalg \mathrm{incl}_1}\downarrow & & \downarrow{\scriptstyle c(f) \amalg q(f)} \\ X \times [0,1] & \xrightarrow{f'} & C(f) \end{array}$$

Since pushouts preserve cofibrations by Corollary 4.3, the right vertical map is a cofibration. The canonical map $X \rightarrowtail X \coprod Y$ is a cofibration by Corollary 4.3 since it is a pushout of $\emptyset \rightarrowtail Y$ along $\emptyset \rightarrowtail X$. Its composition with $c(f) \coprod q(f)$ is the front of $C(f)$, which must now be a cofibration. □

A morphism $f : X \to Y$, in a topological model category, is an *h-cofibration* if it has the homotopy extension property, that is, given a morphism $g : Y \to Z$ and a homotopy $H : X \times [0, 1] \to Z$, with $H_0 = gf$, there exists a homotopy $H' : Y \times [0, 1] \to Z$, with $H'_0 = g$ and $H' \circ (f \times [0, 1]) = H$. This is equivalent to saying that there exists a lift in every commutative diagram

$$\begin{array}{ccc} X \times [0,1] \cup_{X \times \{0\}} Y \times \{0\} & \longrightarrow & Z \\ {\scriptstyle f \square \mathrm{incl}_0}\downarrow & & \downarrow \\ Y \times [0,1] & \longrightarrow & * \end{array}$$

with $*$ the terminal object. This is again equivalent to saying that the pushout product $f\square\,\mathrm{incl}_0$ admits a retraction h. This way, one can see that the functor $W \times (-)$ preserves h-cofibrations for every object W. In **Top**, an h-cofibration is also called a *Hurewicz cofibration*. They are all closed inclusions by [Sch18, Proposition A.31].

One might wonder if a different model structure on **Top** exists that uses homotopy equivalences as weak equivalences instead of weak homotopy equivalences. The Strøm model structure is just that. Recall that a *homotopy equivalence* is a map $f : A \to B$ that has a homotopy inverse g, that is, there exists a map $g : B \to A$ such that gf and fg are homotopic to id_A and id_B. Homotopy equivalences are weak homotopy equivalences. A *Hurewicz fibration* is a map $f : A \to B$ that has the RLP with respect to all inclusions $E \times \{0\} \hookrightarrow E \times [0, 1]$. A Hurewicz fibration is a Serre fibration.

Theorem 4.29 (Strøm model structure on **Top**, [Str72, Theorem 3], [MP11, Theorem 17.1.1])**.** *The category* **Top** *is a model category with weak equivalences, the homotopy equivalences, fibrations the Hurewicz fibrations and cofibrations the Hurewicz cofibrations.*

We will implicitly assume that **Top** has the classical model structure, so let $\mathbf{Top}_\mathrm{s}$ denote the category of spaces with the Strøm model structure. The identity functor $\mathbf{Top}_\mathrm{s} \to \mathbf{Top}$ preserves weak equivalences, fibrations and acyclic fibration. Since the identity is right adjoint to itself, this adjunction is a Quillen adjunction. Therefore the identity functor $\mathbf{Top} \to \mathbf{Top}_\mathrm{s}$ preserves cofibrations and acyclic cofibrations. Interestingly the Strøm model structure on **S** is not cofibrantly generated as proven in [Rap18].

We finish this chapter with the model structure on **sSet**. A simplicial map $f : K \to L$ is a *weak equivalence* if the realization $|f|$ is a weak homotopy equivalence. The map f is a *level-wise injection* if f_p is an injection for every $p \geq 0$. The standard n-simplex is the simplicial set $\Delta[n] = \Delta(-, n)$. The boundary $\partial\Delta[n] \subset \Delta[n]$ is the smallest subcomplex that contains all injections $d_i : [n-1] \to [n]$, with $0 \leq i \leq n$. It is the subcomplex where $\partial\Delta[n]_p \subseteq \Delta(p, n)$ contains the non surjective maps $[p] \to [n]$. The k-th horn $\Lambda^n_k \subset \Delta[n]$ is the smallest subcomplex that contains all injections $d_i : [n-1] \to [n]$, with $0 \leq i \leq n, i \neq k$. It is the subcomplex where $(\Lambda^n_k)_p \subseteq \partial\Delta[n]_p$ contains the maps $[p] \to [n]$ that do not have $k \in [n]$ in their image. Let $I' = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\}$ and $J' = \{\Lambda^n_k \hookrightarrow \Delta[n] \mid n > 0, 0 \leq k \leq n\}$.

The realization of $\Delta[n]$ is the standard topological n-simplex $\Delta^n = |\Delta[n]|$. We can choose a homeomorphism to D^n such that the realization of $\partial\Delta[n] \hookrightarrow \Delta[n]$ is homeomorphic to $S^{n-1} \hookrightarrow D^n$. We can also choose a homeomorphism to $D^{n-1} \times [0, 1]$ such that $\Lambda^n_k \hookrightarrow \Delta[n]$ is homeomorphic to $D^{n-1} \times \{0\} \hookrightarrow D^{n-1} \times [0, 1]$. See [GJ09, Section I.2] and [Hov99, Section 3.2] for more details.

Definition 4.30. A simplicial map $f : K \to L$ is a *Kan fibration* if it is in J'-inj.

Theorem 4.31 (Model structure on **sSet**, [Hov99, Theorem 3.6.5])**.** *The category* **sSet** *is a cofibrantly generated model category with weak equivalences, fibrations the Kan fibrations, cofibrations those maps that have the LLP with respect to all Kan fibrations that are weak equivalences and the sets I' and J' the generating cofibrations and acyclic cofibrations respectively.*

By [GJ09, Theorem 11.3], the cofibrations are precisely the level-wise injections, making every simplicial set cofibrant. The realization preserves weak equivalences and generating (acyclic) cofibrations. Since it is left adjoint, it preserves colimits and, therefore, (acyclic) cofibrations. We get the following Quillen adjunction.

Theorem 4.32 ([Hov99, Theorem 3.6.7]). *The adjunction*

$$|-| : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \mathrm{Sing}\,.$$

is a Quillen adjunction.

5. Model structures on $\mathcal{V}$-spaces

We are ready to construct model structures on the category of $\mathcal{V}$-spaces. As the previous chapter shows, fibrations or cofibrations are usually defined using lifting properties. Therefore we will usually leave one of these classes implicit. The generating (acyclic) cofibrations are also often left implicit. Since it is usually clear from the context which morphisms play the role of weak equivalences, fibrations and cofibrations, Theorem 4.23 can be stated as: *The weak homotopy equivalences and Serre fibrations are part of a cofibrantly generated model structure on* **Top**.

We start by constructing the (absolute) level model structure and the positive level model structure. These are used to construct the (absolute) $\mathcal{V}$-model structure and positive $\mathcal{V}$-model structure. This last model structure will later be lifted to a model structure on the category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ of commutative $\mathcal{V}$-space monoids. Here 'positivity' is a requirement. The level model structures presented in this chapter are also necessary for $\mathbf{Top}^{\mathcal{N}}$. Therefore we will start by considering a general index category $\mathcal{K}$. Let $(\mathcal{K}, \oplus, 0)$ be a symmetric monoidal index category, whose tensor product is continuous, such that $\mathcal{K}(k,l)$ is cofibrant and $\{\mathrm{id}_k\} \to \mathcal{K}(k,k)$ is a cofibration, for all $k, l \in \mathrm{Ob}\,\mathcal{K}$. Let $\mathcal{K}_+ \subset \mathcal{K}$ denote the full subcategory with $\mathrm{Ob}\,\mathcal{K}_+ = \mathrm{Ob}\,\mathcal{K} - \{0\}$. Discrete spaces are cofibrant since we can attach each point using a pushout of $\emptyset \to *$ transfinitely. We can choose which point we attach first arbitrarily. Therefore the inclusion of a basepoint in a discrete space is a cofibration. Thus if $\mathcal{K}$ is discrete, the cofibrancy assumptions are automatically satisfied. For $\mathcal{V}$ the space $\mathcal{V}(n,m)$ is cofibrant for all $n, m \geq 0$ by [Sch18, Proposition 1.1.19]. The *special orthogonal group* $SO(n)$ is the path-connected component of $O(n)$ containing the identity. As shown in [Hat02, Proposition 3D.1], it is a CW-complex constructed by attaching cells to the identity $\mathrm{id}_{\mathbb{R}^n}$. Therefore $\{\mathrm{id}_{\mathbb{R}^n}\} \rightarrowtail SO(n)$ is a cofibration. The other path-connected component $O(n) - SO(n)$ is homeomorphic to $SO(n)$ and, therefore, cofibrant. Thus the inclusion $\{\mathrm{id}_{\mathbb{R}^n}\} \to O(n)$ is a cofibration.

5.1. The level model structures

The level equivalences from Definition 3.8 will be accompanied by level fibrations to form a model structure. Proving this will invoke the recognition theorem (Theorem 4.21).

Definition 5.1. A morphism of $\mathcal{K}$-spaces $f : X \to Y$ is a *level fibration* (*level cofibration*) if all components $f_k : X_k \to Y_k$ are Serre fibrations (cofibrations).

The *cofibrations* in $\mathbf{Top}^{\mathcal{K}}$ must then be those morphisms that have the LLP with respect to level acyclic fibrations. The term cofibration is used for both the level model structure on $\mathbf{Top}^{\mathcal{K}}$ and the model structure on **Top**. We will specify whether we are in **Top** or $\mathbf{Top}^{\mathcal{K}}$ if it is not clear from the context. While cofibrations in $\mathbf{Top}^{\mathcal{K}}$ are level cofibrations, the converse is not true.

The following adjunction is used to define the generating (acyclic) cofibrations. Given $k \in \mathrm{Ob}\,\mathcal{K}$ the *evaluation functor* $\mathrm{ev}_k : \mathbf{Top}^{\mathcal{K}} \to \mathbf{Top}$ is defined by $\mathrm{ev}_k(X) = X_k$ and is right adjoint to the *free functor* F_k defined by $F_kA = \mathcal{K}(k,-) \times A$.

$$F_k : \mathbf{Top} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}^{\mathcal{K}} : \mathrm{ev}_k \,. \tag{5.1}$$

To see this, consider the chain of bijections

$$\mathbf{Top}^{\mathcal{K}}(F_kA, X) \cong \mathbf{Top}^{\mathcal{K}}(\mathcal{K}(k,-), X^A) \cong \mathbf{Top}(*, (X_k)^A) \cong \mathbf{Top}(A, X_k)$$

induced by (2.15), the Yoneda lemma ([Kel05, Section 1.9]) and (2.3). Consider the sets $FI = \{F_ki \mid k \in \mathrm{Ob}\,\mathcal{K}, i \in I\}$ and $FJ = \{F_kj \mid k \in \mathrm{Ob}\,\mathcal{K}, j \in J\}$ in $\mathbf{Top}^{\mathcal{K}}$ and let W_{lev} denote the class of level equivalences. For every $k, l \in \mathcal{K}$ the composite functor $\mathrm{ev}_l \circ F_k : \mathbf{Top} \to \mathbf{Top}$ sends a space A to $\mathcal{K}(k,l) \times A$. Since $\mathcal{K}(k,l)$ is cofibrant, this composite functor preserves (acyclic) cofibrations. Therefore, the morphisms in FI and FJ are level cofibrations.

Lemma 5.2. *Retracts of relative FI- and FJ-cell complexes are level cofibrations*

Proof. By [Hir03, Proposition 10.3.4], the class of cofibrations in a model category must be closed under pushouts, transfinite compositions and retracts. Evaluating a retract of a relative FI- or FJ-cell complex gives a retract of a transfinite composition of pushouts of cofibrations in **Top**, which must be a cofibration. □

The following proposition will ensure that FI and FJ permit the small object argument. Later we use this proposition for spaces other than codomains in I and J.

Proposition 5.3. *Let A be a space. Then F_kA is small relative to the sets FI and FJ.*

Proof. By Lemma 4.17, there exists a cardinal κ such that A is κ-small relative to closed inclusion. Let λ be a κ-filtered ordinal and consider a λ-sequence $E : \lambda \to \mathbf{Top}^{\mathcal{V}}$ written as

$$E_0 \to E_1 \to \cdots \to E_\beta \to \ldots,$$

where $E_\beta \to E_{\beta+1}, \beta+1 < \lambda$, is in FI-cell. For every $k \in \mathrm{Ob}\,\mathcal{K}$ the functor $E_{-,k} : \lambda \to \mathbf{Top}$ is a λ-sequence

$$E_{0,k} \to E_{1,k} \to \cdots \to E_{\beta,k} \to \ldots$$

in **Top**. The maps $E_{\beta,k} \to E_{\beta+1,k}, \beta+1 < \lambda$, are cofibrations by Lemma 5.2 and therefore closed inclusions by [Hov99, Corollary 2.4.6]. Thus $E_{-,k}$ is a λ-sequence in **Top** of closed inclusions. Using adjointness of (5.1) and the fact that A is κ-small relative to closed inclusions, we find a chain of bijections

$$\begin{aligned}
\mathrm{colim}\,\mathbf{Top}^{\mathcal{V}}(F_kA, E) &= \mathrm{colim}_{\beta<\lambda}\,\mathbf{Top}^{\mathcal{V}}(F_kA, E_\beta) \\
&\cong \mathrm{colim}_{\beta<\lambda}\,\mathbf{Top}(A, E_{\beta,k}) \\
&\cong \mathbf{Top}(A, \mathrm{colim}_{\beta<\lambda}(E_{\beta,k})) \\
&= \mathbf{Top}(A, (\mathrm{colim}\,E)_k) \cong \mathbf{Top}^{\mathcal{V}}(F_kA, \mathrm{colim}\,E).
\end{aligned}$$

Thus F_kA is κ-small relative to FI. The argument for FJ is entirely similar. □

Substituting the domains in I and J for A shows that FI and FJ permit the small object argument. If A is compact, it is finite relative to closed inclusions by Lemma 4.17. Since every limit ordinal is κ-filtered, if κ is finite, there would be no restriction on the limit ordinal λ. In this case, F_kA is finite relative to FI and FJ. The domains in I and J are compact, so this would be a slightly altered way to show that FI and FJ permit the small object argument. We now check the remaining conditions in Theorem 4.21.

Theorem 5.4 (Level model structure on $\mathbf{Top}^{\mathcal{K}}$)**.** *The level equivalences and level fibrations are part of a cofibrantly generated model structure on* $\mathbf{Top}^{\mathcal{K}}$*.*

Proof. We will show that the data consisting of W_{lev}, FI and FJ satisfies Theorem 4.21. The category $\mathbf{Top}^{\mathcal{K}}$ is bicomplete by Theorem 2.27, and all identity morphisms are level equivalences. The two-out-of-three property and closedness under retracts of W_{lev} follow from analogous properties in $\mathbf{Top}$. Thus RT1 is satisfied. The class W_{lev} is, in particular, closed under composition. RT2 is satisfied by substituting the domains of I and J in Proposition 5.3.

Let $F_n j \in FJ$ and $p : X \to Y$ be a morphism of $\mathcal{V}$-spaces and consider the commutative square on the left.

$$\begin{array}{ccc} F_nA & \longrightarrow & X \\ {\scriptstyle F_kj}\downarrow & & \downarrow{\scriptstyle p} \\ F_nB & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & X_n \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p_n} \\ B & \longrightarrow & Y_n \end{array}$$

This square is adjoint to the right commutative square. A lift exists in the left square if and only if a lift exists in the right square. One can see that FJ-inj is exactly the class of level fibrations. Replacing $j \in J$ with $i \in I$ shows that FI-inj is precisely the class of level acyclic fibrations. Hence RT4 and RT5 follow from the equality $FI\text{-inj} = W_{\text{lev}} \cap FJ\text{-inj}$. The class of cofibrations is FI-cof.

Since $FI\text{-inj} \subset FJ\text{-inj}$ we have $FJ\text{-cell} \subset FJ\text{-cof} \subset FI\text{-cof}$. Since the functors $\text{ev}_l \circ F_k$ preserve acyclic cofibrations, the morphisms in FJ are level cofibrations that are level equivalences. By [Hir03, Proposition 10.3.4], the class of acyclic cofibrations in a model category must be closed under pushouts and transfinite compositions. Evaluating a relative FJ-cell complex gives a transfinite composition of pushouts of acyclic cofibrations in $\mathbf{Top}$, which must be an acyclic cofibration. Thus RT3 is satisfied since $FJ\text{-cell} \subset W_{\text{lev}}$. □

Since ev_k clearly preserves (acyclic) fibrations, the adjunction $(F_k \dashv \text{ev}_k)$, (5.1), is a Quillen adjunction. The evaluation functors even preserve cofibrations. Retracts of level cofibrations are level cofibrations. By Theorem 5.4, the cofibrations in $\mathbf{Top}^{\mathcal{K}}$ are precisely the retracts of relative FI-cell complexes. The result follows from Lemma 5.2. The constant functor $c : \mathbf{Top} \to \mathbf{Top}^{\mathcal{K}}$ clearly preserves (acyclic) fibrations, hence the adjunction $(\text{colim} \dashv c)$, (2.7), is a Quillen adjunction.

Proposition 5.5. *The level model structure on* $\mathbf{Top}^{\mathcal{K}}$ *is topological.*

Proof. The continuous map $i : \emptyset \to *$ is an element in I and, therefore, a cofibrantion. Thus $F_k i : \emptyset \to \mathcal{K}(k, -)$ is a cofibration in $\mathbf{Top}^{\mathcal{K}}$ making $\mathcal{K}(k, -)$ cofibrant for every $k \in \operatorname{Ob}\mathcal{K}$. Let $\mathcal{G} = \{\mathcal{K}(k, -) \mid k \in \operatorname{Ob}\mathcal{K}\}$ and $\mathcal{Z} = \emptyset$. Then Proposition 4.25 shows that the level model structure is topological. □

Proposition 5.6 ([Sch18, Proposition 1.4.12(iii)])**.** *The boxproduct of* $\mathbf{Top}^{\mathcal{K}}$ *satisfies the pushout product property.*

The positive level model structure on $\mathbf{Top}^{\mathcal{V}}$ is a variation of the (absolute) level model structures where we do not impose any conditions on the components in degree 0. This means that the component f_0 of a morphism $f : X \to Y$ does not have to be a weak equivalence or a fibration for f to be one. In a general $\mathcal{K}$, the object 0 is, of course, just the identity object.

Definition 5.7. A morphism of $\mathcal{K}$-spaces $f : X \to Y$ is a *positive level equivalence* (*positive level fibration*) if all components $f_k : X_k \to Y_k, k \in \operatorname{Ob}\mathcal{K}_+$, are weak homotopy equivalences (Serre fibrations). It is a *positive cofibration* if it has the LLP with respect to all positive level fibrations that are also positive level equivalences.

Let $FI^+ = \{F_k i \mid k \in \operatorname{Ob}\mathcal{K}_+, i \in I\}$ and $FJ^+ = \{F_k j \mid k \in \operatorname{Ob}\mathcal{K}_+, j \in J\}$ in $\mathbf{Top}^{\mathcal{K}}$ and let W^+_{lev} denote the class of positive level equivalences. Since $FI^+ \subset FI$ and $FJ^+ \subset FJ$, these sets permit the small object argument. Proving that the data consisting of W^+_{lev}, FI^+ and FJ^+ satisfies Theorem 4.21 is done in the same manner as was done in Theorem 5.4.

Theorem 5.8 (Positive level model structure on $\mathbf{Top}^{\mathcal{K}}$)**.** *The positive level equivalences and positive level fibrations are part of a cofibrantly generated model structure on* $\mathbf{Top}^{\mathcal{K}}$*.*

When considering the positive level model structure, the functors ev_k and c also preserve (acyclic) fibration. Thus the adjunctions $(F_k \dashv \mathrm{ev}_k)$, $k \in \mathcal{K}_+$, from(5.1), and $(\operatorname{colim} \dashv c)$, from (2.7), are also Quillen adjunctions when $\mathbf{Top}^{\mathcal{K}}$ is equipped with the positive level model structure.

Proposition 5.9. *The positive level model structure on* $\mathbf{Top}^{\mathcal{V}}$ *is topological.*

Proof. Let $\mathcal{G} = \{\mathcal{K}(k,-) \mid k \in \operatorname{Ob}\mathcal{K}_+\}$ and $\mathcal{Z} = \emptyset$. Then Proposition 4.25 shows that the level model structure is topological. □

5.2. Motivating the homotopy colimit

Before defining the $\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$, we look at some properties of the homotopy colimit. Since (2.7) is a Quillen adjunction, we now know that the colimit preserves (acyclic) cofibrations. By Ken Brown's lemma (Proposition 4.7), it preserves weak equivalences between cofibrant objects. However, as mentioned in Section 3.3 and [Dug08, Section 2], the colimit does not preserve weak equivalences in general. This is one motivating property that the homotopy colimit has. For this reason, we consider the following class of morphisms.

Definition 5.10. A morphism of $\mathcal{K}$-spaces $f : X \to Y$ is a *$\mathcal{K}$-equivalence* if the map $\operatorname{hocolim}_{\mathcal{K}} f$ is a weak homotopy equivalence.

When working with cofibrant objects using the colimit is fine. Therefore another motivating property of the homotopy colimit is that it is weakly equivalent to the ordinary colimit for cofibrant objects. These properties can be stated as the following theorems.

Theorem 5.11. *Level equivalences are $\mathcal{K}$-equivalences.*

Proof. Since $\{\mathrm{id}_k\} \to \mathcal{K}(k,k)$ is a cofibration it is an h-cofibration. Taking the product with any space preserves h-cofibrations, and coproducts also preserve h-cofibrations. Therefore $s_i(B_{p-1}(*,\mathcal{K},X)) \hookrightarrow B_p(*,\mathcal{K},X)$ is an h-cofibration making $B_\bullet(*,\mathcal{K},X)$ 'good' in the sense of [Seg74, Definition A.4]. Then by the proof of [Lew82, Corollary 2.4(b)], 'good' implies 'proper' in the sense of [May72, Definition 11.2] in that the maps

$$\bigcup_{0\le i\le p} s_i(B_{p-1}(*,\mathcal{K},X)) \hookrightarrow B_p(*,\mathcal{K},X)$$

are h-cofibrations for all $p \ge 0$. [Lew82, Corollary 2.4(b)] makes use of Lillig's union theorem found in [Die08, Theorem 5.4.5]. A level equivalence of $\mathcal{K}$-spaces $X \to Y$ induces a level equivalence of simplicial spaces $B_\bullet(*,\mathcal{K},X) \to B_\bullet(*,\mathcal{K},Y)$. Then by [May74, Theorem A.4] the realization $X_{h\mathcal{K}} \to Y_{h\mathcal{K}}$ is a weak homotopy equivalence. Here we need to apply the glueing lemma for h-cofibrations and weak homotopy equivalences, [Sch18, Proposition B.6], instead of the one for h-cofibrations and homotopy equivalences. □

Theorem 5.12. *If X is a cofibrant $\mathcal{K}$-space in the level model structure, then the projection*

$$\pi : \operatorname{hocolim}_{\mathcal{K}} X \to \operatorname{colim} X$$

is a weak homotopy equivalence.

Proof. The $\mathcal{K}$-space X is level-wise cofibrant, because ev_k preserves cofibrations. Since every space $\mathcal{K}(k,l)$ is cofibrant and every map $\{\mathrm{id}_k\} \to \mathcal{K}(k,k)$ is a cofibration, an argument, analogous to the proof for [SS19, Lemma 3.6], implies that the bar resolution $\overline{X}$ is cofibrant. By [HV92, Proposition 3.1(5)] the evaluation $\epsilon : \overline{X} \to X$ is a level equivalence. Since (colim $\dashv c$) is a Quillen adjunction and ϵ is a level equivalence between cofibrant objects, the map $\operatorname{colim}(\epsilon) \cong \pi$ is a weak homotopy equivalence. □

In the case that $\mathcal{K}$ is discrete and 'well structured' in the sense of [SS12, Definitions 5.2], the $\mathcal{K}$-equivalences are the weak equivalences of a cofibrantly generated model structure on $\mathbf{Top}^{\mathcal{K}}$ as shown in [SS12, Proposition 6.16]. Important examples are the categories $\mathcal{N}$, in Example 2.22, and $\mathcal{I}$, in Example 2.23. We will not go into this discrete case, but we note that the construction of this model structure is very similar to the upcoming construction of the absolute and positive $\mathcal{V}$-model structure.

5.3. The $\mathcal{V}$-model structure

From this point onward, we will focus on the index category $\mathcal{V}$. The model structures to be built are particular cases of the construction given in [Lin13, Section 15]. The $\mathcal{V}$-equivalences are the weak equivalences in the $\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$. This model structure has the same cofibrations as the level model structure. The $\mathcal{V}$-model structure is constructed by 'adding more weak equivalences' to the level model structure. The fibrations are defined as follows.

Definition 5.13. A morphism of $\mathcal{V}$-spaces $f : X \to Y$ is a *$\mathcal{V}$-fibration* if it is a level fibration and if $X_n \to X_m \times_{Y_m} Y_n$ is a weak homotopy equivalence for every isometry $\mathbb{R}^n \to \mathbb{R}^m$ in $\mathcal{V}$.

Since the $\mathcal{V}$-model structure has the same cofibrations as the level model structure, it will also have the same acyclic fibrations. Therefore the set of generating cofibrations is the same as well. It already permits the small object argument. The following proposition is used for the smallness argument for the set of generating acyclic cofibrations to be built.

Proposition 5.14. *The cofibrant objects in the level model structure of* $\mathbf{Top}^{\mathcal{V}}$ *are small relative to all cofibrations.*

Proof. By Proposition 5.3, both the domains and codomains of FI are small relative to FI-cell. The initial object $\emptyset$ in $\mathbf{Top}^{\mathcal{V}}$ is, in particular, small relative to FI-cell. Then by [Hir03, Corollary 10.4.9], all FI-cell complexes are small relative to FI-cell. Retracts of objects that are small relative to FI-cell are themselves small relative to FI-cell by [Hir03, Proposition 10.4.7]. Every cofibrant object is a retract of an FI-cell complex by [Hir03, Corollary 11.2.2] and objects small relative to FI-cell are small relative to all cofibrations by [Hir03, Proposition 11.2.3]. When put together this implies the result. □

An isometry $\phi \in \mathcal{V}(n,m)$ induces a morphism $\phi^* : F_m(*) \to F_n(*)$ between cofibrant $\mathcal{V}$-spaces. It is a $\mathcal{V}$-equivalence since $(F_m(*))_{h\mathcal{V}} \cong \mathcal{V}(m,-)_{h\mathcal{V}} \cong B(m/\mathcal{V}) \simeq *$ and $(F_n(*))_{h\mathcal{V}} \simeq *$ by [HV92, (4.1)] and Lemma 3.14. Factor ϕ^* through its mapping cylinder.

$$F_m(*) \xrightarrow{c(\phi^*)} C(\phi^*) \xrightarrow{p(\phi^*)} F_n(*).$$

Here $p(\phi^*)$ is a level-wise homotopy equivalence and, therefore, a $\mathcal{V}$-equivalence. Since the two-out-of-three property clearly holds for $\mathcal{V}$-equivalences, the front $c(\phi^*)$ is a $\mathcal{V}$-equivalence. It is also a cofibration by Lemma 4.28. Let $K' = \{c(\phi^*) \mid \phi \in \operatorname{Mor}\mathcal{V}\}$. Since the level model structure is topological, the set $K'\square I$ is a set of cofibrations. The set of generating acyclic cofibrations is $K = FJ \cup (K'\square I)$. Let $W_{\mathcal{V}}$ be the class of $\mathcal{V}$-equivalences.

Proposition 5.15 ([Lin13, Proposition 15.5])**.** *A morphism of $\mathcal{V}$-spaces is a K-injective if and only if it is a $\mathcal{V}$-fibration.*

If $W_{\mathcal{V}}$, FI and K make $\mathbf{Top}^{\mathcal{V}}$ a cofibrantly generated model category, then the $\mathcal{V}$-fibrations are indeed the fibrations in this model category. An $\mathcal{V}$-space X is $\mathcal{V}$-fibrant if $X_n \to X_m$ is a weak homotopy equivalence for every isometry $\mathbb{R}^n \to \mathbb{R}^m$.

Theorem 5.16 ($\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$)**.** *The $\mathcal{V}$-equivalences, $\mathcal{V}$-fibrations and cofibrations are part of a cofibrantly generated model structure on $\mathbf{Top}^{V}$.*

Proof. We will show that the data consisting of $W_{\mathcal{V}}$, FI and K satisfies Theorem 4.21. The category $\mathbf{Top}^{\mathcal{V}}$ is bicomplete. The functoriality of $\operatorname{hocolim}_{\mathcal{V}}$ means that $W_{\mathcal{V}}$ contains all identities and satisfies RT1.

The domains of FJ are cofibrant since all F_n preserve cofibrations. Let $c(\phi^*)\square i \in K'\square I$. It has the domain

$$X = F_m(*) \times D^n \bigcup_{F_m(*)\times S^{n-1}} C(\phi^*) \times S^{n-1}.$$

Since $F_m(*)$ and S^n are cofibrant and $c(\phi^*)$ and i are cofibrations in their respective categories the composite $F_m(S^n) \cong F_m(*) \times S^n \rightarrowtail X$ is a cofibration. Hence X is cofibrant since $F_m(S^n)$ is. By Proposition 5.14 $FJ \cup K$ permits the small object argument, so RT2 is satisfied.

The inclusion K-cell $\subset FI$-cof holds because K is a set of cofibrations. For every generating cofibration we have $(F_m i)_{h\mathcal{V}} \cong B(m/\mathcal{V}) \times i$ which is a cofibration in **Top**. Since hocolim preserves colimits by Corollary 3.12, it must preserve transfinite compositions of pushouts. Therefore it preserves cofibrations. Every morphism in K is a cofibration and a $\mathcal{V}$-equivalence and is sent by hocolim to an acyclic cofibration in **Top**. Thus the class K-cell is sent to acyclic cofibrations in **Top** because hocolim preserves colimits. In particular, we have K-cell $\subset W_{\mathcal{V}}$. Therefore RT3 is satisfied.

The inclusion FI-inj $\subset K$-inj also holds because K is a set of cofibrations. By Theorem 5.11 we also have FI-inj $\subset W_{\text{lev}} \subset W_{\mathcal{V}}$, so RT4 is satisfied. Finally RT5 is satisfied, since $\mathcal{V}$-equivalence that are K-injectives are FI-injectives by [Lin13, Proposition 15.9]. □

The $\mathcal{V}$-fibrations are level fibrations. Since acyclic $\mathcal{V}$-fibrations are precisely the FI-injectives, they are level acyclic fibrations. The evaluation functors ev_n then preserve (acyclic) fibrations. Therefore the adjunction ($F_n \dashv \text{ev}_n$),(5.1), is a Quillen adjunction when $\mathbf{Top}^{\mathcal{V}}$ is equipped with the $\mathcal{V}$-model structure.

Proposition 5.17. *The $\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$ is topological.*

Proof. Let $\mathcal{G} = \{\mathcal{V}(n, -) \mid \mathbb{R}^n \in \operatorname{Ob}\mathcal{V}\}$ and $\mathcal{Z} = K'$. Then Proposition 4.25 shows that the level model structure is topological. □

Recall that $V_+ \subset \mathcal{V}$ is the full subcategory with $\operatorname{Ob}\mathcal{V}_+ = \operatorname{Ob}\mathcal{V} - \{\mathbb{R}^0\}$. Like the positive level model structure, the positive $\mathcal{V}$-model structure is constructed similarly to the (absolute) $\mathcal{V}$-model structure. It will have the $\mathcal{V}$-equivalences as weak equivalences and have the positive cofibrations as cofibrations. The fibrations are defined as follows.

Definition 5.18. A morphism of $\mathcal{V}$-spaces $f : X \to Y$ is a *positive $\mathcal{V}$-fibration* if it is a positive level fibration and if $X_n \to X_m \times_{Y_m} Y_n$ is a weak homotopy equivalence for every isometry $\mathbb{R}^n \to \mathbb{R}^m$ in $\mathcal{V}_+$.

Let $(K')^+ = \{c(\phi^*) \mid \phi \in \operatorname{Mor} \mathcal{V}_+\}$ and let $K^+ = FJ^+ \cup ((K')^+ \square I)$. Since $K^+ \subset K$, this set permits the small object argument. For $n > 0$, the free functor F_n sends cofibrations to positive cofibrations. Hence $F_n(*)$ is positive cofibrant. By Lemma 4.28, the morphisms in $(K')^+$ are positive cofibrations. Since the positive level model structure is topological, the set $(K')^+ \square I$ is a set of positive cofibrations.

Proposition 5.19 ([Lin13, Proposition 15.5])**.** *A morphism of $\mathcal{V}$-spaces is a K^+-injective if and only if it is a positive $\mathcal{V}$-fibration.*

A $\mathcal{V}$-space X is positive $\mathcal{V}$-fibrant if $X_n \to X_m$ is a weak homotopy equivalence for every isometry $\mathbb{R}^n \to \mathbb{R}^m$ in $\mathcal{V}_+$. Applying the proof of [MS02, Lemma 6.4] to the category $\mathcal{V}$ shows that there exists a weak homotopy equivalence $X_{h\mathcal{V}_+} \xrightarrow{\sim} X_{h\mathcal{V}}$. Then by Theorem 5.11, positive level equivalences are $\mathcal{V}$-equivalences. Proving that the data $W_{\mathcal{V}}$, FI^+ and K^+ satisfies Theorem 4.21 is now done in the same manner as in Theorem 5.16.

Theorem 5.20 (Positive $\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$)**.** *The $\mathcal{V}$-equivalences, positive $\mathcal{V}$-fibrations and positive cofibrations are part of a cofibrantly generated model structure on $\mathbf{Top}^{\mathcal{V}}$.*

The adjunction $(F_n \dashv \mathrm{ev}_n), n > 0$, from (5.1), is a Quillen adjunction when $\mathbf{Top}^{\mathcal{V}}$ is equipped with the positive $\mathcal{V}$-model structure.

Proposition 5.21. *The positive $\mathcal{V}$-model structure on $\mathbf{Top}^{\mathcal{V}}$ is topological.*

Proof. Let $\mathcal{G} = \{\mathcal{V}(n, -) \mid \mathbb{R}^n \in \operatorname{Ob} \mathcal{V}_+\}$ and $\mathcal{Z} = (K')^+$. Then Proposition 4.25 shows that the level model structure is topological. □

6. The class of $\mathcal{V}$-equivalences

The $\mathcal{V}$-model structure can be defined in many different ways. Stefan Schwede gives an alternate definition in [Sch18, Section 1.4]. The general tactic is the same: Define a level model structure, define $\mathcal{V}$-equivalences and show that the $\mathcal{V}$-equivalences and the cofibrations of the level model structure form a new model structure. Then by a slight adjustment, one can define the positive variations. The main difference is that Schwede avoids the use of homotopy colimits. Schwede also works in the more general setting of global homotopy theory where model structures depend on a global family of Lie groups $\mathcal{F}$, usually the family of all Lie groups. We will assume that this family is the trivial family $\mathcal{F}_{\text{triv}} = \{*\}$ consisting of the single trivial Lie group. Then the $\mathcal{V}$-model structure should coincide with the $\mathcal{F}_{\text{triv}}$-global model structure in [Sch18, Theorem 1.4.8]. The $\mathcal{F}_{\text{triv}}$-level model structure in [Sch18, Proposition 1.4.3] already coincides with our level model structure. It remains to show that the $\mathcal{F}_{\text{triv}}$-equivalences in [Sch18, Defintion 1.4.4] coincide with our $\mathcal{V}$-equivalences. Many results in [Sch18] implicitly use the family of all Lie groups but can be applied to an arbitrary family enabling us to use them.

6.1. The class of $\mathcal{F}_{\text{triv}}$-equivalences

The following definition describes the alternative equivalence of $\mathcal{V}$-spaces. It is a specific case of [Sch18, Definition 1.4.4] where we assume that the family of Lie groups $\mathcal{F}$ is the trivial one $\mathcal{F}_{\text{triv}} = \{*\}$.

Definition 6.1. A morphism of $\mathcal{V}$-spaces $f : X \to Y$ is an $\mathcal{F}_{\text{triv}}$-*equivalence* if for all $k \geq -1$ and $n \geq 0$ and all commutative diagrams

$$\begin{array}{ccc} S^k & \xrightarrow{\;g\;} & X_n \\ \big\downarrow & & \big\downarrow f_n \\ D^{k+1} & \xrightarrow{\;h\;} & Y_n \end{array}$$

there exists an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and a map $\lambda : D^{k+1} \to X_m$ such that in the diagram

$$\begin{array}{ccccc} S^k & \xrightarrow{\;g\;} & X_n & \xrightarrow{\;\phi_*\;} & X_m \\ \big\downarrow & & \nearrow_{\lambda} & & \big\downarrow f_m \\ D^{k+1} & \xrightarrow{\;h\;} & Y_n & \xrightarrow{\;\phi_*\;} & Y_m \end{array}$$

the upper triangle commutes and the lower triangle commutes up to homotopy relative to S^k. That is, there exists a homotopy $H : D^{k+1} \times [0,1] \to Y_m$ with $H_0 = f_m\lambda$, $H_1 = \phi_* h$ and $H_t(x) = x$ for all $(x,t) \in S^k \times [0,1]$.

The following proposition motivates why one might consider this definition.

Proposition 6.2 ([May99, Section 9.6]). *A continuous map $f : A \to B$ is a weak homotopy equivalence if and only if for all $k \geq -1$ and all diagrams*

$$\begin{array}{ccc} S^k & \longrightarrow & A \\ \big\downarrow & \overset{\lambda}{\nearrow} & \big\downarrow g \\ D^{k+1} & \longrightarrow & B \end{array}$$

where the outer square commutes, the dotted arrow exists such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative to S^k.

Proof. This is the equivalence of (i) and (iii) of the lemma in [May99, Section 9.6]. □

If the $\mathcal{V}$-spaces in Definition 6.1 are constant, then the morphism f is an $\mathcal{F}_{\text{triv}}$-equivalence if and only if the underlying map is a weak homotopy equivalence, by Proposition 6.2. This is a property that $\mathcal{V}$-equivalences have as well. The homotopy colimit of a constant $\mathcal{V}$-space cA is $B\mathcal{V} \times A$ which is weakly equivalent to A since $B\mathcal{V}$ is contractible. Therefore a morphism of constant $\mathcal{V}$-spaces is a $\mathcal{V}$-equivalence if and only if its underlying map is a weak homotopy equivalence. Proposition 6.2 also gives the following corollary.

Corollary 6.3 ([Sch18, Proposition 1.4.7(i)]). *Level equivalences are $\mathcal{F}_{\text{triv}}$-equivalences.*

6.2. The mapping telescope

The mapping telescope can be used to model the homotopy colimit. It is one piece that bridges the gap between $\mathcal{V}$-equivalences and $\mathcal{F}_{\text{triv}}$-equivalences. The mapping telescope of a sequence of maps $X_0 \to X_1 \to \dots$ is an iterated mapping cylinder. To see this, we will also construct the truncated mapping telescope. We restate the pushout diagram for a mapping cylinder in **Top** for easy reference.

$$\begin{array}{ccccc} & & X & \xrightarrow{\;f\;} & Y \\ & & {\scriptstyle \text{incl}_1}\big\downarrow & & \big\downarrow{\scriptstyle q(f)} \\ X & \xrightarrow{\text{incl}_0} & X \times [0,1] & \xrightarrow{\;f'\;} & C(f) \end{array} \qquad c(f) = f' \circ \text{incl}_0 : X \to C(f) \tag{6.1}$$

Maps of the form $\text{incl}_t : X \hookrightarrow X \times [0.1]$ are closed inclusions. Pushouts preserve closed inclusions by [Sch18, Proposition A.13]. Therefore $q(f)$ is a closed inclusion. It is not hard to see that the front of a mapping cylinder in **Top** must also be a closed inclusion. We recall the definition of a strong deformation retract.

Definition 6.4. A *strong deformation retract* is an inclusion $i : A \hookrightarrow B$ such that there exists a map $r : B \to A$ and a homotopy $H : B \times [0,1] \to B$ such that $ri = \text{id}_A$, $H_0 = \text{id}_B$, $H_1 = ir$ and $H_t(i(a)) = i(a)$, for all $a \in A$. The map r is called the *retraction.*

Strong deformation retracts are homotopy equivalences and, therefore, weak homotopy equivalences. Maps of the form $\text{incl}_t : X \to X \times [0,1]$ are strong deformation retracts. Pushouts preserve strong deformation retract as shown in the proof of [Hov99, Proposition 2.4.9]. Therefore $q(f)$ is a strong deformation retract.

Definition 6.5. Let X be an $\mathcal{N}$-space. It is a sequence $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$. The *mapping telescope* or *telescope* $\operatorname{tel} X$ of X is the coequalizer of the diagram

$$\coprod_{n\geq 0} X_n \rightrightarrows \coprod_{n\geq 0} X_n \times [0,1]$$

where for every $n \geq 0$ the top and bottom maps are induced by the maps $x \mapsto (x,1)$ and $x \mapsto (f_n(x), 0)$ respectively.

There is a canonical map from the telescope to the homotopy colimit over $\mathcal{N}$. One identifies $\coprod_{n\geq 0} X_n \times \{0\}$ with $B_0(*, \mathcal{N}, X) \times \Delta^0$ and identifies $\coprod_{n\geq 0} X_n \times [0,1]$ with a subspace of $B_1(*, \mathcal{N}, X) \times \Delta^1$.

Definition 6.6. Let X be an $\mathcal{N}$-space. It is a sequence $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$. For $n \geq 0$ we construct the *n-th truncated mapping telescope* or *n-th truncated telescope* $\operatorname{tel}_n X$ of X inductively. Let $\operatorname{tel}_0 = X_0$ and $i_0 = \operatorname{id}_{X_0}$. The n-th truncated mapping telescope of X, with $n > 0$, is the pushout of the diagram

$$\begin{array}{ccc} X_{n-1} & \xhookrightarrow{c(f_{n-1})} & C(f_{n-1}) \\ {\scriptstyle i_{n-1}}\downarrow & & \downarrow{\scriptstyle q} \\ \operatorname{tel}_{n-1} X & \hookrightarrow & \operatorname{tel}_n X \end{array} \tag{6.2}$$

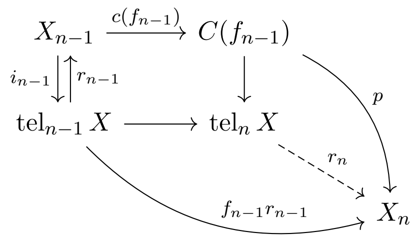
where i_{n-1} is a closed inclusion. The right vertical map denoted by q is a closed inclusion by [Sch18, Proposition A.13]. The composite $i_n = q \circ q(f_{n-1}) : X_n \to \operatorname{tel}_n X$ is a new closed inclusion.

The top map in (6.2) is the front of a mapping telescope and, therefore, a closed inclusion. Again by [Sch18, Proposition A.13], the bottom map $\operatorname{tel}_{n-1} X \hookrightarrow \operatorname{tel}_n X$ must also be a closed inclusion. For an $\mathcal{N}$-space X, we now have a sequence of closed inclusions

$$\operatorname{tel}_0 X \hookrightarrow \operatorname{tel}_1 X \hookrightarrow \operatorname{tel}_2 X \hookrightarrow \dots$$

which is another $\mathcal{N}$-space $\operatorname{tel}_{(-)} X$. This assignment is a functor from $\mathbf{Top}^{\mathcal{N}}$ to itself. The colimit of the $\mathcal{N}$-space $\operatorname{tel}_{(-)} X$ is the union of all truncated telescopes, which is homeomorphic to the telescope of X. Therefore we might as well set $\operatorname{colim}(\operatorname{tel}_{(-)} X) = \operatorname{tel} X$.

Using induction we show that all the maps i_n in Definition 6.6 are strong deformation retracts that admit retractions $r_n : \operatorname{tel}_n X \to X_n$ such that $f_{n-1} r_{n-1} = r_n | \operatorname{tel}_{n-1} X$. We use the following pushout diagram:



Let $r_0 = \operatorname{id}_{X_0}$. Let p be the retraction of the strong deformation retract $q(f_{n-1})$. Then f_{n-1} factors through $C(f_{n-1})$ as $f_{n-1} = p \circ c(f_{n-1})$. If r_{n-1} is the retraction of the strong deformation retract i_{n-1}, then the maps $f_{n-1} r_{n-1}$ and p, together with the universal property of the above diagram, induce a map $r_n : \mathrm{tel}_n X \to X_n$ such that $f_{n-1}r_{n-1} = r_n|_{\mathrm{tel}_{n-1} X}$ and $p = r_n q$. Since i_{n-1} is a strong deformation retract, so is q, and therefore so is $i_n = q \circ q(f_{n-1})$ being a composite of strong deformation retracts. Its retraction is the retraction of q composed with p. This must be r_n by the uniqueness of the universal property. In particular, all i_n and r_n are weak homotopy equivalences.

We get the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{tel}_0 X & \hookrightarrow & \mathrm{tel}_1 X & \hookrightarrow & \mathrm{tel}_2 X & \hookrightarrow & \dots \\
{\scriptstyle r_0}\downarrow\sim & & {\scriptstyle r_1}\downarrow\sim & & {\scriptstyle r_2}\downarrow\sim & & \\
X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots
\end{array} \tag{6.3}$$

Thus the morphism of $\mathcal{N}$-spaces $r : \mathrm{tel}_{(-)} X \to X$ with components $r_n, n \geq 0$, is a level equivalence.

6.3. A comparison of equivalences

Recall from Example 2.24 that we view $\mathcal{N}$ as a subcategory of $\mathcal{V}$ by sending a morphism $n \to m$ to the isometry defined by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0 \dots, 0)$. This way, we can view $\mathbb{R}^n$ as a linear subspace of $\mathbb{R}^m$ by mapping it onto the first n summands. We note that the functor chosen as the inclusion $\mathcal{N} \subset \mathcal{V}$ was chosen mostly for notation and that there is no preferred functor $\mathcal{N} \to \mathcal{V}$. Nonetheless, we interpret $\mathcal{V}$-spaces as $\mathcal{N}$-spaces via this inclusion. When X is a $\mathcal{V}$-space, we can use a functor $F : \mathcal{N} \to \mathcal{V}$ to construct the telescope $\mathrm{tel}(X \circ F)$. If F is the inclusion $\mathcal{N} \subset \mathcal{V}$, we simply write $\mathrm{tel}\, X$.

A functor $\mathcal{N} \to \mathcal{N}$ is an ascending sequence of non-negative integers

$$n_0 \leq n_1 \leq n_2 \leq n_3 \leq \dots.$$

Such a sequence is called an *exhaustive sequence* in $\mathcal{N}$ if for every $n \in \mathrm{Ob}\,\mathcal{N}$ there exists some $m \in \mathrm{Ob}\,\mathcal{N}$ such that $n \leq n_m$. The composite of a functor $\mathcal{N} \to \mathcal{N}$ with the inclusion $\mathcal{N} \subset \mathcal{V}$ is a sequence of inclusions

$$\mathbb{R}^{n_0} \subset \mathbb{R}^{n_1} \subset \mathbb{R}^{n_2} \subset \dots.$$

Such a sequence is called an *exhaustive sequence* in $\mathcal{V}$ if the functor $\mathcal{N} \to \mathcal{N}$ is an exhaustive sequence in $\mathcal{N}$. A sequence of inclusions is exhaustive precisely when every $\mathbb{R}^n \in \mathrm{Ob}\,\mathcal{V}$ admits an isometric embedding in some $\mathbb{R}^{n_m}$. The exhaustive sequences $F : \mathcal{N} \to \mathcal{N} \subset \mathcal{V}$ are analogous to those defined in [Sch18, Definition 1.1.6]. Thus we can use the following result.

Lemma 6.7 ([Sch18, Proposition 1.4.5])**.** *Let $f : X \to Y$ be a morphism of $\mathcal{V}$-spaces. It is an $\mathcal{F}_{\mathrm{triv}}$-equivalence if and only if for all exhaustive sequences $F : \mathcal{N} \to \mathcal{N} \subset \mathcal{V}$ the induced maps $\mathrm{tel}(X \circ F) \to \mathrm{tel}(Y \circ F)$ are weak homotopy equivalences.*

Lemma 6.7 concerns itself with all possible exhaustive sequences. It turns out that we only need an arbitrary one. Since this might as well be the inclusion $\mathcal{N} \subset \mathcal{V}$, we get the following corollary. Its proof is analogous to the proof of [Sch18, Propositions 1.4.5, 1.1.7].

Corollary 6.8 ([Sch18, Propositions 1.4.5, 1.1.7])**.** *Let $f : X \to Y$ be a morphism of $\mathcal{V}$-spaces. It is a $\mathcal{F}_{\mathrm{triv}}$-equivalence if and only if the induced map $\mathrm{tel}\, X \to \mathrm{tel}\, Y$ is a weak homotopy equivalence.*

Corollary 6.9. *If $f : X \to Y$ is a level equivalence of $\mathcal{V}$-spaces, the induced map* $\operatorname{tel} X \to \operatorname{tel} Y$ *is a weak homotopy equivalence.*

This corollary also holds for $\mathcal{N}$-spaces. Via the continuous functor $\mathcal{V} \to \mathcal{N}, \mathbb{R}^n \mapsto n$ we can interpret an $\mathcal{N}$-space as a $\mathcal{V}$-space. Then a morphism of $\mathcal{N}$-spaces is a level equivalence of $\mathcal{N}$-spaces if and only if it is a level equivalence of $\mathcal{V}$-spaces.

Lemma 6.10 ([Dug08, Corollary 14.11])**.** *Let X be an $\mathcal{N}$-space that is level-wise cofibrant. That is, all spaces X_n are cofibrant in* **Top**. *Then there exists a chain of natural weak homotopy equivalences* $\operatorname{tel} X \xleftarrow{\sim} h(X) \xrightarrow{\sim} X_{h\mathcal{N}}$.

Proof. Consider the morphism of $\mathcal{N}$-spaces (6.3). Since r is a level equivalence, it must be an $\mathcal{N}$-equivalence by Theorem 5.11. Thus the induced map

$$r_* : (\operatorname{tel}_{(-)} X)_{h\mathcal{N}} \to X_{h\mathcal{N}}$$

is a weak homotopy equivalence.

Since X is level-wise cofibrant, every front $c(f_{n-1}) : X_{n-1} \to C(f_{n-1}), n > 0$, is a cofibration by Lemma 4.28. Then every closed inclusion $\operatorname{tel}_{n-1} X \to \operatorname{tel}_n X, n > 0$, is a pushout of a cofibration and, therefore, a cofibration itself. Since $\operatorname{tel}_0 X = X_0$ is cofibrant the $\mathcal{N}$-space $\operatorname{tel}_{(-)} X$ is cofibrant in $\mathbf{Top}^{\mathcal{N}}$ by [Dug08, Example 14.9]. Hence the projection

$$\pi : (\operatorname{tel}_{(-)} X)_{h\mathcal{N}} \to \operatorname{colim}(\operatorname{tel}_{(-)} X) = \operatorname{tel} X$$

is a weak homotopy equivalence by Theorem 5.12. □

[DI04, Theorem A.7] tells us that the homotopy colimits of an $\mathcal{N}$-space and its level-wise cofibrant replacement are weakly equivalent. If X is an level-wise cofibrant $\mathcal{N}$-space then so is $\operatorname{tel}_{(-)} X$. We also know that level equivalences in $\mathbf{Top}^{\mathcal{N}}$ induce weak homotopy equivalences on the telescope. Thus the assumption in Lemma 6.10 that X is level-wise cofibrant is redundant.

Corollary 6.11. *Let X be an $\mathcal{N}$-space. Then there exists a chain of natural weak homotopy equivalences* $\operatorname{tel} X \xleftarrow{\sim} h(X) \xrightarrow{\sim} X_{h\mathcal{N}}$.

Let X now be a $\mathcal{V}$-space. The inclusion $\mathcal{N} \subset \mathcal{V}$ induces a natural map $X_{h\mathcal{N}} \to X_{h\mathcal{V}}$. The final step in showing that the $\mathcal{F}_{\text{triv}}$-equivalences are precisely the $\mathcal{V}$-equivalences is to show that this map is a weak equivalence.

Lemma 6.12. *Let X be a $\mathcal{V}$-space. The map $X_{h\mathcal{N}} \to X_{h\mathcal{V}}$ is a weak homotopy equivalence.*

Proof. We can apply the proof of [Lin13, Proposition 9.4]. We only need to show that the $\mathcal{V}^{\text{op}}$-space $B(*, \mathcal{N}, \mathcal{V})$ is level-wise contractible. Evaluated at $\mathbb{R}^n$ this is the space $B(*, \mathcal{N}, \mathcal{V}(n, -)) = \mathcal{V}(n, -)_{h\mathcal{N}}$. The $\mathcal{V}$-space $\mathcal{V}(n, -) \cong F_n(*)$ is clearly cofibrant. The space $\mathcal{V}(n, 0)$ is cofibrant and the maps $\mathcal{V}(n, m) \to \mathcal{V}(n, m+1)$, induced by the inclusions $\mathbb{R}^m \subset \mathbb{R}^{m+1}$, are cofibrations by [Sch18, Proposition 1.1.19]. Then $\mathcal{V}(n, -)$ is cofibrant as an $\mathcal{N}$-space by [Dug08, Example 14.9]. Therefore by Theorem 5.12 we have $\mathcal{V}(n, -)_{h\mathcal{N}} \simeq \operatorname{colim}_{m \in \operatorname{Ob} \mathcal{N}} \mathcal{V}(n, m) \cong \mathcal{V}(n, \infty)$, where $\mathcal{V}(n, \infty)$ is the space containing all isometries $\mathbb{R}^n \to \mathbb{R}^\infty$. This space is contractible by [May77, Lemma I.1.3]. □

Theorem 6.13. *Let $f : X \to Y$ be a morphism of $\mathcal{V}$-spaces. Then f is a $\mathcal{F}_{\text{triv}}$-equivalence if and only if it is a $\mathcal{V}$-equivalence.*

Proof. Corollary 6.11 and Lemma 6.12 provide chains of weak homotopy equivalences that fit in the commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{tel} X & \xleftarrow[\pi]{\sim} & h(X) & \xrightarrow[r_*]{\sim} & X_{h\mathcal{N}} & \xrightarrow{\sim} & X_{h\mathcal{V}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{tel} Y & \xleftarrow[\pi]{\sim} & h(Y) & \xrightarrow[r_*]{\sim} & Y_{h\mathcal{N}} & \xrightarrow{\sim} & Y_{h\mathcal{V}}
\end{array}$$

By the two-out-of-three property, any vertical map is a weak homotopy equivalence if and only if any other vertical map is. This, in particular, holds for the outer vertical maps. Corollary 6.8 then finishes the proof. □

7. Commutative monoids

We are nearly ready to lift the positive $\mathcal{V}$-model structure to commutative $\mathcal{V}$-space monoids. [SS00, Theorem 4.1] gives a criterion to lift a model structure on a symmetric monoidal category $\mathscr{C}$ to the category of monoids $\mathrm{Mon}(\mathscr{C})$. This can be used to lift the $\mathcal{V}$-model structure to the category of $\mathcal{V}$-space monoids $\mathrm{Mon}(\mathbf{Top}^{\mathcal{V}})$. One needs a stronger criterion to lift a model structure to commutative monoids. This is given by [Whi17, Theorem 3.2]. Unfortunately, the absolute $\mathcal{V}$-model structure does not satisfy the hypothesis in this theorem and cannot be lifted to $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. This is because not all acyclic cofibrations in the absolute $\mathcal{V}$-model structure are 'symmetrizable'. [Sch18, Theorem 2.1.13] and the discussion thereafter give more details on this. It is the positive $\mathcal{V}$-model structure that we must use.

There is a complication, however. If M is a commutative $\mathcal{V}$-space monoid and A is a space, then the tensor $M \times A$ is not a commutative $\mathcal{V}$-space monoid in general. Thus we would need to redefine the tensor for $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. This tensor is not very nice to work with, and it is more convenient to consider a simplicial tensor. For this reason, we would like to generalize the notion of enriched, tensored and cotensored categories and topological model categories.

7.1. More on enriched categories

In enriched category theory, Hom-sets are replaced by Hom-objects in a monoidal category. In Section 2.3, we have already seen the case where this monoidal category is $(\mathbf{Top}, \times, *)$. Enriched category theory is a generalization of ordinary category theory since this is just **Set**-enriched category theory. Enriched category theory will allow us to generalize the notion of a topological model category, given in Definition 4.24, to an $\mathscr{E}$-model category for some category $\mathscr{E}$. The following definition specifies what properties the category $\mathscr{E}$ must have.

Definition 7.1. A closed symmetric monoidal category $(\mathscr{E}, \otimes, 1)$ that is also a model category is a *monoidal model category* if the tensor product $\otimes$ satisfies the pushout product property in Definition 4.9 and if for every cofibrant object X the morphism $X \otimes \widetilde{1} \to X \otimes 1$, induced by the cofibrant replacement $\widetilde{1} \xrightarrow{\sim} 1$, is a weak equivalence.

Example 7.2. Since the terminal object $*$ is cofibrant in **Top** and the pushout product property holds by Proposition 4.26, we find that $(\mathbf{Top}, \times, *)$ is a monoidal model category.

Theorem 7.3 ([Hov99, Proposition 4.2.8])**.** *The category* $(\mathbf{sSet}, \times, *)$ *is a monoidal model category with a cofibrant terminal object.*

A category is *based* if it contains an object, called the *zero object*, that is both initial and terminal. If $\mathscr{E}$ has a terminal object $*$, we write $\mathscr{E}_* = */\mathscr{E}$ for the category under the terminal object. It is a based category with zero object the identity on $*$. We often write (X, x) for an object $x : * \to X$ in $\mathscr{E}_*$.

Let $(\mathscr{E}, \otimes, *)$ be a monoidal model category whose terminal object $*$ is the identity object. Then $\mathscr{E}_*$ is a based model category by Proposition 4.8. Let $x : * \to X$ and $y : * \to Y$ be objects in $\mathscr{E}_*$. Let

$$f = (X \otimes y) + (x \otimes Y) : (X \otimes *) \coprod (* \otimes Y) \to X \otimes Y.$$

This induces a morphism $f' : X \coprod Y \xrightarrow{\cong} (X \otimes *) \coprod (* \otimes Y) \xrightarrow{f} X \otimes Y$ where the isomorphism is the coproduct of the right unitor at X and left unitor at Y. Let $X \wedge Y$ be the pushout in the following diagram:

$$\begin{array}{ccc} X \coprod Y & \xrightarrow{f'} & X \otimes Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge Y \end{array}$$

Then let the *smash product* $(X, x) \wedge (Y, y)$ be the pushout of f' along $X \coprod Y \to *$, which is the bottom map in the diagram above. The smash product $\wedge$ defines a functor $\mathscr{E}_* \times \mathscr{E}_* \to \mathscr{E}_*$.

Theorem 7.4 ([Hov99, Proposition 4.2.9])**.** *Let $(\mathscr{E}, \otimes, *)$ be a monoidal model category whose terminal object $*$ is the identity object and is cofibrant. Then $(\mathscr{E}_*, \wedge, * \coprod *)$ is a monoidal model category.*

Corollary 7.5 ([Hov99, Corollary 4.2.10])**.** *The category $(\mathbf{sSet}_*, \wedge, * \coprod *)$ is a monoidal model category.*

With the categories **sSet** and $\mathbf{sSet}_*$ set up, we now look at enriched, tensored and cotensored categories. While these properties can be defined separately, we will follow the conventions in [Hov99] and define them together in Definition 7.10. The following definition will give an alternate definition of an enriched category that allows us to define enriched functors. To distinguish these from Definition 7.10, we call these $\mathscr{E}$-categories and $\mathscr{E}$-functors for some monoidal category $\mathscr{E}$. We will see that a category being enriched, tensored and cotensored over $\mathscr{E}$ as in Definition 7.10 will imply that it is an $\mathscr{E}$-category.

Definition 7.6. Let $(\mathscr{E}, \otimes, 1)$ be a monoidal category. An *$\mathscr{E}$-category $\mathscr{C}$* consists of a class of objects $\operatorname{Ob}\mathscr{C}$, a Hom-object $\operatorname{Map}(X, Y) \in \operatorname{Ob}\mathscr{E}$ for every pair of objects in $\mathscr{C}$, composition morphisms $\operatorname{Map}(Y, Z) \otimes \operatorname{Map}(X, Y) \to \operatorname{Map}(X, Z)$ in $\mathscr{E}$ for every triple of objects in $\mathscr{C}$ and identity morphisms $1 \to \operatorname{Map}(X, X)$ in $\mathscr{E}$ for every object in $\mathscr{C}$, satisfying associativity and unity conditions given in [Bor94, Definition 6.2.1].

An *$\mathscr{E}$-functor* $F : \mathscr{D} \to \mathscr{C}$ between $\mathscr{E}$-categories consists of a function $F : \operatorname{Ob}\mathscr{D} \to \operatorname{Ob}\mathscr{C}$ and morphisms $F : \mathscr{D}(X, Y) \to \mathscr{C}(FX, FY)$ in $\mathscr{E}$ for every pair of objects in $\mathscr{D}$ compatible with composition and identity morphisms as given in [Bor94, Definition 6.2.3].

Example 7.7. Consider $(\mathbf{Set}, \times, *)$. Then (locally small) categories and functors are precisely **Set**-categories and **Set**-functors.

Example 7.8. Consider $(\mathbf{Top}, \times, *)$. Then Definition 2.14 coincides with Definition 7.6, and continuous functors are precisely **Top**-functors.

Example 7.9 ([Bor94, Proposition 6.2.6])**.** Let $(\mathscr{E}, \otimes, 1)$ be a closed symmetric monoidal category. Then the internal-hom $[-, -]$ makes $\mathscr{E}$ itself an $\mathscr{E}$-category. This holds in particular for monoidal model categories.

The following definition is a particular case of Hovey's 'adjunction of two variables' given in [Hov99, Definition 4.1.12]. Defining enrichments, tensors and cotensors this way will make it easier to prove that certain tensors have the pushout product property.

Definition 7.10. Let $(\mathscr{E}, \otimes, 1)$ be a closed symmetric monoidal category and $\mathscr{C}$ a category. Then $\mathscr{C}$ is *enriched, tensored and cotensored* over $\mathscr{E}$ if there exist functors

$$\mathrm{Map} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{E}, \qquad (-) \times (-) : \mathscr{C} \times \mathscr{E} \to \mathscr{C}, \qquad (-)^{(-)} : \mathscr{C} \times \mathscr{E}^{\mathrm{op}} \to \mathscr{C},$$

called the *enrichment*, *tensor* and *cotensor* respectively, such that we have natural bijections

$$\mathscr{C}(X \times K, Y) \cong \mathscr{E}(K, \mathrm{Map}(X, Y)) \cong \mathscr{C}(X, Y^K), \tag{7.1}$$

for $X, Y \in \mathrm{Ob}\,\mathscr{C}, K \in \mathrm{Ob}\,\mathscr{E}$, and natural isomorphisms

$$(X \times K) \times L \cong X \times (K \otimes L), \tag{7.2}$$
$$X \times 1 \cong X, \tag{7.3}$$

for $X \in \mathrm{Ob}\,\mathscr{C}, K, L \in \mathrm{Ob}\,\mathscr{E}$, that satisfy 'coherence axioms' given in [Hov99, Definition 4.1.6]. The tensor and cotensor are sometimes called the *copower* and *power*, respectively.

In [Hov99] such a category $\mathscr{C}$ would be a 'right $\mathscr{E}$-module' [Hov99, Definition 4.1.6] where $\times$ is an 'adjunction of two variables' [Hov99, Definition 4.1.12] and $\mathscr{E}$ is closed symmetric. Using (7.1), one could equivalently use natural isomorphisms $(X^L)^K \cong X^{K \oplus L}$ and $X^1 \cong X$, with corresponding 'coherence axioms', instead of those used in (7.2) and (7.3).

If $\mathscr{C}$ is enriched, tensored and cotensored over $\mathscr{E}$ the enrichment $\mathrm{Map}(-.-)$ makes $\mathscr{C}$ an $\mathscr{E}$-category. The identity morphism $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y)$ corresponds by (7.1) to a morphism $X \times \mathrm{Map}(X, Y) \to Y$. Together with a morphism $Y \times \mathrm{Map}(Y, Z) \to Z$ we obtain $X \times (\mathrm{Map}(X, Y) \otimes \mathrm{Map}(Y, Z)) \to Z$ after applying (7.2). After applying the symmetry isomorphism in $\mathscr{E}$ this corresponds by (7.1) to a composition morphism $\mathrm{Map}(Y, Z) \otimes \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ in $\mathscr{E}$. For $X \in \mathrm{Ob}\,\mathscr{C}$ the isomorphism $X \times 1 \cong X$ in (7.3) corresponds by (7.1) to an identity morphism $1 \to \mathrm{Map}(X, X)$.

Let $\mathscr{C}$ be enriched, tensored and cotensored over $\mathscr{E}$ and consider the natural bijections $\mathscr{C}(X \times K, Y) \cong \mathscr{E}(K, \mathrm{Map}(X, Y)) \cong \mathscr{C}(X, Y^K)$. If we fix two out of the three objects X, Y and K, we can invoke the Yoneda lemma ([Mac78, Section III.2]) to show that the enrichment, tensor and cotensor determine each other uniquely up to isomorphism.

Example 7.11. In a bicomplete category $\mathscr{C}$, the Hom-set and (co)product provide the necessary enrichment and (co)power over **Set** as

$$\mathrm{Map}(X, Y) = \mathscr{C}(X, Y), \qquad X \times S = \coprod_{s \in S} X, \qquad X^S = \prod_{s \in S} X.$$

Example 7.12. Consider $(\mathbf{Top}, \times, *)$. Then Definition 2.15 coincides with Definition 7.10.

Example 7.13 ([Bor94, Proposition 6.5.3])**.** Let $(\mathscr{E}, \otimes, 1)$ be a closed symmetric monoidal category. Then $\mathscr{E}$ is enriched, tensored and cotensored over itself with the monoidal product $(-) \times (-)$ the tensor and the internal-hom $[-, -]$ the enrichment and cotensor. This holds in particular for monoidal model categories.

We focus on the monoidal model categories **sSet** and $\mathbf{sSet}_*$ of simplicial and pointed simplicial sets. Given simplicial sets K and L the internal-hom in **sSet** is the simplicial set $[K, L]$ with set of p-simplices $[K, L]_p = \mathbf{sSet}(K \times \Delta[p], L)$, as shown in [Hov99, Section 3.1] after Remark 3.1.7. If $\mathscr{C}$ is enriched, tensored and cotensored over $\mathbf{sSet}$ we obtain natural bijections

$$\mathscr{C}(X \times K, Y) \cong \mathbf{sSet}(K, \mathrm{Map}(X, Y)) \cong \mathscr{C}(X, Y^K).$$

Fix X and Y and let $K = \Delta[p]$. The bijections above induce the isomorphisms of simplicial sets

$$\begin{aligned}\mathrm{Map}(X, Y) &\cong [*, \mathrm{Map}(X, Y)] \cong \mathbf{sSet}(\Delta[-], \mathrm{Map}(X, Y)) \\ &\cong \mathscr{C}(X \times \Delta[-], Y) \\ &\cong \mathscr{C}(X, Y^{\Delta[-]}),\end{aligned}$$

where the first isomorphism comes from the adjunction of the monoidal product and internal-hom in $\mathbf{sSet}$ induced by the terminal object $*$. Thus given the tensor or cotensor, we can explicitly construct the enrichment. A completely analogous argument holds for the pointed simplicial sets.

Definition 7.14. Let $\mathscr{C}$ be a model category enriched, tensored and cotensored over a monoidal model category $(\mathscr{E}, \otimes, 1)$. Then $\mathscr{C}$ is an *$\mathscr{E}$-model category* if the tensor satisfies the pushout product property in Definition 4.9 and if for every cofibrant object X in $\mathscr{C}$ the morphism $X \times \widetilde{1} \to X \times 1$, induced by the cofibrant replacement $\widetilde{1} \xrightarrow{\sim} 1$, is a weak equivalence.

Example 7.15. Topological model categories as in Definition 4.24 are precisely **Top**-model categories since the terminal object $*$ in **Top** is cofibrant. The category of pointed spaces $(\mathbf{Top}_*, \wedge, * \coprod *)$ is a monoidal model category by Theorem 7.4. We define *pointed topological model categories* to be $\mathbf{Top}_*$-model categories.

Example 7.16. A monoidal model category $\mathscr{C}$ is a $\mathscr{C}$-model category.

Topological model categories have played an essential role in the previous chapters. Definition 7.14 now allows us to define the simplicial variants.

Definition 7.17. *Simplicial model categories* are **sSet**-model categories and *pointed simplicial model categories* are $\mathbf{sSet}_*$-model categories.

Theorem 7.18 ([Hov99, Proposition 4.2.19])**.** *If $\mathscr{C}$ is a simplicial model category, then $\mathscr{C}_*$ is a pointed simplicial model category.*

If $\mathscr{C}$ is a based category, then $\mathscr{C} \cong \mathscr{C}_*$. Thus if a simplicial model category $\mathscr{C}$ is based, with zero object 0, then it is also a pointed simplicial model category. It is valuable to see how the enrichment, tensor and cotensor are constructed over $\mathbf{sSet}_*$. Let $X, Y \in \mathrm{Ob}\,\mathscr{C}$ and $(K, k) \in \mathrm{Ob}\,\mathbf{sSet}_*$ and let $\mathrm{Map}(X, Y)$, $X \times K$ and X^K denote the enrichment, tensor and cotensor over $\mathbf{sSet}$. Since $\mathscr{C}$ is based $\mathrm{Map}(X, Y)$ comes equipped with a basepoint in $\mathrm{Map}_0(X, Y) \cong \mathscr{C}(X, Y)$ which uniquely sends X to Y via the zero object 0. As a pointed simplicial set, $\mathrm{Map}(X, Y)$ also provides the enrichment over $\mathbf{sSet}_*$. The tensor and cotensor over $\mathbf{sSet}_*$, denoted as $X \times (K, k)$ and $X^{(K,k)}$, are the pushout and pullback

$$\begin{array}{ccc} X \times * & \xrightarrow{X \times k} & X \times K \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \times (K, k) \end{array} \qquad\qquad \begin{array}{ccc} X^{(K,k)} & \longrightarrow & X^K \\ \downarrow & & \downarrow{\scriptstyle X^k} \\ 0 & \longrightarrow & X^* \end{array}$$

respectively.

The category $\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure is a simplicial model category. Let X and Y be $\mathcal{V}$-spaces and K a simplicial set. Define the tensor and cotensor as $(X \times K)_n = X_n \times |K|$ and $(Y^K)_n = Y_n^{|K|}$. Then define the enrichment as $\mathrm{Map}_p(X, Y) = \mathbf{Top}^{\mathcal{V}}(X \times \Delta[p], Y)$.

Theorem 7.19. *The category $\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure is a simplicial model category.*

Proof. Let $X \in \mathrm{Ob}\,\mathbf{Top}^{\mathcal{V}}, K, L \in \mathrm{Ob}\,\mathbf{sSet}$. We have $|K|\times|L| \cong |K\times L|$ by [Hov99, Lemma 3.1.8] and $|*| \cong *$ since the realization of a constant simplicial space is homeomorphic to the underlying space. Thus we have $(X \times K) \times L \cong X \times (K \times L)$ and $X \times * \cong X$, since $\mathbf{Top}^{\mathcal{V}}$ is topological by Proposition 5.21. Note that the simplicial tensor and cotensor are just the topological tensor and cotensor after applying the realization. Therefore the functor $(-) \times K : \mathbf{Top}^{\mathcal{V}} \to \mathbf{Top}^{\mathcal{V}}$ is left adjoint. Then $\mathbf{Top}^{\mathcal{V}}$ is enriched, tensored and cotensored over $\mathbf{sSet}$ by [GJ09, Lemma I.2.4]. The terminal simplicial set $*$ is cofibrant by Theorem 7.3. Therefore we only need to check the pushout product property for the tensor. Let $f \in \mathrm{Mor}\,\mathbf{Top}^{\mathcal{V}}$ and $i \in \mathrm{Mor}\,\mathbf{sSet}$. Then $f\Box i = f\Box|i|$. Since the realization preserves (acyclic) cofibrations by Theorem 4.32, the pushout product property is satisfied since $\mathbf{Top}^{\mathcal{V}}$ is again topological by Proposition 5.21. □

7.2. The lift to commutative monoids

Recall that $(\mathbf{Top}^{\mathcal{V}}, \boxtimes, *)$ is a closed symmetric monoidal category. We aim to lift the positive $\mathcal{V}$-model structure to the category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ of commutative $\mathcal{V}$-space monoids. Consider the isometries

$$i_1 : \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^m, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0),$$
$$i_2 : \mathbb{R}^m \to \mathbb{R}^n \oplus \mathbb{R}^m, (x_1, \ldots, x_m) \mapsto (0, \ldots, 0, x_1, \ldots, x_m).$$

Given $\mathcal{V}$-spaces X and Y, the isometries i_1 and i_2 induce a morphism of $\mathcal{V} \times \mathcal{V}$-spaces $(X, Y) \to (X \times Y)(- \oplus -)$ defined by

$$X_n \times Y_m \xrightarrow{(i_1,i_2)} X_{n\oplus m} \times Y_{n\oplus m}.$$

Let

$$\rho_{X,Y} : X \boxtimes Y \to X \times Y$$

be the morphism of $\mathcal{V}$-spaces, given by the universal property of the box product.

Theorem 7.20 ([Sch18, Theorem 1.3.2, Proposition 1.4.7])**.** *Let X and Y be $\mathcal{V}$-spaces.*

1. *The morphism $\rho_{X,Y} : X \boxtimes Y \to X \times Y$ is a $\mathcal{V}$-equivalence.*
2. *The functor $X \boxtimes -$ preserves $\mathcal{V}$-equivalences.*

Proof. By Theorem 6.13, the theorem can be restated using $\mathcal{F}_{\mathrm{triv}}$-equivalences instead of $\mathcal{V}$-equivalences. The proof is then analogous to the proof of [Sch18, Theorem 1.3.2]. Alternatively, statement 1. is a direct corollary of its analogue in [Sch18, Theorem 1.3.2] if one works with Schwede's definition of a 'global equivalence'. In that case, statement 2. is a consequence of [Sch18, Proposition 1.4.7(xiv)]. □

Let X be a $\mathcal{V}$-space. The subset $\mathrm{Aut}(X) \subset \mathbf{Top}^{\mathcal{V}}(X,X)$ of isomorphisms is a group. The symmetry isomorphism $b_{X,X} : X \boxtimes X \cong X \boxtimes X$ induces an action of Σ_n on $X^{\boxtimes n}$. Thus we get a quotient $\mathcal{V}$-space $X^{\boxtimes n}/\Sigma_n$. We obtain a commutative $\mathcal{V}$-space monoid

$$\mathbb{P}(X) = \coprod_{n \geq 0} X^{\boxtimes n}/\Sigma_n.$$

This induces a functor $\mathbb{P} : \mathbf{Top}^{\mathcal{V}} \to \mathcal{C}\mathbf{Top}^{\mathcal{V}}$ that is left adjoint to the forgetful functor $\mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathbf{Top}^{\mathcal{V}}$, as shown in [Sch18, Example 2.1.5]. Along this adjunction, we lift the positive $\mathcal{V}$-model structure. This invokes the use of [Whi17, Theorem 3.2], which requires a long list of hypotheses we will not all state here. In our case, these have all been checked in [Sch18, Theorem 2.1.15(i)].

Definition 7.21. Let $f : M \to N$ be a morphism of commutative $\mathcal{V}$-space monoids. It is a *$\mathcal{V}$-equivalence* (*positive $\mathcal{V}$-fibration*) if the underlying morphism of $\mathcal{V}$-spaces is a $\mathcal{V}$-equivalence (positive $\mathcal{V}$-fibration). It is a *cofibration* if it has the LLP with respect to all $\mathcal{V}$-equivalences that are also positive $\mathcal{V}$-fibrations.

Theorem 7.22 (Positive $\mathcal{V}$-model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, [Sch18, Theorem 2.1.15], [Whi17, Theorem 3.2])**.** *The $\mathcal{V}$-equivalences, positive $\mathcal{V}$-fibrations and cofibrations are part of a cofibrantly generated, left proper model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. A cofibration in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ forgets to an h-cofibration in $\mathbf{Top}^{\mathcal{V}}$.*

The left properness of this model structure is necessary to construct the group completion model structure. Cofibrant commutative $\mathcal{V}$-space monoids being h-cofibrant in $\mathbf{Top}^{\mathcal{V}}$ allows us to replace a fat realization with an ordinary realization later on. The terminal $\mathcal{V}$-space $* \cong \mathcal{V}(0,-)$ is a commutative $\mathcal{V}$-space monoid. Since it is also the identity object with respect to the boxproduct, every commutative $\mathcal{V}$-space monoid M comes with a unique morphism $* \to M$. Therefore $*$ is a zero object making $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ based. Let $\{M_s\}_{s \in S}$ denote a family of commutative $\mathcal{V}$-space monoids, for some set S. If S is finite, we let $\boxtimes_{s \in S} M_s$ denote the iterated boxproduct. It is equal to $*$ if S is empty. The associator makes this unique up to isomorphism. If S is infinite we let the infinite boxproduct be $\boxtimes_{s \in S} M_s = \mathrm{colim}_{S' \subset S, S' \text{finite}}(\boxtimes_{s \in S'} M_s)$. By [Sch18, Example 2.2.22] the boxproduct $\boxtimes_{s \in S} M_s$ is the coproduct of the family $\{M_s\}_{s \in S}$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$.

Let M be a commutative $\mathcal{V}$-space monoid then a map of sets $f : S \to S'$ induces a morphism $f_* : \boxtimes_S M \to \boxtimes_{S'} M$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ as follows. First, consider the isomorphism

$$\boxtimes_S M \cong \boxtimes_{s' \in S'}(\boxtimes_{s \in f^{-1}(s')} M).$$

The $f^{-1}(j)$-fold multiplication $\boxtimes_{s \in f^{-1}(s')} M \to M$, which is the basepoint of M if $f^{-1}(j)$ is empty and the identity if it is a singleton, then induces the morphism $\boxtimes_S M \to \boxtimes_{S'} M$. If K is a simplicial set we write $\boxtimes_K M$ for the functor $[p] \mapsto \boxtimes_{K_p} M$ which is an object in $[\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}]$. Since an object X in $[\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}]$ is a simplicial $\mathcal{V}$-space, its realization $|X|$ is a $\mathcal{V}$-space. Alternatively, their exists an *internal realization* $|X|_{\mathrm{in}}$, in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, of an object X in $[\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}]$, as given before [Sch18, Proposition 2.1.7]. [Sch18, Proposition 2.1.7] shows that $|X|$ and $|X|_{\mathrm{in}}$ are isomorphic as $\mathcal{V}$-spaces. Therefore $|X|$ has the structure of a commutative $\mathcal{V}$-space monoid.

We will now construct the simplicial structures on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. Let M and N be commutative $\mathcal{V}$-space monoids and K a simplicial set. The cotensor $(N^K)_n = N_n^{|K|}$ in $\mathbf{Top}^{\mathcal{V}}$ comes with the structure of a commutative $\mathcal{V}$-space monoid and is taken to be the cotensor over $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. Multiplication is defined by

$$\begin{aligned} \mathbf{Top}(|K|, N_m) \times \mathbf{Top}(|K|, N_n) &\to \mathbf{Top}(|K|, N_{n \oplus m}), \\ (f,g) &\mapsto \mu_{n,m} \circ (f,g), \end{aligned}$$

with μ the multiplication of N. The basepoint of $N_0^{|K|}$ is the constant map from $|K|$ to the basepoint of N_0. Unfortunately, the tensor in $\mathbf{Top}^{\mathcal{V}}$ does not come with the structure of a commutative monoid. In $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, we define the tensor as the realization $M \otimes K = |\boxtimes_K M|$. The enrichment is defined as $\mathrm{Map}_p(M, N) = \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M \otimes \Delta[p], N)$.

We need the following adjunctions to prove that the functors above make $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ a simplicial model category. If we equip the singular complex $\mathrm{Sing}_p(A) = \mathbf{Top}(\Delta^p, A)$ of a space A with the compact-open topology, it defines simplicial space, and we obtain the adjunction

$$|-| : [\Delta^{\mathrm{op}}, \mathbf{Top}] \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top} : \mathrm{Sing}\,. \tag{7.4}$$

If we let $\mathrm{Sing}(X)$ be the simplicial $\mathcal{V}$-space defined by $(\mathrm{Sing}_p(X))_n = \mathrm{Sing}_p(X_n)$, for any $\mathcal{V}$-space X, then we get another adjunction

$$|-| : [\Delta^{\mathrm{op}}, \mathbf{Top}^{\mathcal{V}}] \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top}^{\mathcal{V}} : \mathrm{Sing}\,.$$

Note that for a commutative $\mathcal{V}$-space monoid M we have $\mathrm{Sing}_p(M) = M^{\Delta[p]}$ which is again a commutative $\mathcal{V}$-space monoid. Therefore we finally obtain the adjunction

$$|-| : [\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}] \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{C}\mathbf{Top}^{\mathcal{V}} : \mathrm{Sing}\,. \tag{7.5}$$

Theorem 7.23. *The category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure is a simplicial model category.*

Proof. Let M and N be commutative $\mathcal{V}$-space monoids, and let K be a simplicial set. Since the cotensor in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is defined on the underlying category $\mathbf{Top}^{\mathcal{V}}$ we only need to show by [GJ09, Lemma 2.4] and [Hov99, Lemma 4.2.2] that the functor $(-)\otimes K : \mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is left adjoint to $(-)^K : \mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathcal{C}\mathbf{Top}^{\mathcal{V}}$. We have the following chain of bijections

$$\mathcal{C}\mathbf{Top}^{\mathcal{V}}(\boxtimes_{K_p} M, N^{\Delta[p]}) \cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, \prod_{K_p} N^{\Delta[p]}) \cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, \mathbf{Top}(K_p, N^{\Delta[p]})).$$

The left bijection follows from the fact that the boxproduct is the coproduct in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. Equipping K_p with the discrete topology gives the right bijection. We then get

$$\begin{aligned}\prod_{[p]\in\Delta^{\mathrm{op}}} \mathcal{C}\mathbf{Top}^{\mathcal{V}}(\boxtimes_{K_p} M, N^{\Delta[p]}) &\cong \prod_{[p]\in\Delta^{\mathrm{op}}} \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, \mathbf{Top}(K_p, N^{\Delta[p]})) \\ &\cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, \prod_{[p]\in\Delta^{\mathrm{op}}} \mathbf{Top}(K_p, N^{\Delta[p]})).\end{aligned}$$

For a morphism $[q] \to [p]$ in Δ, we get adjoint squares:

$$\begin{array}{ccc} \boxtimes_{K_p} M & \longrightarrow & N^{\Delta[p]} \\ \downarrow & & \downarrow \\ \boxtimes_{K_q} M & \longrightarrow & N^{\Delta[q]} \end{array} \qquad \begin{array}{ccc} M & \longrightarrow & \mathbf{Top}(K_p, N^{\Delta[p]}) \\ \downarrow & & \downarrow \\ \mathbf{Top}(K_q, N^{\Delta[q]}) & \longrightarrow & \mathbf{Top}(K_p, N^{\Delta[q]}) \end{array}$$

The left square is commutative if and only if the right square is. This induces a bijections

$$[\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}](\boxtimes_K M, N^{\Delta[-]}) \cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, [\Delta^{\mathrm{op}}, \mathbf{Top}](K, N^{\Delta[-]})).$$

The result follows from the bijections

$$\mathcal{C}\mathbf{Top}^{\mathcal{V}}(M \otimes K, N) \cong [\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}](\boxtimes_K M, N^{\Delta[-]}), \tag{7.6}$$

$$\mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, \mathbf{Top}(|K|, N)) \cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, [\Delta^{\mathrm{op}}, \mathbf{Top}](K, N^{\Delta[-]})). \tag{7.7}$$

Here (7.6) comes from the adjunction (7.5), and (7.7) comes from the adjunction (7.4) by evaluating at $n \geq 0$. □

Theorem 7.24. *The category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure is a pointed simplicial model category.*

Proof. In $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ the identity object $\mathcal{V}(0, -) \cong *$ is both initial and terminal. Therefore $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is based and isomorphic to $(\mathcal{C}\mathbf{Top}^{\mathcal{V}})_*$. Theorem 7.23 and Theorem 7.18 finish the proof. □

Let $M \in \mathrm{Ob}\,\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ and $K_* = (K, k) \in \mathrm{Ob}\,\mathbf{sSet}_*$. The forgetful functor $\mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathbf{Top}^{\mathcal{V}}$ creates all limits by [Sch18, corollary 2.1.4]. Therefore the pointed simplicial cotensor in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is defined in the underlying category $\mathbf{Top}^{\mathcal{V}}$ where pullbacks are defined level-wise. Hence we find that $(M^{K_*})_n$ is the space $\mathbf{Top}_*((|K|, |k|), M_n)$ of basepoint preserving maps from $|K|$ to M_n.

Let $[p] \in \Delta$ and $S_p = K_p \setminus \mathrm{im}(k_p)$ be the set K_p without its basepoint. The following diagram of commutative $\mathcal{V}$-space monoids is a pushout diagram.

$$\begin{array}{ccc} M & \xrightarrow{(k_p)_*} & \boxtimes_{K_p} M \\ \downarrow & & \downarrow \\ * & \longrightarrow & \boxtimes_{S_p} M \end{array}$$

since the boxproduct $\boxtimes$ is the coproduct in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. The objects $\boxtimes_{S_p} M$ will form a simplicial object. However, we stress that the sets S_p do not form a simplicial set. A map $\phi : [q] \to [p]$ in Δ induces a map $\phi^* : K_p \to K_q$. The multiplications $\boxtimes_{s' \in (\phi^*)^{-1}(s)} M \to M, s \in S_q$ and the morphism $\boxtimes_{s' \in (\phi^*)^{-1}(k_p)} M \to *$ determine a morphism $\boxtimes_{S_p} M \to \boxtimes_{S_q} M$. Thus we get an object $\boxtimes_S M$ in $[\Delta^{\mathrm{op}}, \mathcal{C}\mathbf{Top}^{\mathcal{V}}]$. Taking the realization preserves pushouts since it is left adjoint. We end up with the following pushout diagram:

$$\begin{array}{ccc} M \otimes * & \xrightarrow{M \otimes k} & M \otimes K \\ \downarrow & & \downarrow \\ * & \longrightarrow & |\boxtimes_S M| \end{array}$$

Hence $M \otimes K_* = |\boxtimes_S M|$.

8. Group completions

Group completions have been defined in several different ways. May's definition in [May74] for H-spaces involves using homology groups. This was motivated by the work of Quillen in [Qui94] and Barrat and Priddy in [BP72]. The map $A \to \Omega(B(A))$, where A is an E_∞-space, B is the classifying space functor, and Ω is the loop space functor, is a group completion as shown in [May74, Theorem 1.6] and [BM05, Theorem 6.5]. Schwede gives a definition for the group completion of commutative $\mathcal{V}$-space monoids in [Sch18, Defintion 2.5.15], and this definition can also be expressed using homology groups as shown in [Sch18, Proposition 2.5.31]. The morphism $\Phi : \mathrm{Gr} \to \mathrm{BOP}$ defined in Example 3.25 is a group completion in the sense of Schwede by [Sch18, Theorem 2.5.33]. After applying the homotopy colimit and taking the set of path-connected components, we obtain the familiar group completion $\mathbb{N} \to \mathbb{Z}$ as shown in [Sch18, Examples 2.3.12, 2.4.2] and [Sch18, Theorem 2.4.13].

Our definition of the group completion of topological monoids involves a variation of the classifying space. This definition and the definitions of the loop and classifying spaces have analogues in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. We show that given a cofibrant commutative $\mathcal{V}$-space monoid M, the morphism $M \to \Omega(B(M)^{\mathcal{V}\text{-fib}})$ is a group completion. To construct group completions for any commutative $\mathcal{V}$-space monoid, we localize the positive $\mathcal{V}$-model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. In this localization, called the group completion model structure, the fibrant replacement is a group completion for every commutative $\mathcal{V}$-space monoid. In the final section, we look at the augmented group completion model structure on the overcategory $\mathcal{C}\mathbf{Top}^{\mathcal{V}}/T$, where T is fibrant in the group completion model structure. We will show that endowing an overcategory with a model structure using Proposition 4.8 commutes with the localization mentioned above.

8.1. The group completion model structure

Consider the functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ that sends a space to its set of path-connected components. The bijections $\pi_0(A \times A') \cong \pi_0(A) \times \pi_0(A')$ and $* \cong \pi_0(*)$ make π_0 a strong monoidal functor. Let A be a *topological monoid*, that is, a monoid in $(\mathbf{Top}, \times, *)$. Then $\pi_0(A)$ is a monoid (in $(\mathbf{Set}, \times, *)$). We call A *grouplike* if $\pi_0(A)$ is a group. Let $B_\bullet(A)$ be the simplicial space with space of p-simplices

$$B_p(A) = \underbrace{A \times \cdots \times A}_{p \text{ times}}.$$

Boundary maps are induced by the multiplication of A and two projections. Degeneracy maps are induced by the unit of A. The realization $B(A) = |B_\bullet(A)|$ is the *classifying space* of A and the fat realization $B^{\mathbf{F}}(A) = ||B_\bullet(A)||$ is the *fat classifying space* of A. A map of topological monoids $A \to A'$ is a *group completion* of A if A' is grouplike and $B^{\mathbf{F}}(A) \to B^{\mathbf{F}}(A')$ is a weak homotopy equivalence.

By definition, the classifying space can be written as the coend $B(A) = \int^{p \in \Delta} B_p(A) \times \Delta^p$. Let $[(a_1, \ldots, a_p), d]$ be a point in $B(A)$ represented by $((a_1, \ldots, a_p), d) \in B_p(A) \times \Delta^p$. If d is a vertex in Δ^p then $[(a_1, \ldots, a_p), d] = [*, *]$, which is the basepoint of $B(A)$ that is also represented by $(*, *) \in B_0(A) \times \Delta^0$. If d is not a vertex in Δ^p then there exists a path from d to a vertex d' in Δ^p that induces a path from $[(a_1, \ldots, a_p), d]$ to $[(a_1, \ldots, a_p), d']$. Thus $B(A)$ is path-connected and $\pi_0(B(A)) \cong *$. An analogous result holds for $B^{\mathbf{F}}$.

Lemma 8.1. *If M is a cofibrant commutative $\mathcal{V}$-space monoid then the unit of $M_{h\mathcal{V}}$ is an h-cofibration.*

Proof. By Theorem 7.22, the unit $* \to M$ is an h-cofibration of $\mathcal{V}$-spaces. Since the homotopy colimit preserves colimits by Corollary 3.12 and tensors by Lemma 3.13, we can argue as in [SS12, Lemma 7.7] and show that it preserves h-cofibrations. We only need to show that the unit $* \to B\mathcal{V}$ is an h-cofibration. The isomorphisms $* \cong \mathcal{V}(0,0)^p$ are clearly h-cofibrations. Therefore $* \to B_p(*, \mathcal{V}, *)$ is an h-cofibration since $B_p(*, \mathcal{V}, *)$ is a disjoint union of $\mathcal{V}(0,0)^p$ and another space. Since realizations preserve colimits and tensors and since these are determined level-wise for simplicial spaces, the map $* \to B\mathcal{V}$ must be an h-cofibration. □

If the unit of a topological monoid A is an h-cofibration, then all degeneracy maps of $B_\bullet(A)$ are h-cofibrations since $A \times (-)$ preserves h-cofibrations. Thus $B_\bullet(A)$ is a 'good' simplicial space in the sense of [Seg74, Definition A.4]. Applying [Seg74, Proposition A.1(iv)] we get a weak homotopy equivalence $B^{\mathbf{F}}(A) \xrightarrow{\sim} B(A)$. Thus if the units of A and A' are h-cofibrations, then $A \to A'$ is a group completion if A' is grouplike and $B(A) \to B(A')$ is a weak homotopy equivalence.

Lemma 8.2. *If $A \to A'$ is a weak homotopy equivalence between topological monoids, then $B^{\mathbf{F}}(A) \to B^{\mathbf{F}}(A')$ is a weak homotopy equivalence. If the units of A and A' are h-cofibrations, then $B(A) \to B(A')$ is a weak homotopy equivalence.*

Proof. The fat realization is a homotopy colimit, as seen in [Seg74, Appendix A]. It, therefore, preserves weak equivalences. The second claim follows from the weak homotopy equivalence $B^{\mathbf{F}}(A) \xrightarrow{\sim} B(A)$. □

By combining these two lemmas, a $\mathcal{V}$-equivalence $M \to N$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ induces a weak homotopy equivalence $B^{\mathbf{F}}(M_{h\mathcal{V}}) \xrightarrow{\sim} B^{\mathbf{F}}(N_{h\mathcal{V}})$ and, if M and N are cofibrant, a weak homotopy equivalence $B(M_{h\mathcal{V}}) \xrightarrow{\sim} B(N_{h\mathcal{V}})$.

Proposition 8.3. *The homotopy colimit functor* $\operatorname{hocolim}_{\mathcal{V}}$ *is a monoidal functor.*

Proof. The proof of [Sch09, Proposition 4.17] can be adapted for the index category $\mathcal{V}$ since it has an initial object. The accompanying natural transformation has as components

$$\mu_{X,Y} : X_{h\mathcal{V}} \times Y_{h\mathcal{V}} \xrightarrow{\cong} (X,Y)_{h(\mathcal{V}\times\mathcal{V})} \to ((-\oplus-)^* X \boxtimes Y)_{h(\mathcal{V}\times\mathcal{V})} \to (X \boxtimes Y)_{h\mathcal{V}} \qquad (8.1)$$

for $\mathcal{V}$-spaces X and Y. Here the first map follows from [HV92, Proposition 3.1(4)]. The second map follows from the boxproduct being constructed as a left Kan extension. The third map is an example of (3.4). □

The map $\mu_{X,Y}$ is a weak homotopy equivalence: Consider the following commutative diagram

$$\begin{array}{ccccc} X_{h\mathcal{V}} \times Y_{h\mathcal{V}} & \xrightarrow{\cong} & (X,Y)_{h(\mathcal{V}\times\mathcal{V})} & \longrightarrow & (X \boxtimes Y)_{h\mathcal{V}} \\ \downarrow & & & & \downarrow \\ \operatorname{colim}_{\mathcal{V}} X \times \operatorname{colim}_{\mathcal{V}} Y & \xrightarrow{\cong} & \operatorname{colim}_{\mathcal{V}\times\mathcal{V}}(X,Y) & \xrightarrow{\cong} & \operatorname{colim}_{\mathcal{V}}(X \boxtimes Y) \end{array}$$

where the upper composite is $\mu_{X,Y}$ and the lower right horizontal map is an isomorphism by [BCS10, Lemma 8.8]. If X and Y are cofibrant, then $X \boxtimes Y$ is cofibrant by the pushout product property given by Proposition 5.6. Hence the vertical maps are weak homotopy equivalences by Theorem 5.12, making $\mu_{X,Y}$ a weak homotopy equivalence. For general X and Y let $\widetilde{X} \xrightarrow{\sim} X$ and $\widetilde{Y} \xrightarrow{\sim} Y$ be cofibrant replacements. Then by Theorem 7.20 $\widetilde{X} \boxtimes \widetilde{Y} \xrightarrow{\sim} X \boxtimes Y$ is a $\mathcal{V}$-equivalence. Thus $\mu_{X,Y}$ is a weak homotopy equivalence since $\mu_{\widetilde{X},\widetilde{Y}}$ is.

Let M be a commutative $\mathcal{V}$-space monoid. The set of path-connected components $\pi_0(M_{h\mathcal{V}})$ is a monoid since both π_0 and $\operatorname{hocolim}_{\mathcal{V}}$ are monoidal functors. The diagram

$$\begin{array}{ccccc} M_{h\mathcal{V}} \times M_{h\mathcal{V}} & \xrightarrow{\mu_{M,M}} & (M \boxtimes M)_{h\mathcal{V}} & \longrightarrow & M_{h\mathcal{V}} \\ \downarrow{\scriptstyle b_{M_{h\mathcal{V}},M_{h\mathcal{V}}}} & & \downarrow{\scriptstyle (b_{M,M})_*} & \nearrow & \\ M_{h\mathcal{V}} \times M_{h\mathcal{V}} & \xrightarrow{\mu_{M,M}} & (M \boxtimes M)_{h\mathcal{V}} & & \end{array}$$

is, in general, not commutative. Consider the additive Grassmannian Gr, for example. It is commutative up to homotopy. Thus $\pi_0(M_{h\mathcal{V}})$ is a commutative monoid.

Definition 8.4. A commutative $\mathcal{V}$-space monoid M is *grouplike* if the monoid $M_{h\mathcal{V}}$ is grouplike, that is, if $\pi_0(M_{h\mathcal{V}})$ is a group. A morphism $M \to N$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is a *group completion* if the map $M_{h\mathcal{V}} \to N_{h\mathcal{V}}$ is a group completion.

We will now construct the loop space using the simplicial 1-sphere, which, together with the classifying space, will motivate our definition of group completion. The simplicial 1-sphere is $S^1 = \Delta[1]/\partial\Delta[1]$. We identify $\Delta[1]_p = \Delta([p],[1])$ with $\{0,\dots,p+1\}$ by sending $\phi : [p] \to [1]$ to the cardinality of the preimage of $0 \in [1]$. The maps in $\partial\Delta[1]_p$ are the constant maps identified with 0 and $p+1$. Then S^1 is the simplicial set whose set of p-simplices S^1_p can be identified with $[p] = \{0,\dots,p\}$. Boundaries d_i and degeneracies s_i are then given by

$$d_i(k) = \begin{cases} k, & \text{if } k = 0,\dots,i \\ k-1, & \text{if } k = i+1,\dots,p \end{cases} \qquad d_p(k) = \begin{cases} k, & \text{if } k = 0,\dots,p-1 \\ 0, & \text{else} \end{cases}$$

$$s_j(k) = \begin{cases} k, & \text{if } k = 0,\dots,j \\ k+1, & \text{if } k = j+1,\dots,p \end{cases}$$

for $i = 0,\dots,p-1$ and $j = 0,\dots,p$. Note $d_i(0) = 0 = s_i(0)$ for all p and i. Also, note that since $S^1_0 = [0]$, there exists a unique basepoint $s : * \to S^1$. Let $S^1_* = (S^1, s)$ denote the pointed simplicial 1-sphere. We remark that the realization $|S^1|$ is the topological 1-sphere and $|S^1_*| = (|S^1|, |s|)$ is the pointed topological 1-sphere. The *loop space* of a based space A is the based space $\Omega(A) = \mathbf{Top}_*(|S^1_*|, A)$ of basepoint preserving maps from the 1-sphere to A. Note that the loop space is defined using the internal-hom of $\mathbf{Top}_*$ and is, therefore, a functor right adjoint to the functor $A \mapsto A \wedge |S^1_*|$ defined by the *suspension* of a based space A.

Let A be a topological monoid with unit $a \in A$. Again expressing the classifying space as the coend $B(A) = \int^{p\in\Delta} B_p(A) \times \Delta^p$, we obtain a map $A \times |\Delta[1]| \cong B_1(A) \times \Delta^1 \to B(A)$. The subspaces $A \times |\partial\Delta[1]|$ and $a \times |\Delta[1]|$ map to the basepoint of $B(A)$. Thus our map factors as a quotient map and a map $A \wedge |S^1_*| \to B(A)$. The *group completion map* of A

$$\eta_A : A \to \Omega(B(A))$$

is the adjoint of this map. If A is grouplike, this is a weak homotopy equivalence by [Hat14, Lemma D.2]. If A is discrete, this will induce the ordinary group completion (in **Set**). Let

$f : A \to A'$ be a map of topological monoids whose units are h-cofibrations. Since classifying spaces are path-connected, $B(f)$ being a weak homotopy equivalence is equivalent to $\Omega(B(f))$ being a weak homotopy equivalence. This is because $\pi_n(\Omega(A)) \cong \pi_{n+1}(A)$. Therefore if f is a group completion, we have a chain of weak homotopy equivalences $A' \xrightarrow{\sim} \Omega(B(A') \xleftarrow{\sim} \Omega(B(A))$. This motivates our definition of group completion.

The loop space and classifying space have analogues in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. Fix a pointed simplicial set (K,k). Then the tensor and cotensor define left and right adjoint functors, respectively, such that

$$\mathcal{C}\mathbf{Top}^{\mathcal{V}}(M \otimes (K,k), N) \cong \mathcal{C}\mathbf{Top}^{\mathcal{V}}(M, N^{(K,k)}).$$

Definition 8.5. The *loop functor* $\Omega : \mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is the functor defined by the cotensor of a commutative $\mathcal{V}$-space monoid with the S^1_*. We get $\Omega(M) = M^{S^1_*}$ for $M \in \mathrm{Ob}(\mathcal{C}\mathbf{Top}^{\mathcal{V}})$.

Note that $(\Omega M)_n$ is the loop space $\Omega(M_n)$ of the based space M_n.

Definition 8.6. The *bar construction* $B : \mathcal{C}\mathbf{Top}^{\mathcal{V}} \to \mathcal{C}\,\mathbf{Top}^{\mathcal{V}}$ is the functor defined by the tensor of a commutative $\mathcal{V}$-space monoid with S^1_*. We get $B(M) = M \otimes S^1_*$ for $M \in \mathrm{Ob}(\mathcal{C}\mathbf{Top}^{\mathcal{V}})$.

From the discussion after Theorem 7.24 we know that $B(M)$ is the realization of some simplicial object $B_\bullet(M)$ with $B_p(M) = \boxtimes_{K_p} M$, where $K_p = S^1_p \backslash [0] = \{1, \dots, p\}$. We could therefore write $B_p(M) = M_1 \boxtimes \cdots \boxtimes M_p$ with $M_i = M$ for $i = 1, \dots, p$. A boundary $d_i : B_p(M) \to B_{p-1}(M), i = 1, \dots, p-1$, is given by the multiplication $M_i \boxtimes M_{i+1} \to M_i$. A boundary $d_i : B_p(M) \to B_{p-1}(M)$, with $i = 0$ or $i = p$, is given by the terminal map $M_i \to *$. A degeneracy $s_i : B_p(M) \to B_{p+1}(M), i = 0, \dots, p$, is given by a unit morphism $* \to M_i$.

Corollary 8.7. *The functors B and Ω form a Quillen adjunction $(B \dashv \Omega)$.*

Proof. By Theorem 7.24, the category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ is a pointed simplicial model category. Then the functor B preserves cofibrations and acyclic cofibrations by the pushout product property since S^1_* is cofibrant. □

Let M be a commutative $\mathcal{V}$-space monoid and let $B(M) \to B(M)^{\mathcal{V}\text{-fib}}$ be a positive $\mathcal{V}$-fibrant replacement in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. The unit of the adjunction $(B \dashv \Omega)$ is $M \to \Omega(B(M))$. Let the *derived unit* be the composite morphism

$$\eta^{\mathcal{V}}_M : M \to \Omega(B(M)) \to \Omega(B(M)^{\mathcal{V}\text{-fib}}) = \Gamma(M).$$

We will show that the derived unit is a group completion if M is cofibrant. For this, we need two lemmas.

For a based $\mathcal{K}$-space X let $X_{h^*\mathcal{K}} = X_{h\mathcal{K}}/B\mathcal{K} \in \mathrm{Ob}\,\mathbf{Top}_*$.

Lemma 8.8 ([SS13, Propositions 4.2, 4.6, 4.10, 4.12])**.** *Let M be a cofibrant commutative $\mathcal{V}$-space monoid.*

(1) There exists a space A and a chain of natural weak homotopy equivalences

$$B(M)_{h\mathcal{V}} \xleftarrow{\sim} A \xrightarrow{\sim} B(M_{h\mathcal{V}}).$$

(2) There exists a weak homotopy equivalence

$$\Omega(M)_{h^*\mathcal{V}} \xrightarrow{\sim} \Omega(M_{h^*\mathcal{V}}).$$

(3) There exists a chain of natural weak homotopy equivalences relating $\Omega(B(M)^{\mathcal{V}\text{-fib}})_{h\mathcal{V}}$ and $\Omega(B(M_{h\mathcal{V}}))$ such that the diagram

$$\begin{array}{ccc} & M_{h\mathcal{V}} & \\ {\scriptstyle (\eta^{\mathcal{V}}_M)_{h\mathcal{V}}} \swarrow & & \searrow {\scriptstyle \eta_{M_{h\mathcal{V}}}} \\ \Omega(B(M)^{\mathcal{V}\text{-fib}})_{h\mathcal{V}} & \overset{\sim}{\dashrightarrow} & \Omega(B(M_{h\mathcal{V}})) \end{array}$$

is commutative up to homotopy, where the dotted arrow is the chain of weak homotopy equivalences mentioned above.

(4) If M is grouplike, then $\eta^{\mathcal{V}}_M$ is a $\mathcal{V}$-equivalence.

This lemma is the $\mathbf{Top}^{\mathcal{V}}$ analogue of [SS13, Propositions 4.2, 4.6, 4.10, 4.12]. Care has to be taken when switching from $\mathbf{sSet}^{\mathcal{I}}$ to $\mathbf{Top}^{\mathcal{I}}$ as explained in [SS13, Appendix C1], and when switching to $\mathbf{Top}^{\mathcal{V}}$. Various conditions in [SS13] can be dropped. We summarize this in the proof.

Proof. The 'flatness' condition in [SS13, Propositions 4.2, 4.10] can be dropped since the functor $X \boxtimes (-)$ preserves $\mathcal{V}$-equivalences for all $\mathcal{V}$-spaces X by Theorem 7.20, making all maps (8.1) weak homotopy equivalence. The 'semistability' conditions in [SS13, Proposition 4.6] can be dropped since the map $X_{h\mathcal{N}} \to X_{h\mathcal{V}}$ is a weak homotopy equivalence for all $\mathcal{V}$-spaces X by Lemma 6.12. The 'level-wise fibrant' condition in [SS13, Proposition 4.6] and the fibrant replacements in [SS13, Propositions 4.6, 4.10] can be dropped since every space is fibrant. □

Lemma 8.9. *Let M be a cofibrant commutative $\mathcal{V}$-space monoid. Then ΩM is grouplike.*

Proof. Since $B\mathcal{V}$ is contractible $\Omega(M)_{h\mathcal{V}} \to \Omega(M)_{h^*\mathcal{V}}$ induces a bijection on the sets of path-connected components, compatible with the monoidal structures. By Lemma 8.8 (2) the map $\omega : \Omega(M)_{h^*\mathcal{V}} \xrightarrow{\sim} \Omega(M_{h^*\mathcal{V}})$ is a weak homotopy equivalence. It is compatible with the monoidal structures: An element in $M_{h^*\mathcal{V}}$ is represented by $(\phi, m, d)_{p,k}$ with ϕ a sequence of $p+1$ composable isometries in $\mathcal{V}$, $m \in M_k$ and $d \in \Delta^p$ for some p and k. Let $(\phi, m, d)_{p,k}$ and $(\phi', m', d')_{p',k'}$ represent two elements a and b in $M_{h^*\mathcal{V}}$, let $\psi = (\phi_p \times \mathrm{id}, \ldots, \phi_0 \times \mathrm{id}, \mathrm{id} \times \phi'_p \ldots, \mathrm{id} \times \phi'_0)$ and consider the multiplication $\mu_{k,k'} : M_k \times M_{k'} \to M_{k \oplus k'}$ and $\mu^*_{M,M} : M_{h^*\mathcal{V}} \times M_{h^*\mathcal{V}} \to M_{h^*\mathcal{V}}$. Then $\mu_{M,M}(a,b) = [\psi, \mu_{k,k'}(m,m'), (d,d')]$. Now $\mu_{M,M}$ induces the maps $\mu_{(\Omega M),(\Omega M)}$ and $\Omega(\mu_{M,M})$. Let $x = [(\phi, f, d)_{p,k}], y = [(\phi', f', d')_{p',k'}] \in \Omega(M)_{h^*\mathcal{V}}$, then $\omega(x) : t \mapsto [(\phi, f(t), d)_{p,k}]$. The diagonal map $|S^1_*| \to |S^1_*| \times |S^1_*|$ induces the map $\gamma : \Omega(M_{h^*\mathcal{V}}) \times \Omega(M_{h^*\mathcal{V}}) \to \Omega(M_{h^*\mathcal{V}} \times M_{h^*\mathcal{V}})$. Showing that ω is compatible with the monoidal structures comes down to showing $\Omega(\mu_{M,M}) \circ \gamma \circ (\omega \times \omega) = \omega \circ \mu_{(\Omega M),(\Omega M)}$. For every $t \in |S^1_*|$ we have

$$\begin{aligned}(\Omega(\mu_{M,M}) \circ \gamma \circ (\omega \times \omega))(x,y)(t) =& \mu_{M,M}(\omega(x)(t), \omega(y)(t)) \\ =& [\psi, \mu_{k,k'} \circ (f(t), f'(t)), (d,d')] \\ =& \omega \circ \mu_{\Omega M, \Omega M}(x,y)(t).\end{aligned}$$

We obtain a chain of bijections

$$\pi_0(\Omega(M)_{h\mathcal{V}}) \cong \pi_0(\Omega(M)_{h^*\mathcal{V}}) \cong \pi_0(\Omega(M_{h^*\mathcal{V}})) \cong \pi_1(M_{h^*\mathcal{V}}, \{B\mathcal{V}\}).$$

The left and middle bijections preserve the monoidal structures induced by M. The right bijection then defines a monoidal operation on $\pi_1(M_{h^*\mathcal{V}}, \{B\mathcal{V}\})$. This operation takes two loops in $M_{h^*\mathcal{V}}$ and multiplies them pointwise using $\mu_{M,M}$. This then commutes with concatenation which defines the usual group operation on fundamental groups. Thus the monoidal structure coincides with the group structure of $\pi_1(M_{h^*\mathcal{V}}, \{B\mathcal{V}\})$ by an Eckmann-Hilton argument making $\pi_0(\Omega(M)_{h\mathcal{V}})$ a group and making ΩM grouplike. □

The following theorem gives us group completions for cofibrant objects. It is the analogue of [SS13, Theorem 1.2].

Theorem 8.10. *Let M be a cofibrant commutative $\mathcal{V}$-space monoid. Then the derived unit $\eta_M^{\mathcal{V}}$ is a group completion.*

Proof. By Lemma 8.9 the codomain $\Gamma(M) = \Omega(B(M)^{\mathcal{V}\text{-fib}})$ of $\eta_M^{\mathcal{V}}$ is grouplike. Let

$$M \rightarrowtail N \xrightarrow{\sim} \Gamma(M) \tag{8.2}$$

be a factorization of $\eta_M^{\mathcal{V}}$ into a cofibration followed by an acyclic fibration. The right morphism in (8.2) becomes a weak equivalence after applying $B^{\mathbf{F}}((-)_{h\mathcal{V}})$ by Lemma 8.2. Since M and N are cofibrant, it suffices to check that the left morphism in (8.2) becomes a weak homotopy equivalence after applying $B((-)_{h\mathcal{V}})$. Since classifying spaces are path-connected, this is reduced to checking that it becomes a weak homotopy equivalence after applying $\Omega(B((-)_{h\mathcal{V}}))$. By Lemma 8.8 (3) this is equivalent to checking that $B(M) \to B(N)$ becomes a $\mathcal{V}$-equivalence after applying $\Omega((-)^{\mathcal{V}\text{-fib}})$.

The remainder of this proof is completely analogous to the proof of [SS13, Proposition 4.13]. We apply B to the right map in (8.2) and compose with the counit of $(B \dashv \Omega)$ at $B(M)^{\mathcal{V}\text{-fib}}$ to obtain

$$B(N) \to B\Omega(B(M)^{\mathcal{V}\text{-fib}}) \to B(M)^{\mathcal{V}\text{-fib}}. \tag{8.3}$$

The composite of $B(M) \to B(N)$ and (8.3) is the positive $\mathcal{V}$-fibrant replacement of $B(M)$, which certainly becomes a $\mathcal{V}$-equivalence after applying $\Omega((-)^{\mathcal{V}\text{-fib}})$. By the two-out-of-three property we only need to check that (8.3) becomes a $\mathcal{V}$-equivalence after applying $\Omega((-)^{\mathcal{V}\text{-fib}})$. The composite of this morphism and derived unit $\eta_N^{\mathcal{V}} : N \to \Omega(B(N)^{\mathcal{V}\text{-fib}})$ can be identified with the composite

$$N \to \Gamma(M) = \Omega(B(M)^{\mathcal{V}\text{-fib}}) \to \Omega((B(M)^{\mathcal{V}\text{-fib}})^{\mathcal{V}\text{-fib}})$$

which is a $\mathcal{V}$-equivalence. The cofibrant commutative $\mathcal{V}$-space monoid N is grouplike since it is $\mathcal{V}$-equivalent to $\Gamma(M)$. Therefore the derived unit $\eta_N^{\mathcal{V}}$ is a $\mathcal{V}$-equivalence by Lemma 8.8 (4). Thus by the two-out-of-three property, (8.3) becomes a $\mathcal{V}$-equivalence after applying $\Omega((-)^{\mathcal{V}\text{-fib}})$. □

We have constructed group completions for cofibrant objects in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. To construct group completions for all objects in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, we will localize the positive $\mathcal{V}$-model structure. Informally this means that we enlarge the class of weak equivalences.

Definition 8.11. Consider the commutative $\mathcal{V}$-space monoid $C_1 = \mathbb{P}(F_1(*))$. We factor the derived unit $\eta_{C_1}^{\mathcal{V}}$ as a cofibration ξ followed by an acyclic fibration in the positive $\mathcal{V}$-model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$:

$$C_1 \overset{\xi}{\rightarrowtail} C_1^{\mathrm{gp}} \xrightarrow{\sim} \Gamma(C_1).$$

Lemma 8.12 ([SS13, Lemma 5.2])**.** *Let M be a commutative $\mathcal{V}$-space monoid that is positive $\mathcal{V}$-fibrant. Then M is grouplike if and only if every morphism $C_1 \to M$ extends to a morphism $C_1^{\mathrm{gp}} \to M$.*

By [Hir03, Theorem 17.6.3] we know that for any $\mathcal{V}$-equivalence $f : M \xrightarrow{\sim} N$, between cofibrant objects N and M, and any positive $\mathcal{V}$-fibrant object W in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ the map $\widetilde{f}^* : \mathrm{Map}(\widetilde{N}, W) \to \mathrm{Map}(\widetilde{M}, W)$ is a weak equivalence in $\mathbf{sSet}$. Lemma 8.12 then motivates us to adjust our model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ so that ξ becomes a weak equivalence and fibrant objects become grouplike. The idea of 'adding weak equivalences' to a model structure can be made precise using a *left Bousfield localization.*

Definition 8.13 ([Hir03, Definition 3.3.1])**.** Let $\mathscr{C}$ be a simplicial model category and S a set of morphisms in $\mathscr{C}$. An object W in $\mathscr{C}$ is *S-local* if it is fibrant and if, for every morphism $g : A \to B$ in S, the induced map

$$\widetilde{g}^* : \mathrm{Map}(\widetilde{B}, W) \twoheadrightarrow \mathrm{Map}(\widetilde{A}, W), \tag{8.4}$$

with $\widetilde{g} : \widetilde{A} \rightarrowtail \widetilde{B}$ a cofibrant replacement of g, is a weak equivalence of simplicial sets. A morphism $f : X \to Y$ in $\mathscr{C}$ is an *S-local equivalence* if, after choosing a cofibrant replacement $\widetilde{f} : \widetilde{X} \rightarrowtail \widetilde{Y}$ of f, the induced maps

$$\widetilde{f}^* : \mathrm{Map}(\widetilde{Y}, W) \twoheadrightarrow \mathrm{Map}(\widetilde{X}, W) \tag{8.5}$$

are weak equivalences of simplicial sets for all S-local objects W. The *left Bousfield localization* of $\mathscr{C}$ with respect to S (if it exists) is a model structure on $\mathscr{C}$ having the same class of cofibrations and the S-local equivalences as the weak equivalences.

The maps (8.4) and (8.5) are fibrations due to [Hov99, Lemma 4.2.2(3)]. By [Hir03, Proposition 3.1.5], we can see that weak equivalences in a model category $\mathscr{C}$ must be S-local equivalences for every set of morphisms S. If a simplicial model category is left proper, the S-local objects are precisely the fibrant objects in the left Bousfield localizations by [Hir03, Proposition 3.4.1]. Left Bousfield localizations do not always exist. By [Hir03, Theorem 4.1.1], we know that they do exist if our simplicial model category $\mathscr{C}$ is left proper and *cellular.*

We briefly explain what a cellular model category is. Cellularity is needed for the existence of left Bousfield localizations, but we will not need it anywhere else and will not go into full detail. All the model categories we need to be cellular are cellular, and we will reference the corresponding proofs. A cellular model category is a cofibrantly generated model category satisfying three additional properties that concern themselves with *compactness*, *smallness* and *effective monomorphisms.* We remark that compactness in this context is not the same as the compactness of a space, although the two concepts are related. We refer to [Hir03, Definition 12.1.1] for the detailed definition.

Our desired model category is the left Bousfield localization of the positive $\mathcal{V}$-model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ with respect to $\{\xi\}$. We usually write ξ-local objects and equivalences instead of $\{\xi\}$-local objects and equivalences.

Theorem 8.14 (Group completion model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$, [SS13, Proposition 5.3], [Hir03, Theorem 4.1.1])**.** *The ξ-local equivalences and cofibrations are part of a cofibrantly generated model structure on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$.*

Proof. The category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure is left proper by Theorem 7.22. [SS13, Proposition A.1] shows that $\mathcal{C}\mathbf{Top}^{\mathcal{I}}$ is cellular. The same proof can be applied to $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. Thus by [Hir03, Theorem 4.1.1], the left Bousfield localization with respect to ξ exists and is the desired model structure. □

We now have two distinct model structures on $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$. To differentiate these, we write $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{pos}}$ and $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ for the category $\mathcal{C}\mathbf{Top}^{\mathcal{V}}$ with the positive $\mathcal{V}$-model structure and the group completion model structure, respectively.

The following theorem is the $\mathbf{Top}^{\mathcal{V}}$ analogue of [SS13, Theorem 1.3]. This theorem explains what the weak equivalences, fibrant objects and fibrant replacements are in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$, and its proof is entirely analogous to the one given for [SS13, Theorem 1.3] after Lemma 5.9 which is concerned with the case $\mathbf{sSet}^{\mathcal{I}}$. In particular, it gives us group completions for every commutative $\mathcal{V}$-space monoid.

Theorem 8.15. *A morphism of commutative $\mathcal{V}$-space monoids $M \to N$ is a weak equivalence in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ if and only if the induced map*

$$B^{\mathbf{F}}(M_{h\mathcal{V}}) \to B^{\mathbf{F}}(N_{h\mathcal{V}})$$

is a weak homotopy equivalence. The fibrant objects in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ are those objects that are fibrant in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{pos}}$ and grouplike. A fibrant replacement $M \to \widehat{M}$ in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ is a group completion.

We give a sketch of the proof. Lemma 8.12 can be used to show that the fibrant objects in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ are positive $\mathcal{V}$-fibrant and grouplike. The cofibrant replacement of M has a group completion by Theorem 8.10. It can be shown that the fibrant replacement in $\mathcal{C}\mathbf{Top}^{\mathcal{V}}_{\text{gp}}$ is $\mathcal{V}$-equivalent to this group completion. The claim about weak equivalences is then a consequence of [Hir03, Theorem 3.2.18(1)].

8.2. Augmented model structures

Let $\mathscr{C}$ be a category and $C \in \operatorname{Ob}\mathscr{C}$. If S is a set of morphisms in $\mathscr{C}$ then let S_C denote the set of morphisms

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \swarrow & \\ & C & \end{array}$$

in $\mathscr{C}/C$ where $f \in S$.

Proposition 8.16 ([Hir21, Theorems 1.20, 1.23, 1.24])**.** *Let $\mathscr{C}$ be a cofibrantly generated model category, with generating cofibrations I and generating acyclic cofibrations J, and let $C \in \operatorname{Ob}\mathscr{C}$. The model structure on $\mathscr{C}/C$ as given in Proposition 4.8 is cofibrantly generated, with generating cofibrations I_C and generating acyclic cofibrations J_C. If $\mathscr{C}$ is cellular, then so is $\mathscr{C}/C$. If C is left proper, then so is $\mathscr{C}/C$.*

Proposition 8.17. *The overcategory $\mathscr{C}/C$ of a simplicial model category $\mathscr{C}$ is a simplicial model category.*

Proof. Let $f : X \to C$ and $g : Y \to C$ be objects in $\mathscr{C}/C$ and K and L be simplicial sets. Let $\operatorname{Map}(X, Y)$, $X \otimes K$ and Y^K denote the enrichment, tensor and cotensor in $\mathscr{C}$ respectively. In $\mathscr{C}/C$ the tensor and cotensor are

$$f \star K : X \otimes K \to X \to C, \qquad\qquad g \diamond K : C \times_{C^K} Y^K \to C,$$

where $f \star K$ is the composite of f and the tensor of X and $K \to *$, and $g \diamond K$ the pullback of g^K along $C \to C^K$. We show that $\mathscr{C}/C(f \star K, g) \cong \mathscr{C}/C(f, g \diamond K)$ is a bijection.

The simplicial set $\mathrm{Map}(f,g) = (\mathscr{C}/C)(f \star \Delta[-], g)$ must then determine the enrichment. Consider the left diagram, which is a morphism in $\mathscr{C}/C(f \star K, g)$:

$$\begin{array}{ccc} X \otimes K & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & C \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & Y^K \\ \downarrow & & \downarrow{\scriptstyle g^K} \\ X^K & \xrightarrow{f^K} & C^K \end{array}$$

The adjointness of the tensor and cotensor in $\mathscr{C}$ uniquely determines the right diagram. The bottom left composite of the right diagram equals the composite $X \xrightarrow{f} C \to C^K$. Therefore there exists a unique morphism $X \to C \times_{C^K} Y^K$ that uniquely determines a morphism in $\mathscr{C}/C(f, g \diamond K)$. Repeating this argument in reverse gives the bijection. Thus $(-) \star K$ is left adjoint. Clearly $f \star * \cong f$ and the isomorphism $(f \star K) \star L \cong f \star (K \times L)$ follows from the commutativity of the diagram:

$$\begin{array}{ccc} (X \otimes K) \otimes L & \xrightarrow{\cong} & X \otimes (K \times L) \\ \downarrow & & \downarrow \\ X \otimes K & \longrightarrow & X \end{array}$$

Therefore by [GJ09, Lemma2.4] the category $\mathscr{C}/C$ is enriched, tensored and cotensored over **sSet**. Let $\phi : f \to g$ and $i : K \to L$ be morphisms in $\mathscr{C}/C$ and **sSet** respectively and let $h : X \to Y$ be the image of ϕ under the forgetful functor $\mathscr{C}/C \to \mathscr{C}$. Since this functor is left adjoint, it preserves colimits. Therefore it sends $\phi \square i$ to $h \square i$. The pushout product property is satisfied since $\mathscr{C}$ is a simplicial model category. Thus $\mathscr{C}/C$ is also a simplicial model category. $\square$

Proposition 8.18 ([Hir03, Theorem 4.1.1(4)]). *The left Bousfield localization of a left proper cellular simplicial model category $\mathscr{C}$ with respect to a set of morphisms S is a simplicial model category with the same enrichment, tensor and cotensor.*

Let $\mathscr{M} = \mathcal{C}\mathbf{Top}^{\mathcal{V}}$, let T be a commutative $\mathcal{V}$-space monoid that is fibrant in $\mathscr{M}_{\mathrm{gp}}$ and consider the overcategory $\mathscr{M}/T$. This category can now inherit either model structure on $\mathscr{M}$, so let $\mathscr{M}_{\mathrm{pos}}/T$ and $\mathscr{M}_{\mathrm{gp}}/T$ denote $\mathscr{M}/T$ with the positive $\mathcal{V}$-model structure and group completion model structure, respectively. By Proposition 8.18, the category $\mathscr{M}_{\mathrm{gp}}$ is a simplicial model category with the same enrichment as $\mathscr{M}_{\mathrm{pos}}$. Then by Proposition 8.17, the categories $\mathscr{M}_{\mathrm{pos}}/T$ and $\mathscr{M}_{\mathrm{gp}}/T$ are simplicial model categories with the same enrichments. In Theorem 8.14, we have seen that $\mathscr{M}_{\mathrm{pos}}$ is a left proper cellular model category. Therefore $\mathscr{M}_{\mathrm{pos}}/T$ is left proper and cellular by Proposition 8.16. Hence left Bousfield localizations of $\mathscr{M}_{\mathrm{pos}}/T$ exist by [Hir03, Theorem 4.1.1(1)]. Let $(\mathscr{M}/T)_\xi$ denote the left Bousfield localization of $\mathscr{M}_{\mathrm{pos}}/T$ with respect to the set $\{\xi\}_T$. By Proposition 8.18, the category $(\mathscr{M}/T)_\xi$ is a simplicial model category with the same enrichment as $\mathscr{M}_{\mathrm{pos}}/T$ and $\mathscr{M}_{\mathrm{gp}}/T$. We aim to show that the model structure on $\mathscr{M}_{\mathrm{gp}}/T$ and $(\mathscr{M}/T)_\xi$ are equal. Since the cofibrations coincide, we only need to worry about the weak equivalences.

Let $h : W \to T$ be an object and $\phi : f \to g$ a morphism in $\mathscr{M}/T$. Let $\widetilde{\phi} : \widetilde{f} \rightarrowtail \widetilde{g}$ be a cofibrant replacement. Then consider the simplicial map

$$(\widetilde{\phi})^* : \mathrm{Map}(\widetilde{g}, h) \to \mathrm{Map}(\widetilde{f}, h). \tag{8.6}$$

Proposition 8.19. *Fibrant objects in $\mathscr{M}_{\mathrm{gp}}/T$ are fibrant in $(\mathscr{M}/T)_\xi$.*

Proof. Let $h : W \to T$ be fibrant in $\mathscr{M}_{\mathrm{gp}}/T$ and $\phi : f \to g$ be a weak equivalence in $\mathscr{M}_{\mathrm{gp}}/T$. Then the map (8.6) is an acyclic fibration in **sSet** by the pushout product property in $\mathscr{M}_{\mathrm{gp}}/T$ by [Hov99, Lemma 4.2.2]. Any morphism in $\{\xi\}_T$ is a weak equivalence in $\mathscr{M}_{\mathrm{gp}}/T$. Therefore h is $\{\xi\}_T$-local in $(\mathscr{M}/T)_\xi$ since (8.6) is a weak equivalence if $\phi \in \{\xi\}_T$. □

Corollary 8.20. *Weak equivalences in* $(\mathscr{M}/T)_\xi$ *are weak equivalences in* $\mathscr{M}_{\mathrm{gp}}/T$.

Proof. Let $\phi : f \to g$ be a weak equivalence in $(\mathscr{M}/T)_\xi$. Then (8.6) is a weak equivalence for all fibrant objects h in $(\mathscr{M}/T)_\xi$. Let f^{gp} and g^{gp} be fibrant replacements of f and g in $\mathscr{M}_{\mathrm{gp}}/T$, then they are fibrant in $(\mathscr{M}/T)_\xi$ by Proposition 8.19. Thus the map (8.6) is a weak equivalence if $h = f^{\mathrm{gp}}$ or $h = g^{\mathrm{gp}}$. Then ϕ is a weak equivalence in $\mathscr{M}_{\mathrm{gp}}/T$ by [Hir03, Proposition 17.7.6(2)]. □

Lemma 8.21. *If* $h : W \to T$ *is fibrant in* $(\mathscr{M}/T)_\xi$*, then* W *is fibrant in* $\mathcal{M}_{\mathrm{gp}}$.

Proof. The fibrant objects in $(\mathscr{M}/T)_\xi$ are precisely the $\{\xi\}_T$-local objects in $\mathscr{M}_{\mathrm{pos}}/T$, and the fibrant objects in $\mathscr{M}_{\mathrm{gp}}$ are the positive $\mathcal{V}$-fibrant objects that are grouplike. The identity on T is the terminal object in $\mathscr{M}/T$. Then since h is fibrant in $\mathscr{M}_{\mathrm{pos}}/T$, it must be a fibration in $\mathscr{M}_{\mathrm{pos}}$. Thus W is positive $\mathcal{V}$-fibrant since T is. By Lemma 8.12, it suffices to show that every morphism $f : C_1 \to W$ in $\mathscr{M}$ extends to a morphism $f' : C_1^{\mathrm{gp}} \to W$.

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & W \\ {\scriptstyle \xi}\big\downarrow & \overset{f'}{\nearrow} & \big\downarrow{\scriptstyle h} \\ C_1^{\mathrm{gp}} & \xrightarrow{g} & T \end{array} \tag{8.7}$$

Let f be such a morphism. By Lemma 8.12 the morphism $hf : C_1 \to T$ extends to a morphism $g : C_1^{\mathrm{gp}} \to T$ since T is positive $\mathcal{V}$-fibrant and grouplike. The unit $* \to T$ is the initial object in $\mathscr{M}/T$, so hf and g are cofibrant objects in $\mathscr{M}/T$ since C_1 and C_1^{gp} are in $\mathscr{M}$. Now ξ determines a cofibration $\xi_* : hf \rightarrowtail g$ between cofibrant objects that is contained in $\{\xi\}_T$, since $g\xi = hf$. This, in turn, determines the acyclic fibration

$$\xi^* : \mathrm{Map}(g, h) \xrightarrow{\sim} \mathrm{Map}(hf, h),$$

since h is $\{\xi\}_T$-local. The map $\emptyset \to *$ is a cofibration in **sSet**. Therefore the lifting property implies that $(\xi^*)_0 : (\mathscr{M}/T)(g, h) \to (\mathscr{M}/T)(hf, h)$ must be a surjection. The morphism $hf \to h$ induced by f, in particular, extends to a morphism $g \to h$ in $\mathscr{M}/T$. This morphism forgets to a morphism $f' : C_1^{\mathrm{gp}} \to W$ in $\mathscr{M}$ that provides a lift in (8.7). Now f extends to f' along ξ making W grouplike. □

Lemma 8.22 ([Hir03, Theorem 3.2.18(1)])**.** *Let* $\mathscr{C}$ *be a left proper cellular simplicial model category and* S *a set of morphisms in* $\mathscr{C}$*. Let* $\mathscr{C}_S$ *denote the left Bousfield localization and let* $\widehat{f} : \widehat{X} \to \widehat{Y}$ *be a fibrant replacement of* $f : X \to Y$ *in* $\mathscr{C}_S$*. Then* f *is a weak equivalence in* $\mathscr{C}_S$ *if and only if* $\widehat{f}$ *is a weak equivalence in* $\mathscr{C}$*.*

Theorem 8.23. *The model structure on* $\mathscr{M}_{\mathrm{gp}}/T$ *is the left Bousfield localization of* $\mathscr{M}_{\mathrm{pos}}/T$ *with respect to the set* $\{\xi\}_T$*, that is,* $\mathscr{M}_{\mathrm{gp}}/T = (\mathscr{M}/T)_\xi$.

Proof. Since $\mathscr{M}_{\mathrm{gp}}/T$ and $(\mathscr{M}/T)_\xi$ have the same classes of cofibrations, it suffices to prove the converse of Corollary 8.20.

Let $f : M \to T$ be an object in $\mathscr{M}/T$ and let $f \to \widehat{f}$ be a fibrant replacement in $(\mathscr{M}/T)_\xi$, with $\widehat{f} : \widehat{M} \to T$. The morphism $M \to \widehat{M}$ is a weak equivalence in $\mathscr{M}_{\mathrm{gp}}$ by

Corollary 8.20 and $\widehat{M}$ is fibrant in $\mathscr{M}_{\mathrm{gp}}$ by Lemma 8.21, hence $M \to \widehat{M}$ is a fibrant replacement in $\mathscr{M}_{\mathrm{gp}}$. Let $\phi : f \to g$ be a morphism in $\mathscr{M}/T$, with $f : M \to T$ and $g : N \to T$, such that ϕ forgets to $h : M \to N$. Let $\widehat{\phi} : \widehat{f} \to \widehat{g}$ be a fibrant replacement in $(\mathscr{M}/T)_\xi$ that forgets to $\widehat{h} : \widehat{M} \to \widehat{N}$. Then $\widehat{h}$ is a fibrant replacement of h in $\mathscr{M}_{\mathrm{gp}}$.

If ϕ is a weak equivalence in $\mathscr{M}_{\mathrm{gp}}/T$, then h is a weak equivalence in $\mathscr{M}_{\mathrm{gp}}$. Therefore $\widehat{h}$ is a $\mathcal{V}$-equivalence by Lemma 8.22 making $\widehat{\phi}$ a weak equivalence in $\mathscr{M}_{\mathrm{pos}}/T$. Thus ϕ is a weak equivalence in $(\mathscr{M}/T)_\xi$ again by Lemma 8.22. $\square$

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