A Tannaka Type Theorem for Quantales

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Overview

The main purpose of this thesis is to state and prove a Tannaka duality type result for *unital quantales*. Our approach is to show that the category SupLat of sup-lattices has enough structure, namely that of a closed tensor category, and that unital quantales are monoid objects in SupLat. Using this, our theorem follows from a much more general theorem in enriched category theory.

In Chapter 1, we will introduce tensor categories and some basic properties and notions, as well as show that SupLat is a closed tensor category. Chapter 2 will cover some enriched category theory that we will need later. Finally, Chapter 3 introduces monoid objects in tensor categories, and shows that unital quantales are monoids in SupLat. Furthermore, we state a general Tannaka duality theorem and derive our main theorem from it, as well as some corollaries.

Chapter 1

Tensor Categories

In mathematics, one often works with categories that allow one to construct from two objects a third one, in a way that behaves nicely. Consider for example Vec_k, the category of vector spaces over a field k. Here we can take the tensor product $-\otimes-$, and this construction behaves nicely in the sense that we have isomorphisms

$$V_1 \otimes (V_2 \otimes V_3) \xrightarrow{\sim} (V_1 \otimes V_2) \otimes V_3, \quad v_1 \otimes (v_2 \otimes v_3) \mapsto (v_1 \otimes v_2) \otimes v_3$$

and

$$V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1, \quad v_1 \otimes v_2 \mapsto v_2 \otimes v_1,$$

as well as

$$k \otimes V \xrightarrow{\sim} V, \quad a \otimes v \mapsto av \quad \text{and} \quad V \otimes k \xrightarrow{\sim} V, \quad v \otimes a \mapsto av.$$

Recall that a commutative monoid is a set M together with an operation $\cdot\colon M \times M \to M$ and an element $1 \in M$ such that for all $l, m, n \in M$ we have $l \cdot (m \cdot n) = (l \cdot m) \cdot n$ and $m \cdot n = n \cdot m$, as well as $1 \cdot m = m \cdot 1 = m$. Comparing this with our earlier observations about Vec_k, it follows that the tensor product turns the set of isomorphism classes of k-vector spaces into a *commutative monoid*, with identity element k.

By replacing the *equalities* in the definition of a commutative monoid by *isomorphisms*, one can "categorify" the notion of a commutative monoid. However, we need to be careful about choosing these isomorphisms. As is standard in category theory, they need to be *natural.* Furthermore, some coherency conditions need to be imposed to ensure our categories behave nicely.

1.1 The Definition

Let $\mathcal{C}$ be a category and $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor. Then we have functors

$$F_1 := - \otimes (- \otimes -) \quad \text{and} \quad F_2 := (- \otimes -) \otimes -$$

An associativity constraint is an isomorphism of functors $\phi\colon F_1 \to F_2$ such that for all objects W, X, Y, Z in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
 & W \otimes (X \otimes (Y \otimes Z)) \arrow[dl, "\mathrm{id}_W \otimes \phi_{X,Y,Z}"'] \arrow[dr, "\phi_{W,X,Y \otimes Z}"] & \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "\phi_{W,X \otimes Y,Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\phi_{W \otimes X,Y,Z}"] \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\phi_{W,X,Y} \otimes \mathrm{id}_Z"] & & ((W \otimes X) \otimes Y) \otimes Z
\end{tikzcd}$$

commutes.

Similarly, we have functors G_1 and G_2 that send a pair $(X, Y) \in \mathcal{C} \times \mathcal{C}$ to $X \otimes Y$ and $Y \otimes X$ respectively. A commutativity constraint is an isomorphism of functors $\psi\colon G_1 \to G_2$ such that for all objects X, Y in $\mathcal{C}$ we have $\psi_{Y,X} \circ \psi_{X,Y} = \mathrm{id}_{X \otimes Y}$.

An associativity constraint ϕ and a commutativity constraint are said to be compatible if for all objects X, Y, Z in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\psi_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[r, "\phi_{Z,X,Y}"] & (Z \otimes X) \otimes Y \\
X \otimes (Y \otimes Z) \arrow[u, "\phi_{X,Y,Z}"] \arrow[r, "\mathrm{id}_X \otimes \psi_{Y,Z}"] & X \otimes (Z \otimes Y) \arrow[r, "\phi_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[u, "\psi_{X,Z} \otimes \mathrm{id}_Y"']
\end{tikzcd}$$

commutes.

An identity object is a pair $(\mathbb{1}, i)$ consisting of an object $\mathbb{1}$ of $\mathcal{C}$ and an isomorphism $i\colon \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ such that the functor $\mathcal{C} \to \mathcal{C}$ given by $X \mapsto \mathbb{1} \otimes X$ is an equivalence of categories.

We are now ready to state the definition of a tensor category.

Definition 1.1. A tensor category is a tuple $(\mathcal{C}, \otimes, \phi, \psi, \mathbb{1}, i)$ consisting of a category $\mathcal{C}$ with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, compatible constraints ϕ and ψ, and an identity object $(\mathbb{1}, i)$.

The connection between tensor categories and commutative monoids becomes even more evident when one considers the following equivalent definition of the identity element of a commutative monoid M. Namely, an element $1 \in M$ such that $1 \cdot 1 = 1$ and the map $M \to M$ given by $m \mapsto 1 \cdot m$ is a bijection.

1.1.1 Examples

As suggested at the start of this chapter, the category Vec_k of vector spaces over a field k is a tensor category. The associativity and commutativity constraints are the obvious maps, and are clearly compatible. An identity object is (k, i) where

$$i\colon k \otimes k \to k, \quad a \otimes b \mapsto 1 \otimes ab.$$

Another example is the category $\mathrm{Rep}_k(G)$ of representations of G over k, with tensor product given by

$$(V, \rho) \otimes (V', \rho') = (V \otimes V', \rho \otimes \rho').$$

An identity object is $(g \mapsto \mathrm{id}_k, i)$, where i is as above.

1.2 Tensor Functors

Given this notion of a category with some additional structure, we should of course consider functors between them that respect this structure.

Definition 1.2. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be tensor categories. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ together with an isomorphism of functors

$$c_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$$

is called a tensor functor if the diagrams

$$\begin{array}{ccccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{c_{X,Y} \otimes \mathrm{id}_{F(Z)}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{c_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z) \\
\phi'_{F(X),F(Y),F(Z)} \uparrow & & & & \uparrow F(\phi_{X,Y,Z}) \\
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\mathrm{id}_{F(X)} \otimes c_{Y,Z}} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{c_{X, Y \otimes Z}} & F(X \otimes (Y \otimes Z))
\end{array}$$

and

$$\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{c_{X,Y}} & F(X \otimes Y) \\
\downarrow \psi'_{F(X),F(Y)} & & \downarrow F(\psi_{X,Y}) \\
F(Y) \otimes F(X) & \xrightarrow{c_{Y,X}} & F(Y \otimes X)
\end{array}$$

commute for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, and whenever $(\mathbb{1}, i)$ is a identity object of $\mathcal{C}$ then $(F(\mathbb{1}), F(i))$ is an identity object of $\mathcal{C}'$. A tensor functor that is also an equivalence of categories is called a tensor equivalence.

1.3 Basic Properties

This section will list and prove a few basic, but nontrivial, properties of tensor categories.

1.3.1 Identity Objects

Recall that in the definition of a (commutative) monoid, the identity element is not assumed to be unique. However, uniqueness easily follows. Given identity elements 1 and $1'$, we have

$$1 = 1 \cdot 1' = 1'.$$

It turns out that the identity object of a tensor category is essentially unique as well. To be precise, we have the following

Proposition 1.3. *Let $\mathcal{C}$ be a tensor category, and suppose $(\mathbb{1}, i)$ and $(\mathbb{1}', i')$ are identity objects. Then there is a unique isomorphism $\iota\colon \mathbb{1} \xrightarrow{\sim} \mathbb{1}'$ such that the diagram*

$$\begin{array}{ccc} \mathbb{1} \otimes \mathbb{1} & \xrightarrow{i} & \mathbb{1} \\ {\scriptstyle \iota\otimes\iota}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ \mathbb{1}' \otimes \mathbb{1}' & \xrightarrow[i']{} & \mathbb{1}' \end{array}$$

commutes.

We will prove this shortly. First, let us make the following observation. As the functor F that is given on objects by $X \mapsto \mathbb{1} \otimes X$ is an equivalence of categories, we can define a natural isomorphism $l_X\colon \mathbb{1} \otimes X \to X$ by requiring that Fl_X equals

$$\mathbb{1} \otimes (\mathbb{1} \otimes X) \xrightarrow{\phi_{\mathbb{1},\mathbb{1},X}} (\mathbb{1} \otimes \mathbb{1}) \otimes X \xrightarrow{i\otimes \mathrm{id}_X} \mathbb{1} \otimes X$$

We will call l_X the left unit constraint. We also have a right unit constraint $r_X\colon X \otimes \mathbb{1} \to X$ given by $r_X := l_X \circ \psi_{X,\mathbb{1}}$.

Proof of Proposition 1.3 (Sketch). Let l_X, r_X and l'_X, r'_X be the left and right unit constraints associated to $(\mathbb{1}, i)$ and $(\mathbb{1}', i')$ respectively. We take ι to be the isomorphism

$$\mathbb{1} \xrightarrow{(l'_{\mathbb{1}})^{-1}} \mathbb{1}' \otimes \mathbb{1} \xrightarrow{r_{\mathbb{1}'}} \mathbb{1}'$$

One can show that $\iota \circ i = i' \circ (\iota \otimes \iota)$. For this and uniqueness, see [2, Prop. 2.2.6]. □

1.3.2 Strictness

As one may have realized themselves by now, the associativity and commutativity constraints, as well as the many diagrams that need to be verified, make the theory of tensor categories quite cumbersome to work with. One case in which all these difficulties fade away is the case when one works with a strict tensor category

Definition 1.4. A tensor category $(\mathcal{C}, \otimes, \phi, \psi, \mathbb{1}, i)$ is called *strict* if for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ we have

- $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$
- $X \otimes Y = Y \otimes X$
- $\mathbb{1} \otimes X = X$

and the associativity, commutativity and (left) unit constraints are the identity.

The following theorem says that for many purposes, we may assume that we are working with a strict tensor category.

Theorem 1.5 (MacLane Strictness)**.** *Let $\mathcal{C}$ be a tensor category. Then there is a strict tensor category $\mathcal{C}^{str}$ and a tensor equivalence $\mathcal{C} \cong \mathcal{C}^{str}$.*

Proof. See [2, Thm. 2.8.5]. □

1.3.3 Coherence

Given a tensor category $\mathcal{C}$ and objects $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{C})$, one can tensor them in many different ways. Namely by ordering the X_i differently, by parenthesizing the products differently and by inserting instances of $\mathbb{1}$. Thankfully, one can always write down an isomorphism between two such products by using the associativity, commutativity and unit constraints. However, on first glance, it seems that there may be more than one such isomorphism. For example, taking $n = 3$, we have isomorphisms

$$X_1 \otimes (X_2 \otimes X_3) \xrightarrow{\phi_{X_1,X_2,X_3}} (X_1 \otimes X_2) \otimes X_3 \xrightarrow{\psi_{X_1 \otimes X_2, X_3}} X_3 \otimes (X_1 \otimes X_2) \xrightarrow{\phi_{X_3,X_1,X_2}} (X_3 \otimes X_1) \otimes X_2$$

and

$$X_1 \otimes (X_2 \otimes X_3) \xrightarrow{\mathrm{id}_{X_1} \otimes \psi_{X_2,X_3}} X_1 \otimes (X_3 \otimes X_2) \xrightarrow{\phi_{X_1,X_3,X_2}} (X_1 \otimes X_3) \otimes X_2 \xrightarrow{\psi_{X_1,X_3} \otimes \mathrm{id}_{X_2}} (X_3 \otimes X_1) \otimes X_2$$

However, these are the same since ϕ and ψ are compatible.

It turns out that for any n, if f and g are isomorphisms that consist of associativity, commutativity and unit constraints, then $f = g$. Hence we can safely identify all such products and write things like

$$\bigotimes_i X_i$$

without any confusion.

1.4 The Internal Hom

Recall the following well-known theorem from commutative algebra

Theorem 1.6 (Tensor-Hom Adjunction)**.** *Let R be a commutative ring*[1] *and M an R-module. Then the functors $-\otimes_R M$ and $Hom_R(M,-)$ from R-mod to itself form an adjoint pair, with the former being a left-adjoint to the latter.*

This means that for R-modules L, M and N, we have a bijection

$$\Phi_{L,N}\colon \mathrm{Hom}_R(L\otimes_R M, N) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(M,N))$$

that is natural in L and N. As a consequence, fixing M and N, we find that Φ is an isomorphism of functors $\mathrm{Hom}_R(-\otimes_R M, N) =: F \to \mathrm{Hom}_R(-, \mathrm{Hom}_R(M,N))$. We say that F is respresented by $\mathrm{Hom}_R(M,N)$, the set of R-linear maps $M \to N$ regarded as an R-module via the rule $(rf)(m) = rf(m)$ for $r \in R$ and $m \in M$. Note that $\mathrm{Hom}_R(M,N)$ can hence be viewed as a "Hom-object" inside of the category R-mod itself.

Let $\mathcal{C}$ be a tensor category. Motivated by the above, we make the following definition

Definition 1.7. Let $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and let F be the contravariant functor that sends an object[2] X of $\mathcal{C}$ to $\mathrm{Hom}(X\otimes Y, Z)$. If F is representable, then we write $\underline{\mathrm{Hom}}(Y,Z)$ for the representing object and call it the *internal Hom*. We call $\mathcal{C}$ closed if it has all internal homs.

Of course, just as in the case of R-mod, we have an adjunction

$$-\otimes X \dashv \underline{\mathrm{Hom}}(X,-),$$

for $X \in \mathrm{Ob}(\mathcal{C})$. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, let

$$\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}(X,Y)\otimes X \to Y$$

be the map corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ under this adjunction. This map can be used to obtain two simple results.

Lemma 1.8. *Let I be a set and let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be objects of a closed tensor category $\mathcal{C}$. Then there is a canonical morphism*

$$\bigotimes_{i\in I} \underline{Hom}(X_i, Y_i) \to \underline{Hom}\Big(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i\Big).$$

Proof. By coherence, we can take the morphism corresponding to

[1] If one wishes to stay true to the motivating example in the beginning of this chapter, simply take R to be a field k.

[2] On morphisms, it is given by $(f\colon X' \to X) \mapsto -\circ(f\otimes \mathrm{id}_Y)$.

$$\left(\bigotimes_{i\in I}\underline{\mathrm{Hom}}(X_i,Y_i)\right)\otimes\left(\bigotimes_{i\in I}X_i\right)$$

$$\big\downarrow\cong$$

$$\bigotimes_{i\in I}(\underline{\mathrm{Hom}}(X_i,Y_i)\otimes X_i)$$

$$\big\downarrow\otimes\,\mathrm{ev}_{X_i,Y_i}$$

$$\bigotimes_{i\in I}Y_i$$

□

Lemma 1.9. *Let X, Y, Z be objects of a closed tensor category $\mathcal{C}$. Then there is a canonical morphism*

$$\underline{Hom}(Y,Z)\otimes\underline{Hom}(X,Y)\to\underline{Hom}(X,Z).$$

Proof. Take the morphism corresponding to

$$(\underline{\mathrm{Hom}}(Y,Z)\otimes\underline{\mathrm{Hom}}(X,Y))\otimes X$$

$$\big\downarrow\phi^{-1}_{\underline{\mathrm{Hom}}(Y,Z),\underline{\mathrm{Hom}}(X,Y),X}$$

$$\underline{\mathrm{Hom}}(Y,Z)\otimes(\underline{\mathrm{Hom}}(X,Y))\otimes X)$$

$$\big\downarrow\mathrm{id}_{\underline{\mathrm{Hom}}(Y,Z)}\otimes\mathrm{ev}_{X,Y}$$

$$\underline{\mathrm{Hom}}(Y,Z)\otimes Y$$

$$\big\downarrow\mathrm{ev}_{Y,Z}$$

$$Z$$

□

1.5 Duals

In algebra, one often encounters the notion of *dual* objects. Sticking to our familiar example, vector spaces, one has associated to a vector space V its dual space $\widehat{V} := \mathrm{Hom}_k(V,k)$. Furthermore, a linear map $f\colon V\to W$ induces a linear map

$$\widehat{f}\colon\widehat{W}\to\widehat{V},\quad g\mapsto g\circ f.$$

Taking this as motivation, we make the following definition

Definition 1.10. Let $\mathcal{C}$ be a tensor category and $X\in\mathrm{Ob}(\mathcal{C})$. Let $\mathbb{1}$ be its identity object. We call

$$\widehat{X} := \underline{\mathrm{Hom}}(X,\mathbb{1})$$

the *dual* of X.

For any $X \in \mathrm{Ob}(\mathcal{C})$, we have a map

$$\mathrm{ev}_X := \mathrm{ev}_{X,\mathbb{1}} \colon \widehat{X} \otimes X \to \mathbb{1}$$

Given a morphism $f\colon X \to X'$, there is a unique morphism $\widehat{f}\colon X' \to X$ making the diagram

$$\begin{array}{ccc} \widehat{X'} \otimes X & \xrightarrow{\widehat{f} \otimes \mathrm{id}_X} & \widehat{X} \otimes X \\ {\scriptstyle \mathrm{id}_{\widehat{X'}} \otimes f} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{ev}_X} \\ \widehat{X'} \otimes X' & \xrightarrow{\mathrm{ev}_{X'}} & \mathbb{1} \end{array}$$

commute. Hence, $\widehat{-}$ is a contravariant functor.

Lemma 1.11. *Let I be a set and let $(X_i)_{i\in I}$ be objects of $\mathcal{C}$. Then there is a canonical morphism*

$$\bigotimes_{i\in I} \widehat{X_i} \to \widehat{\bigotimes_{i\in I} X_i}.$$

Proof. Use Lemma 1.8 with $Y_i = \mathbb{1}$ and use that

$$\bigotimes_{i\in I} \mathbb{1} \cong \mathbb{1}.$$

□

1.6 Sup-Lattices

We finish this chapter with a non-trivial example of a closed tensor category.

Definition 1.12. A sup-lattice L is a partially ordered set that has suprema of arbitrary subsets. A morphism of sup-lattices is an order-preserving map that preserves suprema. That is, a map $f\colon L \to L$ such that

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in L$, and

$$f\Big(\bigvee_{x\in S} x\Big) = \bigvee_{x\in S} f(x)$$

for all subsets $S \subseteq L$. The category SupLat consists of sup-lattices and their morphisms.

It turns out that SupLat is a closed tensor category. Unlike previous examples, this is not a triviality, since there does not seem to be an obvious structure of a tensor category on SupLat.

We begin by considering a candidate for the internal hom associated to two sup-lattices L and L'. The set $\mathrm{Hom}(L, L')$ has a natural sup-lattice structure, given by

$$f \leq g \iff f(x) \leq g(x) \text{ for all } x \in L.$$

Given a subset $S \subseteq \mathrm{Hom}(L, L')$, its supremum is computed pointwise, i.e.

$$\Big(\bigvee_{f \in S} f\Big)(x) = \bigvee_{f \in S} f(x)$$

for $x \in L$. We will suggestively denote this sup-lattice by $\underline{\mathrm{Hom}}(L, L')$.

Next, it will prove useful to view sup-lattices in a more category theoretic light. Given a sup-lattice L, consider the category $\mathcal{L}$ with the underlying set of L as the set of objects, and for $x, y \in L$

$$\mathrm{Hom}(x, y) = \begin{cases} \{\to\}, & \text{if } x \leq y \\ \emptyset, & \text{otherwise.} \end{cases}$$

We can set up a dictionary between sup-lattices L and their corresponding categories $\mathcal{L}$.

Let L and L' be sup-lattices, and $f \colon L \to L'$ a morphism. Then f induces an operation $F \colon \mathcal{L} \to \mathcal{L}'$ as follows

- On objects it is given by f;
- A morphism $x \to y$ is sent to the unique morphism $f(x) \to f(y)$.

Now note that the fact that f is order-preserving precisely means that F is well-defined, and in fact a functor!

Now let us focus on suprema. By definition, given a subset $S \subseteq L$, the supremum of S is defined as an element $x \in L$ such that $s \leq x$ for all $s \in S$, and whenever $y \in L$ is an element such that $s \leq y$ for all $s \in S$, then $x \leq y$. Translating this to the associated category $\mathcal{L}$, this precisely says that x is the *colimit* (or coproduct) of the objects in S, that is

$$\bigvee_{s \in S} s \leftrightsquigarrow \coprod_{s \in S} s.$$

Since $\mathcal{L}$ is skeletal, this implies that x is unique. Note that the colimit of the empty diagram is the initial object, so we make the convention that the supremum of the empty set is the least element.

Saying that a morphism $f \colon L \to L'$ preserves suprema precisely means that the associated functor $F \colon \mathcal{L} \to \mathcal{L}'$ preserves all colimits.[3] By Freyd's adjoint functor theorem[4], it

[3]Note that coequalizers are simply coproducts in $\mathcal{L}$, so F preserves all colimits by a standard result in category theory.

[4]See the Appendix.

follows that F has a right adjoint $F^*\colon \mathcal{L}' \to \mathcal{L}$. Recall that right adjoints preserve limits, so by dualizing our earlier discussion F^* corresponds to an order-preserving infimum preserving map $f^*\colon L' \to L$.

For a sup-lattice $(L, \leq)$, let $(L^{\mathrm{op}}, \leq^{\mathrm{op}})$ be the lattice with underlying set L and

$$x \leq^{\mathrm{op}} y \iff y \leq x$$

for $x, y \in L$.

Lemma 1.13. *L^{op} is a sup-lattice.*

Proof. This is equivalent to showing that L has all infima. Given a subset $S \subseteq L$, write

$$X = \{x \in L : x \leq s \text{ for all } s \in S\}.$$

Then

$$\bigwedge_{x \in S} x = \bigvee_{x \in X} x.$$

□

Hence, we have shown that a morphism $f\colon L \to L'$ induces a morphism $f^{*,\mathrm{op}}\colon L'^{\mathrm{op}} \to L^{\mathrm{op}}$.[5]

Lemma 1.14. *Let L and L' be sup-lattices. Then we have an isomorphism of sup-lattices*

$$-^{*,op}\colon \underline{Hom}(L, L') \to \underline{Hom}(L'^{op}, L^{op}), \quad f \mapsto f^{*,op}$$

that is natural in L and L' in the sense that

$$\begin{array}{ccc} \underline{Hom}(L, L') & \xrightarrow{-^{*,op}} & \underline{Hom}(L'^{op}, L^{op}) \\ {\scriptstyle f'\circ - \circ f}\downarrow & & \downarrow{\scriptstyle f^{op}\circ - \circ f'^{op}} \\ \underline{Hom}(K, K') & \xrightarrow[-^{*,op}]{} & \underline{Hom}(K'^{op}, K^{op}) \end{array}$$

commutes for all morphisms $f\colon L \to L'$ and $f'\colon K \to K'$.

Proof. The diagram and the fact that $-^{*,\mathrm{op}}$ is a bijection are clear from the above discussion. It remains to prove that $-^{*,\mathrm{op}}$ is an isomorphism of sup-lattices, so it suffices to show that it preserves suprema.

Let $S \subseteq \underline{\mathrm{Hom}}(L, L')$. Note that, for a morphism $f\colon L \to L'$, the functors F and F^* being an adjoint pair means precisely that

$$f(x) \leq y \iff x \leq f^*(y)$$

[5] Since morphisms of sup-lattices are not required to preserve infima, we must take opposites to obtain a morphism.

for $x \in L$ and $y \in L'$. Using this, and the fact that f preserves suprema, we get

$$\begin{aligned}\Big(\bigvee_{f\in S} f\Big)^{*,\mathrm{op}}(y) \le x &\iff x \le \Big(\bigvee_{f\in S} f\Big)^{*}(y)\\ &\iff \Big(\bigvee_{f\in S} f\Big)(x) \le y\\ &\iff \bigvee_{f\in S} f(x) \le y\\ &\iff x \le \bigvee_{f\in S} f^{*}(y)\\ &\iff \bigvee_{f\in S} f^{*,\mathrm{op}}(y) \le x.\end{aligned}$$

This precisely says that there is a natural transformation

$$\mathrm{Hom}\Big(\Big(\bigvee_{f\in S} f\Big)^{*,\mathrm{op}}(y), -\Big) \to \mathrm{Hom}\Big(\bigvee_{f\in S} f^{*,\mathrm{op}}(y), -\Big),$$

so by the Yoneda lemma we obtain

$$\Big(\bigvee_{f\in S} f\Big)^{*,\mathrm{op}}(y) \le \bigvee_{f\in S} f^{*,\mathrm{op}}(y).$$

By symmetry, this is in fact an equality. Since y was arbitrary, we obtain

$$\Big(\bigvee_{f\in S} f\Big)^{*,\mathrm{op}} = \bigvee_{f\in S} f^{*,\mathrm{op}}$$

as desired. $\square$

Using the fact that for a morphism $f\colon L \to L'$ we have

$$f(x) \le y \iff x \le f^{*}(y)$$

for all $x \in L$ and $y \in L'$, we can give an explicit description of f^*.[6]

Lemma 1.15. *Let $f\colon L \to L'$ be a morphism and $y \in L'$. Then we have*

$$f^{*}(y) = \bigvee_{f(x)\le y} x.$$

[6]Since $f^* = f^{*,\mathrm{op}}$ as maps of sets, this yields an explicit description of $f^{*,\mathrm{op}}$ as well.

Proof. Since f preserves suprema, we have

$$f\Big(\bigvee_{f(x)\le y} x\Big) = \bigvee_{f(x)\le y} f(x) \le y,$$

which implies

$$\bigvee_{f(x)\le y} x \le f^*(y).$$

For the reverse inequality, note that $f(f^*(y)) \le y$. □

We need one more easy lemma.

Lemma 1.16. *Let J, K and L be sup-lattices. Then we have a natural bijection*

$$Hom(J, \underline{Hom}(K, L)) \to Hom(K, \underline{Hom}(J, L)).$$

Proof. This follows immediately from the fact that a sup-preserving map $J \to \underline{\mathrm{Hom}}(K, L))$ is the same thing as a bi-sup-preserving map $J \times K \to L$. □

We are now ready to define a tensor product on SupLat that realizes our candidate internal hom.

Theorem 1.17. *For K, K', L, L' sup-lattices, $f\colon K \to K'$ and $g\colon L \to L'$, define*

$$K \otimes L = \underline{Hom}(K, L^{op})^{op}$$

and

$$f \otimes g = (g^{*,op} \circ - \circ f)^{*,op}.$$

Then we have an adjunction

$$- \otimes L \dashv \underline{Hom}(L, -).$$

Proof. Let J, K and L be sup-lattices. By Lemma 1.14, we have isomorphisms

$$\begin{aligned}\mathrm{Hom}(J, \underline{\mathrm{Hom}}(K, L)) &\cong \mathrm{Hom}(J, \underline{\mathrm{Hom}}(L^{\mathrm{op}}, K^{\mathrm{op}})) \\ &\cong \mathrm{Hom}(L^{\mathrm{op}}, \underline{\mathrm{Hom}}(J, K^{\mathrm{op}})) \\ &\cong \mathrm{Hom}(\underline{\mathrm{Hom}}(J, K^{\mathrm{op}})^{\mathrm{op}}, L) \\ &= \mathrm{Hom}(J \otimes K, L).\end{aligned}$$

Since these isomorphisms are natural, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(J \otimes K, L) & \xrightarrow{\cong} & \mathrm{Hom}(J, \underline{\mathrm{Hom}}(K, L)) \\ {\scriptstyle g\circ - \circ (f \otimes \mathrm{id}_K)}\downarrow & & \downarrow{\scriptstyle (g\circ -)\circ - \circ f} \\ \mathrm{Hom}(J' \otimes K, L') & \xrightarrow{\cong} & \mathrm{Hom}(J', \underline{\mathrm{Hom}}(K, L')) \end{array}$$

commutes for all sup-preserving maps $f\colon J' \to J$ and $g\colon L \to L'$. □

This leads us to the main result of this section

Theorem 1.18. *$(SupLat, \otimes)$ is a closed tensor category.*

The proof will take up the remainder of this section. The matter of internal homs has already been settled (by construction). Hence, it remains to give compatible associativity and commutativity constraints, as well as an identity object.

Lemma 1.19. *Let J, K and L be sup-lattices. Then we have*

$$J \otimes (K \otimes L) \cong (J \otimes K) \otimes L$$

and

$$K \otimes L \cong L \otimes K.$$

Furthermore, these isomorphisms are natural.

Proof. By Theorem 1.17, we have

$$\begin{aligned} J \otimes (K \otimes L) &= \underline{\mathrm{Hom}}(J, (\underline{\mathrm{Hom}}(K, L^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}} \\ &\cong \underline{\mathrm{Hom}}(J, \underline{\mathrm{Hom}}(K, L^{\mathrm{op}}))^{\mathrm{op}} \\ &\cong \underline{\mathrm{Hom}}(J \otimes K, L^{\mathrm{op}})^{\mathrm{op}} \\ &= (J \otimes K) \otimes L. \end{aligned}$$

For the second statement, applying Lemma 1.14 yields

$$\begin{aligned} K \otimes L &= \underline{\mathrm{Hom}}(K, L^{\mathrm{op}})^{\mathrm{op}} \\ &\cong \underline{\mathrm{Hom}}((L^{\mathrm{op}})^{\mathrm{op}}, K^{\mathrm{op}})^{\mathrm{op}} \\ &\cong \underline{\mathrm{Hom}}(L, K^{\mathrm{op}})^{\mathrm{op}} \\ &= L \otimes K. \end{aligned}$$

Naturality follows from Theorem 1.17 and Lemma 1.14 as well. □

It is straightforward, albeit tedious, to show that these natural isomorphisms form a compatible pair of constraints.

Now we will turn to the matter of identity objects

Lemma 1.20. *Let $\top := \{0, 1\}$ with the obvious ordering. Then $(\top, i)$ is an identity object, where*

$$i\colon \top \otimes \top \xrightarrow{\sim} \top, \quad f \mapsto f(1).$$

Proof. Let L be a sup-lattice, and consider $L \otimes \top = \underline{\mathrm{Hom}}(L, \top^{\mathrm{op}})^{\mathrm{op}}$. For a morphism $f\colon L \to \top^{\mathrm{op}}$, we look at the set $f^{-1}(1)$. Since f sends the least element of L (that is, the supremum of the empty set) to 1, it is non-empty. Using that L is a sup-lattice, we can set

$$\Phi_L(f) := \bigvee_{x \in f^{-1}(1)} x.$$

We now claim that Φ_L is an isomorphism of sup-lattices $L \otimes \top \to L$. To show this, let $S \subseteq L \otimes \top$ be a subset. Then we have

$$\begin{aligned} A := \Big(\bigvee_{f \in S} f\Big)^{-1}(1) = \Big\{x \in L : \Big(\bigvee_{f \in S} f\Big)(x) = 1\Big\} &= \Big\{x \in L : \bigwedge_{f \in S} f(x) = 1\Big\} \\ &= \{x \in L : f(x) = 1 \text{ for some } f \in S\}. \end{aligned}$$

Hence, it follows that

$$\Phi_L\Big(\bigvee_{f \in S} f\Big) = \bigvee_{x \in A} x = \bigvee_{f \in S}\Big(\bigvee_{x \in f^{-1}(1)} x\Big) = \bigvee_{f \in S} \Phi_L(f),$$

so Φ_L is a morphism of sup-lattices. Note that the map

$$\Psi_L\colon L \to L \otimes \top,\ \ x \mapsto \left[y \mapsto \begin{cases} 1, & y \leq x \\ 0, & \text{otherwise} \end{cases}\right]$$

is an inverse of Φ_L on the level of sets. Hence, we conclude that Φ_L is an isomorphism as claimed.

Next, we will show that Φ_L is natural in L. Let $f\colon L \to L'$ be a morphism. Then we need to show that the diagram

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(L, \top^{\mathrm{op}})^{\mathrm{op}} & = & L \otimes \top & \xrightarrow{\Phi_L} & L \\ \big\downarrow{\scriptstyle (-\circ f)^{*,\mathrm{op}}} & & \big\downarrow{\scriptstyle f \otimes \mathrm{id}_\top} & & \big\downarrow{\scriptstyle f} \\ \underline{\mathrm{Hom}}(L', \top^{\mathrm{op}})^{\mathrm{op}} & = & L' \otimes \top & \xrightarrow{\Phi_{L'}} & L' \end{array}$$

commutes. To this end, let $g\colon L \to \top^{\mathrm{op}}$ be a morphism. By Lemma 1.15 we have

$$(- \circ f)^*(g) = \bigvee_{g' \circ f \leq g} g' =: h.$$

Note that h is the unique morphism $L' \to \top^{\mathrm{op}}$ such that

$$\bigvee_{x \in h^{-1}(1)} x = \bigvee_{x \in g^{-1}(1)} f(x),$$

so it follows that

$$(\Phi_{L'} \circ (- \circ f)^{*,\mathrm{op}})(g) = \bigvee_{x \in g^{-1}(1)} f(x).$$

Now since f preserves suprema, we get

$$(f \circ \Phi_L)(g) = f\Big(\bigvee_{x \in g^{-1}(1)} x \Big) = \bigvee_{x \in g^{-1}(1)} f(x),$$

so $f \circ \Phi_L = \Phi_{L'} \circ (- \circ f)^{*,\mathrm{op}}$ as desired. We conclude that the Φ_L induce a natural isomorphism between the functor F given by $L \mapsto L \otimes \top$ and the identity functor, so F is an equivalence of categories.

Finally, note that $\top \otimes \top = \underline{\mathrm{Hom}}(\top, \top^{\mathrm{op}})^{\mathrm{op}}$ consists of two maps. Namely, one, say f, with with $1 \mapsto 0$ and one, say g, with $1 \mapsto 1$. The only nontrivial relation is $f \leq g$. Now i is clearly an isomorphism. □

This proves Theorem 1.18. As a final remark, we will compute duals in SupLat.

Proposition 1.21. *Let L be a sup-lattice. Then*

$$\widehat{L} \cong L^{op}.$$

Proof. We have an isomorphism $\phi \colon \top \to \top^{\mathrm{op}}$ given by $0 \mapsto 1$ and $1 \mapsto 0$. By Lemma 1.20, we have an isomorphism

$$L \xrightarrow{\cong} L \otimes \top \xrightarrow{\mathrm{id}_L \otimes \phi} L \otimes \top^{\mathrm{op}}$$

Hence by Lemma 1.14, this induces an isomorphism

$$\widehat{L} = \underline{\mathrm{Hom}}(L, \top) = (L \otimes \top^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\cong} L^{\mathrm{op}}$$

□

1.6.1 Rigidity

Many types of Tannaka duality theorems, to be discussed later, require the tensor category one works with to be *rigid*. In this subsection we will define this notion and show that SupLat is not rigid, and hence that these theorems are not available in our case.

We will split the definition of rigidity in two, and start with the following definition

Definition 1.22. Let $\mathcal{C}$ be a tensor category. Then $\mathcal{C}$ is called $*$-autonomous if there is a fully faithful functor

$$-^* \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

such that there is a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, Z^*) \cong \mathrm{Hom}(X, (Y \otimes Z)^*).$$

It turns out that SupLat is a $*$-autonomous category.

Theorem 1.23. *SupLat is a $*$-autonomous category.*

Proof. We know that SupLat is a tensor category (see Theorem 1.18). In Lemma 1.14, we proved that the functor given by $L \mapsto L^{\mathrm{op}}$ and $f \mapsto f^{*,\mathrm{op}}$ is contravariant and fully faithful.

Given sup-lattices J, K and L, we have natural isomorphisms

$$\mathrm{Hom}(J \otimes K, L^{\mathrm{op}}) \cong \mathrm{Hom}(J, \underline{\mathrm{Hom}}(K, L^{\mathrm{op}})) \cong \mathrm{Hom}(J, (K \otimes L)^{\mathrm{op}})$$

by Theorem 1.17. □

We can now state the definition of rigidity.

Definition 1.24. A tensor category $\mathcal{C}$ is called rigid if it is $*$-autonomous and we have a natural isomorphism

$$(X \otimes Y)^* \cong X^* \otimes Y^*.$$

Unfortunately, SupLat fails to satisfy this last condition, as we will show

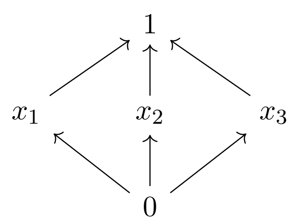
Theorem 1.25. *SupLat is not rigid.*

We prove this by contradiction. Suppose SupLat is rigid. Then for any sup-lattice L we have

$$\underline{\mathrm{Hom}}(L, L) = (L \otimes L^{\mathrm{op}})^{\mathrm{op}} = L^{\mathrm{op}} \otimes L = L \otimes L^{\mathrm{op}} = \underline{\mathrm{Hom}}(L, L)^{\mathrm{op}}.$$

Now let us fix some terminology. Let L be a sup-lattice and let 0 and 1 be its smallest and biggest elements, respectively. Then an element $x \in L$ is called an *atom* if $x \neq 0$ and if $y \in L$ is not 0 then $x \leq y$. Dually, a coatom is an $x \in L$ with $x \neq 1$ and such that $y \leq x$ whenever $y \neq 1$.

Obviously, if $x \in L$ is an atom then x is a coatom in L^{op}. Hence, for any sup-lattice L, we find that $\underline{\mathrm{Hom}}(L, L)$ has as many atoms as coatoms. Now let L be the sup-lattice given by



Consider a sup-preserving map $f\colon L \to L$. Then $f(0) = 0$, and

$$f(x_1) \vee f(x_2) = f(x_1) \vee f(x_3) = f(x_2) \vee f(x_3) = f(1).$$

Hence, f is completely determined by its values at x_1, x_2 and x_3, so we can identify f with the triple $(f(x_1), f(x_2), f(x_3))$. Now it is not very hard to see that $\underline{\mathrm{Hom}}(L, L)$ has precisely 9 coatoms, namely $(x_i, 1, 1)$ and its permutations.

Note that $(x_i, x_i, 0)$ and its permutations are 9 atoms. However, (x_1, x_2, x_3) is also an atom, so SupLat is not rigid.

Chapter 2

Enriched Category Theory

The purpose of this rather technical chapter is to provide the setup to the main theorem of this thesis. we will collect some basic definitions and results from enriched category theory. For a complete treatment, see [6]. Those already familiar with enriched category theory may skip this chapter.

In ordinary category theory, given a category $\mathcal{C}$ and two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have a set, or class, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of arrows $X \to Y$. However, it often turns out that this hom-set carries additional structure. This leads to the notion of *enriched* category theory, where hom-sets are replaced with hom-objects in some suitable category.

2.1 V-Enriched Categories

Definition 2.1. Let V be a tensor category. A V-enriched category, or V-category, $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$ of objects;
- for each ordered pair (X, Y) of objects in $\mathcal{C}$, a hom-object $\underline{\mathrm{Hom}}(X, Y)$ in V;
- for each ordered triple (X, Y, Z) of objects in $\mathcal{C}$, a composition morphism

$$\circ_{X,Y,Z} \colon \underline{\mathrm{Hom}}(Y, Z) \otimes \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(X, Z);$$

- for each object X in $\mathcal{C}$ a morphism $j_X \colon \mathbb{1} \to \underline{\mathrm{Hom}}(X, X)$, called the identity element;

such that the diagrams

$$
\begin{array}{ccc}
\underline{\mathrm{Hom}}(Y,Z)\otimes(\underline{\mathrm{Hom}}(X,Y)\otimes\underline{\mathrm{Hom}}(W,X)) & \xrightarrow{\phi} & (\underline{\mathrm{Hom}}(Y,Z)\otimes\underline{\mathrm{Hom}}(X,Y))\otimes\underline{\mathrm{Hom}}(W,X) \\
 & & \downarrow{\scriptstyle \circ_{X,Y,Z}\otimes\mathrm{id}_{\underline{\mathrm{Hom}}(W,X)}} \\
\downarrow{\scriptstyle \mathrm{id}_{\underline{\mathrm{Hom}}(Y,Z)}\otimes\circ_{W,X,Y}} & & \underline{\mathrm{Hom}}(X,Z)\otimes\underline{\mathrm{Hom}}(W,X) \\
 & & \downarrow{\scriptstyle \circ_{W,X,Z}} \\
\underline{\mathrm{Hom}}(Y,Z)\otimes\underline{\mathrm{Hom}}(W,Y) & \xrightarrow{\circ_{W,Y,Z}} & \underline{\mathrm{Hom}}(W,Z)
\end{array}
$$

and

$$
\begin{array}{ccccc}
\underline{\mathrm{Hom}}(Y,Y)\otimes\underline{\mathrm{Hom}}(X,Y) & \xrightarrow{\circ_{X,Y,Y}} & \underline{\mathrm{Hom}}(X,Y) & \xleftarrow{\circ_{X,X,Y}} & \underline{\mathrm{Hom}}(X,Y)\otimes\underline{\mathrm{Hom}}(X,X) \\
\uparrow{\scriptstyle j_Y\otimes\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}} & \nearrow & & \nwarrow & \uparrow{\scriptstyle \mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}\otimes j_X} \\
\mathbb{1}\otimes\underline{\mathrm{Hom}}(X,Y) & & & & \underline{\mathrm{Hom}}(X,Y)\otimes\mathbb{1}
\end{array}
$$

commute for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Proposition 2.2. *Let V be a closed tensor category. Then V is enriched in itself.*

Proof: (Sketch). In Lemma 1.9 we constructed the composition morphisms. Given $X \in \mathrm{Ob}(V)$, take $j_X : \mathbb{1} \to \underline{\mathrm{Hom}}(X,X)$ to be the map corresponding to $l_X : \mathbb{1} \otimes X \to X$. □

Given V-categories $\mathcal{C}$ and $\mathcal{D}$, we have a V-category $\mathcal{C} \otimes \mathcal{D}$ with

$$\mathrm{Ob}(\mathcal{C}\otimes\mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D}) \quad \text{and} \quad \underline{\mathrm{Hom}}((c,d),(c',d')) = \underline{\mathrm{Hom}}(c,d)\otimes\underline{\mathrm{Hom}}(c',d'),$$

Furthermore, the opposite category $\mathcal{C}^{\mathrm{op}}$ can be given the structure of a V-category. See [6, p. 12] for more details.

2.1.1 Examples

We give some examples of enriched categories that turn op naturally.

- A category enriched in $(\mathrm{Set}, \times)$ is a locally small category.
- A one object category enriched over $(\mathrm{Ab}, \otimes)$ is a ring. A general Ab-enriched category is called a ringoid. Notice the similarity to groupoids.
- In higher category theory, a category enriched over $(\mathrm{Cat}, \times)$ is a 2-category.

2.2 V-Enriched Functors

Of course, having defined V-categories, we need to have a proper notion of morphisms between them.

Definition 2.3. Let V be a tensor category and let $\mathcal{C}$ and $\mathcal{D}$ be V-enriched categories. A V-enriched functor, or simply V-functor, $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- a mapping $\mathrm{Ob}(F)\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$;
- for each ordered pair (X, Y) of objects in $\mathcal{C}$, a morphism

  $$F_{X,Y}\colon \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(FX, FY)$$

  in V;

such that the diagrams

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(Y,Z) \otimes \underline{\mathrm{Hom}}(X,Y) & \xrightarrow{\circ_{X,Y,Z}} & \underline{\mathrm{Hom}}(X,Z) \\ \Big\downarrow{\scriptstyle F_{Y,Z}\otimes F_{X,Y}} & & \Big\downarrow{\scriptstyle F_{X,Z}} \\ \underline{\mathrm{Hom}}(FY,FZ) \otimes \underline{\mathrm{Hom}}(FX,FY) & \xrightarrow[\circ_{FX,FY,FZ}]{} & \underline{\mathrm{Hom}}(FX,FZ) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{1} & & \\ \Big\downarrow{\scriptstyle j_X} & \searrow^{j_{FX}} & \\ \underline{\mathrm{Hom}}(X,X) & \xrightarrow{F_{X,X}} & \underline{\mathrm{Hom}}(FX,FX) \end{array}$$

commute for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

2.2.1 Examples

We will continue with some of the examples from the previous section.

- A Set-functor is simply a functor.
- An Ab-functor between two one object Ab-categories is a ring homomorphism.

2.3 Ends and Enriched Ends

After defining V-categories and V-functors, the obvious next step is figuring out a notion of enriched natural transformations, i.e. morphisms between V-functors. It is not very hard to define such a notion, see [6, p. 9] for example. The problem with this is that it once again leaves us with merely a set, or class, of such natural transformations. This is not in line with the spirit of enriched category theory.

It turns out that the most natural way to define hom-objects corresponding to a pair of V-functors is in terms of ends, a certain categorical construction.

2.3.1 Ordinary Ends

We will first define ends in ordinary category theory. Consider a functor

$$F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}.$$

A wedge $e\colon w \to F$ consists of a $w \in \mathrm{Ob}(\mathcal{D})$ and morphisms $e_X\colon w \to F(X, X)$ indexed by $\mathrm{Ob}(\mathcal{C})$ such that, given a morphism $f\colon X \to Y$, the diagram

$$\begin{array}{ccc} w & \xrightarrow{e_X} & F(X,X) \\ {\scriptstyle e_Y}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}_X, f)} \\ F(Y,Y) & \xrightarrow[F(f,\mathrm{id}_Y)]{} & F(X,Y) \end{array}$$

commutes for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Given a wedge $e\colon w \to F$ and a morphism $f\colon v \to w$, it is easy to see that $ef\colon v \to F$ given by $(ef)_X = (e_X \circ f)$ is a wedge.

An end of F, denoted $\int_{X\in\mathcal{C}} F(X, X)$, is a universal wedge. That is, a wedge

$$e\colon \int_{X\in\mathcal{C}} F(X, X) \to F$$

such that given another wedge $e'\colon w' \to F$, there is a unique morphism $f\colon w' \to \int_{X\in\mathcal{C}} F(X, X)$ such that $e' = ef$.

2.3.2 Enriched Ends

We now turn to enriched ends. The setup is similar as above. Let V be a closed tensor category and $\mathcal{C}$ a V-category. Consider a V-functor

$$F\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to V.$$

As F is contravariant in the first argument, we have a morphism

$$\underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(F(Y, Z), F(X, Z))$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$. This corresponds to a unique morphism

$$\rho_{X,Y,Z}\colon \underline{\mathrm{Hom}}(X, Y) \otimes F(Y, Z) \to F(X, Z).$$

Similarly, using the fact that F is covariant in the second argument, we get morphisms

$$\lambda_{X,Y,Z}\colon \underline{\mathrm{Hom}}(Y, Z) \otimes F(X, Y) \to F(X, Z).$$

Now, a V-wedge $e\colon w \to F$ consists of a $w \in \mathrm{Ob}(V)$ and morphisms $e_X\colon w \to F(X, X)$ such that the diagram

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(X,Y)\otimes w & \xrightarrow{\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}\otimes e_X} & \underline{\mathrm{Hom}}(X,Y)\otimes F(X,X) \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}\otimes e_Y} & & \Big\downarrow{\scriptstyle \lambda_{X,X,Y}} \\
\underline{\mathrm{Hom}}(X,Y)\otimes F(Y,Y) & \xrightarrow[\rho_{X,Y,Y}]{} & F(X,Y)
\end{array}$$

commutes for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Once again, given a V-wedge $e\colon w \to F$ and a morphism $f\colon v \to w$, it is easy to see that $ef\colon v \to F$ given by $(ef)_X = (e_X \circ f)$ is a V-wedge.

A V-end of F, also denoted $\int_{X\in\mathcal{C}} F(X,X)$, is a universal V-wedge. That is, a V-wedge

$$e\colon \int_{X\in\mathcal{C}} F(X,X) \to F$$

such that given another V-wedge $e'\colon w' \to F$, there is a unique morphism

$$f\colon w' \to \int_{X\in\mathcal{C}} F(X,X)$$

such that $e' = ef$.

Remark. One can compute (enriched) ends in terms of equalizers. Hence, if V is complete then enriched ends always exist.

2.4 Enriched Functor Categories

The following result will motivate the construction of V-enriched functor categories.

Proposition 2.4. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be (ordinary) functors. Then*

$$Nat(F,G) = \int_{X\in\mathcal{C}} Hom(FX, GX).$$

Proof. By definition, we have for every object X in $\mathcal{C}$ a map

$$e_X\colon \int_{Y\in\mathcal{C}} \mathrm{Hom}(FY, GY) \to \mathrm{Hom}(FX, GX)$$

of sets. Hence, for every element of $\int_{Y\in\mathcal{C}} \mathrm{Hom}(FY, GY)$ we have a family of morphisms $f_X\colon FX \to GX$ such that for any morphism $g\colon X \to Y$ the diagram

$$\begin{array}{ccc}
F(X) & \xrightarrow{f_X} & G(X) \\
\Big\downarrow{\scriptstyle F(g)} & & \Big\downarrow{\scriptstyle G(g)} \\
F(Y) & \xrightarrow[f_Y]{} & G(Y)
\end{array}$$

commutes. This is precisely the definition of a natural transformation from F to G however. $\square$

With Proposition 2.4 in mind, we will define the V-category $[\mathcal{C}, V]$ of V-valued V-functors from some V-category $\mathcal{C}$.

Proposition 2.5. *Let V be a closed tensor category and $\mathcal{C}$ a V-category. We have a V-category $[\mathcal{C}, V]$ whose*

- *objects are V-functors $\mathcal{C} \to V$;*
- *hom-objects are given by the V-enriched end. That is,*

$$\underline{Nat}(F, G) := \underline{Hom}(F, G) = \int_{X \in \mathcal{C}} \underline{Hom}(FX, GX).$$

Proof: (Sketch). We will construct the composition morphisms and identity elements.

Given V-functors $F, G\colon \mathcal{C} \to V$ and $X \in \mathrm{Ob}(\mathcal{C})$, let $(e_{F,G})_X\colon \underline{\mathrm{Nat}}(F, G) \to \underline{\mathrm{Hom}}(FX, GX)$ be the morphism corresponding to the V-wedge $e_{F,G}\colon \underline{\mathrm{Nat}}(F, G) \to \underline{\mathrm{Hom}}(F-, G-)$. Now let $F, G, H\colon \mathcal{C} \to V$ be V-functors. Then one can check that the morphisms

$$\underline{\mathrm{Nat}}(G, H) \otimes \underline{\mathrm{Nat}}(F, G) \xrightarrow{(e_{G,H})_X \otimes (e_{F,G})_X} \underline{\mathrm{Hom}}(GX, HX) \otimes \underline{\mathrm{Hom}}(FX, GX) \xrightarrow{\circ_{FX,GX,HX}} \underline{\mathrm{Hom}}(FX, HX)$$

form a V-wedge $\underline{\mathrm{Nat}}(G, H) \otimes \underline{\mathrm{Nat}}(F, G) \to \underline{\mathrm{Hom}}(F-, H-)$. Hence, we get a natural morphism

$$\circ_{F,G,H}\colon \underline{\mathrm{Nat}}(G, H) \otimes \underline{\mathrm{Nat}}(F, G) \to \underline{\mathrm{Nat}}(F, H).$$

Similarly, one can show that the identity elements

$$j_{FX}\colon \mathbb{1} \to \underline{\mathrm{Hom}}(FX, FX)$$

form a V-wedge $\mathbb{1} \to \underline{\mathrm{Hom}}(F-, F-)$. Hence, we get a natural morphism

$$j_F\colon \mathbb{1} \to \underline{\mathrm{Nat}}(F, F).$$

$\square$

2.5 The Enriched Yoneda Lemma

Recall the Yoneda Lemma from ordinary category theory

Lemma 2.6 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $X \in Ob(\mathcal{C})$ and $F\colon \mathcal{C} \to Set$ a functor. Then there is a natural bijection*

$$Nat(Hom(X, -), F) \simeq F(X).$$

One can generalize this to the *enriched* Yoneda lemma, or V-Yoneda lemma.

Lemma 2.7 (Enriched Yoneda)**.** *Let V be a complete closed tensor category and $\mathcal{C}$ a V-category. Let $X \in Ob(\mathcal{C})$ and $F : \mathcal{C} \to V$ a V-functor. Then there is a natural isomorphism*

$$\underline{Nat}(\underline{Hom}(X, -), F) := \int_{Y \in \mathcal{C}} \underline{Hom}(\underline{Hom}(X, Y), F(Y)) \simeq F(X).$$

Proof. See [6, section 2.4]. □

Taking $V = \text{Set}$, one recovers the ordinary Yoneda lemma.

Chapter 3

Tannaka Duality for V-modules

Tannaka duality or Tannaka reconstruction theorems are a type of statement in representation theory. They say that in certain cases, given an object A represented on objects of a category $\mathcal{C}$, we can recover A from $\mathrm{End}(F)$, where F is the forgetful functor

$$F\colon \mathrm{Rep}_{\mathcal{C}}(A) \to \mathcal{C}.$$

Here $\mathrm{Rep}_{\mathcal{C}}(A)$ denotes the category of representations of A on objects of $\mathcal{C}$. The first theorem of this type was introduced by Tadao Tannaka, in the context of compact topological groups[1].

In this chapter, we will state and prove a very general Tannaka reconstruction theorem for monoids in a complete closed tensor category V. Using this general theorem, we will prove a Tannaka reconstruction theorem in the case $V = \mathrm{SupLat}$.

3.1 Monoids

We recall the definition of a monoid one more time. It is a set M together with a function $\mu\colon M \times M \to M$ and an element $1 \in M$ such that

$$\mu \circ (\mathrm{id}_M \times \mu) = \mu \circ (\mu \times \mathrm{id}_M)$$

and $\mu(m, 1) = \mu(1, m)$ for all $m \in M$.

We can generalize this notion to a monoid in a tensor category.

Definition 3.1. Let V be a tensor category. A monoid in V is an object M of V together with morphisms $\mu\colon M \otimes M \to M$ and $\eta\colon \mathbb{1} \to M$ such that the diagrams

[1] This theorem is now known as Tannaka-Krein duality.

$$\begin{array}{ccccc}
(M \otimes M) \otimes M & \xleftarrow{\phi} & M \otimes (M \otimes M) & \xrightarrow{\mathrm{id}_M \otimes \mu} & M \otimes M \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}_M} & & & & \downarrow{\scriptstyle \mu} \\
M \otimes M & & \xrightarrow{\qquad\qquad \mu \qquad\qquad} & & M
\end{array}$$

and

$$\begin{array}{ccccc}
\mathbb{1} \otimes M & \xrightarrow{\eta \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes \eta} & M \otimes \mathbb{1} \\
& \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\
& & M & &
\end{array}$$

commute. A morphism of monoids is a morphism $f \colon M \to M'$ such that

$$f \circ \mu = \mu' \circ (f \otimes f) \quad \text{and} \quad f \circ \eta = \eta'.$$

3.1.1 Examples

We will give some simple examples, and finish with monoids in SupLat.

- A monoid in $(\mathrm{Set}, \times)$ is simply a monoid in the classical sense.
- Consider a ring R. Then multiplication in R is given by a biadditive map $R \times R \to R$. Hence, this corresponds to an additive map $R \otimes_{\mathbb{Z}} R \to R$. Therefore, a ring is a monoid in the tensor category $(\mathrm{Ab}, \otimes_{\mathbb{Z}})$.
- Similarly, given a field k, a monoid in $(\mathrm{Vect}_k, \otimes_k)$ is a unital associative k-algebra.
- If $\mathcal{C}$ is a V-category, and $F \colon \mathcal{C} \to V$ a V-functor, then $\underline{\mathrm{End}}(F)$ is a monoid in V.

Let us now turn to monoids in SupLat. It turns out that these are *unital quantales*, see [8].

Definition 3.2. A quantale Q is a sup-lattice with an associative operation $\cdot \colon Q \times Q \to Q$ such that

$$x \cdot \Big(\bigvee_{y \in S} y \Big) = \bigvee_{y \in S} x \cdot y \quad \text{and} \quad \Big(\bigvee_{y \in S} y \Big) \cdot x = \bigvee_{y \in S} y \cdot x$$

for all $x \in Q$ and $S \subseteq Q$. We will call Q unital if it has a unit element 1 for this operation.

Proposition 3.3. *Let Q be a unital quantale. Then Q is a monoid in SupLat.*

Proof. Note that, by definition, the map $\cdot \colon Q \times Q \to Q$ preserves suprema in both arguments. Hence, we obtain a sup-preserving map

$$Q \mapsto \underline{\mathrm{Hom}}(Q, Q), \quad x \mapsto (y \mapsto x \cdot y).$$

By Theorem 1.17, this corresponds to a unique sup-preserving map $\mu \colon Q \otimes Q \to Q$. Next, let 1_Q be the unit element of Q. Then there is a unique sup-preserving map $\eta \colon \top \to Q$ such that $\eta(1) = 1_Q$. The diagrams are now easily verified. $\square$

A morphism between unital quantales Q and Q' is a sup-preserving map $f\colon Q \to Q'$ such that $f(x \cdot y) = f(x) \cdot f(y)$ and $f(1_Q) = 1_{Q'}$.

3.1.2 The Associated V-category

Let M be a monoid in V. We can associate a V-category BM by setting

$$\mathrm{Ob}(BM) = \{\bullet\} \quad \text{and} \quad \underline{\mathrm{Hom}}(\bullet, \bullet) = M,$$

with $\circ_{\bullet,\bullet,\bullet} = \mu$ and $j_\bullet = \eta$.

For example, let R be a ring. In section 4.1.1 we noted that R is a monoid in $(\mathrm{Ab}, \otimes)$. Hence, BR is a one object Ab-category, which also corresponds to a ring by section 3.1.1.

3.2 A General Tannaka Reconstruction Theorem

We now turn to one of the main theorems of this thesis. It is a Tannaka reconstruction theorem for monoids in a complete closed tensor category V. Due to its generality, specializing it to a specific category V takes quite a bit of unwinding of the definitions, which we will see later in the case of $V = \mathrm{SupLat}$. On the other hand, the generality allows us to prove the result very easily, the proof being nothing more than a repeated application of the enriched Yoneda Lemma.

Theorem 3.4. *Let V be a complete closed tensor category and M a monoid in V. Consider the V-category*

$$Mod_M := [BM, V],$$

where BM is the V-category associated to M. Then we have a natural isomorphism

$$\underline{End}(F) \simeq M$$

of monoids in V, where F is the "forgetful functor" $Mod_M \to V$.

Proof. Write $h_\bullet = \underline{\mathrm{Hom}}(\bullet, -)$. Given a V-functor $G\colon BM \to V$, we have

$$\underline{\mathrm{Nat}}(h_\bullet, M) \cong M(\bullet) = FM.$$

by the enriched Yoneda lemma (Lemma 2.7). It follows that we may identify the functors F and $\underline{\mathrm{Nat}}(h_\bullet, -)$. Now we have

$$\begin{aligned}\underline{\mathrm{End}}(F) &:= \underline{\mathrm{Nat}}(F, F) \\ &\cong \underline{\mathrm{Nat}}(\underline{\mathrm{Nat}}(h_\bullet, -), \underline{\mathrm{Nat}}(h_\bullet, -)) \\ &\cong \underline{\mathrm{Nat}}(h_\bullet, h_\bullet) \\ &\cong \underline{\mathrm{Hom}}(\bullet, \bullet) \\ &= M,\end{aligned}$$

where we applied the enriched Yoneda lemma twice. □

Before moving on to $V = \text{SupLat}$, we consider the simplest case $V = \text{Set}$. In this case, a monoid M in V is a monoid in the classical sense. The category

$$\text{Mod}_M = [BM, \text{Set}]$$

can be described as follows. Its objects are pairs (S, ρ), where S is a set and $\rho \colon M \to \text{End}(S)$ is a morphism of monoids. A morphism $f \colon (S, \rho) \to (S', \rho')$ is a map of sets $f \colon S \to S'$ such that for any $m \in M$ the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ {\scriptstyle \rho(m)}\downarrow & & \downarrow{\scriptstyle \rho'(m)} \\ S & \xrightarrow[f]{} & S' \end{array}$$

commutes. Hence, one easily sees that Mod_M is simply the category of sets with a left action of M, and M-equivariant maps. Applying Theorem 3.4 immediately yields the following

Corollary 3.5. *Let M be a monoid. We have an isomorphism of monoids*

$$\underline{End}(F) \simeq M,$$

where $F \colon Mod_M \to Set$ is the forgetful functor.

Corollary 3.6. *Let G be a group. We have a group isomorphism*

$$\underline{Aut}(F) \simeq G,$$

where $F \colon Mod_G \to Set$ is the forgetful functor.

Proof. As G is a group, it follows that so is $\underline{\text{End}}(F)$. Hence,

$$\underline{\text{End}}(F) = \underline{\text{Aut}}(F)$$

□

3.3 A Tannaka Theorem for Unital Quantales

Let us now translate Theorem 3.4 to the case $V = \text{SupLat}$. We already established that monoids in V are unital quantales (Proposition 3.3). The next obvious step is thinking about the category Mod_Q, given a unital quantale Q. To this end, we introduce the following definition.

Definition 3.7. Let Q be a unital quantale. A Q-module M is a sup-lattice together with a bi-sup-preserving map

$$\varphi \colon Q \times M \to M,$$

such that

- for all $q, q' \in Q$ and $m \in M$, we have

$$\varphi(q \cdot q', m) = \varphi(q, \varphi(q', m));$$

- for all $m \in M$, we have

$$\varphi(1, m) = m.$$

We will write $\varphi(q, m) = qm$.

The following Proposition summarizes some basic properties of Q-modules.

Proposition 3.8. *Let Q be a unital quantale and M a Q-module.*

1. *Let 0_Q and 0_M be the smallest elements of Q and M respectively. Then 0_Q acts as zero, i.e.*

$$0_Q m = 0_M$$

for all $m \in M$.

2. *Let $S \subseteq Q$ and $T \subseteq M$. Then we have*

$$\bigvee_{m \in T} \bigvee_{q \in S} qm = \Big(\bigvee_{q \in S} q\Big)\Big(\bigvee_{m \in T} m\Big) = \bigvee_{q \in S} \bigvee_{m \in T} qm.$$

Proof. Both statements follow immediately from the fact that the map $Q \times M \to M$ given by $(q, m) \mapsto qm$ preserves suprema in both arguments. For (1), recall that the least element equals the empty supremum. $\square$

Given Q-modules M and M', a sup-preserving map $f \colon M \to M'$ such that $f(qm) = qf(m)$ for all $q \in Q$ and $m \in M$ is called a Q-linear map. The set $\mathrm{Hom}_Q(M, M')$ of Q-linear maps is a subset of the set $\mathrm{Hom}(M, M')$ of sup-preserving maps. If we can show that $\mathrm{Hom}_Q(M, M')$ is closed under the taking of suprema, then it inherits a sup-lattice structure.

Lemma 3.9. *Let M and M' be Q-modules, and $S \subseteq Hom_Q(M, M')$. Then $\bigvee_{f \in S} f$ is Q-linear.*

Proof. Let $q \in Q$ and $m \in M$. Then

$$\Big(\bigvee_{f \in S} f\Big)(qm) = \bigvee_{f \in S} f(qm) = \bigvee_{f \in S} qf(m) = q \bigvee_{f \in S} f(m) = q\Big(\bigvee_{f \in S} f\Big)(m).$$

$\square$

It does not take much more than this to show that the category $Q\text{Mod}$ of Q-modules and Q-linear maps is a V-category.

Returning to our initial goal, namely thinking about Mod_Q, we note that our choice of names $Q\text{Mod}$ and Mod_Q suggest a connection between these two. Indeed they are essentially the same. To see this, let Q be a unital quantale and BQ its associated V-category. A V-functor $BQ \to V$ then amounts to choosing a sup-lattice M, and a sup-preserving map

$$Q = \underline{\text{Hom}}(\bullet, \bullet) \to \underline{\text{Hom}}(M, M)$$

that respects composition and identity elements. This is precisely the data that constitutes a Q-module however. As for morphisms, we have the following

Lemma 3.10. *Let $F, G\colon BQ \to V$ be V-functors. Then there is an isomorphism*

$$\underline{Nat}(F, G) \simeq \underline{Hom_Q}(F(\bullet), G(\bullet)).$$

Proof. Let

$$e_\bullet \colon \underline{\text{Hom}_Q}(F(\bullet), G(\bullet)) \to \underline{\text{Hom}}(F(\bullet), G(\bullet))$$

be the inclusion. Then for any $f\colon \bullet \to \bullet$ the diagram

$$\begin{array}{ccc} \underline{\text{Hom}_Q}(F(\bullet), G(\bullet)) & \xrightarrow{e_\bullet} & \underline{\text{Hom}}(F(\bullet), G(\bullet)) \\ {\scriptstyle e_\bullet}\downarrow & & \downarrow{\scriptstyle -\circ Ff} \\ \underline{\text{Hom}}(F(\bullet), G(\bullet)) & \xrightarrow{Gf\circ -} & \underline{\text{Hom}}(F(\bullet), G(\bullet)) \end{array}$$

commutes by definition. Hence, we obtain a V-wedge

$$e\colon \underline{\text{Hom}_Q}(F(\bullet), G(\bullet)) \to \underline{\text{Hom}}(F-, G-).$$

It is easy to see that e is universal however, so the result follows. $\square$

Now that we have shown that one can identify $Q\text{Mod}$ and Mod_Q, the last step is to consider the endomorphisms of the forgetful V-functor $F\colon Q\text{Mod} \to \text{SupLat}$. This is done in the next lemma.

Lemma 3.11. *Let Q be a unital quantale. Let L_Q be the sup-lattice consisting of families $f = (f_M)$ indexed over $Q\text{Mod}$, where $f_M\colon M \to M$ is a sup-preserving map, and such that for all Q-linear maps $g\colon M \to M'$ the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f_M} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ M' & \xrightarrow[f_{M'}]{} & M' \end{array}$$

commutes, and such that for $S \subseteq L_Q$ we have

$$\bigvee_{f \in S} f = \Big(\bigvee_{f \in S} f_M\Big)_{M \in QMod}.$$

Then $\underline{End}(F) \cong L_Q$.

Proof. Recall that

$$\underline{\mathrm{End}}(F) = \int_{M \in \mathrm{QMod}} \underline{\mathrm{Hom}}(M, M).$$

For M a Q-module, let $e_M \colon L_Q \to \underline{\mathrm{Hom}}(M, M)$ be the map that sends f to f_M. Then e_M is sup-preserving by definition. Given a Q- linear morphism $g \colon M \to M'$, the diagram

$$\begin{array}{ccc} L_Q & \xrightarrow{e_M} & \underline{\mathrm{Hom}}(M, M) \\ {\scriptstyle e_{M'}}\downarrow & & \downarrow{\scriptstyle g \circ -} \\ \underline{\mathrm{Hom}}(M', M') & \xrightarrow[- \circ g]{} & \underline{\mathrm{Hom}}(M, M') \end{array}$$

commutes, also by definition. Hence, we obtain a V-wedge

$$e \colon L_Q \to \underline{\mathrm{Hom}}(F-, F-).$$

It is not hard to see that e is universal however, so the result follows. □

In fact, we have more

Lemma 3.12. *Let Q be a unital quantale. Then L_Q is a unital quantale, and the isomorphism from the previous lemma is an isomorphism of unital quantales.*

Proof. We can put an operation $- \cdot -$ on L_Q by setting

$$(f \cdot g)_M = g_M \circ f_M$$

for $f, g \in L_Q$. To show this turns L_Q into a quantale, let $S \subseteq L_Q$. Then

$$\begin{aligned} f \cdot \Big(\bigvee_{g \in S} g\Big) = \Big(\Big(\bigvee_{g \in S} g\Big)_M \circ f_M\Big)_M &= \Big(\Big(\bigvee_{g \in S} g_M\Big) \circ f_M\Big)_M \\ &= \Big(\bigvee_{g \in S} (g_M \circ f_M)\Big)_M \\ &= \bigvee_{g \in S} (f \cdot g), \end{aligned}$$

and similarly for

$$\Big(\bigvee_{f \in S} f\Big) \cdot g = \bigvee_{f \in S} (f \cdot g).$$

It is easy to see that

$$1 := (\mathrm{id}_M)_M$$

turns L_Q into a unital quantale, as desired. The fact that the isomorphism from the previous lemma is an isomorphism of unital quantales is obvious. □

We are now ready to state our Tannakian reconstruction theorem for unital quantales.

Theorem 3.13. *Let Q be a unital quantale. We have an isomorphism of unital quantales*

$$L_Q \simeq \underline{End}(F) \simeq Q,$$

where $F\colon QMod \to SupLat$ is the forgetful functor.

Proof. This is clear from Theorem 3.4 and the above discussion. □

3.3.1 Corollaries

In this subsection we derive a few quick but amusing corollaries from Theorem 3.13.

Our first corollary concerns the sup-lattice $\top = \{0 < 1\}$, the identity element of the tensor category SupLat (cf. section 1.6). Note that $\top$ is a (commutative) unital quantale with

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \quad \text{and} \quad 1 \cdot 1 = 1.$$

Now let us think about $\top$-modules. By Proposition 3.8 and the definition of a Q-module, there is a unique $\top$-module structure on any sup-lattice M. Namely

$$0m = 0 \quad \text{and} \quad 1m = m$$

for any $m \in M$. Furthermore, a sup-preserving map $f\colon M \to M'$ between two $\top$-lattices is $\top$-linear, since

$$f(0m) = f(0) = 0 = 0f(m) \quad \text{and} \quad f(1m) = f(m) = 1f(m)$$

for any $m \in M$. Hence, a $\top$-module is nothing more than a sup-lattice. Plugging this into Theorem 3.13, we obtain

Corollary 3.14. *Let $I\colon SupLat \to SupLat$ be the identity functor. Then*

$$L_\top \simeq \underline{End}(I) \simeq \top.$$

In particular, $L_\top$ consists of $(m \mapsto 0)_M$ and $(id_M)_M$.

Proof. Clear from the above discussion. □

The next corollary requires a definition.

Definition 3.15. A unital quantale Q is called *affine* if

$$1_Q = \bigvee_{x \in Q} x.$$

That is, if the unit element equals the largest element.

Taking Q in Theorem 3.13 to be affine yields the following

Corollary 3.16. *Let Q be an affine unital quantale and let $(f_M)_M \in L_Q$. Then for any Q-module M and $m \in M$ we have*

$$f_M(m) \leq m.$$

Proof. By Theorem 3.13 we have an isomorphism of unital quantales $Q \simeq L_Q$. Since an isomorphism of unital quantales preserves the unit as well as suprema, it follows that L_Q is also affine. Hence

$$\Big(\bigvee_{f \in L_Q} f_M \Big)_M = \bigvee_{f \in L_Q} f = (\mathrm{id}_M)_M.$$

It follows that for any Q-module M we have $f_M \leq \mathrm{id}_M$, hence

$$f_M(m) \leq m$$

for any $m \in M$. □

For the final corollary we need one more notion.

Definition 3.17. A quantale Q is called *idempotent* if for all $x \in Q$ we have

$$x \cdot x = x.$$

It turns out that affine (unital) idempotent quantales are very special, as the following lemma shows.

Lemma 3.18. *Let Q be an affine idempotent quantale. Then we have*

$$x \cdot y = x \wedge y$$

for all $x, y \in Q$. That is, the quantale multiplication is the infimum operation.

Proof. We wish to show that

$$q \leq x \text{ and } q \leq y \Longleftrightarrow q \leq x \cdot y.$$

First, if $q \leq x$ and $q \leq y$ then

$$q = q \cdot q \leq x \cdot y.$$

For the converse, if $q \leq x \cdot y$ then

$$q \leq x \cdot y \leq x \cdot 1 = x,$$

and similarly for $q \leq y$. □

Taking Q in Theorem 3.13 to be affine idempotent now yields the following.

Corollary 3.19. *Let Q be an affine idempotent quantale and let $(f_M)_M, (g_M)_M \in L_Q$. Then for any Q-module M and $m \in M$ we have*

$$(g_M \circ f_M)(m) = (f_M \wedge g_M)(m) = \begin{cases} f_M(m), & \textit{if } f_M(m) \leq g_M(m); \\ g_M(m), & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 3.13, we have an isomorphism of unital quantales $L_Q \simeq Q$. Hence, L_Q is also affine and idempotent. By Lemma 3.18, we have

$$g_M \circ f_M = (f \cdot g)_M = (f \wedge g)_M = f_M \wedge g_M.$$

The result now follows. □

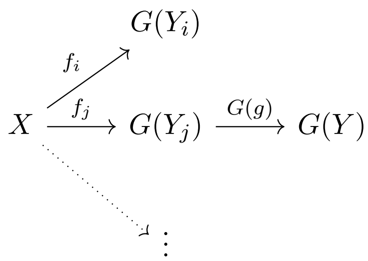
Appendix: Freyd's Adjoint Functor Theorem

It is a well known fact from category theory that if we have an adjoint pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{D}$$

then F preserves all colimits and G preserves all limits. Adjoint functor theorems, popularized by Peter Freyd in [12], state that under certain conditions the converse is true. That is, if F preserves colimits then it is a left adjoint, and if G preserves limits it is a right adjoint.

One such adjoint functor theorem is given in terms of the *Solution Set Condition* (SSC). A functor $G\colon \mathcal{D} \to \mathcal{C}$ satisfies the SSC if for all $X \in \mathrm{Ob}(\mathcal{C})$ there is a family $Y_i \in \mathrm{Ob}(\mathcal{D})$ and morphisms $f_i\colon X \to G(Y_i)$ indexed by a set I, such that any morphism $f\colon X \to G(Y)$ can be written as $f = G(g) \circ f_i$, for some morphism $g\colon Y_i \to Y$.



The theorem is the following

Theorem 3.20. *Let $\mathcal{D}$ be a locally small category that has all small limits. Then a functor $G\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint if and only if G preserves limits and satisfies the SSC.*

The proof is rather involved, and involves comma categories. It turns out in the case of sup-lattices, the SSC is always satisfied.

Lemma 3.21. *Let L and L' be sup-lattices and $f\colon L \to L'$ a sup-preserving map. Then the associated functor $F\colon \mathcal{L} \to \mathcal{L}'$ satisfies the SSC.*

Proof. Translating the SSC to the language of sup-lattices, we see that F satisfies the SSC if and only if for any $x \in L'$ there is a subset $S \subseteq L$ such that if $y \in L$ satisfies $f(y) \leq x$, then $y' \leq y$ for some $y' \in S$.

Now simply take

$$S = \{y \in L : f(y) \leq x\}.$$

□

Corollary 3.22. *Let $f\colon L \to L'$ be a sup-preserving map. Then f is a right adjoint if and only if f preserves infima.*

Proof. Sup-lattices have all infima (limits) by Lemma 1.13. The result follows from Theorem 3.20 and Lemma 3.21. □

By dualizing, we finally obtain the following.

Corollary 3.23. *Let $f\colon L \to L'$ be a sup-preserving map. Then f is a left adjoint, and therefore has a right adjoint.*

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