Fractal dimensions of restricted digit sets for random Lüroth expansions

Dijk, S. van

S.B. van Dijk

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Thesis supervisor: dr. C.C.C.J. Kalle

Leiden University
Mathematical Institute

Abstract

In this thesis we study restricted digit sets for random Lüroth expansions, a type of number expansion for numbers in the unit interval generalising the Lüroth expansions introduced in [Lür83]. By studying a random transformation that generates these expansions and a corresponding iterated function system we find a general formula describing the Hausdorff dimensions of all of these restricted digit sets. We then study a family of two-dimensional fractal sets induced by the skew product representation of this random transformation. In particular, we find conditions under which the box-counting dimension of such a fractal equals the corresponding affinity dimension and use this to find upper and lower bounds.

Contents

1 Introduction

1.1 Historical background

It is a direct consequence of Dirichlet's approximation theorem (1842) that for any real number x there exist infinitely many integer pairs p, q satisfying $\gcd(p, q) = 1$ and $|x - \frac{p}{q}| < \frac{1}{q^2}$. Researchers in the field of Diophantine approximation concern themselves with the study of such approximations of real numbers by rational ones. A particular point of interest in Diophantine approximations is the set of *badly approximable numbers*, which are the numbers x for which a positive constant c exists such that $|x - \frac{p}{q}| > \frac{c}{q^2}$ for every rational $\frac{p}{q}$ with $q \geq 1$.

For any irrational number x a *regular continued fraction expansion* of x is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

for some sequence $a_0, a_1, a_2, ...$ of positive integers called the *digits* of the expansion. The *convergents* of such a regular continued fraction expansion are the rational numbers

$$\frac{p_k}{q_k} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}},$$

for $k \geq 0$. It is known (see [Khi97, Theorem 15]) that all *best approximations* of an irrational number x, defined to be the rationals $\frac{p}{q}$ satisfying $|x - \frac{p}{q}| < |x - \frac{r}{s}|$ for every rational $\frac{r}{s}$ with $0 < s \leq q$ and $\frac{r}{s} \neq \frac{p}{q}$, are convergents of a regular continued fraction expansion of x. Moreover, the badly approximable numbers turn out to be exactly those numbers that have a regular continued fraction expansion with digits $a_0, a_1, a_2, ...$ that are bounded (see [Khi97, Theorem 23]).

More formally, if for any positive integer M we define the set of M-*badly approximable numbers* E_M to be the set of numbers that have a regular continued fraction expansion with digits $a_0, a_1, a_2, ...$ satisfying $a_k \leq M$ for all $k \geq 0$, then the set E of all badly approximable numbers satisfies $E = \bigcup_{M \geq 1} E_M$.

Much research has been done towards quantifying the size of sets of (M-)badly approximable numbers, usually in terms of their Hausdorff dimensions. For instance, it was shown in [Jar29] that E itself has Hausdorff dimension 1 while in [JP18] the first 100 decimal digits of the Hausdorff dimension of the set E_2 were found, improving on earlier estimates found in [Jar29, Goo41, Hen89, Bum06, FN18].

The sets E_M are special cases of *restricted digit sets* for regular continued fraction expansions. For any set A of positive integers, such a set E_A is defined to be the set of numbers that have regular continued fraction expansions for which the digits are all contained in A. As noted in the aforementioned articles these sets typically turn out to be *Cantor sets*, which are nonempty sets that are totally disconnected, compact and devoid of isolated points.

This has sparked a significant interest in researching the dimensions of similar kinds of restricted digit sets for other types of number expansions. In [SF11, MT12, BR22, Zho22] this is done for *Lüroth expansions*, which are number expansions for numbers $x \in [0,1]$ of the form

$$x = \sum_{k=1}^{\infty} \frac{d_k - 1}{\prod_{i=1}^{k} d_i(d_i - 1)},$$

for digits $d_k \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Here the digit ∞ represents a cut-off for the series, making it into a finite sum. These number expansions were introduced by J. Lüroth in 1883 (see [Lür83]). In [KKK90, KKK91, Gan01], a similar but alternating type of number expansion called *alternating Lüroth expansions* is studied. These are number expansions of the form

$$x = \sum_{k=1}^{\infty} (-1)^k \frac{d_k}{\prod_{i=1}^{k} d_i(d_i - 1)},$$

again for digits $d_k \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

In this thesis we study the restricted digit sets for a class of number expansions called *random Lüroth expansions*. These number expansions for numbers $x \in [0,1]$ generalise the Lüroth and alternating Lüroth expansions and are of the form

$$x = \sum_{k=1}^{\infty} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{j=1}^{k} d_j(d_j - 1)},$$

for digits $(s_k, d_k) \in \{0,1\} \times (\mathbb{N}_{\geq 2} \cup \{\infty\})$.

For any set $I \subseteq \mathbb{N}_{\geq 2}$ the corresponding restricted digit set for random Lüroth expansions is the set Λ_I of numbers in $[0,1]$ that have a random Lüroth expansion for which each digit d_k belongs to I. In this thesis we employ tools from the theories of dynamical systems and iterated function systems to find an expression for the Hausdorff dimension of the restricted digit set Λ_I. The first main result of the thesis is that for any $I \subseteq \mathbb{N}_{\geq 2}$ this dimension is given by

$$\dim_{\mathcal{H}} \Lambda_I = \min \Big\{ 1,\ \inf \Big\{ r \ \Big| \ \sum_{d \in I} \Big(\frac{1}{d(d-1)} \Big)^r \leq \frac{1}{2} \Big\} \Big\}.$$

Furthermore, in an attempt to represent two-dimensional restricted digit sets corresponding to restricting both the digits s_k and d_k of random Lüroth expansions we introduce a class of sets in the unit square. These sets F_J^p for $J \subseteq \{0,1\} \times \mathbb{N}_{\geq 2}$ depend on a probability p with which we iteratively choose whether to take s_k to equal 0 or 1 and they turn out to be *box-like sets*, a certain class of two-dimensional fractals (see [Fra12]). When restricting only the digits d_k, the vertical projections of these box-like sets equal the one-dimensional restricted digit sets Λ_I discussed above. The second main result of this thesis provides conditions on J and p for which the box-counting dimension of the box-like set F_J^p equals its *affinity dimension*, a concept introduced by K. Falconer in [Fal88]. We use this result to find certain upper and lower bounds on the box-counting dimension of F_J^p that can be calculated directly from J and p.

1.2 Thesis overview

In Chapter 2 we list the preliminary definitions and results we will need throughout the thesis. More specifically, we define the notions of (discrete-time) dynamical systems and of the box-counting and Hausdorff dimensions of sets. Furthermore, we discuss important results regarding the dimensions of the limit sets of various types of iterated function systems.

In Chapter 3 we discuss Lüroth expansions, alternating Lüroth expansions and generalised Lüroth expansions of numbers in the unit interval, along with the piecewise affine transformations that generate them. Most importantly, we introduce random Lüroth expansions and the random Lüroth transformation, a random system constructed by superimposing the Lüroth and alternating Lüroth transformations.

In Chapter 4 we find a general calculable expression for the Hausdorff and box-counting dimensions of any restricted digit set for random Lüroth expansions. We do this by studying two closed subsystems of the random Lüroth transformation and by analysing the limit sets of corresponding iterated function systems, which turn out to coincide with these restricted digit sets. After discussing implications and examples of these results, we generalise them to a certain class of non-uniform restricted digit sets.

Finally, in Chapter 5 we consider the limit sets of iterated function systems on the unit square induced by the skew product representation of the random Lüroth transformation. These sets are box-like sets, a class of two-dimensional fractals, and in certain cases their vertical projections coincide with the restricted digit sets from Chapter 4. We find conditions under which the box-counting dimensions of these box-like sets equal their affinity dimensions, which we use to find concrete bounds for certain examples.

2 Preliminaries

In this thesis we will use the notations $\mathbb{N} := \mathbb{Z}_{\geq 1}$ and $\mathbb{N}_0 := \mathbb{Z}_{\geq 0}$. Furthermore, for any set A we shall denote the set of m-tuples of elements in A by A^m and the set of one-sided sequences $(x_n)_{n\in\mathbb{N}}$ of elements in A by $A^{\mathbb{N}}$. Familiarity with basic concepts from point set topology and measure theory is assumed throughout. In this chapter we will provide the necessary context from the fields of discrete-time dynamical systems and fractal geometry. More thorough introductions into these respective fields can be found in [DK21] and [Fal04].

2.1 Discrete-time dynamical systems

A dynamical system in discrete time can be defined by considering a space of states equipped with a certain mathematical structure and by letting the time evolution of the system be described by iteration of a self-map on the space that is compatible with its structure. In the field of ergodic theory, the state space is usually a measure space and the evolution is described by a measurable transformation.

Definition 2.1. If $(X, \mathcal{F}, \mu)$ is a measure space and $T : X \to X$ is a measurable transformation, then the quadruple $(X, \mathcal{F}, \mu, T)$ is called a *dynamical system*. The *(forward) orbit* of a point $x \in X$ is the set $\mathcal{O}_T^+(x) := \{T^n x : n \in \mathbb{N}_0\}$, where T^n is used to denote repeated iteration of the map T, i.e. $T^0 x = x$ and

$$T^n x = \underbrace{(T \circ T \circ \cdots \circ T)}_{n \text{ times}}(x) \quad \text{for } n \in \mathbb{N}.$$

2.2 Hausdorff and box-counting dimensions

For any nonempty subset U of $\mathbb{R}^k$ we define the *diameter* of U to be

$$\operatorname{diam}(U) := \sup\{|x - y| : x, y \in U\}.$$

For any $\delta > 0$ a δ*-cover* of $A \subseteq \mathbb{R}^k$ is an at most countable collection $\mathcal{U}$ of subsets of $\mathbb{R}^k$ such that $A \subseteq \bigcup_{U\in\mathcal{U}} U$ and $\operatorname{diam}(U) \leq \delta$ for any $U \in \mathcal{U}$. Now for any subset $A \subseteq \mathbb{R}^k$ and any $s \in \mathbb{R}_{\geq 0}$ we set

$$\mathcal{H}^s_\delta(A) := \inf\Big\{\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^s : \mathcal{U} \text{ is a } \delta\text{-cover of } A\Big\}$$

for any $\delta > 0$ and we define the *s-dimensional Hausdorff measure* of A to be

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A).$$

As explained in Chapter 2 of [Fal04], we have that if $\mathcal{H}^s(A) < \infty$ for some $s \geq 0$, then $\mathcal{H}^t(A) = 0$ for any $t > s$. Hence we can define the *Hausdorff dimension* of A by

$$\dim_{\mathcal{H}} A := \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}.$$

Here we set $\sup \emptyset = 0$. The Hausdorff dimension satisfies the following well-known properties:

1. if $A \subseteq B$ then $\dim_{\mathcal{H}} A \leq \dim_{\mathcal{H}} B$,
2. if A is countable, then $\dim_{\mathcal{H}} A = 0$. Moreover, if A and B are subsets of $\mathbb{R}^k$ such that B is countable, then $\dim_{\mathcal{H}} A = \dim_{\mathcal{H}}(A \cup B) = \dim_{\mathcal{H}}(A \setminus B)$, i.e. adding or removing countable sets does not affect Hausdorff dimension.
3. if A is an open subset of $\mathbb{R}^k$, then $\dim_{\mathcal{H}} A = k$,
4. if A has $\dim_{\mathcal{H}} A < 1$, then A is totally disconnected.

An alternative notion of the dimension of sets in Euclidean space is the box-counting dimension. For any non-empty bounded set $A \subseteq \mathbb{R}^k$ we define

$$N_\delta(A) := \min\{\#\mathcal{U} : \mathcal{U} \text{ is a } \delta\text{-cover of } A\},$$

where $\#S$ denotes the cardinality of a set S. We then define the *lower box-counting dimension* of A by

$$\underline{\dim}_{\mathcal{B}} A := \liminf_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}$$

and the *upper box-counting dimension* of A by

$$\overline{\dim}_{\mathcal{B}} A := \limsup_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

If for a set A the upper and lower box-counting dimensions agree, then the *box-counting dimension* of A is defined by

$$\dim_{\mathcal{B}} A = \lim_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

Both the Hausdorff and the box-counting dimension of a Euclidean set quantify the complexity of its geometrical structure, although the box-counting dimension, when it exists, is slightly more rough than the Haudorff dimension.

Proposition 2.2. *For every non-empty bounded set $A \subseteq \mathbb{R}^k$ we have*

$$\dim_{\mathcal{H}} A \leq \underline{\dim}_{\mathcal{B}} A \leq \overline{\dim}_{\mathcal{B}} A.$$

Proof. See [Fal04, Proposition 3.4]. □

There exist more alternative definitions for the fractal dimension of a Euclidean set, with their values usually in between the Hausdorff and upper box-counting dimensions. Therefore, when Hausdorff and upper box-counting dimensions coincide, all of these notions for the dimension are equal.

2.3 Limit sets of iterated function systems

Let D be a closed subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. A function $\phi : D \to \mathbb{R}^k$ is called a *contraction* if there exists some constant $c < 1$ such that

$$|\phi(x) - \phi(y)| \leq c|x - y|, \quad \text{for every } x, y \in D. \tag{1}$$

A collection $\{\phi_i : i \in I\}$ of injective contractions $\phi_i : D \to \mathbb{R}^k$ for $i \in I$, with I some at most countable index set, is called an *iterated function system (IFS)* on D. For any IFS $\{\phi_i : i \in I\}$ and $n \in \mathbb{N}$ we write

$$\phi_{\mathbf{i}} := \phi_{i_1} \circ \phi_{i_2} \circ ... \circ \phi_{i_n},$$

for any $\mathbf{i} = (i_1, ..., i_n) \in I^n$ and

$$\phi_{\mathbf{i}|_n} := \phi_{i_1} \circ \phi_{i_2} \circ ... \circ \phi_{i_n},$$

for any $\mathbf{i} = (i_1, i_2, i_3, ...) \in I^{\mathbb{N}}$. The *limit set* of an IFS $\{\phi_i : i \in I\}$ on D is then defined to be the set

$$F = \bigcup_{\mathbf{i} \in I^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \phi_{\mathbf{i}|_n}(D).$$

Note that $F = \bigcup_{i \in I} \phi_i(F)$. In fact, if $\{\phi_i : i \in I\}$ is a finite IFS on D, then it was shown by J. Hutchinson in [Hut81] that its limit set F is compact and that it is the unique compact set in D satisfying $F = \bigcup_{i \in I} \phi_i(F)$. For infinite iterated function systems this need not be the case.

The limit set of a finite IFS satisfies the following.

Lemma 2.3. *If F is the limit set of a finite IFS $\{\phi_i : i \in I\}$ on D, then for any non-empty compact subset $E \subseteq D$ that satisfies $\phi_i(E) \subseteq E$ for all $i \in I$ we have $F \subseteq E$.*

Proof. This is a direct consequence of [Fal04, Theorem 9.1]. □

2.3.1 Hausdorff dimensions of self-similar sets

A contraction ϕ on a closed subset $D \subseteq \mathbb{R}^k$ is called a *similarity* if there exists some constant $c < 1$, referred to as the *(similarity) ratio* of ϕ, for which the inequalities given in (1) are equalities, i.e.

$$|\phi(x) - \phi(y)| = c|x - y|, \quad \text{for every } x, y \in D.$$

If all contractions ϕ_i in an IFS $\{\phi_i : i \in I\}$ are similarities, then the limit set F of $\{\phi_i : i \in I\}$ is called a *self-similar set*, as such sets contain similar but smaller copies of themselves. These sets are usually *fractals*, sets that have a nonintegral Hausdorff dimension.

For iterated function systems consisting of similarities, there exist explicit ways to calculate the Hausdorff and box-counting dimensions of the corresponding self-similar set, provided the images of the similarities don't overlap too much. This is captured by the open set condition.

Definition 2.4. An IFS $\{\phi_i : i \in I\}$ satisfies the *open set condition (OSC)* if there exists some bounded and open set $\emptyset \neq U \subseteq D$ such that the sets $\phi_i(U)$ are pairwise disjoint and $\bigcup_{i\in I} \phi_i(U) \subseteq U$.

Proposition 2.5. *Let I be a finite set and let $\{\phi_i : i \in I\}$ be an IFS on $D \subseteq \mathbb{R}^k$ that consists of similarities with ratios $c_i \in (0,1)$ for $i \in I$, and satisfies the OSC. If F is the self-similar set of $\{\phi_i : i \in I\}$, then $\dim_{\mathcal{H}} F = \dim_{\mathcal{B}} F = r$, where r is the unique number that satisfies*

$$\sum_{i\in I} c_i^r = 1.$$

Proof. See [Fal04, Theorem 9.3]. □

2.3.2 Conformal iterated function systems

Note that Proposition 2.5 holds only for finite iterated function systems. For infinite iterated function systems on $\mathbb{R}$ we can say more provided they are conformal.

Definition 2.6. An iterated function system $\{\phi_i : i \in I\}$ on a closed interval $[a,b] \subset \mathbb{R}$ is said to be *conformal* if

(i) $\{\phi_i : i \in I\}$ satisfies the OSC on the interior $U = (a,b)$ of its domain;

(ii) there exists an open connected set $[a,b] \subset V \subset \mathbb{R}$ such that for all $i \in I$ the map ϕ_i extends to a C^1-diffeomorphism on V;

(iii) $\{\phi_i : i \in I\}$ satisfies the *bounded distortion property (BDP)*: there exists some $K \geq 1$ such that $|\phi_{\mathbf{i}}'(x)| \leq K|\phi_{\mathbf{i}}'(y)|$ for any $x, y \in V$, $n \in \mathbb{N}$ and $\mathbf{i} \in I^n$. Here for every $\mathbf{i} \in I^n$, $\phi_{\mathbf{i}}'$ is the derivative of the map $\phi_{\mathbf{i}} = \phi_{i_1} \circ \ldots \circ \phi_{i_n}$.

The notion of conformality is actually more generally defined for iterated function systems on closed connected subsets $D \subset \mathbb{R}^k$, but with some added conditions (see [MU96]).

One of these conditions is that the boundary of D needs to be sufficiently smooth, which is trivially satisfied when D is a closed interval in $\mathbb{R}$. The other condition is that the C^1-diffeomorphisms the contractions ϕ_i extend to in condition (ii) need to be *conformal*, meaning they preserve the angle between any two points. As explained in [Spa22], however, every C^1-diffeomorphism on $\mathbb{R}$ is conformal. Hence we can omit both of these conditions in our definition above.

Definition 2.7. For any IFS $\{\phi_i : i \in I\}$ the *(topological) pressure function* is defined as

$$P(r) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in I^n} \|\phi_{\mathbf{i}}'\|^r, \quad r \in \mathbb{R},$$

where $\|\cdot\|$ denotes the supremum norm.

It is shown in [MU96, Theorem 6.2] that the topological pressure function $P(r)$ of any conformal IFS is nonincreasing in r.

For conformal iterated function systems we have the following result for the dimension of its limit set.

Proposition 2.8. *Let F be the limit set of a conformal IFS $\{\phi_i : i \in I\}$. Then*

$$\dim_{\mathcal{H}} F = \sup\{\dim_{\mathcal{H}} F_J \mid J \subseteq I \text{ finite}\} = \inf\{r \mid P(r) \leq 0\},$$

where for any $J \subseteq I$, F_J denotes the limit set of the IFS $\{\phi_i : i \in J\}$. Furthermore, if $P(r) = 0$ for some r, then r is the only root of P and $\dim_{\mathcal{H}}(F) = r$.

Proof. See [MU96, Theorem 3.15]. □

Hence expressions can be found for the Hausdorff dimension of the limit set of infinite conformal iterated function systems once they are found for their finite counterparts.

Note that for finite systems of similarities we have that the box-counting and Hausdorff dimensions of the corresponding self-similar sets coincide. This is unfortunately not generally the case for the limit sets of infinite conformal iterated function systems, as is proven in [MU96] by means of counterexamples.

2.3.3 Non-autonomous conformal iterated function systems

The iterated function systems that were defined above have been *autonomous*; at each iteration the contractions are taken from the same index set I. In [RGU16] the concept was generalised to *non-autonomous* iterated function systems, defined as follows.

Definition 2.9. A *non-autonomous conformal iterated function system (NCIFS)* on a closed subset $D \subseteq \mathbb{R}$ is a sequence $(\Phi^n)_{n\in\mathbb{N}}$ of conformal iterated function systems $\Phi^n = \{\phi_i^n : i \in I_n\}$ on D, where $\mathbb{I} := (I_n)_{n\in\mathbb{N}}$ is some sequence of at most countable index sets I_n, such that the system is *uniformly contracting*, i.e. there exists some constant $\eta < 1$ such that

$$|(\phi_i^n)'(x)| \leq \eta,$$

for all $n \in \mathbb{N}$, $i \in I_n$ and $x \in D$.

If $\mathbb{I} := (I_n)_{n\in\mathbb{N}}$ is a sequence of at most countable index sets let $\mathbb{I}^k := \prod_{n=1}^k I_n$ denote the collection of k-tuples $(i_1, ..., i_k)$ with $i_n \in I_n$ for each $1 \leq n \leq k$. Similarly, let $\mathbb{I}^{\mathbb{N}} := \prod_{n=1}^\infty I_n$ denote the collection of infinite sequences with $i_n \in I_n$ for each $n \geq 1$. When working with an NCIFS with index sequence $\mathbb{I}$ we use for any $n \in \mathbb{N}$ the notation

$$\phi_{\mathbf{i}|_n} := \phi_{i_1}^1 \circ \phi_{i_2}^2 \circ ... \circ \phi_{i_n}^n,$$

for any sequence $\mathbf{i} = (i_1, i_2, ...) \in \mathbb{I}^{\mathbb{N}}$ and

$$\phi_{\mathbf{i}} := \phi_{i_1}^1 \circ \phi_{i_2}^2 \circ ... \circ \phi_{i_n}^n,$$

for any $\mathbf{i} = (i_1, ..., i_n) \in \mathbb{I}^n$. Note that this same notation was used in the context of an autonomous IFS. It will, however, always be clear from the context whether we are working with an autonomous or a non-autonomous IFS, so confusions should not arise. Just like for autonomous iterated function systems, we can talk about the limit set of an NCIFS.

Definition 2.10. The *limit set* of an NCIFS $(\Phi^n)_{n\in\mathbb{N}} = (\{\phi_i^n : i \in I_n\})_{n\in\mathbb{N}}$ on D with index sequence $\mathbb{I} = (I_n)_{n\in\mathbb{N}}$ is defined as the set

$$F_{\mathbb{I}} := \bigcup_{\mathbf{i}\in\mathbb{I}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \phi_{\mathbf{i}|_n}(D).$$

We define the *lower pressure function* of an NCIFS $(\{\phi_i^n : i \in I_n\})_{n\in\mathbb{N}}$ by

$$\underline{P}(r) := \liminf_{k\to\infty} \frac{1}{k} \log \sum_{\mathbf{i}\in\mathbb{I}^k} \|\phi_{\mathbf{i}}'\|^r, \quad r \geq 0,$$

which takes values in $[-\infty, \infty]$. It was shown in [RGU16, Lemma 2.6] that if the lower pressure function is finite, it is strictly decreasing, i.e. if $r_1 < r_2$ then either $\underline{P}(r_1) = \underline{P}(r_2) \in \{\pm\infty\}$ or $\underline{P}(r_1) > \underline{P}(r_2)$. Note that the lower pressure function is similarly defined as the pressure function of autonomous conformal iterated function systems, but with the limit replaced by a limes inferior. Of course this limes inferior coincides with the limit whenever the latter exists.

An expression for the Hausdorff dimension of the limit set of an NCIFS indexed by an index sequence $\mathbb{I} = (I_n)_{n\in\mathbb{N}}$ exists provided the sets I_n do not grow too fast with n in cardinality. This is captured in the following definition.

Definition 2.11. An index sequence $\mathbb{I} = (I_n)_n$ is said to be of *sub-exponential growth* if I_n is finite for each n and

$$\lim_{n\to\infty} \frac{1}{n} \log \#I_n = 0.$$

Proposition 2.12. *Suppose that $(\Phi^n)_{n\in\mathbb{N}} = (\{\phi_i^n : i \in I_n\})_{n\in\mathbb{N}}$ is a non-autonomous conformal iterated function system such that $\mathbb{I} = (I_n)_n$ is of sub-exponential growth. Then the limit set $F_{\mathbb{I}}$ of $(\Phi^n)_{n\in\mathbb{N}}$ satisfies*

$$\dim_{\mathcal{H}} F_{\mathbb{I}} = \inf\{r \geq 0 : \underline{P}(r) < 0\} = \sup\{r \geq 0 : \underline{P}(r) > 0\}.$$

Proof. See [RGU16, Theorem 1.1]. □

This expression generalises the one for autonomous conformal iterated function systems given in Proposition 2.8, which was instead given in terms of the pressure function.

2.3.4 Self-affine sets and the affinity dimension

In this thesis we will mostly encounter *self-affine sets*, which are the limit sets of autonomous iterated function systems consisting of *affine contractions*, i.e. transformations $\phi : \mathbb{R}^k \supseteq D \to \mathbb{R}^k$ of the form $\phi(x) = L(x) + y$ for some linear contraction $L : D \to \mathbb{R}^k$ and a translation vector $y \in \mathbb{R}^k$.

Definition 2.13. For a contracting and non-singular linear map L the *singular values* are defined as the positive square roots of the eigenvalues of $L^\top L$. If we denote the singular values of L by $1 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k > 0$, then for $0 \leq r \leq k$ the *singular value function* of L is defined by

$$\varphi^r(L) = \alpha_1 \alpha_2 \cdots \alpha_{\lceil r \rceil}^{r - \lceil r \rceil + 1}.$$

For a collection $\{L_i : i \in I\}$ of linear transformations on $\mathbb{R}^k$ and any sequence $\mathbf{i} := (i_1, i_2, ..., i_m) \in I^m$, $m \in \mathbb{N}$, we write

$$L_{\mathbf{i}} := L_{i_1} \circ \cdots \circ L_{i_m},$$

for the repeated matrix multiplication along $\mathbf{i}$, just like we did for compositions of contractions in an iterated function system.

Definition 2.14. Let I be any index set and for any $i \in I$ let $L_i : \mathbb{R}^k \supseteq D \to \mathbb{R}^k$ be a contractive and non-singular linear transformation. The *affinity dimension* of $\{L_i : i \in I\}$ is defined by

$$d(L_i \mid i \in I) := \inf \left\{ r : \sum_{m=1}^{\infty} \sum_{\mathbf{i} \in I^m} \varphi^r(L_{\mathbf{i}}) < \infty \right\},$$

where the second sum runs over all sequences $(i_1, ..., i_m) \in I^m$.

Proposition 2.15. *Let I be some finite index set and for every $i \in I$ let $L_i : \mathbb{R}^k \supseteq D \to \mathbb{R}^k$ be a non-singular linear contraction and $y_i \in \mathbb{R}^k$ a vector. If F is the self-affine set satisfying*

$$F = \bigcup_{i \in I} (L_i(F) + y_i),$$

then $\dim_{\mathcal{H}} F \leq \dim_{\mathcal{B}} F \leq \min\{k, d(L_i | i \in I)\}$.

Proof. See [Fal04, Theorem 9.12]. □

2.3.5 Box-counting dimensions of box-like sets

A particular kind of self-affine sets are the so-called box-like sets. They are the self-affine sets of iterated function systems consisting of affine contractions on the unit square $[0, 1]^2$ whose linear parts are given by (anti-)diagonal matrices. A formal definition is provided below.

The simplest family of box-like sets are the *Bedford-McMullen carpets*, which were introduced in [Bed84] and [McM84]. To construct these sets, we fix integers $n \geq m > 1$ and divide $[0, 1]^2$ into an $m \times n$ grid of rectangles of the same size. An example of such a grid with $m = 3$ and $n = 4$ is shown in Figure 1. We consider the affine transformations that map $[0, 1]^2$ onto each rectangle in the grid while preserving its orientation. For any subset of rectangles in our grid the collection of transformations corresponding to these rectangles forms an IFS.

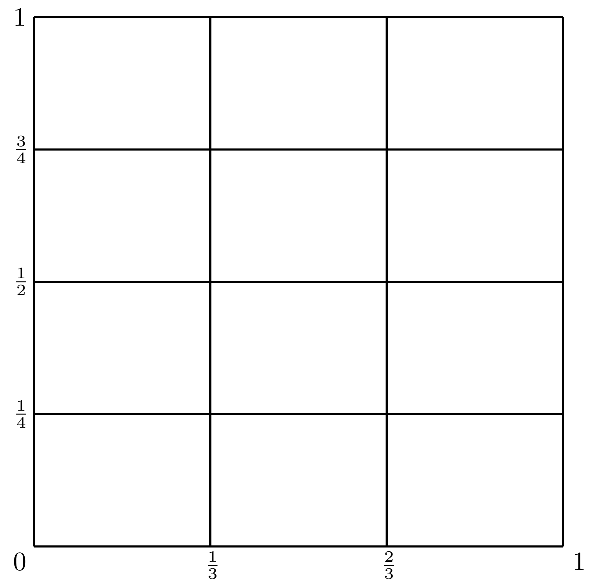


Figure 1: *The rectangle partition corresponding to Bedford-McMullen carpets with $m = 3$ and $n = 4$.*

The self-affine set of such an IFS is then a Bedford-McMullen carpet. Notably, the linear part of each of the affine contractions in this construction is given by the matrix

$$\begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}.$$

Throughout the years, many generalisations of Bedford-McMullen carpets have been studied. In [LG92], *Lalley-Gatzouras carpets* are considered and in [Bar07], *Barański carpets* are introduced, both of which use more general grid-like structures to construct the iterated function systems but still require large restrictions on the linear parts of the contractions. In [FW05], *Feng-Wang carpets* generalise this by allowing arbitrary non-negative diagonal matrices to describe the linear parts of the affine contractions. Finally, and the most relevant for us, in [Fra12] J. Fraser generalised Feng-Wang carpets to general box-like sets by allowing arbitrary contracting (anti-)diagonal matrices.

Definition 2.16. We call a self-affine set *box-like* if it is the limit set of an IFS consisting of affine maps $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\Psi(x) = T \circ S(x) + y,$$

where T is a contracting linear map represented by a diagonal matrix

$$T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

for some $a, b \in (0, 1)$, S is a linear isometry of $\mathbb{R}^2$ such that $J([-1, 1]^2) = [-1, 1]^2$, and $y \in \mathbb{R}^2$ is some translation vector.

In order to study box-like sets we require a condition that is slightly stronger than the previously mentioned open set condition.

Definition 2.17. An IFS $\{\Psi_i : i \in I\}$ on $\mathbb{R}^2$ satisfies the *rectangular open set condition (ROSC)* if there exists a nonempty open rectangle $R = (a,b) \times (c,d)$ in $\mathbb{R}^2$ such that the sets $\Psi_i(R)$ are pairwise disjoint and $\bigcup_{i\in I} \Psi_i(R) \subseteq R$.

Additionally, some of the results below require the box-like set to satisfy the following property.

Definition 2.18. We say a box-like set corresponding to an IFS $\{\Psi_i : i \in I\}$ on $\mathbb{R}^2$ of the form described in Definition 2.16 is of *separated type* if

$$\{i \in I : \Psi_i \text{ maps horizontal lines to vertical lines}\} = \emptyset.$$

For $j \in \{1,2\}$ let $\pi_j : \mathbb{R}^2 \to \mathbb{R}$ denote the projection onto the j-th coordinate and let $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}^2$ be defined by $\chi_1(x) = (x,0)$ and $\chi_2(x) = (0,x)$.

Proposition 2.19. *Let $\{\Psi_i : i \in I\}$ be a finite IFS on $\mathbb{R}^2$ of the form described in Definition 2.16 such that its box-like limit set F is of separated type. For $j \in \{1,2\}$ define*

$$\psi_i^j := \pi_j \circ \Psi_i \circ \chi_j, \qquad i \in I.$$

Then for each $j \in \{1,2\}$, the set $\pi_j(F) \subseteq [0,1]$ is the limit set of the IFS $\{\psi_i^j : i \in I\}$ on $[0,1]$.

Proof. See [Fra12, Lemma 2.7]. □

Not much is currently known about the conditions under which the Hausdorff and box-counting dimensions of self-affine sets, and hence also those of box-like sets, coincide. In fact, there are quite simple examples of Bedford-McMullen carpets that satisfy the ROSC for which the box-counting dimension is strictly larger than the Hausdorff dimension (see [Fal04, Example 9.11]).

The following results yield conditions under which the box-counting dimension of a box-like set equals the affinity dimension of the linear parts of the corresponding IFS.

Proposition 2.20. *Consider a finite IFS $\{\Psi_i : i \in I\}$ on $\mathbb{R}^2$ of the form described in Definition 2.16 and let F denote its box-like limit set. Suppose $\{\Psi_i : i \in I\}$ satisfies the ROSC. If $\dim_{\mathcal{B}} \pi_1(F) = \dim_{\mathcal{B}} \pi_2(F) = 1$, then*

$$\dim_{\mathcal{B}} F = d(L_i \mid i \in I),$$

where the L_i is the linear part of the affine contraction Ψ_i for each $i \in I$.

Proof. See [Fra12, Corollary 2.5] □

Proposition 2.21. *Consider a finite IFS* $\{\Psi_i : i \in I\}$ *on* $\mathbb{R}^2$ *of the form described in Definition 2.16. Suppose this IFS satisfies the ROSC and let* F *denote its box-like limit set. Furthermore, assume that for each* $i \in I$ *the largest singular value for the linear part* L_i *of* Ψ_i *corresponds to the contraction in the horizontal direction. If* $\dim_{\mathcal{B}} \pi_1(F) = 1$*, then*

$$\dim_{\mathcal{B}} F = d(L_i \mid i \in I),$$

Proof. See [Fra12, Corollary 2.6]. □

3 Number systems and Lüroth transformations

Number systems allow us to represent real numbers by sequences of *digits*, chosen from an at most countable set $\mathcal{A}$ of symbols we refer to as an *alphabet*. Preferably these sequences reflect some mathematical structure of the numbers they represent. Typically, the digits in such a sequence correspond to terms in a type of series expansion of the number they represent. Two canonical examples of such number systems are the *binary system* and the *decimal system*, where the digits are taken from the respective alphabets $\{0,1\}$ and $\{0,1,...,9\}$ and sequences $(d_k)_k \in \{0,1\}^{\mathbb{N}}$ and $(\bar{d}_k)_k \in \{0,1,...,9\}^{\mathbb{N}}$ of digits represent a number $x \in [0,1]$ in these systems if

$$x = \sum_{k=1}^{\infty} \frac{d_k}{2^k} \quad \text{and} \quad x = \sum_{k=1}^{\infty} \frac{\bar{d}_k}{10^k}, \quad \text{respectively.}$$

Such series expansions representing numbers in an interval X can often be generated by a certain discrete-time dynamical system $(X, \mathcal{F}, \mu, T)$ and a corresponding partition $\mathcal{P} = \{P_a\}_{a \in \mathcal{A}}$ of X into subintervals. A sequence $(d_k)_k \in \mathcal{A}^{\mathbb{N}}$ of digits then represents a number $x \in X$ if it holds for all $k \in \mathbb{N}$ that $d_k = a$ whenever $T^{k-1}x \in P_a$. This allows many properties of such number expansions to be investigated by analysing the orbits of numbers in the corresponding dynamical system.

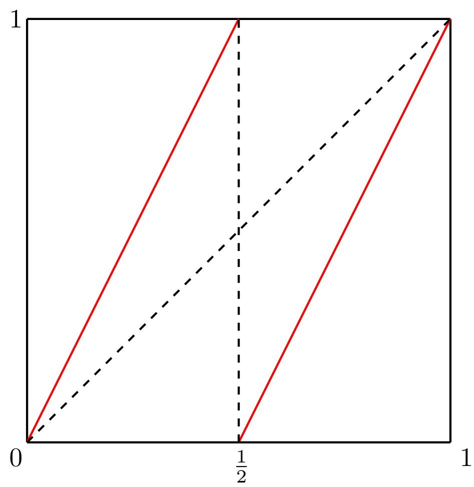


Figure 2: *The doubling map T_D.*

The binary system is a straightforward example of this. For each $x \in [0,1]$ we can generate a digit sequence $(d_k(x))_k \in \{0,1\}^{\mathbb{N}}$ for a binary expansion of x using the *doubling map*. This map $T_D : [0,1] \to [0,1]$ is defined by $T_D x := 2x \mod 1$ and is displayed in Figure 2. These binary digits for x are then generated by setting $d_k(x) = 0$ if $T_D^{k-1}x \in [0, \frac{1}{2})$ and $d_k(x) = 1$ if $T_D^{k-1}x \in [\frac{1}{2}, 1]$ for $k \in \mathbb{N}$.

In this thesis we will consider series expansions for numbers in $[0,1]$ that can be generated by a random dynamical system of so-called Lüroth transformations.

3.1 Lüroth expansions and alternating Lüroth expansions

It was shown by J. Lüroth in [Lür83] that any real number $x \in [0,1]$ can be written as a series expansion of the form

$$x = \sum_{k=1}^{\infty} \frac{d_k - 1}{\prod_{i=1}^{k} d_i(d_i - 1)}, \tag{2}$$

for some sequence $(d_k)_{k\in\mathbb{N}}$ of digits in $\overline{\mathbb{N}}_{\geq 2} := \mathbb{N}_{\geq 2} \cup \{\infty\}$. These series expansions are known as *Lüroth expansions* and the digits for such expansions can be generated by the *Lüroth transformation* $T_L : [0,1] \to [0,1]$, which is defined by

$$T_L x = \begin{cases} d(d-1)x - (d-1), & \text{if} \quad x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

This transformation is shown in Figure 3.

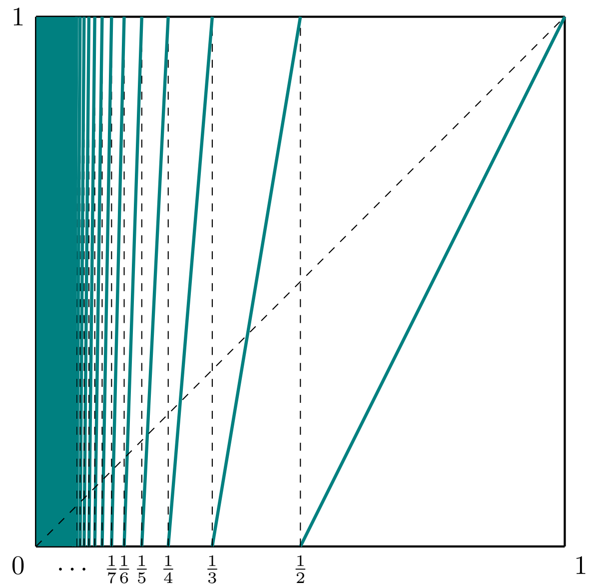


Figure 3: *The Lüroth transformation T_L.*

For each $x \in [0,1]$, the map T_L generates the digit sequence $(d_k(x))_k$ for a Lüroth expansion for x given by

$$d_k(x) = \begin{cases} d, & \text{if} \quad T_L^{k-1}x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ \infty, & \text{if} \quad T_L^{k-1}x = 0, \end{cases} \tag{3}$$

for $k \in \mathbb{N}$. Here the digit ∞ represents a cut-off for the expansion in (2), making the series a finite sum.

With this definition of T_L, the only number in $[0,1]$ that is ever sent into 0 is 0 itself, so the only finite Lüroth expansion generated by (3) is $(d_k(0))_k = (\infty^\infty)$.

Usually (see for instance [JDV69]) one instead considers the map $\overline{T}_L$ given by

$$\overline{T}_L x = \begin{cases} 2x-1, & \text{if} \quad x \in [\frac{1}{2}, 1], \\ d(d-1)x - (d-1), & \text{if} \quad x \in [\frac{1}{d}, \frac{1}{d-1}),\ d \in \mathbb{N}_{\geq 3}, \\ 0, & \text{if} \quad x = 0, \end{cases}$$

which for each $d \in \mathbb{N}_{\geq 2}$ equals T_L on the open interval $(\frac{1}{d}, \frac{1}{d-1})$ but maps $\frac{1}{d}$ to 0 instead of 1. For any $x \in [0,1]$ the map $\overline{T}_L$ also generates a digit sequence $(\overline{d}_k(x))_k$ for a Lüroth expansion of x, which is given by

$$\overline{d}_k(x) = \begin{cases} 2, & \text{if} \quad \overline{T}_L^{k-1} x \in [\frac{1}{2}, 1], \\ d, & \text{if} \quad \overline{T}_L^{k-1} x \in [\frac{1}{d}, \frac{1}{d-1}),\ d \in \mathbb{N}_{\geq 3}, \\ \infty, & \text{if} \quad \overline{T}_L^{k-1} x = 0. \end{cases} \tag{4}$$

In his article, J. Lüroth proved that all Lüroth expansions for numbers $x \in [0,1]$ are described by (3) and (4), and that for only countably many numbers the sequences $(d_k(x))_k$ and $(\overline{d}_k(x))_k$ differ, meaning that all but countably many numbers in $[0,1]$ have a unique Lüroth expansion.

To see this, we define the set

$$\Gamma_L := \{x \in [0,1] \mid T_L^k x \in \{0,1\} \text{ for some } k \in \mathbb{N}\},$$

and note that since every branch of T_L is of the affine form $ax+b$ with $a, b \in \mathbb{Z}$, we necessarily have $\Gamma_L \subseteq \mathbb{Q}$, which implies that Γ_L is countable. Now for any $x \in (0,1) \setminus \Gamma_L$ we have $T_L^{k-1} x = \overline{T}_L^{k-1} \in \bigcup_{d \in \mathbb{N}_{\geq 2}} (\frac{1}{d}, \frac{1}{d-1})$ for every $k \in \mathbb{N}$, implying that $d_k(x) = \overline{d}_k(x)$ for every $k \in \mathbb{N}$. After all, if we had $T_L^{k-1} x = \frac{1}{d}$ for some $k \in \mathbb{N}$ and $d \in \mathbb{N}_{\geq 2}$, then $T_L^k x = 1$, meaning $x \in \Gamma_L$. Indeed for any $x \in [0,1] \setminus \Gamma_L$ the Lüroth expansion described by (3) is its unique Lüroth expansion.

For $x \in \Gamma_L \setminus \{0,1\}$, however, there must exist some $m \in \mathbb{N}$ and $d \in \mathbb{N}_{\geq 2}$ such that $T_L^m x = \frac{1}{d}$. For any $1 \leq k \leq m$ we must then have $T_L^{k-1} x \in \bigcup_{d \in \mathbb{N}_{\geq 2}} (\frac{1}{d}, \frac{1}{d-1})$ and so $d_k(x) = \overline{d}_k(x)$. However, $T_L^m x = \frac{1}{d} \in (\frac{1}{d+1}, \frac{1}{d}]$ then yields $d_k(x) = d+1$ while $\overline{T}_L^m x = T_L^m x = \frac{1}{d} \in [\frac{1}{d}, \frac{1}{d-1})$ yields $\overline{d}_k(x) = d$. Furthermore, for any $k \geq m+2$ we have $T_L^{k-1} x = 1$ and $\overline{T}_L^{k-1} x = 0$, and so $d_k(x) = 2$ while $\overline{d}_k(x) = \infty$. Hence x then has the two distinct Lüroth expansions described by the digit sequences $(d_1, ..., d_m, d_{m+1}, \infty^\infty)$ and $(d_1, ..., d_m, d_{m+1}+1, 2^\infty)$ for some unique $m \in \mathbb{N}$ and $d_1, ..., d_{m+1} \in \mathbb{N}_{\geq 2}$.

In [KKK90] another type of number expansions for numbers $x \in [0,1]$ similar to Lüroth expansions was introduced, namely those of the form

$$x = \sum_{k=1}^{\infty} (-1)^k \frac{d_k}{\prod_{i=1}^k d_i(d_i - 1)}. \tag{5}$$

Due to these expansions being alternating series, they are called *alternating Lüroth expansions*, and the digits d_k of these expansions are elements of the same alphabet $\overline{\mathbb{N}}_{\geq 2} := \mathbb{N}_{\geq 2} \cup \{\infty\}$ as those of Lüroth expansions. These digits are generated in a similar manner by the *alternating Lüroth transformation* $T_A : [0,1] \to [0,1]$, which is defined by

$$T_A x = 1 - T_L x, \quad x \in [0,1].$$

This transformation is displayed in Figure 4.

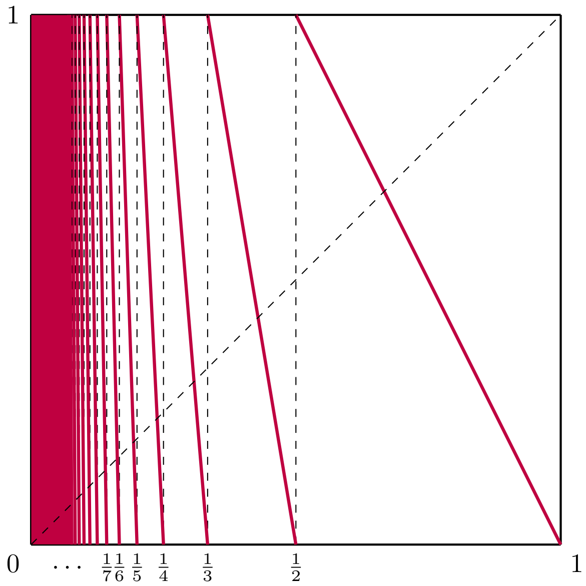


Figure 4: *The alternating Lüroth transformation* T_A.

Again for any $x \in [0,1]$ the map T_A generates the digit sequence $(d_k(x))_k$ for an alternating Lüroth expansion for x given by

$$d_k(x) = \begin{cases} d, & \text{if} \quad T_A^{k-1}x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ \infty, & \text{if} \quad T_A^{k-1}x = 0, \end{cases} \tag{6}$$

while the map $\overline{T}_A := 1 - \overline{T}_L$ generates a second (in most cases identical) alternating Lüroth expansion with digits given by

$$\overline{d}_k(x) = \begin{cases} 2, & \text{if} \quad \overline{T}_A^{k-1}x \in [\frac{1}{2}, 1], \\ d, & \text{if} \quad \overline{T}_A^{k-1}x \in [\frac{1}{d}, \frac{1}{d-1}),\ d \in \mathbb{N}_{\geq 3}, \\ \infty, & \text{if} \quad \overline{T}_A^{k-1}x = 0. \end{cases} \tag{7}$$

Much like for Lüroth expansions, for all but countably many $x \in [0,1]$ the number expansion of x generated by (6) is its unique alternating Lüroth expansion, whereas countably many $x \in [0,1]$ have two distinct alternating Lüroth expansions of the forms $(d_1, ..., d_{m-1}, d_m, 2, \infty, 2, \infty, ...)$ and $(d_1, ..., d_{m-1}, d_m + 1, \infty, 2, \infty, 2, ...)$.

3.2 GLS transformations and number expansions

In [BBDK96] the family $\{T_{(\mathcal{P}_L,\varepsilon)} : \varepsilon \in \{0,1\}^{\mathbb{N}_{\geq 2}}\}$ of $(\mathcal{P}_L,\varepsilon)$-*GLS transformations* $T_{(\mathcal{P}_L,\varepsilon)}$ was introduced, where $\mathcal{P}_L := \left\{(\frac{1}{d}, \frac{1}{d-1}] : d \in \mathbb{N}_{\geq 2}\right\}$ is the *standard Lüroth partition*. For any sequence $\varepsilon \in \{0,1\}^{\mathbb{N}_{\geq 2}}$, $T_{(\mathcal{P}_L,\varepsilon)}$ is defined to be the transformation on $[0,1]$ that for any $d \in \mathbb{N}_{\geq 2}$ equals T_{ε_d} on the partition element $(\frac{1}{d}, \frac{1}{d-1}]$. Here we will from now on often write $T_0 := T_L$ and $T_1 := T_A$. For the constant orientation sequence $\varepsilon = (0)_{d\geq 2}$ the transformation $T_{(\mathcal{P}_L,\varepsilon)}$ is then exactly the Lüroth transformation $T_L = T_0$, whereas $\varepsilon = (1)_{d\geq 2}$ yields the alternating Lüroth transformation $T_A = T_1$. See Figure 5 below for an example of a $(\mathcal{P}_L,\varepsilon)$-GLS transformation.

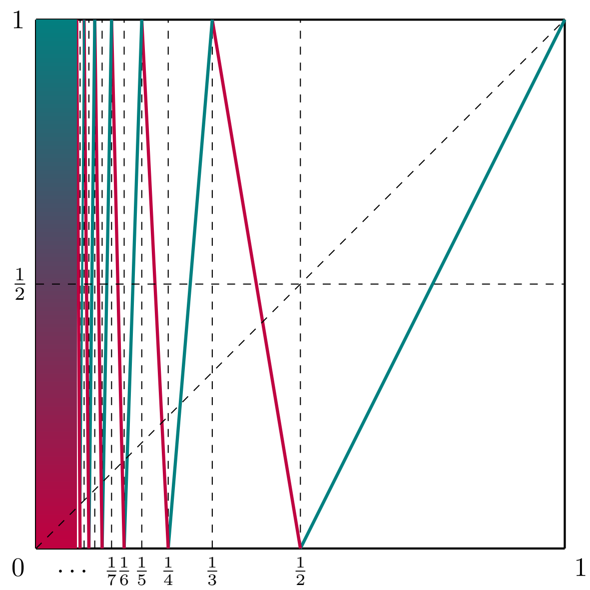


Figure 5: *The $(\mathcal{P}_L,\varepsilon)$-GLS transformation corresponding to the alternating sequence $\varepsilon = (\varepsilon_2, \varepsilon_3, \varepsilon_4, ...) = (0,1,0,1,0,...)$.*

For any $x \in [0,1]$ and $\varepsilon \in \{0,1\}^{\mathbb{N}_{\geq 2}}$ the transformation $T_{(\mathcal{P}_L,\varepsilon)}$ then generates a number expansion for x of the form

$$x = \sum_{k=1}^{\infty} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{j=1}^{k} d_j(d_j - 1)}, \tag{8}$$

with digits $(s_k, d_k) \in \{0,1\} \times \mathbb{N}_{\geq 2}$ for all $k \in \mathbb{N}$. The digit sequence $((s_k, d_k))_k = ((s_k(x), d_k(x)))_k$ for this expansion of x is generated by setting

$$(s_k, d_k) = (\varepsilon_d, d) \quad \text{if} \quad T^{k-1}_{(\mathcal{P}_L,\varepsilon)} x \in (\tfrac{1}{d}, \tfrac{1}{d-1}], \ d \in \mathbb{N}_{\geq 2},$$

for each $k \in \mathbb{N}$. Note that the digits $(d_k)_k$ are generated in the same way the digits for (alternating) Lüroth expansions were generated in (3) and (6), whereas the digits $(s_k)_k$ represent the orientation of the affine branch that governs the k-th step in the orbit of x. In other words, s_k is the element of $\{0,1\}$ that satisfies $T^k_{(\mathcal{P}_L,\varepsilon)} x = T_{s_k}(T^{k-1}_{(\mathcal{P}_L,\varepsilon)} x)$.

As expected, taking $\varepsilon = (0)_{d\geq 2}$ yields $s_k = 0$ for every $k \in \mathbb{N}$, in which case the number expansions in (8) are exactly the Lüroth expansions from (2). Similarly, taking $\varepsilon = (1)_{d\geq 2}$ yields exactly the alternating Lüroth expansions from (5). Note, however, that not all number expansions of the form in (8) can be generated by a GLS-transformation $T_{(\mathcal{P}_L,\varepsilon)}$. After all, any choice of orientation sequence ε fixes the orientation of each branch of the map, which subsequently puts restrictions on the generated expansions; the digits (s_k, d_k) generated by $T_{(\mathcal{P}_L,\varepsilon)}$ are limited to the set $\{(\varepsilon_d, d) : d \in \mathbb{N}_{\geq 2}\}$ meaning that if for any $k, k' \in \mathbb{N}$ we have $d_k = d_{k'}$ we must then also have that $s_k = s_{k'}$.

One way we can generate *all* number expansions of the form in (8) is by superimposing the Lüroth transformation T_L and the alternating Lüroth transformation T_A. This is captured by the *random Lüroth transformation* introduced in [KM22].

3.3 Random Lüroth expansions

Fix some $p \in (0, 1)$ and consider the dynamical system on $[0, 1]$ governed by the *random Lüroth transformation*

$$Tx = \{T_L x, T_A x \; ; \; p, 1-p\}.$$

At each iteration this transformation flips a coin to decide whether to apply the Lüroth transformation T_L or the alternating Lüroth transformation T_A; with probability p the outcome of the coin is 0 and we have $Tx = T_0 x = T_L x$, while with probability $1-p$ the outcome is 1 and so $Tx = T_1 x = T_A x$. The graph of this transformation is displayed in Figure 6 by superimposing those of the Lüroth and alternating Lüroth transformations.

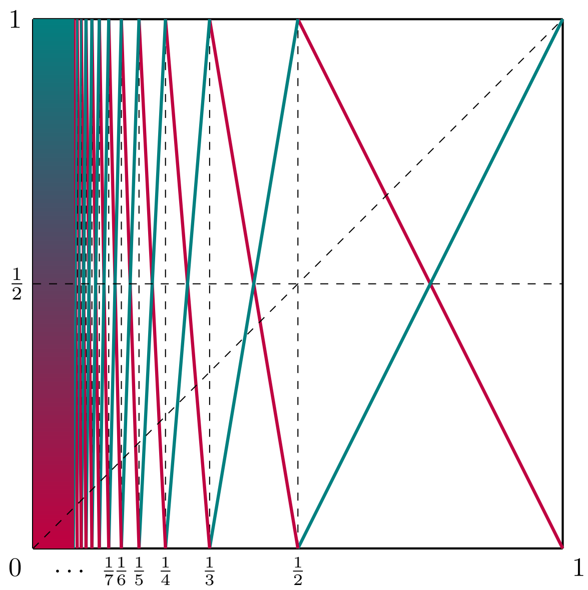


Figure 6: *The random Lüroth transformation.*

We will represent this idea by taking coin flip sequences $\omega = (\omega_1, \omega_2, ...)$ in the sequence space $\{0,1\}^{\mathbb{N}}$ equipped with the p-Bernoulli measure μ_p and considering the orbit of a number x when at time step k we apply the map T_{ω_k}. To keep track of this in a neat way, we introduce for $x \in [0,1]$, $\omega \in \{0,1\}^{\mathbb{N}}$ and $k \in \mathbb{N}$ the notation

$$T_\omega^k x := (T_{\omega_k} \circ T_{\omega_{k-1}} \circ ... \circ T_{\omega_1})x,$$

such that $T_\omega^k x = T_{\omega_k}(T_\omega^{k-1}x)$ for any k. For each $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ this then produces a unique sequence $((s_k, d_k))_k := ((s_k(\omega,x), d_k(\omega,x)))_k$ of digits in $\{0,1\} \times \overline{\mathbb{N}}_{\geq 2}$ given by

$$(s_k(\omega,x), d_k(\omega,x)) = \begin{cases} (\omega_k, d), & \text{if } T_\omega^{k-1}x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ (\omega_k, \infty), & \text{if } T_\omega^{k-1}x = 0, \end{cases} \tag{9}$$

for any $k \in \mathbb{N}$, which satisfies

$$x = \sum_{k=1}^{\infty} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{j=1}^{k} d_j(d_j - 1)}.$$

Hence this method produces the number expansions of the form (8), which we will from now on aptly refer to as *random Lüroth expansions*. For the sake of notational convenience we shall denote the alphabet of these expansions by $\mathcal{D}_\infty := \{0,1\} \times \overline{\mathbb{N}}_{\geq 2}$ and will often refer to the sequences $((s_k, d_k))_k$ themselves as random Lüroth expansions. For some random Lüroth expansion $((s_k, d_k))_k$ we will refer to the digits $s_k \in \{0,1\}$ as the *orientational digits* and to $d_k \in \overline{\mathbb{N}}_{\geq 2}$ as the *Lüroth digits*.

In much the same way as we saw for (alternating) Lüroth expansions, for each $(\omega, x) \in \{0,1\} \times [0,1]$ an alternative (not necessarily distinct) random Lüroth expansion $((\overline{s}_k(\omega,x), \overline{d}_k(\omega,x)))_k \in \mathcal{D}_\infty^{\mathbb{N}}$ is generated by the map

$$\overline{T}x = \{\overline{T}_L x, \overline{T}_A x\ ;\ p, 1-p\},$$

by setting

$$(\overline{s}_k(\omega,x), \overline{d}_k(\omega,x)) = \begin{cases} (\omega_k, 2), & \text{if } \overline{T}_\omega^{k-1}x \in [\frac{1}{2}, 1], \\ (\omega_k, d), & \text{if } \overline{T}_\omega^{k-1}x \in [\frac{1}{d}, \frac{1}{d-1}),\ d \in \mathbb{N}_{\geq 3}, \\ (\omega_k, \infty), & \text{if } \overline{T}_\omega^{k-1}x = 0, \end{cases} \tag{10}$$

for any $k \in \mathbb{N}$. By the same reasoning as before, the only numbers in $[0,1]$ for which some $\omega \in \{0,1\}^{\mathbb{N}}$ exists for which the random Lüroth expansions $(s_k(\omega,x), d_k(\omega,x))_k$ and $(\overline{s}_k(\omega,x), \overline{d}_k(\omega,x))_k$ do not coincide necessarily satisfy $T_\omega^{k-1}x \in \{0,1\}$ for some $k \in \mathbb{N}$. These are therefore all contained in the set

$$\Gamma := \bigcup_{\omega \in \{0,1\}^{\mathbb{N}}} \{x \in [0,1] \mid T_\omega^{k-1}x \in \{0,1\} \text{ for some } k \in \mathbb{N}\}. \tag{11}$$

Since again at any iteration the random Lüroth transformation applies some affine branch with integer coefficients, it can only ever send rational numbers into $\{0,1\}$, and so we have $\Gamma \subseteq \mathbb{Q}$. In particular, Γ is countable.

An alternative way to represent the random Lüroth transformation is by means of the *skew product transformation* $R_p : [0,1]^2 \to [0,1]^2$ defined by

$$R_p(w,x) := (\xi_p(w), T_{\alpha_p(w)}x), \tag{12}$$

where we define $\xi_p : [0,1] \to [0,1]$ and $\alpha_p : [0,1] \to \{0,1\}$ by

$$\xi_p(w) := \begin{cases} \frac{w}{p}, & \text{if } w \in [0,p), \\ \frac{w-p}{1-p}, & \text{if } w \in [p,1], \end{cases} \tag{13}$$

and

$$\alpha_p(w) := \begin{cases} 0, & \text{if } w \in [0,p), \\ 1, & \text{if } w \in [p,1], \end{cases} \tag{14}$$

for any $w \in [0,1]$, where $[0,1]$ is equipped with the Lebesgue measure. The numbers $w \in [0,1]$ then correspond injectively to coin flip sequences $(\alpha(\xi^{k-1}w))_k$ in $\{0,1\}^{\mathbb{N}}$ in a way that preserves probability. This means that for any $w \in [0,1]$ there exists an $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^{k-1}x = \pi_2(R_p^{k-1}(w,x))$ for any $k \in \mathbb{N}$.

Even though the two different representations describe the same random dynamical system, the more direct notation T_ω will prove to be more convenient when studying the dynamics of the system directly, whereas the skew product transformation R_p will allow us to study a class of box-like sets in $[0,1]^2$ induced by the system.

4 Hausdorff dimensions of restricted digit sets

4.1 Restricted digit sets

Our main goal in this chapter is to describe the Hausdorff dimensions of the sets of numbers in $[0,1]$ that have random Lüroth expansions $((s_k,d_k))_k \in \mathcal{D}_\infty^{\mathbb{N}}$ with certain restrictions imposed on the Lüroth digits $(d_k)_k$.

In order to simplify our analysis of these sets, we will henceforth consider only numbers in $X := [0,1] \setminus \Gamma$, with Γ as defined in (11). For the numbers in X all random Lüroth expansions are generated by (9), as those generated by (9) and (10) will coincide. Since we will mostly be concerned with Hausdorff dimensions of sets of numbers in $[0,1]$ and, since Γ is countable, any subset Y of $[0,1]$ has the same Hausdorff dimension as the corresponding subset $Y \setminus \Gamma$ of $[0,1] \setminus \Gamma$, we may restrict to subsets of X without affecting our results.

A consequence of restricting to $X = [0,1] \setminus \Gamma$ is that we need not concern ourselves with numbers that have finite random Lüroth expansions (that is, random Lüroth expansions $((s_k,d_k))_k$ for which $d_k = \infty$ for some $k \in \mathbb{N}$), as these are the numbers that are sent into 0 after some finite iteration of the random Lüroth transformation and hence they are contained in Γ. We may therefore consider from now on the reduced alphabet $\mathcal{D} := \{0,1\} \times \mathbb{N}_{\geq 2}$ rather than the entire alphabet $\mathcal{D}_\infty = \{0,1\} \times \overline{\mathbb{N}}_{\geq 2}$.

For each $x \in X$ we define the set

$$\mathcal{L}_x := \left\{ ((s_k,d_k))_k \in \mathcal{D}^{\mathbb{N}} \;\middle|\; x = \sum_{k=1}^{\infty} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{j=1}^{k} d_j(d_j-1)} \right\},$$

of random Lüroth expansions of x. We can now formalise the idea of imposing restrictions on the Lüroth digits of random Lüroth expansions by means of the following definition.

Definition 4.1. For any nonempty subset $I \subseteq \mathbb{N}_{\geq 2}$ we define the set

$$\Lambda_I := \left\{ x \in X \;\middle|\; \exists ((s_k,d_k))_k \in \mathcal{L}_x \text{ s.t. } d_k \in I \;\forall k \in \mathbb{N} \right\},$$

of numbers that have at least one random Lüroth expansion for which all of the Lüroth digits are exclusively in I. We will refer to Λ_I as the *restricted digit set* corresponding to the *restriction set* I.

One thing we can immediately say about these sets is that if $I \subseteq I' \subseteq \mathbb{N}_{\geq 2}$, then we have $\Lambda_I \subseteq \Lambda_{I'}$, and hence $\dim_{\mathcal{H}} \Lambda_I \leq \dim_{\mathcal{H}} \Lambda_{I'} \leq 1$.

Take any $I \subseteq \mathbb{N}_{\geq 2}$. From (9) and (10) we can see that the numbers in $[0,1]$ that have at least one random Lüroth expansion $((s_k,d_k))_k$ satisfying $d_k \in I$ for every $k \in \mathbb{N}$ are exactly those for which there exists some $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^{k-1} x \in \bigcup_{d \in I} (\frac{1}{d}, \frac{1}{d-1}]$ for all $k \in \mathbb{N}$ or $T_\omega^{k-1} x \in \bigcup_{d \in I} [\frac{1}{d}, \frac{1}{d-1})$ for all $k \in \mathbb{N}$. Hence putting restrictions on the digits in the random Lüroth expansions equates to choosing a number of intervals in the random Lüroth system and finding out which numbers remain in these intervals forever after repeated iteration of T_ω. This is formalised in the following lemma.

Lemma 4.2. *Let I be some subset of $\mathbb{N}_{\geq 2}$. Then we have*

$$\begin{aligned}\Lambda_I &= \Big\{x \in X \;\Big|\; \exists\, \omega \in \{0,1\}^{\mathbb{N}} \textit{ s.t. } T_\omega^{k-1}x \in \bigcup_{d\in I}[\tfrac{1}{d}, \tfrac{1}{d-1}] \;\forall k \in \mathbb{N}\Big\} \\ &= \Big\{x \in [0,1] \;\Big|\; \exists\, \omega \in \{0,1\}^{\mathbb{N}} \textit{ s.t. } T_\omega^{k-1}x \in \bigcup_{d\in I}(\tfrac{1}{d}, \tfrac{1}{d-1}) \;\forall k \in \mathbb{N}\Big\},\end{aligned}$$

Proof. Take any $x \in \Lambda_I$. Then there exists some random Lüroth expansion $((s_k, d_k))_k \in \mathcal{L}_x$ such that $d_k \in I$ for all $k \in \mathbb{N}$. Hence, defining $\omega \in \{0,1\}^{\mathbb{N}}$ by $\omega_k := s_k$ for any $k \in \mathbb{N}$, $((s_k, d_k))_k$ equals the unique random Lüroth expansion $((s_k(\omega, x), d_k(\omega, x))_k$ generated by (9), and so $T_\omega^{k-1}x \in (\frac{1}{d_k}, \frac{1}{d_k-1}] \subset \bigcup_{d\in I}[\frac{1}{d}, \frac{1}{d-1}]$ for any $k \in \mathbb{N}$.

Now take any $x \in X$ for which there exists some $\omega \in \{0,1\}^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$ we have $T_\omega^{k-1}x \in \bigcup_{d\in I}[\frac{1}{d}, \frac{1}{d-1}]$. Since we have $x \in X = [0,1] \setminus \Gamma$ we have $T_\omega^k x \notin \{0,1\}$ and so in particular $T_\omega^{k-1}x \notin \{\frac{1}{d} : d \in \mathbb{N}\}$ for any $k \in \mathbb{N}$. It immediately follows that $T_\omega^{k-1}x \in \bigcup_{d\in I}(\frac{1}{d}, \frac{1}{d-1})$ for every $k \in \mathbb{N}$.

Lastly, take any $x \in [0,1]$ such that there exists some $\omega \in \{0,1\}^{\mathbb{N}}$ satisfying $T_\omega^{k-1}x \in \bigcup_{d\in I}(\frac{1}{d}, \frac{1}{d-1})$ for all $k \in \mathbb{N}$. Since we have $0, 1 \notin \bigcup_{d\in I}(\frac{1}{d}, \frac{1}{d-1})$ it follows that $x \in X$. Moreover, the random Lüroth expansion $((s_k(\omega, x), d_k(\omega, x)))_k$ generated by (9) satisfies $d_k(\omega, x) \in I$ for each k. Hence we have $x \in \Lambda_I$. The above now implies that

$$\begin{aligned}\Lambda_I &\subseteq \Big\{x \in X \;\Big|\; \exists\, \omega \in \{0,1\}^{\mathbb{N}} \text{ s.t. } T_\omega^{k-1}x \in \bigcup_{d\in I}[\tfrac{1}{d}, \tfrac{1}{d-1}] \;\forall k \in \mathbb{N}\Big\} \\ &\subseteq \Big\{x \in [0,1] \;\Big|\; \exists\, \omega \in [0,1] \text{ s.t. } T_\omega^{k-1}x \in \bigcup_{d\in I}(\tfrac{1}{d}, \tfrac{1}{d-1}) \;\forall k \in \mathbb{N}\Big\} \\ &\subseteq \Lambda_I,\end{aligned}$$

and so the three sets coincide. □

This allows us to find expressions for the Hausdorff dimensions of our restricted digit sets directly from studying the dynamics of the random Lüroth transformation. We shall do this by considering two closed subsystems of the random Lüroth transformation on the respective intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, both of which turn out to be deterministic.

4.2 Subsystems of the random Lüroth transformation

For a subinterval $J \subseteq [0,1]$ we can consider a *subsystem* of the random Lüroth transformation by considering a transformation on J that does the following: for any $x \in J$, if we have $T_0x, T_1x \in J$ then it applies T_0 with probability p and T_1 with probability $1-p$, while if we have $T_ix \in J$ for only one $i \in \{0,1\}$ it applies T_i with probability 1. Hence when considering orbits of a number under this new transformation it takes the same coin flip sequences as the random Lüroth transformation, but then manipulates them to ensure that the orbit is contained entirely within J. Of course such a subsystem is well-defined only if for each $x \in J$ there exists some $i \in \{0,1\}$ such that $T_i x \in J$.

Note that for any $x \in [0,1]$ and any $\beta \in \{0,1\}$ we have $T_i x < \frac{1}{2}$ if and only if $T_{1-i} x = 1 - T_i x > \frac{1}{2}$, while otherwise we have $T_0 x = T_1 x = \frac{1}{2}$. It follows inductively that for any $x \in [0,1]$ there exist coin flip sequences $\omega, \omega' \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^k x \in [0, \frac{1}{2}]$ and $T_{\omega'}^k x \in [\frac{1}{2}, 1]$ for each $k \in \mathbb{N}$. In particular, each number in $[0, \frac{1}{2}]$ has at least one orbit under the random Lüroth transformation that is entirely contained in $[0, \frac{1}{2}]$, and analogously for $[\frac{1}{2}, 1]$. Hence we can consider two subsystems of the random Lüroth transformation on these respective intervals.

4.2.1 A deterministic subsystem on $[\frac{1}{2}, 1]$

We will first consider the subsystem of the random Lüroth transformation on the subinterval $[\frac{1}{2}, 1]$, which is shown in Figure 7. This subsystem is a special case of the *c-random Lüroth transformations* introduced in [KM22], which are defined as subsystems of the random Lüroth transformation on the intervals $[c, 1]$ for $0 \leq c \leq \frac{1}{2}$. It was shown that for any $0 \leq c \leq \frac{1}{2}$ this subsystem is well-defined and closed on $[c, 1]$.

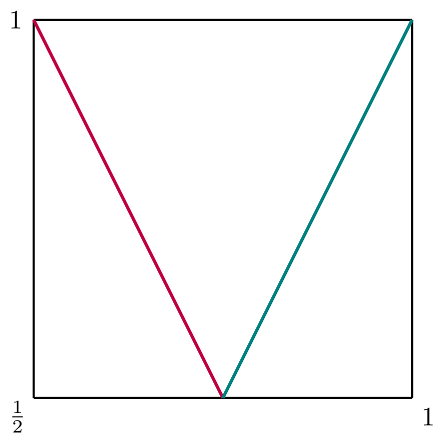


Figure 7: *The closed subsystem of the random Lüroth transformation on* $[\frac{1}{2}, 1]$.

Note that for any $x \in [0,1]$ that is never sent to $\frac{1}{2}$ by any composition of Lüroth and alternating Lüroth transformations, which is particularly the case for any number in $X = [0,1] \backslash \Gamma$, the coin flip sequence $\omega \in \{0,1\}^{\mathbb{N}}$ satisfying $T_\omega^k x \in [\frac{1}{2}, 1]$ for each $k \in \mathbb{N}$ is unique. Therefore, for almost every number in $[\frac{1}{2}, 1]$ this subsystem chooses which of the two transformations to apply deterministically. In particular, for each $x \in X \cap [\frac{1}{2}, 1]$ there exists a unique sequence $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^{k-1} x \in (\frac{1}{2}, 1)$ for each $k \in \mathbb{N}$. This implies the following.

Proposition 4.3. *For any* $I \subseteq \mathbb{N}_{\geq 2}$ *satisfying* $2 \in I$ *we have* $\dim_{\mathcal{H}} \Lambda_I = 1$.

Proof. Let $I \subseteq \mathbb{N}_{\geq 2}$ be any restriction set containing the digit 2. We then have $\Lambda_{\{2\}} \subseteq \Lambda_I$. By Lemma 4.2, $\Lambda_{\{2\}}$ consists of those numbers in $[0,1]$ for which there exists some $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^{k-1}x \in (\frac{1}{2}, 1)$ for each $k \in \mathbb{N}$. This was argued in the above to be the case for any number in $X \cap [\frac{1}{2}, 1]$ and so we have $X \cap [\frac{1}{2}, 1] \subseteq \Lambda_{\{2\}} \subseteq \Lambda_I$, which implies

$$1 = \dim_{\mathcal{H}}[\tfrac{1}{2}, 1] = \dim_{\mathcal{H}}(X \cap [\tfrac{1}{2}, 1]) \leq \dim_{\mathcal{H}} \Lambda_I \leq 1,$$

and hence $\dim_{\mathcal{H}} \Lambda_I = 1$. □

Note that in terms of the random Lüroth expansions themselves we now have that every number in $X \cap [\frac{1}{2}, 1] = [\frac{1}{2}, 1] \setminus \Gamma$ has a unique random Lüroth expansion $((s_k, d_k))_k$ such that $d_k = 2$ for every $k \in \mathbb{N}$. In fact, each number in $\Gamma \cap [\frac{1}{2}, 1]$ also has such a random Lüroth expansion (generated by (10)), but which may not be unique. Hence every $x \in [\frac{1}{2}, 1]$ has some random Lüroth expansion $((s_k, d_k))_k$ such that $d_k = 2$ for every $k \in \mathbb{N}$. In other words, for every $x \in [\frac{1}{2}, 1]$ there exists a sequence $(s_k)_k \in \{0,1\}^{\mathbb{N}}$ such that

$$x = \sum_{n \geq 1} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{1 + s_n}{2^n}.$$

We now know that the restricted digit set corresponding to any restriction set that allows the digit 2 has Hausdorff dimension 1. Therefore we will now turn our attention to the restricted digit sets Λ_I for restriction sets $I \subseteq \mathbb{N}_{\geq 3}$. We will be able to study this case by examining instead the closed subsystem of the random Lüroth transformation on $[0, \frac{1}{2}]$.

4.2.2 A subsystem and an iterated function system on $[0, \frac{1}{2}]$

We consider now the subsystem of the random Lüroth transformation on the subinterval $[0, \frac{1}{2}]$, as shown in Figure 8 below. Just like for the subsystem on $[\frac{1}{2}, 1]$ we have that for any $x \in X \cap [0, \frac{1}{2}]$ the sequence $\omega \in \{0,1\}^{\mathbb{N}}$ keeping the orbit of x entirely contained within $[0, \frac{1}{2}]$ is unique. Hence this subsystem is deterministic as well.

In much the same way the existence of a closed subsystem of the random Lüroth transformation on $[\frac{1}{2}, 1]$ implied Proposition 4.3, the existence of this subsystem on $[0, \frac{1}{2}]$ implies the following.

Proposition 4.4. *Every $x \in X \cap [0, \frac{1}{2}]$ has a unique random Lüroth expansion $((s_k, d_k))_k$ such that $d_k \geq 3$ for every $k \in \mathbb{N}$. In particular, we have*

$$\dim_{\mathcal{H}} \Lambda_{\mathbb{N}_{\geq 3}} = 1.$$

Proof. Take any $x \in X \cap [0, \frac{1}{2}]$ and let $\omega \in \{0,1\}^{\mathbb{N}}$ be the unique sequence satisfying $T_\omega^{k-1}x \in [0, \frac{1}{2}]$ for each k. In particular, as we have $x \in X = [0,1] \setminus \Gamma$, we have $T_\omega^{k-1}x \neq 0$ and hence $T_\omega^{k-1}x \in (0, \frac{1}{2}] = \bigcup_{d \in \mathbb{N}_{\geq 3}} [\frac{1}{d}, \frac{1}{d-1}]$ for each $k \in \mathbb{N}$, so by Lemma 4.2 we have $x \in \Lambda_{\mathbb{N}_{\geq 3}}$. Hence x has a random Lüroth expansion satisfying $d_k \geq 3$ for every $k \in \mathbb{N}$. This implies that $X \cap [0, \frac{1}{2}] \subseteq \Lambda_{\mathbb{N}_{\geq 3}}$ and hence

$$1 = \dim_{\mathcal{H}}[0, \tfrac{1}{2}] = \dim_{\mathcal{H}}(X \cap [0, \tfrac{1}{2}]) \leq \dim_{\mathcal{H}} \Lambda_{\mathbb{N}_{\geq 3}} \leq 1,$$

yielding $\dim_{\mathcal{H}} \Lambda_{\mathbb{N}_{\geq 3}} = 1$. □

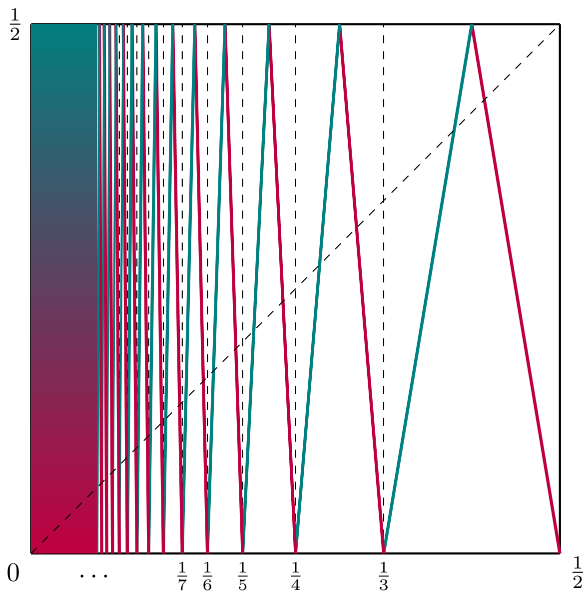


Figure 8: *The closed subsystem of the random Lüroth transformation on* $[0, \frac{1}{2}]$.

The uniqueness of the random Lüroth expansions with digits in $\mathbb{N}_{\geq 3}$ is in stark contrast with the fact that each number in $(0, 1]$ has uncountably many different random Lüroth expansions. It leads to think that more interesting results will follow once we add restrictions on top of the removal of the digit 2.

We will proceed to construct a family of iterated function systems induced by this subsystem on $[0, \frac{1}{2}]$. We will see that the corresponding limit sets coincide with the restricted digit sets Λ_I for $I \subseteq \mathbb{N}_{\geq 3}$. This will allow us to apply the theory of iterated function systems to acquire expressions for the fractal dimensions of these restricted digit sets.

In order to define these iterated function systems, we will need to introduce a notation for each affine branch of the random Lüroth transformation. Hence for any $d \in \mathbb{N}_{\geq 2}$ we define the affine mappings $T_{0,d}, T_{1,d} : \mathbb{R} \to \mathbb{R}$ by

$$T_{0,d}x = d(d-1)x - (d-1) \quad \text{and} \quad T_{1,d}x = 1 - T_{0,d}x = d - d(d-1)x,$$

such that

$$T_L x = T_0 x = \begin{cases} T_{0,d}x, & \text{if } x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$T_A x = T_1 x = \begin{cases} T_{1,d}x, & \text{if } x \in (\frac{1}{d}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 2}, \\ 1, & \text{if } x = 0. \end{cases}$$

For any $(s, d) \in \{0, 1\} \times \mathbb{N}_{\geq 3}$ the map $T_{s,d}$ is an affine bijection from $\mathbb{R}$ to $\mathbb{R}$, allowing us to define the maps $\phi_{s,d} : [0, \frac{1}{2}] \to \mathbb{R}$ by setting

$$\phi_{s,d} = T_{s,d}^{-1}|_{[0,\frac{1}{2}]}, \quad (s, d) \in \{0, 1\} \times \mathbb{N}_{\geq 3}.$$

For each $x \in [0, \frac{1}{2}]$ we then have

$$\phi_{0,d}(x) = \frac{1}{d(d-1)}x + \frac{1}{d} \quad \text{and} \quad \phi_{1,d}(x) = \frac{1}{d-1} - \frac{1}{d(d-1)}x.$$

These affine maps $\phi_{s,d}$ for $(s,d) \in \mathcal{D}$ are shown in Figure 9.

Note now that

$$\phi_{0,d}([0, \tfrac{1}{2}]) = [\tfrac{1}{d}, \tfrac{1}{d} + \tfrac{1}{2d(d-1)}] \subset [0, \tfrac{1}{2}],$$

and

$$\phi_{1,d}([0, \tfrac{1}{2}]) = [\tfrac{1}{d-1} - \tfrac{1}{2d(d-1)}, \tfrac{1}{d-1}] = [\tfrac{1}{d} + \tfrac{1}{2d(d-1)}, \tfrac{1}{d-1}] \subset [0, \tfrac{1}{2}],$$

and so $\{\phi_{s,d} : (s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}\}$ is a family of injective self-maps on $[0, \frac{1}{2}]$.

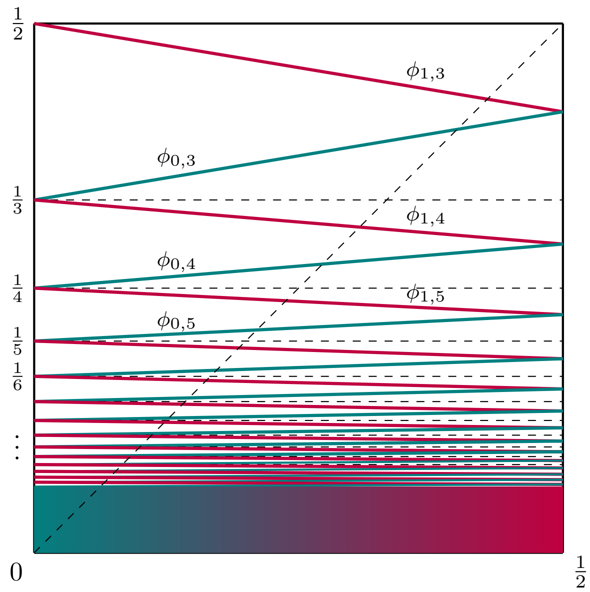


Figure 9: *The affine maps $\phi_{s,d}$ for $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$.*

More importantly, the maps satisfy the following.

Lemma 4.5. *For any $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$ the map $\phi_{s,d}$ is a contractive similarity with ratio $c_{s,d} := \frac{1}{d(d-1)}$.*

Proof. Take any $d \in \mathbb{N}_{\geq 3}$. Then for any $x, y \in [0, \frac{1}{2}]$ we have

$$\begin{aligned} |\phi_{0,d}(x) - \phi_{0,d}(y)| &= \left| \frac{1}{d(d-1)}x + \frac{1}{d} - \frac{1}{d(d-1)}y - \frac{1}{d} \right| \\ &= \left| \frac{1}{d(d-1)}(x-y) \right| = \frac{1}{d(d-1)}|x-y|, \end{aligned}$$

and

$$|\phi_{1,d}(x) - \phi_{1,d}(y)| = \left| -\frac{1}{d(d-1)}x + \frac{1}{d-1} + \frac{1}{d(d-1)}y - \frac{1}{d-1} \right|$$
$$= \left| -\frac{1}{d(d-1)}(x-y) \right| = \frac{1}{d(d-1)}|x-y|.$$

Since we have $\frac{1}{d(d-1)} < 1$ for any $d \geq 3$, the claim follows. □

Therefore for each $I \subseteq \mathbb{N}_{\geq 3}$ the collection $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ is an iterated function system on $[0, \frac{1}{2}]$ of similarities, and so we can study its self-similar set. In fact, these self-similar sets are exactly the restricted digit sets we are interested in.

Proposition 4.6. *For any restriction set $I \subseteq \mathbb{N}_{\geq 3}$ the self-similar set of the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ on $[0, \frac{1}{2}]$ coincides with the restricted digit set Λ_I.*

Proof. Take any $I \subseteq \mathbb{N}_{\geq 3}$ and let F_I denote the self-similar set of the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$. First off, let $x \in \Lambda_I$. By Lemma 4.2 there exists some $\omega \in \{0,1\}^{\mathbb{N}}$ such that $T_\omega^{k-1}x \in \bigcup_{d \in I}[\frac{1}{d}, \frac{1}{d-1}]$ for all $k \in \mathbb{N}$. Since

$$T_\omega^k x = \begin{cases} T_{0,d}(T_\omega^{k-1}x), & \text{if } T_\omega^{k-1}x \in [\frac{1}{d}, \frac{1}{d} + \frac{1}{2d(d-1)}],\ d \in \mathbb{N}_{\geq 3}, \\ T_{1,d}(T_\omega^{k-1}x), & \text{if } T_\omega^{k-1}x \in [\frac{1}{d} + \frac{1}{2d(d-1)}, \frac{1}{d-1}],\ d \in \mathbb{N}_{\geq 3}, \end{cases}$$

for any $k \in \mathbb{N}$, there must exist some sequence $((s_k, d_k))_k$ in $\{0,1\} \times \mathbb{N}_{\geq 3}$ such that $T_\omega^k x = T_{s_k,d_k}(T_\omega^{k-1}x)$ for any k. Here we note that if we have $x \in \{\frac{1}{d}, \frac{1}{d} + \frac{1}{2d(d-1)} : d \in \mathbb{N}_{\geq 3}\}$, then $T_{0,d}(T_\omega^{k-1}x) = T_{1,d}(T_\omega^{k-1}x)$ holds, so there is some ambiguity in how we choose to construct this sequence $((s_k, d_k))_k$. A natural choice, of course, is to take $s_k = \omega_k$ everywhere. Iteratively, we then get for all k that $T_\omega^k x = (T_{s_k,d_k} \circ ... \circ T_{s_1,d_1})x$, and so

$$x = (T_{s_1,d_1}^{-1} \circ ... \circ T_{s_k,d_k}^{-1})(T_\omega^k x) = (\phi_{s_1,d_1} \circ ... \circ \phi_{s_k,d_k})(T_\omega^k x).$$

Since we have $T_\omega^k x \in [0, \frac{1}{2}]$ for any k, this implies that $x \in F_I$, yielding $\Lambda_I \subseteq F_I$.

Conversely, take some $x \in F_I$. Then there exist sequences $((s_k, d_k))_k$ in $\{0,1\} \times I$ and $(y_k)_k$ in $[0, \frac{1}{2}]$ such that $x = (\phi_{s_1,d_1} \circ ... \circ \phi_{s_n,d_n})(y_n)$, or equivalently $y_k = (T_{s_k,d_k} \circ ... \circ T_{s_1,d_1})x$ for every $k \in \mathbb{N}$. Note that by the injectivity of the maps $\phi_{s,d}$, we must have that $y_{k-1} = \phi_{s_k,d_k}(y_k)$, where we write $y_0 = x$, and so we must have

$$y_{k-1} \in \phi_{s_k,d_k}([0, \tfrac{1}{2}]) = \begin{cases} [\frac{1}{d_k}, \frac{1}{d_k} + \frac{1}{2d_k(d_k-1)}], & \text{if } s_k = 0, \\ [\frac{1}{d_k} + \frac{1}{2d_k(d_k-1)}, \frac{1}{d_k-1}], & \text{if } s_k = 1, \end{cases}$$

for every k. Hence, if we define $\omega \in \{0,1\}^{\mathbb{N}}$ by setting $\omega_k = s_k$ for every k, then $y_k = T_{s_k,d_k}(y_{k-1}) = T_{s_k}(y_{k-1}) = T_{\omega_k}(y_{k-1})$, and so inductively we get that

$$y_k = T_{\omega_k}(y_{k-1}) = T_{\omega_n}(T_{\omega_{k-1}}(y_{k-2})) = ... = (T_{\omega_k} \circ ... \circ T_{\omega_1})y_0 = T_\omega^k x,$$

and so in particular $T_\omega^{k-1}x = y_{k-1} \in [\frac{1}{d_k}, \frac{1}{d_k-1}] \subseteq \bigcup_{d \in I}[\frac{1}{d}, \frac{1}{d-1}]$, for every k. Note that, since $0, 1 \notin [\frac{1}{d}, \frac{1}{d-1}]$ for any $d \in \mathbb{N}_{\geq 3}$, this also implies that $T_\omega^{k-1}x \notin \{0,1\}$ for every $k \in \mathbb{N}$, and so $x \in [0,1] \setminus \Gamma = X$. It follows that $x \in \Lambda_I$ and hence we conclude that $F_I = \Lambda_I$. □

Because of this, we will now be able to find expressions for the Hausdorff dimension of the restricted digit sets Λ_I by applying the dimension theory for self-similar sets to the corresponding IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$.

4.3 Main results

Dividing the random Lüroth transformations into two deterministic subsystems on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ has supplied us with the tools to find calculable expressions for the Hausdorff dimension of every restricted digit set for random Lüroth transformations.

In Proposition 4.3 we already found that any restricted digit set Λ_I corresponding to a restriction set $I \subseteq \mathbb{N}_{\geq 2}$ that includes the digit 2 has Hausdorff dimension 1. For finite restriction sets $I \subseteq \mathbb{N}_{\geq 3}$ we find the following.

Theorem 4.7. *For any finite set $I \subseteq \mathbb{N}_{\geq 3}$ we have $\dim_{\mathcal{H}} \Lambda_I = r$, where r is the unique real number that satisfies*

$$\sum_{d \in I} \Big(\frac{1}{d(d-1)}\Big)^r = \frac{1}{2}.$$

Proof. Take any finite set $I \subseteq \mathbb{N}_{\geq 3}$ and consider the finite iterated function system $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$. The self-similar set of this IFS equals the restricted digit set Λ_I by Proposition 4.6.

We will show that $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ satisfies the OSC as in Definition 2.4. To this end, consider the open subset $(0, \frac{1}{2})$ of $[0, \frac{1}{2}]$. For any $d \in I$ we have $\phi_{0,d}((0, \frac{1}{2})) = (\frac{1}{d}, \frac{1}{d} + \frac{1}{2d(d-1)})$ and $\phi_{1,d}((0, \frac{1}{2})) = (\frac{1}{d} + \frac{1}{2d(d-1)}, \frac{1}{d-1})$, and so the sets $\phi_{s,d}((0, \frac{1}{2}))$ are pairwise disjoint. Furthermore, we have

$$\begin{aligned}\bigcup_{(s,d) \in \{0,1\} \times I} \phi_{s,d}((0, \tfrac{1}{2})) &= \bigcup_{d \in I} \Big((\tfrac{1}{d}, \tfrac{1}{d} + \tfrac{1}{2d(d-1)}) \cup (\tfrac{1}{d} + \tfrac{1}{2d(d-1)}, \tfrac{1}{d-1}) \Big) \\ &\subset \bigcup_{d \in I} (\tfrac{1}{d}, \tfrac{1}{d-1}) \subset (0, \tfrac{1}{2}),\end{aligned}$$

and hence $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ satisfies the OSC.

Since the similarity ratios $c_{s,d}$ of each $\phi_{s,d}$ satisfy $c_{s,d} = \frac{1}{d(d-1)} \in (0,1)$ by Lemma 4.5, it follows from Proposition 2.5 that $\dim_{\mathcal{H}} \Lambda_I = r$, where r is the unique number that satisfies $\sum_{(s,d) \in \{0,1\} \times I} c_{s,d}^r = 1$. Since $c_{0,d} = c_{1,d} = \frac{1}{d(d-1)}$ for each $d \in I$, we have for each r that

$$\sum_{(s,d) \in \{0,1\} \times I} c_{s,d}^r = \sum_{d \in I} (c_{0,d}^r + c_{1,d}^r) = \sum_{d \in I} 2\Big(\frac{1}{d(d-1)}\Big)^r = 2\sum_{d \in I} \Big(\frac{1}{d(d-1)}\Big)^r.$$

Therefore $\dim_{\mathcal{H}} \Lambda_I = r$, where r is the unique number that satisfies

$$2\sum_{d \in I} \Big(\frac{1}{d(d-1)}\Big)^r = 1,$$

or equivalently,

$$\sum_{d \in I} \Big(\frac{1}{d(d-1)}\Big)^r = \frac{1}{2}.$$ □

With this result proven, we can generalise to the case of infinite restriction sets $I \subseteq \mathbb{N}_{\geq 3}$ by showing that the iterated function systems describing our restricted digit sets are conformal.

Theorem 4.8. *For any (possibly infinite) restriction set* $I \subseteq \mathbb{N}_{\geq 3}$ *we have*

$$\begin{aligned}\dim_{\mathcal{H}} \Lambda_I &= \sup\{\dim_{\mathcal{H}} \Lambda_J \mid J \subseteq I \textit{ finite}\} \\ &= \inf\Big\{r \;\Big|\; \sum_{d\in I}\Big(\frac{1}{d(d-1)}\Big)^r \leq \frac{1}{2}\Big\}.\end{aligned}$$

Proof. Note that this result reduces to that of Theorem 4.7 when I is finite. Assume therefore that I is infinite. Recall from Proposition 4.6 that Λ_I is the limit set of the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$. We will begin by proving that $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ is conformal.

First off, the method used in the proof for Theorem 4.7 to show that the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ satisfies the OSC on the interior $U = (0, \frac{1}{2})$ of its domain $[0, \frac{1}{2}]$ easily extends to the case where I is infinite. Hence axiom (i) of Definition 2.6 is satisfied.

For axiom (ii) note that each map $\phi_{s,d} = T_{s,d}^{-1}|_{[0,\frac{1}{2}]}$ extends to the map $T_{s,d}^{-1}$ on all of $\mathbb{R}$, which is, as an affine map on $\mathbb{R}$, trivially a C^1-diffeomorphism.

Lastly, take any $n \in \mathbb{N}$ and any $(\mathbf{s},\mathbf{d}) = ((s_k,d_k))_{k=1}^n \in (\{0,1\} \times I)^n$. Then there exists some real constant C such that for any $x \in [0, \frac{1}{2}]$,

$$\phi_{(\mathbf{s},\mathbf{d})}(x) = (\phi_{s_1,d_1} \circ ... \circ \phi_{s_n,d_n})(x) = \Big(\prod_{k=1}^n (-1)^{s_k}\frac{1}{d_k(d_k-1)}\Big)x + C,$$

and so we have

$$|\phi'_{(\mathbf{s},\mathbf{d})}(x)| = \Big|\prod_{k=1}^n (-1)^{s_k}\frac{1}{d_k(d_k-1)}\Big| = \prod_{k=1}^n \frac{1}{d_k(d_k-1)}.$$

As this does not depend on x, we see that $|\phi'_{(\mathbf{s},\mathbf{d})}(x)| = |\phi'_{(\mathbf{s},\mathbf{d})}(y)|$ for any $x, y \in [0, \frac{1}{2}]$ and so the bounded distortion property is satisfied with $K = 1$.

We may conclude that $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ is conformal. It therefore follows from Proposition 2.8 that

$$\dim_{\mathcal{H}} \Lambda_I = \sup\{\dim_{\mathcal{H}} \Lambda_J \mid J \subseteq I \text{ finite}\}.$$

Now label the elements of I in some arbitrary order, i.e. write $I = \{d_1, d_2, d_3, ...\}$. Also define the sequence $(r_n)_n$ in $\mathbb{R}$ by $r_n := \dim_{\mathcal{H}} \Lambda_{\{d_1,...,d_n\}}$. Then since $J \subseteq J'$ implies $\dim_{\mathcal{H}} \Lambda_J \leq \dim_{\mathcal{H}} \Lambda_{J'}$, it follows that $(r_n)_n$ is an increasing sequence. For every finite subset $J \subseteq I$ there exists some $n \in \mathbb{N}$ such that $J \subseteq \{d_1, ..., d_n\}$ and hence $\dim_{\mathcal{H}} \Lambda_J \leq \dim_{\mathcal{H}} \Lambda_{\{d_1,...,d_n\}} = r_n$. This implies that

$$\dim_{\mathcal{H}} \Lambda_I = \sup\{\dim_{\mathcal{H}} \Lambda_{\{d_1,...,d_n\}} \mid n \in \mathbb{N}\} = \lim_{n\to\infty} r_n =: r.$$

By Theorem 4.7 we have for every $n \in \mathbb{N}$ that $\sum_{k=1}^n (\frac{1}{d_k(d_k-1)})^{r_n} = \frac{1}{2}$. Since we have $\frac{1}{d_k(d_k-1)} \in (0,1)$ for any k, it follows that for any $n \in \mathbb{N}$, the map $s \mapsto \sum_{k=1}^n (\frac{1}{d_k(d_k-1)})^s$ is strictly decreasing. For any n we have $r_n \leq r$ and so this implies that $\sum_{k=1}^n (\frac{1}{d_k(d_k-1)})^r \leq \sum_{k=1}^n (\frac{1}{d_k(d_k-1)})^{r_n} = \frac{1}{2}$. Therefore we have

$$\sum_{d \in I} \left(\frac{1}{d(d-1)}\right)^r = \sum_{k=1}^{\infty} \left(\frac{1}{d_k(d_k-1)}\right)^r = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{d_k(d_k-1)}\right)^r \leq \frac{1}{2}.$$

Lastly, for any $s < r$ there exists some $n \in \mathbb{N}$ such that $s \leq r_n$ and so

$$\begin{aligned}\sum_{d \in I} \left(\frac{1}{d(d-1)}\right)^s &= \sum_{k=1}^{\infty} \left(\frac{1}{d_k(d_k-1)}\right)^s \geq \sum_{k=1}^{\infty} \left(\frac{1}{d_k(d_k-1)}\right)^{r_n} \\ &= \sum_{k=1}^{n} \left(\frac{1}{d_k(d_k-1)}\right)^{r_n} + \sum_{k=n+1}^{\infty} \left(\frac{1}{d_k(d_k-1)}\right)^{r_n} \\ &= \frac{1}{2} + \sum_{k=n+1}^{\infty} \left(\frac{1}{d_k(d_k-1)}\right)^{r_n} > \frac{1}{2}.\end{aligned}$$

We conclude that $\dim_{\mathcal{H}} \Lambda_I = r = \inf\{s \mid \sum_{d\in I}(\frac{1}{d(d-1)})^s \leq \frac{1}{2}\}$. □

Remark 4.9. For $I \subseteq \mathbb{N}_{\geq 3}$ we now found the expression for the Hausdorff dimension of a restricted digit set Λ_I in Theorem 4.8 by considering the Hausdorff dimensions of restricted digit sets Λ_J for finite subsets $J \subseteq I$. Alternatively, Proposition 2.8 shows we could have worked with the pressure function formalism $\dim_{\mathcal{H}} \Lambda_I = \inf\{r \geq 0 \mid P(r) \leq 0\}$ instead, where P is the topological pressure function of the conformal IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$. We will show that this leads to the same expression $\dim_{\mathcal{H}} \Lambda_I = \inf\{r \mid \sum_{d\in I}(\frac{1}{d(d-1)})^r \leq \frac{1}{2}\}$.

Take any $n \in \mathbb{N}$ and any sequence $(\mathbf{s},\mathbf{d}) = ((s_1,d_1),...,(s_n,d_n)) \in (\{0,1\} \times I)^n$. In the proof of Theorem 4.8 it was shown that then $|\phi'_{(\mathbf{s},\mathbf{d})|_n}(x)| = \prod_{k=1}^n \frac{1}{d_k(d_k-1)}$ for each $x \in [0, \frac{1}{2}]$ and so we have $\|\phi'_{(\mathbf{s},\mathbf{d})|_n}\| = \prod_{k=1}^n \frac{1}{d_k(d_k-1)}$. The pressure function of the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ is therefore given by

$$\begin{aligned}P(r) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{(\mathbf{s},\mathbf{d})\in(\{0,1\}\times I)^n} \|\phi'_{(\mathbf{s},\mathbf{d})}\|^r \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{(\mathbf{s},\mathbf{d})\in(\{0,1\}\times I)^n} \prod_{k=1}^{n} \left(\frac{1}{d_k(d_k-1)}\right)^r \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(2^n \sum_{\mathbf{d}\in I^n} \prod_{k=1}^{n} \left(\frac{1}{d_k(d_k-1)}\right)^r \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(2^n \prod_{k=1}^{n} \sum_{d\in I} \left(\frac{1}{d(d-1)}\right)^r \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(2 \sum_{d\in I} \left(\frac{1}{d(d-1)}\right)^r \right)^n \\ &= \log \left(2 \sum_{d\in I} \left(\frac{1}{d(d-1)}\right)^r \right).\end{aligned}$$

It follows that $P(r) \leq 0$ holds if and only if $2\sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^r \leq 1$, or equivalently $\sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^r \leq \frac{1}{2}$. Hence we indeed find the same expression

$$\dim_{\mathcal{H}} \Lambda_I = \inf\{r \mid P(r) \leq 0\} = \inf\left\{r \;\middle|\; \sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^r \leq \frac{1}{2}\right\}.$$

4.4 Consequences and examples

The main results we have proven above turn the problem of calculating the dimensions of the restricted digit sets Λ_I from one requiring topological arguments into one we can approach with tools from calculus. We will now proceed to discuss a few special cases and concrete examples, along with some bounds the expressions we found yield for more general cases.

First off, the expression found in Theorem 4.7 for finite sets $I \subseteq \mathbb{N}_{\geq 3}$ implies the following.

Corollary 4.10. *For any finite set* $I \subseteq \mathbb{N}_{\geq 3}$ *we have* $0 < \dim_{\mathcal{H}} \Lambda_I < 1$.

Proof. By Theorem 4.7 we have that $\dim_{\mathcal{H}} \Lambda_I = r$ where r is the unique real number satisfying $\sum_{d\in I}(\frac{1}{d(d-1)})^r = \frac{1}{2}$. Recall that the map $r \mapsto \sum_{d\in I}(\frac{1}{d(d-1)})^r$ is strictly decreasing. Note therefore that

$$\sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^0 = \sum_{d\in I} 1 \geq 1 > \frac{1}{2} = \sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^r,$$

and so we have $\dim_{\mathcal{H}} \Lambda_I = r > 0$. On the other hand, we have

$$\sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^1 < \sum_{d\in \mathbb{N}_{\geq 3}} \frac{1}{d(d-1)} = \frac{1}{2} = \sum_{d\in I}\left(\frac{1}{d(d-1)}\right)^r,$$

and hence $\dim_{\mathcal{H}} \Lambda_I = r < 1$. □

Hence the restricted digit sets corresponding to finite restriction sets in $\mathbb{N}_{\geq 3}$ have nonintegral Hausdorff dimensions, making them fractal sets.

Take any (possibly infinite) restriction set $I \subseteq \mathbb{N}_{\geq 2}$. If we have $2 \in I$, then $\dim_{\mathcal{H}} \Lambda_I = 1 > 0$ by Proposition 4.3. If instead we have $2 \notin I$, then combining Theorem 4.8 with Corollary 4.10 yields

$$\dim_{\mathcal{H}} \Lambda_I = \sup\{\dim_{\mathcal{H}} \Lambda_J \mid J \subseteq I \text{ finite}\} > 0.$$

It follows that every restricted digit set Λ_I for random Lüroth transformations has a strictly positive Hausdorff dimension. This particularly implies that no matter what restriction we put on the Lüroth digits $(d_k)_k$ (as long as this restriction is the same for every digit d_k), there are always uncountably many numbers in $[0, 1]$ that have a random Lüroth expansion satisfying this restriction.

In order to find a more specific positive lower bound for the restricted digit sets, we consider the case where I contains one single digit d, which is a case in which the dimension of the corresponding restricted digit set can easily be calculated directly.

Corollary 4.11. *For any $d \in \mathbb{N}_{\geq 2}$ we have*

$$\dim_{\mathcal{H}} \Lambda_{\{d\}} = \frac{\log 2}{\log\big(d(d-1)\big)}.$$

Proof. Firstly, for $d = 2$ it follows from Corollary 4.3 that

$$\dim_{\mathcal{H}} \Lambda_{\{2\}} = 1 = \frac{\log 2}{\log 2} = \frac{\log 2}{\log\big(2(2-1)\big)}.$$

Now take any $d \in \mathbb{N}_{\geq 3}$. By Theorem 4.7 we have $\dim_{\mathcal{H}} \Lambda_{\{0,1\}\times\{d\}} = r$, where r is the unique number that satisfies $(\frac{1}{d(d-1)})^r = \frac{1}{2}$, or equivalently $(d(d-1))^r = 2$. It follows that

$$\dim_{\mathcal{H}} \Lambda_{\{d\}} = r = \frac{\log 2}{\log\big(d(d-1)\big)}. \qquad \square$$

Example 4.12. We have

$$\begin{aligned}
\dim_{\mathcal{H}} \Lambda_{\{2\}} &= 1,\\
\dim_{\mathcal{H}} \Lambda_{\{3\}} &= \frac{\log 2}{\log 6} \approx 0.38685,\\
\dim_{\mathcal{H}} \Lambda_{\{4\}} &= \frac{\log 2}{\log 12} \approx 0.27894,\\
\dim_{\mathcal{H}} \Lambda_{\{5\}} &= \frac{\log 2}{\log 20} \approx 0.23138,
\end{aligned}$$

and so forth. ♢

Note that if $d, \bar{d} \in \mathbb{N}_{\geq 2}$ are such that $d > \bar{d}$, then

$$\dim_{\mathcal{H}} \Lambda_{\{d\}} = \frac{\log 2}{\log\big(d(d-1)\big)} < \frac{\log 2}{\log\big(\bar{d}(\bar{d}-1)\big)} = \dim_{\mathcal{H}} \Lambda_{\{\bar{d}\}},$$

so the sequence $\big(\dim_{\mathcal{H}} \Lambda_{\{d\}}\big)_{d\in\mathbb{N}_{\geq 2}}$ is decreasing in d. Moreover, we have

$$\lim_{d\to\infty} \dim_{\mathcal{H}} \Lambda_{\{d\}} = \lim_{d\to\infty} \frac{\log 2}{\log\big(d(d-1)\big)} = 0.$$

For any $d \in I \subseteq \mathbb{N}_{\geq 2}$ we have $\Lambda_{\{d\}} \subseteq \Lambda_I$ and hence the monotonicity of Hausdorff dimensions now immediately implies the following.

Corollary 4.13. *Take any $I \subseteq \mathbb{N}_{\geq 2}$ and let $d_{min} := \min I$. Then we have*

$$\frac{\log 2}{\log\big(d_{min}(d_{min}-1)\big)} \leq \dim_{\mathcal{H}} \Lambda_I \leq 1.$$

For every restriction set $I \subseteq \mathbb{N}_{\geq 2}$ this yields a strictly positive lower bound for the Hausdorff dimension of Λ_I.

Intuitively it seems that infinite restriction sets I should lead to restricted digit sets Λ_I with large Hausdorff dimensions. After all, in the cases $I = \mathbb{N}_{\geq 2}$ and $I = \mathbb{N}_{\geq 3}$ the dimension even equals 1. The following corroborates this for restricted digit sets corresponding to removing only finitely many digits.

Corollary 4.14. *If $I = \mathbb{N}_{\geq 2} \setminus S$ for some finite set $S \subseteq \mathbb{N}_{\geq 3}$, then we have*

$$\dim_{\mathcal{H}} \Lambda_I \geq \tfrac{1}{2}.$$

Proof. If $2 \in I$ then $\dim_{\mathcal{H}} \Lambda_I = 1 > \frac{1}{2}$. Suppose $2 \notin I$. For any $d \in \mathbb{N}_{\geq 2}$ we have $d > d-1$ and so we have $(\frac{1}{d(d-1)})^r \geq (\frac{1}{d^2})^r$ for any $r \leq 1$. In particular, we have for any $r \leq \frac{1}{2}$ that

$$\Big(\frac{1}{d(d-1)}\Big)^r \geq \Big(\frac{1}{d^2}\Big)^r \geq \Big(\frac{1}{d^2}\Big)^{1/2} = \frac{1}{d}.$$

It follows that

$$\sum_{d\in I} \Big(\frac{1}{d(d-1)}\Big)^r \geq \sum_{d\in I} \frac{1}{d} = \sum_{d=3}^{\infty} \frac{1}{d} - \sum_{d\in S} \frac{1}{d} = \infty,$$

for any $r \leq \frac{1}{2}$. Theorem 4.8 now yields

$$\dim_{\mathcal{H}} \Lambda_I = \inf \Big\{ r \;\Big|\; \sum_{d\in I} \Big(\frac{1}{d(d-1)}\Big)^r \leq \frac{1}{2} \Big\} \geq \frac{1}{2}. \qquad \square$$

The same argument cannot be used for more general infinite restriction sets I. After all, if we have $I = \mathbb{N}_{\geq 3} \setminus S$ for some *infinite* set $S \subseteq \mathbb{N}_{\geq 3}$, then both $\sum_{d=3}^{\infty} \frac{1}{d}$ and $\sum_{d\in S} \frac{1}{d}$ diverge and so the difference $\sum_{d=3}^{\infty} \frac{1}{d} - \sum_{d\in S} \frac{1}{d}$ is ill-defined.

One may wonder whether every infinite restriction set I leads to a restricted digit set Λ_I of Hausdorff dimension 1. We conclude this paragraph by showing that this is not the case by means of a counterexample.

Example 4.15. Take any function $f : \mathbb{N} \to \mathbb{N}_{\geq 2}$ satisfying $f(n) \geq n^2 + 1$ for all $n \in \mathbb{N}$ and consider the restriction set $I = \{f(n) : n \in \mathbb{N}_{\geq 3}\}$. Then

$$\begin{aligned}
\sum_{d\in I} \Big(\frac{1}{d(d-1)}\Big)^{1/2} < \sum_{d\in I} \Big(\frac{1}{(d-1)^2}\Big)^{1/2} &= \sum_{n=3}^{\infty} \frac{1}{f(n)-1} \leq \sum_{n=3}^{\infty} \frac{1}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4} = \frac{\pi^2}{6} - \frac{5}{4} \approx 0.3949,
\end{aligned}$$

and so $\sum_{d\in I} \Big(\frac{1}{d(d-1)}\Big)^{1/2} < \frac{1}{2}$. Applying Theorem 4.8 yields $\dim_{\mathcal{H}} \Lambda_I \leq \frac{1}{2}$. By Corollary 4.13 we also have that $\dim_{\mathcal{H}} \Lambda_I \geq \frac{\log 2}{\log(f_{\min}(f_{\min}-1))}$, where we define $f_{\min} := \min_{n\geq 3} f(n)$. Hence we find the bounds

$$0 < \frac{\log 2}{\log(f_{\min}(f_{\min}-1))} \leq \dim_{\mathcal{H}} \Lambda_I \leq \frac{1}{2}.$$

Note that Λ_I has a nonintegral Hausdorff dimension and is therefore a fractal.

If, for example, we have $f(3) = 3^2 + 1 = 10$, then $f_{\min} = f(3) = 10$ and so

$$0.15404 \approx \frac{\log 2}{\log 90} \leq \dim_{\mathcal{H}} \Lambda_I \leq \frac{1}{2}. \qquad \diamondsuit$$

4.5 Non-uniform restricted digit sets

We finish this chapter by discussing a generalisation of the restricted digit sets for random Lüroth expansions we have seen thus far. In these restricted digit sets the restrictions we have put on the Lüroth digits $(d_k)_k$ of a random Lüroth expansion $((s_k, d_k))_k$ were uniformly described by the same restriction set $I \subseteq \mathbb{N}_{\geq 2}$ for each $k \in \mathbb{N}$. We can expand this idea by considering different restrictions on d_k for different values of k by means of non-uniform restricted digit sets.

Definition 4.16. A *restriction sequence* is a sequence $\mathbb{I} = (I_k)_{k\in\mathbb{N}}$ of subsets $I_k \subseteq \mathbb{N}_{\geq 2}$. For any restriction sequence $\mathbb{I} = (I_k)_k$ we define the corresponding *non-uniform restricted digit set*

$$\Lambda_{\mathbb{I}} := \Big\{ x \in X \;\Big|\; \exists ((s_k, d_k))_k \in \mathcal{L}_x \text{ s.t. } d_k \in I_k \;\forall k \in \mathbb{N} \Big\}.$$

Of course when taking $I_k = I$ for every $k \in \mathbb{N}$ and some $I \subseteq \mathbb{N}_{\geq 2}$ we just get $\Lambda_{\mathbb{I}} = \Lambda_I$ and so non-uniform restricted digit sets generalise restricted digit sets.

We have seen that for each set $I \subseteq \mathbb{N}_{\geq 3}$ the collection $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$ is a conformal iterated function system of affine similarities. The following is therefore easy to prove.

Lemma 4.17. *Take any sequence $\mathbb{J} = (J_k)_{k\in\mathbb{N}}$ with $J_k := \{0,1\} \times I_k$ for some $I_k \subseteq \mathbb{N}_{\geq 3}$ for every $k \in \mathbb{N}$. Then the sequence*

$$(\Phi^k)_{k\in\mathbb{N}} := (\{\phi_{s,d} : (s,d) \in J_k\})_{k\in\mathbb{N}},$$

is a non-autonomous conformal iterated function system on $[0, \frac{1}{2}]$.

Proof. It follows from the proof of Theorem 4.8 that $\{\phi_{s,d} : (s,d) \in J_k\}$ is a conformal iterated function system on $[0, \frac{1}{2}]$ for each $k \in \mathbb{N}$. It therefore only remains to show that $(\Phi^k)_{k\in\mathbb{N}}$ is uniformly contracting. To prove this, we set $\eta := \frac{1}{6} < 1$ and recall that $\phi_{s,d}$ is an affine map with slope $(-1)^s \frac{1}{d(d-1)}$. For any $x \in [0, \frac{1}{2}]$ we therefore have

$$|\phi'_{s,d}(x)| = \left| (-1)^s \frac{1}{d(d-1)} \right| = \frac{1}{d(d-1)} \leq \frac{1}{3 \cdot 2} = \frac{1}{6} = \eta.$$

for every $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$. Hence $(\Phi^k)_{k\in\mathbb{N}}$ is uniformly contracting regardless of our choice of $\mathbb{J}$. We conclude that $(\Phi^k)_{k\in\mathbb{N}}$ is an NCIFS on $[0, \frac{1}{2}]$. □

The following is now immediate.

Proposition 4.18. *Take any restriction sequence $\mathbb{I} = (I_k)_{k\in\mathbb{N}}$ with $I_k \subseteq \mathbb{N}_{\geq 3}$. Then we have*

$$\Lambda_{\mathbb{I}} = \Big\{ x \in X \;\Big|\; \exists\, \omega \in \{0,1\}^{\mathbb{N}} \text{ s.t. } T_\omega^{k-1} x \in \bigcup_{d \in I_k} [\tfrac{1}{d}, \tfrac{1}{d-1}] \;\forall k \in \mathbb{N} \Big\}.$$

Moreover, if we define $\mathbb{J} := (J_k)_{k\in\mathbb{N}}$ by setting $J_k := \{0,1\} \times I_k$ for each $k \in \mathbb{N}$, then $\Lambda_{\mathbb{I}}$ equals the limit set of the NCIFS $(\Phi^k)_{k\in\mathbb{N}} = (\{\phi_{s,d} : (s,d) \in J_k\})_{k\in\mathbb{N}}$.

The proof of Proposition 4.18 is omitted as it follows the proofs of Lemma 4.2 and Proposition 4.6 verbatim but with sequences $((s_k, d_k))_k$ in $\mathbb{J}^{\mathbb{N}}$ instead of $(\{0,1\} \times I)^{\mathbb{N}}$. This proposition leads to the following result.

Theorem 4.19. *For any $k \in \mathbb{N}$ let $I_k \subseteq \mathbb{N}_{\geq 3}$ be some finite set such that the restriction sequence $\mathbb{I} = (I_k)_{k\in\mathbb{N}}$ is of sub-exponential growth. Then we have*

$$\begin{aligned}\dim_{\mathcal{H}} \Lambda_{\mathbb{I}} &= \inf\Big\{r \geq 0 \;\Big|\; \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big) < 0\Big\} \\ &= \sup\Big\{r \geq 0 \;\Big|\; \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big) > 0\Big\}.\end{aligned}$$

Proof. By Proposition 4.18 the non-uniform restricted digit set $\Lambda_{\mathbb{I}}$ coincides with the limit set of the NCIFS $(\Phi^k)_{k\in\mathbb{N}} = (\{\phi_{s,d} : (s,d) \in J_k\})_{k\in\mathbb{N}}$, where $J_k := \{0,1\} \times I_k$ for each $k \in \mathbb{N}$. Note that $\#J_k = 2\#I_k$ and so we have

$$\frac{1}{k}\log \#J_k = \frac{1}{k}\log 2\#I_k = \frac{1}{k}\log 2 + \frac{1}{k}\log \#I_k \xrightarrow{k\to\infty} 0,$$

where by Definition 2.11 we have $\lim_{k\to\infty} \frac{1}{k}\log \#I_k = 0$ by the assumption that $\mathbb{I}$ has sub-exponential growth. Hence $\mathbb{J} := (J_k)_k$ is itself of sub-exponential growth as well. By Proposition 2.12 we have

$$\dim_{\mathcal{H}} \Lambda_{\mathbb{I}} = \inf\{r \mid \underline{P}(r) < 0\} = \sup\{r \mid \underline{P}(r) > 0\},$$

where $\underline{P}$ is the lower pressure function of the NCIFS $(\Phi^k)_{k\in\mathbb{N}}$. For any $k \in \mathbb{N}$, any sequence $(\mathbf{s},\mathbf{d}) = ((s_1,d_1),...,(s_n,d_n)) \in \mathbb{J}^n$ and any $x \in [0, \frac{1}{2}]$ we have $|\phi'_{(\mathbf{s},\mathbf{d})}(x)| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}$ and hence we find that $\|\phi'_{(\mathbf{s},\mathbf{d})}\| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}$. The lower pressure function is therefore given by

$$\begin{aligned}\underline{P}(r) &= \liminf_{n\to\infty} \frac{1}{n}\log \sum_{(\mathbf{s},\mathbf{d})\in\mathbb{J}^n} \|\phi'_{(\mathbf{s},\mathbf{d})}\|^r \\ &= \liminf_{n\to\infty} \frac{1}{n}\log\left(\sum_{(\mathbf{s},\mathbf{d})\in\mathbb{J}^n}\prod_{k=1}^{n}\Big(\frac{1}{d_k(d_k-1)}\Big)^r\right) \\ &= \liminf_{n\to\infty} \frac{1}{n}\log\left(2^n\sum_{\mathbf{d}\in\mathbb{I}^n}\prod_{k=1}^{n}\Big(\frac{1}{d_k(d_k-1)}\Big)^r\right) \\ &= \liminf_{n\to\infty} \frac{1}{n}\log\left(\prod_{k=1}^{n} 2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\right) \\ &= \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big),\end{aligned}$$

and so the desired expression follows. □

Note that this expression generalises what we found for uniform restricted digit sets in Theorems 4.7 and 4.8. We can use this expression to find the following bounds for the Hausdorff dimension of certain non-uniform restricted digit sets.

Proposition 4.20. *If $\mathbb{I} = (I_k)_{k\in\mathbb{N}}$ is a sequence of finite sets $I_k \subseteq \mathbb{N}_{\geq 3}$ that is of sub-exponential growth, then we have*

$$\inf_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k}) \leq \dim_{\mathcal{H}} \Lambda_{\mathbb{I}} \leq \sup_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k}).$$

Proof. Recall from Theorem 4.7 that for each k, $\dim_{\mathcal{H}} \Lambda_{I_k}$ equals the unique number r satisfying $2\sum_{d\in I_k}(\frac{1}{d(d-1)})^r = 1$. Recall that for any finite collection $I \subseteq \mathbb{N}_{\geq 3}$ the function $r \mapsto 2\sum_{d\in I}(\frac{1}{d(d-1)})^r$ is decreasing in r. It follows that for any $r > \sup_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k})$ we have $2\sum_{d\in I_k}(\frac{1}{d(d-1)})^r < 1$ for every $k \in \mathbb{N}$ and so $\log(2\sum_{d\in I_k}(\frac{1}{d(d-1)})^r) < 0$, which in turn implies

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big) \leq 0.$$

Hence by Theorem 4.19 we have $\dim_{\mathcal{H}} \Lambda_{\mathbb{I}} \leq \sup_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k})$. Similarly we have for $r \leq \inf_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k})$ that $2\sum_{d\in I_k}(\frac{1}{d(d-1)})^r \geq 1$ for every k, implying that $\log(2\sum_{d\in I_k}(\frac{1}{d(d-1)})^r) \geq 0$ for every k and hence

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big) \geq 0.$$

In fact, since the lower pressure function is strictly decreasing when it is finite, we must have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\Big(2\sum_{d\in I_k}\Big(\frac{1}{d(d-1)}\Big)^r\Big) > 0,$$

for any $r < \inf_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k})$. Hence we also have $\dim_{\mathcal{H}} \Lambda_{\mathbb{I}} \geq \inf_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k})$. □

Combining this result with Corollary 4.11 yields the following.

Corollary 4.21. *Consider any function $f : \mathbb{N} \to \mathbb{N}_{\geq 3}$ and define the restriction sequence $\mathbb{I} = (I_k)_k = (\{f(k)\})_k$, such that the corresponding non-uniform restricted digit set equals*

$$\Lambda_{\mathbb{I}} = \Big\{x \in X \;\Big|\; \exists((s_k, d_k))_k \in \mathcal{L}_x \;\; s.t.\; d_k = f(k)\; \forall k \in \mathbb{N}\Big\}.$$

Setting $d_{min} := \min_{k\in\mathbb{N}} f(k)$ and $d_{sup} := \sup_{k\in\mathbb{N}} f(k)$, we then have

$$\frac{\log 2}{\log(d_{sup}(d_{sup}-1))} \leq \dim_{\mathcal{H}} \Lambda_{\mathbb{I}} \leq \frac{\log 2}{\log(d_{min}(d_{min}-1))},$$

where we set the lower bound to be 0 when $d_{sup} = \infty$ (i.e. when $f(k) \to \infty$).

Proof. We have $\#I_k = 1$ for each $k \in \mathbb{N}$ and so $\mathbb{I}$ trivially grows sub-exponentially. By Proposition 4.20 we have

$$\inf_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k}) \leq \dim_{\mathcal{H}} \Lambda_{\mathbb{I}} \leq \sup_{k\in\mathbb{N}}(\dim_{\mathcal{H}} \Lambda_{I_k}).$$

By Corollary 4.11 we have $\dim_{\mathcal{H}} \Lambda_{I_k} = \frac{\log 2}{\log(f(k)(f(k)-1))}$ for each $k \in \mathbb{N}$ and so the statement follows. □

5 Box-like sets induced by the skew product

In the previous chapter we found that the subsystem of the random Lüroth transformation on $[0, \frac{1}{2}]$ induces a collection of affine iterated function systems on $[0, \frac{1}{2}]$. We found that the restricted digit sets Λ_I for $I \subseteq \mathbb{N}_{\geq 3}$ coincided with the limit sets of these iterated function systems, which we were able to use to study their Hausdorff dimensions.

In a similar fashion, we will now study the limit sets of an iterated function system on the unit square $[0, 1]^2$ that is induced by the skew product transformation R_p, which we recall to be defined on $[0, 1]^2$ as

$$R_p(w, x) = (\xi_p(w), T_{\alpha_p(w)}(x)), \quad (w, x) \in [0, 1]^2,$$

for any $p \in (0, 1)$, with the functions ξ_p and α_p as defined in (13) and (14) respectively.

We will see that these limit sets are special cases of box-like sets and that the vertical projection of these box-like sets is related to the restricted digit sets we studied in the previous chapter. This will allow us to use what we found out about them to study the box-counting dimensions of these box-like sets.

5.1 An iterated function system on $[0, 1]^2$

Consider the two monotone branches $\xi_{0,p} : [0, p] \to [0, 1]$ and $\xi_{p,1} : [p, 1] \to [0, 1]$ of the map ξ_p defined in (13), given by $\xi_{p,0}(w) = \frac{w}{p}$ and $\xi_{p,1}(w) = \frac{w-p}{1-p}$. Since these branches are bijective, we can define the maps $\psi^1_{p,0} : [0, 1] \to [0, p]$ and $\psi^1_{p,1} : [0, 1] \to [p, 1]$ by $\psi^1_{p,s} := \xi^{-1}_{p,s}$ for $s \in \{0, 1\}$. For any $w \in [0, 1]$ we have

$$\psi^1_{p,0}(w) = pw \quad \text{and} \quad \psi^1_{p,1}(w) = (1 - p)w + p.$$

These maps are shown in Figure 10 below.

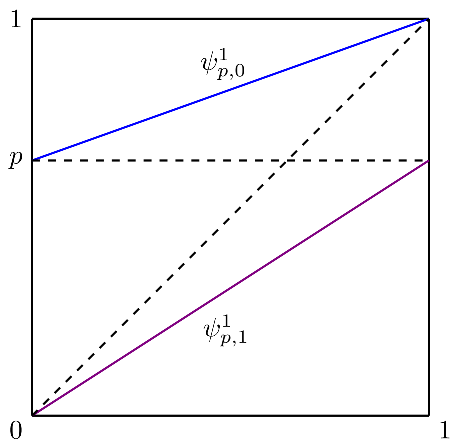


Figure 10: *The affine maps $\psi^1_{p,s}$ for $s \in \{0, 1\}$.*

Similarly, consider once again the affine branches $T_{s,d} : [\frac{1}{d}, \frac{1}{d-1}] \to [0,1]$ for $(s,d) \in \mathcal{D} = \{0,1\} \times \mathbb{N}_{\geq 2}$ of the random Lüroth transformation given by $T_{0,d} = d(d-1)x - (d-1)$ and $T_{1,d}x = 1 - T_{0,d}x = -d(d-1)x + d$. Since these maps are also bijective, we can define the maps $\psi^2_{s,d} : [0,1] \to [\frac{1}{d}, \frac{1}{d-1}]$ by $\psi^2_{s,d} := T^{-1}_{s,d}$ for $(s,d) \in \mathcal{D}$. Note that for every $(s,d) \in \mathcal{D}$ we have $\psi^2_{s,d}|_{[0,\frac{1}{2}]} = \phi_{s,d}$ with the contraction $\phi_{s,d}$ on $[0, \frac{1}{2}]$ as defined in Chapter 4. For every $x \in [0,1]$ we have

$$\psi^2_{0,d}(x) = \frac{x}{d(d-1)} + \frac{1}{d} \quad \text{and} \quad \psi^2_{1,d}(x) = \frac{1}{d-1} - \frac{x}{d(d-1)},$$

for every $d \in \mathbb{N}_{\geq 2}$. The maps $\psi^2_{s,d}$ are shown in Figure 11 below.

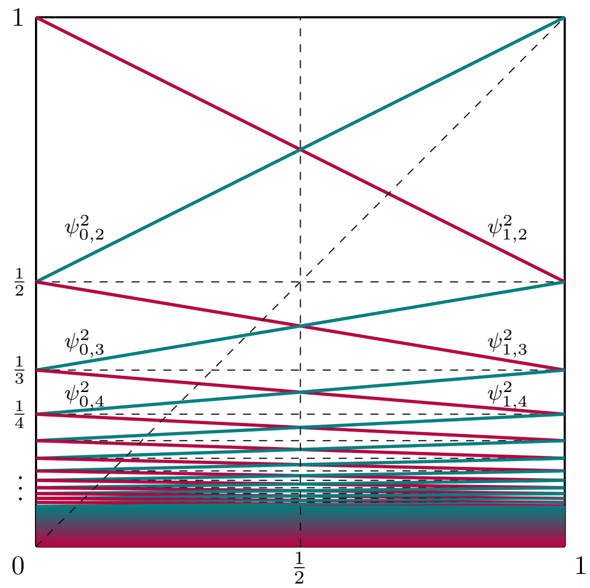


Figure 11: *The affine maps* $\psi^2_{s,d}$ *on* $[0,1]$ *for* $(s,d) \in \mathcal{D}$.

Note that for any $(s,d) \in \mathcal{D}$ the maps $\psi^1_{p,s}$ and $\psi^2_{s,d}$ are affine similarity contractions, where the maps $\psi^1_{p,0}$ and $\psi^1_{p,1}$ have respective ratios p and $1-p$, while for each $(s,d) \in \mathcal{D}$ the map $\psi^2_{s,d}$ has similarity ratio $\frac{1}{d(d-1)}$, all of which are in $(0,1)$.

Now we define the family of maps $\Psi^p_{s,d} : [0,1]^2 \to [0,1]^2$ for $(s,d) \in \mathcal{D}$ by putting

$$\Psi^p_{s,d}(w,x) := \big(\psi^1_{p,s}(w), \psi^2_{s,d}(x)\big), \quad (w,x) \in [0,1]^2.$$

For each $(s,d) \in \mathcal{D}$ define the linear map $L^p_{s,d} : \mathbb{R}^2 \to \mathbb{R}^2$ and the vector $y^p_{s,d} \in \mathbb{R}^2$ by

$$L^p_{0,d} = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{d(d-1)} \end{pmatrix}, y^p_{0,d} = \begin{pmatrix} 0 \\ \frac{1}{d} \end{pmatrix}, \ L^p_{1,d} = \begin{pmatrix} 1-p & 0 \\ 0 & -\frac{1}{d(d-1)} \end{pmatrix} \text{ and } y^p_{1,d} = \begin{pmatrix} p \\ \frac{1}{d-1} \end{pmatrix}.$$

Then for each $(w,x) \in [0,1]^2$ we see that $\Psi^p_{s,d}(w,x) = L^p_{s,d}(w,x) + y^p_{s,d}$. For $(w_1,x_1),(w_2,x_2) \in [0,1]^2$ we therefore have that

$$\begin{aligned} \left|\Psi^p_{s,d}(w_1,x_1) - \Psi^p_{s,d}(w_2,x_2)\right| &= \left|L^p_{s,d}(w_1,x_1) + y^p_{s,d} - L^p_{s,d}(w_2,x_2) - y^p_{s,d}\right| \\ &= \left|L^p_{s,d}(w_1,x_1) - L^p_{s,d}(w_2,x_2)\right| \\ &\leq \|L^p_{s,d}\| \cdot |(w_1,x_1) - (w_2,x_2)|, \end{aligned}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^2$. Recall that the operator norm of a real 2×2-matrix A equals the square root of the largest eigenvalue of $A^\top A$, and so we have $\|L^p_{0,d}\| = \max\left\{p, \frac{1}{d(d-1)}\right\} < 1$ and $\|L^p_{1,d}\| = \max\left\{1-p, \frac{1}{d(d-1)}\right\} < 1$ for any $d \in \mathbb{N}_{\geq 2}$. It follows that $\Psi^p_{s,d}$ is a contraction for each $(s,d) \in \mathcal{D}$, and hence that for any $J \subseteq \mathcal{D}$, the family $\{\Psi^p_{s,d} : (s,d) \in J\}$ is an iterated function system on $[0,1]^2$.

Also note that the determinant of the diagonal matrix $L^p_{s,d}$ is nonzero for any $(s,d) \in \mathcal{D}$, meaning $L^p_{s,d}$ is non-singular. Proposition 2.15 therefore implies the following.

Lemma 5.1. *For any finite subset $J \subseteq \mathcal{D}$ the limit set $F^p_J \subset [0,1]^2$ of the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ is self-affine, and satisfies*

$$\dim_{\mathcal{H}} F^p_J \leq \dim_{\mathcal{B}} F^p_J \leq d(L^p_{s,d} \mid (s,d) \in J).$$

As noted in Chapter 2, we do not currently have the tools to find out whether the Hausdorff and box-counting dimensions of these self-affine sets coincide. Nor are there many general methods available for calculating or approximating the Hausdorff dimensions for self-affine sets.

Instead we will focus on applying the results from [Fra12] to find conditions under which the equality $\dim_{\mathcal{B}} F^p_J = d(L^p_{s,d} \mid (s,d) \in J)$ holds. We will then use the definition of the affinity dimension to find bounds (and sometimes explicit values) for $\dim_{\mathcal{B}} F^p_J$, which by the lemma above will give upper bounds for the Hausdorff dimension.

5.2 Box-like sets and projections

Above we constructed the affine contractions $\Psi^p_{s,d}$ on $[0,1]^2$ for every $(s,d) \in \mathcal{D}$. We saw that for any finite subset $J \subset \mathcal{D}$ the Hausdorff and box-counting dimensions of the self-affine set of the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ are bounded from above by the affinity dimension $d(L^p_{s,d} \mid (s,d) \in J)$, where $L^p_{s,d}$ is the linear part of $\Psi^p_{s,d}$.

We will now argue that these self-affine sets are in fact box-like. Furthermore, we shall see that in certain cases the vertical projection of these sets coincides with the restricted digit sets we considered in the previous chapter. This will help us find certain conditions under which the box-counting dimension of these box-like sets actually equals the affinity dimension.

Recall that box-like sets in $[0,1]^2$ are generally constructed by dividing the unit square into a collection of rectangles and by then considering iterated function systems consisting of affine maps that map $[0,1]^2$ onto these rectangles.

In our setting we have a partition of $[0,1]^2$ into the rectangles $[0,p] \times [\frac{1}{d}, \frac{1}{d-1}]$ and $[p,1] \times [\frac{1}{d}, \frac{1}{d-1}]$ for $d \in \mathbb{N}_{\geq 2}$. Indeed for any $d \in \mathbb{N}_{\geq 2}$ we have

$$\Psi^p_{0,d}([0,1]^2) = \psi^1_{p,0}([0,1]) \times \psi^2_{0,d}([0,1]) = [0,p] \times [\tfrac{1}{d}, \tfrac{1}{d-1}],$$

while

$$\Psi^p_{1,d}([0,1]^2) = \psi^1_{p,1}([0,1]) \times \psi^2_{1,d}([0,1]) = [p,1] \times [\tfrac{1}{d}, \tfrac{1}{d-1}].$$

This partition of $[0,1]^2$ into rectangles, each one labeled by the digits $(s,d) \in \mathcal{D}$ corresponding to the map $\Psi^p_{s,d}$ sending $[0,1]^2$ into it, is shown in Figure 12.

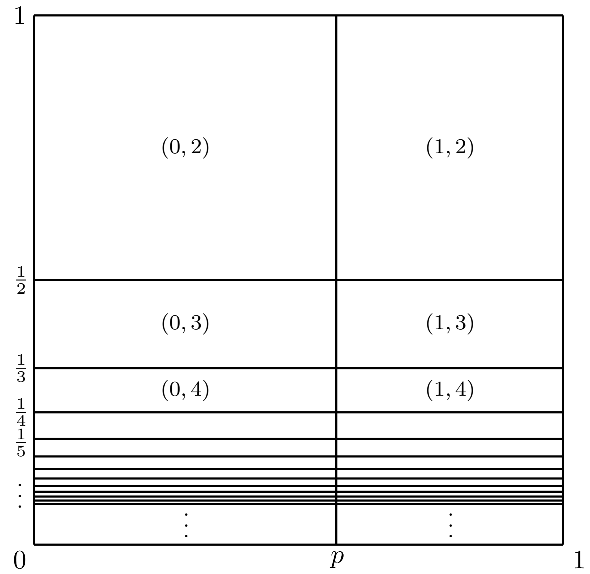


Figure 12: *The rectangles* $[0,p] \times [\frac{1}{d}, \frac{1}{d-1}]$ *and* $[p,1] \times [\frac{1}{d}, \frac{1}{d-1}]$ *for* $d \in \mathbb{N}_{\geq 2}$.

For any $(s,d) \in \mathcal{D}$, if we define the linear maps $T^p_{s,d}, S^p_{s,d} : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$T^p_{0,d} = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{d(d-1)} \end{pmatrix}, S^p_{0,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^p_{1,d} = \begin{pmatrix} 1-p & 0 \\ 0 & \frac{1}{d(d-1)} \end{pmatrix}, S^p_{1,d} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then we have $L^p_{s,d} = T^p_{s,d} \circ S^p_{s,d}$ and hence $\Psi^p_{s,d}(w,x) = T^p_{s,d} \circ S^p_{s,d}(w,x) + y^p_{s,d}$ for any $(w,x) \in [0,1]^2$.

For each (s,d) the map $T^p_{s,d}$ is a diagonal matrix with diagonal components in $(0,1)$ and $S^p_{s,d}$ is easily seen to be a linear isometry on $\mathbb{R}^2$ that keeps $[-1,1]^2$ invariant. By Definition 2.16 therefore, for any $J \subseteq \mathcal{D}$ the self-affine set F^p_J of the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ is in fact box-like. Furthermore, we have the following.

Lemma 5.2. *For any $J \subseteq \mathcal{D}$ the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ satisfies the rectangular open set condition and its box-like limit set F^p_J is of separated type.*

Proof. For any $d \in \mathcal{D}$ we have

$$\Psi^p_{0,d}((0,1)^2) = (0,p) \times (\tfrac{1}{d}, \tfrac{1}{d-1}) \quad \text{and} \quad \Psi^p_{1,d}((0,1)^2) = (p,1) \times (\tfrac{1}{d}, \tfrac{1}{d-1}).$$

Hence, for any subset $J \subseteq \mathcal{D}$, the sets $\Psi^p_{s,d}((0,1)^2)$ for $(s,d) \in J$ are mutually disjoint and we have $\bigcup_{(s,d)\in J} \Psi^p_{s,d}((0,1)^2) \subseteq (0,1)^2$. Taking $R = (0,1)^2$ in Definition 2.17, it follows that the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ satisfies the ROSC.

For any $(s,d) \in \mathcal{D}$ the linear part $L^p_{s,d}$ of $\Psi^p_{s,d}$ is a diagonal matrix, and so $\Psi^p_{s,d}$ sends horizontal lines to horizontal lines, implying that for each $J \subseteq \mathcal{D}$ the box-like set F^p_J corresponding to $\{\Psi^p_{s,d} : (s,d) \in J\}$ is of separated type by Definition 2.18. □

Recall that for $j \in \{1,2\}$, π_j denotes the projection onto the j-th coordinate. The previous lemma implies the following.

Lemma 5.3. *For any finite subset $J \subseteq \mathcal{D}$ the sets $\pi_1(F^p_J)$ and $\pi_2(F^p_J)$ are the self-affine sets corresponding to the respective iterated function systems $\{\psi^1_{p,s} : s \in \pi_1(J)\}$ and $\{\psi^2_{s,d} : (s,d) \in J\}$ on $[0,1]$.*

Proof. Since for any finite $J \subseteq \mathcal{D}$ the box-like set F^p_J is of separated type by Lemma 5.2, this is a direct consequence of Proposition 2.19. After all, if for $j \in \{1,2\}$ the maps χ_1 and χ_2 are defined as above Proposition 2.19, then for any $(s,d) \in J$ and any $x, w \in [0,1]$ we have

$$(\pi_1 \circ \Psi^p_{s,d} \circ \chi_1)(w) = (\pi_1 \circ \Psi^p_{s,d})(w,0) = \pi_1((\psi^1_{p,s}(w), \psi^2_{s,d}(0))) = \psi^1_{p,s}(w),$$

and

$$(\pi_2 \circ \Psi^p_{s,d} \circ \chi_2)(x) = (\pi_2 \circ \Psi^p_{s,d})(0,x) = \pi_2((\psi^1_{p,s}(0), \psi^2_{s,d}(x))) = \psi^2_{s,d}(x).$$

Note that the iterated function systems $\{\psi^1_{p,s} : (s,d) \in J\}$ and $\{\psi^1_{p,s} : s \in \pi_1(J)\}$ share the same limit set, as duplicate contractions do not add anything to the limit set. □

We can use this to find the dimensions of the projections of our box-like sets. In particular, we can find conditions under which these dimensions equal 1, which will allow us to apply Propositions 2.20 and 2.21. First off, we find the following straightforward result for the projection onto the first coordinate.

Lemma 5.4. *For any finite $J \subseteq \mathcal{D}$ we have*

$$\dim_{\mathcal{B}} \pi_1(F^p_J) = \dim_{\mathcal{H}} \pi_1(F^p_J) = \begin{cases} 0, & \text{if } \pi_1(J) = \{0\} \text{ or } \pi_1(J) = \{1\}, \\ 1, & \text{if } \pi_1(J) = \{0,1\}. \end{cases}$$

Proof. By Lemma 5.3 the set $\pi_1(F_J^p)$ is the self-affine set corresponding to the IFS $\{\psi_{p,s}^1 : s \in \pi_1(J)\}$ for any finite $J \subseteq \mathcal{D}$. Recall that $\psi_{p,0}^1$ and $\psi_{p,1}^1$ are similarity contractions with respective ratios $c_0 = p$ and $c_1 = 1-p$. Furthermore, since $\psi_{p,0}^1((0,1)) = (0,p)$ and $\psi_{p,1}^1((0,1)) = (p,1)$, the IFS $\{\psi_{p,s}^1 : s \in \pi_1(J)\}$ satisfies the OSC. It follows from Proposition 2.5 that $\dim_{\mathcal{B}} \pi_1(F_J^p) = \dim_{\mathcal{H}} \pi_1(F_J^p) = r$, where r is the unique number satisfying

$$\sum_{s \in \pi_1(J)} c_s^r = 1.$$

Therefore, if $\pi_1(J) = \{s\}$ for some $s \in \{0,1\}$, it follows that $c_s^r = 1$ and so since $c_s \in (0,1)$, we then have $\dim_{\mathcal{H}} \pi_1(F_J^p) = r = 0$. If instead $\pi_1(J) = \{0,1\}$, then $p^r + (1-p)^r = c_0^r + c_1^r = 1$, implying that $\dim_{\mathcal{H}} \pi_1(F_J^p) = r = 1$. □

Moreover, we find that the projection of our box-like sets onto the second coordinate satisfies the following properties.

Proposition 5.5. *For any finite $J \subseteq \mathcal{D}$ we have*

$$\dim_{\mathcal{B}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \pi_2(F_J^p).$$

Furthermore, if $2 \notin \pi_2(J)$, then

$$\dim_{\mathcal{B}} \pi_2(F_J^p) < 1,$$

while if instead $\{0,1\} \times \{2\} \subseteq J$, then

$$\dim_{\mathcal{B}} \pi_2(F_J^p) = 1.$$

Before we prove this proposition, we will first prove the following lemma, relating the vertical projection of our box-like sets to the restricted digit sets we studied in the previous chapter.

Lemma 5.6. *If $2 \notin \pi_2(J)$, then $\pi_2(F_J^p)$ is contained in $[0, \frac{1}{2}]$, and coincides with the limit set of the IFS $\{\phi_{s,d} : (s,d) \in J\}$ on $[0, \frac{1}{2}]$. In particular, if J is of the form $J = \{0,1\} \times I$ for some finite set $I \subseteq \mathbb{N}_{\geq 3}$, then*

$$\dim_{\mathcal{B}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \Lambda_I.$$

Proof. Consider some finite set $J \subseteq \{0,1\} \times \mathbb{N}_{\geq 3}$. By Lemma 5.3 the set $\pi_2(F_J^p)$ coincides with the limit set F of the IFS $\{\psi_{s,d}^2 : (s,d) \in J\}$ on $[0,1]$. Note that this limit set satisfies $\bigcup_{(s,d) \in J} \psi_{s,d}^2(F) = F$. For any $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$ we have $\psi_{s,d}^2([0,1]) = [\frac{1}{d}, \frac{1}{d-1}] \subset [0, \frac{1}{2}]$, and so $F = \bigcup_{(s,d) \in J} \psi_{s,d}^2(F) \subseteq [0, \frac{1}{2}]$. In particular, we have $F = \bigcup_{(s,d) \in J} \psi_{s,d}^2\big|_{[0,\frac{1}{2}]}(F)$. Note now that, by definition,

$$\psi_{s,d}^2\big|_{[0,\frac{1}{2}]} = T_{s,d}^{-1}\big|_{[0,\frac{1}{2}]} = \phi_{s,d},$$

for every $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$. Hence we have that $F = \bigcup_{(s,d) \in J} \phi_{s,d}(F)$, meaning $\pi_2(F_J^p) = F$ equals the limit set of the IFS $\{\phi_{s,d} : (s,d) \in J\}$ on $[0, \frac{1}{2}]$.

If J is now of the form $\{0,1\} \times I$ for some finite set $I \subseteq \mathbb{N}_{\geq 3}$, then $\pi_2(F_J^p)$ equals the limit set of the IFS $\{\phi_{s,d} : (s,d) \in \{0,1\} \times I\}$, which by Proposition 4.6 coincides with the restricted digit set Λ_I. □

Proof of Proposition 5.5. Take any finite subset $J \subseteq \mathcal{D}$. Then by Lemma 5.3 the set $\pi_2(F_J^p)$ is the limit set of the IFS $\{\psi_{s,d}^2 : (s,d) \in J\}$.

First consider the case $2 \notin \pi_2(J)$. By Lemma 5.6 the projection $\pi_2(F_J^p)$ equals the limit set of the finite IFS $\{\phi_{s,d} : (s,d) \in J\}$ on $[0, \frac{1}{2}]$, which we recall to be an IFS satisfying the OSC, consisting of similarities with corresponding similarity ratios $c_{s,d} := \frac{1}{d(d-1)}$. By Proposition 2.5 we have $\dim_{\mathcal{B}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \pi_2(F_J^p) = r$, where r is the unique number in $[0,1]$ satisfying

$$\sum_{(s,d)\in J} \Big(\frac{1}{d(d-1)}\Big)^r = 1.$$

For any $d \in \pi_2(J)$ there exist at most 2 numbers $s \in \{0,1\}$ such that $(s,d) \in J$ and so $\sum_{(s,d)\in J} \frac{1}{d(d-1)} \leq 2\sum_{d\in\pi_2(J)} \frac{1}{d(d-1)}$. Furthermore, $\pi_2(J)$ is a finite subset of $\mathbb{N}_{\geq 3}$ and so $\sum_{d\in\pi_2(J)} \frac{1}{d(d-1)} < \sum_{d\in\mathbb{N}_{\geq 3}} \frac{1}{d(d-1)} = \frac{1}{2}$. It follows that $\sum_{(s,d)\in J} \frac{1}{d(d-1)} < 1$ and so necessarily $\dim_{\mathcal{B}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \pi_2(F_J^p) < 1$.

Now we consider the case $\{0,1\} \times \{2\} \subseteq J$. Note that $\bigcup_{s\in\{0,1\}} \psi_{s,2}^2([0,1]) = [\frac{1}{2}, 1]$, meaning we cannot assume that $\pi_2(F_J^p) \subseteq [0, \frac{1}{2}]$ as we did in the previous case, and hence we cannot reduce to an IFS satisfying the OSC as above. Note however that since we have that $\pi_2(F_J^p) = \bigcup_{(s,d)\in J} \psi_{s,d}^2(\pi_2(F_J^p))$ and that $\psi_{s,d}^2([0,1]) \subset [0, \frac{1}{2}]$ for every $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 3}$, it follows that

$$\pi_2(F_J^p) \cap [\tfrac{1}{2}, 1] = \Big[\bigcup_{(s,d)\in J} \psi_{s,d}^2(\pi_2(F_J^p))\Big] \cap [\tfrac{1}{2}, 1] = \bigcup_{s\in\{0,1\}} \psi_{s,2}^2(\pi_2(F_J^p)).$$

Furthermore, it follows that $\pi_2(F_J^p) \cap [\frac{1}{2}, 1]$ is a compact set in $[0,1]$ satisfying $\bigcup_{s\in\{0,1\}} \psi_{s,2}^2\big(\pi_2(F_J^p)) \cap [\frac{1}{2}, 1]\big) \subseteq \pi_2(F_J^p) \cap [\frac{1}{2}, 1]$, and so by Lemma 2.3 the limit set of the IFS $\{\psi_{0,2}^2, \psi_{1,2}^2\}$ is contained in $\pi_2(F_J^p) \cap [\frac{1}{2}, 1]$ and hence in $\pi_2(F_J^p)$. Note now that

$$\psi_{0,2}^2([\tfrac{1}{2}, 1]) = [\tfrac{3}{4}, 1] \quad \text{and} \quad \psi_{1,2}^2([\tfrac{1}{2}, 1]) = [\tfrac{1}{2}, \tfrac{3}{4}],$$

and so $[\frac{1}{2}, 1]$ is a compact set satisfying $\bigcup_{s\in\{0,1\}} \psi_{0,2}^2([\frac{1}{2}, 1]) = [\frac{1}{2}, 1]$. By the uniqueness of the limit set of a finite IFS, $[\frac{1}{2}, 1]$ is the limit set of $\{\psi_{0,2}^2, \psi_{1,2}^2\}$. We conclude that $[\frac{1}{2}, 1] \subseteq \pi_2(F_J^p)$ and hence $\dim_{\mathcal{B}} \pi_2(F_J^p) = \dim_{\mathcal{H}} \pi_2(F_J^p) = 1$. $\square$

Note that Proposition 5.5 does not cover the case where $(s,2) \in J$ only for one $s \in \{0,1\}$. This is because in this case the IFS $\{\psi_{s,d}^2 : (s,d) \in J\}$ cannot be reduced to an IFS that satisfies the OSC and additionally, unlike in the case $\{0,1\} \times \{2\} \subseteq J$, here $\pi_2(F_J^p) \cap [\frac{1}{2}, 1] = \psi_{s,2}^2(\pi_2(F_J^p))$ does not necessarily contain $[\frac{1}{2}, 1]$. Hence we cannot use the same methods we used above to argue that $\pi_2(F_J^p)$ must contain an interval. Therefore our methods of calculating the dimension of $\pi_2(F_J^p)$ fall short in this case.

5.3 The box-counting dimension of the box-like set F_J^p

Now that we have found certain conditions under which the horizontal and vertical projections of the box-like set F_J^p in $[0,1]^2$ generated by the finite iterated function system $\{\Psi^p_{s,d} : (s,d) \in J\}$, $J \subset \mathcal{D}$, have box-counting dimension 1, we can apply Propositions 2.20 and 2.21 to equate the box-counting dimension of F^p_J to the affinity dimension of the linear parts of $\Psi^p_{s,d}$ for $(s,d) \in J$.

Theorem 5.7. *For any finite subset $J \subseteq \mathcal{D}$ satisfying $\{0,1\} \times \{2\} \subseteq J$ we have*

$$\dim_{\mathcal{B}} F^p_J = d(L^p_{s,d} \mid (s,d) \in J),$$

where for each $(s,d) \in J$, $L^p_{s,d}$ is the linear part of $\Psi^p_{s,d}$.

Proof. Take any finite $J \subseteq \mathcal{D}$ such that $\{0,1\} \times \{2\} \subseteq J$. Recall that F^p_J is the box-like limit set of the finite IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ on $[0,1]^2$, which by Lemma 5.2 satisfies the ROSC. Since we have $\{0,1\} \times \{2\} \subseteq J$, it follows from Proposition 5.5 that $\dim_{\mathcal{B}} \pi_2(F^p_J) = 1$. Furthermore, we have $\pi_1(J) = \{0,1\}$, so Lemma 5.4 tells us that $\dim_{\mathcal{B}} \pi_1(F^p_J) = 1$. Hence by Proposition 2.20 the box-counting dimension of F^p_J equals the affinity dimension $d(L^p_{s,d} \mid (s,d) \in J)$. □

For results where we do not necessarily demand that $\{0,1\} \times \{2\} \subseteq J$, we will still need to have $\pi_1(J) = \{0,1\}$ in order to satisfy the conditions of Proposition 2.21. Moreover, we will then require the linear parts of our contractions to have their largest singular value be the one corresponding to the contraction in the horizontal direction. This means we will then need to put a bound on the probability p with which we choose the orientation of the branches of the random Lüroth transformation to ensure that $p, 1-p \geq \frac{1}{d(d-1)}$ for any $d \in \pi_2(J)$. For any $J \subseteq \mathcal{D}$ we will from now on write $d_{\min} := \min \pi_2(J)$.

Theorem 5.8. *Take any finite $J \subseteq \mathcal{D}$ such that $\pi_1(J) = \{0,1\}$. Take any $p \in [\frac{1}{d_{min}(d_{min}-1)}, 1 - \frac{1}{d_{min}(d_{min}-1)}]$. Then we have*

$$\dim_{\mathcal{B}} F^p_J = d(L^p_{s,d} \mid (s,d) \in J),$$

where for each $(s,d) \in J$, $L^p_{s,d}$ is the linear part of $\Psi^p_{s,d}$.

Proof. By Lemma 5.2, the IFS $\{\Psi^p_{s,d} : (s,d) \in J\}$ satisfies the ROSC and its box-like limit set F^p_J is of separated type. For any $d \in \mathbb{N}_{\geq 2}$ the singular values of the linear part $L^p_{0,d}$ of $\Psi^p_{0,d}$ are p and $\frac{1}{d(d-1)}$, whereas those of $L^p_{1,d}$ are $1-p$ and $\frac{1}{d(d-1)}$. Furthermore, the respective singular values p and $1-p$ correspond to the contractions in the horizontal direction.

By assumption we have $p, 1-p \in [\frac{1}{d_{\min}(d_{\min}-1)}, 1 - \frac{1}{d_{\min}(d_{\min}-1)}]$. For any $d \in \pi_2(J)$ we have

$$\frac{1}{d(d-1)} \leq \frac{1}{d_{\min}(d_{\min}-1)} \leq p, 1-p \leq 1 - \frac{1}{d_{\min}(d_{\min}-1)} \leq 1 - \frac{1}{d(d-1)}.$$

Moreover, it follows that $p, 1-p \geq \frac{1}{d(d-1)}$ for any $d \in \pi_2(J)$. Therefore for each $(s,d) \in J$ the largest singular value of the linear part $L^p_{s,d}$ of $\Psi^p_{s,d}$ is the one corresponding to the contraction in the horizontal direction.

Finally, since we have $\pi_1(J) = \{0,1\}$ by assumption, Lemma 5.4 tells us that $\dim_{\mathcal{B}} \pi_1(F^p_J) = 1$. All conditions of Proposition 2.21 are met, and so we conclude that $\dim_{\mathcal{B}} F^p_J = d(L^p_{s,d} \mid (s,d) \in J)$. □

5.4 Bounds for the affinity dimension

The results of Theorems 5.7 and 5.8 yield a set of conditions on $J \subset \mathcal{D}$ and $p \in (0,1)$ under which the box-counting dimension of the box-like set F_J^p equals the affinity dimension of the linear parts of the corresponding iterated function system $\{\Psi_{s,d}^p : (s,d) \in J\}$. To demonstrate the power of these theorems, we will now proceed to further describe this affinity dimension and consider simple cases where we can either calculate it directly or use it to find bounds on the dimension of F_J^p. We begin with a general approach.

Recall that in Definition 2.14 we defined the affinity dimension of a collection $\{L_i : i \in I\}$ of contractive, non-singular linear transformations $\mathbb{R}^k \to \mathbb{R}^k$ to be

$$d(L_i : i \in I) = \inf \Big\{ r : \sum_{m=1}^{\infty} \sum_{\mathbf{i} \in I^m} \varphi^r(L_{\mathbf{i}}) < \infty \Big\},$$

where for $0 \leq r \leq k$, φ^r is the singular value function described in Definition 2.13. Therefore we will need to consider the singular values of the compositions $L_{(\mathbf{s},\mathbf{d})}^p = L_{s_1,d_1}^p \circ ... \circ L_{s_m,d_m}^p$ for sequences $((s_j,d_j))_{j=1}^m \in \mathcal{D}^m$, $m \in \mathbb{N}$.

Recall that for any $p \in (0,1)$ and $d \in \mathbb{N}_{\geq 2}$ we have

$$L_{0,d}^p = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{d(d-1)} \end{pmatrix}, \quad \text{and} \quad L_{1,d}^p = \begin{pmatrix} 1-p & 0 \\ 0 & -\frac{1}{d(d-1)} \end{pmatrix}.$$

If for any $(\mathbf{s},\mathbf{d}) := ((s_j,d_j))_{j=1}^m \in \mathcal{D}^m$, $m \in \mathbb{N}$, we write

$$\kappa_{(\mathbf{s},\mathbf{d})} := \#\{1 \leq j \leq m : s_j = 1\} = \sum_{j=1}^m s_j,$$

then we have

$$L_{(\mathbf{s},\mathbf{d})}^p = \begin{pmatrix} (1-p)^{\kappa_{(\mathbf{s},\mathbf{d})}} p^{m-\kappa_{(\mathbf{s},\mathbf{d})}} & 0 \\ 0 & (-1)^{\kappa_{(\mathbf{s},\mathbf{d})}} \prod_{j=1}^m \frac{1}{d_j(d_j-1)} \end{pmatrix}.$$

The singular values of a diagonal matrix are the absolute values of the diagonal components and so it follows that those of the composition $L_{(\mathbf{s},\mathbf{d})}^p$ are given by $(1-p)^{\kappa_{(\mathbf{s},\mathbf{d})}} p^{m-\kappa_{(\mathbf{s},\mathbf{d})}}$ and $\prod_{j=1}^m \frac{1}{d_j(d_j-1)}$ for any sequence $(\mathbf{s},\mathbf{d}) \in \mathcal{D}^m$.

In the case $p \in [\frac{1}{d_{\min}(d_{\min}-1)}, 1-\frac{1}{d_{\min}(d_{\min}-1)}]$ we always have $(1-p)^{\kappa_{(\mathbf{s},\mathbf{d})}} p^{m-\kappa_{(\mathbf{s},\mathbf{d})}} \geq \prod_{j=1}^m \frac{1}{d_j(d_j-1)}$, meaning the singular value function can then be determined consistently. Otherwise which singular value is the largest may differ between different sequences $(\mathbf{s},\mathbf{d})$ in J^m, making a direct analysis difficult.

For any finite $J \subseteq \mathcal{D}$ and $p \in (0,1)$, we will from now on use the notations $p_{\min} := \min\{p, 1-p\}$, $p_{\max} := \max\{p, 1-p\}$ and $d_{\max} := \max \pi_2(J)$, along with the notation $d_{\min} := \min \pi_2(J)$ we introduced earlier. It will always be clear from the context to which set J and probability p these correspond. For the purposes of this chapter, we will restrict ourselves to the case $p \in [\frac{1}{d_{\min}(d_{\min}-1)}, 1-\frac{1}{d_{\min}(d_{\min}-1)}]$. In this case we find the following bounds for the affinity dimension.

Proposition 5.9. *Take any $p \in (0,1)$ and any finite set $J \subseteq \mathcal{D}$ with $\#J \geq 2$ such that $p \in [\frac{1}{d_{min}(d_{min}-1)}, 1 - \frac{1}{d_{min}(d_{min}-1)}]$. If $\#J \leq \frac{1}{p_{min}}$ then we have*

$$\frac{\log \#J}{\log \frac{1}{p_{min}}} \leq d(L^p_{s,d} \mid (s,d) \in J) \leq 1 + \frac{\log \#J - \log \frac{1}{p_{max}}}{\log(d_{min}(d_{min}-1))},$$

while if instead $\#J > \frac{1}{p_{min}}$ then

$$1 + \frac{\log \#J - \log \frac{1}{p_{min}}}{\log(d_{max}(d_{max}-1))} \leq d(L^p_{s,d} \mid (s,d) \in J) \leq 1 + \frac{\log \#J - \log \frac{1}{p_{max}}}{\log(d_{min}(d_{min}-1))}.$$

Proof. Take any $m \in \mathbb{N}$ and $(\mathbf{s},\mathbf{d}) = ((s_1,d_1),...,(s_m,d_m)) \in J^m$. Then the singular value function of $L^p_{(\mathbf{s},\mathbf{d})}$ is given by

$$\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) = \begin{cases} \left((1-p)^{\kappa(\mathbf{s},\mathbf{d})} p^{m-\kappa(\mathbf{s},\mathbf{d})}\right)^r, & \text{if } 0 \leq r \leq 1, \\ (1-p)^{\kappa(\mathbf{s},\mathbf{d})} p^{m-\kappa(\mathbf{s},\mathbf{d})} \left(\prod_{k=1}^m \frac{1}{d_k(d_k-1)}\right)^{r-1}, & \text{if } 1 < r \leq 2. \end{cases}$$

Let us consider the upper bound first. Note that $(1-p)^{\kappa(\mathbf{s},\mathbf{d})} p^{m-\kappa(\mathbf{s},\mathbf{d})} \leq p^m_{\max}$ whereas $\prod_{k=1}^m \frac{1}{d_k(d_k-1)} \leq (\frac{1}{d_{\min}(d_{\min}-1)})^m$. For any $1 < r \leq 2$ we therefore find

$$\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) \leq p^m_{\max}\Big(\frac{1}{d_{\min}(d_{\min}-1)}\Big)^{m(r-1)}.$$

Noting that $\#(J^m) = (\#J)^m$, it follows that

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{(\mathbf{s},\mathbf{d}) \in J^m} \varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) &\leq \sum_{m=1}^{\infty} \sum_{(\mathbf{s},\mathbf{d}) \in J^m} p^m_{\max}\Big(\frac{1}{d_{\min}(d_{\min}-1)}\Big)^{m(r-1)} \\ &= \sum_{m=1}^{\infty} (\#J)^m p^m_{\max}\Big(\frac{1}{d_{\min}(d_{\min}-1)}\Big)^{m(r-1)} \\ &= \sum_{m=1}^{\infty} \Big(\frac{\#J \cdot p_{\max}}{(d_{\min}(d_{\min}-1))^{r-1}}\Big)^m. \end{aligned}$$

The right hand side converges if $\#J \cdot p_{\max}\Big(\frac{1}{d_{\min}(d_{\min}-1)}\Big)^{r-1} < 1$, or equivalently

$$r > 1 + \frac{\log \#J + \log p_{\max}}{\log(d_{\min}(d_{\min}-1))} = 1 + \frac{\log \#J - \log \frac{1}{p_{\max}}}{\log(d_{\min}(d_{\min}-1))}.$$

By assumption we have $\#J \geq 2 \geq \frac{1}{p_{\max}}$ and hence $1 + \frac{\log \#J - \log \frac{1}{p_{\max}}}{\log(d_{\min}(d_{\min}-1))} \geq 1$. Therefore we see that the series $\sum_{m=1}^{\infty} \sum_{(\mathbf{s},\mathbf{d}) \in J^m} \varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big)$ converges if $r > 1 + \frac{\log \#J - \log \frac{1}{p_{\max}}}{\log(d_{\min}(d_{\min}-1))}$. It follows that

$$\begin{aligned} d(L^p_{s,d} \mid (s,d) \in J) &= \inf\Big\{r : \sum_{m=1}^{\infty} \sum_{(\mathbf{s},\mathbf{d}) \in J^m} \varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) < \infty\Big\} \\ &\leq 1 + \frac{\log \#J - \log \frac{1}{p_{\max}}}{\log(d_{\min}(d_{\min}-1))}. \end{aligned}$$

For the lower bounds, we note that $(1-p)^{\kappa(\mathbf{s},\mathbf{d})}p^{m-\kappa(\mathbf{s},\mathbf{d})} \geq p_{\min}^m$. For any $0 \leq r \leq 1$ we have

$$\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) = \big((1-p)^{\kappa(\mathbf{s},\mathbf{d})}p^{m-\kappa(\mathbf{s},\mathbf{d})}\big)^r \geq p_{\min}^{mr},$$

and so

$$\sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) \geq \sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m} p_{\min}^{mr} = \sum_{m=1}^{\infty}(\#J\cdot p_{\min}^r)^m.$$

The right hand side diverges if $\#J\cdot p_{\min}^r \geq 1$, or equivalently $r \leq \frac{\log \#J}{\log \frac{1}{p_{\min}}}$. It follows that the series $\sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big)$ diverges if $0 \leq r \leq \min\big\{1, \frac{\log \#J}{\log \frac{1}{p_{\min}}}\big\}$.

If we assume that $\#J \leq \frac{1}{p_{\min}}$, then we have $\frac{\log \#J}{\log \frac{1}{p_{\min}}} \leq 1$ and so the series diverges for every $r \leq \frac{\log \#J}{\log \frac{1}{p_{\min}}}$, implying that

$$d(L^p_{s,d} \mid (s,d)\in J) = \inf\Big\{r : \sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) < \infty\Big\} \geq \frac{\log \#J}{\log \frac{1}{p_{\min}}}.$$

Finally, suppose $\#J > \frac{1}{p_{\min}}$. By the above the series $\sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big)$ diverges if $r \leq 1$. We have $\prod_{k=1}^{m}\frac{1}{d_k(d_k-1)} \geq (\frac{1}{d_{\max}(d_{\max}-1)})^m$, so any $1 < r \leq 2$ we find that

$$\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) \geq p_{\min}^m\Big(\frac{1}{d_{\max}(d_{\max}-1)}\Big)^{m(r-1)}.$$

Hence

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big) &\geq \sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m} p_{\min}^m\Big(\frac{1}{d_{\max}(d_{\max}-1)}\Big)^{m(r-1)}\\ &= \sum_{m=1}^{\infty}\Big(\frac{\#J\cdot p_{\min}}{(d_{\max}(d_{\max}-1))^{r-1}}\Big)^m.\end{aligned}$$

Similarly as before the right hand side diverges if

$$r \leq 1 + \frac{\log \#J + \log p_{\min}}{\log(d_{\max}(d_{\max}-1))} = 1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))}.$$

By the assumption $\#J > \frac{1}{p_{\min}}$ we have $1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))} > 1$ and so the series $\sum_{m=1}^{\infty}\sum_{(\mathbf{s},\mathbf{d})\in J^m}\varphi^r\big(L^p_{(\mathbf{s},\mathbf{d})}\big)$ also diverges if

$$1 \leq r < 1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))}.$$

We conclude that then

$$d(L^p_{s,d} \mid (s,d)\in J) \geq 1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))}. \qquad \square$$

5.5 Consequences and examples

Note that any finite set $J \subseteq \mathcal{D}$ and probability p that satisfy the conditions of Theorem 5.8 also satisfy those of Proposition 5.9. Hence combining the two results immediately leads to the following bounds for the box-counting dimension of the box-like set F_J^p.

Corollary 5.10. *Take any finite $J \subseteq \mathcal{D}$ such that $\pi_1(J) = \{0,1\}$ and any $p \in [\frac{1}{d_{min}(d_{min}-1)}, 1 - \frac{1}{d_{min}(d_{min}-1)}]$. If $\#J \leq \frac{1}{p_{min}}$ then we have*

$$\frac{\log \#J}{\log \frac{1}{p_{min}}} \leq \dim_{\mathcal{B}} F_J^p \leq \Big\{2,\ 1 + \frac{\log \#J - \log \frac{1}{p_{max}}}{\log(d_{min}(d_{min}-1))}\Big\},$$

while if instead $\#J > \frac{1}{p_{min}}$ then

$$1 + \frac{\log \#J - \log \frac{1}{p_{min}}}{\log(d_{max}(d_{max}-1))} \leq \dim_{\mathcal{B}} F_J^p \leq \min\Big\{2,\ 1 + \frac{\log \#J - \log \frac{1}{p_{max}}}{\log(d_{min}(d_{min}-1))}\Big\}.$$

Corollary 5.11. *For any finite $J \subseteq \mathcal{D}$ satisfying $\pi_1(J) = \{0,1\}$ and any $p \in [\frac{1}{d_{min}(d_{min}-1)}, 1 - \frac{1}{d_{min}(d_{min}-1)}]$ we have $\dim_{\mathcal{B}} F_J^p > 0$.*

As there is no upper bound on the possible values of $\#J$, one may wonder whether the lower bound found in Corollary 5.10 for the case $\#J > \frac{1}{p_{\min}}$ breaks when J is taken to be too large. After all, as a subset of $[0,1]^2$ the dimension of F_J^p cannot possibly be larger than 2, so something is definitely wrong if there exists some J for which $\frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))} > 1$.

This is, however, never the case as the value of $\#J$ puts a lower bound on the possible values of $d_{\max}$. To see why this is true, we take any $k \geq 2$ and proceed to construct a set $J \subset \mathcal{D}$ satisfying $\#J = k$ with the smallest possible value of $d_{\max}$. We can do this by using a sort of greedy algorithm, where we start by setting $J = \emptyset$ and at each iteration we keep adding an element $(s,d) \in \mathcal{D} \setminus J$ to J with the lowest possible value for d, until J has k elements. When k is an even number this uniquely leads us to the set

$$J = \big\{(0,2),(1,2),(0,3),(1,3),...,(0,\tfrac{k}{2}+1),(1,\tfrac{k}{2}+1)\big\} = \{0,1\} \times \{2,...,\tfrac{k}{2}+1\big\},$$

while if k is odd, we may get either

$$J = \big\{(0,2),(1,2),(0,3),(1,3),...,(0,\lceil\tfrac{k}{2}\rfloor),(1,\lceil\tfrac{k}{2}\rceil),(0,\lceil\tfrac{k}{2}\rceil+1)\big\},$$

or

$$J = \big\{(0,2),(1,2),(0,3),(1,3),...,(0,\lceil\tfrac{k}{2}\rfloor),(1,\lceil\tfrac{k}{2}\rceil),(1,\lceil\tfrac{k}{2}\rceil+1)\big\}.$$

In either case we have $d_{\max} \geq \frac{k}{2} + 1$. Therefore, for any finite $J \subseteq \mathcal{D}$ we have

$$\begin{aligned}\log(d_{\max}(d_{\max}-1)) &\geq \log((d_{\max}-1)^2) = 2\log(d_{\max}-1)\\ &\geq 2\log\tfrac{\#J}{2} = 2(\log\#J - \log 2).\end{aligned}$$

At the same time we have $p_{\min} \leq \frac{1}{2}$ and so $\log \frac{1}{p_{\min}} \geq \log 2$. In the case $\#J = 2$ we therefore have

$$1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))} = 1 + \frac{\log 2 - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))} \leq 1 + \frac{\log 2 - \log 2}{\log(d_{\max}(d_{\max}-1))} = 1,$$

while in the case $\#J > 2$ we find that

$$1 + \frac{\log \#J - \log \frac{1}{p_{\min}}}{\log(d_{\max}(d_{\max}-1))} \leq 1 + \frac{\log \#J - \log 2}{\log(d_{\max}(d_{\max}-1))} \leq 1 + \frac{\log \#J - \log 2}{2(\log \#J - \log 2)} = \frac{3}{2}.$$

Hence this lower bound never exceeds 2.

Note that the upper bound given in Corollary 5.10 is only interesting when $\frac{\log \#J - \log \frac{1}{p_{\max}}}{\log(d_{\min}(d_{\min}-1))} < 1$. After all, the box-like sets F_J^p are all contained in the unit square $[0,1]^2$ and so we already know their box-counting dimensions will not exceed 2. As J can be taken as large as we want without necessarily bounding $d_{\min}$, there are indeed many sets for which this upper bound won't tell us anything new. However, the following shows that this bound can get arbitrarily close to 1 in infinitely many cases.

Corollary 5.12. *Fix any $k \geq 2$ and any $p \in (0,1)$. Then for any $\varepsilon > 0$ there exists some $J \subseteq \mathcal{D}$ such that $\#J = k$ and $\dim_{\mathcal{B}} F_J^p \leq 1 + \varepsilon$.*

Proof. Take any $k > 2$, $p \in (0,1)$ and $\varepsilon > 0$. Note that $\frac{1}{d(d-1)} \to 0$ and $1 - \frac{1}{d(d-1)} \to 1$ as $d \to \infty$. As such, there must exist some $\bar{d} \in \mathbb{N}_{\geq 2}$ such that $p \in [\frac{1}{d(d-1)}, 1 - \frac{1}{d(d-1)}]$ for each $d \geq \bar{d}$. Similarly, we have that

$$\frac{\log k - \log \frac{1}{p_{\max}}}{\log(d(d-1))} \longrightarrow 0 \quad \text{as} \quad d \to \infty,$$

and so there also exists some $d' \in \mathbb{N}_{\geq 2}$ such that $\frac{\log k - \log \frac{1}{p_{\max}}}{\log(d(d-1))} < \varepsilon$ for any $d \geq d'$. Hence any set $J \subseteq \mathcal{D}$ satisfying $\pi_1(J) = \{0,1\}$, $\#J = k$ and $d_{\min} \geq \max\{d', \bar{d}\}$ will suffice. □

To get more intuition behind the bounds we found in Corollary 5.10, we will now consider the case where $p = \frac{1}{2}$. This is the simplest case as we have $p_{\min} = p_{\max} = \frac{1}{2}$ and the condition $p \in [\frac{1}{d_{\min}(d_{\min}-1)}, 1 - \frac{1}{d_{\min}(d_{\min}-1)}]$ is satisfied for every $J \subseteq \mathcal{D}$. This is also the case where the bounds given in Corollary 5.10 are the tightest, as $p_{\min}$ is maximal while $p_{\max}$ is minimal. These bounds are given by the following.

Corollary 5.13. *Take any finite subset $J \subseteq \mathcal{D}$ such that $\pi_1(J) = \{0,1\}$. Then*

(i) if $\#J = 2$, we have $\dim_{\mathcal{B}} F_J^{1/2} = 1$;

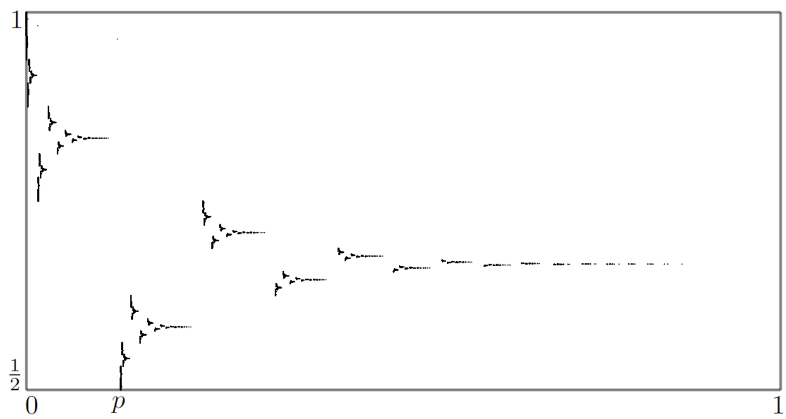
(ii) if $\#J > 2$, we have

$$1 + \frac{\log \#J - \log 2}{\log(d_{max}(d_{max}-1))} \leq \dim_{\mathcal{B}} F_J^{1/2} \leq \min\left\{2,\ 1 + \frac{\log \#J - \log 2}{\log(d_{min}(d_{min}-1))}\right\}.$$

Note that in particular we have $\dim_{\mathcal{B}} F_J^{1/2} > 1$ whenever $J \subseteq \mathcal{D}$ is such that $\pi_1(J) = \{0,1\}$ and $\#J > 2$. Together with Corollary 5.12 this implies that there are infinitely many sets J for which the box-like set $F_J^{1/2}$ has a nonintegral box-counting dimension in $(1,2)$.

We will now conclude this chapter by discussing some concrete examples of the box-like sets we have introduced, giving bounds on their box-counting dimensions where possible and showing approximate images of the sets that were generated using a chaos game algorithm.

Example 5.14. We begin by considering the case where the restriction set is given by $J = \{0,1\} \times \{2\}$. In this case we have $d_{\min} = 2$, and so the condition $p \in [\frac{1}{d_{\min}(d_{\min}-1)}, 1 - \frac{1}{d_{\min}(d_{\min}-1)}]$ is satisfied if and only if $p = \frac{1}{2}$, in which case Corollary 5.13 tells us that $\dim_{\mathcal{B}} F_J^{1/2} = 1$.



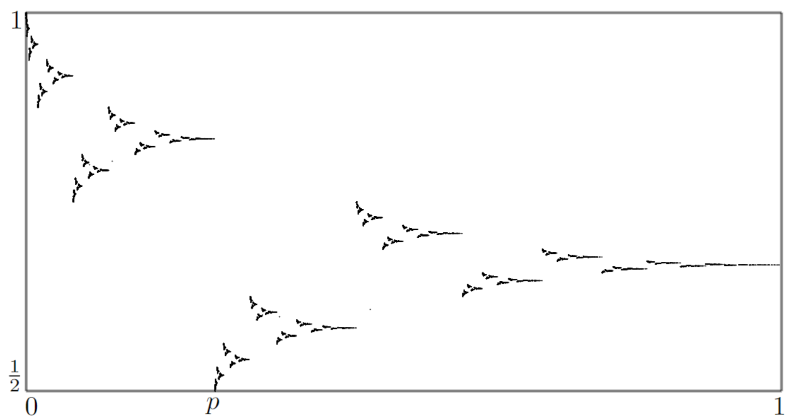


Figure 13: The box-like set F_J^p for $J = \{0,1\} \times \{2\}$ and $p = \frac{1}{8}, \frac{1}{4}$ respectively.

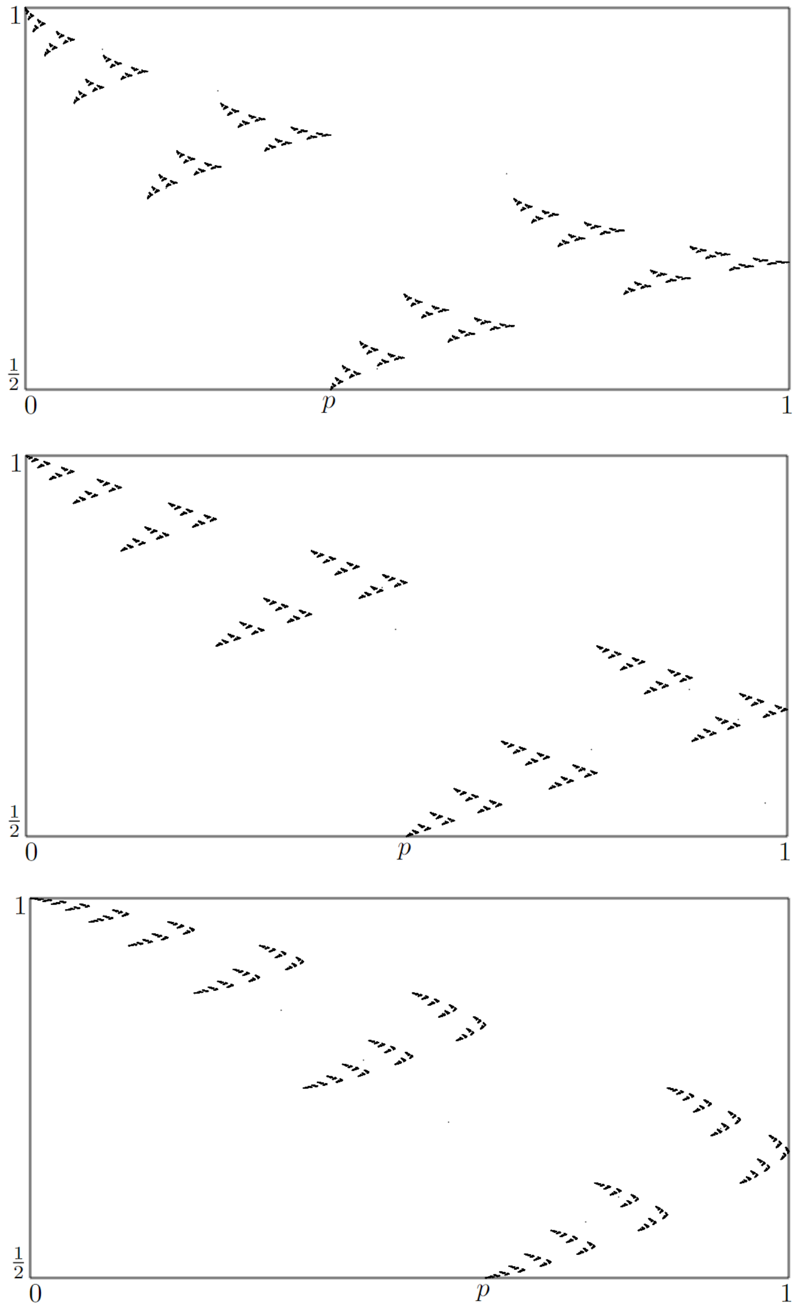


Figure 14: The box-like set F_J^p for $J = \{0, 1\} \times \{2\}$ and $p = \frac{2}{5}, \frac{1}{2}, \frac{3}{5}$ respectively.

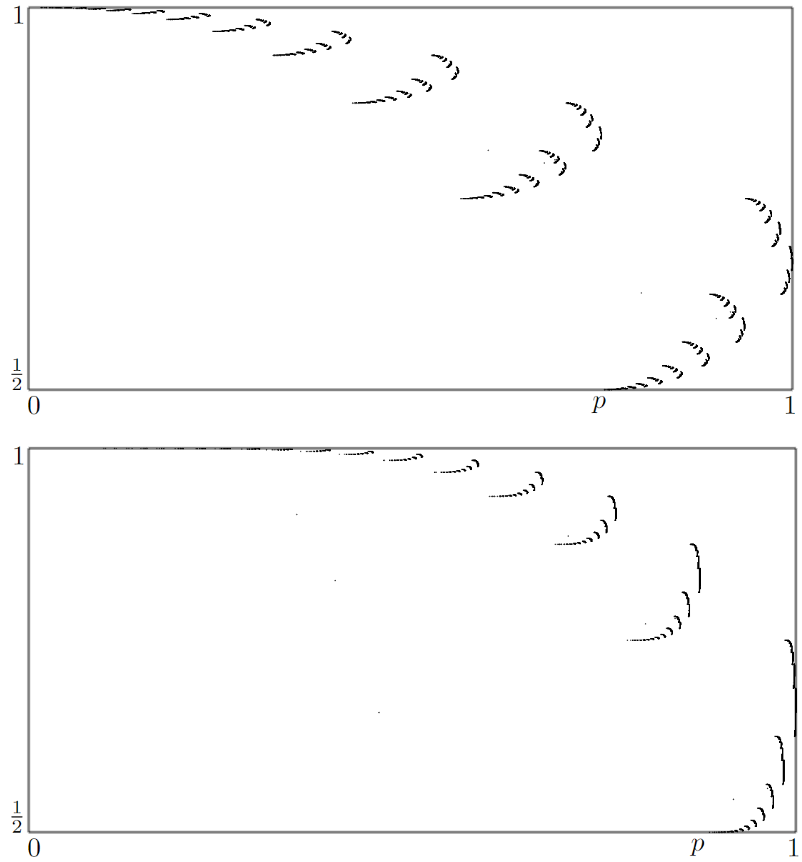


Figure 15: The box-like set F_J^p for $J = \{0, 1\} \times \{2\}$ and $p = \frac{3}{4}, \frac{7}{8}$ respectively.

For any general probability p, Lemma 5.4 and Proposition 5.5 tell us that $\dim_{\mathcal{B}} \pi_1(F_J^p) = \dim_{\mathcal{B}} \pi_2(F_J^p) = 1$. Furthermore, Theorem 5.7 implies that the box-counting dimension of F_J^p itself equals the affinity dimension $d(L_{0,2}^p, L_{1,2}^p)$.

Unfortunately the methods employed above to find concrete bounds do not apply when p equals anything other than $\frac{1}{2}$. This is because if $p \neq \frac{1}{2}$, the singular value $(1-p)^{\kappa_{(\mathbf{s},\mathbf{d})}} p^{m-\kappa_{(\mathbf{s},\mathbf{d})}}$ of $L_{(\mathbf{s},\mathbf{d})}^p$ will be greater or smaller than its other singular value $\frac{1}{2^m}$ depending on the sequence $(\mathbf{s}, \mathbf{d}) \in (\{0, 1\} \times \{2\})^m$, making analysis of the affinity dimension difficult. Since $\kappa_{(\mathbf{s},\mathbf{d})}$ is defined to count the amount of times the digit s_k in a sequence $(\mathbf{s}, \mathbf{d})$ equals 1, it might be possible to find out more about the affinity dimension using combinatorial arguments.

Looking at Figures 13, 14 and 15, however, it does seem that there is some sort of continuity in how the sets F_J^p change when varying only the value of p. It was shown in [Fal88, LG92] that the dimension of a smoothly parametrized family of self-affine sets does not necessarily change continuously with the parameters. It would be interesting to see whether this is the case here. ♢

Since the sharpest bounds are found when $p = \frac{1}{2}$, we will now restrict ourselves to this case and study a handful of examples of box-like sets $F_J^{1/2}$ for certain restriction sets $J \subset \{0,1\} \times \mathbb{N}_{\geq 2}$.

Example 5.15. Consider the two restriction sets $J_0 := \{(0,2),(1,2),(0,3)\}$ and $J_1 := \{(0,2),(1,2),(1,3)\}$. We have $\#J_0 = \#J_1 = 3$ and both sets have $d_{\min} = 2$ and $d_{\max} = 3$. Hence Corollary 5.13 gives the same bounds

$$1.22629 \leq 1 + \frac{\log 3 - \log 2}{\log 6} \leq \dim_{\mathcal{B}} F_{J_\alpha}^{1/2} \leq 1 + \frac{\log 3 - \log 2}{\log 2} \leq 1.58497,$$

for $\alpha \in \{0,1\}$. As seen in Figures 16 and 17 below both sets bear a slight visual resemblance to a somewhat distorted version of the famous Sierpiński triangle, which is displayed in Figure 18.

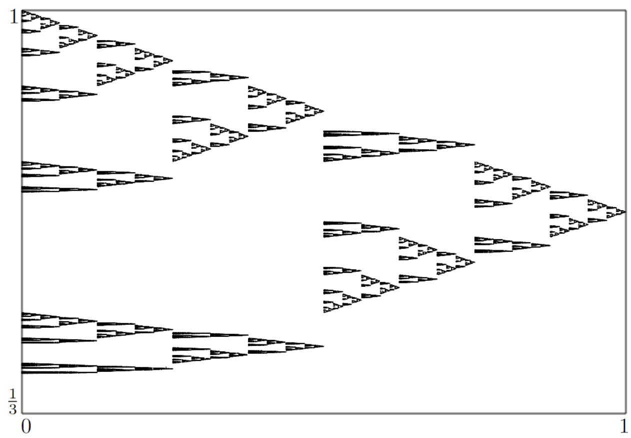


Figure 16: The box-like set $F_{J_0}^{1/2}$ for $J_0 = \{(0,2),(1,2),(0,3)\}$.

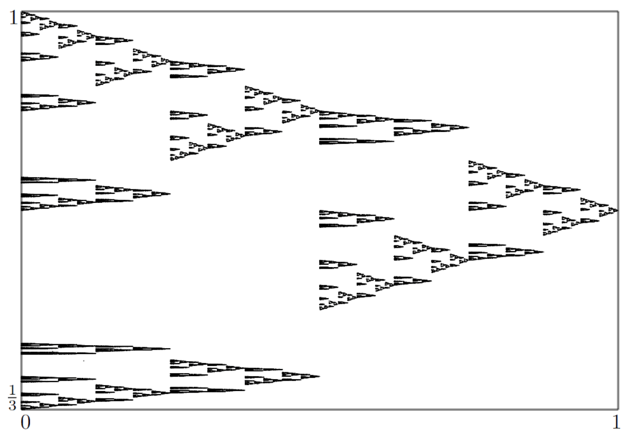


Figure 17: The box-like set $F_{J_1}^{1/2}$ for $J_1 = \{(0,2),(1,2),(1,3)\}$.

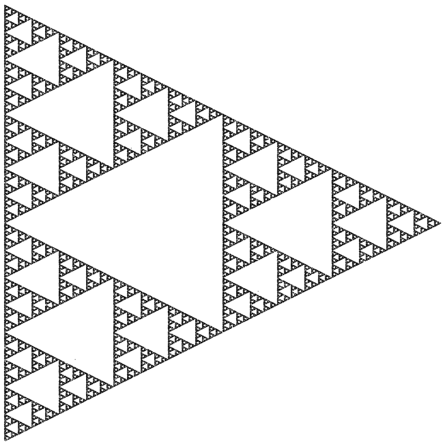


Figure 18: A rendering of the Sierpiński triangle for reference.

We will see in the next example that this resemblance is lost completely when combining the two sets into the restriction set $J = \{0,1\} \times \{2,3\}$. $\diamondsuit$

Example 5.16. We will now consider the box-like sets $F^{1/2}_{J_d}$ for the restriction sets $J_d := \{0,1\} \times \{d, d+1\}$ for $d \in \mathbb{N}_{\geq 2}$. For any d we have $\#J_d = 4$ and hence $\log \#J_d = \log 4 = 2\log 2$, meaning Corollary 5.13 yields the bounds

$$1 + \frac{\log 2}{\log((d+1)d)} \leq \dim_{\mathcal{H}} F^{1/2}_{J_d} \leq 1 + \frac{\log 2}{\log(d(d-1))}.$$

We will work out these bounds up to five decimals for the following examples, with accompanying images in Figures 19, 20 and 21 for the respective cases $d = 2$, $d = 3$ and $d = 4$.

- $J_2 = \{0,1\} \times \{2,3\}$ yields

$$1.38685 \leq 1 + \frac{\log 2}{\log 6} \leq \dim_{\mathcal{B}} F^{1/2}_{J_2} \leq 1 + \frac{\log 2}{\log 2} = 2,$$

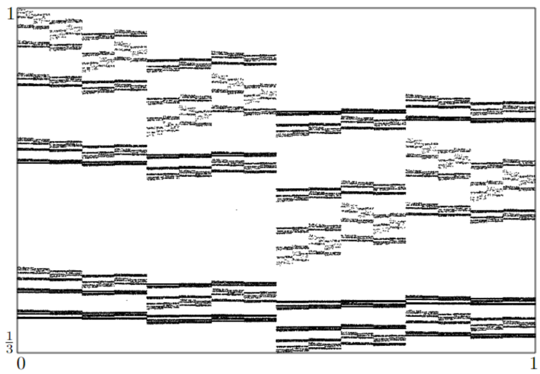


Figure 19: The box-like set $F^{1/2}_{J_2}$ for $J_2 = \{0,1\} \times \{2,3\}$.

- $J_3 = \{0, 1\} \times \{3, 4\}$ yields

$$1.27894 \leq 1 + \frac{\log 2}{\log 12} \leq \dim_{\mathcal{B}} F^{1/2}_{J_3} \leq 1 + \frac{\log 2}{\log 6} \leq 1.38686,$$

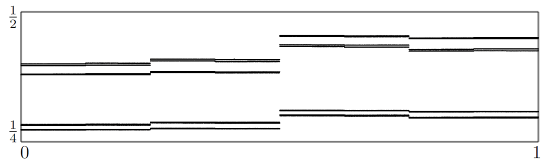


Figure 20: The box-like set $F^{1/2}_{J_3}$ for $J_3 = \{0, 1\} \times \{3, 4\}$.

- $J_4 = \{0, 1\} \times \{4, 5\}$ yields

$$1.23137 \leq 1 + \frac{\log 2}{\log 20} \leq \dim_{\mathcal{B}} F^{1/2}_{J_4} \leq 1 + \frac{\log 2}{\log 12} \leq 1.27895,$$

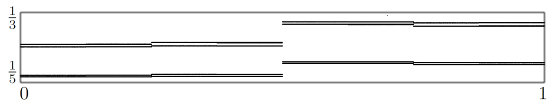


Figure 21: The box-like set $F^{1/2}_{J_4}$ for $J_4 = \{0, 1\} \times \{4, 5\}$.

- $J_5 = \{0, 1\} \times \{5, 6\}$ yields

$$1.20379 \leq 1 + \frac{\log 2}{\log 30} \leq \dim_{\mathcal{B}} F^{1/2}_{J_5} \leq 1 + \frac{\log 2}{\log 20} \leq 1.23138,$$

- $J_6 = \{0, 1\} \times \{6, 7\}$ yields

$$1.18544 \leq 1 + \frac{\log 2}{\log 42} \leq \dim_{\mathcal{B}} F^{1/2}_{J_6} \leq 1 + \frac{\log 2}{\log 30} \leq 1.20380,$$

- $J_7 = \{0, 1\} \times \{7, 8\}$ yields

$$1.17219 \leq 1 + \frac{\log 2}{\log 56} \leq \dim_{\mathcal{B}} F^{1/2}_{J_7} \leq 1 + \frac{\log 2}{\log 42} \leq 1.18545,$$

- $J_8 = \{0, 1\} \times \{8, 9\}$ yields

$$1.16207 \leq 1 + \frac{\log 2}{\log 72} \leq \dim_{\mathcal{B}} F^{1/2}_{J_8} \leq 1 + \frac{\log 2}{\log 56} \leq 1.17220,$$

- $J_9 = \{0, 1\} \times \{9, 10\}$ yields

$$1.15403 \leq 1 + \frac{\log 2}{\log 90} \leq \dim_{\mathcal{B}} F^{1/2}_{J_9} \leq 1 + \frac{\log 2}{\log 72} \leq 1.16208,$$

As expected the bounds on $\dim_{\mathcal{B}} F^{1/2}_{J_d}$ keep getting tighter the larger the value of d, while its value also approaches 1. In particular we have

$$1 < \dim_{\mathcal{B}} F^{1/2}_{J_{d+1}} \leq \dim_{\mathcal{B}} F^{1/2}_{J_d},$$

for every $d \in \mathbb{N}_{\geq 2}$. ♢

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