# An asymptotic bound for the Barroero-Widmer constant

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# Index of Notation

### Topology and measures

$\mathrm{bd}(X)$ Boundary of $X$

$B(Z)$ Fiberwise boundary of $Z$; $\{(T,x) \in \mathbb{R}^{m+n} : x \in \mathrm{bd}(Z_T)\}$

$\mathrm{cl}(X), \overline{X}$ Topological closure of $X$

$C(Z)$ Fiberwise closure of $Z$; $\{(T,x) \in \mathbb{R}^{m+n} : x \in \mathrm{cl}(Z_T)\}$

$\mathcal{L}^j$,$\mathrm{Vol}_j$ $j$-dimensional Lebesgue measure

$\mathcal{H}^j$ $j$-dimensional Hausdorff measure

$\dim(X)$ O-minimal dimension of $X$; maximal dimension of a cell contained in $X$

$\dim_{\mathcal{H}}(X)$ Hausdorff dimension of $X$; $\inf\{j \in \mathbb{R}_{>0} : \mathcal{H}^j(X) = 0\}$

$\overline{B_k}(c,r)$ Closed ball in $\mathbb{R}^k$ with center $c$ and radius $r$

$S^{k-1}(c,r)$ $(k-1)$-sphere in $\mathbb{R}^k$ with center $c$ and radius $r$; the boundary of $\overline{B_k}(c,r)$

$D(c,r)$ Open disk in $\mathbb{C}$ with center $c$ and radius $r$

### Linear Algebra

$A_I$ Subspace of $\mathbb{R}^n$ spanned by standard basis vectors $(e_i)_{i\in I}$

$d(\Lambda)$ Determinant of a lattice $\Lambda$ in $\mathbb{R}^n$; absolute value of the determinant of a matrix representing any basis of $\Lambda$

$\pi_I$ Projection $\mathbb{R}^n \to \mathbb{R}^{|I|}$ sending $(x_1, x_2, \ldots, x_n)$ to $(x_i)_{i\in I}$

$\pi_k, k \in \mathbb{Z}_{>0}$ Projection $\mathbb{R}^n \to \mathbb{R}^{n-k}$ removing the last $k$ coordinates

$\pi_W$ Orthogonal projection onto a subspace $W$ of $\mathbb{R}^n$; usually $W = A_I$

$V_j(X)$ Sum of the volumes $\mathrm{Vol}_j(\pi_{A_I}(X))$ for all $I \subseteq \{1, 2, \ldots, n\}$ of cardinality $j$

$V_j'(X)$ Supremum of the volume $\mathrm{Vol}_j(\pi(X))$ as $\pi$ varies over the orthogonal projections on $j$-dimensional subspaces of $\mathbb{R}^n$

### Simplicial complexes

$[a_0, a_1, \ldots, a_p]$ $p$-simplex with vertices $a_0, a_1, \ldots, a_p$ in $\mathbb{R}^n$ with $n \geq p$

$\dot{s}$ Interior of simplex $s$

ba($s$) Barycenter of simplex $s$; arithmetic mean of the vertices of $s$

ba($K$) Barycentric subdivision of complex $K$

$|K|$ Polyhedron spanned by complex $K$; union of simplices in $K$

$K_p$ Set of $p$-simplices of complex $K$

$\mathrm{Ch}_p(K)$ $p$-th chain group of $K$; free abelian group generated by oriented $p$-simplices of $K$

$\delta_p$ Boundary homomorphism $\mathrm{Ch}_p(K) \to \mathrm{Ch}_{p-1}(K)$

$B_p(K)$ Set of $p$-boundaries of $K$; $\mathrm{im}(\delta_{p+1})$

$Z_p(K)$ Set of $p$-chains of $K$; $\ker(\delta_p)$

$H_p(K)$ $p$-th homology group of $K$; $Z_p(K)/B_p(K)$

$b_i(K)$ $i$-th Betti number of $K$; rank of $H_p(K)$. Later also defined for several types of semialgebraic sets

**Miscellaneous**

$2^X$ Power set of $X$

$\mathrm{Reali}(\sigma, \mathbb{R}^n)$ $\{x \in \mathbb{R}^n : \mathrm{sign}(P(x)) = \sigma(P)$ for all $P$ in the domain of $\sigma\}$

$\mathrm{Reali}_t(\sigma)$ $\{x \in \mathrm{Reali}(\sigma, \mathbb{R}^n) : |x| \leq \frac{1}{t}, |P(x)| \geq t$ for all $P \in \sigma^{-1}(\{-1, 1\})\}$

$d_Z$ Maximal degree of polynomials in a semialgebraic description of $Z$ over $\mathbb{R}$

$p_Z$ Number of polynomials in a semialgebraic description of $Z$ over $\mathbb{R}$

# Introduction

Lattices are commonly seen discrete sets in $\mathbb{R}^n$: Let $(v_1, v_2, \ldots, v_n)$ be a basis of the vector space $\mathbb{R}^n$. The lattice $\Lambda \subset \mathbb{R}^n$ with the same basis consists of all points of the form $\sum_{i=1}^n a_i v_i$ with integer coefficients $a_i$. A lattice splits $\mathbb{R}^n$ into copies of the 'fundamental parallelepiped' with sides $v_1, v_2, \ldots, v_n$, and hence has a covolume $d(\Lambda) = |\det(v_i)_{1\leq i\leq n}|$. So when we have a bounded, measurable set $X \subseteq \mathbb{R}^n$, it makes sense to divide the volume $\mathrm{Vol}_n(X)$ by $d(\Lambda)$ to get an estimate for the number of lattice points contained in $X$.

If $X$ has a reasonably regular shape, this estimate usually is not too far off. But what is the error? For the case $\Lambda = \mathbb{Z}^n$ we have a classical result by Davenport [5]. If $X$ is closed, and we have some positive integer $h$ such that each line parallel to a coordinate axis intersects $X$, as well as the projections of $X$ to all coordinate subspaces, in at most $h$ disjoint intervals, then we have

$$|\operatorname{Vol}_n(X) - |X \cap \mathbb{Z}^n|| \leq \sum_{j=0}^{n-1} h^j V_j(X)$$

where $V_j(X)$ is the sum of the volumes of the projections of $X$ on all $j$-dimensional coordinate subspaces of $\mathbb{R}^n$, and $V_0(X) = 1$. While this is a nice result, it is relatively specific, and may be hard to use: Finding such an $h$ can be difficult. Furthermore, we want to consider $X$ that are not closed as well as other lattices $\Lambda$ than $\mathbb{Z}^n$.

There exist a variety of bounds like this which also hold up for other lattices, but a lot of these are restricted in their use: Either the conditions on the sets are specific or hard to verify, similar to Davenport's Lemma, or the bound is only nontrivial if the volume is large compared to the diameter. In 2013, Fabrizio Barroero and Martin Widmer released a paper ([1]) proposing a generalization fixing these problems. Their generalization applies not to single sets $X \subseteq \mathbb{R}^n$, but to sets $Z \subseteq \mathbb{R}^{n+m}$, which we consider as parametrized families of bounded sets $Z_T = \{x \in \mathbb{R}^n : (T, x) \in Z\}$. It does not apply to all such families, but only those that are contained in some o-minimal structure over $\mathbb{R}$.

O-minimal structures are collections of sets that contain certain basic sets and are closed under logical definitions: The essential example of such a structure is that of all semialgebraic sets, which are sets defined by polynomial equations and inequalities. In fact, as mentioned in the preface of [6], the study of o-minimal structures started out with the observation that a lot of properties of semialgebraic sets follow from a few basic axioms. Although the subject originated in model theory, these structures have since been used in a variety of fields, including real algebraic and real analytic geometry, as well as number theory.

When expanding our view to more general lattices $\Lambda$, we also need to use the successive minima of $\Lambda$ to formulate the dependence of the error bound on the lattice: The successive minima of $\Lambda$ are the values $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{>0}$ such that each $\lambda_i$ is the minimal value $\lambda$ for which there exists $i$ linearly independent elements of $\Lambda$ of (Euclidean) norm at most $\lambda$.

The Barroero-Widmer theorem (Theorem BW-1.3) states that if $Z \subseteq \mathbb{R}^{m+n}$ is contained in an o-minimal structure and the $Z_T$ are bounded, then there exists a constant $c_Z \in \mathbb{R}$ depending only on $Z$ such that for each lattice $\Lambda$ in $\mathbb{R}^n$ and each $T \in \mathbb{R}^m$ we have

$$\left| \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} - |Z_T \cap \Lambda| \right| \leq c_Z \sum_{j=0}^{n} \frac{V_j(Z_T)}{\prod_{i=1}^{j} \lambda_i}$$

where the $\lambda_i$ are the successive minima of $\Lambda$. This bound is optimal to some extent: Barroero and Widmer give an example right after the theorem [1][p. 5, Theorem 1.3] where there is a lower bound of the same form. However, their proof does not give any bounds on the size of the constant $c_Z$. In this thesis, we will prove the following result:

**Theorem 0.1.** *If $Z$ is semialgebraic, then*

$$c_Z \leq p_Z^{O(n^4)} \max(d_Z, 2)^{O(n(n^3+mn)^3)}$$

*where $p_Z$ is the number of polynomials involved in defining $Z$, and $d_Z$ is the maximum of their degrees.*

To prove this, we will need to dive into a variety of topics: The o-minimal structures themselves, as well as some basic geometry of numbers, measure theory, and linear algebra to prove the Barroero-Widmer theorem. These topics, along with a proof of Davenport's lemma (Lemma 1.3), are in chapter 1. Chapter 2 contains a proof of the Barroero-Widmer theorem, which is adapted in such a way that we get an initial upper bound for the constant $c_Z$. However, this bound still involves two unknown constants that are based on the number of connected components of certain sets occurring in the proof.

To bound these constants, we restrict ourselves to the case where $Z$ is semialgebraic. In chapter 3 we will investigate a variety of ways to decompose semialgebraic sets into simpler sets, as well as a way of describing these sets, mostly following [2]. This leads to two key results: One helps to generate these descriptions for more complicated sets, while the other bounds the number of connected components of a set based on such a description. In the final chapter we then use these results to describe the various sets occurring in the proof of the Barroero-Widmer theorem and bound the two unknown constants from our earlier upper bound, and thus obtain the final bound in Theorem 0.1.

# 1 Theoretical background

## 1.1 Davenport's Lemma

The main result we will be covering concerns a generalization of Davenport's Lemma [5], so we will first take a look at the original lemma. Before we get to the lemma itself, we need some notation.

**Definition 1.1.** Let $n \in \mathbb{N}$, and let $I$ be a nonempty subset of $\{1, 2, \ldots, n\}$. We define $A_I$ as the subspace of $\mathbb{R}^n$ spanned by the standard basis vectors $(e_i)_{i \in I}$, and $\pi_{A_I} : \mathbb{R}^n \to \mathbb{R}^n$ as the orthogonal projection onto this subspace.

**Definition 1.2.** Let $n \in \mathbb{N}$, and let $X \subseteq \mathbb{R}^n$ be compact. Then we define $V_j(X)$ for each $1 \leq j \leq n$ as the sum of the volumes of the $\pi_{A_I}(X)$ for all $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = j$. We also define $V_0(X) = 1$ by convention.

With this basis we can get to the actual lemma.

**Lemma 1.3.** *Let $X \subseteq \mathbb{R}^n$ be compact, and let $h \in \mathbb{N}$ be given such that for each nonempty subset $I \subseteq \{1, 2, \ldots, n\}$, each line parallel to a coordinate axis intersects $\pi_{A_I}$ in a set of points that consists of at most $h$ intervals.*
*Then*

$$||X \cap \mathbb{Z}^n| - \mathrm{Vol}_n(X)| \leq \sum_{j=0}^{n-1} h^{n-j} V_j(X).$$

*Proof.* We follow Davenport's proof as given in [5]. Suppose $X$ and $h$ satisfy the conditions of the lemma. As $X$ is compact, each of the $\pi_{A_I}(X)$ with $I \subseteq \{1, 2, \ldots, n\}$ nonempty has a measurable characteristic function. We will write $f(x_{i_1}, x_{i_2}, \ldots, x_{i_j})$ for the characteristic function of $\pi_{A_I}(X)$ when $I = \{i_1, i_2, \ldots, i_j\}$ and $i_1 < i_2 < \ldots < i_j$. Furthermore, we know that the measure of each $\pi_{A_I}(X)$ is given by integrating this characteristic function over all the $x_{i_j}$; as $X$ is compact, we can also take finite integration bounds here.

Similarly $|X \cap \mathbb{Z}^n|$ is given by the repeated summation $\sum_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n)$ where the $x_i$ vary over the integers. As $A$ is compact there are only finitely many non-zero terms. In particular, given that these characteristic functions are bounded, we can interchange these integrals and summations as needed. With this notation, we can rewrite the conclusion of the lemma as

$$\left| \int dx_1 \int dx_2 \ldots \int f(x_1, \ldots, x_n) dx_n - \sum_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n) \right|$$

$$\leq \sum_{j=0}^{n-1} h^{n-j} \sum_{i_1 < i_2 < \ldots < i_j} \int dx_{i_1} \int dx_{i_2} \ldots \int f(x_{i_1}, x_{i_2}, \ldots, x_{i_j}) dx_{i_j}$$

with the integral on the right being taken as 1 when $j = 0$. We proceed the proof by induction on $n$. For $n = 1$ the only term on the right is $h$, while the difference $|\int f(x_1)dx_1 - \sum_{x_1} f(x_1)|$ will be at most the number of intervals $X$ consists of, which by assumption is at most $h$. So for $n = 1$ we are done.

Now suppose the lemma holds for $n = m - 1$. For any $c \in \mathbb{R}$ we have that the intersection $X \cap \{x_1 = c\}$, seen as a subset of $\mathbb{R}^{m-1}$, satisfies the conditions of the lemma, so by the induction hypothesis we get

$$\left| \int dx_2 \int dx_3 \dots \int dx_m f(x_1, x_2, \dots, x_m) - \sum_{x_2, x_3, \dots, x_m} f(x_1, x_2, \dots, x_m) \right|$$

$$\leq \sum_{r=0}^{m-2} h^{m-1-r} \sum_{2 \leq i_1 < i_2 < \dots < i_r} \int dx_{i_1} \int dx_{i_2} \dots \int f(x_1, x_{i_1}, x_{i_2}, \dots, x_{i_r}) dx_{i_r}$$

for every real value $x_1$. Integrating this with respect to $x_1$ and passing the integral through the absolute value and the summations, we find the inequality

$$\left| \int dx_1 \int dx_2 \dots \int dx_m f(x_1, x_2, \dots, x_m) - \sum_{x_2, x_3, \dots, x_m} \int f(x_1, x_2, \dots, x_m) dx_1 \right|$$

$$\leq \sum_{r=0}^{m-2} h^{m-1-r} \sum_{2 \leq i_1 < i_2 < \dots < i_r} \int dx_1 \int dx_{i_1} \dots \int f(x_1, x_{i_1}, x_{i_2}, \dots, x_{i_r}) dx_{i_r}.$$

Writing $j = r + 1$ and defining $(k_1, k_2, \dots, k_j) = (1, i_1, \dots, i_r)$ the right-hand side of this becomes

$$\sum_{j=1}^{m-1} h^{m-j} \sum_{1 = k_1 < k_2 < \dots < k_j} \int dx_{k_1} \int dx_{k_2} \dots \int f(x_{k_1}, x_{k_2}, \dots, x_{k_j}) dx_{k_j}.$$

If instead of fixing the first variable we fix the other $m-1$ variables, the $n = 1$ case provides us with the inequality

$$\left| \int f(x_1, x_2, \dots, x_m) dx_1 - \sum_{x_1} f(x_1, x_2, \dots, x_m) \right| \leq h f(x_2, x_3, \dots, x_m).$$

Summing this over $x_2, x_3, \dots, x_m$, we find that

$$\left| \sum_{x_2, x_3, \dots, x_m} \int f(x_1, x_2, \dots, x_m) dx_1 - \sum_{x_1, x_2, \dots, x_m} f(x_1, x_2, \dots, x_m) \right| \leq h \sum_{x_2, x_3, \dots, x_m} f(x_2, x_3, \dots, x_m)$$

which by the result for $n = m - 1$ is itself at most

$$h \int dx_2 \int dx_3 \ldots \int f(x_2, x_3, \ldots, x_m) dx_m$$

$$+h \sum_{j=0}^{m-2} h^{m-1-j} \sum_{2 \leq i_1 < i_2 < \ldots < i_j} \int dx_{i_1} \int dx_{i_2} \ldots \int f(x_2, x_3, \ldots, x_m) dx_{i_j}.$$

These two terms combine to get

$$\sum_{j=0}^{m-1} h^{m-j} \sum_{2 \leq i_1 < i_2 < \ldots < i_j} \int dx_{i_1} \int dx_{i_2} \ldots \int f(x_2, x_3, \ldots, x_m) dx_{i_j}$$

as the first term is the summation term for $j = m - 1$. Combining the two inequalities gives

$$\left| \int dx_1 \int dx_2 \ldots \int dx_m f(x_1, x_2, \ldots, x_m) - \sum_{x_1, x_2, x_3, \ldots, x_m} f(x_1, x_2, \ldots, x_m) \right|$$

$$\leq \sum_{j=1}^{m-1} h^{m-j} \sum_{1 = i_1 < i_2 < \ldots < i_j} \int dx_{i_1} \int dx_{i_2} \ldots \int f(x_{i_1}, x_{i_2}, \ldots, x_{i_j}) dx_{i_j}$$

$$+ \sum_{j=0}^{m-1} h^{m-j} \sum_{2 \leq i_1 < i_2 < \ldots < i_j} \int dx_{i_1} \int dx_{i_2} \ldots \int f(x_2, x_3, \ldots, x_m) dx_{i_j}.$$

The left-hand side is what we were looking for. The right-hand side is as well, as the first summation consists of all the terms where 1 is part of the sequence $(i_1, i_2, \ldots, i_j)$, while the second consists of the term with $j = 0$ and the terms with $i_1 \geq 2$. Hence the lemma holds for $n = m$ as well, completing the proof. □

## 1.2 Semialgebraic sets

The basic objects of study in real algebraic geometry are algebraic sets, which are the zero sets of ideals of polynomials over $\mathbb{R}$. However, we will focus on a larger collection of sets: The semialgebraic sets.

**Definition 1.4.** A set $S \subseteq \mathbb{R}^n$ is semialgebraic if it can be written as a finite union $\bigcup_{i=1}^m S_i$, where the $S_i$ are all of the form

$$\{x \in \mathbb{R}^n : f_1(x) = f_2(x) = \ldots = f_k(x) = 0, g_1(x) > 0, g_2(x) > 0, \ldots, g_\ell(x) > 0\}$$

for some polynomials $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_\ell \in \mathbb{R}[x_1, x_2, \ldots, x_n]$.

It is not hard to see that these sets form a Boolean algebra.
We can make the relation with algebraic sets more explicit: We can replace every inequality $g_i(x) > 0$ with the statement $\exists c \in \mathbb{R} : c^2 g_i(x) - 1 = 0$, as this equation is equivalent with $g_i(x) = \frac{1}{c^2} > 0$. If we do this for all inequalities, we find that semialgebraic sets are finite unions of sets of the form

$$\{x \in \mathbb{R}^n : \exists c \in \mathbb{R}^m : f_1(x,c) = f_2(x,c) = \ldots = f_k(x,c) = 0\}$$

for some $m \geq 0$ and polynomials $f_1, f_2, \ldots, f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_m]$. This means that semialgebraic sets can be obtained from algebraic sets by projecting certain coordinates away. This begs the question: Are all coordinate projections of algebraic sets semialgebraic? As semialgebraic sets are coordinate projections of algebraic sets and algebraic sets are semialgebraic, this is equivalent to the question "Are coordinate projections of semialgebraic sets semialgebraic?". The answer is yes, as shown in [9]:

**Theorem 1.5** (Tarski-Seidenberg)**.** *If $\pi_k : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is the projection removing the last $k$ coordinates, and $S$ is a semialgebraic set, then $\pi_k(S)$ is also semialgebraic.*

As the coordinate order is not relevant here, it follows that all projections of semialgebraic sets are semialgebraic: We will use this result in the next subsection.

## 1.3 O-minimal structures

This subsection is based loosely on [1, Section 3] and [6]. A structure on $\mathbb{R}$ is a sequence $(D_n)_{n\in\mathbb{N}}$ where each $D_n$ is a family of subsets of $\mathbb{R}^n$ which essentially is 'closed under defining sets with logical formulas'. That is, suppose we have a first-order formula $\Phi$ in a language that only has constants for the elements of the various $D_i$, and as predicates only has the $\in$ symbol and equality. Then if $(x_1, x_2, \ldots, x_n)$ are the free variables of $\Phi$, the set $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \Phi(x_1, \ldots, x_n)\}$ is in $D_n$.

**Definition 1.6.** A structure on $\mathbb{R}$ is a sequence $D = (D_n)_{n\in\mathbb{N}}$, with $D_n \subseteq 2^{\mathbb{R}^n}$, satisfying:
(1): $D_n$ is a Boolean algebra for all $n$. In other words, each $D_n$ is closed under complements, binary unions and binary intersections.
(2): If $A \in D_n$, then $A \times \mathbb{R} \in D_{n+1}$ and $\mathbb{R} \times A \in D_{n+1}$
(3): $\{(x_1, x_2, \ldots, x_n) : x_i = x_j\} \in D_n$ for all $1 \leq i < j \leq n$.
(4): If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection deleting the last coordinate, then for all $A \in D_{n+1}$ we have $\pi(A) \in D_n$.

If $D$ is a structure, we call any set $X$ that belongs to $D_n$ for some $n \geq 1$ definable in $D$, referring to the interpretation mentioned before. While structures are a useful concept, they are too basic for our purposes: We want to keep the ordered field structure of $\mathbb{R}$. On the other hand we do not want every set to be definable, as we would prefer pathological cases like the Cantor set to be excluded. This leads to the definition of o-minimal structures expanding $(\mathbb{R}, <, +, \cdot)$.

**Definition 1.7.** An o-minimal structure expanding the ordered field $(\mathbb{R}, <, +, \cdot)$ is a structure $D$ on $\mathbb{R}$ satisfying the following further axioms:
(5): $\{(x_1, x_2) : x_1 < x_2\} \in D_2$
(6): $\{(x_1, x_2, x_3) : x_3 = x_1 + x_2\} \in D_3$
(7): $\{(x_1, x_2, x_3) : x_3 = x_1 \cdot x_2\} \in D_3$
(8): $D_1$ consists of all finite unions of points and open intervals.

We will soon see that any o-minimal structure expanding $(\mathbb{R}, <, +, \cdot)$ already contains all semialgebraic sets. First we need to get some basic administration out of the way.

**Remark 1.8.** *Let $D = (D_n)_{n\in\mathbb{N}}$ be an arbitrary o-minimal structure expanding the ordered field $(\mathbb{R}, <, +, \cdot)$. From now on, we call a set $X \subseteq \mathbb{R}^n$ definable if it is definable in $D$.*

We derive the following basic facts quite easily:

**Lemma 1.9.** *(i) If $A \in D_m$, $B \in D_n$, then $A \times B \in D_{m+n}$.*

*(ii) If $A \in D_n$, and $\sigma$ is a permutation of $n$ coordinates, then $\sigma(A) \in D_n$.*

*(iii) If $A \in D_n$, and $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is any projection omitting $n - m$ coordinates, then $\pi(A) \in D_m$.*

*(iv) If $A \in D_{m+n}$ and $T \in \mathbb{R}^m$, then the fiber $A_T = \{x \in \mathbb{R}^n, (T, x) \in A\}$ is definable.*

*(v) All semialgebraic sets are definable.*

*Proof.* (i) follows from axioms (1) and (2) as $A \times B = (A \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B)$.
For (ii) we note that $\sigma(A)$ is the projection on the first $n$ coordinates of the intersection of $\mathbb{R}^n \times A$ with $\{(x, y) \in \mathbb{R}^{2n} : x_i = y_{\sigma i} \text{ for } i = 1, \ldots, n\}$, so it is definable by (3) and (4). For (iii), note that we can simply use (ii) to make the first $m$ coordinates those that we want to preserve, after which (4) allows projecting the extra coordinates away. For (iv) we note $A_T = A \cap (\{T\} \times \mathbb{R}^n)$, and use (1) and (2).
Finally, for (v) we first observe that all finite unions and intersections of definable sets are definable since binary unions and intersections are. From this it follows that if for any polynomial $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ the sets $\{x \in \mathbb{R}^n : f(x) = 0\}$ and $\{x \in \mathbb{R}^n : f(x) > 0\}$ are definable, all semialgebraic sets are. As the graphs of addition and multiplication are definable in $\mathbb{R}^3$, by using (ii) and (2) it follows that (the graphs of) any binary sums and binary products of variables are definable in every $\mathbb{R}^n$. We can extend this to all finite sums and finite products of variables: This just requires introducing some helper variables and then projecting them away. For example, the set
$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_k = x_{k+1}\}$ is definable as it is the projection of

$$\{(x_1, x_2, \ldots, x_{n+k-2}) \in \mathbb{R}^{n+k-2} : x_{n+1} = x_1 + x_2, x_{n+2} = x_{n+1} + x_3, \ldots,$$

$$x_{n+k-2} = x_{n+k-3} + x_{k-1}, x_{k+1} = x_{n+k-2} + x_k\}$$

on the first $n$ coordinates, and we can do something similar for products. (8) gives that constants are definable, so multiplication by constants is also possible and hence the graph $\Gamma(f) = \{(x, y) \in \mathbb{R}^{n+1} : y = f(x)\}$ of any polynomial $f$ is definable. The sets we were looking for are then simply the projections of $\{(x, y, z) \in \mathbb{R}^{n+2} : y = f(x), z = 0, y * z\}$, where $*$ is either $=$ or $>$, on the first $n$ coordinates. □

**Corollary 1.10.** *The semialgebraic sets over $\mathbb{R}$ form the smallest o-minimal structure expanding $(\mathbb{R}, <, +, \cdot)$.*

*Proof.* By the Tarski-Seidenberg theorem projections of semialgebraic sets are semialgebraic, and the other axioms of o-minimal structures expanding $(\mathbb{R}, <, +, \cdot)$ are easily checked. So the semialgebraic sets form an o-minimal structure expanding $(\mathbb{R}, <, +, \cdot)$. By (v) of Lemma 1.9 we see that they are contained in every such structure, so the semialgebraic sets form the smallest one. □

We can also define endomorphisms of vector spaces, as the set $\{(\Psi, x, y) : \Psi \in \text{End}(\mathbb{R}^n), x, y \in \mathbb{R}^n, y = \Psi(x)\}$ is definable: Let $M$ be the matrix such that $\Psi(x) = M_\Psi x$ for all $x \in \mathbb{R}^n$. Then write $M_\Psi$ as an element of $\mathbb{R}^{n^2}$ by writing it as $(M_{\Psi,11}, M_{\Psi,12}, \ldots, M_{\Psi,nn})$, and $y = \Psi(x)$ is equivalent to the statement that for all $i = 1, 2, \ldots, n$ we have $y_i = \sum_{j=1}^{n} M_{\Psi,ij} x_j$. Now the set we wanted to define can simply been written as $\{(M, x, y) \in \mathbb{R}^{n^2+2n} : y_i = \sum_{j=1}^{n} M_{ij} x_j \text{ for } i = 1, \ldots, n\}$, which is definable as we have just described it in polynomial equations.

In the proof of Lemma 1.9 we mentioned that the graphs of all polynomials are definable. Functions with definable graphs are generally useful (for example, we can use them in logical formulas to define definable sets), so we get the following definition.

**Definition 1.11.** Let $X$ be a definable set. A function $f : X \to \mathbb{R}^m$ is called definable if its graph $\Gamma(f) = \{(x, y) : x \in X, y \in \mathbb{R}^m : y = f(x)\}$ is a definable set.

For any definable $X$ define $C(X)$ as the set of definable continuous functions from $X$ to $\mathbb{R}$ (under the Euclidean topology). Next define $C_\infty(X)$ as $C(X) \cup \{\infty, -\infty\}$, where the latter two are seen as constant functions to the extended reals $\mathbb{R} \cup \{\infty, -\infty\}$. For functions $f, g \in C_\infty(X)$ where $f(x) < g(x)$ for all $x \in X$ (also written $f < g$) we define the function interval $(f, g)_X$ as the set of points between the function graphs:

$$(f, g)_X = \{(x, r) \in X \times \mathbb{R} : f(x) < r < g(x)\}.$$

This function interval is a definable set. By looking at functions in $C_\infty(X)$ rather than $C(X)$ we include unbounded intervals, allowing us to decompose the cylinder $X \times \mathbb{R}$ above $X$: Let $f_1, f_2, \ldots, f_n \in C(X)$ satisfy $f_1 < f_2 < \ldots < f_n$. Then we can partition $X \times \mathbb{R}$ into the graphs $\Gamma(f_1), \Gamma(f_2), \ldots, \Gamma(f_n)$ along with the function intervals $(-\infty, f_1)_X, (f_1, f_2)_X, \ldots, (f_n, \infty)_X$. This decomposition is what leads to the definition of cells.

**Definition 1.12.** For any sequence $(i_1, i_2, \ldots, i_n)$ of zeroes and ones, a $(i_1, i_2, \ldots, i_n)$-cell is a subset of $\mathbb{R}^n$ obtained through the following recursion:

(i) A $(0)$-cell is a singleton $\{r\} \subset \mathbb{R}$, and a $(1)$-cell is a nonempty open interval $(a, b) \subseteq \mathbb{R}$.

(ii) An $(i_1, i_2, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where $X$ is an $(i_1, i_2, \ldots, i_n)$-cell, and an $(i_1, i_2, \ldots, i_n, 1)$-cell is a function interval $(f, g)_X$ with $f, g \in C_\infty(X)$, $f < g$, and $X$ an $(i_1, i_2, \ldots, i_n)$-cell.

In general, we call $X \subseteq \mathbb{R}^n$ a cell if it is an $(i_1, i_2, \ldots, i_n)$-cell for some sequence $(i_1, i_2, \ldots, i_n)$.

Cells are a particularly nice kind of definable sets. For example, we have the following [6][p. 51,Prop. 2.9 + p.59,ex. 7]

**Lemma 1.13.** *Cells are connected sets under the Euclidean topology.*

If we apply the observation we just made about decompositions to the situation where $X$ is an $(i_1, i_2, \ldots, i_n)$-cell, we find that $\{\Gamma(f_1), \Gamma(f_2), \ldots, \Gamma(f_n), (-\infty, f_1)_X, (f_1, f_2)_X, \ldots, (f_n, \infty)_X\}$ is a partition of the cylinder $X \times \mathbb{R}$ into cells. This leads to the following definition.

**Definition 1.14.** A cylindrical decomposition (also known as a cell decomposition, or just a decomposition) of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many cells, such that for any projection $\pi_k$ obtained by removing the last $k$ coordinates for some $1 \leq k \leq n-1$, the projections $\pi_k(C)$ of the cells in the decomposition form a decomposition of $\mathbb{R}^k$.

This is the higher-dimensional analogue of the partition of $X \times \mathbb{R}$ we just had. For example, a decomposition of $\mathbb{R}$ is simply a partition of $\mathbb{R}$ into finitely many points (which are graphs of functions from the single point space $\mathbb{R}^0$ to $\mathbb{R}$) and the intervals that remain after removing these points (which correspond to the function intervals in that partition). Moving up further dimensions we find stacks of cells decomposing the cylinder $C \times \mathbb{R}$ for each cell $C$ of a decomposition one level lower.
One of the main features of cell decompositions is the cell decomposition theorem [6, p. 52]. The main portion of it we will be using is the following result:

**Proposition 1.15.** *For every definable set $X \subseteq \mathbb{R}^n$ there exists a decomposition of $\mathbb{R}^n$ in which $X$ is a finite union of cells.*

The full result allows for simultaneous decomposition of any finite number of definable sets, as well as partitioning the domain of a definable function such that it becomes continuous on each cell. We now define the dimension of a definable set.

**Definition 1.16.** We define the dimension $\dim(X)$ of a definable set $X \subseteq \mathbb{R}^n$ as the maximal $d$ such that there exist $i_1, i_2, \ldots, i_n \in \{0, 1\}$ with $\sum_{j=1}^n i_j = d$ such that $X$ contains an $(i_1, i_2, \ldots, i_n)$-cell. We also define $\dim(\emptyset) = -\infty$.

We have a couple of relevant results on dimension here.

**Proposition 1.17.** *Let $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ be definable. Then the following results hold:*

*(i) If there exists a definable bijection $X \to Y$, then $\dim X = \dim Y$.*

*(ii) If $\dim(X) = 0$, then $X$ is finite.*

*(iii) $\dim(\mathrm{bd}(X)) < m$, where $\mathrm{bd}(X)$ is the boundary of $X$ in the Euclidean topology.*

*Proof.* For part (i), refer to [6][p. 64, Proposition 1.3(ii)]. To get part (ii), take a cell decomposition of $X$. As $\dim(X) = 0$ that decomposition can only contain $(0, 0, \ldots, 0)$-cells, which are single points. As there are only finitely many cells in a decomposition, this implies $X$ is finite. For part (iii), refer to [6][p. 68, Corollary 1.10]. □

By definition, cell decompositions are preserved under projections that remove coordinates from the end. However, moving to fibers also preserves decompositions: [6, p. 60, Proposition 3.5]

**Proposition 1.18.** *Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the projection to the first $m$ coordinates, and let $C$ be a cell in $\mathbb{R}^{m+n}$. Then for every $T \in \mathbb{R}^m$ the fiber $C_T$ is a cell in $\mathbb{R}^n$. Furthermore, given a decomposition of $\mathbb{R}^{m+n}$, the fibers $C_T$ of all cells in this decomposition form a decomposition of $\mathbb{R}^n$.*

We will mainly need this to get a uniform bound:

**Corollary 1.19.** *For each definable family $Z \subseteq \mathbb{R}^{m+n}$ there is an integer $M_Z$ such that for each $T \in \mathbb{R}^m$ the fiber $Z_T$ can be partitioned into at most $M_Z$ cells. In particular each fiber $Z_T$ has at most $M_Z$ connected components.*

*Proof.* By Proposition 1.15 we can decompose $\mathbb{R}^{m+n}$ so that $Z$ is partitioned into cells, and by Proposition 1.18 there exist decompositions of $\mathbb{R}^n$ with the same number of cells partitioning the $Z_T$. Hence we can take $M_Z$ to be the number of cells in an arbitrary decomposition of $\mathbb{R}^{m+n}$ partitioning $Z$. The last statement then follows as cells are connected. □

There is one further fact about definable sets we will need: The existence of definable choice functions.

**Proposition 1.20.** *If $Z \subseteq \mathbb{R}^{m+n}$ is definable, and $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection on the first $m$ coordinates, there is a definable function $f : \pi(Z) \to \mathbb{R}^n$ such that $\Gamma(f) \subseteq Z$.*

That is, we can simultaneously choose an element from each fiber $Z_T$ in a definable way.

*Proof.* We construct such an $f$ by induction on $n$. Let $n = 1$ and let $T \subseteq \mathbb{R}^m$ be arbitrary. If $Z_T$ has a minimum, we take

$$f(T) = \min(Z_T). \tag{1}$$

Otherwise $Z_T$ is infinite. As $Z_T$ is a definable subset of $\mathbb{R}$ and hence is a finite union of open intervals and points, $Z_T$ being infinite implies it contains an open interval. Let $(u_T, v_T)$ be the interval closest to $-\infty$, that is, let $u_T = \inf(Z_T)$ and $v_T = \sup(x \in \mathbb{R} : (u_T, x) \subseteq Z_T)$. We have four cases depending on whether $u$ or $v$ are actually in $\mathbb{R}$:

$$f(T) = \begin{cases} 0 & u_T = -\infty, v_T = \infty & (2) \\ v_T - 1 & u_T = -\infty, v_T \in \mathbb{R} & (3) \\ u_T + 1 & u_T \in \mathbb{R}, v_T = \infty & (4) \\ \dfrac{u_T + v_T}{2} & u_T, v_T \in \mathbb{R} & (5) \end{cases}$$

Clearly these cases cover all potential situations, and each gives a point $f(T) \in Z_T$, so we have defined a choice function $f$. To show $f$ is definable we note that the statement '$y = f(T)$' is equivalent to

$$\begin{aligned} &(y \in Z_T \wedge \forall z : (z < y \implies z \notin Z_T)) && \text{Case (1)} \\ &\vee \forall z : (z \in Z_T \wedge y = 0) && \text{Case (2)} \\ &\vee \forall z : ((z < y + 1 \implies z \in Z_T) \wedge (z > y + 1 \implies \exists x : x < z \wedge x \notin Z_T)) && \text{Case (3)} \\ &\vee \forall z : (z \in Z_T \iff z > y - 1) && \text{Case (4)} \\ &\vee (\exists u_T \forall z : ((z \leq u_T \wedge z \notin Z_T) \vee (u_T < z \wedge z < 2y - u_T \wedge z \in Z_T) \\ &\quad \vee (\exists x : 2y - u_T < x \wedge x < z \wedge x \notin Z_T)) && \text{Case (5)} \end{aligned}$$

The specific formulation was chosen to minimize the number of quantifiers by avoiding references to $u_T$ and $v_T$ where possible, as this helps improve the bound later. This equivalent formulation uses only polynomial equations and inequalities as well as statements of the form '$x \in Z_T$' in its atomic formulas. Therefore $\Gamma(f) = \{(T, y) \in \mathbb{R}^{m+n} : T \in \pi(Z), y = f(T)\}$ is definable.

Now suppose $n = k + 1$ for some $k \in \mathbb{Z}_{\geq 1}$. Let $\pi_1 : \mathbb{R}^{m+k+1} \to \mathbb{R}^{m+k}$ be the projection omitting the last coordinate, and let $\pi' : \mathbb{R}^{m+k} \to \mathbb{R}^m$ be the projection omitting the last $k$ coordinates. Note that $\pi = \pi' \circ \pi_1$, and that $\pi_1(Z)$ is definable as $Z$ is. By the induction hypothesis we have constructed choice functions $f_k$ assigning each $T \in \pi'(\pi_1(Z)) = \pi(Z)$ a point in $\pi'(\pi_1(Z))_T \subseteq \mathbb{R}^k$ and $f_1$ assigning each point $(T, x_1, \ldots, x_k) \in \pi_1(Z)$ a point in $Z_{(T,x_1,\ldots,x_k)}$. Then we define $f_{k+1} : \pi(Z) \to \mathbb{R}^{k+1}$ as $f_{k+1}(T) = (f_k(T), f_1(T, f_k(T)))$. As $T \in \pi(Z) = \pi'(\pi_1(Z))$ and $f_k(T) \in \pi'(\pi_1(Z))_T$, we see that $(T, f_k(T)) \in \pi_1(Z)$, so $f_{k+1}$ is well-defined and the point $(T, f_{k+1}(T))$ indeed lies in $Z$. Furthermore, $f_{k+1}$ is clearly definable as $f_k$ and $f_1$ are, completing the induction. □

In particular, the proof provides a recursive construction of a formula for such a choice function. The 1-dimensional case gives a formula of the form $\bigvee_{1 \leq i \leq 5} \Phi_{1,k}(T, y)$, with $\Phi_{1,k}(T, y)$ the part corresponding to case $(k)$. The general case then follows by picking each coordinate $x_i$ in turn from $\pi_{n-i}(Z)_{(T,x_1,\ldots,x_{i-1})}$, which is 1-dimensional, until we finally choose from $Z_{(T,x_1,\ldots,x_{n-1})}$. Writing $\Phi_{i,k}(T, y)$ for the 5 parts of the formula corresponding to the choice of the $i$-th coordinate, we find that $y = f(T)$ is equivalent to $\bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq k \leq 5} \Phi_{i,k}(T, y)$.

## 1.4 Geometry of numbers

This section is a fast and very basic introduction to the geometry of numbers. A more thorough explanation can be found in a variety of sources, e.g. [3]. Geometry of numbers is an area of mathematics primarily concerned with lattice points in $\mathbb{R}^n$, so we should first recall the definition of lattices.

**Definition 1.21.** A (full) lattice in $\mathbb{R}^n$ is an additive group $\Lambda$ of the form

$$\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \ldots + \mathbb{Z}v_n$$

with $\{v_1, v_2, \ldots, v_n\}$ forming a basis of $\mathbb{R}^n$. We call $\{v_1, v_2, \ldots, v_n\}$ a basis of $\Lambda$ and define the determinant $d(\Lambda) = |\det(v_1, v_2, \ldots, v_n)|$.

This determinant is equal to the volume of a fundamental parallelepiped
$P = \{x \in \mathbb{R}^n : \exists \alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1) : x = \sum_{i=1}^n \alpha_i v_i\}$, and is independent of the choice of basis. Furthermore, it is quite clear that a fundamental parallelepiped is a full system of representatives for the quotient $\mathbb{R}^n/\Lambda$, in other words, it contains one element of each equivalence class. For this reason we also call $d(\Lambda)$ the covolume $\mathrm{Vol}_n(\mathbb{R}^n/\Lambda)$ of $\Lambda$.

We often want to count the number of lattice points in some subset of $\mathbb{R}^n$. For this a particular kind of structured set is useful:

**Definition 1.22.** A central symmetric convex body in $\mathbb{R}^n$ is a closed bounded convex subset $K$ of $\mathbb{R}^n$ that is symmetric around 0 and has 0 as an interior point.

Let $K$ be a central symmetric convex body. As 0 is an interior point of $K$, $K$ contains some open ball around 0. Scaling $K$ with $\lambda \in \mathbb{R}_{>0}$, it follows that each point in $\mathbb{R}^n$ is contained in $\lambda K$ for certain $\lambda \in \mathbb{R}_{>0}$. It is not hard to prove that the set of $\lambda$ for which this holds is in fact an interval of the form $[\lambda_{x,K}, \infty)$ for some positive real $\lambda_{x,K}$. In particular this allows us to define the following:

**Definition 1.23.** Let $\Lambda$ be a lattice and let $K$ be a central symmetric convex body, both in $\mathbb{R}^n$. For each $1 \leq i \leq n$, we define the $i$-th successive minimum of $\Lambda$ with respect to $K$ as the minimal $\lambda_i \in \mathbb{R}_{>0}$ such that $\lambda_i K$ contains $i$ linearly independent points of $\Lambda$ If $K$ is not specified, we take $K$ to be the closed unit ball.

To see that the successive minima are well-defined, first note we can list the set of lattice points in increasing order of the $\lambda_{x,K}$ as $\Lambda$ is discrete and $K$ closed and bounded. We can then simply go through the list and create a set $n$ of linearly independent points by repeatedly picking the first lattice point on the list that is linearly independent of the previous ones; this produces a linearly independent sequence of lattice points $(x_1, x_2, \ldots, x_n)$ which have minimal $\lambda_{x_i,K}$ at each stage, meaning that $\lambda_i = \lambda_{x_i,K}$ for all $i$. The best-known application of successive minima is probably Minkowski's Second Convex Body Theorem.

**Theorem 1.24** (Minkowski's Second Convex Body Theorem)**.** *Let $\Lambda$ be a lattice in $\mathbb{R}^n$, let $K$ be a central symmetric convex body in $\mathbb{R}^n$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the successive minima of $\Lambda$ with respect to $K$. Then*

$$\frac{2^n}{n!}\operatorname{Vol}_n(\mathbb{R}^n/\Lambda) \leq \operatorname{Vol}(K) \cdot \prod_{i=1}^{n} \lambda_i \leq 2^n \operatorname{Vol}_n(\mathbb{R}^n/\Lambda).$$

This is proven in chapter 8 of [3], with the actual theorem being Theorem V on page 218. Though the sequence $(x_1, x_2, \ldots, x_n)$ of points we constructed while showing the successive minima are well-defined is linearly independent, it does not have to be a basis for $\Lambda$. In the case where $K$ is the closed unit disk, we can get somewhat close:

**Lemma 1.25.** *Let $\Lambda$ be a lattice in $\mathbb{R}^n$ with successive minima $\lambda_1, \lambda_2, \ldots, \lambda_n$ (with regards to the closed unit ball), and let $|.|$ be the Euclidean norm. Then there exists a basis $v_1, v_2, \ldots, v_n$ of $\Lambda$ such that $|v_i| \leq i\lambda_i$ for $i = 1, \ldots, n$.*

This is in fact a weaker form of [3, Lemma 8,p. 135], which gives us that there is a basis with $|v_1| = \lambda_1$ and $|v_i| \leq \frac{i}{2}\lambda_i$ for $i = 2, \ldots, n$.

## 1.5 Measure Theory

Besides the usual $j$-dimensional Lebesgue measure, which we denote as $\operatorname{Vol}_j$ (or as $\mathcal{L}^j$ in the case of a Lebesgue integral), the proof of Theorem BW-1.3 also uses the $j$-dimensional Hausdorff measure $\mathcal{H}^j$. We will give a brief introduction, omitting some proofs. For a more detailed introduction, see for instance [7][Chapter 2].

**Definition 1.26.** Let $A \subseteq \mathbb{R}^n$, and let $j \in \mathbb{R}_{\geq 0}$. For any $\delta \in \mathbb{R}_{\geq 0}$, define

$$\mathcal{H}^j_\delta(A) = \nu_j \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(C_i)^j : A \subseteq \bigcup_{i=1}^{\infty} C_i, \operatorname{diam}(C_i) \leq \delta\right\}$$

where the $C_i$ are allowed to vary along all subsets of $\mathbb{R}^n$ of diameter at most $\delta$, and $\nu_j$ is a normalization constant. We define the (normalized) $j$-dimensional Hausdorff measure $\mathcal{H}^j(A)$ as $\lim_{\delta\to 0} \mathcal{H}^j_\delta(A)$.

Let $K_j$ be the $j$-dimensional hypercube $\{\sum_{i=1}^j \mu_i e_i : 0 \leq \mu_1, \mu_2, \ldots, \mu_j \leq 1\}$ with the $e_i$ the standard basis vectors in $\mathbb{R}^n$. The normalization constant $\nu_j$ is chosen such that $\mathcal{H}^j(K_j) = 1 = \operatorname{Vol}_j(K_j)$: This implies $\nu_j$ is the $n$-dimensional Lebesgue measure of the $n$-dimensional ball with diameter 1, and ensures $\mathcal{H}^j$ and $\operatorname{Vol}_j$ agree on all measurable subsets of $\mathbb{R}^j$. In particular we have the following result:

**Proposition 1.27.** *Let $1 \leq j \leq n$, and let $A \subseteq \mathbb{R}^n$ be $j$-Hausdorff measurable. Suppose $\phi \in \operatorname{End}(\mathbb{R}^n)$ has operator norm $c$. Then $\mathcal{H}^j(\phi(A)) \leq c\mathcal{H}^j(A)$. In particular, if $\phi$ is an orthogonal projection we have $\mathcal{H}^j(\phi(A)) \leq \mathcal{H}^j(A)$, and if $\phi \in O_n(\mathbb{R})$ we have $\mathcal{H}^j(\phi(A)) = \mathcal{H}^j(A)$.*

*Proof.* The first part follows from [7][p. 97, Theorem 2.8] by noting that linear endomorphisms are Lipschitz continuous with Lipschitz constant equal to their operator norm. The first special case follows as orthogonal projections have operator norm 1. The other special case follows as for $\phi \in O_n(\mathbb{R})$ both $\phi$ and $\phi^{-1}$ have operator norm 1. □

The Hausdorff measure also induces a notion of dimension:

**Definition 1.28.** Let $A \subseteq \mathbb{R}^n$. We define the Hausdorff dimension of $A$ as

$$\dim_{\mathcal{H}}(A) = \inf\{j \in \mathbb{R}_{\geq 0} : \mathcal{H}^j(A) = 0\}.$$

The Hausdorff measure certainly cares about Hausdorff dimension: It turns out that $\mathcal{H}^j(A)$ can only be a nonzero real number if $j = \dim_{\mathcal{H}}(A)$.

**Proposition 1.29.** *Let $A \subseteq \mathbb{R}^n$, and let $\dim_{\mathcal{H}}(A) = d$. Then for all $j > d$ we have $\mathcal{H}^j(A) = 0$, and for all $j < d$ we have $\mathcal{H}^j(A) = \infty$.*

The Hausdorff dimension turns out to agree with our notion of dimension of definable sets, as we have [1][p. 14, Proposition 5.2]:

**Proposition 1.30.** *Let $A \subseteq \mathbb{R}^n$ be a nonempty definable set. Then $\dim(A) = \dim_{\mathcal{H}}(A)$. If $A$ is also bounded and $d = \dim(A)$, then $A$ is $j$-Hausdorff measurable for all $d \leq j \leq n$, and $\mathcal{H}^d(A) < \infty$.*

This immediately leads to the following lemma:

**Lemma 1.31.** *If $A \subseteq \mathbb{R}^n$ is definable, then $\mathrm{Vol}_n(\mathrm{bd}(A)) = 0$.*

*Proof.* If $A$ is empty, this is trivial. Otherwise we know from Proposition 1.17 that $\dim(\mathrm{bd}(A)) < n$, so by Proposition 1.30 $\dim_{\mathcal{H}}(\mathrm{bd}(A)) < n$. By Proposition 1.29 this means that $\mathrm{Vol}_n(\mathrm{bd}(A)) = \mathcal{H}^n(\mathrm{bd}(A)) = 0$. □

# 2 The Barroero-Widmer Theorem

Recall that $\pi_{A_I}$ is the orthogonal projection on the coordinate space $A_I$ generated by the standard basis vectors $(e_i)_{i\in I}$, and $V_j(Z_T)$ is the sum of the volumes of the $\pi_{A_I}(Z_T)$ with $|I| = j$. The main result in this thesis is about the following theorem by Barroero and Widmer [1, Theorem 1.3]:

**Theorem BW-1.3.** *Fix an o-minimal structure $D$. Let $m$ and $n$ be positive integers, let $\Lambda \subset \mathbb{R}^n$ be a lattice with successive minima $\lambda_1, \ldots, \lambda_n$, and let $Z \subseteq \mathbb{R}^{m+n}$ be a definable family of which the fibers $Z_T$ are bounded for all $T \in \mathbb{R}^m$. Then there exists a constant $c_Z \in \mathbb{R}$ depending only on $Z$ such that*

$$\left| |Z_T \cap \Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} \right| \leq c_Z \sum_{j=0}^{n-1} \frac{V_j(Z_T)}{\prod_{i=1}^{j} \lambda_i}.$$

Our goal in this thesis is bounding the constant $c_Z$ in this theorem in the case where $D$ is the family of semialgebraic sets. We will first look at the proof to see where this constant arises. We will see that in several lemmas we need the assumption that the fibers $Z_T$ are all compact. As the theorem itself assumes they are bounded, we only need the additional assumption that they are closed. So for now assume the $Z_T$ are closed; we will deal with the general case at the end.

We start with [1][Lemma 4.1]. This lemma involves a definition based on Davenport's lemma (Lemma 1.3), which BW-1.3 generalizes.

**Definition 2.1.** Let $X \subseteq \mathbb{R}^n$ and let $h$ be a positive integer. Then $h$ is a Davenport constant for $X$ if for every nonempty $I \subseteq \{1, 2, \ldots, n\}$ the intersection of any line parallel to a coordinate axis and $\pi_{A_I}(X)$ can be expressed as a union of at most $h$ disjoint intervals.

That is, $h$ is a Davenport constant for $X$ if it satisfies the main condition of Davenport's lemma. We can now formulate the lemma.

**Lemma BW-4.1.** *Let $Z \subseteq \mathbb{R}^{m+n}$ be a definable family. There exists a natural number $M_Z$ depending only on $Z$ such that for every $T \in \mathbb{R}^m$ and endomorphism $\psi$ of $\mathbb{R}^n$ the number $M_Z$ is a Davenport constant for $\psi(Z_T)$.*

*Proof.* As a consequence of the fact noted below Corollary 1.10 the set $V = \{(\psi, T, x, y) \in \mathbb{R}^{n^2+2n+m} : x = \psi(y), y \in Z_T\}$ is definable, so its projection $W = \{(\psi, T, x) \in \mathbb{R}^{n^2+m+n} : x \in \psi(Z_T)\}$ is definable as well, as the formula $x \in \psi(Z_T)$ is equivalent to

$$\exists (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : (T, y_1, y_2, \ldots, y_n) \in Z, x_i = \sum_{j=1}^{n} \psi_{ij} y_j \text{ for } i = 1, 2, \ldots, n.$$

For $I$ a nonempty subset of $\{1, 2, \ldots, n\}$, let $\pi'_{A_I}$ be the endomorphism of $\mathbb{R}^{n^2+m+n}$ sending $(\psi, T, x)$ to $(\psi, T, \pi_{A_I}(x))$. A line in $A_I$ parallel to a coordinate axis is determined by the $|I|-1$ values of the constant coordinates, so we will write it as $(\ell_i)_{i\in I\setminus\{i_0\}}$ for some $i_0 \in I$.

For each choice of $I$ and $i_0 \in I$, now define

$$B^{I,i_0} = \{((\ell_i)_{i\in I\setminus\{i_0\}}, \psi, T, x) \in \mathbb{R}^{n^2+m+n+|I|-1} : (\psi, T, x) \in \pi'_{A_I}(W), \ell_i = x_i \text{ for } i \in I\setminus\{i_0\}\}.$$

These are clearly definable sets, and a fiber $B^{I,i_0}_{(\ell_i),\psi,T}$ is exactly the intersection of the line $(\ell_i)_{i\in I\setminus\{i_0\}}$ parallel to $e_{i_0}$ in $A_I$ with $\pi'_{A_I}(W)_{\psi,T} = \pi_{A_I}(W_{\psi,T}) = \pi_{A_I}(\psi(Z_T))$. In fact, all of these fibers can be viewed in a single context.

**Lemma 2.2.** *Each fiber $B^{I,i_0}_{(\ell_i),\psi,T}$ is homeomorphic to some fiber $W_{(\phi,T,x_1,\dots,x_{n-1})}$ of $W$.*

*Proof of Lemma 2.2.* Let $F = B^{I,i_0}_{(\ell_i),\psi,T}$ be given, and define $j = |I|$. Let $\sigma \in S_n$ such that $\sigma(I) = \{n-j+1, n-j+2, \dots, n\}$ and in particular $\sigma(i_0) = n$. Let $\phi_\sigma$ be the endomorphism of $\mathbb{R}^n$ that sends $e_i$ to $e_{\sigma(i)}$ for each standard basis vector $e_i$.

Let $y = (y_1, y_2, \dots, y_n) \in F$, and let $z = (z_1, z_2, \dots, z_n)$ such that $z = \phi_\sigma(y)$. As $y \in (\pi_{A_I} \circ \psi)(Z_T)$, it follows that $z \in (\phi_\sigma \circ \pi_{A_I} \circ \psi)(Z_T)$, and hence $z \in W_{(\phi_\sigma\circ\pi_{A_I}\circ\psi,T)}$. Furthermore, as $y$ lies in the image of $\pi_{A_I}$, we have $y_i = 0$ for all $i \notin I$. Hence $z_i = 0$ for all $i \le n-j$. As $y_i = \ell_i$ for all $i \in I\setminus\{i_0\}$, we also have $z_i = \ell_{\sigma^{-1}(i)}$ for all $n-j+1 \le i \le n-1$.

Clearly $\phi_\sigma$ is continuous, as it is just a coordinate permutation, and invertible as it has inverse $\phi_{\sigma^{-1}}$. So $\phi_\sigma$ induces a homeomorphism between $B^{I,i_0}_{(\ell_i),\psi,T}$ and its image $G$. By our earlier discussion, $G$ is a subset of

$$H = \left\{ z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z \in W_{(\phi_\sigma\circ\pi_{A_I}\circ\psi,T)}, z_i = 0 \text{ for all } i \le n-j \\ z_i = \ell_{\sigma^{-1}(i)} \text{ for all } n-j+1 \le i \le n-1 \end{array} \right\}.$$

We first prove $G = H$. As all $z_i$ with $1 \le i \le n-1$ are already predetermined, we just need to prove each value of $z_n \in \mathbb{R}$ occurs in $\phi_\sigma(F)$. But clearly for any choice of $z_n \in \mathbb{R}$ there is a point $y$ in $F$ with $y_{i_0} = z_n$, and that point has $\phi_\sigma(y)_n = z_n$.

So $F$ is homeomorphic to $H$. But $H$ is clearly itself homeomorphic to the fiber $W_{(\phi_\sigma\circ\pi_{A_I}\circ\psi,T,x_1,x_2,\dots,x_{n-1})}$ satisfying $x_i = 0$ for $i \le n-j$ and $x_i = \ell_{\sigma^{-1}(i)}$ as all coordinates in $H$ except the last one are already predetermined. Hence $F$ is homeomorphic to a fiber $W_{(\phi,T,x_1,\dots,x_{n-1})}$ of $W$. □

We continue with the proof of Lemma BW-4.1. Using Corollary 1.19 we can find a uniform bound $M_Z$ for the number of connected components of all fibers $W_{(\phi,T,x_1,\dots,x_{n-1})}$. As each $B^{I,i_0}_{(\ell_i),\psi,T}$ is homeomorphic to some fiber of that form, each $B^{I,i_0}_{(\ell_i),\psi,T}$ has at most $M_Z$ connected components. This implies that any line parallel to a coordinate axis in $A_I$ intersects $\pi_{A_I}(\psi(Z_T))$ in at most $M_Z$ disjoint intervals for any nonempty $I \subseteq \{1, 2, \dots, n\}$, so $M_Z$ is a Davenport constant for $\psi(Z_T)$ for all $T \in \mathbb{R}^m$. □

We now follow chapter 2 from [1]. Here we find three lemmas that in combination form a chain of inequalities, starting with the Davenport constant we just got from Lemma BW-4.1. Using Lemma 1.25 we get a basis $(v_1, v_2, \ldots, v_n)$ of $\Lambda$ such that $|v_i| \leq i\lambda_i$ for $i = 1, \ldots, n$. Let $\Psi$ be the endomorphism of $\mathbb{R}^n$ sending this basis to the standard basis. With this basis and endomorphism now given, we can formulate the lemmas.

**Lemma BW-2.1.** *Let $C \subseteq \mathbb{R}^n$ be a compact set, and let $h$ be a Davenport constant for $\Psi(C)$. Then*

$$\left| |C \cap \Lambda| - \frac{\mathrm{Vol}_n(C)}{d(\Lambda)} \right| \leq \sum_{j=0}^{n-1} h^{n-j} V_j(\Psi(C)).$$

*Proof.* Clearly $\Psi(C \cap \Lambda) = \Psi(C) \cap \mathbb{Z}^n$ and $\mathrm{Vol}_n(\Psi(C)) = |\det(\Psi)| \mathrm{Vol}_n(C)$. As $\Psi^{-1}$ is represented by the matrix with columns $v_1, v_2, \ldots, v_n$, clearly $|\det(\Psi)| = \frac{1}{|\det(\Psi^{-1})|} = \frac{1}{d(\Lambda)}$. As $C$ is compact, so is $\Psi(C)$, so we can apply Davenport's Lemma (Lemma 1.3) to $\Psi(C)$. This gives

$$\left| |C \cap \Lambda| - \frac{\mathrm{Vol}_n(C)}{d(\Lambda)} \right| = ||\Psi(C) \cap \mathbb{Z}^n| - \mathrm{Vol}_n(\Psi(C))| \leq \sum_{j=0}^{n-1} h^{n-j} V_j(\Psi(C))$$

as required. □

**Lemma BW-2.2.** *Suppose $C \subseteq \mathbb{R}^n$ is compact. Then for $j = 1, \ldots, n-1$ we have*

$$V_j(\Psi(C)) \leq \sum_{I \subseteq \{1,2,\ldots,n\}, |I|=j} \frac{2^j \cdot \mathrm{Vol}_j(C^I)}{B_j \cdot \prod_{i=1}^{j} \lambda_i}$$

*where $B_j$ is the volume of the $j$-dimensional unit ball and $C^I$ is the orthogonal projection of $C$ to the subspace of $\mathbb{R}^n$ spanned by $(v_i)_{i \in I}$.*

*Proof.* For each $I \subseteq \{1, 2, \ldots, n\}$, let $\Lambda_I$ be the sublattice of $\Lambda$ spanned by $(v_i)_{i \in I}$, and let $W_I$ be the subspace of $\mathbb{R}^n$ spanned by those same vectors. The orthogonal projection of $\Psi(C)$ to the subspace $A_I$ of $\mathbb{R}^n$ spanned by the standard basis vectors $(e_i)_{i \in I}$ is $\Psi(C^I)$. Using a similar argument as in the proof of Lemma BW-2.1 we find that $\mathrm{Vol}_j(\Psi(C^I)) = \frac{\mathrm{Vol}_j(C^I)}{d(\Lambda_I)}$. As clearly the successive minima of $\Lambda_I$ (with respect to an orthonormal basis of $W_I$) are at least those of $\Lambda$, Minkowski's Second Theorem (Theorem 1.24) gives us that $d(\Lambda_I) \geq \frac{B_j}{2^j} \prod_{i=1}^{j} \lambda_i$. So we have

$$V_j(\Psi(C)) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \mathrm{Vol}_j(\Psi(C^I)) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \frac{\mathrm{Vol}_j(C^I)}{d(\Lambda_I)} \leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \frac{2^j \cdot \mathrm{Vol}_j(C^I)}{B_j \cdot \prod_{i=1}^{j} \lambda_i}$$

finishing the proof. □

**Lemma BW-2.4.** *Suppose $C \subseteq \mathbb{R}^n$ is compact. Then for any $j \in \{1, 2, \ldots, n-1\}$ and $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = j$ we have*

$$\mathrm{Vol}_j(C^I) \le \left( j^{3/2} \frac{n! 2^n}{B_n} \right)^j V'_j(C)$$

*where $V'_j(C)$ is the supremum of the volumes of the orthogonal projections of $C$ on all $j$-dimensional subspaces of $\mathbb{R}^n$.*

Note that this supremum is finite because $C$ is compact.

*Proof.* Let $j, I$ be as in the lemma, and write $\overline{I}$ for the complement of $I$ in $\{1, 2, \ldots, n\}$. For $i = 1, 2, \ldots, n$, let $v'_i$ be the $i$-th row of the matrix of $\Psi$. As $\Psi \circ \Psi^{-1}$ is the identity matrix, it is clear that the $v'_i$ form a basis of $\mathbb{R}^n$ such that $v_i \cdot v'_j$ is the Kronecker delta $\delta_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. This basis also induces the basis $(v'_i)_{i \in I}$ for $W_{\overline{I}}^{\perp}$.

Let $\widehat{C^I}$ be the orthogonal projection of $C$ on $W_{\overline{I}}^{\perp}$, and let $\pi$ be the linear map $W_I \to W_{\overline{I}}^{\perp}$ sending each point in $W_I$ to its orthogonal projection on $W_{\overline{I}}^{\perp}$. If $x, y \in W_I$, then $x - y \in W_I$. If we also have $\pi(x) = \pi(y)$, then $x - y \in \ker(\pi) \subseteq W_{\overline{I}}$. But then $x - y \in W_I \cap W_{\overline{I}} = \{0\}$, so $x = y$, which means $\pi$ is injective. As $W_I$ and $W_{\overline{I}}^{\perp}$ both have dimension $j$, we can pick orthonormal bases $(w_1, w_2, \ldots, w_j)$ for $W_I$ and $(z_1, z_2, \ldots, z_j)$ for $W_{\overline{I}}^{\perp}$; then we can represent $\pi$ using a $j \times j$ matrix with respect to these bases, which is invertible as $\pi$ is injective.

For each $x \in C^I$ we have that $x = z + y$ with $z \in C, y \in W_{\overline{I}}$, and $\pi(x) = x + y'$ for some $y' \in W_{\overline{I}}$. Hence $\pi(x) = z + (y + y')$ lies in $\widehat{C^I}$, so $\pi(C^I) \subseteq \widehat{C^I}$. By definition $\mathrm{Vol}_j(\widehat{C^I}) \le V'_j(C)$, so we want to bound the volume of $C^I$ in terms of the volume of $\widehat{C^I}$. As we just proved $C^I \subseteq \pi^{-1}(\widehat{C^I})$, we want to bound the determinant of the matrix representing $\pi^{-1}$.

Let $x \in W_I$, and write $x = \sum_{i \in I} a_i v_i$. As $x - \pi(x) \in W_{\overline{I}}$, we have that $(x - \pi(x)) \cdot v'_i = 0$ for all $i \in I$. Hence $a_i = x \cdot v'_i = \pi(x) \cdot v'_i$ for all $i \in I$, and we have

$$|x| \le \sum_{i \in I} |a_i||v_i| \le \sum_{i \in I} |\pi(x)||v'_i||v_i|. \tag{6}$$

Now fix an $i$ with $1 \le i \le n$, and use Gram-Schmidt orthonormalization to construct an orthonormal basis $(u_1, u_2, \ldots, u_n)$ of $\mathbb{R}^n$ such that the $(v_k)_{k \ne i}$ are linear combinations $v_k = \sum_{\ell=1}^{n-1} \xi_{k\ell} u_\ell$ of the first $n-1$ basis elements. By definition $v'_i$ is $\frac{1}{\det(\Psi^{-1})} = \frac{1}{d(\Lambda)}$ times the vector of cofactors $M_{ki}$, where $M$ is the matrix corresponding to $\Psi^{-1}$. Writing $u_k = (u_{k1}, u_{k2}, \ldots, u_{kn})$, we get

$$\left(v_{km}\right)_{1 \le k, m \le n, k \ne i} = \left(\xi_{k\ell}\right)_{k \ne i, 1 \le \ell \le n-1} \cdot \left(u_{\ell m}\right)_{1 \le \ell \le n-1, 1 \le m \le n}.$$

Hence each $M_{ki}$ is equal to $\det\left(\xi_{k\ell}\right)_{k\neq i,1\leq \ell\leq n-1}$ times the corresponding cofactor $N_{kn}$, with $N$ the matrix with columns $u_1, u_2, \ldots, u_n$. By using Hadamard's inequality and the fact that the $\xi_{k\ell}$ are the coefficients of the $v_k$ with respect to an orthonormal basis, it is clear that

$$\det\left(\xi_{k\ell}\right)_{k\neq i,1\leq \ell\leq n-1} \leq \prod_{k\neq i}\sqrt{\sum_{\ell=1}^{n-1}\xi_{k\ell}^2} = \prod_{k\neq i}|v_k|.$$

As $((N_{kn})_{1\leq k\leq n}) = \det(N)^{-1}u_n = u_n$ because the $u_k$ are an orthonormal basis and hence $N$ is an orthogonal matrix, it follows that

$$|v_i'| = \frac{1}{d(\Lambda)}\cdot|((M_{ki})_{1\leq k\leq n})| \leq \frac{1}{d(\Lambda)}\cdot\prod_{k\neq i}|v_k|\cdot|((N_{kn})_{1\leq k\leq n})| = \frac{\prod_{k\neq i}|v_k|}{d(\Lambda)}.$$

Using this inequality, our assumption that $|v_k| \leq k\lambda_k$ for all $1\leq k\leq n$, and Minkowski's Second Theorem (Theorem 1.24) we get

$$|v_i||v_i'| \leq \frac{\prod_{k=1}^n|v_k|}{d(\Lambda)} \leq \frac{n!\prod_{k=1}^n\lambda_k}{d(\Lambda)} \leq \frac{n!2^n}{B_n}$$

for all $1\leq i\leq n$. Combining this with inequality (6) gives $|x| \leq j\frac{n!2^n}{B_n}\cdot|\pi(x)|$. As $x$ was arbitrary, this means the operator norm of $\pi^{-1}$ is at most $j\frac{n!2^n}{B_n}$. Now write $\pi^{-1}$ as a matrix $(a_{ik})_{1\leq i,k\leq j}$ with respect to the orthonormal bases $(w_i)_{1\leq i\leq j}$ and $(z_i)_{1\leq i\leq j}$ we defined earlier. Clearly the operator norm of $\pi^{-1}$ bounds the $|a_{ik}|$ from above, so using Hadamard's inequality we find that

$$|\det(\pi^{-1})| \leq \prod_{i=1}^{j}\sqrt{\sum_{k=1}^{j}a_{ik}^2} \leq \prod_{i=1}^{j}\sqrt{j\cdot\left(j\frac{n!2^n}{B_n}\right)^2} = \left(j^{3/2}\cdot\frac{n!2^n}{B_n}\right)^j.$$

Hence

$$\mathrm{Vol}_j(C^I) \leq \mathrm{Vol}_j(\pi^{-1}(\widehat{C^I})) \leq \left(j^{3/2}\cdot\frac{n!2^n}{B_n}\right)^j \mathrm{Vol}_j(\widehat{C^I}) \leq \left(j^{3/2}\cdot\frac{n!2^n}{B_n}\right)^j V_j'(C)$$

completing the proof. □

Combining lemmas BW-2.1, BW-2.2 and BW-2.4 in our situation with the fact that $M_Z$ is a Davenport constant for all $\Psi(Z_T)$, we find that

$$\left||Z_T\cap\Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)}\right| \leq \sum_{j=0}^{n-1}M_Z^{n-j}V_j\Psi(Z_T) \leq \sum_{j=0}^{n-1}M_Z^{n-j}\sum_{I\subseteq\{1,2,\ldots,n\},|I|=j}\frac{2^j\cdot\mathrm{Vol}_j(Z_T^I)}{B_j\cdot\prod_{i=1}^j\lambda_i}$$

$$\leq \sum_{j=0}^{n-1}M_Z^{n-j}\sum_{I\subseteq\{1,2,\ldots,n\},|I|=j}\frac{2^j\cdot\left(j^{3/2}\frac{n!2^n}{B_n}\right)^j V_j'(Z_T)}{B_j\cdot\prod_{i=1}^j\lambda_i} = \sum_{j=0}^{n-1}M_Z^{n-j}\binom{n}{j}\frac{2^j\cdot\left(j^{3/2}\frac{n!2^n}{B_n}\right)^j V_j'(Z_T)}{B_j\cdot\prod_{i=1}^j\lambda_i}.$$

This is close to the result in the theorem, but we want the $V_j(Z_T)$ rather than the $V'_j(Z_T)$ in that summation. To solve this we have the next proposition, which is the main innovation of [1]. Note that we use $\mathrm{cl}(Z_T)$ for the (topological) closure of $Z_T$.

**Proposition BW-6.1.** *Let $Z \subseteq \mathbb{R}^{m+n}$ be a definable family such that the fibers $Z_T$ are bounded, and let $j$ be an integer with $0 \leq j \leq n-1$. Then there exists a constant $B_Z$ depending only on $Z$ such that $V'_j(\mathrm{cl}(Z_T)) \leq B_Z V_j(Z_T)$ for all $T \in \mathbb{R}^m$.*

*Proof.* Clearly $B_Z = 1$ works when $j = 0$ or $Z = \emptyset$, so assume $1 \leq j \leq n-1$ and $Z$ is nonempty. For any $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = j$, let $A_I$ be the subspace of $\mathbb{R}^n$ generated by the standard basis vectors $(e_i)_{i \in I}$, and let $\pi_{A_I}$ be the orthogonal projection $\mathbb{R}^n \to A_I$. As this projection is continuous, Lemma 1.31 implies $\mathrm{Vol}_j(\pi_{A_I}(\mathrm{cl}(Z_T))) = \mathrm{Vol}_j(\mathrm{cl}(\pi_{A_I}(Z_T))) = \mathrm{Vol}_j(\pi_{A_I}(Z_T))$. As this holds for all choices of $I$, we also have $V_j(\mathrm{cl}(Z_T)) = V_j(Z_T)$. Hence the inequality we want to prove is equivalent to $V'_j(\mathrm{cl}(Z_T)) \leq B_Z V_j(\mathrm{cl}(Z_T))$, so we can without loss of generality assume that all the $Z_T$ are closed. In that case we need to prove $V'_j(Z_T) \leq B_Z V_j(Z_T)$ for all $T \in \mathbb{R}^m$.

Let $O_n(\mathbb{R})$ be the orthogonal group. By identifying a $\phi \in O_n(\mathbb{R})$ with the coefficients of the matrix representing $\phi$ in terms of the standard basis, we can view $O_n(\mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$.

**Lemma 2.3.** *There exists a definable set $Z'_j \subseteq \mathbb{R}^{n^2+m+n}$ such that*

*(i)* $\dim(Z'_{j,(\phi,T)}) \leq j$ *for all* $(\phi, T) \in \mathbb{R}^{n^2+m}$.

*(ii)* $Z'_{j,(\phi,T)} \subseteq Z_T$ *for all* $(\phi, T) \in \mathbb{R}^{n^2+m}$.

*(iii)* $V'_j(Z_T) \leq \sup_{\phi \in O_n(\mathbb{R})} \mathcal{H}^j(Z'_{j,(\phi,T)})$ *for all* $T \in \mathbb{R}^m$.

*Proof.* We will construct a $Z'_j$ satisfying these requirements. Define

$$S = \{(\phi, T, y) \in \mathbb{R}^{n^2+m+n} : \phi \in O_n(\mathbb{R}), y \in \phi(Z_T)\}.$$

This is simply the set $W$ from the proof of Proposition BW-4.1, except that we add the restriction that $\phi$ is orthogonal. Since this can be expressed in $\binom{n}{2}$ quadratic equations in the coefficients of $\phi$, we see that $S$ is definable. By definition $S_{(\phi,T)} = \phi(Z_T)$ for each $(\phi, T) \in O_n(\mathbb{R}) \times \mathbb{R}^m$, so $S_{(\phi,T)} \subseteq \phi(Z_T)$ for all $(\phi, T) \in \mathbb{R}^{n^2+m}$.

Let $\pi_{n-j}$ be the projection $\mathbb{R}^{n^2+m+n} \to \mathbb{R}^{n^2+m+j}$ omitting the last $n-j$ coordinates. By Proposition 1.20 we can construct a definable function $f_j : \pi_{n-j}(S) \to \mathbb{R}^{n-j}$ such that the graph $\Gamma(f_j)$ of $f_j$ is contained in $S$. This immediately implies $\Gamma(f_j)_{(\phi,T)} \subseteq S_{(\phi,T)} \subseteq \phi(Z_T)$ for each $(\phi, T) \in \mathbb{R}^{n^2+m}$. Furthermore, by definition of $f_j$ the projection $\pi_{n-j}|_{\Gamma(f_j)}$ is a definable bijection from $\Gamma(f_j)$ to $\pi_{n-j}(S)$, and it induces definable bijections $\Gamma(f_j)_{(\phi,T)} \to \pi_{n-j}(S)_{(\phi,T)}$. This implies by Proposition 1.17, part (i), that $\dim(\Gamma(f_j)_{(\phi,T)}) = \dim(\pi_{n-j}(S)_{(\phi,T)})$ for all $(\phi, T) \in \mathbb{R}^{n^2+m}$. We now define

$$Z'_j = \{(\phi, T, x) \in \mathbb{R}^{n^2+m+n} : \phi \in O_n(\mathbb{R}), \phi(x) \in \Gamma(f_j)_{(\phi,T)}\}.$$

Parts (i) and (ii) of the lemma follow pretty easily: For each $(\phi, T) \in \mathbb{R}^{n^2+m}$ we have

$$\phi(Z'_{j,(\phi,T)}) = \Gamma(f_j)_{(\phi,T)} \subseteq \phi(Z_T)$$

and hence $Z'_{j,(\phi,T)} \subseteq Z_T$ as well. Now note that $\pi_{n-j}(S)_{(\phi,T)} \subseteq \mathbb{R}^j$. Hence the equality $\phi(Z'_{j,(\phi,T)}) = \Gamma(f_j)_{(\phi,T)}$ implies as each $\phi \in O_n(\mathbb{R})$ is bijective and definable that

$$\dim(Z'_{j,(\phi,T)}) = \dim(\Gamma(f_j)_{(\phi,T)}) = \dim(\pi_{n-j}(S)_{(\phi,T)}) \leq j$$

for all $(\phi, T) \in O_n(\mathbb{R}) \times \mathbb{R}^m$. For other choices of $\phi \in \mathbb{R}^{n^2}$ we find that $Z'_{j,(\phi,T)}$ is empty and hence has dimension $-\infty < j$.

To prove part (iii), we first need the following lemma:

**Lemma 2.4.** *Let $X \subseteq \mathbb{R}^{p+n}$ be definable, such that the fibers $X_a \subseteq \mathbb{R}^n$ are bounded and have dimension at most $j \geq 1$. Then there exist positive real constants $E_I$ only dependent on $X$ such that*

$$\mathcal{H}^j(X_a) \leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} E_I \operatorname{Vol}_j(\pi_I(X_a))$$

*where $\pi_I$ is the projection $\mathbb{R}^n \to \mathbb{R}^j$ sending $(x_1, x_2, \ldots, x_n)$ to $(x_i)_{i \in I}$.*

*Proof.* If $\dim(X_a) \leq 0$ we have $\mathcal{H}^j(X_a) = 0$, in which case the result is always true. For any $a$ with $\dim(X_a) \geq 1$ we get from [1][p. 15-16, Prop. 5.6 and Thm 5.7] that

$$\mathcal{H}^j(X_a) \leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \int_{\mathbb{R}^j} |\pi_I^{-1}(y) \cap X_a| d\mathcal{L}^j y.$$

So what remains to be proven is that each of these Lebesgue integrals is at most $E_I \operatorname{Vol}_j(X_a)$ for some constant $E_I$. Let $R^I = \{(a, y, x) \in \mathbb{R}^{p+j+n} : (a, x) \in X, y = \pi_I(x)\}$. By Corollary 1.19 there exists an $E_I$ such that each fiber $R^I_{a,y}$ has at most $E_I$ connected components. As $R^I_{(a,y)} = \pi_I^{-1}(y) \cap X_a$, when $\dim(R_{a,y}) \leq 0$ this implies $|\pi_I^{-1}(y) \cap X_a| \leq E_I$ by part (ii) of Proposition 1.17. As $\pi_I|_{X_a}$ is definable, by [6][p. 56, Corollary 1.6(ii)] we find that $P_I = \{y \in \mathbb{R}^j : \dim(\pi_I^{-1}(y) \cap X_a) \geq 1\}$ is definable and $\dim(P_I) \leq \dim(X_a) - 1 \leq j - 1$. Hence $P_I$ has measure 0, which means

$$\mathcal{H}^j(X_a) \leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \int_{\mathbb{R}^j} |\pi_I^{-1}(y) \cap X_a| d\mathcal{L}^j y = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \int_{\pi_I(X_a) \setminus P_I} |\pi_I^{-1}(y) \cap X_a| d\mathcal{L}^j y$$

$$\leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \int_{\pi_I(X_a) \setminus P_I} E_I d\mathcal{L}^j y = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} E_I \operatorname{Vol}_j(\pi_I(X_a) \setminus P_I) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} E_I \operatorname{Vol}_j(\pi_I(X_a)).$$

□

Now let $T \in \mathbb{R}^m$ be arbitrary, let $U$ be any $j$-dimensional subspace of $\mathbb{R}^n$, and let $(u_1, u_2, \ldots, u_j)$ be an orthonormal basis for $U$. Then let $\phi \in O_n(\mathbb{R})$ such that $\phi(u_i) = e_i$ for each $1 \leq i \leq j$. Define $\pi_U$ as the orthogonal projection of $\mathbb{R}^n$ on $U$ and $\tilde{\pi}$ as the orthogonal projection on the span of the standard basis vectors $(e_i)_{1 \leq i \leq j}$. Clearly $\phi \circ \pi_U = \tilde{\pi} \circ \phi$. As $U$ and $\phi(U)$ are $j$-dimensional, $\mathrm{Vol}_j$ and $\mathcal{H}^j$ agree on their subsets. As $\phi \in O_n(\mathbb{R})$, Proposition 1.27 implies that

$$\mathrm{Vol}_j(\pi_U(Z_T)) = \mathrm{Vol}_j(\phi \circ \pi_U(Z_T)) = \mathrm{Vol}_j(\tilde{\pi} \circ \phi(Z_T)) = \mathrm{Vol}_j(\tilde{\pi}(S_{(\phi,T)})).$$

So

$$V_j'(Z_T) = \sup_U(\mathrm{Vol}_j(\pi_U(Z_T))) \leq \sup_{\phi \in O_n(\mathbb{R})} (\mathrm{Vol}_j(\tilde{\pi}(S_{(\phi,T)}).$$

Now fix $\phi \in O_n(\mathbb{R})$. For any $A \subseteq \mathbb{R}^{n^2+m+n}$ we have that $\pi_{n-j}(A)_{(\phi,T)} = \{(x_1, x_2, \ldots, x_j) | \exists x_{j+1}, x_{j+2}, \ldots, x_n : (\phi, T, x_1, x_2, \ldots, x_n) \in A\}$ while $\tilde{\pi}(A_{(\phi,T)}) = \{(x_1, x_2, \ldots, x_j, 0, 0, \ldots, 0) \in \mathbb{R}^n | \exists x_{j+1}, x_{j+2}, \ldots, x_n : (\phi, T, x_1, x_2, \ldots, x_n) \in A\}$. So as we have already shown that $\pi_{n-j}(S)_{(\phi,T)} = \pi_{n-j}(\Gamma(f_j))_{(\phi,T)}$, it follows that $\tilde{\pi}(S_{(\phi,T)}) = \tilde{\pi}(\Gamma(f_j)_{(\phi,T)})$. So by using Proposition 1.27 we find

$$\mathrm{Vol}_j(\tilde{\pi}(S_{(\phi,T)}) = \mathrm{Vol}_j(\tilde{\pi}(\Gamma(f_j)_{(\phi,T)})) = \mathcal{H}^j(\tilde{\pi}(\Gamma(f_j)_{(\phi,T)})) \leq \mathcal{H}^j(\Gamma(f_j)_{(\phi,T)})$$

as the $j$-dimensional Lebesgue and Hausdorff measures agree on $\mathbb{R}^j$. As $Z'_{j,(\phi,T)} = \phi(\Gamma(f_j)_{(\phi,T)})$ and $\phi \in O_n(\mathbb{R})$, we have $\mathcal{H}^j(\Gamma(f_j)_{(\phi,T)}) = \mathcal{H}^j(Z'_{j,(\phi,T)})$. Hence we have

$$V_j'(Z_T) \leq \sup_{\phi \in O_n(\mathbb{R})} (\mathrm{Vol}_j(\tilde{\pi}(S_{(\phi,T)}) \leq \sup_{\phi \in O_n(\mathbb{R})} \mathcal{H}^j(Z'_{j,(\phi,T)})$$

which is exactly (iii), finishing the proof of Lemma 2.3. □

Now let $Z'_j$ as in Lemma 2.3 be given. For each $I \subseteq \{1, 2, \ldots, n\}$, define the projection $\pi_I : \mathbb{R}^n \to \mathbb{R}^j$ sending $(x_1, x_2, \ldots, x_n)$ to $(x_i)_{i \in I}$. As the $Z_T$ are bounded, properties (i) and (ii) of Lemma 2.3 imply that we can use Lemma 2.4 on $Z'_j$. This shows that there exist constants $E_I$ such that

$$\mathcal{H}^j(Z'_{j,(\phi,T)}) \leq \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} E_I \, \mathrm{Vol}_j(\pi_I(Z'_{j,(\phi,T)}))$$

for every $(\phi, T) \in \mathbb{R}^{n^2+m}$. Define $B_Z$ as the maximum of the $E_I$ as $I$ varies over the nonempty proper subsets of $\{1, 2, \ldots, n\}$. It is clear that $\mathrm{Vol}_j(\pi_{A_I}(Z'_{j,(\phi,T)})) = \mathrm{Vol}_j(\pi_I(Z'_{j,(\phi,T)}))$ for all $(\phi, T) \in \mathbb{R}^{n^2+m}$, so we have

$$V_j'(Z_T) \leq \sup_{\phi \in O_n(\mathbb{R})} \mathcal{H}^j(Z'_{j,(\phi,T)}) \leq \sup_{\phi \in O_n(\mathbb{R})} \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} E_I \, \mathrm{Vol}_j(\pi_I(Z'_{j,(\phi,T)}))$$

$$\leq \sup_{\phi \in O_n(\mathbb{R})} B_Z \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=j}} \mathrm{Vol}_j(\pi_{A_I}(Z'_{j,(\phi,T)})) = \sup_{\phi \in O_n(\mathbb{R})} B_Z V_j(Z'_{j,(\phi,T)})$$

$$\leq \sup_{\phi \in O_n(\mathbb{R})} B_Z V_j(Z_T) = B_Z V_j(Z_T)$$

for all $T \in \mathbb{R}^m$, finishing the proof of Proposition BW-6.1. □

We assumed earlier that the $Z_T$ are all closed, so $\mathrm{cl}(Z_T) = Z_T$ for all $T$. Hence we can use this proposition to make the last links in our inequality chain, which gives

$$\left| |Z_T \cap \Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} \right| \leq \sum_{j=0}^{n-1} M_Z^{n-j} \binom{n}{j} \frac{2^j \cdot \left(j^{3/2} \frac{n!2^n}{B_n}\right)^j V_j'(Z_T)}{B_j \cdot \prod_{i=1}^j \lambda_i}$$

$$\leq \sum_{j=0}^{n-1} M_Z^{n-j} \binom{n}{j} \frac{2^j \cdot \left(j^{3/2} \frac{n!2^n}{B_n}\right)^j B_Z V_j(Z_T)}{B_j \cdot \prod_{i=1}^j \lambda_i} \leq c_Z \sum_{j=0}^{n-1} \frac{V_j(Z_T)}{\prod_{i=1}^j \lambda_i}$$

where

$$c_Z = \max_{0 \leq j \leq n-1} M_Z^{n-j} \binom{n}{j} \frac{\left(j^{3/2} \cdot n! 2^{n+1}\right)^j B_Z}{B_j \cdot B_n^j} \tag{7}$$

proving the theorem. However, this was all under the assumption that the fibers $Z_T$ were closed. If they are not closed, then we need to look at what we will call the 'fiberwise closure' and the 'fiberwise boundary', which are respectively the sets

$$C(Z) = \{(T, x) : T \in \mathbb{R}^m, x \in \mathrm{cl}(Z_T)\},$$
$$B(Z) = \{(T, x) : T \in \mathbb{R}^m, x \in \mathrm{bd}(Z_T)\}.$$

In the o-minimal context these are definable sets with closed fibers, so the proof does apply to $C(Z)$ and $B(Z)$. Combining this with Lemma 1.31 giving us that $\mathrm{Vol}_n(\mathrm{bd}(Z_T)) = 0$ and hence $\mathrm{Vol}_n(\mathrm{cl}(Z_T)) = \mathrm{Vol}_n(Z_T)$, we get that

$$\left| |Z_T \cap \Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} \right| \leq \left| |\,\mathrm{cl}(Z_T) \cap \Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} \right| + |\,\mathrm{bd}(Z_T) \cap \Lambda|$$

$$= \left| |\,\mathrm{cl}(Z_T) \cap \Lambda| - \frac{\mathrm{Vol}_n(\mathrm{cl}(Z_T))}{d(\Lambda)} \right| + \left| |\,\mathrm{bd}(Z_T) \cap \Lambda| - \frac{\mathrm{Vol}_n(\mathrm{bd}(Z_T))}{d(\Lambda)} \right|$$

$$\leq c_{C(Z)} \sum_{j=0}^{n-1} \frac{V_j(\mathrm{cl}(Z_T))}{\prod_{i=1}^j \lambda_i} + c_{B(Z)} \sum_{j=0}^{n-1} \frac{V_j(\mathrm{bd}(Z_T))}{\prod_{i=1}^j \lambda_i} \leq (c_{C(Z)} + c_{B(Z)}) \sum_{j=0}^{n-1} \frac{V_j(Z_T)}{\prod_{i=1}^j \lambda_i}$$

so the constant $c_Z = c_{C(Z)} + c_{B(Z)}$ will do in that case, finishing the proof of Theorem BW-1.3.

While this proof gives us a formula for the constant $c_Z$, it is not quite complete: We still need to find $M_Z$ and $B_Z$ in order to actually calculate $c_Z$. Furthermore, we need to know whether $Z$ is fiberwise closed, and in the case it is not, we would need a method to calculate the same bounds for the fiberwise closure and fiberwise boundary of $Z$. By looking at the proofs of Lemma BW-4.1 and Lemma BW-6.1 to find out where these constants come from, it is clear that we need information on the number of connected components of various sets related to $Z$: The set $W$ in the former lemma, the sets $R^I$ in the latter.

# 3 Decomposing semialgebraic sets

## 3.1 Cell decomposition

To be able to bound the constants $M_Z$ and $B_Z$ appearing in our bound for $c_Z$, we need information on the structure of certain definable sets. However, it turns out this gets too complicated for general o-minimal structures. Because of this we restrict our view to the o-minimal structure of semialgebraic sets. We already know that as the semialgebraic sets form an o-minimal structure, we have cell decompositions for them. We will look at a construction for them in the semialgebraic case, as well as how we can find decompositions with additional structure. The general concept of this first construction is from [4], though the proof has been adapted from [2][Chapter 5.1].

**Definition 3.1.** Let $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \dots, X_n]$ and let $S \subseteq \mathbb{R}^n$. We call $\mathcal{P}$ sign invariant on $S$ if $\operatorname{sign}(P(x))$ is the same for all $x \in S$. We also call $S$ a $\mathcal{P}$-invariant set in this case, and call a cell decomposition of $\mathbb{R}^n$ $\mathcal{P}$-invariant if each cell in the decomposition is $\mathcal{P}$-invariant.

We will prove the following:

**Theorem 3.2.** *Let $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \dots, X_n]$ be finite. Then there exists a $\mathcal{P}$-invariant cell decomposition of $\mathbb{R}^n$.*

*Proof.* We construct a decomposition by induction on $n$. To do so, we first need that the roots of a polynomial vary continuously in terms of the coefficients. The specific statement we need is [2][p. 178, Theorem 5.12]; the proof can be found there.

**Proposition 3.3** (Continuity of roots)**.** *Let $P \in \mathbb{R}[X_1, \dots, X_n]$, and let $S$ be a semialgebraic subset of $\mathbb{R}^{n-1}$ such that the degree of $P(x', X_n)$ is constant for $x'$ varying over $S$. Let $x \in S$ be arbitrary, and let $z_1, z_2, \dots, z_k$ be the roots of $P(x, X_n)$ in $\mathbb{C}$ with respective multiplicities $\mu_1, \mu_2, \dots, \mu_k$. Fix $r > 0$ such that the disks $D(z_i, r)$ of radius $r$ around the $z_i$ in $\mathbb{C}$ are disjoint. Then there exists a open neighborhood $U$ of $x$ in $S$ such that for each $x' \in U$, the roots of $P(x', X_n)$ in each disk $D(z_i, r)$ have total multiplicity $\mu_i$.*

The idea of the proof is to first observe it is enough to check it for monic $P$, then prove it for $(X_n - z)^\mu$ and show that the coefficients of the coprime factors vary continuously in terms of those of the product.

**Definition 3.4.** Let $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \dots, X_n]$ and $S \subseteq \mathbb{R}^{n-1}$. We call $\mathcal{P}$ delineable over $S$ if for any $P, Q \in \mathcal{P}$ the following three quantities remain constant as $x$ varies over $S$:

(i) The $X_n$-degree of $P(x, X_n)$,

(ii) The number of distinct roots of $P(x, X_n)$ in $\mathbb{C}$,

(iii) The $X_n$-degree of $\gcd(P(x, X_n), Q(x, X_n))$.

To get a cell decomposition of $\mathbb{R}^n$ that is $\mathcal{P}$-invariant, the key is finding a set of projection polynomials $\operatorname{Proj}_{X_n}(\mathcal{P})$ such that if a cell $S \subseteq \mathbb{R}^{n-1}$ is $\operatorname{Proj}_{X_n}(\mathcal{P})$-invariant, then $\mathcal{P}$ is delineable over $S$. We then apply the following proposition.

**Proposition 3.5.** *Let $S$ be a semialgebraically connected region in $\mathbb{R}^{n-1}$, and let $\mathcal{P}$ be a finite set of polynomials in $\mathbb{R}[X_1, X_2, \ldots, X_n]$ that is delineable over $S$. Let $\mathcal{P}'$ be the subset of $\mathcal{P}$ consisting of all polynomials that are not identically $0$ on $S \times \mathbb{R}$. Then there exist continuous semialgebraic functions $\zeta_1, \zeta_2, \ldots, \zeta_m : S \to \mathbb{R}$ such that for each $x \in S$ the roots of $\prod_{P \in \mathcal{P}'} P(x, X_n)$ in $\mathbb{R}$ are $\zeta_1(x) < \zeta_2(x) < \ldots < \zeta_m(x)$. Furthermore, the multiplicity of $\zeta_i(x)$ as a root of any $P \in \mathcal{P}'$ is the same for all $x \in S$.*

*Proof.* We prove the theorem for $|\mathcal{P}'| = 2$; the result for $|\mathcal{P}'| = 1$ follows by taking $\mathcal{Q} = \mathcal{P}' \cup \{1\}$, and for other values of $|\mathcal{P}'|$ we get the theorem by repeatedly replacing $P, Q \in \mathcal{P}'$ with $PQ$. So let $P, Q \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ be two polynomials that are not identically 0 on $S \times \mathbb{R}$. Let $x \in S$ be arbitrary, and let $z_1, z_2, \ldots, z_k$ be the distinct roots of $PQ(x, X_n)$ in $\mathbb{C}$ with respective multiplicities $\mu_1, \mu_2, \ldots, \mu_k$ as roots of $P$ and $\nu_1, \nu_2, \ldots, \nu_k$ as roots of $Q$. Let $r > 0$ be such that the disks $D(z_i, r)$ are disjoint.

Note that the coefficients of $P(x', X_n)$ and $Q(x', X_n)$ vary continuously with $x'$. As the degrees of $P(x, X_n)$ and $Q(x, X_n)$, as well as their numbers of distinct roots in $\mathbb{C}$, do not depend on $x$, we can use continuity of roots (Proposition 3.3). This gives us that there is a neighborhood $U$ of $x$ such that for any $x'$ in $U$ each disk $D(z_i, r)$ contains a single root $p_i$ of $P(x', X_n)$ with multiplicity $\mu_i$ and a single root $q_i$ of $Q(x', X_n)$ of multiplicity $\nu_i$. The degree of the gcd of $P(x, X_n)$ and $Q(x, X_n)$ is $\sum_{i=1}^{k} \min(\mu_i, \nu_i)$ and remains constant on $S$. So whenever $\min(\mu_i, \nu_i) > 0$ it must hold that $p_i = q_i$ and the multiplicity of $p_i$ as a root of $\gcd(P(x', X_n), Q(x', X_n))$ is $\min(\mu_i, \nu_i)$. In particular $PQ(x', X_n)$ has exactly one root $r_i$ in each $D(z_i, r)$ for all $x' \in U$.

If $z_i \in \mathbb{R}$, then also $r_i \in \mathbb{R}$, as otherwise its conjugate $\overline{r_i}$ would be another root of $PQ(x', X_n)$ in the same disk. Similarly, if $z_i \notin \mathbb{R}$ we must have $r_i \notin \mathbb{R}$, as the disk $D(z_i, r)$ would otherwise intersect $D(\overline{z_i}, r)$. So the number of roots of $PQ(x', X_n)$ in $\mathbb{R}$, as well as their relative order and multiplicity, is constant as $x'$ varies over $U$. As $S$ is semialgebraically connected this implies that these quantities are constant over $S$ as well.

Hence the functions $\zeta_i : S \to \mathbb{R}$ mapping $x' \in S$ to the $i$-th smallest root of $PQ(x', X_n)$ in $\mathbb{R}$ are well-defined, and the roots they map to have constant multiplicity as roots of $P$ and $Q$. As $y = \zeta_i(x)$ is equivalent to the formula

$$x \in S \wedge \exists X_1, X_2, \ldots, X_k : X_1 < X_2 < \ldots < X_k \wedge \bigwedge_{j=1}^{k} PQ(x, X_j) = 0 \wedge y = X_i,$$

the functions $\zeta_i$ are semialgebraic (as we have just shown it to be definable in the structure of semialgebraic sets), and by repeating the above argument for arbitrarily small $r$ it follows that they are continuous. □

We use Proposition 3.5 to split the cylinder $S \times \mathbb{R}$ into the graphs of the $\zeta_i$ and the sectors into which the cylinder is divided by these graphs. These are by definition cells, and as a polynomial can only change sign at one of its roots, it follows that each of the cells is $\mathcal{P}$-invariant. So if we have such a projection operator, we can reduce the problem of creating a $\mathcal{P}$-invariant decomposition of $\mathbb{R}^n$ to creating a $\text{Proj}_{X_n}(\mathcal{P})$-invariant decomposition of $\mathbb{R}^{n-1}$. To find one, we need the following definition.

**Definition 3.6.** Let $P, Q \in \mathbb{R}[X]$ with $\deg(P) = p$, $\deg(Q) = q$, and let $j$ be an integer such that $0 \leq j \leq \min(p, q)$. Let $M_j$ be the matrix corresponding to the linear map $m_j : \mathbb{R}^{p+q-2j} \to \mathbb{R}^{p+q-j}$ sending $(u_{q-j-1}, u_{q-j-2}, \ldots, u_0, v_{p-j-1}, v_{p-j-2}, \ldots, v_0)$ to $(a_{p+q-j-1}, \ldots, a_0)$ such that

$$\sum_{i=0}^{p+q-j-1} a_i X^i = P(X) \cdot \sum_{i=0}^{q-j-1} u_i X^i + Q(X) \cdot \sum_{i=0}^{p-j-1} v_i X^i.$$

Then we define the $j$-th principal subresultant coefficient $s_j(P, Q)$ to be the determinant of the matrix formed by the first $p + q - 2j$ rows of $M_j$.

Concretely $m_j$ takes any pair $U, V \in \mathbb{R}[X]$ with $\deg(U) < q - j$ and $\deg(V) < p - j$ and sends the coefficient vector of the pair $(U, V)$ to that of $PU + QV$, which by the polynomial version of Bézout's identity implies that the kernel of $m_j$ is non-trivial if and only if $\deg(\gcd(P, Q)) > j$. So clearly $\deg(\gcd(P, Q)) > j$ implies that $s_i(P, Q) = 0$ for all $i \leq j$. As proven in [8, p. 261, Lemma 7.7.8], the converse also holds. Hence we have the following proposition.

**Proposition 3.7.** *Let $P, Q \in \mathbb{R}[X]$ be two nonzero polynomials. Then* $\deg(\gcd(P, Q))$ *is equal to the minimal $j$ such that $s_j(P, Q) \neq 0$.*

We also have the following basic result on the number of distinct roots of a $P \in \mathbb{R}[X]$ in $\mathbb{C}$:

**Lemma 3.8.** *Let $P \in \mathbb{R}[X]$ be nonzero. Then the number of distinct roots of $P$ in $\mathbb{C}$ is* $\deg(P) - \deg(\gcd(P, P'))$.

*Proof.* Write $P = a \prod_{i=1}^{k} (X - z_i)^{\mu_i}$, with the $z_i$ the distinct roots of $P$ in $\mathbb{C}$. Then $P' = a \sum_{i=1}^{k} \mu_i (X - z_i)^{\mu_i - 1} \prod_{j \neq i} (X - z_j)^{\mu_j}$, so clearly $\gcd(P, P') = \prod_{i=1}^{k} (X - z_i)^{\mu_i - 1}$ has degree $\deg(P) - k$. □

With this, we can construct the projection set required.

**Definition 3.9.** Let $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_n]$. For each $P \in \mathcal{P}$ let $t_k(P)$ be the remainder of the division of $P$ by $X_n^k$. Consider $\mathcal{P}$ as a set of polynomials in $X_n$ with coefficients in $\mathbb{R}[X_1, X_2, \ldots, X_{n-1}]$. We define $\text{Proj}_{X_n}(\mathcal{P})$ as the set containing for each $P, Q \in \mathcal{P}$:

- The nonzero coefficients of $P$,
- $s_j(t_k(P), t_k(P)')$ for each $0 \leq j \leq k \leq \deg(P)$,
- $s_j(t_k(P), t_\ell(Q))$ for each $k \leq \deg(P), \ell \leq \deg(Q)$, and $j \leq \min(k, \ell)$.

Note that we can exclude all constant polynomials from the projection if we want, as they are sign invariant over $\mathbb{R}^{n-1}$.

**Proposition 3.10.** *Let* $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_n]$. *Then if* $S \subseteq \mathbb{R}^{n-1}$ *is* $\mathrm{Proj}_{X_n}(\mathcal{P})$*-invariant,* $\mathcal{P}$ *is delineable over* $S$.

*Proof.* Let $P, Q \in \mathcal{P}$. The sign invariance of the coefficients of $P$ and $Q$ in $S$ implies that $P(x, X_n)$ and $Q(x, X_n)$ have constant degree when $x$ varies over $S$. So if $P(x, X_n)$ or $Q(x, X_n)$ is the zero polynomial for any $x \in S$, this is true for all $x \in S$, in which case the delineability conditions are trivial. So assume neither is the zero polynomial. Let $k$ be the degree of $P(x, X_n)$ and $\ell$ the degree of $Q(x, X_n)$ for any $x \in S$. The number of distinct roots of $P(x, X_n)$ in $\mathbb{C}$ is $k - \deg(\gcd(P(x, X_n), \frac{\partial}{\partial X_n} P(x, X_n))$ by Lemma 3.8. That in turn depends on which of the $s_j(P(x, X_n), \frac{\partial}{\partial X_n} P(x, X_n))$ are 0 by Proposition 3.7, and as each $s_j(t_k(P(x, X_n), t_k(P(x, X_n))')$ is sign invariant on $S$, this is independent of the choice of $x$. Similarly the sign invariance of the $s_j(t_k(P(x, X_n)), t_\ell(Q(x, X_N)))$ on $S$ implies the invariance of the degree of $\gcd(P(x, X_n), Q(x, X_n))$, so $\mathcal{P}$ is delineable over $S$. □

This, together with our earlier remarks, finishes the proof of Theorem 3.2. □

## 3.2 Stratification

A problem with cell decompositions is that cell adjacency is not always nice: The boundary of a cell might contain only part of a cell. To solve this, we take a look at sign conditions and their realizations.

**Definition 3.11.** Let $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_k]$ be a set of polynomials. We define a sign condition on $\mathcal{P}$ as a function $\sigma : \mathcal{P} \to \{-1, 0, 1\}$, and a weak sign condition on $\mathcal{P}$ as a function $\tau : \mathcal{P} \to \{\{-1, 0\}, \{0\}, \{0, 1\}\}$. The weak sign condition $\overline{\sigma}$ corresponding to $\sigma$ is the weak sign condition defined by $\overline{\sigma}(P) = \{\sigma(P), 0\}$ for all $P \in \mathcal{P}$.
The realization of $\sigma$ on a set $S \subseteq \mathbb{R}^k$ is the set

$$\mathrm{Reali}(\sigma, S) = \{x \in S : \mathrm{sign}(P(x)) = \sigma(P) \text{ for all } P \in \mathcal{P}\},$$

and similarly the realization of $\overline{\sigma}$ on $S$ is the set

$$\mathrm{Reali}(\overline{\sigma}, S) = \{x \in S : \mathrm{sign}(P(x)) \in \overline{\sigma}(P) \text{ for all } P \in \mathcal{P}\}.$$

In the univariate case, the realizations of sign conditions can be rather simple in structure, as long as the set of polynomials is closed under taking derivatives:

**Lemma 3.12** (Thom's Lemma)**.** *Let* $\mathcal{P} \subseteq \mathbb{R}[X]$ *be a finite set of polynomials that is closed under taking derivatives, and let* $\sigma$ *be a sign condition on* $\mathcal{P}$. *Then* $\mathrm{Reali}(\sigma, \mathbb{R})$ *is either empty, a point, or an open interval. Furthermore:*

- *If* $\mathrm{Reali}(\sigma, \mathbb{R})$ *is empty, then* $\mathrm{Reali}(\overline{\sigma}, \mathbb{R})$ *is either empty or a point.*
- *If* $\mathrm{Reali}(\sigma, \mathbb{R})$ *is a point, then* $\mathrm{Reali}(\overline{\sigma}, \mathbb{R})$ *is that point.*

- *If* $\mathrm{Reali}(\sigma, \mathbb{R})$ *is an open interval, then* $\mathrm{Reali}(\overline{\sigma}, \mathbb{R})$ *is the closure of that interval.*

*Proof.* We induct on $p = |\mathcal{P}|$. For $p = 0$ there is only the empty sign condition, which is realized in the open interval $\mathbb{R}$. The corresponding weak sign condition is again the empty sign condition, and the closure of $\mathbb{R}$ is $\mathbb{R}$, so the lemma holds when $p = 0$. Now let $\mathcal{P}$ be of size $p > 0$ and assume we already have the result for sets of size $p - 1$. Let $P$ be a polynomial of maximal degree in $\mathcal{P}$, let $\sigma$ be a sign condition on $\mathcal{P}$, and let $\tau$ be the restriction of $\sigma$ to $\mathcal{P} \setminus \{P\}$. As $P$ has maximal degree, $\mathcal{P} \setminus \{P\}$ is still closed under taking non-constant derivatives. So by the induction hypothesis $\mathrm{Reali}(\tau, \mathbb{R})$ is empty, a point, or an open interval.

In the first two cases it is clear that the same holds for $\mathrm{Reali}(\sigma, \mathbb{R}) \subseteq \mathrm{Reali}(\tau, \mathbb{R})$. If $\mathrm{Reali}(\tau, \mathbb{R})$ is an open interval, we know that $P' \in \mathcal{P} \setminus \{P\}$ has constant sign on that interval. If that sign is 0, $P$ is constant, in which case $\mathrm{Reali}(\sigma, \mathbb{R})$ is either $\mathrm{Reali}(\tau, \mathbb{R})$ or empty, and we are done. Otherwise $P$ is either strictly increasing or strictly decreasing on that interval, and the result follows easily. □

This gives us information on the closure of sets defined by a set of univariate polynomials, as long as that set is closed under differentiation. However, to fix our issue with cell boundaries, we have to look at multivariate polynomials as well. So we first generalize Thom's Lemma to cylinders of a cell decomposition.

**Definition 3.13.** Let $P \in \mathbb{R}[X_1, X_2, \ldots, X_k]$. We call $P$ quasi-monic in $X_k$ if when writing $P$ as

$$P(X_1, X_2, \ldots, X_k) = \sum_{i=0}^{d} P_i(X_1, X_2, \ldots, X_{k-1}) X_k^i$$

the leading coefficient $P_d(X_1, X_2, \ldots, X_{k-1})$ is a constant.

From this point onward we will use $\overline{S}$ as an alternate notation for the (topological) closure of $S$.

**Proposition 3.14** (Generalized Thom's Lemma). *Let* $\mathcal{P} \subseteq \mathbb{R}[X_1, \ldots, X_k]$ *be a finite set of polynomials that is closed under differentiation with respect to* $X_k$ *and is quasi-monic in* $X_k$*. Let* $S$ *and* $S'$ *be semialgebraically connected subsets of* $\mathbb{R}^{k-1}$ *with* $S' \subseteq \overline{S}$ *that are both sign invariant for* $\mathrm{Proj}_{X_k}(\mathcal{P})$*. Let* $\zeta_1, \zeta_2, \ldots, \zeta_n : S \to \mathbb{R}$ *and* $\zeta'_1, \ldots, \zeta'_{n'} : S' \to \mathbb{R}$ *be continuous functions describing the roots of the polynomials in* $\mathcal{P}$ *above respectively* $S$ *and* $S'$*.*

*Then we have that*

*(i) Each* $\zeta_i$ *extends continuously to a* $\overline{\zeta_i}$ *defined on* $S \cup S'$*, and the restriction of* $\overline{\zeta_i} |_{S'}$ *is one of the* $\zeta'_j$*.*

*(ii) Each of the* $\zeta'_j$ *is a restriction* $\overline{\zeta_i} |_{S'}$ *as in (i).*

*Furthermore, let $\sigma$ be a sign condition on $\mathcal{P}$, and let $\overline{\sigma}$ be the corresponding weak sign condition. Then*

*(iii)* $\mathrm{Reali}(\sigma, S \times \mathbb{R})$ *is either empty, the graph of one of the* $\zeta_i$*, or one of the sectors into which these graphs divide* $S \times \mathbb{R}$*.*

*(iv)* $\mathrm{Reali}(\overline{\sigma}, S' \times \mathbb{R})$ *is the graph of a* $\zeta_i'$ *or the closure in* $S' \times \mathbb{R}$ *of one of the sectors into which these graphs divide* $S' \times \mathbb{R}$*.*

*If* $\mathrm{Reali}(\sigma, S \times \mathbb{R})$ *is nonempty, we also have*

*(v)* $\overline{\mathrm{Reali}(\sigma, S \times \mathbb{R})} \cap S \times \mathbb{R} = \mathrm{Reali}(\overline{\sigma}, S \times \mathbb{R})$*,*

*(vi)* $\overline{\mathrm{Reali}(\sigma, S \times \mathbb{R})} \cap S' \times \mathbb{R} = \mathrm{Reali}(\overline{\sigma}, S' \times \mathbb{R})$*.*

To prove this, we are going to need two further lemmas.

**Lemma 3.15.** *Let $f : (0,1) \to \mathbb{R}$ be a continuous, bounded, and semialgebraic map. Then $f$ has a continuous semialgebraic extension $\overline{f} : [0,1) \to \mathbb{R}$.*

**Lemma 3.16.** *Let $S \subseteq \mathbb{R}^k$ be semialgebraic, and let $x \in \overline{S}$. Then there exists a continuous semialgebraic map $\gamma : [0,1) \to \overline{S}$ with $\gamma(0) = x$ and $\gamma((0,1)) \subseteq S$.*

Proofs of these lemmas can be found in [2][p. 104, Lemma 3.21 and Theorem 3.22]; the core idea involves looking in a larger field obtained by adding an infinitesimal to $\mathbb{R}$.

*Proof of Proposition 3.14.* Let $\zeta_i$ be arbitrary, and let $x' \in S'$. By definition there is a $P \in \mathcal{P}$ such that $P(x, \zeta_i(x)) = 0$ for all $x \in S$. As $\mathcal{P}$ is closed under differentiation and the multiplicity of this root is constant as $x$ varies, we can choose $P$ such that $P(x, X_k)$ always has a simple root at $\zeta_i(x)$. As $P$ is quasi-monic with respect to $X_k$, we can write $P(x, X_k) = a_p X_k^p + a_{p-1}(x) X_k^{p-1} + \ldots + a_0(x)$. The Cauchy bound on roots gives us that $|\zeta_i(x)| \leq 1 + \max_{0 \leq i < p} \left|\frac{a_i(x)}{a_p}\right|$. Now consider $M = 1 + \max_{0 \leq i < p} \left|\frac{a_i(x')}{a_p}\right|$. By continuity of roots (Proposition 3.3) we find that there is a semialgebraic open neighborhood $U$ of $x'$ in $\overline{S}$ such that $|\zeta_i(x)| \leq M + 1$.

From Lemma 3.16 we know that there is a continuous $\gamma : [0,1)$ to $U$ such that $\gamma(0) = x'$ and $\gamma(x) \in S \cap U$ for all $x \in (0,1)$. As $\zeta_i \circ \gamma$ is a continuous bounded function $f : (0,1) \to \mathbb{R}$, by Lemma 3.15 it extends continuously to a function $\overline{f} : [0,1) \to \mathbb{R}$. So the point $(x', \overline{f}(0))$ is in the closure of the graph of $\zeta_i$.

For $x \in S$, define the sign condition $\tau_x$ which sends each polynomial $Q \in \{P, P', \ldots, P^{(p)}\}$ to the sign of $Q(x, \zeta_i(x))$. By the $\mathrm{Proj}_{X_k}(\mathcal{P})$-invariance of $S$ all $\tau_x$ must be equal, so let $\tau = \tau_x$ for an arbitrary $x \in S$. As each point in the graph of $\zeta_i$ satisfies $\tau$, any point $(x', x_k')$ in the closure of this graph must satisfy $\overline{\tau}$ by continuity of $\zeta_i$. Thom's Lemma gives that the set of $x_k'$ such that $(x', x_k')$ satisfies $\overline{\tau}$ is either the closure of an open interval or at most a point. As $\overline{\tau}(P) = 0$ and $P(x', X_k)$ is not the zero polynomial, the former is not possible, so there is at most one $x'_k$ with $(x', x'_k)$ in the closure of the graph of $\zeta_i$. As $(x', \overline{f}(0))$ is in this closure, we have a unique continuous extension of $\zeta_i$ to $x'$, and hence we have a unique continuous extension $\overline{\zeta_i}$ of $\zeta_i$ to $S \cup S'$. Furthermore, as $P(x', \overline{\zeta_i}(x')) = 0$, the extension agrees with a $\zeta'_j$ on $S'$, proving (i).

Now let $\zeta'_i$ be given with a $P \in \mathcal{P}$ and $x' \in S'$ such that $P(x', X_k)$ has a simple root at $X_k = \zeta'_i(x')$. Without loss of generality the derivative of $P(x', X_k)$ is positive at $X_k = \zeta'_i(x')$, and the same must hold in some neighborhood $V$ of $(x', \zeta'_i(x'))$ in $\overline{S} \times \mathbb{R}$. There exists an open interval $(\zeta'_i(x') - m, \zeta'_i(x') + m)$ such that for all $y$ in the interval, $P(x', y) < 0$ if $y < \zeta'_i(x')$ and $P(x', y) > 0$ if $y > \zeta'_i(x')$, so $P(x', \zeta'_i(x') - m')P(x', \zeta'_i(x') + m') < 0$ for all $0 < m' < m$.

Let $U$ be the set of all $u \in \overline{S}$ such that $P(u, \zeta'_i(x') - m)P(u, \zeta'_i(x') + m) < 0$ and such that the derivative of $P(u, y)$ is positive for all $y \in [\zeta'_i(x') - m, \zeta'_i(x') + m]$. Then $U$ is semialgebraic, open, and contains $x'$. The Intermediate Value Theorem then implies that $P(u, X_k)$ has a simple root in $(\zeta'_i(x') - m, \zeta'_i(x') + m)$ for all $u \in U$, and the function $\xi$ mapping $u \in U$ to this root is continuous. $U \cap S$ must be nonempty since $x' \in \overline{S}$, so $\xi$ must extend continuously to a $\zeta_j$ on $S$ and $\overline{\zeta_j} \mid_{S'} = \zeta'_i$, proving (ii).

Now (iii) and (iv) follow pretty easily: By $\mathrm{Proj}_{X_k}(\mathcal{P})$-invariance of $S$ and $S'$ the realizations must be unions of cells, which are exactly the graphs and sectors described. Let $x \in S$ be arbitrary. Thom's Lemma gives us that the realization of $\sigma$ in $\{x\} \times \mathbb{R}$ is empty, a point, or an open interval. As at least one polynomial has a root at each graph of a $\zeta_i$, this implies that $\mathrm{Reali}(\sigma, S \times \mathbb{R})$ is either empty, a graph $\zeta_i$, or a single sector. Similarly we find that $\mathrm{Reali}(\sigma, S' \times \mathbb{R})$ is empty, the graph of a $\zeta'_i$, or the closure of a sector in $S' \times \mathbb{R}$.

This leaves the last two items. (v) follows as the closure of a graph is simply that graph, and the closure of a sector in $S \times \mathbb{R}$ is the union of that sector with its bounding graphs. For (vi), note that if $\sigma$ is realized on $S \times \mathbb{R}$ in a graph $\zeta_i$, then clearly both sides are equal to the $\zeta'_i$ it extends to on $S'$. If $\sigma$ is realized on a sector in $S \times \mathbb{R}$, then $\mathrm{Reali}(\overline{\sigma}, S' \times \mathbb{R})$ is the closed sector bounded by the extensions of the bounding graphs of that sector to $S'$. Clearly $\overline{\mathrm{Reali}(\sigma, S \times \mathbb{R})} \cap S' \times \mathbb{R}$ cannot be just a graph, as $\overline{\mathrm{Reali}(\sigma, S' \times \mathbb{R})}$ contains a sector, so it must be the same closed sector $\mathrm{Reali}(\overline{\sigma}, S' \times \mathbb{R})$. □

This allows us to augment our cell decompositions in a way that the boundary of each cell consists of cells.

**Definition 3.17.** Let $\mathcal{P}^* = (\mathcal{P}_i)_{1 \leq i \leq k}$ be a sequence of sets of nonzero polynomials $\mathcal{P}_i \subseteq \mathbb{R}[X_1, X_2 \ldots, X_i]$. We call $\mathcal{P}^*$ a stratifying family for $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_k]$ if

- $\mathcal{P} \subseteq \mathcal{P}_k$,
- $\mathcal{P}_i$ is quasi-monic with regards to $X_i$, and is closed under taking $X_i$-derivatives for $i = 1, 2, \ldots, k$,

- $\text{Proj}_{X_{i+1}}(\mathcal{P}_{i+1}) \subseteq \mathcal{P}_i$ for $i = 1, 2, \dots, k-1$.

In general, if $\mathcal{P}^*$ satisfies the latter two conditions we call $\mathcal{P}^*$ a stratifying family.

**Proposition 3.18.** *Let $\mathcal{P}^*$ be a stratifying family for $\mathcal{P}$, and let $S_i$ be the family of nonempty semialgebraic sets that are the realizations of sign conditions on $\bigcup_{j\le i}\mathcal{P}_j$ in $\mathbb{R}^i$. Then the $S_i$ are the layers of a $\mathcal{P}$-invariant cell decomposition of $\mathbb{R}^k$, of which the closure of each cell is a union of that cell with cells of lower dimension.*

*Proof.* We use induction on $k$. The case $k = 0$ is trivial. Now suppose the proposition holds for $k-1$. Then $S_{k-1}$ is a $\text{Proj}_{X_k}(\mathcal{P}_k)$-invariant decomposition of $\mathbb{R}^{k-1}$, so by following the proof of Theorem 1.14 we can extend this to a $\mathcal{P}_k$-invariant decomposition of $\mathbb{R}^k$. In this decomposition each element of $S_k$ will be a union of cells; to prove the first part of this proposition, it is enough to show that each element of $S_k$ is a single cell.

Let $\sigma$ be a sign condition on $\bigcup_{j\le k}\mathcal{P}_j$ with $C = \text{Reali}(\sigma, \mathbb{R}^k) \in S_k$. As $C$ is nonempty, the restriction of $\sigma$ to $\bigcup_{j\le k-1}\mathcal{P}_j$ has a nonempty realization $C' \in S_{k-1}$, and $C \subseteq C' \times \mathbb{R}$. By the induction hypothesis $C'$ is a cell, and $\mathcal{P}_k$ is delineable over $C'$. Let $\zeta_1, \zeta_2, \dots, \zeta_m$ be the functions delineating the roots of polynomials in $\mathcal{P}_k$ in $C' \times \mathbb{R}$ as in Proposition 3.5.

The cells of our decomposition making up $C' \times \mathbb{R}$ are the graphs of the $\zeta_i$ and the sectors into which these graphs divide $C' \times \mathbb{R}$. $C$ is a union of some of these cells, and we need to prove it is just a single cell. Let $x \in C'$ be arbitrary. The cells divide $\{x\}\times\mathbb{R}$ into the points $(x, \zeta_i(x))$ and the open intervals between them. By Thom's Lemma $\text{Reali}(\sigma, \{x\}\times\mathbb{R})$ must be a point or an open interval.

If $\text{Reali}(\sigma, \{x\} \times \mathbb{R})$ is a point, then $C$ is the graph of some $\zeta_i$. If $\text{Reali}(\sigma, \{x\} \times \mathbb{R})$ is an open interval, then $C$ must contain a sector. If the graph of $\zeta_i$ is one of the graphs bounding this sector, then there is some $P \in \mathcal{P}_k$ that has no roots in this sector, while it is 0 on the graph of $\zeta_i$. Hence the graph of $\zeta_i$ is not part of $C$. That means that $C$ cannot contain multiple adjacent sectors either, so $C$ is a single sector. In either case $C$ is a single cell of this decomposition, which finishes the proof of the first part of the proposition.

To continue with the statement about the closure, let $\pi_1$ be the projection $\mathbb{R}^k \to \mathbb{R}^{k-1}$ obtained by omitting the last coordinate. Then as $\pi_1(C) = C'$ we find that $\pi_1(\overline{C})$ is contained in $\overline{C'}$, so $\overline{C} \subseteq \overline{C'} \times \mathbb{R}$. By the induction hypothesis $\overline{C'}$ is a union of $C'$ with cells of lower dimension. Let $D$ be one of these lower-dimensional cells. Using the Generalized Thom's Lemma we find that $\overline{C}$ intersects the cylinder $D\times\mathbb{R}$ in $\text{Reali}(\overline{\sigma}, D\times\mathbb{R})$, which is a union of cells. Let $E$ be such a cell. If $E$ is a graph, we have $\dim(E) = \dim(D) < \dim(C') \le \dim(C)$. If $E$ is a sector, then $\dim(E) = \dim(D)+1 < \dim(C')+1$, and as $E$ is part of the boundary of $C$, $C$ must also be a sector, so $\dim(C) = \dim(C')+1$. So $E$ has lower dimension than $C$.

By using the Generalized Thom's Lemma again we find that $\overline{C}\cap C'\times\mathbb{R} = \text{Reali}(\overline{\sigma}, C'\times\mathbb{R})$. If $C$ is a graph, this is just $C$, and if $C$ is a sector, then this is the union of $C$ with the graphs that bound it, which each have dimension $\dim(C) - 1$. As $\overline{C}$ is the union of $\overline{C} \cap C' \times \mathbb{R}$ with all the intersections $\overline{C} \cap D \times \mathbb{R}$, this finishes the proof. □

**Theorem 3.19.** *Let $S$ be a semialgebraic set defined by sign conditions on polynomials in the finite set $\mathcal{P} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_k]$. Then there exists a linear automorphism $u : \mathbb{R}^k \to \mathbb{R}^k$ and a stratifying family $\mathcal{P}^*$ for $\{P \circ u : P \in \mathcal{P}\}$.*

*Proof.* We use induction on $k$. For $k = 1$ we take $\mathcal{P}_1$ as the closure of $\mathcal{P}$ under taking derivatives of nonconstant polynomials. This clearly gives a stratifying family for $\mathcal{P}$, so we can just take the identity morphism for $u$. Now suppose the result holds for $k-1$. We first prove that there is a linear automorphism that makes $\mathcal{P}$ quasi-monic.

Let $v$ be an linear automorphism of the type $(x_1, x_2, \ldots, x_k) \mapsto (x_1 + a_1 x_k, x_2 + a_2 x_k, \ldots, x_k)$, and let $P \in \mathcal{P}$ be arbitrary. Define $P_h$ as the homogeneous polynomial that is the sum of all monomials of $P$ of degree $\deg(P)$. Then $P \circ v = P_h(a_1, a_2, \ldots, a_{k-1}, 1) X_k^{\deg(P)} + Q$ where $Q$ has $X_k$-degree lower than $\deg(P)$. Let $R = \prod_{P \in \mathcal{P}} P_h$. If we can choose the $a_i$ such that $R(a_1, a_2, \ldots, a_{k-1}, 1)$ is nonzero, then all the $P \circ v$ are quasi-monic in $X_k$. But we can do this: As the nonzero homogeneous polynomial $R$ cannot be divisible by the inhomogeneous factor $X_k - 1$, $R(X_1, X_2, \ldots, X_{k-1}, 1)$ is not the zero polynomial, so it does not vanish everywhere.

So let $v$ be a linear automorphism such that the $P \circ v$ are quasi-monic for $P \in \mathcal{P}$. Let $\mathcal{P}_k$ be the closure of the $P \circ v$ under taking nonzero derivatives with respect to $X_k$. By the induction hypothesis we have a linear automorphism $w$ and a stratifying family $\mathcal{Q}^*$ for $\{Q \circ w : Q \in \mathrm{Proj}_{X_k}(\mathcal{P}_k)\}$. Define $w \times e : \mathbb{R}^k \to \mathbb{R}^k$ as the linear automorphism given by $(x_1, x_2, \ldots, x_{k-1}, x_k) \mapsto (w(x_1, x_2, \ldots, x_{k-1}), x_k)$. Then taking $\mathcal{P}_i = \mathcal{Q}_i$ for all $1 \leq i \leq k-1$ and $u = (w \times e) \circ v$ gives an automorphism and stratifying family for $\{P \circ u : P \in \mathcal{P}\}$ as required. □

We now continue by showing that we have a homeomorphism from any closed and bounded semialgebraic set to a simplicial complex, which simplifies the structure even further.

**Definition 3.20.** For any non-negative integers $p \leq q$, let $a_0, a_1, \ldots, a_p$ be a set of $p+1$ points in $\mathbb{R}^q$ that are affinely independent, that is, such that the $a_i - a_0$ are linearly independent. Define the $p$-simplex $s = [a_0, a_1, \ldots, a_p]$ as the $p$-dimensional set

$$\left\{ \sum_{i=0}^{p} \lambda_i a_i : \sum_{i=0}^{p} \lambda_i = 1, \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } 0 \leq i \leq p \right\}.$$

The $a_i$ are called the vertices of $s$. A face of $s$ is any simplex with as its vertices a subset of the $a_i$; this includes the empty set, which is a simplex of dimension $-1$. We write $f \preceq s$ if $f$ is a face of $s$. We define the open simplex $\dot{s}$ as the interior of $s$, which is

$$\left\{ \sum_{i=0}^{p} \lambda_i a_i : \sum_{i=0}^{p} \lambda_i = 1, \lambda_i \in \mathbb{R}_{>0} \text{ for all } 0 \leq i \leq p \right\}.$$

The barycenter $\mathrm{ba}(s)$ of $s$ is the arithmetic mean $\sum_{i=0}^{p} \frac{x_p}{p+1}$ of its vertices, and is clearly contained in $\dot{s}$. A simplicial complex $K \subseteq \mathbb{R}^k$ is a finite set of simplices in $\mathbb{R}^k$ that is closed under taking faces, for which every two simplices in it intersect in a common face (which may be the empty face).
The barycentric subdivision $\mathrm{ba}(K)$ of a simplicial complex $K$ is the complex that consists of all simplices of the form $[\mathrm{ba}(s_0), \mathrm{ba}(s_1), \ldots, \mathrm{ba}(s_p)]$ with $s_0 \prec s_1 \prec \ldots \prec s_p$. Finally, we define the polyhedron $|K|$ spanned by $K$ as the union of the simplices in $K$; note that $|\mathrm{ba}(K)| = |K|$.

Note that simplicial complexes are semialgebraic. In fact, the polynomials defining a simplex are linear. We will now show that closed and bounded semialgebraic sets can be triangulated: That is, they are homeomorphic to the polyhedron of a simplicial complex.

**Theorem 3.21** (Triangulation)**.** *Let $S \subseteq \mathbb{R}^k$ be a closed and bounded semialgebraic set, and let $S_1, \ldots, S_m$ be semialgebraic subsets of $S$. Then there exists a simplicial complex $K$ and a semialgebraic homeomorphism $f : |K| \to S$ such that each $S_i$ is a union of images of open simplices of $K$.*

*Proof.* We use induction on $k$. For $k = 1$ we have that $S$ is a disjoint union of the semialgebraic sets $\bigcap_{i=1}^{m} S_i'$ with $S_i' \in \{S_i, S \setminus S_i\}$ for all $i$. Each of these intersections is a disjoint union of points and open intervals, so we can also write $S$ as a disjoint union of points and open intervals. Now let $K$ be the set consisting of the closures of these open intervals and the singleton sets containing these points. We will prove $K$ is a simplicial complex.

As $S$ is closed and bounded, each of these intervals has two boundary points in $\mathbb{R}$, and the boundary points must be among the points included in the singletons. As we wrote $S$ as a disjoint union of the open intervals and points, the closures of the open intervals intersect in at most a point, and if they do said point must be in one of the singletons. Hence $K$ is a simplicial complex. By definition $|K| = S$, so the identity map $|K| \to S$ is a triangulation of $S$. Furthermore, each of the $\bigcap_{i=1}^{m} S_i'$ is a union of open simplices of $K$, so the same holds for each $S_i$.

Now suppose that $k > 1$ and that the theorem holds for $k - 1$. We know that we can make a linear change of variables such that $S$ and the $S_i$ are unions of strata with respect to a stratifying set of polynomials $\mathcal{P}$. Taking the appropriate cylindrical decomposition, we find that $\mathbb{R}^{k-1}$ decomposes into cells $C_i$ with continuous functions $\zeta_{i,j} : C_i \to \mathbb{R}$ describing the roots of $\mathcal{P}$ above $C_i$. Let $\pi_k$ be the projection of $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$ forgetting the last coordinate. As $S$ is closed and bounded, so is $\pi_k(S)$, and $\pi_k(S)$ is the union of some of the $C_i$.

By the induction hypothesis we have a simpicial complex $K$ and a homeomorphism $f : |K| \to \pi_k(S)$ such that each cell $C_i \subseteq \pi_k(S)$ is a union of images of open simplices of $K$. So the cylinders $C_i \times \mathbb{R}$ above these cells also split into cylinders $f(\dot{s}) \times \mathbb{R}$ for simplices $s \in K$. We will triangulate each cell above an $f(\dot{s})$ that lies in $S$ individually; as each $S_i$ is a union of such cells, the second part of the theorem will follow directly.

We now first triangulate the parts $S \cap f(\dot{s}) \times \mathbb{R}$ with complexes $L_s$. Let $s \in K$ be a simplex $[a_0, a_1, \ldots, a_p]$, and let $\zeta : f(\dot{s}) \to \mathbb{R}$ be one of the root functions such that the graph of $\zeta$ lies in $S$. Using the Generalized Thom's Lemma we can continuously extend $\zeta$ to a function $\overline{\zeta}$ defined on the closure of $f(\dot{s})$, which is $f(s)$. Now define the new simplex $s_\zeta = [b_0, \ldots, b_p]$ with $b_i = (a_i, \overline{\zeta}(a_i))$ and add this simplex and its faces to $L_s$. We define the homeomorphism $g_\zeta$ from $s_\zeta$ to the graph of $\overline{\zeta}$ by $g_\zeta(\sum_{i=0}^p \lambda_i b_i) = (y, \overline{\zeta}(y))$ with $y = f(\sum_{i=0}^p \lambda_i a_i)$. Clearly this restricts to a homeomorphism from $\dot{s_\zeta}$ to the graph of $\zeta$.

Let $\zeta' : f(\dot{s}) \to \mathbb{R}$ be a different root function of which the graph is contained in $S$. We do not want $s_\zeta = s_{\zeta'}$, which means that we need to prove that one of the $b_i$ is different from the corresponding $b'_i$ in $s_{\zeta'}$. This means proving $\overline{\zeta}(a_i) \neq \overline{\zeta'}(a_i)$ for some $i$, but unfortunately this does not have to be true. To solve this problem we replace $K$ by the barycentric subdivision $\mathrm{ba}(K)$. Each of the $p$-simplices in this subdivision that has part of $\dot{s}$ as its interior has $\mathrm{ba}(s)$ as a vertex, and as $\mathrm{ba}(s) \in \dot{s}$ we have $\overline{\zeta}(\mathrm{ba}(s)) = \zeta(\mathrm{ba}(s))$, which does differ for each $\zeta$.

Of course the cells of the decomposition that are part of $S$ do not need to just be graphs of root functions $\zeta$. As $S$ is bounded, we have no unbounded cells, so all that remains are the sectors between consecutive root functions on the same cylinder. So let $\zeta < \zeta'$ be two consecutive root functions above $\dot{s}$ such that the sector between them is part of $S$. The sector $P$ between $s_\zeta = [b_0, b_1, \ldots, b_p]$ and $s_{\zeta'} = [b'_0, b'_1, \ldots, b'_p]$ can be written as

$$P = \bigcup_{i=0}^{p} \left| [b_0, b_1, \ldots, b_i, b'_i, b'_{i+1}, \ldots, b'_p] \right|$$

(where we can ignore any cases where $b_i = b'_i$, as they occur as faces of other simplices in this decomposition anyway). Taking all these simplices and their faces as part of our complex $K_s$, what remains is defining our homeomorphism from $P$ to the sector between $\zeta$ and $\zeta'$. As $S$ is closed, the graphs of $\zeta$ and $\zeta'$ are in $S$ as well, so we have $g_\zeta$ and $g_{\zeta'}$ already defined. Let $x = \sum_{i=0}^p \lambda_i a_i \in s$, $y = f(x)$, and let $\mu, \nu \geq 0$ with $\mu + \nu = 1$. We define $g_P(y, \mu g_\zeta(y) + \nu g_{\zeta'}(y)) = (y, \mu\overline{\zeta}(y) + \nu\overline{\zeta'}(y))$; this agrees with $g_\zeta$ and $g_{\zeta'}$ on $s_\zeta$ and $s_{\zeta'}$, so this all glues together to a homeomorphism $g_s : |L_s| \to \overline{S \cap f(\dot{s}) \times \mathbb{R}}$

What remains is proving that the $L_s$ and $g_s$ just defined can be glued together to form a single simplicial complex and homeomorphism. For this we need to check that two simplices can only intersect in a face, and that the homeomorphisms $g_s$ agree on intersections of simplices. It is enough to check this for adjacent cylinders, which are always of the form $g(\dot{s}) \times \mathbb{R}$ and $g(\dot{t}) \times \mathbb{R}$ with $t$ a face of $s$. If $\xi$ is a root function $g(\dot{t}) \to \mathbb{R}$, then $t_\xi$ intersecting $|L_s|$ implies that $t_\xi$ is in $L_s$. Indeed, the Generalized Thom's Lemma gives us that there is a root function $\zeta : g(\dot{t}) \to \mathbb{R}$ for which $\overline{\zeta}$ restricts to $\xi$ on $g(\dot{t})$, so $t_\xi$ is a face of $s_\zeta$ and $g_t$ and $g_s$ agree on $t_\xi$.

That leaves the sectors; we know that the sectors themselves behave nicely, but as the decomposition of sectors into simplices is dependent on a choice of vertex order, the simplices might not be defined in the same way on the intersection. As $K$ has finitely many vertices, we can define some total order on the vertices of $K$ and order the vertices of each simplex of $K$ according to that order. This forces the decompositions of sectors to agree on intersections, so the homeomorphisms combine to the triangulation we wanted. □

## 3.3 Algebraic Topology

One of our goals is estimating the number of connected components of certain semialgebraic sets. In the context of semialgebraic sets it is more natural to look at semialgebraic connected components instead, but as we will be looking over $\mathbb{R}$, these are the same.

**Definition 3.22.** A semialgebraic set $S$ is semialgebraically connected if it cannot be written as a union of two nonempty disjoint semialgebraic open subsets of $S$.

**Proposition 3.23.** *A semialgebraic set $S \subseteq \mathbb{R}^n$ is semialgebraically connected if and only if it is connected.*

*Proof.* If $S$ is connected, then $S$ is not a union of any two nonempty disjoint open subsets of $S$, let alone open semialgebraic subsets, so then $S$ is semialgebraically connected as well. For the other direction, suppose $S = O_1 \cup O_2$ with $O_1$ and $O_2$ open and disjoint. Let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^n$ that is $S$-invariant, and let $C$ be an arbitrary cell of the decomposition. $C$ is the disjoint union of $C \cap O_1$ and $C \cap O_2$, which are both open in $C$. As each cell is connected by Lemma 1.13 $C \cap O_1$ or $C \cap O_2$ must be empty. Hence $O_1$ and $O_2$ are both unions of cells, which are semialgebraic. So if $S$ is disconnected, it is also semialgebraically disconnected, completing the proof. □

We now need to introduce some elements of algebraic topology. We first introduce simplicial homology, as this will allow us to define homology for certain semialgebraic sets later. Let $K$ be a simplicial complex. We write $K_p$ for the set of $p$-simplices of $K$. Let $S = \{a_0, a_1, \ldots, a_p\}$ be the vertices of some $p$-simplex $s$. Any total ordering on $S$ is uniquely described by writing the vertices in ascending order. Call two such orderings equivalent if the corresponding ascending sequences differ by an even permutation. We define an oriented $p$-simplex as a $p$-simplex paired with an equivalence class of this relation, and write $s = [a_0, \ldots, a_p]$ for the oriented $p$-simplex that corresponds to the regular simplex $s = [a_0, \ldots, a_p]$ paired with the equivalence class of the ordering with $a_0 < a_1 < \ldots < a_p$, and $-s$ for the oriented $p$-simplex $[a_1, a_0, a_2, \ldots, a_p]$.

The $p$-th chain group of $K$, $\mathrm{Ch}_p(K)$, is then defined as the free abelian group generated by the oriented $p$-simplices of $K$. As $K_p$ is finite these have finite rank, and $\mathrm{Ch}_p(K) = 0$ when $K_p = \emptyset$. We also define $\mathrm{Ch}_p(K)$ as 0 for negative integers $p$. We then define the boundary of a $p$-simplex as

$$\delta_p([a_0, \ldots, a_p]) = \sum_{i=0}^{p} (-1)^i [a_0, \ldots, a_{i-1}, \hat{a_i}, a_{i+1}, \ldots, a_p]$$

with $\hat{a}_i$ meaning $a_i$ is omitted. This induces boundary homomorphisms $\delta_p : \mathrm{Ch}_p(K) \to \mathrm{Ch}_{p-1}(K)$ by setting $\delta_p(\sum_{i=1}^k n_i s_i) = \sum_{i=1}^k n_i \delta_p(s_i)$ for all $n_i \in \mathbb{Z}$, $s_i \in K_p$. Now note that for all $p$ we have

$$\delta_{p-1}(\delta_p([a_0, \ldots, a_n])) = \sum_{i=0}^{p} (-1)^i \delta_{p-1}([a_0, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_p]).$$

By definition of $\delta_{p-1}$ this becomes a double summation of terms of the form $(-1)^k [a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_p]$, where $k = i + j$ if the second removed vertex preceeds the first one in the original sequence, and $k = i+j-1$ otherwise. Each $[a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_p]$ appears twice, once with $a_i$ removed first and once with $a_j$ removed first, so we find that the summation is 0. So $\delta_{p-1} \circ \delta_p$ is 0 on all oriented $p$-simplices, which means it is the zero map. This means that the $(\mathrm{Ch}_p(K), \delta_p)$ form a chain complex.

We define the subgroups $B_p(K) = \mathrm{im}(\delta_{p+1})$, whose elements are called $p$-boundaries, and $Z_p(K) = \ker(\delta_p)$, whose elements are called $p$-cycles. It is quite easy to show that $\delta_p \circ \delta_{p+1} = 0$ for all $p$, so each $p$-boundary is a $p$-cycle, which means we can define the $p$-th homology group $H_p(K) = Z_p(K)/B_p(K)$. We define the $p$-th Betti number of $K$, written $b_p(K)$, as the rank of $H_p(K)$.

The Betti numbers provide information on topological properties of $|K|$. We care primarily about the following result.

**Proposition 3.24.** *Let $K$ be a nonempty simplicial complex. Then $b_0(K)$ is the number of connected components of $|K|$.*

This is usually proven through cohomology, see for example [2][p. 216, Proposition 6.5]. We will base our homology for semialgebraic sets on simplicial homology, which allows us to use the 0-th homology group to find the number of connected components of semialgebraic sets. Let $S$ be a closed and bounded semialgebraic set. By Theorem 3.21 $S$ is homeomorphic to $|K|$ for some simplicial complex $K$, so we make the following definition.

**Definition 3.25.** Let $S \subset \mathbb{R}^n$ be a closed and bounded semialgebraic set, and $K$ a simplicial complex such that $|K|$ is homeomorphic to $S$. Then we define the homology groups of $S$ as $H_p(S) = H_p(K)$ for all $p \in \mathbb{Z}$.

This is well-defined, as we have the following result from algebraic topology:

**Proposition 3.26.** *Let $K, L$ be semialgebraically homeomorphic simplicial complexes in $\mathbb{R}^k$. Then $H_p(K)$ is isomorphic to $H_p(L)$ for all $p \in \mathbb{Z}$.*

This is for instance proven in [2][p. 225, Theorem 6.21]. While this gives us a homology for semialgebraic sets that are both closed and bounded, we want to define it for more general collections of sets. To do so, we first generalize this proposition.

**Definition 3.27.** Let $S, T \subseteq \mathbb{R}^k$ be two semialgebraic sets, and let $f, g : S \to T$ be two continuous maps. A semialgebraic homotopy from $f$ to $g$ is a semialgebraic continuous map $F : S \times [0,1] \to T$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in S$. This gives an equivalence relation on the continuous maps from $S$ to $T$, which we denote $f \sim g$. We call $S$ and $T$ semialgebraically homotopy equivalent if there exist semialgebraic continuous maps $f : S \to T$ and $g : T \to S$ such that $f \circ g \sim \mathrm{Id}_T$, $g \circ f \sim \mathrm{Id}_S$: this is an equivalence relation on the semialgebraic subsets of $\mathbb{R}^k$.

If $f : S \to T$ is a semialgebraic homeomorphism, then $f$ and $f^{-1}$ induce a semialgebraic homotopy equivalence between $S$ and $T$, so this is a more general definition. With the more general definition also comes a more general proposition:

**Proposition 3.28.** *Let $S, T \subseteq \mathbb{R}^k$ be closed and bounded semialgebraic sets which are semialgebraically homotopy equivalent. Then $H_p(S)$ is isomorphic to $H_p(T)$ for all $p \in \mathbb{Z}$.*

This is [2][p. 242, Theorem 6.42], and is effectively proven by showing that $H_p$ is a functor from the category of simplicial complexes (with as arrows continuous maps between their polyhedra) to the category of abelian groups which sends homotopic maps to the same group homomorphism. We mostly concern ourselves with a specific type of homotopy equivalence:

**Definition 3.29.** Let $U \subseteq S \subseteq \mathbb{R}^k$ be two semialgebraic sets, and let $U$ be closed. A semialgebraic deformation retraction from $S$ to $U$ is a continuous semialgebraic function $\gamma : S \times [0,1] \to S$ such that

- $\gamma(s, 0) = s$ for all $s \in S$,
- $\gamma(u, t) = u$ for all $u \in U$, $t \in [0,1]$,
- $\gamma(s, 1) \in U$ for all $s \in S$.

**Proposition 3.30.** *Let $U \subseteq S \subset \mathbb{R}^k$ be semialgebraic sets such that there exists a semialgebraic deformation retraction from $S$ to $U$. Then $S$ and $U$ are semialgebraically homotopy equivalent.*

*Proof.* If $\gamma$ is a semialgebraic deformation retraction from $S$ to $U$, let $f : S \to U$ be the map defined by $s \mapsto \gamma(s, 1)$. It is not hard to see that $f$, together with the inclusion map $\iota : U \to S$, forms a semialgebraic homotopy equivalence between $S$ and $U$. □

We also need the following result.

**Proposition 3.31.** *Let $S \subseteq \mathbb{R}^k$, $T \subseteq \mathbb{R}^m$ be two semialgebraic sets, and let $f : S \to T$ be a semialgebraic continuous function. There exists a partition of $T$ into finitely many sets $T_1, T_2, \ldots, T_r$ such that for each $i \in \{1, 2, \ldots, r\}$ and $x \in T_i$ there exists a semialgebraic homeomorphism $\theta_x : T_i \times f^{-1}(x) \to f^{-1}(T_i)$ such that $f \circ \theta_x$ is the projection map $T_i \times f^{-1}(x) \to T_i$. Furthermore, the $\theta_x$ can be taken such that $\theta_x(x, y) = y$ for all $y \in f^{-1}(x)$.*

*Proof.* This proposition, except for the last statement, is a weaker version of [2][p. 201, Theorem 5.46]. To prove the last statement, first note that $\theta_x^{-1}(f^{-1}(x)) = \{x\} \times f^{-1}(x)$. Hence by composing $\theta_x^{-1}|_{f^{-1}(x)}$ with the projection to $f^{-1}(x)$ we find a semialgebraic homeomorphism $\xi_x : f^{-1}(x) \to f^{-1}(x)$. Now consider the map $\theta'_x$ sending $(t, y) \in T_i \times f^{-1}(x)$ to $\theta_x(t, \xi_x(y)) \in f^{-1}(T_i)$: As $\theta_x$ and $\xi_x$ are semialgebraic homeomorphisms, so is $\theta'_x$. We find that $f(\theta'_x(t, y)) = f(\theta_x(t, \xi_x(y))) = t$ for all $(t, y) \in T_i \times f^{-1}(x)$ by definition of $\theta_x$, so $\theta'_x$ also satisfies the requirements the proposition has on $\theta_x$. As $\theta'_x(x, y) = \theta_x(x, \xi_x(y)) = \theta_x(\theta_x^{-1}(y)) = y$ for all $y \in f^{-1}(x)$, this proves the final statement. □

We will primarily use the following consequence:

**Proposition 3.32.** *Let $S \subseteq \mathbb{R}^k$ be a semialgebraic set and let $f : S \to (0, \infty)$ be a continuous semialgebraic function. Then there exists a $t \in \mathbb{R}_{>0}$ such that for each $t' \in (0, t]$ there exists a semialgebraic deformation retraction from $S$ to $f^{-1}([t', \infty))$.*

*Proof.* Let $T_1, T_2, \ldots, T_r$ be the pieces of a decomposition of $\mathbb{R}_{>0}$ as in Proposition 3.31. Clearly one of these $T_i$ contains an interval of the form $(0, u)$: Let $A$ be this $T_i$, and define $t = \frac{u}{2}$. Then $(0, t] \subseteq A$. Now let $t' \in (0, t]$ be given. By Proposition 3.31 there exists a semialgebraic homeomorphism $\theta_{t'} : A \times f^{-1}(t') \to f^{-1}(A)$ such that $\theta_{t'}(t', y) = y$ for all $y \in f^{-1}(t')$ and $f(\theta_{t'}(a, y)) = a$ for all $(a, y) \in A \times f^{-1}(t')$. Let $\pi$ be the projection from $A \times f^{-1}(t')$ to $f^{-1}(t')$. We now define the following function $S \times [0, 1] \to S$:

$$\gamma_{t'}(x, s) = \begin{cases} x \text{ if } f(x) \geq t' \\ \theta_{t'}((1-s)f(x) + st', \pi(\theta_{t'}^{-1}(x))) \text{ if } f(x) \leq t' \end{cases}$$

This is well-defined: If $f(x) \leq t'$ then $f(x) \in (0, t'] \subseteq A$, so $\theta_{t'}^{-1}(x)$ exists, and the cases agree when $f(x) = t'$, as then the second branch evaluates to $\theta_{t'}(t', \pi(\theta_{t'}^{-1}(x))) = \theta_{t'}(\theta_{t'}^{-1}(x)) = x$. It is also clear that $\gamma_{t'}$ is semialgebraic and continuous. We claim that $\gamma_{t'}$ is a deformation retraction from $S$ to $f^{-1}([t', \infty))$.

If $x \in f^{-1}([t', \infty))$ we have $f(x) \geq t'$, so $\gamma_{t'}(x, s) = x$ for all $s \in [0, 1]$. If $f(x) \leq t'$, then $\gamma_{t'}(x, 0) = \theta_{t'}(f(x), \pi(\theta_{t'}^{-1}(x)))$, but $\theta_{t'}^{-1}(x) = (f(x), \pi(\theta_{t'}^{-1}(x)))$, so this is just $x$. Hence $\gamma_{t'}(x, 0) = x$ for all $x \in S$. If $f(x) \leq t'$ we also find that $\gamma_{t'}(x, 1) = \theta_{t'}(t', \pi(\theta_{t'}^{-1}(x)) \in f^{-1}(t')$, so $f(\gamma_{t'}(x, 1)) = t'$. So $\gamma_{t'}(x, 1) \in f^{-1}([t', \infty))$ for all $x \in S$. Hence $\gamma_{t'}$ is indeed a deformation retraction from $S$ to $f^{-1}([t', \infty))$, completing the proof. □

Using Proposition 3.32 we deduce the following result, which allows us to expand the definition of homology to arbitrary closed semialgebraic sets:

**Definition 3.33.** We define $\overline{B_k}(0, r)$ as the closed ball of radius $r$ around 0 in $\mathbb{R}^k$.

**Proposition 3.34.** *Let $S \subseteq \mathbb{R}^k$ be a closed semialgebraic set. Define $S_r = S \cap \overline{B_k}(0, r)$ for all $r > 0$,. Then there exists a $t > 0$ in $\mathbb{R}$ such that for all $t' \geq t$ there exists a semialgebraic deformation retraction from $S$ to $S_{t'}$.*

*Proof.* Consider the function $f : S \setminus \{0\} \to (0, \infty)$ defined by $f(x) = \frac{1}{|x|}$. This is clearly semialgebraic and continuous, and $S_r \setminus \{0\} = f^{-1}([\frac{1}{r}, \infty))$, so by Proposition 3.32 there exists a $t > 0$ such that there exist deformation retractions from $S \setminus \{0\}$ to $S_{t'} \setminus \{0\}$ for all $t' \in (0, t]$. If $0 \notin S$, then this is enough. If $0 \in S$, then we also find $0 \in S_r$ for all $r > 0$, so these deformation retractions can be extended to deformation retractions from $S$ to $S_{t'}$ by setting $\gamma(0, s) = 0$ for all $s \in [0, 1]$. □

Let $S \subseteq \mathbb{R}^k$ be a closed semialgebraic set. Then the $S_r = S \cap \overline{B_k}(0, r)$ are all closed and bounded semialgebraic sets. The existence of the semialgebraic deformation retractions from Proposition 3.34 implies that all $S_{t'}$ with $t' \geq t$ are homotopy equivalent to $S$, so also to each other. This means that the $S_{t'}$ must all have isomorphic homology groups by Proposition 3.28. As the $S_r$ form an increasing family of subsets of $S$ with union $S$ of which the homology eventually grows constant (up to isomorphism), and there exist deformation retractions from $S$ to each $S_{t'}$ with $t' \geq t$, this motivates the following definition.

**Definition 3.35.** Let $S \subseteq \mathbb{R}^k$ be a closed semialgebraic set, and let $t > 0$ be such that for any $t' \geq t$ there is a semialgebraic deformation retraction from $S$ to $S_{t'} = S \cap \overline{B_k}(0, t')$. We define the homology groups of $S$ as $H_p(S) = H_p(S_t)$.

Note that these groups are only defined up to isomorphism: If we want specific groups instead, we can always let $s$ be the supremum of such $t$, and either take the homology groups of $S_{s/2}$ if $s$ is finite, or those of $S_1$ if $s = \infty$. We also find the expected generalization of Proposition 3.28:

**Corollary 3.36.** *Let $T \subseteq S \subseteq \mathbb{R}^k$ be closed semialgebraic sets which are semialgebraically homotopy equivalent. Then $H_p(S)$ is isomorphic to $H_p(T)$ for all $p \in \mathbb{Z}$.*

*Proof.* This immediately follows from Proposition 3.28 using the definition of homology for closed semialgebraic sets: Let $t, t' \in \mathbb{R}_{>0}$ be such that by definition $H_p(S) = H_p(S_t)$ and $H_p(T) = H_p(T_{t'})$. As we have deformation retractions from $S$ to $T$, from $S$ to $S_t$ and from $T$ to $T_{t'}$, it follows that $S_t$ and $T_{t'}$ are homotopy equivalent. Proposition 3.28 implies that the $H_p(S) = H_p(S_t)$ are isomorphic to the $H_p(T) = H_p(T_{t'})$. □

We will now define homology for realizations of sign conditions in a similar way: Let $\sigma$ be a sign condition on a finite subset $\mathcal{P}$ of $\mathbb{R}[X_1, \ldots, X_k]$. If $\mathcal{P}$ is empty, then $\mathrm{Reali}(\sigma, \mathbb{R}^k) = \mathbb{R}^k$, which is closed and hence already has a homology, so we may assume $\mathcal{P}$ is nonempty. Let $\mathrm{Reali}_t(\sigma)$ be the set of $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ satisfying

$$|x| \leq \frac{1}{t} \wedge \bigwedge_{P \in \sigma^{-1}(0)} P(x) = 0 \wedge \bigwedge_{P \in \sigma^{-1}(1)} P(x) \geq t \wedge \bigwedge_{P \in \sigma^{-1}(-1)} P(x) \leq -t.$$

Clearly $\mathrm{Reali}_t(\sigma)$ is a semialgebraic closed and bounded subset of $\mathbb{R}^k$, so we have already defined homology groups for $\mathrm{Reali}_t(\sigma)$.

**Proposition 3.37.** *Let $\sigma$ be a sign condition on a finite nonempty subset of $\mathbb{R}[X_1, X_2, \ldots, X_k]$. Then there exists a $t \in \mathbb{R}_{>0}$ such that for all $t' \in (0, t]$ there exists a semialgebraic deformation retraction from $\mathrm{Reali}(\sigma, \mathbb{R}^k)$ to $\mathrm{Reali}_{t'}(\sigma)$.*

*Proof.* Consider the semialgebraic continuous function $f : \text{Reali}(\sigma, \mathbb{R}^k) \to \mathbb{R}_{>0}$ sending $x = (x_1, \ldots, x_k)$ to $\min(\frac{1}{|x|}, \min_{P \in \sigma^{-1}(\{-1,1\})} |P(x_1, \ldots, x_k)|)$. As $\mathcal{P}$ is nonempty, this is well-defined at 0. Note that $f^{-1}([t, \infty)) = \text{Reali}_t(\sigma)$ for all $t > 0$. Proposition 3.32 then gives that there exists a $t \in \mathbb{R}_{>0}$ such that for any $t' \in (0, t]$ we have a semialgebraic deformation retraction from $\text{Reali}(\sigma, \mathbb{R}^k)$ to $\text{Reali}(\sigma, \mathbb{R}^k) \cap g^{-1}([t', \infty)) = \text{Reali}_{t'}(\sigma)$ as required. □

This shows the $\text{Reali}_t(\sigma)$ are similar in nature to the $S_t$: They form an increasing family of closed and bounded semialgebraic sets with union $\text{Reali}(\sigma, \mathbb{R}^k)$, of which the homology grows constant up to isomorphism as $t$ approaches 0, which all receive a deformation retraction from $\text{Reali}(\sigma, \mathbb{R}^k)$. This motivates the following definition:

**Definition 3.38.** Let $\sigma$ be a sign condition on a finite nonempty subset of $\mathbb{R}[X_1, X_2, \ldots, X_k]$, and let $t \in \mathbb{R}_{>0}$ be such that for all $t' \in (0, t]$ there exists a deformation retraction from $\text{Reali}(\sigma, \mathbb{R}^k)$ to $\text{Reali}_{t'}(\sigma)$. We define the homology groups of $\sigma$ as $H_p(\text{Reali}(\sigma, \mathbb{R}^k)) = H_p(\text{Reali}_t(\sigma))$.

When $S$ is either a closed semialgebraic set or the realization of a sign condition, we also define the Betti number $b_i(S)$ as the rank of $H_i(S)$ for all $i \in \mathbb{Z}$. A well-known result from algebraic topology is the Mayer-Vietoris Theorem, which in simplicial homology looks like this:

**Theorem 3.39** (Mayer-Vietoris)**.** *Let $K_1, K_2$ be two subcomplexes of a simplicial complex $K$. Then there exists an exact sequence*

$$\cdots \to H_n(K_1 \cap K_2) \to H_n(K_1) \oplus H_n(K_2) \to H_n(K_1 \cup K_2) \to H_{n-1}(K_1 \cap K_2) \to \cdots .$$

We can extract some bounds on Betti numbers from this sequence:

**Proposition 3.40.** *Let $S_1, S_2$ be two closed semialgebraic sets. Then the following inequalities hold for all $i \in \mathbb{Z}$:*

$$\begin{aligned} b_i(S_1) + b_i(S_2) &\leq b_i(S_1 \cap S_2) + b_i(S_1 \cup S_2), \\ b_i(S_1 \cap S_2) &\leq b_i(S_1) + b_i(S_2) + b_{i+1}(S_1 \cup S_2), \\ b_i(S_1 \cup S_2) &\leq b_i(S_1) + b_i(S_2) + b_{i-1}(S_1 \cap S_2). \end{aligned}$$

*Proof.* It follows from the definition of the homology of a closed semialgebraic set that we only need to prove the case where $S_1$ and $S_2$ are bounded. By Theorem 3.21 we can find simplicial complexes $K, K_1, K_2$ such that $K_1$ and $K_2$ are subcomplexes of $K$, $|K_i|$ is homeomorphic to $S_i$ for $i = 1, 2$, and $|K_1 \cap K_2|$ is homeomorphic to $S_1 \cap S_2$. So up to isomorphism we have $H_p(S_i) = H_p(K_i)$ for $i = 1, 2$, $H_p(S_1 \cap S_2) = H_p(K_1 \cap K_2)$, and $H_p(S_1 \cup S_2) = H_p(K_1 \cup K_2)$, so the Mayer-Vietoris sequence for $K_1$ and $K_2$ induces one for $S_1$ and $S_2$. The exactness of this Mayer-Vietoris sequence implies that the rank of any group in the sequence is at most the sum of that of the neighboring groups. As that is exactly what the inequalities we want to prove express, we are done. □

We will also need the Oleinik-Petrovski/Thom/Milner bound.

**Theorem 3.41.** *Let $S \subseteq \mathbb{R}^k$ be an algebraic set defined by polynomials of degree at most $d$. Let $b(S)$ be the sum of the Betti numbers of $S$. Then*

$$b(S) \leq d(2d-1)^{k-1}.$$

A proof is given in [2][p. 273, Theorem 7.25].

We are going to prove Theorem 7.32 from [2]:

**Theorem 3.42.** *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}[X_1, X_2, \ldots, X_k]$ with $|\mathcal{P}| \leq s$, $\max\{\deg(P) : P \in \mathcal{P}\} \leq d$ and $\dim(Z(\mathcal{Q})) = k'$. For each sign condition $\sigma$ for $\mathcal{P}$, let $b_i(\sigma)$ be the $i$-th Betti number of the realization $\mathrm{Reali}(\sigma, Z(\mathcal{Q}))$ of $\sigma$ over the zero set $Z(\mathcal{Q}) = \{x \in \mathbb{R}^k : Q(x) = 0$ for all $Q \in \mathcal{Q}\}$. Finally, let $b_i(\mathcal{Q}, \mathcal{P})$ be the sum of the $b_i(\sigma)$. Then we have*

$$b_i(\mathcal{Q}, \mathcal{P}) \leq \sum_{i=0}^{k'-i} \binom{s}{i} 4^i d(2d-1)^{k-1}$$

*for all $i$.*

*Proof.* To prove this theorem, we first need a proposition allowing us to shift between Betti numbers of intersections and unions. Let $S_1, S_2, \ldots, S_s$ be closed semialgebraic subsets of some closed semialgebraic set $T \subseteq \mathbb{R}^k$ of dimension $k'$.

**Proposition 3.43.** *For each $0 \leq i \leq k'$, we have*

$$b_i\left(\bigcup_{j=1}^{s} S_j\right) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subseteq \{1,2,\ldots,s\} \\ |J|=j}} b_{i+1-j}\left(\bigcap_{j \in J} S_j\right), \tag{8}$$

$$b_i\left(\bigcap_{j=1}^{s} S_j\right) \leq \sum_{j=0}^{k'-i} \binom{s}{j} b_{k'}(T) + \sum_{\substack{J \subseteq \{1,2,\ldots,s\} \\ |J|=j}} b_{i+j-1}\left(\bigcup_{j \in J} S_j\right). \tag{9}$$

*Proof.* We first prove (8) by induction on $s$. For $s = 1$ the inequality becomes $b_i(S_1) \leq b_i(S_1)$, which trivially holds. Now suppose the inequality holds for $s-1$. We use Proposition 3.40 to get $b_i\left(\bigcup_{j=1}^{s} S_j\right) \leq b_i\left(\bigcup_{j=1}^{s-1} S_j\right) + b_i(S_s) + b_{i-1}\left(\bigcup_{j=1}^{s-1} S_j \cap S_s\right)$. Applying the induction hypothesis to $\bigcup_{j=1}^{s-1} S_j$ gives

$$b_i\left(\bigcup_{j=1}^{s-1} S_j\right) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subseteq \{1,2,\ldots,s-1\} \\ |J|=j}} b_{i+1-j}\left(\bigcap_{j \in J} S_j\right).$$

Similarly, applying the induction hypothesis to $\bigcup_{j=1}^{s-1}(S_j \cap S_s)$ we find that

$$b_{i-1}\left(\bigcup_{j=1}^{s-1}(S_j \cap S_s)\right) \leq \sum_{j=1}^{i} \sum_{\substack{J \subseteq \{1,2,\ldots,s-1\} \\ |J|=j}} b_{i-j}\left(\bigcap_{j \in J \cup \{s\}} S_j\right).$$

Adding these two inequalities together, then adding the term $b_i(S_s)$ (which is still missing on the right as $J$ is always nonempty) gives the required inequality for $s$.

Now we prove (9), again using induction on $s$. For $s = 1$ and $i < k'$ the inequality becomes $b_i(S_1) \leq 2b_{k'}(T) + b_i(S_1)$, which clearly holds. For $i = k'$ we need to prove $b_{k'}(S_1) \leq b_{k'}(T)$. If $\dim(S_1) < k'$, then $b_{k'}(S_1) = 0$, so the inequality is trivial. Otherwise let $V$ be the closure of $T \setminus S_1$. Then $V \cap S_1$ is the boundary of $S_1$, which has dimension less than $k'$ by Proposition 1.17. This implies that $b_{k'}(V \cap S_1) = 0$, so using Proposition 3.40 we find that

$$b_{k'}(S_1) + b_{k'}(V) \leq b_{k'}(S_1 \cup V) + b_{k'}(S_1 \cap V) = b_{k'}(T) + 0,$$

so clearly $b_{k'}(S_1) \leq b_{k'}(T)$. The induction step proceeds similarly to that of the other part of the lemma as $b_i\left(\bigcap_{j=1}^{s} S_j\right) \leq b_i\left(\bigcap_{j=1}^{s-1} S_j\right) + b_i(S_s) + b_{i+1}\left(\bigcap_{j=1}^{s-1} S_j \cup S_s\right)$; the main difference is that we also need to deal with the terms that are multiples of $b_{k'}(T)$. The upper bound for $b_i\left(\bigcap_{j=1}^{s-1} S_j\right)$ gets an extra term $\sum_{j=0}^{k'-i} \binom{s-1}{j} b_{k'}(T)$ while the upper bound for $b_{i+1}\left(\bigcap_{j=1}^{s-1}(S_j \cup S_s)\right)$ gets a term $\sum_{j=0}^{k'-i-1} \binom{s-1}{j} b_{k'}(T) = \sum_{j=1}^{k'-i} \binom{s-1}{j-1} b_{k'}(T)$, and adding these together gives $\sum_{j=0}^{k'-i} \binom{s}{j} b_{k'}(T)$ as required, completing the proof. □

We now return to the setting of Theorem 3.42. Let $\sigma$ be any sign condition on $\mathcal{P}$. We use Proposition 3.32 on the function $f_\sigma : \mathrm{Reali}(\sigma, Z(\mathcal{Q})) \to (0, \infty)$ defined by $f(x) = \min_{P \in \sigma^{-1}(\{-1,1\})} |P(x_1, \ldots, x_k)|$: This gives that there exists a $t_\sigma$ such that for all $t \in (0, t_\sigma]$ there exists a semialgebraic deformation retraction from $\mathrm{Reali}(\sigma, Z(\mathcal{Q}))$ to $f_\sigma^{-1}([t, \infty))$.

Similarly, by using Proposition 3.37 on the sign condition $\tau = \sigma \cup \{(Q, 0) : Q \in \mathcal{Q}\}$ we find a $u_\sigma$ such that for all $t \in (0, u_\sigma]$ there exists a semialgebraic deformation retraction from $\mathrm{Reali}(\tau, \mathbb{R}^k) = \mathrm{Reali}(\sigma, Z(\mathcal{Q}))$ to $\mathrm{Reali}_t(\tau) = \mathrm{Reali}_t(\sigma) \cap Z(\mathcal{Q})$: In particular the homology of $\mathrm{Reali}(\sigma, Z(\mathcal{Q}))$ is defined as that of $\mathrm{Reali}_{u_\sigma}(\sigma) \cap Z(\mathcal{Q})$, so $b_i(\sigma) = b_i(\mathrm{Reali}_t(\sigma) \cap Z(\mathcal{Q}))$ for all $t \leq u_\sigma$.

Let $u$ be the minimum of the $t_\sigma$ and $u_\sigma$ over all sign conditions $\sigma$ on $\mathcal{P}$. Write $\mathcal{P} = \{P_1, P_2, \ldots, P_s\}$, and define

$$S_j = \{x \in Z(\mathcal{Q}) : P_j(x)^2(P_j(x)^2 - u^2) \geq 0\}$$

for each $1 \leq j \leq s$. Let $S$ be the intersection of the $S_j$. We will show $S$ has the $b_i(\mathcal{Q}, \mathcal{P})$ as its Betti numbers, and then bound the $b_i(S)$.

**Lemma 3.44.** *For all $i$*

$$b_i(S) = b_i(\mathcal{Q}, \mathcal{P}).$$

*Proof.* $S$ is the union of the disjoint, closed, and semialgebraic sets $f_\sigma^{-1}([u, \infty))$ as $\sigma$ varies over all sign conditions on $\mathcal{P}$. By using induction and Proposition 3.40 we find that the Betti numbers of a union of disjoint closed semialgebraic sets are the sums of the corresponding Betti numbers of those sets, so $b_i(S) = \sum_\sigma b_i(f_\sigma^{-1}([u,\infty)))$. As both the $\mathrm{Reali}_u(\sigma) \cap Z(\mathcal{Q})$ and $f_\sigma^{-1}([u,\infty))$ are closed semialgebraic deformation retracts of $\mathrm{Reali}(\sigma, Z(\mathcal{Q}))$, they are semialgebraically homotopy equivalent, so it follows from Proposition 3.36 that they have isomorphic homology groups, so they also have the same Betti numbers. This implies that

$$b_i(S) = \sum_\sigma b_i(f_\sigma^{-1}([u,\infty))) = \sum_\sigma b_i(\mathrm{Reali}_u(\sigma) \cap Z(\mathcal{Q})) = \sum_\sigma b_i(\sigma) = b_i(\mathcal{Q}, \mathcal{P})$$

as required. □

We need some intermediate sets to bound the $b_i(S)$. Define

$$T_j = \{x \in Z(\mathcal{Q}) : P_j(x)^2(P_j(x)^2 - u^2) = 0\}.$$

For nonempty $J \subseteq \{1, 2, \ldots, s\}$ define $V_J = \bigcup_{j \in J} T_j$ and $W_J = \bigcup_{j \in J} S_j$.

**Lemma 3.45.** *For all $i \in \mathbb{Z}$*

$$b_i(V_J) \leq (4^{|J|} - 1)d(2d-1)^{k-1}.$$

*Proof.* Using (8) on the $T_j$ of which $V_J$ is the union, we get

$$b_i(V_J) \leq \sum_{j=1}^{i+1} \sum_{\substack{K \subseteq J \\ |K| = j}} b_{i+1-j}\left(\bigcap_{k \in K} T_k\right).$$

Clearly we can bound this from above by

$$\sum_{K \subseteq J} b\left(\bigcap_{k \in K} T_k\right)$$

writing $b(Z)$ for the sum of all Betti numbers of $Z$. For each $\ell \leq |J|$ this summation includes $\binom{|J|}{\ell}$ terms corresponding to $\ell$-ary intersections of $T_k$. Each $T_k$ splits as a disjoint union of the three pieces where $P_k(X) = 0$, $P_k(X) = u$, and $P_k(X) = -u$, so these intersections split as disjoint unions of $3^\ell$ algebraic sets. Using Theorem 3.41 we get that for each of these algebraic sets the sum of its Betti numbers is at most $d(2d-1)^{k-1}$, so their disjoint union has a sum of Betti numbers of at most $3^\ell d(2d-1)^{k-1}$. Therefore we have

$$b_i(V_J) \leq \sum_{K \subseteq J} b\left(\bigcap_{k \in K} T_k\right) \leq \sum_{j=1}^{|J|} \binom{|J|}{j} 3^j d(2d-1)^{k-1} \leq (4^{|J|} - 1)d(2d-1)^{k-1}.$$

□

**Lemma 3.46.** *For all $i \in \mathbb{Z}$ we have*

$$b_i(W_J) \le (4^{|J|} - 1)d(2d-1)^{k-1} + b_i(Z(\mathcal{Q})).$$

*Proof.* Consider the set

$$F_J = \{x \in Z(\mathcal{Q}) : \bigwedge_{j \in J} P_j(x)^2(P_j(x)^2 - u^2) \le 0 \vee \bigvee_{j \in J} P_j(x)^2(P_j(x)^2 - u^2) = 0\}.$$

We see that $F_J \cup W_J = Z(\mathcal{Q})$ and $F_J \cap W_J = V_J$, so Proposition 3.40 gives that

$$b_i(W_J) \le b_i(W_J) + b_i(F_J) \le b_i(F_J \cup W_J) + b_i(F_J \cap W_J) = b_i(Z(\mathcal{Q})) + b_i(V_J)$$

and the required inequality now follows from Lemma 3.45. □

To finish the proof of Theorem 3.42, we now prove that $b_i(S) \le \sum_{i=0}^{k'-i} d(2d-1)^{k-1}\binom{s}{i}4^i$. Using (9) on the $S_i$ gives that

$$b_i(S) \le \sum_{j=0}^{k'-i} \binom{s}{j} b_{k'}(Z(\mathcal{Q})) + \sum_{\substack{J \subseteq \{1,2,\ldots,s\} \\ |J| = j}} b_{i+j-1}\left(\bigcup_{j \in J} S_j\right)$$

$$= \sum_{j=0}^{k'-i} \binom{s}{j} b_{k'}(Z(\mathcal{Q})) + \sum_{\substack{J \subseteq \{1,2,\ldots,s\} \\ |J| = j}} b_{i+j-1}(W_j).$$

By Lemma 3.46 this is at most

$$\sum_{j=0}^{k'-i} \binom{s}{j} (b_{k'}(Z(\mathcal{Q})) + (4^j - 1)d(2d-1)^{k-1} + b_{i+j-1}(Z(\mathcal{Q}))).$$

As $i + j - 1 < k'$, by Theorem 3.41 we have that

$$b_{k'}(Z(\mathcal{Q})) + b_{i+j-1}(Z(\mathcal{Q})) \le b(Z(\mathcal{Q})) \le d(2d-1)^{k-1}.$$

Together with Lemma 3.44 this gives us that

$$b_i(\mathcal{Q}, \mathcal{P}) = b_i(S) \le \sum_{j=0}^{k'-i} \binom{s}{j} ((4^j - 1)d(2d-1)^{k-1} + d(2d-1)^{k-1}) = \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}$$

completing the proof. □

Theorem 3.42 is one of two key results we need to construct our bound for the Barroero-Widmer constant. For the other result, we first need some way to measure the complexity of semialgebraic sets.

**Definition 3.47.** The language $\mathcal{L}_{\mathbb{R}}$ of ordered fields using coefficients from $\mathbb{R}$ consists of the following formulas:

- All formulas of the form $P(X_1, X_2, \ldots, X_n) = 0$ or $P(X_1, X_2, \ldots, X_n) > 0$ with $P$ a polynomial in $\mathbb{R}[X_1, X_2, \ldots, X_n]$ and $n$ a non-negative integer; these are known as the atoms.
- If $\Phi$ and $\Psi$ are formulas in $\mathcal{L}_{\mathbb{R}}$, then so are $\neg\Phi$, $\Phi \vee \Psi$, $\Phi \wedge \Psi$, $\Phi \implies \Psi$, and $\Phi \Leftrightarrow \Psi$.
- If $\Phi$ is a formula in $\mathcal{L}_{\mathbb{R}}$ with free variable $X$, then $\exists X \Phi(X)$ and $\forall X \Phi(X)$ are also formulas in $\mathcal{L}_{\mathbb{R}}$.

**Definition 3.48.** We define a (semialgebraic) description to be quantifier-free formula $\Phi(X_1, X_2, \ldots, X_r)$ in the language $\mathcal{L}_{\mathbb{R}}$. For a semialgebraic description $\Phi(X_1, X_2, \ldots, X_r)$ we define $\text{Reali}(\Phi, \mathbb{R}^r)$ to be the set
$\{x \in \mathbb{R}^r : \Phi(x)\}$. We define $p_\Phi$ as the number of unique polynomials $P_i$ appearing in $\Phi$, and $d_\Phi$ as the maximum of the degrees of the $P_i$.

If $A$ is a semialgebraic subset of $\mathbb{R}^r$, we call a formula $\Phi$ a semialgebraic description of $A$ if it is a semialgebraic description with $\text{Reali}(\Phi, \mathbb{R}^r) = A$. If we are looking at a fixed semialgebraic description $\Phi_A$ of $A$, we usually write $p_A$ and $d_A$ instead of $p_{\Phi_A}$ and $d_{\Phi_A}$.

From the definition of a semialgebraic set (Definition 1.4) it is clear that every semialgebraic set has a description. Recall that for any two semialgebraic descriptions $\Phi$ and $\Omega$ we have that $\Phi$ is equivalent to $\Omega$ over $\mathbb{R}$ if and only if $\text{Reali}(\Phi, \mathbb{R}^r) = \text{Reali}(\Omega, \mathbb{R}^r)$.

While semialgebraic descriptions are easy to modify for simple set operations, we encounter trouble when taking projections. Projections introduce existential quantifiers, which are not allowed in descriptions. However, we can still deal with this at the cost of increasing the number of polynomials and their degrees exponentially in terms of the number of eliminated quantifiers. This is where the second key result comes in:

**Theorem 3.49** (Local Quantifier Elimination).
*Let $\Omega(y_1, \ldots, y_\ell) = (Q_1 x^{[1]}) \ldots (Q_\nu x^{[\nu]})\Phi(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ be a first-order formula where $\Phi$ is a semialgebraic description, each $Q_i$ a quantifier, each $x^{[i]}$ a block of $k_i$ variables, and $k = \sum_{i=1}^{\nu} k_i$. Let $A = \text{Reali}(\Omega, \mathbb{R}^\ell)$. Then there exists a description $\Omega'$ of $A$ with $p_{\Omega'} \leq p_\Omega^{\prod_{i=1}^{\nu}(k_i+1)} d_\Omega^{\ell \prod_{i=1}^{\nu} O(k_i)}$ and $d_{\Omega'} \leq d_\Omega^{\prod_{i=1}^{\nu} O(k_i)}$.*

*Proof.* We use the Local Quantifier Elimination algorithm [2, Algorithm 14.28] on $\Omega$. This gives an output formula $\Omega'$ which the algorithm guarantees to be a semialgebraic description equivalent to $\Omega$ over $\mathbb{R}$. As $\Omega'$ is equivalent to $\Omega$ over $\mathbb{R}$, we find that $\text{Reali}(\Omega', \mathbb{R}^\ell) = A$, and the complexity discussion afterwards proves the bounds on $p_{\Omega'}$ and $d_{\Omega'}$. □

As the Local Quantifier Elimination algorithm requires a significant amount of further theory to fully understand, we do not explain it in detail. The main idea is this: Similar to how you can reduce the problem of cell decomposition step by step, reducing the dimension by one each step, like we do in the proof of Theorem 3.2, this algorithm reduces the problem one quantifier block at a time. For larger blocks this can be significantly more efficient. This is already accomplished in the Quantifier Elimination algorithm found at [2][p. 591, Algorithm 14.21]. However, Local Quantifier Elimination then uses the structure of $\Omega$ in building the new formula rather than giving a fixed structure to the output, which allows it to reduce the exponent of $p_\Omega$ in the bound for $p_{\Omega'}$ by a factor $\ell + 1$.

# 4 Asymptotic bounds

We now return to the central theorem of this thesis: The Barroero-Widmer Theorem.

**Theorem BW-1.3.** *Fix an o-minimal structure $D$. Let $m$ and $n$ be positive integers, let $\Lambda \subset \mathbb{R}^n$ be a lattice with successive minima $\lambda_1, \dots, \lambda_n$, and let $Z \subseteq \mathbb{R}^{m+n}$ be a definable family of which the fibers $Z_T$ are bounded for all $T \in \mathbb{R}^m$. Then there exists a constant $c_Z \in \mathbb{R}$ depending only on $Z$ such that*

$$\left| |Z_T \cap \Lambda| - \frac{\mathrm{Vol}_n(Z_T)}{d(\Lambda)} \right| \leq c_Z \sum_{j=0}^{n-1} \frac{V_j(Z_T)}{\prod_{i=1}^{j} \lambda_i}.$$

In this section, we are going to give an explicit asymptotic bound for the constant $c_Z$ in the case that $D$ is the structure of semialgebraic sets. Recall from (7) that in the case where the fibers $Z_T$ are closed we can set

$$c_Z = \max_{0 \leq j \leq n-1} M_Z^{n-j} \binom{n}{j} \frac{\left(2 \cdot j^{3/2} \cdot n! 2^n\right)^j B_Z}{B_j \cdot B_n^j}$$

with $B_j$ respectively $B_n$ the volume of the $j$- respectively $n$-dimensional unit sphere, $M_Z$ a constant obtained from the proof of Lemma BW-4.1, and $B_Z$ a constant obtained from the proof of Proposition BW-6.1. Furthermore, recall that in the general case we have $c_Z = c_{C(Z)} + c_{B(Z)}$, with $C(Z)$ the fiberwise closure and $B(Z)$ the fiberwise boundary of $Z$. To give an asymptotic bound for $c_Z$ in the general case, we will therefore look to bound $M_Z$ and $B_Z$ in terms of $Z$, and then replace $Z$ with $C(Z)$ and $B(Z)$ in these bounds.

From now on, assume we have been given a semialgebraic description of $Z$. We will construct bounds on $M_Z$, $B_Z$ and $c_Z$ in terms of $p_Z$, $d_Z$, $m$ and $n$, which will result in the following theorem:

**Theorem 4.1.** *The Barroero-Widmer constant $c_Z$ is at most*

$$p_Z^{O(n^4)} \max(d_Z, 2)^{O(n(n^3+mn)^3)}.$$

*Proof.* This bound will be established by using the description of $Z$ to get descriptions of various intermediate sets. As in describing the very first of these sets (either $B(Z)$ and $C(Z)$ or $V$, depending on whether or not $Z$ has closed fibers) we have to add various quadratic polynomials, in the following we will assume that $d_Z \geq 2$ for simplicity of notation.

## 4.1 Bounding $M_Z$

To get a bound for $M_Z$, we first recall that we identify an endomorphism $\psi \in \mathrm{End}(\mathbb{R}^n)$ with the $n^2$ coefficients of the matrix representing it with respect to the standard basis. Recall the set $W = \{(\psi, T, x) \in \mathbb{R}^{n^2+m+n} : x \in \psi(Z_T)\}$, which is an $n$-fold projection of $V = \{(\psi, T, x, y) \in \mathbb{R}^{n^2+m+2n} : x = \psi(y), y \in Z_T\}$. From the proof of Lemma BW-4.1 we know that for $M_Z$ we can take an upper bound for the number of connected components of all fibers $W_{(\psi,T,x_1,\dots,x_{n-1})}$. Hence the following proposition gives an upper bound for $M_Z$:

**Proposition 4.2.** *Each fiber of the form $W_{(\psi,T,x_1,\dots,x_{n-1})}$ has at most $(p_Z+n)^{n+1}d_Z^{O(n^3+mn)}$ connected components.*

*Proof.* We first construct a semialgebraic description of $W$. By assumption we have a description $\Phi_Z$ of $Z$. Then we can define a description $\Phi_V$ of $V$ as

$$\Phi_V(\psi, T, x, y) = \Phi_Z(T, y) \wedge \bigwedge_{i=1}^{n} \left( x_i - \sum_{j=1}^{n} \psi_{ij} y_j = 0 \right).$$

Clearly this description satisfies $p_V = p_Z + n$ and $d_V = \max(d_Z, 2)$. To get from $V$ to $W$ we use Local Quantifier Elimination (Theorem 3.49) to give us a description of $W$ with

$$p_W \leq (p_Z+n)^{n+1} d_Z^{(n^2+n+m)O(n)} = (p_Z+n)^{n+1} d_Z^{O(n^3+mn)}, \tag{10}$$

$$d_W \leq d_Z^{O(n)}. \tag{11}$$

To finish the proof we need the following lemma.

**Lemma 4.3.** *Let $A$ be a semialgebraic subset of $\mathbb{R}$ with description $\Phi_A$. Then $A$ has at most $p_A d_A + 1$ connected components.*

*Proof.* Let $N$ be the set of roots of the nonzero polynomials used in $\Phi_A$. Then clearly $|N| \leq p_A d_A$. As these polynomials are continuous, they cannot change sign except at these roots. The sign of the zero polynomial is constant, so the signs of all the polynomials are constant on each of the at most $p_A d_A + 1$ open intervals that $\mathbb{R} \setminus N$ is split into by the roots. In particular the truth of $\Phi_A$ will not change on these intervals. As it will not change on the $p_A d_A$ roots in $N$ either, we find that $A$ is the union of some of these intervals and roots, of which there are at most $2p_A d_A + 1$. However, if consecutive pieces are chosen, they will belong to the same connected component. Hence $A$ has at most $p_A d_A + 1$ connected components. □

This bound is sharp, as equality holds in the case where $|N| = p_A d_A$ and $A = \mathbb{R} \setminus N$. By substituting the parameters into the defining polynomials, our description of $W$ is actually a description of each fiber $W_{(\psi,T,x_1,\dots,x_{n-1})}$ as well. Using the lemma on the fibers shows that each fiber has at most

$$p_W d_W + 1 \leq (p_Z+n)^{n+1} d_Z^{O(n^3+mn)+O(n)} + 1 = (p_Z+n)^{n+1} d_Z^{O(n^3+mn)}$$

connected components as required. □

## 4.2 Bounding $B_Z$

Getting a bound for $B_Z$ takes some additional steps, but the central principle remains the same: We construct descriptions for the various sets involved in the proof of Proposition BW-6.1.

**Lemma 4.4.** *We can take a value of $B_Z$ that is at most*

$$n^n(p_Z+n)^{8n^3}d_Z^{O((n^3+mn)^3)}.$$

*Proof.* We start with $S=\{(\phi,T,x)\in\mathbb{R}^{n^2+m+n}:\phi\in O_n(\mathbb{R}),x\in\phi(Z_T)\}$. This is simply $W$ with the requirement that $M_\phi M_\phi^\top=I_n$, with $M_\phi$ the matrix $(\phi_{ij})_{1\leq i,j\leq n}$. This can be expressed in $\frac{n(n+1)}{2}$ quadratic equations as this is an equation of symmetric matrices, and taking the conjunction of $\Phi_W$ with them gives a description $\Phi_S$ of $S$ with $p_S=p_W+\frac{n(n+1)}{2}$ and $d_S=d_W$. As $\frac{n(n+1)}{2}<2^{n(n+1)}\leq d_Z^{O(n^3+mn)}$ and $d_Z^{O(n^3+mn)}=2\cdot d_Z^{O(n^3+mn)}$, we find using (10) and (11) that

$$p_S\leq(p_Z+n)^{n+1}d_Z^{O(n^3+mn)}+d_Z^{O(n^3+mn)}=(p_Z+n)^{n+1}d_Z^{O(n^3+mn)},\tag{12}$$

$$d_S=d_W\leq d_Z^{O(n)}.\tag{13}$$

We next move to the graphs of the definable choice functions $f_j:\pi_{n-j}(S)\to\mathbb{R}^{n-j}$.

**Lemma 4.5.** *We can construct a semialgebraic description of $\Gamma(f_j)$ with*

$$p_{\Gamma(f_j)}\leq np_S^{8(n-1)}d_S^{(n^2+m+n)^2O(n)},$$
$$d_{\Gamma(f_j)}\leq d_S^{O(n)}.$$

*Proof.* As discussed after the proof of Proposition 1.20, $(\phi,T,y)\in\Gamma(f_j)$ is equivalent to $\bigwedge_{1\leq i\leq n-j}\bigvee_{1\leq k\leq 5}\Phi_{i,k}(\phi,T,y)$. However, this is not quite a description yet, as the $\Phi_{i,k}$ contain quantifiers and refer to projections $\pi_{i-1}(S)$. To solve the latter problem, using Local Quantifier Elimination to eliminate the projections gives a description of each $\pi_{i-1}(S)$ with

$$p_{\pi_{i-1}(S)}\leq p_S^{i-1}d_S^{(n^2+m+n+1-i)O(i)}\leq p_S^{n-1}d_S^{(n^2+m+n)O(n)},$$
$$d_{\pi_{i-1}(S)}\leq d_S^{O(i)}\leq d_S^{O(n)}$$

by Theorem 3.49. We then substitute these descriptions into the $\Phi_{i,k}$ to get equivalent formulae $\Phi'_{i,k}$.

Using basic logical manipulations we can move the remaining quantifiers in each $\Phi'_{i,k}$ to the front to get equivalent formulas $\Phi''_{i,k}$. When $k\in\{1,2,4\}$ this gives a single quantifier $\forall z$, for $k=3$ we get the two quantifiers $\forall z\exists x$, and for $k=5$ we get the three quantifiers $\exists u_T\forall z\exists x$. For example, we take

$$\Phi''_{i,5}=\exists u\forall z\exists x:((z-u\geq 0\wedge z\notin\pi_{i-1}(S)_{(\phi,T,y_1,\dots,y_{n-i})})$$

$$\vee(z - u > 0 \wedge 2y_{n-i+1} - u - z > 0 \wedge z \in \pi_{i-1}(S)_{(\phi,T,y_1,\ldots,y_{n-i})})$$
$$\vee(2y_{n-i+1} - u - x < 0 \wedge z - x > 0 \wedge x \notin \pi_{i-1}(S)_{(\phi,T,y_1,\ldots,y_{n-i})})).$$

Local Quantifier Elimination can turn each of these formulas into a semialgebraic description $\Psi_{i,k}$. Each $\Phi''_{i,k}$ includes either $p_{\pi_{i-1}(S)}$ or $2p_{\pi_{i-1}(S)}$ equations and inequalities, as well as a couple of linear equations and inequalities we can safely ignore as $p_{\pi_{i-1}(S)} \geq p_S \geq n^{n+1}$. The worst case is clearly $k = 5$ as it has a triple quantifier, which means

$$\begin{aligned} p_{\Psi_{i,k}} &\leq (2p_{\pi_{i-1}(S)})^8 d_{\pi_{i-1}(S)}^{(n^2+m+n+1-i)O(1)} \leq p_S^{8(n-1)} d_S^{(n^2+m+n)^2 O(n)}, \\ d_{\Psi_{i,k}} &\leq d_{\pi_{i-1}(S)}^{O(1)} \leq d_S^{O(n)}. \end{aligned}$$

As $(\phi, T, y) \in \Gamma(f_j)$ is equivalent to $\bigwedge_{1\leq i\leq n-j} \bigvee_{1\leq k\leq 5} \Psi_{i,k}(\phi, T, y)$, this gives a description of $\Gamma(f_j)$ with

$$\begin{aligned} p_{\Gamma(f_j)} &\leq \sum_{1\leq i\leq n-j, 1\leq k\leq 5} p_{\Psi_{i,k}}, \\ d_{\Gamma(f_j)} &\leq \max_{1\leq i\leq n-j, 1\leq k\leq 5} d_{\Psi_{i,k}}. \end{aligned}$$

As the bounds we just found for the $p_{\Psi_{i,k}}$ and $d_{\Psi_{i,k}}$ are independent of $i$ and $k$, these bounds for $p_{\Gamma(f_j)}$ and $d_{\Gamma(f_j)}$ become

$$\begin{aligned} p_{\Gamma(f_j)} &\leq 5(n-j) p_S^{8(n-1)} d_S^{(n^2+m+n)^2 O(n)} \leq n p_S^{8(n-1)} d_S^{(n^2+m+n)^2 O(n)}, \\ d_{\Gamma(f_j)} &\leq d_S^{O(n)} \end{aligned}$$

finishing the proof. □

To get a description for $Z'_j = \{(\phi, T, x) \in \mathbb{R}^{n^2+m+n} : \phi \in O_n(\mathbb{R}), \phi(x) \in \Gamma(f_j)_{(\phi,T)}\}$ we just need to replace each $x_i$ in the description of $(\phi, T, x) \in \Gamma(f_j)$ with the quadratic expression $\phi(x)_i = \sum_{k=1}^n \phi_{ik} x_k$. Indeed, $(\phi, T, \sum_{k=1}^n \phi_{ik} x_k) \in \Gamma(f_j)$ already implies that $\phi \in O_n(\mathbb{R})$. So we get the following corollary:

**Corollary 4.6.** *We can construct a semialgebraic description of* $Z'_j$ *with*

$$\begin{aligned} p_{Z'_j} &= p_{\Gamma(f_j)}, \\ d_{Z'_j} &= 2d_{\Gamma(f_j)}. \end{aligned}$$

What we need in the end is a bound on the number of connected components of the $(\phi, T, y)$ fibers of the sets $R^I = \{(\phi, T, y, x) \in \mathbb{R}^{n^2+m+j+n} : \phi \in O_n(\mathbb{R}), \phi(x) \in \Gamma(f_j)_{(\phi,T)}, y = \pi_I(x)\}$ where $I \subseteq \{1, 2, \ldots, n\}$ and $|I| = j$. We first prove that such a bound can be generated independently of the choice of $I$.

**Lemma 4.7.** *For each* $1 \leq j \leq n-1$ *and each* $I \subseteq \{1, 2, \ldots, n\}$ *with* $|I| = j$ *the* $(\phi, T, y)$-*fibers of* $R^I$ *are homeomorphic to* $(\phi, T, x_1, \ldots, x_j)$-*fibers of a semialgebraic set* $U^I$ *with* $p_{U^I} = p_{Z'_j}$ *and* $d_{U^I} = d_{Z'_j}$.

*Proof.* Consider the permutation $\sigma_I$ of $\{1, 2, \ldots, n\}$ that sends $\{1, 2, \ldots, j\}$ to the elements of $I$ in increasing order, while sending $\{j+1, \ldots, n\}$ to the complement of $I$ in increasing order. Now define $U^I = \{(\phi, T, x) \in \mathbb{R}^{n^2+m+n} : (\phi, T, \sigma_I(x)) \in Z'_j\}$. Define the function $f_I : U^I \to R^I$ given by $f_I(\phi, T, x) = (\phi, T, (x_i)_{i \leq j}, (x_{\sigma(i)})_{1 \leq i \leq n})$; this is well-defined by definition of $\sigma_I$.

As $f_I$ simply duplicates and reorders the $x$-coordinates, it is a homeomorphism. Its restrictions to fibers $U^I_{(\phi,T,x_1,\ldots,x_j)}$ are homeomorphisms mapping these to the fibers $R^I_{(\phi,T,y)}$. Hence it suffices to look at the fibers $U^I_{(\phi,T,x_1,\ldots,x_j)}$. Note that for each $I$ we have that $p_{U^I} = p_{Z'_j}$ and $d_{U^I} = d_{Z'_j}$, as we can create a description for $U^I$ by permuting the variables of a description of $Z'_j$. Then each fiber of the form $U^I_{(\phi,T,c_1,\ldots,c_j)}$ is homeomorphic to the corresponding fiber $R^I_{(\phi,T,c_1,\ldots,c_j)}$ of $R^I$: after all, we can explicitly give such a homeomorphism $f_I : R^I_{(\phi,T,c_1,\ldots,c_j)} \to U^I_{(\phi,T,c_1,\ldots,c_j)}$ by $f_I(x_1, x_2, \ldots, x_n) = (x_i)_{i \notin I}$, which simply is deleting the coordinates that are constant in the $R^I$-fiber. □

Lemma 4.7 implies we can use bounds for the number of connected components of the $U^I$-fibers instead, and as the descriptions of the $U^I$ have fixed size and degree for all $I$ with $|I| = j$, we will get a single bound $E_j$. The fibers $U^I_{(\phi,T,x_1,\ldots,x_j)}$ we are interested in are subsets of $\mathbb{R}^{n-j}$ defined by sign conditions on $p_{Z'_j}$ polynomials of degree at most $d_{Z'_j}$. So Theorem 3.42 gives us that the number of connected components of such a fiber is at most

$$\sum_{0 \leq i \leq n-j} \binom{p_{Z'_j}}{i} 4^i d_{Z'_j} (2d_{Z'_j} - 1)^{n-j-1} \leq (8 p_{Z'_j} d_{Z'_j})^{n-j} \leq (8 p_{Z'_j} d_{Z'_j})^{n-1}.$$

Using the estimates from Corollary 4.6, Lemma 4.5, and inequalities (12) and (13), and collecting the constant factors in the big-$O$ exponents, we find that we can take a value of $E_j$ that is at most

$$8^{n-1} p_{Z'_j}^{n-1} d_{Z'_j}^{n-1} \leq n^{n-1} p_S^{8(n-1)^2} d_S^{8(n^2+m+n)^2(n-1)O(n)} d_S^{(n-1)O(n)}$$

$$\leq n^n p_S^{8(n-1)^2} d_S^{O((n^3+mn)^2)} \leq n^n (p_Z + n)^{8n^3} d_Z^{O((n^3+mn)^3)}.$$

As this bound applies to all $E_j$ and $B_Z$ is the maximum of the $E_j$, it follows that

$$B_Z \leq n^n (p_Z + n)^{8n^3} d_Z^{O((n^3+mn)^3)}$$

completing the proof. □

## 4.3 Bounding $c_Z$

As all bounds so far are still using the assumption that $Z$ has closed fibers, we now have to move to the fiberwise closure and fiberwise boundary.

**Proposition 4.8.** *Let $Z \subseteq \mathbb{R}^{m+n}$ be a semialgebraic family with $d_Z \geq 2$. Let $C(Z)$ be its fiberwise closure and $B(Z)$ its fiberwise boundary. Then*

$$M_{C(Z)} \leq p_Z^{2(n+1)^2} d_Z^{O(n(n^3+mn))},$$
$$B_{C(Z)} \leq n^n p_Z^{16(n+1)n^3} d_Z^{O(n(n^3+mn)^3)},$$

*and the same bounds hold for $M_{B(Z)}$ and $B_{B(Z)}$.*

*Proof.* We first construct semialgebraic descriptions of $C(Z)$ and $B(Z)$.

**Lemma 4.9.** *Let $Z \subseteq \mathbb{R}^{m+n}$ be a semialgebraic family. There exist semialgebraic descriptions for the fiberwise closure $C(Z)$ and the fiberwise boundary $B(Z)$ such that*

$$p_{C(Z)} = p_{B(Z)} \leq p_Z^{2(n+1)} d_Z^{O(n(m+n))},$$
$$d_{C(Z)} = d_{B(Z)} \leq d_Z^{O(n)}.$$

*Proof.* $C(Z)$ is the set

$$\{(T, x) \in \mathbb{R}^{m+n} : \forall z \in \mathbb{R}_{>0} \exists y \in \mathbb{R}^n : \|x - y\| < z \wedge (T, y) \in Z\}$$

$$= \{(T, x) \in \mathbb{R}^{m+n} : \forall z \in \mathbb{R} \exists y \in \mathbb{R}^n : z \leq 0 \vee (\|x - y\| < z \wedge (T, y) \in Z\}.$$

Hence $C(Z)$ is the realization of a formula with two blocks of quantifiers of sizes 1 and $n$ which uses $p_Z + 2$ polynomials of degree at most $\max(d_Z, 2) = d_Z$. Using Local Quantifier Elimination (Theorem 3.49) we find that there is a description of $C(Z)$ with

$$p_{C(Z)} \leq (p_Z + 2)^{2(n+1)} d_Z^{O(n(m+n))},$$
$$d_{C(Z)} \leq d_Z^{O(n)}.$$

As $d_Z \geq 2$, $p_Z \geq 1$, we have that

$$(p_Z + 2)^{2(n+1)} d_Z^{O(n(m+n))} \leq (3p_Z)^{2(n+1)} d_Z^{O(n(m+n))} = p_Z^{2(n+1)} 9^{n+1} d_Z^{O(n(m+n))}.$$

The factor $9^{n+1}$ can be absorbed into the power of $d_Z$, so

$$p_{C(Z)} \leq p_Z^{2(n+1)} d_Z^{O(n(m+n))}.$$

This finishes the proof of the bounds for $C(Z)$. $Z$ and its complement $\overline{Z}$ are described by the same polynomials, so the same bounds apply for $C(\overline{Z})$. As $B(Z) = C(Z) \cap C(\overline{Z})$ we find $p_{B(Z)} \leq 2p_{C(Z)}$ and $d_{B(Z)} = d_{C(Z)}$. Our bound for $p_{C(Z)}$ contains a factor $d_Z^{O(n(m+n))}$, so the factor 2 falls out there, finishing the proof. □

Using this description of $C(Z)$ in the bound for $M_Z$ we constructed, we find that

$$M_{C(Z)} \leq (p_Z^{2(n+1)} d_Z^{O(n(m+n))} + n)^{n+1} d_Z^{O(n(n^3+mn))}.$$

As $n \leq 2^n \leq p_Z^{2(n+1)} d_Z^{O(n(m+n))}$ and $2 \cdot p_Z^{2(n+1)} d_Z^{O(n(m+n))} = p_Z^{2(n+1)} d_Z^{O(n(m+n))}$, we can just remove the term $n$ in the base of the first factor. This gives

$$M_{C(Z)} \leq p_Z^{2(n+1)^2} d_Z^{O(n(m+n)(n+1))} d_Z^{O(n(n^3+mn))} = p_Z^{2(n+1)^2} d_Z^{O(n(n^3+mn))}.$$

We similarly find that

$$B_{C(Z)} \leq n^n (p_Z^{2(n+1)} d_Z^{O(n(m+n))} + n)^{8n^3} d_Z^{O(n(n^3+mn)^3)} = n^n (p_Z^{2(n+1)} d_Z^{O(n(m+n))})^{8n^3} d_Z^{O(n(n^3+mn)^3)}$$

$$= n^n p_Z^{16(n+1)n^3} d_Z^{O(n(n^3+mn)^3)}.$$

As $p_{B(Z)} = p_{C(Z)}$ and $d_{B(Z)} = d_{C(Z)}$, we have the same bounds for $M_{B(Z)}$ and $B_{B(Z)}$. □

Substituting the bounds for $M_{C(Z)}$ and $B_{C(Z)}$ in our earlier expression (7) for $c_Z$, we find that

$$c_{C(Z)} \leq \max_{0 \leq j \leq n-1} M_{C(Z)}^{n-j} \binom{n}{j} \frac{(j^{3/2} n! 2^{n+1})^j B_{C(Z)}}{B_n^j B_j}$$

$$\leq \max_{0 \leq j \leq n-1} p_Z^{2(n-j)(n+1)^2} d_Z^{(n-j)O(n(n^3+mn)} \binom{n}{j} \frac{(j^{3/2} n! 2^{n+1})^j n^n p_Z^{16(n+1)n^3} d_Z^{O(n(n^3+mn)^3)}}{B_n^j B_j}$$

$$\leq \max_{0 \leq j \leq n-1} n^n p_Z^{16n^4+18n^3+4n^2+2n} d_Z^{O(n(n^3+mn)^3)} \cdot \frac{j^{3j/2} (n!)^{j+1} 2^{(n+1)j}}{j!(n-j)! B_n^j B_j}.$$

We can estimate $B_i$, the volume of the $i$-dimensional unit ball, from below by taking the volume of the hyperoctahedron $\{x \in \mathbb{R}^i : \sum_{k=1}^i |x_k| \leq 1\}$ inscribed in that ball. The hyperoctahedron has volume $\frac{2^i}{i!}$, which means that $c_{C(Z)}$ is at most

$$\max_{0 \leq j \leq n-1} n^n p_Z^{O(n^4)} d_Z^{O(n(n^3+mn)^3)} \cdot \frac{j^{3j/2} (n!)^{2j+1}}{(n-j)!}.$$

Estimating the numerator of the fraction from above by setting $j = n$, we find this is at most

$$\max_{0 \leq j \leq n-1} n^n p_Z^{O(n^4)} d_Z^{O(n(n^3+mn)^3)} \cdot \frac{n^{3n/2} (n!)^{2n+1}}{(n-j)!} = n^n p_Z^{O(n^4)} d_Z^{O(n^4(n^2+m)^3)} \cdot n^{3n/2} (n!)^{2n}.$$

Noting that $n! \leq n^n$ and $n \leq 2^n \leq d_Z^n$, this is bounded from above by

$$p_Z^{O(n^4)} d_Z^{O(n(n^3+mn)^3)} d_Z^{n^2+3n^2/2+2n^3} = p_Z^{O(n^4)} d_Z^{O(n(n^3+mn)^3)}.$$

As the bounds we have for $M_{B(Z)}$ and $B_{B(Z)}$ are the same, we also have $c_{B(Z)} \leq p_Z^{O(n^4)} d_Z^{O(n(n^3+mn)^3)}$. A factor 2 can be absorbed into the power of $d_Z$, so the same bound holds for $c_Z = c_{C(Z)} + c_{B(Z)}$, proving Theorem 4.1. □

We can also derive the following corollary:

**Corollary 4.10.** *If the fibers $Z_T$ are closed, we can reduce the bound from Theorem 4.1 to*

$$p_Z^{O(n^3)} \max(d_Z, 2)^{O((n^3+mn)^3)}.$$

*Proof.* This follows directly from the proof of Theorem 4.1: If the fibers $Z_T$ are known to be closed, we do not need to calculate bounds for $c_{C(Z)}$ and $c_{B(Z)}$, but can instead bound $c_Z$ directly. As replacing $M_Z$ and $B_Z$ with $M_{C(Z)}$ and $B_{C(Z)}$ increases the exponents of $p_Z$ and $d_Z$ in our bounds by a factor $n$, skipping this step reduces the final exponents of $p_Z$ and $d_Z$ in the bound for $c_Z$ by a factor $n$, which gives the bound in this corollary. □

# Conclusion

In this thesis we have proven an asymptotic bound for the Barroero-Widmer constant that is only singly exponential in $m$ and $n$, which is somewhat remarkable given that algorithms in this field have a tendency to be doubly exponential. Furthermore, if an actual semi-algebraic family is provided, we can calculate an exact bound, as the algorithms used in computing our bound are explicit, and hence descriptions for all sets involved can actually be constructed given a description of the family $Z$.

Of course, that does not mean there is no room for improvement. The asymptotic bound could be tightened by either using better estimates or more efficient algorithms. Alternatively, an exact bound may be found by replacing Local Quantifier Elimination with an algorithm with a known exact complexity bound. Furthermore, our bound only works for semialgebraic families, and there are plenty of other o-minimal structures to consider. But hopefully this thesis provides a good starting point for further investigation, as well as a bound that is already useful in its own right.

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