Algebraic subproduct systems of the Hopf algebra O(SUq(2))
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THESIS

submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
MATHEMATICS

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Leiden, The Netherlands, July 12, 2022

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July 12, 2022

Abstract

We study the Hopf algebra $\mathcal{O}(SU_q(2))$, which is a q-deformed analogue of the Lie group $SU(2)$. In particular, we study its co-representations, and show how to construct a subproduct system E_m out of the determinant $\det(\rho)$ of a co-representation ρ. We also show how this determinant can be constructed from a braiding σ. We also study the quadratic algebra $\bigoplus_{m=0}^{\infty} E_m$ that is constructed from the subproduct system using the non-commutative Nullstellensatz, and calculate its Hilbert series.

Contents

Introduction

One of the major breakthroughs in physics of the past century has been quantum mechanics, which has since found many applications in the technology we use today.

However, finding and solving the quantum mechanical equations that describe specific systems has proven to be notoriously difficult to do analytically.

The theory of Lie Groups and Lie Algebras has been essential in describing and analytically solving many quantum mechanical systems, the prototypical examples of which are the famous works by Pauli and Dirac, where the spin of an electron is described by a representation of the Lie Algebra $\mathfrak{su}(2)$.

When trying to describe and solve a specific quantum mechanical system related to magnetism, physicists found the Bethe ansatz. This solution did not directly involve Lie Groups or Lie Algebras, but introduced the concept of a “Quantum Group”. Since then, several related theories of “Quantum Groups” have been developed, several of which involve the concept of a Hopf algebra. A Hopf algebra is a structure that generalises a group, as outlined in section 1.1.

Besides their applications in physics, the theory of Hopf algebras has been of interest to mathematicians for a long time because of their applications in many fields ranging from category theory and algebraic topology to representation theory and, indeed, mathematical physics.

In this thesis, we restrict ourselves to the study of the Hopf algebra $\mathcal{O}(SU_q(2))$, which is a certain q-deformed Hopf-algebra analogue of the well-known Lie Group $SU(2)$.

An extensive description of this Hopf algebra and related Hopf algebras can be found in [5]. A study of Hopf algebras and their interaction with quadratic algebras can be found in [6]. The article [1] describes several properties and constructions from the Lie Group $SU(2)$, including the construction of quadratic algebras related to representations of $SU(2)$. The results in [1] are expanded upon in [3].

Our main goal is to investigate up to what extend some of the constructions in [1] can be generalised to the Hopf algebra $\mathcal{O}(SU_q(2))$. Besides this, we try to show analogues with the theory of quadratic algebras from [6] and compare our results with the results in [3].

Chapter 1 contains the preliminaries that are needed for Chapter 2, which contains the main research and results. In particular, section 1.1 describes how a (Lie) group is related to a Hopf algebra and introduces the algebra $\mathcal{O}(SU_q(2))$, section 1.2 describes how the representation theory of (Lie) groups translates to the Hopf algebra setting, section 1.3 describes how this representation theory applies to $\mathcal{O}(SU_q(2))$ and section 1.4 describes Clebsch–Gordan coefficients of $\mathcal{O}(SU_q(2))$, which wil be one of our most important tools in Chapter 2.

Chapter 2 contains our main results. Section 2.3 describes how the co-representations of $\mathcal{O}(SU_q(2))$ give rise to a subproduct system E_m, in which the determinant $\det(\rho)$ will play a central role. Section 2.2 investigates the properties of this determinant. In particular, section 2.2.2 shows how the determinant can be constructed from a certain solution of the braid equation. Finally, section 2.4 shows how the non-commutative Nullstellensatz (c.f. section 2.1) gives rise to a quadratic algebra starting from E_m, and compares these results to the theory in [6].

Chapter 1

Preliminaries

In this chapter the concept of a Hopf algebra is introduced. After discussing basic definitions, the construction of a Hopf algebra from a Lie Group is described. Then we give a basic introduction to related Hopf algebras and Hopf co-modules (also known as Hopf co-representations). For more background knowledge on Hopf algebras and their (co-)representations, we refer to the books [6], [7], [4] and [5].

1.1 Groups and the related Hopf algebras

In this section, a Hopf algebras will be introduced and the construction of a Hopf algebra from a group is discussed.

Definition 1.1.1. *A* group $(G, \cdot, 1, \iota)$ *is a quadruple with a set* G*, an associative binary operation* $\cdot \colon G \times G \to G$*, a unit* $1 \in G$ *where* $\forall g \in G,\ 1 \cdot g = g = g \cdot 1$*, and an inversion map* $\iota \colon G \to G$ *where* $g \cdot \iota(g) = \iota(g) \cdot g = 1$ *for all* $g \in G$*.*

Definition 1.1.2. *For vector spaces* V, W*, the* twisting map *or* flip $\tau_{V \otimes W} \colon V \otimes W \to W \otimes V$ *is given by* $v \otimes w \mapsto w \otimes v$*.*

Definition 1.1.3. *An (unital, associative)* algebra $(\mathcal{A}, \mu, \eta)$ *is a triple where* $\mathcal{A}$ *is a vector space over a field* $\mathbb{K}$*,* $\mu \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *is an associative bilinear binary operation and* $\eta \colon \mathbb{K} \to \mathcal{A}$ *is a linear map that satisfies* $\mu(\eta(1), a) = \mu(a, \eta(1)) = a$ *for all* $a \in \mathcal{A}$*.*

By the universal property of the tensor product, the bilinear map $\mu \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ factors through a linear map $\mu \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

Example 1.1.4. *The ground field* $\mathbb{K}$ *is an algebra when we set* $\eta = \mathrm{id}_{\mathbb{K}}$ *and the multiplication* $\mu(a, b) = ab$ *is just the multiplication of the field.*

Example 1.1.5. *Let S be a set and $\mathcal{A}$ an algebra. Then $\mathcal{A}^S = \mathrm{Map}(S, \mathcal{A}) = \{f : S \to \mathcal{A}\}$ is an algebra under pointwise operations. So the multiplication is given by $\mu_{\mathcal{A}^S}(f, g)(s) = \mu_{\mathcal{A}}(f(s), g(s))$, the unit is given by $\eta_{\mathcal{A}^S}(\alpha)(s) = \eta_{\mathcal{A}}(\alpha)$, and the vector space structure on $\mathcal{A}^S$ is constructed similarly.*
Of special interest are the indicator functions $\{\mathbf{1}_a\}_{a\in S}$, where $\mathbf{1}_a$ takes the value 1 on a and 0 on all other elements of S. We can formally write $f = \sum_{a\in S} f(a)\mathbf{1}_a$.

Many often encountered algebras arise as subalgebras of $\mathbb{K}^S = \mathrm{Map}(S, \mathbb{K})$. For example, if S is a topological space, we could look at $C(S) \subseteq \mathbb{K}^S$, the space of continuous functions. If S is a differentiable manifold, we could look at $C^\infty(S) \subseteq C(S)$, the space of infinitely differentiable functions. And if S has the structure of an algebraic variety, we could look at $\mathcal{O}_S(S) \subseteq C^\infty(S)$, the space of polynomial/algebraic functions on S.

A Hopf algebra arises when the set S has the additional structure of a group.

Lemma 1.1.6. *Let $(G, \cdot, 1, \iota)$ be a group, and $(\mathcal{A}, \mu, \eta)$ be a subalgebra of $\mathbb{K}^G$.*
The element $1 \in G$ induces a map $\varepsilon : \mathcal{A} \to \mathbb{K}$ given by $\varepsilon(f) = f(1)$.
The map $\iota : G \to G$ induces a map $S : \mathcal{A} \to \mathcal{A}$ given by $S(f) = \sum_{g\in G} f(\iota(g))\mathbf{1}_g$.
The map $\cdot : G \times G \to G$ gives $\Delta : \mathcal{A} \to \mathbb{K}^G \otimes \mathbb{K}^G$, $f \mapsto \sum_{a,b\in G} f(a \cdot b)\mathbf{1}_a \otimes \mathbf{1}_b$.

Assuming the image of Δ is contained in $\mathcal{A} \otimes \mathcal{A} \subseteq \mathbb{K}^G \otimes \mathbb{K}^G$, the data $(\mathcal{A}, \mu, \eta, \Delta, \varepsilon, S)$ satisfy the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mu\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mu} & \mathcal{A} \end{array} \tag{1.1}$$

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{\Delta\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathcal{A}\otimes\mathcal{A} & \xleftarrow{\Delta} & \mathcal{A} \end{array} \tag{1.2}$$

$$\begin{array}{ccccc} \mathbb{K}\otimes\mathcal{A} & \xrightarrow{\eta\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} & \xleftarrow{\mathrm{id}\otimes\eta} & \mathcal{A}\otimes\mathbb{K} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle\cong} & \\ & & \mathcal{A} & & \end{array} \tag{1.3}$$

$$\begin{array}{ccccc} \mathbb{K}\otimes\mathcal{A} & \xleftarrow{\varepsilon\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\varepsilon} & \mathcal{A}\otimes\mathbb{K} \\ & \nwarrow{\scriptstyle\cong} & {\scriptstyle\Delta}\uparrow & {\scriptstyle\cong}\nearrow & \\ & & \mathcal{A} & & \end{array} \tag{1.4}$$

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\quad\eta\quad} & \mathcal{A} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathbb{K}\otimes\mathbb{K} & \xrightarrow{\eta\otimes\eta} & \mathcal{A}\otimes\mathcal{A} \end{array} \tag{1.5}$$

$$\begin{array}{ccc} \mathbb{K} & \xleftarrow{\quad\varepsilon\quad} & \mathcal{A} \\ {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\mu} \\ \mathbb{K}\otimes\mathbb{K} & \xleftarrow{\varepsilon\otimes\varepsilon} & \mathcal{A}\otimes\mathcal{A} \end{array} \tag{1.6}$$

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\quad\mathrm{id}\quad} & \mathbb{K} \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\varepsilon} & \\ & \mathcal{A} & \end{array} \tag{1.7}$$

$$\begin{array}{ccccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\quad\mu\quad} & \mathcal{A} & \xrightarrow{\quad\Delta\quad} & \mathcal{A}\otimes\mathcal{A} \\ \downarrow{\scriptstyle\Delta\otimes\Delta} & & & & {\scriptstyle\mu\otimes\mu}\uparrow \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & & \xrightarrow{\mathrm{id}\otimes\tau_{\mathcal{A}\otimes\mathcal{A}}\otimes\mathrm{id}} & & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \end{array} \tag{1.8}$$

$$\begin{array}{ccccc} & \mathcal{A}\otimes\mathcal{A} & \xrightarrow{S\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} & \\ {\scriptstyle\Delta}\nearrow & & & & \searrow{\scriptstyle\mu} \\ \mathcal{A} & \xrightarrow{\quad\varepsilon\quad} & \mathbb{K} & \xrightarrow{\quad\eta\quad} & \mathcal{A} \\ {\scriptstyle\Delta}\searrow & & & & \nearrow{\scriptstyle\mu} \\ & \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes S} & \mathcal{A}\otimes\mathcal{A} & \end{array} \tag{1.9}$$

The proof of this lemma consists of straightforward computations. A few remarks on this lemma:

- The assumption that the image of Δ is contained in $\mathcal{A} \otimes \mathcal{A}$ is not always true. It is true in the following important cases:
  - $\mathcal{A} = \mathbb{K}^G$,
  - G is a compact topological group and $\mathcal{A} = C(G)$,
  - G is a compact classical Lie group and $\mathcal{A}$ are the polynomial functions on G.

  It is not true in general when the above groups are not compact.

- Commutativity of the diagram (1.1) expresses associativity of the algebra $(\mathcal{A}, \mu, \eta)$ and commutativity of (1.3) expresses its unital property.

- Commutativity of the diagrams (1.2) and (1.4) is referred to as 'co-associativity' and 'co-identity' respectively. A tuple $(\mathcal{A}, \varepsilon, \Delta)$ that satisfies these properties is called a *co-algebra*.

- Commutativity of the diagrams (1.5)-(1.8) expresses how the algebra and co-algebra structures nicely cooperate. A tuple $(\mathcal{A}, \mu, \eta, \Delta, \varepsilon)$ satisfying (1.1)-(1.8) is called a *bialgebra*.

We see that a group gives rise to a bialgebra with additional structure given by the map $S\colon \mathcal{A} \to \mathcal{A}$, which arises from the inversion on the group. A structure with these properties thus generalises a group, giving rise to the following definition:

Definition 1.1.7. *A* Hopf algebra *is a sextuple $(\mathcal{A}, \mu, \eta, \Delta, \varepsilon, S)$ satisfying the 9 relations (1.1)-(1.9).*

Most examples of Hopf algebras in this thesis have the additional structure of a Hopf $*$-algebra. In particular, when G is a group, the Hopf-algebra $\mathcal{A} = \mathbb{C}^G$ with the map $*\colon \mathcal{A} \to \mathcal{A}$, $f^*(a) = (f(a))^*$ (where $z \mapsto z^* \in \mathbb{C}$ is complex conjugation) is a Hopf-*-algebra.

Definition 1.1.8. *A* *-algebra *is an algebra $(\mathcal{A}, \eta, \mu)$ endowed with a involutive antihomomorphism $*\colon \mathcal{A} \to \mathcal{A}$, i.e. an additive map such that*

$$* \circ \mu = \mu \circ (* \otimes *) \circ \tau_{\mathcal{A} \otimes \mathcal{A}} \text{ and } * \circ \eta = \eta \circ *,$$

where the last $*\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation.
A *-co-algebra *is a co-algebra $\mathcal{A}$ with an additive map $*\colon \mathcal{A} \to \mathcal{A}$ such that*

$$\Delta \circ * = (* \otimes *) \circ \Delta \text{ and } \varepsilon \circ * = * \circ \varepsilon$$

A Hopf *-algebra *is a Hopf algebra with both the structure of a *-algebra and a *-co-algebra.*

Example 1.1.9. *Consider the group*

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \alpha\alpha^* + \beta\beta^* = 1 \right\},$$

of 2×2 *complex-valued unitary matrices. This is a classical Lie group, and we can consider* $\mathcal{O}(SU(2))$*, the space of complex polynomials on* $SU(2)$*.*

This is most easily thought of as the ring

$$\mathcal{O}(SU(2)) = \mathbb{C}[\alpha, \beta, \alpha^*, \beta^*]/(\alpha\alpha^* + \beta\beta^* - 1) \tag{1.10}$$

together with the antilinear conjugation map

$$*: \alpha \mapsto \alpha^*, \alpha^* \mapsto \alpha, \beta \mapsto \beta^*, \beta^* \mapsto \beta.$$

The elements $\alpha, \beta, \alpha^*, \beta^*$ *map a matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)$ *to the following values:*

$$\alpha \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a, \qquad \beta^* \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = -b,$$
$$\beta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = c, \qquad \alpha^* \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = d.$$

The function ε *can be seen to map* α, β, β^* *and* α^* *to the following values*

$$\varepsilon(\alpha) = \alpha \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1, \qquad \varepsilon(\beta^*) = \beta^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 0,$$
$$\varepsilon(\beta) = \beta \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 0, \qquad \varepsilon(\alpha^*) = \alpha^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1.$$

Note that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)$ *we have*

$$S(\alpha) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \alpha \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \right) = \alpha \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = d = \alpha^* \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

and thus $S(\alpha) = \alpha^*$*. Similarly we can deduce*

$$S(\alpha) = \alpha^*, \qquad S(\beta^*) = -\beta^*,$$
$$S(\beta) = -\beta, \qquad S(\alpha^*) = \alpha.$$

Furthermore, we can deduce that

$$\begin{aligned}\Delta(\alpha) &= \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SU(2)} \alpha\left(\begin{smallmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{smallmatrix}\right) \mathbf{1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \otimes \mathbf{1}_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)} \\ &= \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SU(2)} (aa' + bc') \mathbf{1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \otimes \mathbf{1}_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)} \\ &= \left(\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)} a \mathbf{1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \right) \otimes \left(\sum_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SU(2)} a' \mathbf{1}_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)} \right) \\ &\quad + \left(\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)} b \mathbf{1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \right) \otimes \left(\sum_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SU(2)} c' \mathbf{1}_{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)} \right) \\ &= \alpha \otimes \alpha + (-\beta^*) \otimes \beta \\ &= \alpha \otimes \alpha - \beta^* \otimes \beta \end{aligned}$$

and in an analogous manner we get $\Delta(\beta) = \alpha^* \otimes \beta + \beta \otimes \alpha$. *The values of* $\Delta(\alpha^*)$ *and* $\Delta(\beta^*)$ *can then be deduced by requiring compatibility with the antilinear conjugation map* $*$*:*

$$\Delta(\beta^*) = (\Delta(\beta))^* = \alpha \otimes \beta^* + \beta^* \otimes \alpha^* \quad \Delta(\alpha^*) = (\Delta(\alpha))^* = \alpha^* \otimes \alpha^* - \beta \otimes \beta^*$$

The above identities can be written shortly as

$$\begin{aligned} \varepsilon \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ S \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} &= \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}, \\ \Delta \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} &= \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \otimes \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \\ &= \begin{bmatrix} \alpha \otimes \alpha - \beta^* \otimes \beta & -(\alpha \otimes \beta^* + \beta^* \otimes \alpha^*) \\ \beta \otimes \alpha + \alpha^* \otimes \beta & \alpha^* \otimes \alpha^* - \beta \otimes \beta^* \end{bmatrix}. \end{aligned}$$

The described structures turn $\mathcal{O}(SU(2))$ *into a Hopf *-algebra.*

In this thesis we mainly focus on non-commutative Hopf *-algebras related to the Hopf *-algebra introduced above. Because all Hopf algebras that are constructed from a group as in Lemma 1.1.6 are commutative, these non-commutative Hopf algebras do not directly correspond to a group. Nevertheless, we prefer to think of them as coming from a group-like object.

Notation 1.1.10. *For a field $\mathbb{K}$ and symbols $a, b, \ldots, d$, write $\mathbb{K}\langle a, b, \ldots, d\rangle$ for the free, associative, non-commutative algebra generated by $a, b, \ldots, d$ over $\mathbb{K}$.*

Example 1.1.11. *For a parameter $q \in (0,1) \subseteq \mathbb{R}$, the quantum $SU(2)$ coordinate algebra $\mathcal{O}(SU_q(2))$ is given by*

$$\mathcal{O}(SU_q(2)) = \mathbb{C}\langle \alpha, \beta, \alpha^*, \beta^* \rangle / I,$$

where I is the two-sided ideal generated by the relations

$$\alpha\beta = q\beta\alpha, \qquad \beta^*\alpha^* = q\alpha^*\beta^*, \tag{1.11}$$

$$\alpha\beta^* = q\beta^*\alpha, \qquad \beta\alpha^* = q\alpha^*\beta, \tag{1.12}$$

$$\beta\beta^* = \beta^*\beta, \qquad \alpha\alpha^* = \alpha^*\alpha + (1-q^2)\beta\beta^*, \tag{1.13}$$

$$1 = \alpha^*\alpha + \beta^*\beta, \qquad 1 = \alpha\alpha^* + q^2\beta\beta^*. \tag{1.14}$$

On this algebra, there is a conjugation antiautomorphism

$$*\colon \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2)), \alpha \mapsto \alpha^*, \alpha^* \mapsto \alpha, \beta \mapsto \beta^*, \beta^* \mapsto \beta, \tag{1.15}$$

*which turns this algebra into a Hopf *-algebra as in Definition 1.1.8.*

Under $*$, the equations in (1.11) are interchanged, and likewise the equations in (1.12), whereas both equations in (1.13) are invariant under $*$. When $q = 1$, the equations (1.11)-(1.13) give commutativity, and the equation (1.14) becomes the equation in (1.10), thereby re-obtaining $\mathcal{O}(SU(2))$ from example 1.1.9.

The Hopf algebra maps are summarised by

$$\varepsilon \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.16}$$

$$S \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} = \begin{bmatrix} \alpha^* & \beta^* \\ -q\beta & \alpha \end{bmatrix} \tag{1.17}$$

$$\begin{aligned} \Delta \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} &= \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \otimes \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \\ &= \begin{bmatrix} \alpha\otimes\alpha - q\beta^*\otimes\beta & -q(\alpha\otimes\beta^* + \beta^*\otimes\alpha^*) \\ \beta\otimes\alpha + \alpha^*\otimes\beta & \alpha^*\otimes\alpha^* - q\beta\otimes\beta^* \end{bmatrix} \end{aligned} \tag{1.18}$$

It is a simple but tedious calculation to show that the equations (1.1)-(1.9) are *satisfied. For example, to show that the top of (1.9) is satisfied, we calculate*

$$\begin{aligned}\mu \circ (S \otimes \mathrm{id}) \circ \Delta \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} &= \mu \left(S \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \otimes \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \right) \\ &= \mu \left(\begin{bmatrix} \alpha^* & \beta^* \\ -q\beta & \alpha \end{bmatrix} \otimes \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix} \right) \\ &= \mu \left(\begin{bmatrix} \alpha^* \otimes \alpha + \beta^* \otimes \beta & -q\alpha^* \otimes \beta^* + \beta^* \otimes \alpha^* \\ -q\beta \otimes \alpha + \alpha \otimes \beta & q^2\beta \otimes \beta^* + \alpha \otimes \alpha^* \end{bmatrix} \right) \\ &= \begin{bmatrix} \alpha^*\alpha + \beta^*\beta & -q\alpha^*\beta^* + \beta^*\alpha^* \\ -q\beta\alpha + \alpha\beta & q^2\beta\beta^* + \alpha\alpha^* \end{bmatrix}.\end{aligned}$$

Using the equations (1.11) and (1.14), we see that this is equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \eta \circ \varepsilon \begin{bmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{bmatrix}.$$

This shows that $\mu \circ (S \otimes \mathrm{id}) \circ \Delta(r) = \eta \circ \varepsilon(r)$ *when* $r \in \{\alpha, -q\beta^*, \beta, \alpha^*\}$*, and henceforth it is true for all of* $\mathcal{O}(SU_q(2))$*.*

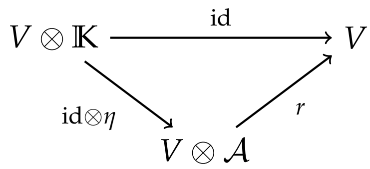
We will now move on to the co-representations of the Hopf algebra $\mathcal{O}(SU_q(2))$.

1.2 Co-modules

(Lie) groups, algebras, and rings in general can be studied via their representations or modules, and the theory of representations has been an active field of study on its own. In this thesis, all (co-)representations are assumed to be finite-dimensional when the converse is not explicitly stated. More about co-modules can be found in e.g. [5, Chapters 4 & 11]. We start with the definition of a *module*.

Definition 1.2.1. *A (right)* representation *or* module *of an algebra* $(\mathcal{A}, \mu, \eta)$ *over a field* $\mathbb{K}$ *is a vector space* V *together with a linear map* $r\colon V \otimes \mathcal{A} \to V$ *which satisfies the following diagrams:*

$$\begin{array}{ccc} V \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{r \otimes \mathrm{id}} & V \otimes \mathcal{A} \\ {\scriptstyle \mathrm{id} \otimes \mu} \downarrow & & \downarrow {\scriptstyle r} \\ V \otimes \mathcal{A} & \xrightarrow{r} & V \end{array} \tag{1.19}$$



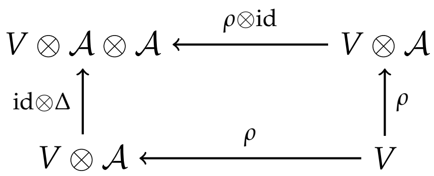
(1.20)

A representation r is irreducible *if* $V \neq 0$ *and there are no non-trivial vector subspaces* $W \subseteq V$ *such that* $r(W \otimes \mathcal{A}) \subseteq W$.

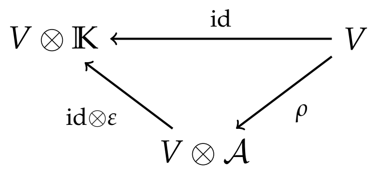
Diagram (1.19) is referred to as *associativity* and (1.20) as *unit* of the module.

A co-module is a structure dual to a module. Recall that a co-algebra is a triple $(\mathcal{A}, \Delta, \varepsilon)$ satisfying diagrams (1.2) and (1.4).

Definition 1.2.2. *A (right)* co-representation *or* co-module *of a co-algebra* $(\mathcal{A}, \Delta, \varepsilon)$ *over a field* $\mathbb{K}$ *is a vector space* V *together with a linear map* $\rho\colon V \to V \otimes \mathcal{A}$ *which satisfies the following diagrams:*



(1.21)



(1.22)

Diagram (1.21) is referred to as co-associativity and (1.22) as co-unit.

The notions of irreducible representations and unitary representations are also present in the dual setting.

Definition 1.2.3. *A* co-invariant subspace *of a co-representation* $\rho\colon V \to V \otimes \mathcal{A}$ *is a subspace* $W \subseteq V$ *such that* $\rho(W) \subseteq W \otimes \mathcal{A}$.
An irreducible co-representation *is a co-representation* $\rho\colon V \to V \otimes \mathcal{A}$ *such that* $V \neq 0$ *and there are no non-trivial co-invariant subspaces.*

Unitary co-representations are best described using a basis of V. The following procedure is described with more detail in [5, Section 11.1.1].

Definition 1.2.4. *The* matrix coefficients *of a co-representation* $\rho\colon V \to V \otimes \mathcal{A}$ *with respect to a basis* $(e_i)_{i=1}^n$ *are the elements* $\rho_{ij} \in \mathcal{A}$ *such that*

$$\rho(e_j) = \sum_i e_i \otimes \rho_{ij}. \tag{1.23}$$

These satisfy $\varepsilon(\rho_{ij}) = \mathbf{1}_{i=j}$ *and* $\Delta(\rho_{ij}) = \sum_k \rho_{ik} \otimes \rho_{kj}$ *for any* i, j*. Conversely, for any* $\rho_{ij} \in \mathcal{A}$ *that satisfy these conditions, formula (1.23) gives a co-representation.*

Definition 1.2.5. *A* unitary co-representation *of a Hopf *-algebra* $\mathcal{A}$ *onto a complex inner product space* V *is a co-representation* $\rho\colon V \to V \otimes \mathcal{A}$ *such that for an orthonormal basis* e_i *of* V*, the matrix coefficients* ρ_{ij} *satisfy* $S(\rho_{ij}) = \rho_{ji}^*$*, which implies* $\mu(\sum_k \rho_{ki}^* \otimes \rho_{kj}) = \mathbf{1}_{i=j}$*.*

A unitarisable co-representation *of a Hopf *-algebra onto a complex vector space* V *is a co-representation for which we can find an inner product on* V *for which the co-representation is unitary.*

In the case of group representations, unitary representations are characterised by them leaving an inner product invariant: If $\langle\cdot,\cdot\rangle_V$ is an inner product on V and $\rho\colon G \times V \to V$ is a (left) group representation, then $\langle \rho(g, v), w\rangle_V = \langle v, \rho(g^{-1}, w)\rangle_V$. A similar characterisation exists in this case:

Lemma 1.2.6. *When* $\rho\colon V \to V \otimes \mathcal{A}$ *is a co-representation of the Hopf-*-algebra* $\mathcal{A}$ *that is unitary with respect to an inner product* $\langle\cdot,\cdot\rangle_V$*, then we have*

$$\langle \rho(a), b \otimes 1\rangle_{V\otimes\mathcal{A}} = \langle a \otimes 1, (1 \otimes S)(\rho(b))\rangle_{V\otimes\mathcal{A}},$$

where $\langle\cdot,\cdot\rangle_{V\otimes\mathcal{A}}\colon (V \otimes \mathcal{A}) \times (V \otimes \mathcal{A}) \to \mathcal{A}$ *is given by*

$$\langle v \otimes s, w \otimes t\rangle_{V\otimes\mathcal{A}} = \langle v, w\rangle_V s^* t.$$

Proof. It suffices to prove it for the basis elements. Let e_μ, e_ν be arbitrary, and recall that $t_{\nu\mu}^* = S(t_{\mu\nu})$ because ρ is unitary, and $\langle e_\mu, e_\eta\rangle_V = \mathbf{1}_{\mu=\eta}$

because the basis e_μ is orthonormal w.r.t. $\langle\cdot,\cdot\rangle_V$. Then

$$\begin{aligned}\langle\rho(e_\mu), e_\nu\otimes 1\rangle_{V\otimes\mathcal{A}} &= \left\langle \sum_\eta e_\eta\otimes t_{\eta\mu}, e_\nu\otimes 1\right\rangle_{V\otimes\mathcal{A}} \\ &= \sum_\eta \langle e_\eta, e_\nu\rangle_V t^*_{\eta\mu} \\ &= \sum_\eta \mathbf{1}_{\eta=\nu} t^*_{\eta\mu} = t^*_{\nu\mu} \\ &= S(t_{\mu\nu}) = \sum_\eta \mathbf{1}_{\mu=\eta} S(t_{\eta\nu}) \\ &= \sum_\eta \langle e_\mu, e_\eta\rangle_V S(t_{\eta\nu}) \\ &= \left\langle e_\mu\otimes 1, \sum_\eta e_\eta\otimes S(t_{\eta\nu})\right\rangle_{V\otimes\mathcal{A}} \\ &= \left\langle e_\mu\otimes 1, (1\otimes S)\left(\sum_\eta e_\eta\otimes t_{\eta\nu}\right)\right\rangle_{V\otimes\mathcal{A}} \\ &= \langle e_\mu\otimes 1, (1\otimes S)(\rho(e_\nu))\rangle_{V\otimes\mathcal{A}}.\end{aligned}$$

□

Corollary 1.2.7. *If $\rho\colon V\to V\otimes\mathcal{A}$ is unitary and $W\subseteq V$ is a co-invariant subspace, then $W^\perp$ is also a co-invariant subspace.*

Proof. Let $v\in W^\perp$ and $w\in W$ be arbitrary. Because W is co-invariant, we have $\rho(w)\in W\otimes\mathcal{A}$ and thus also $(1\otimes S)(\rho(w))\in W\otimes\mathcal{A}$. Because $v\in W^\perp$ we have that $\langle v\otimes 1, (1\otimes S)(\rho(w))\rangle_{V\otimes\mathcal{A}} = 0$. By the previous lemma,

$$0 = \langle v\otimes 1, (1\otimes S)(\rho(w))\rangle_{V\otimes\mathcal{A}} = \langle\rho(v), w\otimes 1\rangle_{V\otimes\mathcal{A}}$$

and thus $\rho(v)\perp w\otimes 1$ for all $w\in W$, i.e. $\rho(v)\in W^\perp\otimes\mathcal{A}$. □

Notation 1.2.8. *Given a bialgebra $(\mathcal{A},\mu,\eta,\Delta,\varepsilon)$ and two co-representations $\rho_V\colon V\to V\otimes\mathcal{A}$ and $\rho_W\colon W\to W\otimes\mathcal{A}$, their tensor product co-representation is the co-representation $\rho_{V\otimes W}\colon V\otimes W\to(V\otimes W)\otimes\mathcal{A}$ defined by*

$$V\otimes W\xrightarrow{\rho_V\otimes\rho_W}(V\otimes\mathcal{A})\otimes(W\otimes\mathcal{A})\xrightarrow{E}(V\otimes W)\otimes\mathcal{A},$$

where E is given by $E(v\otimes a_1\otimes w\otimes a_2) = v\otimes w\otimes\mu(a_1,a_2)$. This co-representation is sometimes also denoted by $\rho_V\otimes\rho_W$.

Notation 1.2.9. *Given a co-algebra* $(\mathcal{A}, \Delta, \varepsilon)$ *and two co-representations* $\rho_V \colon V \to V \otimes \mathcal{A}$ *and* $\rho_W \colon W \to W \otimes \mathcal{A}$*, their sum co-representation* $\rho_V \oplus \rho_W \colon V \oplus W \to (V \oplus W) \otimes \mathcal{A}$ *is induced by the identification* $(V \oplus W) \otimes \mathcal{A} = V \otimes \mathcal{A} \oplus W \otimes \mathcal{A}$*,*

$$(v \oplus 0) \otimes a_1 + (0 \oplus w) \otimes a_2 \leftrightarrow (v \otimes a_1) \oplus (w \otimes a_2).$$

Given a representation of a (Lie) group, we can construct a co-representation of the corresponding Hopf algebra $\mathbb{K}^G$:

Lemma 1.2.10. *Let* G *be a group,* V *a* $\mathbb{K}$*-vector space and* $r \colon V \times G \to V$ *a (right) group representation, i.e.* $r(r(v,g),h) = r(v, g \cdot h)$*. Then* r *induces a (right) co-representation* $\rho_r \colon V^* \to V^* \otimes \mathbb{K}^G$ *given by*

$$\sigma \mapsto \sum_{g \in G} \lambda_g^\sigma \otimes \mathbf{1}_g,$$

where $\lambda_g^\sigma \colon V \to \mathbb{K}$ *is the linear form* $v \mapsto \sigma(r(v,g))$*.*

When V *is finite-dimensional, the co-representation* ρ_r *is irreducible if and only if* r *is irreducible.*

Often, there is an interesting algebra $\mathcal{A} \subseteq \mathbb{K}^G$, such as $\mathcal{O}(G)$ or $C^\infty(G)$, such that the image of ρ_r is contained in $V^* \otimes \mathcal{A}$. The function ρ_r can then be viewed as a co-representation of $\mathcal{A}$.

Proof. To check that (1.22) is satisfied, note that

$$(\mathrm{id} \otimes \varepsilon) \circ \rho_r(\sigma) = \sum_{g \in G} \lambda_g^\sigma \otimes \varepsilon(\mathbf{1}_g) = \lambda_1^\sigma \otimes 1.$$

From $\lambda_1^\sigma(v) = \sigma(r(1,v)) = \sigma(v)$ we get $\lambda_1^\sigma = \sigma$ and thus

$$(\mathrm{id} \otimes \varepsilon) \circ \rho_r(\sigma) = \sigma \otimes 1.$$

To check that (1.21) is satisfied, note that

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \rho_r(\sigma) &= \sum_{g \in G} \lambda_g^\sigma \otimes \Delta(\mathbf{1}_g) \\ &= \sum_{g \in G} \sum_{a,b \in G} \mathbf{1}_g(a \cdot b) \lambda_g^\sigma \otimes \mathbf{1}_a \otimes \mathbf{1}_b \\ &= \sum_{a,b \in G} \lambda_{(a \cdot b)}^\sigma \otimes \mathbf{1}_a \otimes \mathbf{1}_b\end{aligned}$$

and

$$(\rho_r \otimes \mathrm{id}) \circ \rho_r(\sigma) = \sum_{g \in G} \rho_r(\lambda_g^\sigma) \otimes \mathbf{1}_g = \sum_{g \in G} \sum_{h \in G} \lambda_h^{\lambda_g^\sigma} \otimes \mathbf{1}_h \otimes \mathbf{1}_g.$$

Note that

$$\lambda_h^{\lambda_g^\sigma}(v) = \lambda_g^\sigma(r(v,h)) = \sigma(r(r(v,h),g)) = \sigma(r(v, h \cdot g)) = \lambda_{(h \cdot g)}^\sigma(v).$$

This gives $\lambda_h^{\lambda_g^\sigma} = \lambda_{(h \cdot g)}^\sigma$ from which we conclude that $(\mathrm{id} \otimes \Delta) \circ \rho_r(\sigma) = (\rho_r \otimes \mathrm{id}) \circ \rho_r(\sigma)$ and therefore (1.21) is satisfied.

For the final part of the lemma, let $W \subseteq V$ be a linear subspace, and let $W^\perp = \{\omega \in V^* \colon \omega(w) = 0 \ \forall w \in W\}$. Because V is finite-dimensional, it is known that $W = (W^\perp)^\perp = \{w \in V \colon \omega(w) = 0 \ \forall \omega \in W^\perp\}$. We will see that W is an invariant subspace under r if and only if $W^\perp$ is a co-invariant subspace under ρ_r. The statement regarding irreducibility then follows directly.

Note that $\rho_r(W^\perp) \subseteq W^\perp \otimes \mathbb{K}^G$ if and only if

$$\forall w \in W^\perp, \sum_{g \in G} \lambda_g^w \otimes \mathbf{1}_g \in W^\perp \otimes \mathbb{K}^G,$$

thus if and only if for all $\omega \in W^\perp$ and $g \in G$ we have $\lambda_g^\omega \in W^\perp$.

Now assume $W^\perp$ is a co-invariant subspace, i.e. $\rho_r(W^\perp) \subseteq W^\perp \otimes \mathbb{K}^G$, and let $w \in W$ be arbitrary. Then for any $\omega \in W^\perp$ and $g \in G$ we have $\lambda_g^\omega \in W^\perp$ and thus $\lambda_g^\omega(w) = 0$. But $\lambda_g^\omega(w) = \omega(r(w,g))$, so $r(w,g) \in W$. Therefore, $r(W \times G) \subseteq W$, i.e. W is an invariant subspace.

Now assume W is an invariant subspace, i.e. $r(W \times G) \subseteq W$, and let $\omega \in W^\perp$ and $g \in G$ be arbitrary. Then for any $w \in W$ it holds that $r(w,g) \in W$ and thus $\omega(r(w,g)) = 0$ per definition of $W^\perp$. But $\omega(r(w,g)) = \lambda_g^\omega(w)$, so $\lambda_g^\omega \in W^\perp$. Hence, $\rho_r(W^\perp) \subseteq W^\perp \otimes \mathbb{K}^G$, i.e. $W^\perp$ is a co-invariant subspace. □

This construction can be applied to the irreducible representations of $SU(2)$ to obtain all the irreducible co-representations of $\mathcal{O}(SU(2))$. Recall that the irreducible representations of $SU(2)$ are given on spaces of homogeneous polynomials in two variables, as found* in e.g. [2]:

*The given reference classifies all left group representations. The right representations can be obtained using the anti-automorphism $SU(2) \to SU(2), g \mapsto g^{-1}$.

Theorem 1.2.11. *Let $P_n(\mathbb{C}) = \mathbb{C}[X,Y]_n$ be the complex vector space of homogeneous polynomials in X and Y of degree n. The map*

$$r_n \colon P_n(\mathbb{C}) \times SU(2) \to P_n(\mathbb{C})$$

given by $p \mapsto p(g(X,Y))$ is an irreducible representation of $SU(2)$, and every irreducible representation of $SU(2)$ is equivalent to a representation of this form.

Explicitly, for $g = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \in SU(2)$ and $p = \sum_{k=0}^n p_k X^{n-k} Y^k \in \mathbb{C}[X,Y]_n$ we have

$$\begin{aligned} r_n(p,g) &= \sum_{k=0}^n p_k (aX - b^*Y)^{n-k}(bX + a^*Y)^k \\ &= \sum_{k=0}^n p_k \left(\sum_{i=0}^{n-k} \binom{n-k}{i} (aX)^{n-k-i}(-b^*Y)^i \right) \left(\sum_{j=0}^{k} \binom{k}{j} (bX)^{k-j}(a^*Y)^j \right) \\ &= \sum_{k=0}^n p_k \sum_{r=0}^n \left(\sum_l \binom{n-k}{r-l}\binom{k}{l} (a)^{n-k-(r-l)}(-b^*)^{r-l}(b)^{k-l}(a^*)^l \right) X^{n-r}Y^r \\ &=: \sum_{r,k=0}^n p_k\, \varpi^g_{k,r}\, X^{n-r}Y^r \end{aligned} \tag{1.24}$$

where l runs over all integers such that all exponents are non-negative. The second equality follows from Newton's binomium, and the third equality uses the identity

$$\left(\sum_{i=0}^a s_i Y^i\right)\left(\sum_{j=0}^b t_j Y^j\right) = \sum_{r=0}^{a+b}\left(\sum_{l=\max\{0,r-a\}}^{\min\{r,b\}} s_{r-l}t_l\right) Y^r.$$

To apply lemma 1.2.10 to the representations in theorem 1.2.11, choose the basis $(\mathcal{X}^{n-i}\mathcal{Y}^i)_{i=0}^n$ for $\mathbb{C}[\mathcal{X},\mathcal{Y}]_n := P_n(\mathbb{C})^*$ that sends the basis of $P_n(\mathbb{C})$ to the values

$$\mathcal{X}^{n-i}\mathcal{Y}^i(X^{n-j}Y^j) = \begin{cases} \frac{1}{\binom{n}{i}} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.25}$$

Using (1.24) gives

$$\lambda_g^{\mathcal{X}^{n-i}\mathcal{Y}^i}(p) = \mathcal{X}^{n-i}\mathcal{Y}^i(r_n(p,g)) = \sum_{r,k=0}^n p_k \varpi^g_{k,r}\mathcal{X}^{n-i}\mathcal{Y}^i(X^{n-r}Y^r) = \sum_{k=0}^n p_k \varpi^g_{k,i}\frac{1}{\binom{n}{i}}.$$

Now $p_k = \binom{n}{k}\mathcal{X}^{n-k}\mathcal{Y}^k(p)$ gives

$$\lambda_g^{\mathcal{X}^{n-i}\mathcal{Y}^i} = \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n}{i}} \varpi^g_{k,i}\mathcal{X}^{n-k}\mathcal{Y}^k.$$

Expanding $\varpi^g_{k,i}$ and using that $a = \alpha(g)$ and $b = \beta(g)$ etc. we finally obtain that the co-representation from lemma 1.2.10 is given by

$$\mathcal{X}^{n-i}\mathcal{Y}^i \mapsto \sum_{k=0}^{n} \mathcal{X}^{n-k}\mathcal{Y}^k \otimes \left(\sum_l \frac{\binom{k}{l}\binom{n-k}{i-l}\binom{n}{k}}{\binom{n}{i}} (\alpha)^{n-k-(i-l)}(-\beta^*)^{i-l}(\beta)^{k-l}(\alpha^*)^l \right).$$

Introducing notation for the matrix coefficients of the right hand side, this becomes

$$\mathcal{X}^{n-i}\mathcal{Y}^i \mapsto \sum_{k=0}^{n} \mathcal{X}^{n-k}\mathcal{Y}^k \otimes \tau^{(n)}_{ki} \tag{1.26}$$

Note that this co-representation can indeed be seen as a co-representation of $\mathcal{O}(SU(2))$ instead of $\mathbb{C}^{SU(2)}$.

There is another way to look at these co-representations, which uses the notion of algebra co-representation. Algebra co-representations are often not finite-dimensional, which is why we do not extensively cover them in this thesis. This viewpoint is used in [5].

Definition 1.2.12. *An* algebra co-representation $\rho_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V} \otimes \mathcal{A}$ *of a Hopf-algebra* $(\mathcal{A}, \mu, \eta, \Delta, \varepsilon, S)$ *onto an algebra* $(\mathcal{V}, m, e)$ *is a co-representation such that* $(m \otimes \mathrm{id}_{\mathcal{A}}) \circ \rho_{\mathcal{V}\otimes\mathcal{V}} = \rho_{\mathcal{V}} \circ m$ *and* $(e \otimes \mathrm{id}_{\mathcal{A}}) \circ \eta = \rho \circ e$*, where* $\rho_{\mathcal{V}\otimes\mathcal{V}}$ *is as in Notation 1.2.8.*

The co-representations given in (1.26) can be combined into a co-representation on $\mathbb{C}[\mathcal{X},\mathcal{Y}] = \bigoplus_{n=0}^{\infty} \mathbb{C}[\mathcal{X},\mathcal{Y}]_n$. This can be shown to be an algebra co-representation, hence all matrix coefficients can be deduced from the case $n = 1$ and the algebra co-representation properties. Note that $n = 1$ is given by

$$\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{X} \otimes \alpha + \mathcal{Y} \otimes \beta \\ \mathcal{X} \otimes (-\beta^*) + \mathcal{Y} \otimes \alpha^* \end{pmatrix} \tag{1.27}$$

from which we can deduce that e.g.

$$\begin{aligned} \rho(\mathcal{X}^2) = \rho(m(\mathcal{X} \otimes \mathcal{X})) &= (m \otimes \mathrm{id}_{\mathcal{A}}) \circ \rho_{\mathbb{C}[\mathcal{X},\mathcal{Y}]\otimes\mathbb{C}[\mathcal{X},\mathcal{Y}]}(\mathcal{X} \otimes \mathcal{X}) \\ &= (m \otimes \mathrm{id}_{\mathcal{A}}) \circ E((\mathcal{X} \otimes \alpha + \mathcal{Y} \otimes \beta) \otimes (\mathcal{X} \otimes \alpha + \mathcal{Y} \otimes \beta)) \\ &= \mathcal{X}^2 \otimes \alpha^2 + \mathcal{X}\mathcal{Y} \otimes \alpha\beta + \mathcal{Y}\mathcal{X} \otimes \beta\alpha + \mathcal{Y}^2 \otimes \beta^2. \end{aligned}$$

where E is as in Notation 1.2.8.

1.3 Irreducible co-modules of $\mathcal{O}(SU_q(2))$

Following [5], the irreducible co-representations of $\mathcal{O}(SU_q(2))$ are described using definition 1.2.12. We should keep in mind that these co-representations are very similar to the co-representations of $\mathcal{O}(SU(2))$ described in the previous section.

Definition 1.3.1. *The algebra $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]$ is defined as $\mathbb{C}\langle\mathcal{X},\mathcal{Y}\rangle/(\mathcal{X}\mathcal{Y}-q\mathcal{Y}\mathcal{X})$ where $\mathbb{C}\langle\mathcal{X},\mathcal{Y}\rangle$ is the free algebra on $\mathcal{X}$ and $\mathcal{Y}$ (c.f. Notation 1.1.10) and $(\mathcal{X}\mathcal{Y}-q\mathcal{Y}\mathcal{X})$ is the two-sided ideal generated by $\mathcal{X}\mathcal{Y}-q\mathcal{Y}\mathcal{X}$.*
The vector space $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n$ is the subspace of homogeneous elements of degree n.

In the literature, the algebra $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]$ is known as the *quantum plane*. Some new notation is also required.

Notation 1.3.2. *The* q-binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ *are the numbers (dependent on q) such that in $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]$ we have the identity*

$$(\mathcal{X}+\mathcal{Y})^n = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \mathcal{Y}^{n-k}\mathcal{X}^k. \tag{1.28}$$

Furthermore, in this thesis the notation $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q^{-2}}$ is used.
Sometimes, for $n \in \mathbb{Z}$ the notation $[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$ is used.

Of course, formula (1.28) holds for any two elements in an algebra that satisfy the same commutation relation as $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]$. The reason for $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ to have q^{-2} instead of q is because for algebras of the form $\mathcal{A}\otimes\mathcal{B}$, and elements $\mathcal{X},\mathcal{Y}\in\mathcal{A}$ and $\alpha,\beta\in\mathcal{B}$ that commute up to a factor of q, we have $(\mathcal{X}\otimes\alpha+\mathcal{Y}\otimes\beta)^n = \sum_{k=0}^{n}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\mathcal{X}^{n-k}\mathcal{Y}^k\otimes\alpha^{n-k}\beta^k$. More about the q-binomial coefficients and related notions can be found in [5, Chapter 2].

Theorem 1.3.3. *Consider the function*

$$\varphi_1\colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_1 \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_1 \otimes \mathcal{O}(SU_q(2))$$

given by

$$\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix} \mapsto [\mathcal{X}\ \mathcal{Y}] \otimes \begin{bmatrix}\alpha & -q\beta^* \\ \beta & \alpha^*\end{bmatrix} = \begin{pmatrix}\mathcal{X}\otimes\alpha+\mathcal{Y}\otimes\beta \\ \mathcal{X}\otimes(-q\beta^*)+\mathcal{Y}\otimes\alpha^*\end{pmatrix} \tag{1.29}$$

This function can be extended to an (infinite-dimensional) algebra co-representation

$$\varphi\colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}] \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}] \otimes \mathcal{O}(SU_q(2))$$

using the relations from definition 1.2.12. The restrictions

$$\varphi_n \colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n \otimes \mathcal{O}(SU_q(2))$$

are finite-dimensional irreducible co-representations, and any finite-dimensional irreducible co-representation of $\mathcal{O}(SU_q(2))$ is equivalent to a co-representation of this form.

Proof. See [5, Chapter 4.2.3-4.2.5]. □

A few remarks:

- When $q \to 1$, equation (1.29) exactly agrees with equation (1.27).
- In [5], the algebra $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]$ is identified with the subalgebra of $\mathcal{O}(SU_q(2))$ generated by α and β^* via $\mathcal{X} \leftrightarrow \alpha$, $\mathcal{Y} \leftrightarrow -q\beta^*$. Then the co-action of φ can be identified with the co-action of Δ. Indeed, we have $\alpha(-q\beta^*) = q(-q\beta^*)\alpha$ and
$$\Delta\begin{pmatrix} \alpha \\ -q\beta^* \end{pmatrix} = \begin{pmatrix} \alpha \otimes \alpha + (-q\beta^*) \otimes \beta \\ \alpha \otimes (-q\beta^*) + (-q\beta^*) \otimes \alpha^* \end{pmatrix}$$
which agrees with equation (1.29).
- One can explicitly compute the matrix coefficients for these co-representations. This is described in [5, Chapter 4.2.3-4.2.4], with respect to the basis
$$f_i^{(l)} = \sqrt{\left[\begin{smallmatrix} 2l \\ l+i \end{smallmatrix}\right]}\,\mathcal{X}^{l-i}\mathcal{Y}^{l+i} \tag{1.30}$$
where $l = \frac{n}{2} \in \frac{1}{2}\mathbb{N}$ and i runs from $-l$ to l in integer steps. With this basis, we have
$$t_{ij}^{(l)} = \sum_{\mu} \frac{\left[\begin{smallmatrix} l-i \\ \mu \end{smallmatrix}\right]\left[\begin{smallmatrix} l+i \\ l+j-\mu \end{smallmatrix}\right]\sqrt{\left[\begin{smallmatrix} 2l \\ l+i \end{smallmatrix}\right]}}{\sqrt{\left[\begin{smallmatrix} 2l \\ l+j \end{smallmatrix}\right]}} q^{-\mu(\mu+i-j)}\alpha^{l-i-\mu}(-q\beta^*)^{\mu}\beta^{i-j+\mu}(\alpha^*)^{l+j-\mu}$$
where μ sums over all integers for which the exponents are all positive, and $\varphi_{2l}\left(f_j^{(l)}\right) = \sum_i f_i^{(l)} \otimes t_{ij}^{(l)}$.
- With respect to the basis $(\mathcal{X}^{n-i}\mathcal{Y}^i)_{i=0}^n$ of $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n$, and after relabeling the indices via $n \leftrightarrow 2l$, $i \leftrightarrow j+l$, $j \leftrightarrow i+l$ and $l \leftrightarrow l+j-\mu$, the matrix coefficients are given by
$$\tau_{ij}^{(n)} = \sum_{l} \frac{\left[\begin{smallmatrix} n-j \\ i-l \end{smallmatrix}\right]\left[\begin{smallmatrix} j \\ l \end{smallmatrix}\right]\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]}{\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]} q^{(l-i)(j-l)}\alpha^{n-j-(i-l)}(-q\beta^*)^{i-l}\beta^{j-l}(\alpha^*)^{l} \tag{1.31}$$

where $\varphi_n(\mathcal{X}^{n-j}\mathcal{Y}^j) = \sum_i \mathcal{X}^{n-i}\mathcal{Y}^i \otimes \tau_{ij}^{(n)}$. Note the similarity to the coefficients found at (1.26).

Lemma 1.3.4. *The matrix coefficients $t_{ij}^{(l)}$ considered above satisfy $S\left(t_{ij}^{(l)}\right) = \left(t_{ji}^{(l)}\right)^*$, i.e. the co-representations φ_n is unitary with respect to the inner product induced by the basis $f_i^{(l)}$.*

Proof. See [5, Chapter 4.3.1 Proposition 16]. □

Example 1.3.5. *The co-representation $\varphi_2\colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_2 \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_2 \otimes \mathcal{O}(SU_q(2))$ can be deduced as follows:*

$$\begin{aligned}
\varphi_2(\mathcal{X}^2) =& \varphi(m(\mathcal{X}\otimes\mathcal{X})) = m\otimes \mathrm{id}_{\mathcal{O}(SU_q(2))}\left(\varphi\otimes\varphi(\mathcal{X}\otimes\mathcal{X})\right)\\
=& m\otimes \mathrm{id}_{\mathcal{O}(SU_q(2))}\left(E((\mathcal{X}\otimes\alpha+\mathcal{Y}\otimes\beta)\otimes(\mathcal{X}\otimes\alpha+\mathcal{Y}\otimes\beta))\right)\\
=& m\otimes \mathrm{id}_{\mathcal{O}(SU_q(2))}\Big(\mathcal{X}\otimes\mathcal{X}\otimes\alpha\alpha+\mathcal{X}\otimes\mathcal{Y}\otimes\alpha\beta\\
&+\mathcal{Y}\otimes\mathcal{X}\otimes\beta\alpha+\mathcal{Y}\otimes\mathcal{Y}\otimes\beta\beta\Big)\\
=& \mathcal{X}^2\otimes\alpha\alpha+(1+q^{-2})\mathcal{X}\mathcal{Y}\otimes\alpha\beta+\mathcal{Y}^2\otimes\beta\beta
\end{aligned}$$

The values of $\varphi_2(\mathcal{X}\mathcal{Y})$ and $\varphi_2(\mathcal{Y}^2)$ are deduced similarly, which gives

$$\varphi_2\begin{pmatrix}\mathcal{X}^2\\ \mathcal{X}\mathcal{Y}\\ \mathcal{Y}^2\end{pmatrix} = \begin{pmatrix}\mathcal{X}^2\otimes\alpha\alpha & +(1+q^{-2})\mathcal{X}\mathcal{Y}\otimes\alpha\beta & +\mathcal{Y}^2\otimes\beta\beta\\ \mathcal{X}^2\otimes\alpha(-q\beta^*) & +\mathcal{X}\mathcal{Y}\otimes(\alpha\alpha^*-\beta\beta^*) & +\mathcal{Y}^2\otimes\beta\alpha^*\\ \mathcal{X}^2\otimes(-q\beta^*)(-q\beta^*) & +(1+q^{-2})\mathcal{X}\mathcal{Y}\otimes(-q\beta^*)\alpha^* & +\mathcal{Y}^2\otimes\alpha^*\alpha^*\end{pmatrix}$$

Similarly, the co-representation $\varphi_3\colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_3 \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_3 \otimes \mathcal{O}(SU_q(2))$ can be deduced to be given by $\varphi_3\begin{pmatrix}\mathcal{X}^3\\ \mathcal{X}^2\mathcal{Y}\\ \mathcal{X}\mathcal{Y}^2\\ \mathcal{Y}^3\end{pmatrix} =$

$$\begin{pmatrix}\mathcal{X}^3\otimes\alpha^3 & +(1+q^{-2}+q^{-4})\mathcal{X}^2\mathcal{Y}\otimes\alpha^2\beta & +(1+q^{-2}+q^{-4})\mathcal{X}\mathcal{Y}^2\otimes\alpha\beta^2 & +\mathcal{Y}^3\otimes\beta^3\\ -q\mathcal{X}^3\otimes\alpha^2\beta^* & +\mathcal{X}^2\mathcal{Y}\otimes(\alpha^2\alpha^*-(1+q^{-2})\alpha\beta\beta^*) & +\mathcal{X}\mathcal{Y}^2\otimes((1+q^{-2})\alpha\beta\alpha^*-q^{-1}\beta^2\beta^*) & +\mathcal{Y}^3\otimes\beta^2\alpha^*\\ q^2\mathcal{X}^3\otimes\alpha(\beta^*)^2 & +\mathcal{X}^2\mathcal{Y}\otimes(\beta(\beta^*)^2-q(1+q^{-2})\alpha\beta^*\alpha^*) & +\mathcal{X}\mathcal{Y}^2\otimes(\alpha(\alpha^*)^2-(1+q^{-2})\beta\beta^*\alpha^*) & +\mathcal{Y}^3\otimes\beta(\alpha^*)^2\\ -q^3\mathcal{X}^3\otimes(\beta^*)^3 & +q^2(1+q^{-2}+q^{-4})\mathcal{X}^2\mathcal{Y}\otimes(\beta^*)^2\alpha^* & -q(1+q^{-2}+q^{-4})\mathcal{X}\mathcal{Y}^2\otimes\beta^*(\alpha^*)^2 & +\mathcal{Y}^3\otimes(\alpha^*)^3\end{pmatrix}$$

Indeed, these expressions are the same as the expressions that can be calculated using formula (1.31).

1.4 Clebsch–Gordan coefficients of $\mathcal{O}(SU_q(2))$ co-modules

Per [5, Remark 1 below Theorem 4.14], any co-representation

$$\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$$

can be written as a direct sum $\rho \cong \oplus_i \varphi_{n_i}$, where φ_{n_i} are the irreducible co-representations given in Theorem 1.3.3. In other words, finite-dimensional co-representations of $\mathcal{O}(SU_q(2))$ are completely reducible.

In particular, for any n_1, n_2 the co-representation $\varphi_{n_1} \otimes \varphi_{n_2}$ as in Notation 1.2.8 can be written as such, as summarised by the following result found in [5]:

Lemma 1.4.1. *Given n_1, n_2, there is an isomorphism*

$$C\colon \bigoplus_{k=0}^{\min\{n_1,n_2\}} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k+|n_1-n_2|} \xrightarrow{\sim} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_1} \otimes \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_2}$$

such that the following square diagram commutes:

$$\begin{array}{ccc} \bigoplus_{k=0}^{\min\{n_1,n_2\}} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k+|n_1-n_2|} & \xleftrightarrow{\quad C \quad} & \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_1} \otimes \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_2} \\ \Big\downarrow \bigoplus_{k=0}^{\min\{n_1,n_2\}} \varphi_{2k+|n_1-n_2|} & & \Big\downarrow \varphi_{n_1} \otimes \varphi_{n_2} \\ \left(\bigoplus_{k=0}^{\min\{n_1,n_2\}} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k+|n_1-n_2|}\right) \otimes \mathcal{O}(SU_q(2)) & \xleftrightarrow{C \otimes \mathrm{id}_{\mathcal{O}(SU_q(2))}} & \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_1} \otimes \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{n_2} \otimes \mathcal{O}(SU_q(2)) \end{array}$$

Explicitly, for the basis vectors $f_j^{(l_1)} \in \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2l_1}$ and $f_j^{(l_2)} \in \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2l_2}$ and $f_j^{(l)} \in \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2l}$ as in (1.30), such that $\left(f_i^{(l_1)} \otimes f_j^{(l_2)}\right)_{i=-l_1,\, j=-l_2}^{l_1,\, l_2}$ is a basis of the right-hand side and $\left((f_j^{(l)})_{j=-l}^{l}\right)_{l=|l_1-l_2|}^{l_1+l_2}$ is a basis of the left-hand side, we *have that*[†]

$$C(f_m^{(l)}) = \sum_{j+k=m} C_q(l_2, l_1, l; k, j, m)\, f_j^{(l_1)} \otimes f_k^{(l_2)}.$$

where the expression for $C_q(l_1, l_2, l; j, k, m)$ can be found in [5, Chapter 3 Formulas (51)-(53)].

The general formula for $C_q(l_1, l_2, l; j, k, m)$ will not be written out in this thesis; it will only be used for a few special cases.

Proof. This proof will sketch the arguments used in [5], and fill in a few missing gaps.

The proof starts with two Hopf algebras[‡] introduced in [5, Chapter 3.1.1 and 3.1.2]: the algebra $U_q(sl_2)$ generated by E, K, F and the closely related $\breve{U}_q(sl_2)$ that is generated by $\breve{E}, \breve{K}, \breve{F}$ (which [5] just writes as E, K, F). There is an injective map $U_q(sl_2) \hookrightarrow \breve{U}_q(sl_2)$ given by $\begin{pmatrix} E \\ K \\ L \end{pmatrix} \mapsto \begin{pmatrix} \breve{E}\breve{K} \\ \breve{K}^2 \\ \breve{K}^{-1}\breve{F} \end{pmatrix}$.

By restriction of q to $\mathbb{R}$ and with an additional $*$-map, the Hopf-algebras $U_q(su_2)$ and $\breve{U}_q(su_2)$ are constructed out of $U_q(sl_2)$ and $\breve{U}_q(sl_2)$.

By [5, Chapter 4.4.1, Theorem 21], there is a dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{O}(SL_q(2))$ and $U_q(sl_2)$, which also gives a dual pairing between $\mathcal{O}(SU_q(2))$ and $U_q(su_2)$.

In [5, Chapter 4.4.2] it is explained how any *co-representation* φ of $\mathcal{O}(SU_q(2))$ gives rise to a *representation* $\hat{\varphi}$ of $U_q(su_2)$. Explicitly for the co-representation φ_{2l} given on the basis $f_i^{(l)}$ by $\varphi_{2l}\left(f_j^{(l)}\right) = \sum_i f_i^{(l)} \otimes t_{ij}^{(l)}$, we have $\hat{\varphi_{2l}}(x) f_j^{(l)} = \sum_i f_i^{(l)} \langle x, t_{ij}^{(l)} \rangle$ for all $x \in U_q(su_2)$. For the generators E, K, F we have

$$\begin{aligned} \hat{\varphi_{2l}}(E) f_j^{(l)} &= q^{-j} \sqrt{[l-j+1]_q [l+j]_q} f_{j-1}^{(l)} \\ \hat{\varphi_{2l}}(K) f_j^{(l)} &= q^{-2j} f_j^{(l)} \\ \hat{\varphi_{2l}}(F) f_j^{(l)} &= q^{j+1} \sqrt{[l+j+1]_q [l-j]_q} f_{j+1}^{(l)} \end{aligned} \tag{1.32}$$

[†]Calculations in specific cases (See the footnote in the proof of Lemma 2.2.3) show that this formula might be incorrect. In [5] it seems to be implied that the correct formula is

$$C(f_m^{(l)}) = \sum_{j+k=m} C_q(l_1, l_2, l; j, k, m) f_j^{(l_1)} \otimes f_k^{(l_2)}.$$

[‡]These algebras are Hopf-algebra analogues of the Lie algebra $\mathfrak{sl}(2)$ of the group $SL(2)$. In particular, $U_q(sl_2)$ should be thought of as (a quantum analogue of) the universal enveloping algebra of the lie-algebra $\mathfrak{sl}(2)$.

Using the map $f_j^{(l)} \leftrightarrow \mathbf{e}_{-j}$, this representation agrees with the representation of $\breve{U}_q(sl_2)$ found in [5, Chapter 3.2.3, Theorem 13]:

$$\begin{aligned} T_l(\breve{E})\mathbf{e}_m &= \sqrt{[l+m+1]_q[l-m]_q}\mathbf{e}_{m+1} \\ T_l(\breve{K})\mathbf{e}_m &= q^m\mathbf{e}_m \\ T_l(\breve{F})\mathbf{e}_m &= \sqrt{[l-m+1]_q[l+m]_q}\mathbf{e}_{m-1} \end{aligned} \tag{1.33}$$

The powers of q in (1.32) and (1.33) agree after applying the aforementioned inclusion map $U_q(sl_2) \hookrightarrow \breve{U}_q(sl_2)$.

The representation theory of $\breve{U}_q(su_2)$ is described in [5, Chapter 3.4], which states

$$\mathbf{e}_m^l = \sum_{j+k=m} C_q(l_1, l_2, l; j, k, m)\ \mathbf{e}_j \otimes \mathbf{e}_k.$$

This result can now be applied to the co-representations of $\mathcal{O}(SU_q(2))$, by pulling it back via the map $f_j^{(l)} \leftrightarrow \mathbf{e}_{-j}$, which gives

$$C(f_j^{(l)}) = \sum_{j+k=m} C_q(l_1, l_2, l; -j, -k, -m)\ f_j^{(l_1)} \otimes f_k^{(l_2)}.$$

By [5, Chapter 3.4.4, formula (70)], we have

$$C_q(l_1, l_2, l; -j, -k, -m) = C_q(l_2, l_1, l; k, j, m).$$

□

Chapter 2

Algebraic subproduct systems

In this chapter, an algebraic subproduct system $E_n \subseteq V^{\otimes n}$ is constructed for a unitary co-representation $\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$ onto a complex inner product space V, similar to the constructions in [1, Chapter 2].

2.1 The non-commutative Nullstellensatz, an introduction

The famous (commutative) Hilbert Nullstellensatz is a fundamental result in commutative algebra that relates the algebraic notion of an ideal $I \subseteq R$ in a ring R to the geometric notion of a variety $V(I) \subseteq \mathrm{Spec}(R)$.

The classical version of this statement takes $R = \mathbb{C}[X_1, \ldots, X_n]$ in which case $\mathrm{Spec}(R)$ is known as n-dimensional affine space, and an ideal generated by a polynomial p corresponds to the set of zeroes of p.

In non-commutative algebra, an analogous statement has been described in [8], where an ideal $I \subseteq \mathbb{C}\langle X_1, \ldots, X_n\rangle$ is related to a structure known as a subproduct system. This thesis considers algebraic subproduct systems as opposed to the more general, analytical notion of subproduct systems of Hilbert spaces.

Definition 2.1.1. *An* algebraic subproduct system *is a collection of finite-dimensional complex inner product spaces* $\{E_n\}_{n\geq 0}$ *and linear isometries* $\iota_{k,m}\colon E_{k+m} \hookrightarrow E_k \otimes E_m$ *that satisfy the following three conditions:*

i. $E_0 = \mathbb{C}$,

ii. $\iota_{0,m}\colon E_m \to \mathbb{C} \otimes E_m$ *and* $\iota_{m,0}\colon E_m \to E_m \otimes \mathbb{C}$ *are the canonical identifications* $v \mapsto 1 \otimes v$ *and* $v \mapsto v \otimes 1$, *iii.* $(\mathrm{id}_{E_k} \otimes \iota_{l,m}) \circ \iota_{k,l+m} = (\iota_{k,l} \otimes \mathrm{id}_{E_m}) \circ \iota_{k+l,m}$.

The more general subproduct systems used in the literature, notably in [8] differ from this definition in that the E_n are C^*-correspondences instead of finite-dimensional inner product spaces. The definition used here is obtained when the "correspondences" E_n are over the trivial C^*-algebra $\mathbb{C}$, and required to be finite-dimensional.

The noncommutative Nullstellensatz describes how for a fixed inner product space $V \cong \mathbb{C}^n$, a subproduct system $(E_k)_{k\geq 0}$ with $E_1 \subseteq V$ relates to a homogeneous two-sided ideal $I \subseteq \mathbb{C}\langle X_1, \ldots, X_n\rangle$. To state it, we need the following simple but crucial observation:

Lemma 2.1.2. *Given an n-dimensional complex inner product space V with orthonormal basis $\{e_1, \ldots, e_n\}$, there is a natural algebra isomorphism between the tensor algebra $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ and the polynomial algebra $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, which is obtained by identifying the algebra generators $(e_i)_{i=1}^n$ with the generators $(X_i)_{i=1}^n$.*

Notation 2.1.3. *For a polynomial $p \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$, the image under the identification of Lemma 2.1.2 is written $p(\mathbf{e})$, whereas for a vector $v \in T(V)$, we use the notation $\mathbf{X}(v)$ for the corresponding element in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$.*

For example, when $v = 1 + 2e_1 \otimes e_2 + 3e_4 \otimes e_3 \otimes e_2 \in T(V)$ we have

$$\mathbf{X}(v) = 1 + 2X_1X_2 + 3X_4X_3X_2$$

and similarly for $p = 1 + 2X_1X_2 + 3X_4X_3X_2 \in \mathbb{C}\langle X_1, X_2, X_3, X_4\rangle$ we have

$$p(\mathbf{e}) = 1 + 2e_1 \otimes e_2 + 3e_4 \otimes e_3 \otimes e_2.$$

The noncommutative nullstellensatz can now be stated:

Theorem 2.1.4. *(*Noncommutative Nullstellensatz*) There is an inclusion-reversing bijective correspondence between subproduct systems $(E_m)_{m=0}^{\infty}$ with $E_m \subseteq (V)^{\otimes m}$ and homogeneous two-sided proper ideals $I \subseteq \mathbb{C}\langle X_1, \ldots, X_k\rangle$. Under this correspondence, a subproduct system $(E_m)_{m=0}^{\infty}$ corresponds to the ideal*

$$I_{E_m} = \mathrm{Span}\{\mathbf{X}(v) \colon \exists m \text{ such that } v \in (E_m)^{\perp} \subseteq V^{\otimes m}\},$$

and an ideal $I \subseteq \mathbb{C}\langle X_1, \ldots, X_k\rangle$ corresponds to the subproduct system $(E_m^I)_{m=0}^{\infty}$ where

$$E_m^I = \{p(\mathbf{e}) \colon p \in I \text{ is homogeneous of degree } m\}^{\perp} \subseteq V^{\otimes m},$$

where the maps $\iota_{k,m} \colon E_{k+m}^I \hookrightarrow E_k^I \otimes E_m^I$ are induced by the natural isomorphism $V^{\otimes(k+m)} \cong V^{\otimes k} \otimes V^{\otimes m}$.

Proof. This is [8, Proposition 7.2] □

In section 2.3, for each co-representation $\varphi\colon V \to V \otimes \mathcal{O}(SU_q(2))$ we construct a subproduct system related to the subspace $\det(\rho)$.

Under the non-commutative Nullstellensatz, this subspace becomes an ideal generated by quadratic polynomials, which puts it in the context of quadratic algebras as found in e.g. [6, Chapter 4]. The following section will study $\det(\rho)$ in greater detail, and section 2.4 will give more insight in the related quadratic algebras for irreducible ρ.

2.2 The determinant of a co-representation

This section describes and studies the determinant $\det(\rho)$ of a co-representation, which will play a central role in the construction of the subproduct system.

Definition 2.2.1. *The* subspace of co-invariant elements $\mathrm{CoInv}(\rho) \subseteq V$ *of a co-representations* $\rho\colon V \to V \otimes \mathcal{A}$ *is the set* $\{v \in V : \rho(v) = v \otimes 1\}$.

Note the similarity and subtle difference between this definition and the definition of co-invariant subspaces (definition 1.2.3). In particular, $\mathrm{CoInv}(\rho)$ is a co-invariant subspace.

Definition 2.2.2. *The* determinant $\det(\rho) \subseteq V \otimes V$ *of an* $\mathcal{A}$*-co-representation*

$$\rho\colon V \to V \otimes \mathcal{A}$$

is the space

$$\mathrm{CoInv}(\rho^{\otimes k}) \subseteq V^{\otimes k},$$

where $k \in \mathbb{N}_{\geq 1}$ *is the minimal number such that there exists some co-representation* $\varphi\colon V \to V \otimes \mathcal{A}$ *with* $\mathrm{CoInv}(\varphi) = \{0\}$ *and* $\dim_{\mathbb{C}}(V) = k$.

In the case that $\mathcal{A} = \mathcal{O}(SU_q(2))$, we have that $k = 2$ for the co-representation φ_1 from Theorem 1.3.3.

Another example is when $\mathcal{A} = \mathcal{O}(SU(3))$ is a function Hopf-algebra on the classical Lie group $SU(3)$, in which case $k = 3$ for the co-representation $\varphi\colon \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathcal{O}(SU_q(3))$ obtained from the fundamental group action $\mathbb{C}^3 \times SU(3) \to \mathbb{C}^3$ through Lemma 1.2.10.

This thesis will from now on only focus on the case that $\mathcal{A} = \mathcal{O}(SU_q(2))$, so $\det(\rho) = \mathrm{CoInv}(\rho \otimes \rho)$. In this case, the name "determinant" has been chosen because of the following observation:

Lemma 2.2.3. *When* $\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$ *is an irreducible co-representation, we have that* $\det(\rho)$ *is a 1-dimensional subspace of* $V \otimes V$.

In particular, for the co-representation

$$\varphi_1\colon \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_1 \to \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_1 \otimes \mathcal{O}(SU_q(2))$$

we have $\det(\rho) = \mathrm{Span}(\mathcal{X} \otimes \mathcal{Y} - q\mathcal{Y} \otimes \mathcal{X})$

Proof. Assume without loss of generality that ρ is the $(n+1)$-dimensional irreducible co-representation φ_n from Theorem 1.2.11. By Lemma 1.4.1, the co-representation $\rho \otimes \rho$ decomposes as $\rho \otimes \rho \cong \bigoplus_{k=0}^{n} \varphi_{2k}$.

All co-invariant subspaces of $\rho \otimes \rho$ are therefore given by

$$\bigoplus_{k \in S} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k} \subseteq \mathbb{C}[\mathcal{X},\mathcal{Y}]_n \otimes \mathbb{C}[\mathcal{X},\mathcal{Y}]_n,$$

where the $\subseteq$-identification is as in Lemma 1.4.1, and $S \subseteq \{0,1,\ldots,n\}$ is any subset. It follows that $\mathrm{CoInv}(\rho \otimes \rho)$ must be of this form as well. Because for $k \neq 0$ the elements in $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k}$ are not co-invariant under φ_{2k}, it follows that k can only be 0. We thus obtain that

$$\begin{aligned}\mathrm{CoInv}(\rho \otimes \rho) &= \bigoplus_{k \in \{0\}} \mathbb{C}_q[\mathcal{X},\mathcal{Y}]_{2k} \\ &= \mathrm{Span}(f_0^{(0)}) \\ &= \mathrm{Span}\left(\sum_{j+k=0} C_q\left(\frac{n}{2},\frac{n}{2},0;k,j,0\right) f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})}\right).\end{aligned}$$

In [5, Chapter 3.4.3 just below formula (67)] we find $C_q(l_1,l_2,0;j,-j,0) = \frac{(-1)^{l_1-j}q^j}{\sqrt{[2l_1+1]}}$ when $l_1 = l_2$, so we see that *

$$\mathrm{CoInv}(\rho \otimes \rho) = \mathrm{Span}\left(\sum_{j=-\frac{n}{2}}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-j}q^j}{\sqrt{[n+1]_q}} f_j^{(\frac{n}{2})} \otimes f_{-j}^{(\frac{n}{2})}\right). \tag{2.1}$$

*The cautious reader might have noticed that here we used the formula for $C_q(\frac{n}{2},\frac{n}{2},0;j,-j,0)$ instead of $C_q(\frac{n}{2},\frac{n}{2},0;-j,j,0)$. There must be some miscalculation somewhere, either here or in the proof of Theorem 1.2.11. Calculations for small n do show that this formula should be used for $\det(\rho)$, and not the one using $C_q(l_2,l_1,0;-j,j,0)$.

Writing this in the basis $\mathcal{X}^{n-k}\mathcal{Y}^k$ of $\mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n$ and dropping a constant factor of $\sqrt{\frac{(-q)^n}{[n+1]_q}}$, we get

$$\det(\rho) = \mathrm{CoInv}(\rho\otimes\rho) = \mathrm{Span}\left(\sum_{j=0}^{n}(-q)^j \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right] \mathcal{X}^{n-j}\mathcal{Y}^j \otimes \mathcal{X}^j\mathcal{Y}^{n-j}\right).$$

For small n, this gives

$$\det(\varphi_1) = \mathrm{Span}\Big(\mathcal{X}\otimes\mathcal{Y} - q\mathcal{Y}\otimes\mathcal{X}\Big), \tag{2.2}$$

$$\det(\varphi_2) = \mathrm{Span}\Big(\mathcal{X}^2\otimes\mathcal{Y}^2 - q(q^{-2}+1)\mathcal{X}\mathcal{Y}\otimes\mathcal{X}\mathcal{Y} + q^2\mathcal{Y}^2\mathcal{X}^2\Big), \tag{2.3}$$

$$\begin{aligned}\det(\varphi_3) = \mathrm{Span}\Big(&\mathcal{X}^3\otimes\mathcal{Y}^3 - q(q^{-4}+q^{-2}+1)\mathcal{X}^2\mathcal{Y}\otimes\mathcal{X}\mathcal{Y}^2 \\ &+ q^2(q^{-4}+q^{-2}+1)\mathcal{X}\mathcal{Y}^2\otimes\mathcal{X}^2\mathcal{Y} - q^3\mathcal{Y}^3\otimes\mathcal{X}^3\Big),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\det(\varphi_4) = \mathrm{Span}\Big(&\mathcal{X}^4\otimes\mathcal{Y}^4 - q(q^{-6}+q^{-4}+q^{-2}+1)\mathcal{X}^3\mathcal{Y}\otimes\mathcal{X}\mathcal{Y}^3 \\ &+ q^2(q^{-8}+q^{-6}+2q^{-4}+q^{-2}+1)\mathcal{X}^2\mathcal{Y}^2\otimes\mathcal{X}^2\mathcal{Y}^2 \\ &- q^3(q^{-6}+q^{-4}+q^{-2}+1)\mathcal{X}\mathcal{Y}^3\otimes\mathcal{X}^3\mathcal{Y} + q^4\mathcal{Y}^4\otimes\mathcal{X}^4\Big).\end{aligned} \tag{2.5}$$

It is a simple but tedious verification to show that these vectors are indeed co-invariant under these respective co-representations as found in Example 1.3.5. □

We can also describe the determinant of a reducible co-representation of $\mathcal{O}(SU_q(2))$:

Lemma 2.2.4. *Let $\rho\colon V \to V\otimes\mathcal{O}(SU_q(2))$ be a reducible unitary co-representation. Let $V = \bigoplus_i(\oplus_{j=1}^{n_i} V_{ij})$ be a decomposition such that for each V_{ij} there is a bijective orthogonal map $p_{ij}\colon V_i \to V_{ij}$ that intertwines $\rho|_{V_{ij}}$ with the map $\varphi_i\colon V_i \to V_i\otimes\mathcal{O}(SU_q(2))$ from Theorem 1.3.3, i.e. the diagram*

$$\begin{array}{ccc} V_i & \xrightarrow{\;\;\varphi_i\;\;} & V_i\otimes\mathcal{O}(SU_q(2)) \\ {\scriptstyle p_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle p_{ij}\otimes\mathrm{id}} \\ V & \xrightarrow{\;\;\rho\;\;} & V\otimes\mathcal{O}(SU_q(2)) \end{array}$$

commutes. Then

$$\det(\rho) = \bigoplus_i \left(\bigoplus_{j_1,j_2=1}^{n_i} (p_{ij_1} \otimes p_{ij_2})(\det(\varphi_i)) \right)$$

Proof. Similarly to the proof of Lemma 2.2.3, the determinant can be found by decomposing $\rho \otimes \rho$ into irreducible co-representations via Lemma 1.4.1, and then the trivial irreducible co-representations of this decomposition constitute the determinant. Clearly,

$$\rho \otimes \rho = \bigoplus_{i_1,j_1,i_2,j_2} \rho|_{V_{i_1 j_1}} \otimes \rho|_{V_{i_2 j_2}} \cong \bigoplus_{i_1,j_1,i_2,j_2} \varphi_{i_1} \otimes \varphi_{i_2}$$

and thus

$$\operatorname{CoInv}(\rho \otimes \rho) = \bigoplus_{i_1,j_1,i_2,j_2} \operatorname{CoInv}\left(\rho|_{V_{i_1 j_2}} \otimes \rho|_{V_{i_2 j_2}} \right). \tag{2.6}$$

Thanks to Lemma 1.4.1, we have

$$\varphi_{i_1} \otimes \varphi_{i_2} \cong \bigoplus_{k=0}^{\min\{i_1,i_2\}} \varphi_{2k+|i_1-i_2|}.$$

In this decomposition, the trivial co-representation, denoted φ_0, is in the direct sum only when $i_1 = i_2$. And when $i_1 = i_2 = i$, we have that $\operatorname{CoInv}(\varphi_{i_1} \otimes \varphi_{i_2}) = \det(\varphi_i)$, so we obtain that

$$\operatorname{CoInv}\left(\rho|_{V_{i_1 j_2}} \otimes \rho|_{V_{i_2 j_2}} \right) = \begin{cases} (p_{ij_1} \otimes p_{ij_2})(\det(\varphi_i)) & \text{if } i_1 = i_2 = i, \\ \{0\} & \text{if } i_1 \neq i_2. \end{cases} \tag{2.7}$$

Combining (2.6) and (2.7) gives the desired result. □

For example, take the vector space V as

$$\begin{aligned} V_1 &= \mathbb{C}_q[\mathcal{X}_1, \mathcal{Y}_1]_1 = \operatorname{Span}(\mathcal{X}_1, \mathcal{Y}_1), \\ V_2 &= \mathbb{C}_q[\mathcal{X}_2, \mathcal{Y}_2]_1 = \operatorname{Span}(\mathcal{X}_2, \mathcal{Y}_2), \\ V_3 &= \mathbb{C}_q[\mathcal{X}_3, \mathcal{Y}_3]_2 = \operatorname{Span}(\mathcal{X}_3^2, \mathcal{X}_3\mathcal{Y}_3, \mathcal{Y}_3^2), \\ V &= V_1 \oplus V_2 \oplus V_3 = \operatorname{Span}(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, \mathcal{X}_3^2, \mathcal{X}_3\mathcal{Y}_3, \mathcal{Y}_3^2) \end{aligned}$$

where the V_i are as in Definition 1.3.1, and the co-representation ρ as

$$\begin{aligned} \varphi_{11} &\colon V_1 \to V_1 \otimes \mathcal{O}(SU_q(2)), \\ \varphi_{12} &\colon V_2 \to V_2 \otimes \mathcal{O}(SU_q(2)), \\ \varphi_{23} &\colon V_3 \to V_3 \otimes \mathcal{O}(SU_q(2)), \\ \rho = \varphi_{11} \oplus \varphi_{12} \oplus \varphi_{23} &\colon V \to V \otimes \mathcal{O}(SU_q(2)), \end{aligned}$$

where the representation φ_{ij} agrees with the representation φ_i from Theorem 1.3.3. Then

$$\begin{aligned}\det(\rho) &= \mathrm{CoInv}(\varphi_{11}\otimes\varphi_{11})\oplus\mathrm{CoInv}(\varphi_{11}\otimes\varphi_{12})\oplus\\ &\quad\ \mathrm{CoInv}(\varphi_{12}\otimes\varphi_{11})\oplus\mathrm{CoInv}(\varphi_{12}\otimes\varphi_{12})\oplus\\ &\quad\ \mathrm{CoInv}(\varphi_{23}\otimes\varphi_{23})\\ &= \mathrm{Span}\Big((\mathcal{X}_1\otimes\mathcal{Y}_1-q\mathcal{Y}_1\otimes\mathcal{X}_1),\ (\mathcal{X}_1\otimes\mathcal{Y}_2-q\mathcal{Y}_1\otimes\mathcal{X}_2),\\ &\qquad (\mathcal{X}_2\otimes\mathcal{Y}_1-q\mathcal{Y}_2\otimes\mathcal{X}_1),\ (\mathcal{X}_2\otimes\mathcal{Y}_2-q\mathcal{Y}_2\otimes\mathcal{X}_2),\\ &\qquad \Big(\mathcal{X}_3^2\otimes\mathcal{Y}_3^2-q(1+q^{-2})\mathcal{X}_3\mathcal{Y}_3\otimes\mathcal{X}_3\mathcal{Y}_3+q^2\mathcal{Y}_3^2\mathcal{X}_3^2\Big)\Big).\end{aligned}$$

2.2.1 Temperley–Lieb vectors

In this section we investigate a property known as being "Temperley–Lieb", as found in [3, Definition 1.2]. Temperley–Lieb vectors and algebras are of interest in several mathematical fields, including braid theory and quantum groups.

Definition 2.2.5. *Let V be a complex finite-dimensional inner product space V. A vector $\delta\in V\otimes V$ is called* Temperley-Lieb *when the orthogonal projection $e\colon V\otimes V\to \mathrm{Span}(\delta)$ satisfies*

$$(e\otimes 1)(1\otimes e)(e\otimes 1)=\frac{1}{\lambda}e\otimes 1$$

for some $\lambda\in\mathbb{R}_{>0}$

We can show that $\det(\varphi_n)$ is Temperley-Lieb:

Lemma 2.2.6. *Let ρ be as in Lemma 2.2.3. There exists a $\lambda\in\mathbb{R}_{>0}$ such that the ortogonal projection*

$$e\colon V\otimes V\to\det(\rho)$$

onto the subspace $\det(\rho)$ satisfies $(e\otimes 1)(1\otimes e)(e\otimes 1)=\frac{1}{\lambda}e\otimes 1$. In other words, $\det(\rho)$ is Temperley-Lieb.

Proof. This is most easily proven by direct computation. Recall that by Lemma 1.3.4, the inner product on V is induced by the basis $f_j^{(\frac{n}{2})}$. By the proof of Lemma 2.2.3, $\det(\rho)=\mathrm{Span}(\delta)$ where

$$\delta=\sum_{j=-\frac{n}{2}}^{\frac{n}{2}}\frac{(-1)^{\frac{n}{2}-j}q^j}{\sqrt{[n+1]_q}}f_j^{(\frac{n}{2})}\otimes f_{-j}^{(\frac{n}{2})}=:\sum_{j=-\frac{n}{2}}^{\frac{n}{2}}\delta_j\, f_j^{(\frac{n}{2})}\otimes f_{-j}^{(\frac{n}{2})}. \tag{2.8}$$

Note that $\|\delta\|^2 = 1$, as by the geometric series we have

$$\|\delta\|^2 = \frac{1}{[n+1]_q}(q^{-n} + q^{2-n} + \cdots + q^{n-2} + q^n) = \frac{1}{[n+1]_q} q^{-n} \left(\sum_{k=0}^{n} q^{2k} \right)$$
$$= \frac{1}{[n+1]_q} q^{-n} \frac{1-q^{2(n+1)}}{1-q^2} = \frac{1}{[n+1]_q} \frac{q^{-(n+1)} - q^{n+1}}{q^{-1}-q} = 1. \tag{2.9}$$

On an element $f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})}$, the projection e is given by

$$e\left(f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})}\right) = \frac{\left\langle f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})}, \delta \right\rangle}{\|\delta\|^2} = (\delta_a \mathbf{1}_{a=-b})\delta.$$

Hence, for a general basis element $f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}$ of $V \otimes V \otimes V$, we have

$$(e \otimes 1)\left(f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}\right) = (\delta_a \mathbf{1}_{a=-b})\delta \otimes f_c^{(\frac{n}{2})}.$$

Writing $\mu_{ab} = \delta_a \mathbf{1}_{a=-b}$ and applying $(1 \otimes e)$ now gives

$$\begin{aligned}(1 \otimes e)(e \otimes 1)\left(f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}\right) &= \mu_{ab} \sum_j \delta_j f_j^{(\frac{n}{2})} \otimes e\left(f_{-j}^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}\right) \\ &= \mu_{ab} \sum_j (\delta_j \delta_{-j} \mathbf{1}_{c=j}) f_j^{(\frac{n}{2})} \otimes \delta \\ &= (\mu_{ab}\delta_c\delta_{-c}) f_c^{(\frac{n}{2})} \otimes \delta.\end{aligned}$$

Applying $(e \otimes 1)$ to the result gives

$$\begin{aligned}&(e \otimes 1)(1 \otimes e)(e \otimes 1)\left(f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}\right) \\ &\qquad = \mu_{ab}\delta_c\delta_{-c} \sum_j \delta_j e\left(f_c^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})}\right) \otimes f_{-j}^{(\frac{n}{2})} \\ &\qquad = \mu_{ab}\delta_c\delta_{-c} \sum_j (\delta_j \delta_c \mathbf{1}_{c=-j})\delta \otimes f_{-j}^{(\frac{n}{2})} \\ &\qquad = \mu_{ab}\left(\delta_c\delta_{-c}\right)^2 \delta \otimes f_c^{(\frac{n}{2})} \\ &\qquad = \left(\delta_c\delta_{-c}\right)^2 (e \otimes 1)\left(f_a^{(\frac{n}{2})} \otimes f_b^{(\frac{n}{2})} \otimes f_c^{(\frac{n}{2})}\right).\end{aligned}$$

Finally, $\delta_j = \frac{(-1)^{\frac{n}{2}-j} q^j}{\sqrt{[n+1]_q}}$ gives us that $\delta_c\delta_{-c} = \frac{(-1)^n}{[n+1]_q}$ is independent of c, so the lemma holds for $\lambda = \left([n+1]_q\right)^2 = \left(\frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}\right)^2$. □

2.2.2 The determinant and braids

This section, together with section 2.2.3, will be dedicated to proving the following theorem:

Theorem 2.2.7. *For each unitary finite-dimensional co-representation* $\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$ *there exists a map* $\sigma\colon V \otimes V \to V \otimes V$ *such that*

$$\det(\rho) = (\mathrm{Inv}(\sigma))^{\perp} \tag{2.10}$$

and

$$(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma). \tag{2.11}$$

Equation (2.11) is known as the *braid equation* or *Yang-Baxter equation*, and is central to the study of a broad class of Hopf algebras. First, the simpler case when ρ is an irreducible co-representation of $\mathcal{O}(SU_q(2))$ will be studied. In particular, we have the following result:

Theorem 2.2.8. *Given an irreducible co-representation*

$$\varphi_n\colon V \to V \otimes \mathcal{O}(SU_q(2)),$$

a map $\sigma\colon V \otimes V \to V \otimes V$ *satisfies (2.10) and (2.11) if and only if* σ *expressed on the basis (1.30) has the form*

$$\sigma\left(f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})}\right) = \begin{cases} \sum_{i=-\frac{n}{2}}^{\frac{n}{2}} \left(\mathbf{1}_{j=i} - (-q)^{j-i} x_i\right) f_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} & \text{if } j = -k, \\ f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})} & \text{if } j \neq -k, \end{cases} \tag{2.12}$$

where for each j*, the numbers* x_k *satisfy the equation*

$$1 + x_j x_{-j} = \sum_m x_m. \tag{2.13}$$

The proof of 2.2.8 starts with the following observation:

Lemma 2.2.9. *When* $\varphi_n\colon V \to V \otimes \mathcal{O}(SU_q(2))$ *is irreducible as in Theorem 1.3.3, any* $\sigma\colon V \otimes V \to V \otimes V$ *that satisfies equation (2.10) is of the form*

$$\sigma\left(f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})}\right) = \begin{cases} \sum_{i=-\frac{n}{2}}^{\frac{n}{2}} \left(\mathbf{1}_{j=i} - (-q)^{j-i} x_i\right) f_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} & \text{if } j = -k, \\ f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})} & \text{if } j \neq -k, \end{cases}$$

where $(x_k)_{k=-\frac{n}{2}}^{\frac{n}{2}} \in V$ *is an arbirary non-zero vector.*

Visually, the function σ above is given by

$$\sigma\left(f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})}\right) = \begin{cases} \sum_k \sigma_{jk} f_k^{(\frac{n}{2})} \otimes f_{-k}^{(\frac{n}{2})} & \text{if } k = -j, \\ f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})} & \text{if } k \neq -j, \end{cases}$$

where σ_{jk} is given by

- If n is odd and $\dim_{\mathbb{C}}(V)$ is even:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & \sigma_{\frac{5}{2}\frac{5}{2}} & \sigma_{\frac{3}{2}\frac{5}{2}} & \sigma_{\frac{1}{2}\frac{5}{2}} & \sigma_{-\frac{1}{2}\frac{5}{2}} & \sigma_{-\frac{3}{2}\frac{5}{2}} & \sigma_{-\frac{5}{2}\frac{5}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2}\frac{3}{2}} & \sigma_{\frac{3}{2}\frac{3}{2}} & \sigma_{\frac{1}{2}\frac{3}{2}} & \sigma_{-\frac{1}{2}\frac{3}{2}} & \sigma_{-\frac{3}{2}\frac{3}{2}} & \sigma_{-\frac{5}{2}\frac{3}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2}\frac{1}{2}} & \sigma_{\frac{3}{2}\frac{1}{2}} & \sigma_{\frac{1}{2}\frac{1}{2}} & \sigma_{-\frac{1}{2}\frac{1}{2}} & \sigma_{-\frac{3}{2}\frac{1}{2}} & \sigma_{-\frac{5}{2}\frac{1}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{1}{2}} & \sigma_{\frac{3}{2},-\frac{1}{2}} & \sigma_{\frac{1}{2},-\frac{1}{2}} & \sigma_{-\frac{1}{2},-\frac{1}{2}} & \sigma_{-\frac{3}{2},-\frac{1}{2}} & \sigma_{-\frac{5}{2},-\frac{1}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{3}{2}} & \sigma_{\frac{3}{2},-\frac{3}{2}} & \sigma_{\frac{1}{2},-\frac{3}{2}} & \sigma_{-\frac{1}{2},-\frac{3}{2}} & \sigma_{-\frac{3}{2},-\frac{3}{2}} & \sigma_{-\frac{5}{2},-\frac{3}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{5}{2}} & \sigma_{\frac{3}{2},-\frac{5}{2}} & \sigma_{\frac{1}{2},-\frac{5}{2}} & \sigma_{-\frac{1}{2},-\frac{5}{2}} & \sigma_{-\frac{3}{2},-\frac{5}{2}} & \sigma_{-\frac{5}{2},-\frac{5}{2}} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & 1-x_{\frac{5}{2}} & x_{\frac{5}{2}}q^{-1} & -x_{\frac{5}{2}}q^{-2} & x_{\frac{5}{2}}q^{-3} & -x_{\frac{5}{2}}q^{-4} & x_{\frac{5}{2}}q^{-5} & \ddots \\ \ddots & x_{\frac{3}{2}}q & 1-x_{\frac{3}{2}} & x_{\frac{3}{2}}q^{-1} & -x_{\frac{3}{2}}q^{-2} & x_{\frac{3}{2}}q^{-3} & -x_{\frac{3}{2}}q^{-4} & \ddots \\ \ddots & -x_{\frac{1}{2}}q^{2} & x_{\frac{1}{2}}q & 1-x_{\frac{1}{2}} & x_{\frac{1}{2}}q^{-1} & -x_{\frac{1}{2}}q^{-2} & x_{\frac{1}{2}}q^{-3} & \ddots \\ \ddots & x_{-\frac{1}{2}}q^{3} & -x_{-\frac{1}{2}}q^{2} & x_{-\frac{1}{2}}q & 1-x_{-\frac{1}{2}} & x_{-\frac{1}{2}}q^{-1} & -x_{-\frac{1}{2}}q^{-2} & \ddots \\ \ddots & -x_{-\frac{3}{2}}q^{4} & x_{-\frac{3}{2}}q^{3} & -x_{-\frac{3}{2}}q^{2} & x_{-\frac{3}{2}}q & 1-x_{-\frac{3}{2}} & x_{-\frac{3}{2}}q^{-1} & \ddots \\ \ddots & x_{-\frac{5}{2}}q^{5} & -x_{-\frac{5}{2}}q^{4} & x_{-\frac{5}{2}}q^{3} & -x_{-\frac{5}{2}}q^{2} & x_{-\frac{5}{2}}q & 1-x_{-\frac{5}{2}} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- If n is even and $\dim_{\mathbb{C}}(V)$ is odd:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & \sigma_{2,2} & \sigma_{1,2} & \sigma_{0,2} & \sigma_{-1,2} & \sigma_{-2,2} & \ddots \\ \ddots & \sigma_{2,1} & \sigma_{1,1} & \sigma_{0,1} & \sigma_{-1,1} & \sigma_{-2,1} & \ddots \\ \ddots & \sigma_{2,0} & \sigma_{1,0} & \sigma_{0,0} & \sigma_{-1,0} & \sigma_{-2,0} & \ddots \\ \ddots & \sigma_{2,-1} & \sigma_{1,-1} & \sigma_{0,-1} & \sigma_{-1,-1} & \sigma_{-2,-1} & \ddots \\ \ddots & \sigma_{2,-2} & \sigma_{1,-2} & \sigma_{0,-2} & \sigma_{-1,-2} & \sigma_{-2,-2} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & 1-x_2 & x_2q^{-1} & -x_2q^{-2} & x_2q^{-3} & -x_2q^{-4} & \ddots \\ \ddots & x_1q & 1-x_1 & x_1q^{-1} & -x_1q^{-2} & x_1q^{-3} & \ddots \\ \ddots & -x_0q^2 & x_0q & 1-x_0 & x_0q^{-1} & -x_0q^{-2} & \ddots \\ \ddots & x_{-1}q^3 & -x_{-1}q^2 & x_{-1}q & 1-x_{-1} & x_{-1}q^{-1} & \ddots \\ \ddots & -x_{-2}q^4 & x_{-2}q^3 & -x_{-2}q^2 & x_{-2}q & 1-x_{-2} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Proof of Lemma 2.2.9. Recall that by Lemma 1.3.4, $f_j^{(\frac{n}{2})}$ is an orthonormal basis of V, and by equation (2.1), we have

$$\det(\varphi_n) = \operatorname{Span}\left(\sum_j (-q)^j f_j^{(\frac{n}{2})} \otimes f_{-j}^{(\frac{n}{2})}\right).$$

For the remainder of this proof, introduce the shorthand notation $f_{jk} = f_j^{(\frac{n}{2})} \otimes f_k^{(\frac{n}{2})}$. We easily find that

$$\det(\varphi_n)^{\perp} = \operatorname{Span}\left((f_{jk})_{k \neq -j}, (q f_{j,-j} + f_{j+1,-j-1})\right).$$

A rigorous proof of this can be found in the proof of Lemma 2.4.4.

Now $\mathrm{Inv}(\sigma) = \det(\varphi_n)^\perp$ implies that $\sigma(f_{jk}) = f_{jk}$ when $k \neq -j$, and also $\sigma(qf_{j,-j} + f_{j+1,-j-1}) = qf_{j,-j} + f_{j+1,-j-1}$. Writing $\sigma(f_{j,-j}) = \sum_k \sigma_{jk} f_{k,-k}$, the last relation gives

$$q\sigma_{jk} + \sigma_{j+1,k} = \begin{cases} q & \text{if } j = k, \\ 1 & \text{if } j+1 = k, \\ 0 & \text{else.} \end{cases}$$

which is equivalent to

$$\sigma_{j+1,k} = \begin{cases} q(1-\sigma_{jk}) & \text{if } k = j, \\ 1 - q\sigma_{jk} & \text{if } k = j+1, \\ -q\sigma_{jk} & \text{else.} \end{cases}$$

After choosing $x_k = 1 - \sigma_{kk}$, we see that the above inductive relation is uniquely solved by $\sigma_{jk} = \mathbf{1}_{j=k} - (-q)^{j-k} x_k$. □

Having solved equation (2.10), the next equation of interest is equation (2.11). Consider the following lemma

Lemma 2.2.10. *For a finite-dimensional vector space V with basis $(f_j)_{j\in I}$ and a map*

$$\sigma \colon V \otimes V \to V \otimes V$$

that satisfies $\sigma(f_j \otimes f_k) = f_j \otimes f_k$ when $k \neq -j$, and

$$\sigma(f_j \otimes f_{-j}) = \sum_{k\in I} \sigma_{jk} f_k \otimes f_{-k},$$

the braid equation (2.11) is equivalent to the set of equations

$$\sigma_{jj}\sigma_{-j,-j}(\sigma_{-j,-j} - \sigma_{jj}) = \sum_{m\neq j} \sigma_{jm}\sigma_{mj} - \sum_{m\neq -j} \sigma_{-jm}\sigma_{m,-j} \qquad \forall j \in I, \quad (2.14)$$

$$\sigma_{jk}\sigma_{-j,-j}(1 - \sigma_{jj}) = \sum_{m\neq j} \sigma_{jm}\sigma_{mk} \qquad j \neq k, \quad (2.15)$$

$$\sigma_{ij}\sigma_{-j,-j}(1 - \sigma_{jj}) = \sum_{m\neq j} \sigma_{im}\sigma_{mj} \qquad i \neq j, \quad (2.16)$$

$$\sigma_{ik} - \sigma_{ij}\sigma_{jk}\sigma_{-j,-j} = \sum_{m\neq j} \sigma_{im}\sigma_{mk} \qquad i \neq j, j \neq k. \quad (2.17)$$

Proof. Introduce the shorthand notations $f_{ijk} = f_i \otimes f_j \otimes f_k$ and $\check{\jmath} = -j$. The braid equation can be investigated on the basis elements f_{ijk}. The following cases can be distinguised:

1. $i \neq \check{j}, j \neq \check{k}$,
2. $i = \check{j}, j = \check{k}$,
3. $i = \check{j}, j \neq \check{k}$ or $i \neq \check{j}, j = \check{k}$.

In the first case, we have

$$(1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)(f_{ijk}) = f_{ijk} = (\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1)(f_{ijk}).$$

For the second case, we can calculate

$$\begin{aligned}
&(1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)(f_{j\check{j}j}) \\
&\qquad =(1 \otimes \sigma)(\sigma \otimes 1)\left(\sigma_{\check{j}\check{j}} f_{j\check{j}j} + \sum_{m\in I} \mathbf{1}_{m\neq \check{j}} \sigma_{\check{j}m} f_{jm\check{m}}\right) \\
&\qquad =(1 \otimes \sigma)\left(\sum_{k\in I} \sigma_{\check{j}\check{j}}\sigma_{jk} f_{k\check{k}j} + \sum_{m\in I} \mathbf{1}_{m\neq \check{j}} \sigma_{\check{j}m} f_{jm\check{m}}\right) \\
&\qquad =\sum_{k\in I} \mathbf{1}_{k\neq j} \sigma_{\check{j}\check{j}}\sigma_{jk} f_{k\check{k}j} + \sigma_{jj}\sigma_{\check{j}\check{j}}\sigma_{jk} f_{jk\check{k}} + \left(\sum_{m\in I} \mathbf{1}_{m\neq \check{j}} \sigma_{\check{j}m}\sigma_{mk}\right) f_{jk\check{k}}
\end{aligned}$$

and similarly

$$\begin{aligned}
&(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1)(f_{j\check{j}j}) \\
&\qquad =(\sigma \otimes 1)(1 \otimes \sigma)\left(\sigma_{jj} f_{j\check{j}j} + \sum_{m\in I} \mathbf{1}_{m\neq j} \sigma_{jm} f_{m\check{m}j}\right) \\
&\qquad =(\sigma \otimes 1)\left(\sum_{k\in I} \sigma_{jj}\sigma_{\check{j}k} f_{jk\check{k}} + \sum_{m\in I} \mathbf{1}_{m\neq j} \sigma_{jm} f_{m\check{m}j}\right) \\
&\qquad =\sum_{k\in I} \mathbf{1}_{k\neq \check{j}} \sigma_{jj}\sigma_{\check{j}k} f_{jk\check{k}} + \sigma_{jj}\sigma_{\check{j}\check{j}}\sigma_{jk} f_{k\check{k}j} + \left(\sum_{m\in I} \mathbf{1}_{m\neq j} \sigma_{jm}\sigma_{mk}\right) f_{k\check{k}j}.
\end{aligned}$$

Equating the coefficient for $f_{j\check{j}j}$ gives

$$\sigma_{jj}\sigma_{\check{j}\check{j}}\sigma_{\check{j}\check{j}} + \sum_{m\in I} \mathbf{1}_{m\neq \check{j}} \sigma_{\check{j}m}\sigma_{m\check{j}} = \sigma_{jj}\sigma_{\check{j}\check{j}}\sigma_{jj} + \sum_{m\in I} \mathbf{1}_{m\neq j} \sigma_{jm}\sigma_{mj},$$

which is equivalent to equation (2.14). Equating the coefficients for $f_{k\check{k}j}$ gives (when $k \neq j$) that

$$\sigma_{jk}\sigma_{\check{j}\check{j}} = \sigma_{jk}\sigma_{\check{j}\check{j}}\sigma_{jj} + \sum_{m\in I} \mathbf{1}_{m\neq j} \sigma_{jm}\sigma_{mk},$$

which is equivalent to equation (2.15). Lastly, note that equating the coefficients for $f_{jk\check{k}}$ gives exactly the same, but with $\check{j}$ instead of j.

For case 3, let $i \neq j$ and note that $(1 \otimes \sigma)(f_{i\check{i}j}) = f_{i\check{i}j}$ so the braid equation (2.11) is $(1 \otimes \sigma)(\sigma \otimes 1)(f_{i\check{i}j}) = (\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1)(f_{i\check{i}j})$. Now

$$\begin{aligned}(1 \otimes \sigma)(\sigma \otimes 1)(f_{i\check{i}j}) &= (1 \otimes \sigma)\left(\sigma_{ij} f_{j\check{j}j} + \sum_{k \in I} \mathbf{1}_{k \neq j} \sigma_{ik} f_{k\check{k}j}\right) \\ &= \sum_{k \in I} \sigma_{ij} \sigma_{\check{j}k} f_{jk\check{k}} + \mathbf{1}_{k \neq j} \sigma_{ik} f_{k\check{k}j}\end{aligned}$$

and

$$\begin{aligned}&(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1)(f_{i\check{i}j}) \\ &\quad=(\sigma \otimes 1)\left(\sum_{k \in I} \sigma_{ij} \sigma_{\check{j}k} f_{jk\check{k}} + \sum_{m \in I} \mathbf{1}_{m \neq j} \sigma_{im} f_{m\check{m}j}\right) \\ &\quad= \sum_{k \in I} \mathbf{1}_{k \neq \check{j}} \sigma_{ij} \sigma_{\check{j}k} f_{jk\check{k}} + \left(\sigma_{ij} \sigma_{\check{j}\check{j}} \sigma_{jk} + \sum_{m \in I} \mathbf{1}_{m \neq j} \sigma_{im} \sigma_{mk}\right) f_{k\check{k}j}.\end{aligned}$$

We see that the coefficients for $f_{jk\check{k}}$ already agree for $k \neq \check{j}$. Equating the coefficients for $f_{j\check{j}j}$ gives

$$\sigma_{ij} \sigma_{\check{j}\check{j}} = \sigma_{ij} \sigma_{\check{j}\check{j}} \sigma_{jj} + \sum_{m \in I} \mathbf{1}_{m \neq j} \sigma_{im} \sigma_{mj}$$

which is equivalent to equation (2.16), and equating the coefficient for $f_{k\check{k}j}$ gives for $k \neq j$ that

$$\sigma_{ik} = \sigma_{ij} \sigma_{\check{j}\check{j}} \sigma_{jk} + \sum_{m \in I} \mathbf{1}_{m \neq j} \sigma_{im} \sigma_{mk}$$

which is equivalent to equation (2.17). We can show that the braid equation for $f_{\check{j}i\check{i}}$ gives exactly the same equations, which completes the last case. □

Combining the results of Lemmas 2.2.9 and 2.2.10, we obtain the following result, which concludes the proof of Theorem 2.2.8:

Lemma 2.2.11. *When σ is as in Lemma 2.2.9, the equations (2.14)-(2.17) are all equivalent to*

$$\forall j, \sum_m x_m = 1 + x_j x_{-j}.$$

Proof. Note that if $m \neq i, j$, we have that

$$\sigma_{im}\sigma_{mj} = (-q)^{i-j} x_m x_j.$$

For (2.14) we find that

$$\begin{aligned}
(1 + x_j x_{-j} - (x_j + x_{-j}))(x_j - x_{-j}) &= (1 - x_j)(1 - x_{-j})(x_j - x_{-j}) \\
&= \sigma_{jj}\sigma_{-j,-j}(\sigma_{-j,-j} - \sigma_{jj}) \\
&\overset{(2.14)}{=} \sum_{m \neq j} \sigma_{jm}\sigma_{mj} - \sum_{m \neq -j} \sigma_{-jm}\sigma_{m,-j} \\
&= \left(\sum_m x_m x_j - x_j x_j\right) - \left(\sum_m x_m x_{-j} - x_{-j} x_{-j}\right) \\
&= (x_j - x_{-j}) \left(\sum_m x_m\right) - (x_j^2 - x_{-j}^2) \\
&= (x_j - x_{-j}) \left(\sum_m x_m - (x_j + x_{-j})\right),
\end{aligned}$$

which is equivalent to $x_j - x_{-j} = 0$ or $\sum_m x_m = 1 + x_j x_{-j}$. For (2.17) with $i = k \neq j$ we obtain

$$\begin{aligned}
1 - x_i + x_i \left(x_j x_{-j} - x_j\right) &= (1 - x_i) - x_j x_i (1 - x_{-j}) \\
&= \sigma_{ii} - \sigma_{ij}\sigma_{ji}\sigma_{-j,-j} \\
&\overset{(2.17)}{=} \sum_{m \neq j} \sigma_{im}\sigma_{mi} \\
&= \sigma_{ii}^2 + \sum_{m \neq i} \sigma_{im}\sigma_{mi} - \sigma_{ij}\sigma_{ji} \\
&= (1 - x_i)^2 + \sum_{m \neq i} x_m x_i - x_j x_i \\
&= 1 - 2x_i + x_i^2 + x_i \left(\sum_m x_m - x_i - x_j\right) \\
&= 1 - x_i + x_i \left(\sum_m x_m - 1 - x_j\right),
\end{aligned}$$

which is equivalent to $x_i = 0$ or $\sum_m x_m = 1 + x_j x_{-j}$

For (2.17) with $i \neq k$, as well as for (2.15) and (2.16), first note that for

$i \neq j$ we have

$$\begin{aligned}
\sum_m \sigma_{im}\sigma_{mj} &= \sigma_{ii}\sigma_{ij} + \sigma_{ij}\sigma_{jj} + \sum_{m\neq i,j} \sigma_{im}\sigma_{mj} \\
&= \sigma_{ij}(\sigma_{ii} + \sigma_{jj}) + \sum_{m\neq i,j} (-q)^{i-j} x_m x_j \\
&= \sigma_{ij}\left((1-x_i) + (1-x_j) - \left(\sum_m x_m - x_i - x_j\right)\right) \\
&= \sigma_{ij}\left(2 - \sum_m x_m\right). \qquad (2.18)
\end{aligned}$$

Now (2.17) with $i \neq k$ gives

$$\begin{aligned}
\sigma_{ik}(1 + x_j - x_j x_{-j}) &= -(-q)^{i-k} x_k (1 + x_j(1 - x_{-j})) \\
&= -(-q)^{i-k} x_k - (-q)^{i-k} x_j x_k \sigma_{-j,-j} \\
&= \sigma_{ik} - \sigma_{ij}\sigma_{jk}\sigma_{-j,-j} \\
&\overset{(2.17)}{=} \sum_{m\neq j} \sigma_{im}\sigma_{mk} \\
&\overset{(2.18)}{=} \sigma_{ik}\left(2 - \sum_m x_m\right) - \sigma_{ij}\sigma_{jk} \\
&= \sigma_{ik}\left(2 - \sum_m x_m\right) - (-q)^{i-k} x_j x_k \\
&= \sigma_{ik}\left(2 - \sum_m x_m + x_j\right),
\end{aligned}$$

which is equivalent to $\sigma_{ik} = 0$ or $\sum_m x_m = 1 + x_j x_{-j}$. Lastly, from (2.18) one sees that (2.15) immediately becomes

$$\begin{aligned}\sigma_{jk}(x_j - x_j x_{-j}) &= \sigma_{jk}\sigma_{-j,-j}(1-\sigma_{jj}) \\ &\overset{(2.15)}{=} \sum_{m\neq j} \sigma_{jm}\sigma_{mk} \\ &\overset{(2.18)}{=} \sigma_{jk}\left(2 - \sum_m x_m\right) - \sigma_{jj}\sigma_{jk} \\ &= \sigma_{jk}\left(2 - \sum_m x_m - (1-x_j)\right) \\ &= \sigma_{jk}\left(x_j + 1 - \sum_m x_m\right),\end{aligned}$$

which is equivalent to $\sigma_{jk} = 0$ or $\sum_m x_m = 1 + x_j x_{-j}$. Similarly (2.16) is equivalent to

$$\begin{aligned}\sigma_{ij}(x_j - x_j x_{-j}) &= \sigma_{ij}\sigma_{-j,-j}(1-\sigma_{jj}) \\ &\overset{(2.16)}{=} \sum_{m\neq j} \sigma_{im}\sigma_{mj} \\ &\overset{(2.18)}{=} \sigma_{ij}\left(2 - \sum_m x_m - \sigma_{jj}\right),\end{aligned}$$

which is equivalent to $\sigma_{ij} = 0$ or $\sum_m x_m = 1 + x_j x_{-j}$.

We conclude that (2.14)-(2.17) are satisfied if and only if $x_k = 0$ for all k, or $\sum_m x_m = 1 + x_j x_{-j}$ for all j. Because 2.2.9 requires x_k to be nonzero, it follows that $\sum_m x_m = 1 + x_j x_{-j}$ must hold for all j. □

We can now try to solve equation (2.13). The following statement says that there are $\lceil \frac{n}{2} \rceil$ degrees of freedom in finding a solution:

Lemma 2.2.12. *When x_k is given for all $k > 0$, equation (2.13) has a solution except when we are in the very specific case that n is odd, all x_k are nonzero, $\sum_{k>0} \frac{1}{x_k} = 1$ and $\sum_{k>0} x_k \neq 1$.*

Note that instead of all $k > 0$, we could take any subset of the index set that only contains at most one of $\{k, -k\}$ for each k.

Proof. Let x_k for $k > 0$ be arbitrary.

1. If $\exists j$ such that $x_j = 0$, then $x_k x_{-k} = x_j x_{-j} = 0$ for all k. Furthermore, (2.13) becomes $\sum_m x_m = 1$. The numbers $x_k, k \leq 0$ can now be chosen such that these two requirements are satisfied: if n is odd, we require $x_0 = 0$ and for all $k > 0$ where $x_k \neq 0$ we require $x_{-k} = 0$. The other x_{-k} for $k > 0$ must be chosen such that $\sum_m x_m = 1$, e.g. $x_{-j} = 1 - \sum_{k>0} x_k$ and $x_{-k} = 0$ for $k \neq j$.

2. The next case is when $\forall j, x_j \neq 0$, $\sum_{k>0} \frac{1}{x_k} = 1$ and n is even. Then $x_{-j} = \frac{x_0^2}{x_j}$ and (2.13) becomes

$$1 + x_0^2 = \sum_{k>0} x_k + x_0 + \sum_{k>0} \frac{x_0^2}{x_k} = x_0 + \sum_{k>0} x_k + x_0^2.$$

   Hence we require $x_0 = 1 - \sum_{k>0} x_k$ and $x_{-k} = \frac{x_0^2}{x_k}$.

3. The next case is when $\forall j, x_j \neq 0$, $\sum_{k>0} \frac{1}{x_k} = 1$ and n is odd. If furthermore $\sum_{k>0} x_k = 1$, then for an arbitrary λ we can set $x_{-k} = \frac{\lambda}{x_k}$ such that (2.13) becomes

$$\sum_{k>0} x_k + \sum_{k>0} \frac{\lambda}{x_k} = 1 + \lambda,$$

   which is clearly satisfied. If $\sum_{k>0} x_k \neq 1$, the requirement that $x_k x_{-k} = x_m x_{-m}$ for all k, m gives a similar equation to the one above, which cannot be satisfied in this case.

4. The next case is when $\forall j, x_j \neq 0, \sum_{k>0} \frac{1}{x_k} \neq 1$ and n is odd. In this case, when we choose $x_{-k} = \frac{-\left(\frac{1-\sum_{m>0} x_m}{1-\sum_{m>0} \frac{1}{x_m}}\right)}{x_k}$, equation (2.13) becomes

$$\sum_{k>0} x_k + \sum_{k<0} \frac{-\left(\frac{1-\sum_{m>0} x_m}{1-\sum_{m>0} \frac{1}{x_m}}\right)}{x_k} = 1 - \frac{1-\sum_{m>0} x_m}{1-\sum_{m>0} \frac{1}{x_m}},$$

   which is true.

5. The last case is when $\forall j, x_j \neq 0, \sum_{k>0} \frac{1}{x_k} \neq 1$ and n is even. Then we must choose $x_{-k} = \frac{x_0^2}{x_k}$ and (2.13) becomes

$$1 + x_0^2 = x_0^2 \left(\sum_{k>0} \frac{1}{x_k} \right) + x_0 + \sum_{k>0} x_k,$$

which is equivalent to

$$x_0^2\left(\sum_{k>0}\frac{1}{x_k}-1\right)+x_0+\left(\sum_{k>0}x_k-1\right)=0.$$

Solving this for x_0 gives

$$x_0=\frac{1}{2\left(1-\sum_{k>0}\frac{1}{x_k}\right)}\pm\sqrt{\left(\frac{1}{2\left(1-\sum_{k>0}\frac{1}{x_k}\right)}\right)^2-\left(\frac{1-\sum_{m>0}x_m}{1-\sum_{m>0}\frac{1}{x_m}}\right)}.$$

This concludes the cases to be considered for Lemma 2.2.12. $\square$

We can now look at some specific examples:

1. We can choose $x_j=0$ for all j except one specific $k\neq 0$. Then (2.13) gives $x_k=1$, and thus σ has the following form:

$$\sigma\left(f_i^{(\frac{n}{2})}\otimes f_j^{(\frac{n}{2})}\right)=\begin{cases} f_i^{(\frac{n}{2})}\otimes f_j^{(\frac{n}{2})} & \text{if } -j\neq i,\\ f_i^{(\frac{n}{2})}\otimes f_{-i}^{(\frac{n}{2})}-(-q)^{i-k}f_k^{(\frac{n}{2})}\otimes f_{-k}^{(\frac{n}{2})} & \text{if } -j=i\neq k,\\ 0 & \text{if } -j=i=k.\end{cases}$$

   In particular, σ satisfies $\sigma\circ\sigma=\sigma$ and has nontrivial kernel.

2. For odd n (so $\dim_{\mathbb{C}}(\mathbb{C}_q[\mathcal{X},\mathcal{Y}]_n)$ is even), we can take

$$x_k=\operatorname{sgn}(k)=\begin{cases}1 & k>0,\\ -1 & k<0.\end{cases}$$

   Then (2.13) states that $1+1(-1)=0=\sum_{m>0}x_m+x_{-m}$, which is clearly true. Now σ has the form (as in Lemma 2.2.10)

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & \sigma_{\frac{5}{2}\frac{5}{2}} & \sigma_{\frac{3}{2}\frac{5}{2}} & \sigma_{\frac{1}{2}\frac{5}{2}} & \sigma_{-\frac{1}{2}\frac{5}{2}} & \sigma_{-\frac{3}{2}\frac{5}{2}} & \sigma_{-\frac{5}{2}\frac{5}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2}\frac{3}{2}} & \sigma_{\frac{3}{2}\frac{3}{2}} & \sigma_{\frac{1}{2}\frac{3}{2}} & \sigma_{-\frac{1}{2}\frac{3}{2}} & \sigma_{-\frac{3}{2}\frac{3}{2}} & \sigma_{-\frac{5}{2}\frac{3}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2}\frac{1}{2}} & \sigma_{\frac{3}{2}\frac{1}{2}} & \sigma_{\frac{1}{2}\frac{1}{2}} & \sigma_{-\frac{1}{2}\frac{1}{2}} & \sigma_{-\frac{3}{2}\frac{1}{2}} & \sigma_{-\frac{5}{2}\frac{1}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{1}{2}} & \sigma_{\frac{3}{2},-\frac{1}{2}} & \sigma_{\frac{1}{2},-\frac{1}{2}} & \sigma_{-\frac{1}{2},-\frac{1}{2}} & \sigma_{-\frac{3}{2},-\frac{1}{2}} & \sigma_{-\frac{5}{2},-\frac{1}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{3}{2}} & \sigma_{\frac{3}{2},-\frac{3}{2}} & \sigma_{\frac{1}{2},-\frac{3}{2}} & \sigma_{-\frac{1}{2},-\frac{3}{2}} & \sigma_{-\frac{3}{2},-\frac{3}{2}} & \sigma_{-\frac{5}{2},-\frac{3}{2}} & \ddots \\ \ddots & \sigma_{\frac{5}{2},-\frac{5}{2}} & \sigma_{\frac{3}{2},-\frac{5}{2}} & \sigma_{\frac{1}{2},-\frac{5}{2}} & \sigma_{-\frac{1}{2},-\frac{5}{2}} & \sigma_{-\frac{3}{2},-\frac{5}{2}} & \sigma_{-\frac{5}{2},-\frac{5}{2}} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \ddots & 0 & q^{-1} & -q^{-2} & q^{-3} & -q^{-4} & q^{-5} & \ddots \\ \ddots & q & 0 & q^{-1} & -q^{-2} & q^{-3} & -q^{-4} & \ddots \\ \cdots & -q^2 & q & 0 & q^{-1} & -q^{-2} & q^{-3} & \cdots \\ \cdots & -q^3 & q^2 & -q & 2 & -q^{-1} & q^{-2} & \cdots \\ \ddots & q^4 & -q^3 & q^2 & -q & 2 & -q^{-1} & \ddots \\ \ddots & -q^5 & q^4 & -q^3 & q^2 & -q & 2 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

3. We can request σ to be symmetric, i.e. $\sigma_{ij} = \sigma_{ji}$ for all $i \neq j$. It follows that $x_i = (-q)^{2(i-j)} x_j$, which is uniquely solved by

$$x_i = \lambda(-q)^{2i}$$

for an arbitrary λ. Now (2.13) becomes $\lambda\left(\sum_m (-q)^{2m}\right) = 1 + \lambda^2$ which is solved by

$$\lambda = \frac{\sum_m (-q)^{2m}}{2} \pm \sqrt{\left(\frac{\sum_m (-q)^{2m}}{2}\right)^2 - 1}.$$

For $n = 1$ this means

$$\lambda = -\frac{q+q^{-1}}{2} \pm \sqrt{\left(\frac{q+q^{-1}}{2}\right)^2 - 1} = -\frac{q+q^{-1} \mp (q-q^{-1})}{2} = -q^{\mp 1},$$

in which case σ has the form

$$\begin{pmatrix} \sigma_{\frac{1}{2}\frac{1}{2}} & \sigma_{-\frac{1}{2}\frac{1}{2}} \\ \sigma_{\frac{1}{2},-\frac{1}{2}} & \sigma_{-\frac{1}{2},-\frac{1}{2}} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1-q^2 & q \\ q & 0 \end{pmatrix} & \text{if } \lambda = -q, \\ \begin{pmatrix} 0 & q^{-1} \\ q^{-1} & 1-q^{-2} \end{pmatrix} & \text{if } \lambda = -q^{-1}, \end{cases}$$

where the σ_{ik} are as in Lemma 2.2.10. For $n \geq 2$ the formula for λ cannot be simplified much further. The conclusion is that for every n, there are exactly two symmetric solutions of (2.13).

2.2.3 The determinant and braids for reducible co-representations

In this section, suppose that $\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$ is not irreducible. For this case, we have the following result:

Lemma 2.2.13. *Let $\rho\colon V \to V \otimes \mathcal{O}(SU_q(2))$ be a reducible unitary co-representation. Let $V = \bigoplus_i (\oplus_{j=1}^{n_i} V_{ij})$ be a decomposition with maps $p_{ij}\colon V_i \to V_{ij}$ as in Lemma 2.2.4. For each i let $\sigma^i\colon V_i \otimes V_i \to V_i \otimes V_i$ satisfy (2.10), (2.11) and $\sigma^i \circ \sigma^i = \sigma^i$, and define $\sigma\colon V \to V$ via*

$$\sigma|_{V_{i_1 j_1} \otimes V_{i_2 j_2}} = \begin{cases} (p_{i_1 j_1} \otimes p_{i_2 j_2}) \circ \sigma^i \circ (p_{i_1 j_1} \otimes p_{i_2 j_2})^{-1} & \text{if } i_1 = i_2, \\ \mathrm{id}_{V_{i_1 j_1} \otimes V_{i_2 j_2}} & \text{if } i_1 \neq i_2. \end{cases}$$

This map satisfies the equations (2.10), (2.11) and $\sigma \circ \sigma = \sigma$.

Proof. To check (2.10), we start with equation (2.7). From this equation it follows that for $i_1 \neq i_2$ we have

$$\begin{aligned}\operatorname{Inv}(\sigma|_{V_{i_1j_1}\otimes V_{i_2j_2}}) &= V_{i_1j_1} \otimes V_{i_2j_2} \\ &= (p_{i_1j_1} \otimes p_{i_2j_2}) \left(\{0\}^\perp\right) \\ &= (p_{i_1j_1} \otimes p_{i_2j_2}) \left((\operatorname{CoInv}(\varphi_{i_1} \otimes \varphi_{i_2}))^\perp\right),\end{aligned}$$

whereas if $i_1 = i_2$, then because σ^i satisfies (2.10) we directly obtain

$$\begin{aligned}\operatorname{Inv}(\sigma|_{V_{i_1j_1}\otimes V_{i_2j_2}}) &= \operatorname{Inv}\left((p_{i_1j_1} \otimes p_{i_2j_2}) \circ \sigma^i \circ (p_{i_1j_1} \otimes p_{i_2j_2})^{-1}\right) \\ &= (p_{i_1j_1} \otimes p_{i_2j_2}) \left(\operatorname{Inv}(\sigma^i)\right) \\ &= (p_{i_1j_1} \otimes p_{i_2j_2}) \left((\operatorname{CoInv}(\varphi_{i_1} \otimes \varphi_{i_2}))^\perp\right).\end{aligned}$$

Hence

$$\operatorname{Inv}(\sigma) = \bigoplus_{i_1,j_1,i_2,j_2} \left((p_{i_1j_1} \otimes p_{i_2j_2}) \left((\operatorname{CoInv}(\varphi_{i_1} \otimes \varphi_{i_2}))^\perp\right)\right).$$

Because the maps $p_{i_1j_1}$ are orthogonal, it quickly follows that ρ and σ satisfy (2.10).

To check (2.11), consider it restricted to the subspace $V_{i_1j_1} \otimes V_{i_2j_2} \otimes V_{i_3j_3}$ for arbitrary $i_1, j_1, i_2, j_2, i_3, j_3$ and introduce the notation $\sigma_{12} = \sigma|_{V_{i_1j_1}\otimes V_{i_2j_2}}$, $\sigma_{23} = \sigma|_{V_{i_2j_2}\otimes V_{i_3j_3}}$ and $p_k = p_{i_kj_k}$ for $k = 1, 2, 3$. Then (2.11) reads

$$(\sigma_{12} \otimes 1)(1 \otimes \sigma_{23})(\sigma_{12} \otimes 1) = (1 \otimes \sigma_{23})(\sigma_{12} \otimes 1)(1 \otimes \sigma_{23}).$$

Note that

$$\sigma_{12} \otimes 1 = \begin{cases} 1 \otimes 1 \otimes 1 & \text{if } i_1 \neq i_2, \\ (p_1 \otimes p_2 \otimes p_3)(\sigma^i \otimes 1)(p_1^{-1} \otimes p_2^{-1} \otimes p_3^{-1}) & \text{if } i_1 = i_2 = i, \end{cases}$$

$$1 \otimes \sigma_{23} = \begin{cases} 1 \otimes 1 \otimes 1 & \text{if } i_2 \neq i_3, \\ (p_1 \otimes p_2 \otimes p_3)(1 \otimes \sigma^i)(p_1^{-1} \otimes p_2^{-1} \otimes p_3^{-1}) & \text{if } i_2 = i_3 = i. \end{cases}$$

We can distinguish the following cases:

1. $i_1 \neq i_2 \neq i_3$,
2. $i_1 = i_2 = i_3$,

3. $i_1 = i_2 \neq i_3$ or $i_1 \neq i_2 = i_3$.

In the first case, (2.11) reads $(1 \otimes 1 \otimes 1)^3 = (1 \otimes 1 \otimes 1)^3$ which is clearly true. In the second case, (2.11) reads

$$(p_1 \otimes p_2 \otimes p_3)(\sigma^i \otimes 1)(1 \otimes \sigma^i)(\sigma^i \otimes 1)(p_1^{-1} \otimes p_2^{-1} \otimes p_3^{-1}) = \\ (p_1 \otimes p_2 \otimes p_3)(1 \otimes \sigma^i)(\sigma^i \otimes 1)(\sigma^i \otimes 1)(p_1^{-1} \otimes p_2^{-1} \otimes p_3^{-1}),$$

which is true because σ^i satisfies (2.11). In the last case, say $i_1 = i_2 \neq i_3$, we have $1 \otimes \sigma_{23} = 1 \otimes 1 \otimes 1$ so (2.11) reduces to $(\sigma_{12} \otimes 1)(\sigma_{12} \otimes 1) = (\sigma_{12} \otimes 1)$ which is true because $\sigma^i \circ \sigma^i = \sigma^i$. Finally, $\sigma \circ \sigma = \sigma$ holds per construction on every space $V_{i_1 j_1} \otimes V_{i_2 j_2}$, and therefore also on $V \otimes V$. □

Note that in the previous proof, $\sigma^i \circ \sigma^i = \sigma^i$ is only required when $\rho \not\cong \oplus_j \varphi_i$, i.e. when the decomposition of ρ contains non-equivalent co-representations.

The previous result allows us to prove Theorem 2.2.7

Proof of Theorem 2.2.7. Combining Theorem 2.2.8 and Lemma 2.2.13, we only have to show whether equation (2.13) has solutions such that $\sigma \circ \sigma = \sigma$, i.e. $\sum_m \sigma_{im}\sigma_{mj} = \sigma_{ij}$ for all i, j. Consider the following cases:

- $(i \neq j)$ Equation (2.18) states that

$$\sigma_{ij} = \sigma_{ij}\left(2 - \sum_m x_m\right),$$

so $\sigma_{ij} = \sum_m \sigma_{im}\sigma_{mj}$ if and only if $\sigma_{ij} = 0$ (equivalently, $x_j = 0$) or $\sum_m x_m = 1$.

- $(i = j)$ We can derive

$$\sum_m \sigma_{im}\sigma_{mi} = \sigma_{ii}^2 + \sum_{m \neq i} x_m x_i = 1 - 2x_i + x_i^2 + x_i\left(\sum_m x_m - x_i\right) \\ = 1 - x_i + x_i\left(\sum_m x_m - 1\right),$$

so $\sigma_{ii} = \sum_m \sigma_{im}\sigma_{mi}$ if and only if $x_i(\sum_m x_m - 1) = 0$, i.e. if and only if $\sum_m x_m = 1$ or $x_i = 0$.

Hence, because not all x_i should be zero, these are equivalent to $\sum_m x_m = 1$. Solutions of (2.13) where also $\sum_m x_m = 1$ do exist for all n. The first example given above Lemma 2.2.13 is one possibility. □

2.3 The subproduct systems

This section describes the construction of an $\mathcal{O}(SU_q(2))$-co-equivariant subproduct system E_m from a unitary finite-dimensional co-representation

$$\rho\colon V \to V \otimes \mathcal{O}(SU_q(2)).$$

Definition 2.3.1. *An* $\mathcal{O}(SU_q(2))$-co-equivariant *(algebraic) subproduct system* $\{E_m\}_{m\geq 0}$ *is an (algebraic) subproduct system with for each* $n > 0$ *a co-action*

$$\rho_n\colon E_m \to E_m \otimes \mathcal{O}(SU_q(2))$$

such that the following diagram commutes:

$$\begin{array}{ccc} E_{k+m} & \xrightarrow{\rho_{k+m}} & E_{k+m} \otimes \mathcal{O}(SU_q(2)) \\ \downarrow{\scriptstyle \iota_{k,m}} & & \downarrow{\scriptstyle \iota_{k,m} \otimes \mathrm{id}_{\mathcal{O}(SU_q(2))}} \\ E_k \otimes E_m & \xrightarrow{\rho_k \otimes \rho_m} & E_k \otimes E_m \otimes \mathcal{O}(SU_q(2)) \end{array} \tag{2.19}$$

The construction of the subproduct system related to the co-representation ρ starts by making subspaces $K_m \subseteq V^{\otimes m}$ for $m \geq 2$.

Definition 2.3.2. *For any* $m \geq 2$ *and* $1 \leq i < m$*, we define the maps* $\blacktriangle_m(i)\colon V^{\otimes m} \to V^{\otimes m} \otimes \mathcal{O}(SU_q(2))$ *via*

$$\blacktriangle_m(i) = 1^{\otimes(i-1)} \otimes (\rho \otimes \rho) \otimes 1^{\otimes(m-1-i)},$$

where $1\colon V \to V \otimes \mathcal{O}(SU_q(2))$ *is the co-representation* $v \mapsto v \otimes 1$ *and the tensor products are as in Notation 1.2.8.*

We define the spaces K_m *as* $K_m = \sum_{i=1}^{m-1} \mathrm{CoInv}(\blacktriangle_m(i))$*. For convenience, we also define* $K_m(i) := \mathrm{CoInv}(\blacktriangle_m(i))$*.*

Note that $\det(\rho) = \mathrm{CoInv}(\blacktriangle_2(1)) = K_2(1) = K_2$.

Lemma 2.3.3. *For all* $m > 2$*, we have*

$$K_m(i) = V^{\otimes(i-1)} \otimes \det(\rho) \otimes V^{\otimes(m-1-i)},$$

and thus

$$K_m = \sum_{i=1}^{m-1} V^{\otimes(i-1)} \otimes \det(\rho) \otimes V^{\otimes(m-1-i)}.$$

Proof. This follows from the following observation:

Claim: *For finite-dimensional vector spaces W, W', W'' and a co-representation $\rho_W \colon W \to W \otimes \mathcal{A}$ we have*

$$\mathrm{CoInv}(1_{W'} \otimes \rho_W) = W' \otimes \mathrm{CoInv}(\rho_W)$$

and

$$\mathrm{CoInv}(\rho_W \otimes 1_{W''}) = \mathrm{CoInv}(\rho_W) \otimes W''$$

When this claim is true, the lemma directly follows by looking at Definition 2.3.2: First apply the first part of the claim with $\rho_W = \rho \otimes \rho$ and $W' = V^{\otimes(i-1)}$, then apply the second part with $\rho_W = 1^{\otimes(i-1)} \otimes \rho \otimes \rho$ and $W'' = V^{\otimes(m-1-i)}$.

Proof of Claim: Let $\sum_i w_i' \otimes w_i \in W' \otimes W$ be arbitrary. Assume w.l.o.g. that all w_i' are linearly independent, and write $\rho_W(w_i) = \sum_j w_{ij} \otimes a_{ij}$. Using Notation 1.2.8, we obtain

$$\begin{aligned}
(1_{W'} \otimes \rho_W)\left(\sum_i w_i' \otimes w_i\right) &= E\left(\sum_i 1_{W'}(w_i') \otimes \rho_W(w_i)\right) \\
&= \sum_i E\left(w_i' \otimes 1 \otimes \left(\sum_j w_{ij} \otimes a_{ij}\right)\right) \\
&= \sum_i \sum_j w_i' \otimes w_{ij} \otimes (1 \cdot a_{ij}) \\
&= \sum_i w_i' \otimes \left(\sum_j w_{ij} \otimes a_{ij}\right) \\
&= \sum_i w_i' \otimes \rho_W(w_i)
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_i w_i' \otimes w_i \in \mathrm{CoInv}(1_{W'} \otimes \rho_W) && \Leftrightarrow \\
&(1_{W'} \otimes \rho_W)\left(\sum_i w_i' \otimes w_i\right) = \sum_i w_i' \otimes \rho_W(w_i) = \left(\sum_i w_i' \otimes w_i\right) \otimes 1 && \Leftrightarrow \\
\forall i, \quad &\rho_W(w_i) = w_i \otimes 1 && \Leftrightarrow \\
&\sum_i w_i' \otimes w_i \in V \otimes \mathrm{CoInv}(\rho_W),
\end{aligned}$$

where for the second equivalence we used linear independence of the w_i'. This proves half of the claim. The other half is proven analogously. □

For a unitary co-representation ρ, we can now define the system of subspaces $E_m \subseteq V^{\otimes n}$.

Definition 2.3.4. *The* subproduct system induced by ρ, $(E_m)_{m\geq 0}$ *is given by* $E_0 = \mathbb{C}$, $E_1 = V$ *and for* $m \geq 2$ *define*

$$E_m = (K_m)^{\perp} = \{\nu \in V^{\otimes n} \colon \forall v \in K_m,\ \langle \nu, v\rangle_{E_m} = 0\}.$$

The following follows immediately from lemma 2.3.3

Lemma 2.3.5. *For* $k < m$ *we have* $E_m \subsetneq E_k \otimes E_{m-k}$, *and for* $k_1 \neq k_2$ *we have* $E_m = \left(E_{k_1} \otimes E_{m-k_1}\right) \cap \left(E_{k_2} \otimes E_{m-k_2}\right)$. *In particular,* $E_m = \cap_k E_k \otimes E_{m-k}$.

Proof. First, note that $((K_k)^{\perp} \otimes (K_{m-k})^{\perp})^{\perp} = K_k \otimes V^{\otimes(m-k)} + V^{\otimes k} \otimes K_{m-k}$, so lemma 2.3.3 gives

$$\begin{aligned}(E_k \otimes E_{m-k})^{\perp} &= ((K_k)^{\perp} \otimes (K_m)^{\perp})^{\perp} \\ &= K_k \otimes V^{\otimes(m-k)} + V^{\otimes k} \otimes K_{m-k} \\ &= \sum_{i\neq k} V^{\otimes(i-1)} \otimes \det(\rho) \otimes V^{\otimes(m-1-i)}.\end{aligned}$$

From

$$K_m = \sum_{i=1}^{m-1} V^{\otimes(i-1)} \otimes \det(\rho) \otimes V^{\otimes(m-1-i)}$$

it is immediate that

$$(E_k \otimes E_{m-k})^{\perp} \subsetneq K_m, \tag{2.20}$$

and

$$(E_{k_1} \otimes E_{m-k_1})^{\perp} + (E_{k_2} \otimes E_{m-k_2})^{\perp} = K_m. \tag{2.21}$$

Both results follow from (2.20) and (2.21) by taking perpendiculars again. □

Corollary 2.3.6. *The system* $(E_m)_{m\geq 0}$, *together with the maps*

$$\iota_{k,m} \colon E_{k+m} \to E_k \otimes E_m$$

that are induced by the canonical maps $\tilde{\iota}_{k,m} \colon V^{\otimes(k+m)} \to V^{\otimes k} \otimes V^{\otimes m}$, *forms an algebraic subproduct system.*

Proof. By lemma 2.3.5, the maps $\iota_{k,m}$ are well-defined. The conditions i-iii of definition 2.1.1 are clearly true for $(V^{\otimes n})$ and $\tilde{\iota}_{k,m}$, and therefore also true for E_m and $\iota_{k,m}$. □

Corollary 2.3.7. *The subproduct system $(E_m)_{m\geq 0}$ from Corollary 2.3.6 together with the maps*
$$\rho_m = \rho^{\otimes m}|_{E_m} \colon E_m \to E_m \otimes \mathcal{O}(SU_q(2))$$
is $\mathcal{O}(SU_q(2))$-co-equivariant.
Here $\rho^{\otimes m} \colon V^{\otimes m} \to V^{\otimes m} \otimes \mathcal{O}(SU_q(2))$ is as in Notation 1.2.8.

Proof. First, we have to prove that $\rho_m \colon E_m \to E_m \otimes \mathcal{O}(SU_q(2))$ is well-defined, i.e. that E_m is co-invariant under $\rho^{\otimes m}$. We start by proving co-invariance of K_m under $\rho^{\otimes m}$. Co-invariance of K_m can be proven by noting that if $\delta \in \det(\rho)$ and

$$v = e_{\mu_1} \otimes \cdots \otimes e_{\mu_{m-2}} \otimes \delta \in K_m(m-1),$$

then $\rho(e_{\mu_k}) = \sum_{\nu_k} e_{\nu_k} \otimes t_{\nu_k \mu_k}$ and $\rho(\delta) = \delta \otimes 1$ so

$$\rho^{\otimes m}(v) = \sum_{\nu_1 \ldots \nu_{m-2}} e_{\nu_1} \otimes \cdots \otimes e_{\nu_{m-2}} \otimes \delta \otimes \left(\prod_{k=1}^{m-2} t_{\nu_k \mu_k} \right) \in K_m(m-1) \otimes \mathcal{O}(SU_q(2)).$$

so $K_m(m-1)$ is co-invariant under $\rho^{\otimes m}$. In a similar manner, we can show that $K_m(i)$ is co-invariant for all other i, and thus $K_m = \sum_i K_m(i)$ is also co-invariant under $\rho^{\otimes m}$. Now Corollary 1.2.7 gives that $E_m = (K_m)^\perp$ is co-invariant under $\rho^{\otimes m}$ as well, i.e. ρ_m is well-defined.

Diagram (2.19) states that

$$\iota_{k,m}(\rho^{\otimes(k+m)}|_{E_{k+m}}(v)) = (\rho^{\otimes k}|_{E_k} \otimes \rho^{\otimes m}|_{E_m})(\iota_{k,m}(v)).$$

for all $v \in E_{k+m}$. In $V^{\otimes(k+m)} \supseteq E_{k+m}$ we have $\iota_{k,m}(v) = v$ and $\rho^{\otimes(k+m)} = \rho^{\otimes k} \otimes \rho^{\otimes m}$, so diagram (2.19) clearly commutes.

Hence, both conditions of Definition 2.3.1 are satisfied. □

We conclude this section by calculating the dimensions of the vector spaces E_m for irreducible $\rho \colon V \to V \otimes \mathcal{O}(SU_q(2))$, analogously to [1, Lemma 3.3]. Before we start the proof, we have a small lemma

Lemma 2.3.8. *For $\alpha \in \mathbb{R}$ with $|\alpha| > 0$, the recurrence relation for $(a_k)_{k=0}^\infty$ with $a_k \in \mathbb{R}$ given by*
$$a_{k+2} = \alpha a_{k+1} - a_k, \tag{2.22}$$
is solved by
$$a_k = \frac{a_0}{t - t^{-1}}(t^{k+1} - t^{-(k+1)}), \tag{2.23}$$
where t satisfies $t + t^{-1} = \alpha$.

Proof. This result can be found near [3, Lemma 1.6]. It is easily checked: Assuming (2.23), we have

$$\begin{aligned} \alpha a_{k+1} - a_k &= \frac{a_0}{t - t^{-1}} \left(\alpha t^{k+2} - \alpha t^{-(k+2)} - t^{k+1} + t^{-(k+1)} \right) \\ &= \frac{a_0}{t - t^{-1}} \left(t^{k+2}(\alpha - t^{-1}) - t^{-(k+2)}(\alpha - t) \right) \\ &= \frac{a_0}{t - t^{-1}} \left(t^{k+2}(t) - t^{-(k+2)}(t^{-1}) \right) \\ &= a_{k+2}. \end{aligned}$$

□

In [1] the dimensions of E_m are calculated by introducing the maps G_m.

Definition 2.3.9. *For the irreducible co-representation $\varphi_n \colon V \to V \otimes \mathcal{O}(SU_q(2))$ from Theorem 1.3.3, and $m \geq 1$ define the maps $\overline{G}_m \colon V^{\otimes(m-1)} \to K_{m+1}$ by*

$$\overline{G}_m \left(\bigotimes_{i=1}^{m-1} v_i \right) = \sum_{k=0}^{m-1} \lambda_k \left(\bigotimes_{i=1}^{k} v_i \right) \otimes \delta \otimes \left(\bigotimes_{i=k+1}^{m-1} v_i \right), \tag{2.24}$$

where $\delta \in \det(\varphi_n)$ is as in (2.8), $\lambda = \frac{(-1)^n}{[n+1]_q} = \delta_s \delta_{-s}$ $\forall s$ and the $\lambda_k \in \mathbb{R}$ satisfy the recurrence relation

$$\lambda_k = -\frac{\lambda_{k-1}}{\lambda} - \lambda_{k-2}, \tag{2.25}$$

with $\lambda_{-1} = 0$ and $\lambda_0 \neq 0$ arbitrary.
Let $G_m \colon E_{m-1} \to K_{m+1}$ be the restriction of $\overline{G}_m$ to E_{m-1}.

Note that the recurrence relation (2.25) is of the form (2.22).

Remark 2.3.10. *Note that $\overline{G}_m$ satisfies the recursive relation*

$$\begin{aligned} \overline{G}_1(1) &= \lambda_0 \delta \\ \overline{G}_m &= \overline{G}_{m-1} \otimes \mathrm{id}_V + \lambda_{m-1}(\mathrm{id}_{V^{\otimes(m-1)}} \otimes \overline{G}_1). \end{aligned} \tag{2.26}$$

Because $E_m \subseteq E_{m-1} \otimes V$ per Lemma 2.3.5, G_m also satisfies this relation.

Remark 2.3.11. *We can show that the maps $\overline{G}_m$ and G_m satisfy*

$$\begin{array}{ccc} E_{m-1} & \xrightarrow{\varphi_n^{\otimes(m-1)}} & E_{m-1} \otimes \mathcal{O}(SU_q(2)) \\ {\scriptstyle G_m}\downarrow & & \downarrow{\scriptstyle G_m \otimes \mathrm{id}} \\ K_{m+1} & \xrightarrow{\varphi_n^{\otimes(m+1)}} & K_{m+1} \otimes \mathcal{O}(SU_q(2)) \end{array}$$

because both $\overline{G}_m\left(\varphi_n^{\otimes(m-1)}\left(\bigotimes_{i=1}^{m-1} f_{\mu_i}^{(\frac{n}{2})}\right)\right)$ *and* $\varphi_n^{\otimes(m+1)}\left(\overline{G}_m\left(\bigotimes_{i=1}^{m-1} f_{\mu_i}^{(\frac{n}{2})}\right)\right)$ *are equal to*

$$\sum_{k=0}^{m-1}\sum_{\nu_{1,k}\ldots\nu_{m-1,k}=-\frac{n}{2}}^{\frac{n}{2}} \lambda_k\left(\bigotimes_{i=1}^{k} f_{\nu_{i,k}}^{(\frac{n}{2})}\right)\otimes\delta\otimes\left(\bigotimes_{i=k+1}^{m-1} f_{\nu_{i,k}}^{(\frac{n}{2})}\right)\otimes\left(\prod_{i=1}^{m-1} t_{\nu_{i,k}\mu_i}\right).$$

Lemma 2.3.12. *For all* $v\in E_{m-1}\otimes V, w\in E_m$ *we have*

$$\langle (G_m\otimes \mathrm{id}_V)(v), G_{m+1}(w)\rangle = 0.$$

Proof. First, note that by (2.26) we have that

$$\begin{aligned}\big\langle (G_m\otimes \mathrm{id}_V)(v), G_{m+1}(w)\big\rangle =&\big\langle (G_m\otimes \mathrm{id}_V)(v), (G_m\otimes \mathrm{id}_V)(w)\big\rangle\\ &+\lambda_m\big\langle (G_m\otimes \mathrm{id}_V)(v), w\otimes\delta\big\rangle. \qquad (2.27)\end{aligned}$$

We will investigate both terms separately. Write $v=\sum_i v_i\otimes f_i^{(\frac{n}{2})}$ and note that (2.26) gives

$$\begin{aligned}\big\langle (G_m\otimes \mathrm{id}_V)(v), w\otimes\delta\big\rangle &= \left\langle \sum_i G_m(v_i)\otimes f_i^{(\frac{n}{2})}, w\otimes\delta\right\rangle\\ &=\sum_i\left\langle (G_{m-1}\otimes \mathrm{id}_V)(v_i)\otimes f_i^{(\frac{n}{2})}, w\otimes\delta\right\rangle\\ &\quad+\sum_i\lambda_{m-1}\left\langle v_i\otimes\delta\otimes f_i^{(\frac{n}{2})}, w\otimes\delta\right\rangle.\end{aligned}$$

Note that the image of G_{m-1} is K_m and $w\in E_m=K_m^{\perp}$, so

$$\left\langle (G_{m-1}\otimes \mathrm{id}_V)(v_i)\otimes f_i^{(\frac{n}{2})}, w\otimes\delta\right\rangle = 0.$$

Furthermore, from $\delta=\sum_j \delta_j f_j^{(\frac{n}{2})}\otimes f_{-j}^{(\frac{n}{2})}$ and $\delta_j\delta_{-j}=\lambda$ we obtain

$$\begin{aligned}\sum_i\langle v_i\otimes\delta\otimes f_i^{(\frac{n}{2})}, w\otimes\delta\rangle &= \sum_{ijk}\delta_j\delta_k\left\langle v_i\otimes f_j^{(\frac{n}{2})}\otimes f_{-j}^{(\frac{n}{2})}\otimes f_i^{(\frac{n}{2})}, w\otimes f_k^{(\frac{n}{2})}\otimes f_{-k}^{(\frac{n}{2})}\right\rangle\\ &=\sum_{ijk}\delta_j\delta_k\left\langle v_i\otimes f_j^{(\frac{n}{2})}, w\right\rangle\left\langle f_{-j}^{(\frac{n}{2})}, f_k^{(\frac{n}{2})}\right\rangle\left\langle f_i^{(\frac{n}{2})}, f_{-k}^{(\frac{n}{2})}\right\rangle\\ &=\sum_i\delta_i\delta_{-i}\left\langle v_i\otimes f_i^{(\frac{n}{2})}, w\right\rangle\\ &=\lambda\langle v, w\rangle.\end{aligned}$$

In conclusion, we have

$$\big\langle (G_m \otimes \mathrm{id}_V)(v), w \otimes \delta \big\rangle = \lambda \lambda_{m-1} \langle v, w \rangle. \tag{2.28}$$

Next, consider $\langle G_m(v), G_m(w) \rangle$ for $v, w \in E_{m-1}$. Again using (2.26), we obtain

$$\begin{aligned} \langle G_m(v), G_m(w) \rangle =& \langle (G_{m-1} \otimes \mathrm{id}_V)(v), (G_{m-1} \otimes \mathrm{id}_V)(w) \rangle \\ &+ \lambda_{m-1}^2 \langle v \otimes \delta, w \otimes \delta \rangle \\ &+ \lambda_{m-1} \langle (G_{m-1} \otimes \mathrm{id}_V)(v), w \otimes \delta \rangle \\ &+ \lambda_{m-1} \langle v \otimes \delta, (G_{m-1} \otimes \mathrm{id}_V)(w) \rangle. \end{aligned}$$

Because $\langle \delta, \delta \rangle = 1$ per (2.9) and using (2.28), this is equal to

$$\langle (G_{m-1} \otimes \mathrm{id}_V)(v), (G_{m-1} \otimes \mathrm{id}_V)(w) \rangle + (\lambda_{m-1}^2 + 2\lambda_{m-1} \lambda \lambda_{m-2}) \langle v, w \rangle.$$

By induction and because $\langle G_1(1), G_1(1) \rangle = \lambda_0^2$ and $\lambda_{-1} = 0$ we obtain

$$\langle G_m(v), G_m(w) \rangle = \left(\sum_{k=0}^{m-1} \lambda_k^2 + 2\lambda \lambda_k \lambda_{k-1} \right) \langle v, w \rangle.$$

Per (2.25) and by rearranging the sum, we have

$$\begin{aligned} \left(\sum_{k=0}^{m-1} \lambda_k^2 + 2\lambda \lambda_k \lambda_{k-1} \right) &= \left(\sum_{k=0}^{m-1} \lambda \lambda_{k+1} \lambda_k + \lambda_k^2 + \lambda \lambda_k \lambda_{k-1} \right) - \lambda \lambda_m \lambda_{m-1} \\ &= -\lambda \lambda_m \lambda_{m-1}. \end{aligned}$$

In conclusion, we find

$$\langle G_m(v), G_m(w) \rangle = -\lambda \lambda_m \lambda_{m-1} \langle v, w \rangle. \tag{2.29}$$

Combining (2.27) with (2.28), (2.29) and (2.25), we find that

$$\begin{aligned} \langle (G_m \otimes \mathrm{id}_V)(v), G_{m+1}(w) \rangle &= -\lambda \lambda_m \lambda_{m-1} \langle v, w \rangle + \lambda \lambda_m \lambda_{m-1} \langle v, w \rangle \\ &= 0 \end{aligned}$$

□

Corollary 2.3.13. *The vector space dimensions* $\dim_{\mathbb{C}}(E_m)$ *satisfy the recurrence relation*

$$\begin{aligned} \dim_{\mathbb{C}}(E_0) &= 1, \\ \dim_{\mathbb{C}}(E_1) &= \dim_{\mathbb{C}}(V), \\ \dim_{\mathbb{C}}(E_{m+1}) &= \dim_{\mathbb{C}}(V) \dim_{\mathbb{C}}(E_m) - \dim_{\mathbb{C}}(E_{m-1}) \end{aligned}$$

which can be solved with Lemma 2.3.8.

Proof. It is clear that $\dim_{\mathbb{C}}(E_0) = 1$ and $\dim_{\mathbb{C}}(E_1) = \dim_{\mathbb{C}}(V)$. We will show that

$$(K_m \otimes V) \oplus G_m(E_{m-1}) = K_{m+1}.$$

Note that $G_m(E_{m+1}) \subseteq K_{m+1}$ and $K_m \otimes V \subseteq K_{m+1}$ so what remains is $(K_m \otimes V) + G_m(E_{m-1}) \supseteq K_{m+1}$ and $(K_m \otimes V) \cap G_m(E_{m-1}) = \{0\}$. Now for $v \in K_{m+1}$, Lemma 2.3.3 gives that

$$\begin{aligned} v &= \sum_{i=1}^{m} \left(\bigotimes_{k=1}^{i-1} v_k^i \right) \otimes \delta \otimes \left(\bigotimes_{k=i}^{m-1} v_k^i \right) \\ &= \left(\bigotimes_{k=1}^{m-1} v_k^m \right) \otimes \delta + \sum_{i=1}^{m-1} \left(\bigotimes_{k=1}^{i-1} v_k^i \right) \otimes \delta \otimes \left(\bigotimes_{k=i}^{m-1} v_k^i \right) \\ &=: \overline{v} \otimes \delta + v_{K_m}. \end{aligned}$$

Note that we can write $\overline{v} = \overline{v_K} + \overline{v_E}$ with $\overline{v_K} \in K_{m-1}$ and $\overline{v_E} \in E_{m-1}$. However, $\overline{v_K} \otimes \delta \in K_m \otimes V$, so we can put $\overline{v_K} \otimes \delta$ inside v_{K_m} and assume w.l.o.g. that $\overline{v} \in E_{m-1}$. We find that

$$\begin{aligned} v - G_m\left(\frac{1}{\lambda_{m-1}} \overline{v} \right) =& \overline{v} \otimes \delta + v_{K_m} - \overline{v} \otimes \delta \\ & - \sum_{i=0}^{m-2} \frac{\lambda_i}{\lambda_{m-1}} \left(\bigotimes_{k=0}^{i} v_{k+1}^m \right) \otimes \delta \otimes \left(\bigotimes_{k=i+1}^{m-1} v_{k+1}^m \right) \\ \in& K_m \otimes V. \end{aligned}$$

Therefore we have proven that

$$K_{m+1} \subseteq K_m \otimes V + G_m(E_{m-1}). \tag{2.30}$$

To prove that $(K_m \otimes V) \cap G_m(E_{m-1}) = \{0\}$, we show that for arbitrary $v \in K_m \otimes V$ and $w \in E_{m-1}$ we have $\langle v, G_m(w) \rangle = 0$ such that $v = G_m(w)$ if and only if $v = G_m(w) = 0$. Using (2.30) for $m-1$ we find that

$$v = v_{K_{m-1}} + (G_{m-1} \otimes \mathrm{id}_V)(\overline{v_E}),$$

where $v_{K_{m-1}} \in K_{m-1} \otimes V^{\otimes 2}$ and $\overline{v_E} \in E_{m-1} \otimes V$. Now

$$\langle v, G_m(w) \rangle = \langle v_{K_{m-1}}, G_m(w) \rangle + \langle (G_{m-1} \otimes \mathrm{id}_V)(\overline{v_E}), G_m(w) \rangle.$$

By Lemma 2.3.12, the second term is 0. Using induction starting with $K_1 \otimes V = \{0\}$ and $\langle v, G_1(w) \rangle = 0$ for all $v \in K_1 \otimes V$ and $w \in E_0$, the first term is also 0. Therefore,

$$(K_m \otimes V) \cap G_m(E_{m-1}) = \{0\}.$$

To show injectivity of G_m, note that (2.29) states that $G_m(v) = 0 \Leftrightarrow \langle G_m(v), G_m(v)\rangle = -\lambda\lambda_m\lambda_{m-1}\langle v, v\rangle = 0$. And $\lambda\lambda_m\lambda_{m-1}$ is nonzero because of Lemma 2.3.8 (the conditions are easily checked), so indeed G_m is injective. Hence it follows that $\dim_{\mathbb{C}}(G_m(E_{m-1})) = \dim_{\mathbb{C}}(E_{m-1})$ and thus

$$\begin{aligned}\dim(E_{m+1}) &= \dim(V^{\otimes m+1}) - \dim(K_{m+1})\\ &= \dim(V)^{m+1} - (\dim(K_m)\dim(V) + \dim(E_{m-1}))\\ &= \dim(V)^{m+1} - (\dim(V^{\otimes m}) - \dim(E_m))\dim(V) - \dim(E_{m-1})\\ &= \dim(V)\dim(E_m) - \dim(E_{m-1}).\end{aligned}$$

□

Note that [3, Lemma 1.6] states that the result in Corollary 2.3.13 holds when $\det(\rho)$ is Temperley-Lieb. We have shown this in Lemma 2.2.6, so our results do agree with [3, Lemma 1.6] and generalise [1, Lemma 3.3].

2.4 The quadratic algebras

In this chapter we do not look at each E_m separately, but consider the space $A(\rho) := \bigoplus_{m=0}^{\infty} E_m$ instead. We can turn $A(\rho)$ into an algebra by identifying $E_m \cong V^{\otimes m}/K_m$ such that

$$A(\rho) \cong \left(\bigoplus_{m=0}^{\infty} V^{\otimes m}\right) \Big/ \left(\bigoplus_{m=2}^{\infty} K_m\right)$$

is a quotient of the tensor algebra of V. This is a quadratic algebra in the sense of [6, Section 4.1]:

Definition 2.4.1. *A* quadratic algebra *A is an* $\mathbb{N}$*-graded associative algebra with the properties:*

- $A_0 = \mathbb{C}$,
- *A is generated (as an algebra) by* A_1*, so* $A \cong (\bigoplus_{n=0}^{\infty} A_1^{\otimes n})/R(A)$ *for some two-sided ideal of relations* $R(A)$,
- *The ideal of relations* $R(A)$ *is generated by elements from* $A_1 \otimes A_1$.

This algebra can also be obtained by applying the non-commutative nullstellensatz, Theorem 2.1.4, to our subproduct system. Then we find that

$$I_{E_m} = \mathrm{Span}\{\mathbf{X}(v) \colon \exists m \text{ such that } v \in K_m\} = \bigoplus_{m=2}^{\infty} \mathbf{X}(K_m).$$

From Lemma 2.3.3 it thus follows that $I_{E_m} \subseteq \mathbb{C}\langle X_0, \ldots, X_n \rangle$ is the two-sided ideal generated by $\mathbf{X}(\det(\rho))$, and the algebra $A(\rho)$ can then be obtained as $A(\rho) \cong \mathbb{C}\langle X_0, \ldots, X_n \rangle / I_{E_m}$. Explicitly, $A(\rho)$ can be described by the following definition:

Definition 2.4.2. *The algebra $A(\rho) = \bigoplus_{m=0}^{\infty} E_m$ together with the algebra structure described above is an (infinite-dimensional) algebra generated by elements $\left(f_i^{(\frac{n}{2})}\right)_{i=-\frac{n}{2}}^{\frac{n}{2}}$ with multiplication $a, b \mapsto a \otimes b$ and unit $\eta \colon \mathbb{C} \xrightarrow{\sim} E_0$ subject to the relation*

$$0 = \sum_{i=-\frac{n}{2}}^{\frac{n}{2}} q^i f_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})}.$$

Note that for $n = 1$, we re-obtain the algebra $\mathbb{C}_q[\mathcal{X}, \mathcal{Y}]$ from Definition 1.3.1, together with the co-action $\varphi \colon \mathbb{C}_q[\mathcal{X}, \mathcal{Y}] \to \mathbb{C}_q[\mathcal{X}, \mathcal{Y}] \otimes \mathcal{O}(SU_q(2))$ from Theorem 1.3.3. Because $\mathbb{C}_q[\mathcal{X}, \mathcal{Y}]$ is known as the "Quantum Plane", for other n the space $A(\rho)$ with the map

$$\bigoplus_{m=0}^{\infty} \rho_m \colon A(\rho) \to A(\rho) \otimes \mathcal{O}(SU_q(2))$$

can be thought of as a more general notion of "Quantum Plane". In [6, Chapter 4.1] it is argued that *any* quadratic algebra can play the role of a Quantum Plane. We now describe the construction of the algebras $A(\rho)^!$, $A(\rho)^{(d)}$ and $A(\rho)^{!(d)}$ from the algebra $A(\rho)$.

Definition 2.4.3. *For a quadratic algebra $A = (\bigoplus_{n=0}^{\infty} A_1^{\otimes n})/R(A)$, the* dual algebra *$A^!$ is given by*

$$A^! = \left(\bigoplus_{n=0}^{\infty} (A_1^*)^{\otimes n} \right) / (R(A)^{\perp})$$

where A_1^* is the dual of A_1 and $R(A)^{\perp}$ is the ideal generated by those elements $r \in A_1^* \otimes A_1^*$ such that $r(a) = 0$ for all $a \in R(A) \cap A_1 \otimes A_1$.

Lemma 2.4.4. *For the algebra $A(\rho)$ from Definition 2.4.2, the dual $A(\rho)^!$ is generated by elements $\left(f_i^{(\frac{n}{2})}\right)_{i=-\frac{n}{2}}^{\frac{n}{2}}$ subject to the relations*

$$f_i^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})} = 0 \qquad \text{if } i \neq -j,$$
$$q f_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} + f_{i+1}^{(\frac{n}{2})} \otimes f_{-(i+1)}^{(\frac{n}{2})} = 0 \qquad \text{for } i \in \left\{ -\frac{n}{2}, \ldots, \frac{n}{2} - 1 \right\}.$$

Alternatively, we have $A(\rho) = 1\mathbb{C} \oplus V \oplus \tau\mathbb{C}$ with the relations

$$\begin{aligned} 0 &= f_i^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})} && \text{if } i \neq -j, \\ (-q)^{-i}\tau &= f_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} && \text{for all } i, \\ 0 &= f_i^{(\frac{n}{2})} \otimes \tau = \tau \otimes f_i^{(\frac{n}{2})} = \tau \otimes \tau && \text{for all } i. \end{aligned}$$

For $n = 1$, this algebra coincides with the one in [6, Section 2.2].

Proof. Clearly, $E_1 = E_1^*$ because E_1 is an inner product space. Now we just have to calculate $\det(\varphi_n)^\perp \subseteq V \otimes V$, which has already been used in the proof of Lemma 2.2.9. First, note that

$$\dim_{\mathbb{C}}\left(\det(\varphi_n)^\perp\right) = \dim_{\mathbb{C}}(V^{\otimes 2}) - \dim_{\mathbb{C}}(\det(\varphi_n)) = (n+1)^2 - 1 = n(n+2).$$

Furthermore, there are $n(n+1)$ vectors of the form $f_i^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})}$ $(i \neq j)$ and $f_i^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})} \perp \det(\varphi_n)$, because these vectors do not appear as terms in (2.1). There are n vectors of the form $qf_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} + f_{i+1}^{(\frac{n}{2})} \otimes f_{-(i+1)}^{(\frac{n}{2})}$ with $-\frac{n}{2} \leq i < \frac{n}{2}$. Using (2.8) we can calculate that

$$\begin{aligned} \left\langle qf_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} + f_{i+1}^{(\frac{n}{2})} \otimes f_{-(i+1)}^{(\frac{n}{2})}, \delta \right\rangle &= q\delta_i + \delta_{i+1} \\ &= \frac{(-1)^{\frac{n}{2}}}{\sqrt{[n+1]_q}} \left(q(-1)^{-i}q^i + (-1)^{-(i+1)}q^{i+1}\right) \\ &= \frac{(-1)^{\frac{n}{2}-i}}{\sqrt{[n+1]_q}} \left(q^{i+1} - q^{i+1}\right) \\ &= 0. \end{aligned}$$

We have thus found $n(n+1) + n = n(n+2)$ linearly independent vectors in $V \otimes V$ that belong to the $n(n+2)$-dimensional subspace $\det(\varphi_n)^\perp \subseteq V \otimes V$, so these vectors must span $\det(\varphi_n)^\perp$.

When we introduce

$$\tau := (-q)^{\frac{n}{2}} f_{\frac{n}{2}}^{(\frac{n}{2})} \otimes f_{-\frac{n}{2}}^{(\frac{n}{2})},$$

the relation $qf_i^{(\frac{n}{2})} \otimes f_{-i}^{(\frac{n}{2})} + f_{i+1}^{(\frac{n}{2})} \otimes f_{-(i+1)}^{(\frac{n}{2})} = 0$ gives

$$f_i \otimes f_{-i} = (-q)^{-1} f_{i+1} \otimes f_{-(i+1)} = (-q)^{-i}\tau$$

by induction. Thus, for any i we can choose $j \neq -i$ for which it follows that

$$f_i^{(\frac{n}{2})} \otimes \tau = (-q)^{-j} \left(f_i^{(\frac{n}{2})} \otimes f_j^{(\frac{n}{2})} \right) \otimes f_{-j}^{(\frac{n}{2})} = 0.$$

Similarly $\tau \otimes f_i^{(\frac{n}{2})} = 0$ and $\tau \otimes \tau = 0$ □

Definition 2.4.5. *For a quadratic algebra* $A = (\bigoplus_{n=0}^{\infty} A_1^{\otimes n})/R(A)$, *the* d-th Quantum symmetric power $A^{(d)}$ *is given by*

$$A^{(d)} = \bigoplus_{n=0}^{\infty} A_{dn}$$

and the d-th quantum exterior power $A^{!(d)}$ *is given by* $A^{!(d)} = (A^!)^{(d)}$

Remark 2.4.6. *Per [6, Section 4.10],* $A^{(d)}$ *is again a quadratic algebra. Note that* $(A^!)^{(d)}$ *is in general not the same as* $(A^{(d)})^!$*. In most cases,* $(A^!)^{(d)}$ *is more relevant than* $(A^{(d)})^!$*.*

Lemma 2.4.7. *For the algebra* $A(\rho)$ *from Definition 2.4.2, the d-th quantum symmetric power* $A(\rho)^{(d)}$ *is an algebra generated by elements* $(f_{i_1 \dots i_d})_{i_1, \dots i_d = -\frac{n}{2}}^{\frac{n}{2}}$ *subject to the relations*

$$0 = \sum_{i_k = -\frac{n}{2}}^{\frac{n}{2}} q^i f_{i_1 \dots i_d} \otimes f_{j_1 \dots (-i_k) \dots j_d} \qquad \text{for all } k, i_1 \dots \hat{\imath}_k \dots i_d, j_1 \dots \hat{\jmath}_k \dots j_d.$$

Here $(-i_k)$ *appears where* j_k *would have been, and* $\hat{\imath}_k$ *resp.* $\hat{\jmath}_k$ *means that* i_k *resp.* j_k *is omitted.*

Proof. This quickly follows from Lemma 2.3.3 when we define

$$f_{i_1 \dots i_d} := \bigotimes_{k=1}^{d} f_{i_k}^{(\frac{n}{2})} = f_{i_1}^{(\frac{n}{2})} \otimes \dots \otimes f_{i_d}^{(\frac{n}{2})}.$$

□

Lemma 2.4.8. *For the algebra* $A(\rho)$ *from Definition 2.4.2, the d-th quantum exterior power* $A(\rho)^{!(d)}$ *is equal to* $\tau\mathbb{C}$ *for* $d = 2$ *and* $\{0\}$ *for* $d > 2$*.*

Proof. This quickly follows from the last part in Lemma 2.4.4. □

We can also calculate the "Hilbert series" for the graded algebra $A(\rho)$. By Corollary 2.3.13 and Lemma 2.3.8, we have that $\dim_{\mathbb{C}}(E_k) = \frac{t^{k+1}-t^{-(k+1)}}{t-t^{-1}}$ where $t + t^{-1} = n + 1$. We can now calculate the series, which converges for $|x| < \min\{|t|, |t^{-1}|\}$:

$$\begin{aligned}
HS_{E_m}(x) &= \sum_{k=0}^{\infty} \dim_{\mathbb{C}}(E_k)x^k \\
&= \frac{1}{t-t^{-1}} \sum_{k=0}^{\infty} (t^{k+1} - t^{-(k+1)})x^k \\
&= \left(\frac{1}{t-t^{-1}}\right)\left(\frac{1}{t^{-1}-x} - \frac{1}{t-x}\right) \\
&= \frac{1}{(t-x)(t^{-1}-x)}
\end{aligned}$$

Conclusion & further research

In this thesis we have studied the Hopf algebra $\mathcal{O}(SU_q(2))$. We have seen in section 2.3 how we can construct a subproduct system E_m from a co-representation of $\mathcal{O}(SU_q(2))$, with a natural co-action $\rho\colon E_m \to E_m \otimes \mathcal{O}(SU_q(2))$. We investigated the structure of this subproduct system in two ways: At the end of section 2.3 we constructed (for irreducible co-representations) a decomposition

$$K_{m+1} = (K_m \otimes V) \oplus G_m(E_{m-1})$$

which allowed us to calculate the dimension of E_m, and in section 2.2 we investigated the subspace $\det(\rho) \subseteq V \otimes V$, which plays a central role in the construction of the subproduct system E_m. In particular, we have shown how $\det(\rho)$ can be constructed as the orthogonal complement of the invariant elements of a braiding σ. Finally, in section 2.4 we showed analogues of the quantum plane $\mathbb{C}_q[\mathcal{X}, \mathcal{Y}]$ that could be constructed from E_m, for irreducible co-representations. These generalised quantum planes do admit a natural co-action of $\mathcal{O}(SU_q(2))$.

However, there is an inconsistency somewhere between the proof of Lemma 1.4.1 and the proof of Lemma 2.2.3, as noted in the footnotes of these proofs. These lemmas are main building blocks of this thesis. Therefore, either trying to prove these lemmas in a different way, or finding the inconsistency is one main problem that has been left open for further research. Some other minor problems for further research are how to generalise the decomposition $K_{m+1} = (K_m \otimes V) \oplus G_m(E_{m-1})$ of section 2.3 and the quantum plane analogues of section 2.4 for reducible co-representations. Other problems to consider are generalising other parts of [1] to the $\mathcal{O}(SU_q(2))$-setting, or generalising the results of this thesis to other Hopf algebras.

Acknowledgements

I am very grateful to my thesis supervisor, Francesca Arici, who provided me with continuous support, clear descriptions of the scope of the project and extensive high quality feedback throughout the course of my thesis. Furthermore, I would like to thank Jannetje Driessen for her great support and remarks during the last phase of my thesis, and Thomas Wink for listening to me every time I was stuck on a technical computation. Lastly, I would like to thank Bram Mesland for his willingness to supervise me during Francesca's leave.

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