Money laundering & the d'Alembert

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Contents

1 Introduction

The client of a criminal defence lawyer is accused of money laundering in casinos, but according to the client himself he just won 10,000 euros each year for five years in a row by playing simultaneously on red/black and low/high in roulette at Holland Casino. He followed the so-called d'Alembert strategy, which, according to him, made it possible to win that amount of money. This thesis takes a look at this case from a probabilistic perspective.

Much work has already been done by Prof. dr. Richard D. Gill, a Dutch-British professor emeritus of mathematical statistics at Leiden University. Richard Gill is known for his work in forensic statistics and for his work in fighting injustice, among other things. More information on Gill can be found on his web page: `https://pub.math.leidenuniv.nl/~gillrd`.

Gill performed R simulations for the games played by the client. He also constructed a Markov chain model for these games, which enabled him to derive exactly the distribution of the client's (final) capital. Zero outcomes are disregarded so that the red/black-games and low/high-games are independent, and it is assumed that the odds are 36:37 in favour of the house. Gill also performed a forensic probabilistic assessment of the client's story. According to Gill it is most likely that the client lost more than he won and the client's chance of having a net gain of 10,000 euros each year for five successive years is negligible. See [1] for more information on Gill's work and for some nice background information, histograms and bar charts. Gill's work forms an important source of inspiration for this thesis.

Let us now outline the structure of this thesis. This thesis starts with an introductory chapter (chapter 1). Chapters 2 and 3 form the main body of this thesis. The thesis ends with a short conclusion (chapter 4). In addition, a reference list and an appendix are provided at the end of the thesis.

Now we will describe in more depth the purpose of each chapter. In chapter 2 we explain how the roulette game works, what the d'Alembert strategy is, and how it can be used in roulette. We lay out the Holland Casino rules for roulette, and we explore probabilistic properties of the d'Alembert system.

We show in chapter 3 that the two simultaneous games (red/black and low/high) are not independent, but that they can be approximated fairly well by an independent coupling of these games. We construct such a coupling and we show that the number of differences between the original games and the coupling follow a binomial distribution. We also try to quantify the differences between these processes in terms of capital, both theoretically and numerically (based on simulations). In doing this, we assume the use of the d'Alembert strategy. Moreover, we formulate Markov models for the capital/stake-levels in these games, and we use these models to get insights in the distributional and probabilistic properties of capital and stake. We also run simulations and compare these simulations with the theoretical results. We end the third chapter with a forensic statistical analysis of our criminal case. The likelihood ratio plays a central role in this analysis.

We end this thesis with a concluding chapter 4. This conclusion summarizes the main results of our thesis and it poses some new research questions.

In addition, at the end of the thesis we provide a list of references and an appendix. The first section of the appendix provides preliminary knowledge on discrete-time Markov chains and their properties. This section hugely follows chapter 12 of [2]. The second section deals with the asymptotic properties of a test statistic used in chapter 3. The third section provides the MATLAB codes used for this thesis.

2 Roulette

2.1 Explanation of the roulette game

This subsection is based on [3].

Usually a roulette table has 37 or 38 numbers: 1-36 and one or two zeros (0 and 00). The zeros are green. The numbers 1, 3, 5, 7, 9, 12, 14, 16, 18, 19, 21, 23, 25, 27, 30, 32, 34, 36 are red. The other numbers are black. From now on we assume that we are playing with 37 numbers. A roulette wheel and roulette table look like this:

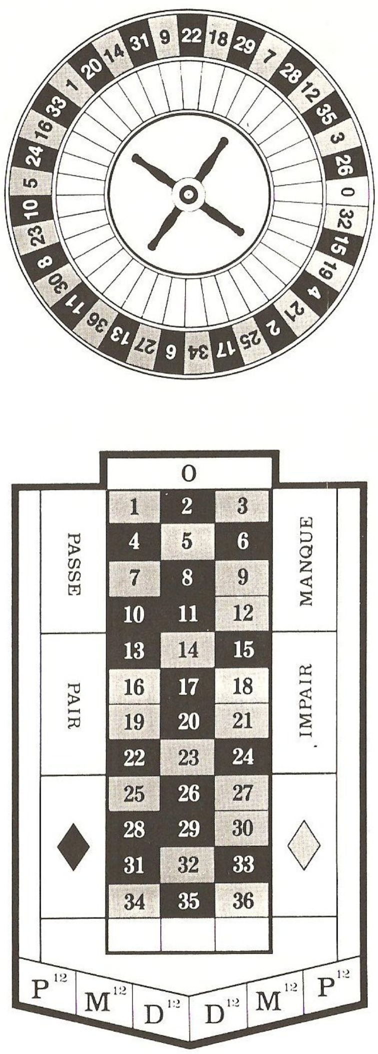

Figure 1: Roulette wheel and roulette table. Source: Wikipedia EN.

For every positive integer n, let $[n]$ denote the set of all positive integers up to and including n and let $[n]_0 := \{0\} \cup [n]$. Let us represent the numbers of roulette by the set $[36]_0 = \{0, 1, 2, \ldots, 36\}$.

According to the rules of roulette, the player can bet on certain subsets of $[36]_0$ by placing a token on the roulette table. The player can place a token in an area, on the line between two areas, or on the corner between several areas. One can also place a token on designated fields for playing on odd, even, red, black, and so on. The playable subsets of $[36]_0$ are, in terms of the roulette table, the following: a single number, two adjacent numbers, a row, the group 012, the group 023, a 2×2-submatrix of the roulette table, the group 0123, two adjacent rows, the group 1-12, the group 13-24, the group 25-36, first column, second column, third column, low (1-18), high (19-36), even (excluding zero), odd, red, black, two adjacent dozens (1-24 or 13-36), two adjacent columns. In Holland Casino it is only allowed to play on subsets of at most 18 numbers. (See [4].)

In each round the outcome of the bet is decided by the roulette wheel. We assume that the wheel is unbiased. The outcome U follows a discrete uniform distribution: $U \sim \mathrm{UNIF}([36]_0)$. When the player bets a stake s on an allowed subset $V \subseteq [36]_0$, she is paid

$$\Delta C = \left(\frac{36}{|V|} \mathbb{1}_V(U) - 1 \right) s,$$

where $|V|$ denotes the cardinality of V. All rounds are assumed to be independent from one another.

2.2 Roulette in Holland Casino

2.2.1 Holland Casino

Holland Casino is a state-owned gambling company in the Netherlands. See [5]. This company was founded as *De Nationale Stichting tot Exploitatie van Casinospelen* on January 22, 1974. The Dutch government granted Holland Casino its casino license on December 17, 1975. Since then, Holland Casino is the only company in the Netherlands with a casino license. The reason for this is that the Dutch government wants to control the Dutch gambling market in order to prevent addiction, crime, fraud, and so on. Holland Casino opened its first casino on October 1, 1976, in Zandvoort. See [6].

2.2.2 The rules of roulette in Holland Casino

When you play French roulette in Holland Casino and you bet on red/black, low/high, or odd/even, the zero will not be treated as a regular loss. When zero comes up, you have to choose between two options:

- You share half of your stake with the house.
- You put your stake 'en prison'.

When you choose the second option, a next turn of the wheel will decide whether you lose or win your stake. If zero comes up a second time, your stake will lose half of its worth and you have again the choice between sharing your stake or putting it 'en prison'. You lose your stake when the zero comes up a third time. See [4].

For simplicity, we start our analysis by assuming that a zero outcome is treated as a regular loss. After this simplified analysis, we let go of this assumption and we will treat a zero outcome in the same way Holland Casino does. However, we will make an assumption on the behaviour of the player: the player is not too risk-seeking and will always choose the first option of sharing half of the stake (this option has higher expectation and lower risk, so the utility of this option is higher for players who are not too risk-seeking).

2.3 The d'Alembert system

The *d'Alembert* system is a well-known and established casino betting system that has already been used in the 18th century [7]. It is characterised by a set of rules.

The player starts with a unit bet. After every loss, the player increases the bet by one unit. After every win, the player decreases the bet by one unit, unless the size of the bet is already one unit.

This strategy is very popular because people believe that eventually the numbers of wins and losses will equalize, leaving them with a net gain equal to the number of wins. However, the problem with this strategy is of course that you risk running out of money before these numbers equalize, and that there is a positive probability that these numbers will never equalize.

We will formalize these statements in a couple of theorems.

Assume that in roulette the zero is treated as a regular loss. Also assume that the player is playing only on red/black, low/high, or odd/even. Assume that the player uses the d'Alembert system to play roulette

Theorem 2.1. *Suppose that at time $k \in \mathbb{N}_{\geq 0}$ the numbers of wins and losses equalize. Then the net gain at time k equals the number of wins.*

Proof. This proof follows pages 289-291 of [3].

We would like to terminate the game after an initial win, or as soon as the numbers of wins and losses equalize.

Let $V_1, V_2, \dots$ be random variables s.t. $V_i = +1$ when the player wins in round i and $V_i = -1$ when the player loses. Moreover, define $W_k := \sum_{i=1}^{k} V_i, k \in \mathbb{N}_0$, where we use the convention that the empty sum equals zero.

Denote the stake that is going to be played in round $n+1$ by S_n. By following the d'Alembert strategy, we would get the following stakes

$$\hat{S}_n = 1 - \sum_{l=1}^{n} V_l, \quad n \in \mathbb{N}_0.$$

We immediately see that the numbers of wins and losses equalize when the stake (which could be played in the next round) becomes one again.

Since we would like to terminate the game after an initial win, or as soon as the numbers of wins and losses equalize, we would like a stopping rule given by the following stopping time:

$$T := \mathbb{1}(V_1 = 1) + \min\{n \geq 2 \mid \hat{S}_n = 1\} \cdot \mathbb{1}(V_1 = -1).$$

The stakes that we actually play thus become $S_n = \hat{S}_n \cdot \mathbb{1}(n < T),\ n \in \mathbb{N}_0$.

Given the event $\{n \leq T\}$ the capital level C_n in round n becomes

$$\begin{aligned} C_n &= C_0 + \sum_{l=1}^{n} S_{l-1} V_l = C_0 + \sum_{l=1}^{n} \hat{S}_{l-1} V_l \\ &= C_0 + \sum_{l=1} \left(1 - \sum_{k=1}^{l-1} V_k\right) V_l \\ &= C_0 + \sum_{l=1}^{n} V_l - \sum_{1 \leq k < l \leq n} V_k V_l \\ &= C_0 + \sum_{l=1}^{n} V_l - \frac{1}{2}\left(\left(\sum_{l=1}^{n} V_l\right)^2 - \sum_{l=1}^{n} V_l^2\right) \\ &= C_0 + W_n - \frac{1}{2} W_n^2 + \frac{1}{2} n. \end{aligned} \tag{1}$$

When $T = 1$, we have $W_T = V_1 = 1$, and when $2 \leq T < \infty$, we have that the numbers of wins and losses are equalized at time T. Therefore, on the event $\{T < \infty\}$, we have

$$C_T = C_0 + (1 \vee (T/2)).$$

This shows us that we are left with a net gain equal to the number of wins when the numbers of wins and losses equalize. $\square$

Theorem 2.2. *There is a positive probability that the numbers of wins and losses will never equalize.*

Proof. Adopt the notations from the proof of the previous theorem. By assumption the outcomes of the roulette wheel are i.i.d., so $V_1, V_2, \dots$ are i.i.d. as well. Moreover,

$$V_1 = \begin{cases} +1, & \text{with probability } \frac{18}{37}, \\ -1, & \text{with probability } \frac{19}{37}. \end{cases}$$

Notice that the numbers of wins and losses equalize when for some $k \in \mathbb{N}_1$ it holds that $W_k = 0$. Also notice that $W = (W_k)_{k=0}^{\infty}$ is an asymmetric random walk starting from zero.

Obviously, this random walk W is an irreducible Markov chain, and we denote its n-step transition probabilities by $p_{ij}^{[n]}$ with $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. Notice that $p_{00}^{[0]} = 1$ and notice that for all $n \geq 1$:

$$p_{00}^{[n]} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \left(\frac{18}{37}\right)^{\frac{n}{2}} \left(\frac{19}{37}\right)^{\frac{n}{2}} \binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even}. \end{cases} \tag{2}$$

Recall *Stirling's formula*:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as} \quad n \to \infty.$$

By applying this formula, we find:

$$\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}} \quad \text{as} \quad m \to \infty \tag{3}$$

This is worked out in detail in Example 1.59 from [2].

By using (2) and (3) we find

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{00}^{[n]} &= p_{00}^{[0]} + \sum_{n \in 1+2\mathbb{N}_0} p_{00}^{[n]} + \sum_{n \in 2\mathbb{N}_1} p_{00}^{[n]} \\
&\overset{(2)}{=} 1 + \sum_{n \in 1+2\mathbb{N}_0} \left(\frac{18}{37}\right)^{\frac{n}{2}} \left(\frac{19}{37}\right)^{\frac{n}{2}} \binom{n}{\frac{n}{2}} \\
&= 1 + \sum_{m=1}^{\infty} \left(\frac{18}{37}\right)^{m} \left(\frac{19}{37}\right)^{m} \binom{2m}{m} \\
&\overset{(3)}{\approx} \sum_{m=1}^{\infty} \left(\frac{18}{37}\right)^{m} \left(\frac{19}{37}\right)^{m} \frac{4^m}{\sqrt{\pi m}} \\
&= \sum_{m=1}^{\infty} \left(4 \cdot \frac{18}{37} \cdot \frac{19}{37}\right)^{m} \frac{1}{\sqrt{\pi m}} \\
&\leq \sum_{m=1}^{\infty} \left(4 \cdot \frac{18}{37} \cdot \frac{19}{37}\right)^{m} \\
&< \infty,
\end{aligned} \tag{4}
$$

where the approximation is justified because we are only interested in whether the series diverges to ∞, and where the geometric series $\sum_{m=1}^{\infty} \left(4 \cdot \frac{18}{37} \cdot \frac{19}{37}\right)^m$ converges because $4 \cdot \frac{18}{37} \cdot \frac{19}{37} < 1$.

By applying Theorem A.4 on (4) and by using the fact that W is an irreducible Markov chain, we find that W is transient. Therefore, there is a positive probability that W never returns to zero. This shows us that there is a positive probability that the numbers of wins and losses will never equalize. □

3 Analysis of money laundering case

3.1 Introduction

A criminal defence lawyer says that his client is accused of laundering money. According to the client himself, he won approximately 10000 euro each year for five years in a row by playing red/black and low/high at roulette at Holland Casino. The client said he used the d'Alembert system.

The client plays two simultaneous games: red/black and low/high. For each game the following parameters are set. The initial capital is 25 units, where each unit amounts to 50 euro. The maximum number of rounds is set to 21. Also, the client quits the game if his capital falls below 16 units.

Throughout this chapter we assume that the roulette player is playing on red/black and low/high, and that she is using the d'Alembert system with the parameters mentioned above.

3.2 (In)dependence of the simultaneous games

3.2.1 The dependence of the two simultaneous games

The two simultaneous games have the following numbers of possible outcomes.

	red	black	zero	
low	9	9	0	18
high	9	9	0	18
zero	0	0	1	1
	18	18	1	37

Table 1: This table describes the number of possibilities for each pair of outcomes of the two simultaneous games

This suggests that the two games are 'approximately' independent.

Suppose that the roulette wheel has no zero. This gives rise to the following probabilities:

$$\begin{aligned}
\mathbb{P}(\text{red}) = \mathbb{P}(\text{black}) = \mathbb{P}(\text{low}) = \mathbb{P}(\text{high}) &= \frac{18}{36}, \\
\mathbb{P}(\text{red and low}) &= \frac{9}{36} = \mathbb{P}(\text{red}) \cdot \mathbb{P}(\text{low}), \\
\mathbb{P}(\text{red and high}) &= \frac{9}{36} = \mathbb{P}(\text{red}) \cdot \mathbb{P}(\text{high}), \\
\mathbb{P}(\text{black and low}) &= \frac{9}{36} = \mathbb{P}(\text{black}) \cdot \mathbb{P}(\text{low}), \\
\mathbb{P}(\text{black and high}) &= \frac{9}{36} = \mathbb{P}(\text{black}) \cdot \mathbb{P}(\text{high}).
\end{aligned}$$

Moreover, all the rounds are independent. This shows us that in the case of no zero the two games are indeed independent. However, the zero turns the two games into dependent games, as can be seen by

$$\mathbb{P}(\text{zero and (low or high)}) = 0 \neq \frac{1}{37} \cdot \frac{36}{37} = \mathbb{P}(\text{zero}) \cdot \mathbb{P}(\text{low or high}).$$

3.2.2 An independent coupling of the two games

The question is how well these two dependent games can be approximated by independent games.

We will use the following numerical identifications

$$\begin{aligned}
\text{zero} &= 0; \\
\text{red} = \text{low} &= 1; \\
\text{black} = \text{high} &= 2.
\end{aligned}$$

Denote the red/black process by $X = (X_t)_{t=1}^n$ and the low/high process by $Y = (Y_t)_{t=1}^n$, where n is some convenient time horizon.

As we have seen before, the dependence of the red/black process X and the low/high process Y is caused by the possibility of a zero outcome, since in each round, zero comes up in the red/black process if and only if at the same time zero comes up in the low/high process.

This observation can be used to construct an independent coupling of X, Y. What we can do to turn these processes into an independent coupling is the following:

- Replace the ‘old zero’ in one of the processes by an independent ‘new zero’. That is, flip an independent (unfair) coin to decide for one of the processes in which rounds there is a zero outcome and for which rounds there is no zero outcome.
- Consider for each round whether this coin tells us if there is a zero or not. If there is a (new) zero, then this zero will be the new outcome and this new outcome will replace the old outcome. If there is no (new) zero, then keep the old outcome when this old outcome is not (an old) zero. When the old outcome is zero, then get rid of this outcome by flipping another (fair) coin to decide the new nonzero outcome.

In a word formula:

$$\begin{aligned}&\text{new outcome}\\ &= \text{first coin (0 or 1)} \times \begin{cases}\text{old outcome,} & \text{when old outcome} \neq 0\\ \text{second coin (1 or 2)} & \text{when old outcome} = 0\end{cases}\end{aligned}$$

This story inspires us to formally construct two processes $\hat{X}$, $\hat{Y}$ that are very similar to X, Y. To this end, we put

$$\begin{aligned}\hat{X}_t &:= X_t,\\ \hat{Y}_t &:= N_t(Y_t \mathbb{1}(Y_t \neq 0) + Z_t \mathbb{1}(Y_t = 0)),\\ \hat{X} &:= (\hat{X}_t)_{t=1}^n,\\ \hat{Y} &:= (\hat{Y}_t)_{t=1}^n,\end{aligned} \tag{5}$$

where $N_t \overset{\text{iid}}{\sim} \text{Bern}(36/37)$ are independent of X, Y, and where $Z_t - 1 \overset{\text{iid}}{\sim} \text{Bern}(1/2)$ are independent of $X, Y, N = (N_t)_{t \geq 1}$. We will show that $(\hat{X}, \hat{Y})$ is an independent coupling of (X, Y)

Theorem 3.1. *The processes $\hat{X}$ and $\hat{Y}$ satisfy the following properties:*

(i) $\hat{X}$ and $\hat{Y}$ are independent.

(ii) $\hat{X}, \hat{Y}$ have the same marginal distributions as X, Y.

Proof. Upon careful inspection of (5), we see that $\hat{Y}_t = 0$ if and only if $N_t = 0$. Therefore, we have for all $m \in [2]_0$:

$$\begin{aligned}\mathbb{P}(\hat{X}_t = m, \hat{Y}_t = 0) &= \mathbb{P}(X_t = m, N_t = 0) = \mathbb{P}(X_t = m)\mathbb{P}(N_t = 0)\\ &= \mathbb{P}(\hat{X}_t = m)\mathbb{P}(\hat{Y}_t = 0),\end{aligned} \tag{6}$$

where the second equality holds because X_t and N_t are independent.

By carefully looking at (5) we also see that

$$\forall j \in [2]: \quad \hat{Y}_t = j \iff (N_t = 1 \wedge Y_t = j) \vee (N_t = 1 \wedge Y_t = 0 \wedge Z_t = j). \quad (7)$$

From this follows that for all $j \in [2]$:

$$\begin{aligned} \mathbb{P}(\hat{Y}_t = j) &= \mathbb{P}(N_t = 1, Y_t = j) + \mathbb{P}(N_t = 1, Y_t = 0, Z_t = j) \\ &= \mathbb{P}(N_t = 1)\mathbb{P}(Y_t = j) + \mathbb{P}(N_t = 1)\mathbb{P}(Y_t = 0)\mathbb{P}(Z_t = j) \\ &= \frac{36}{37} \cdot \frac{18}{37} + \frac{36}{37} \cdot \frac{1}{37} \cdot \frac{1}{2} = \frac{18}{37} \\ &= \mathbb{P}(Y_t = j), \end{aligned} \quad (8)$$

where the second equality holds because Y_t, N_t, Z_t are independent. From (8) follows immediately that $\mathbb{P}(\hat{Y}_t = 0) = 1/37 = \mathbb{P}(Y_t = 0)$. Obviously, we also have for all $m \in [2]_0$:

$$\mathbb{P}(\hat{X}_t = m) = \mathbb{P}(X_t = m) = \frac{18}{37} \quad (9)$$

From (7) follows that for all $m, j \in [2]$:

$$\begin{aligned} \mathbb{P}[\hat{X}_t = m, \hat{Y}_t = j] =& \mathbb{P}[N_t = 1 \wedge (X_t, Y_t) = (m, j)] \\ &+ \mathbb{P}[N_t = 1 \wedge Z_t = j \wedge (X_t, Y_t) = (m, 0)] \\ =& \mathbb{P}[N_t = 1] \cdot \mathbb{P}[(X_t, Y_t) = (m, j)] \\ &+ \mathbb{P}[N_t = 1] \cdot \mathbb{P}[Z_t = j] \cdot \mathbb{P}[(X_t, Y_t) = (m, 0)] \\ =& \frac{36}{37} \cdot \frac{9}{37} + \frac{36}{37} \cdot \frac{1}{2} \cdot 0 = \frac{18}{37} \cdot \frac{18}{37} \\ =& \mathbb{P}[\hat{X}_t = m] \cdot \mathbb{P}[\hat{Y}_t = j], \end{aligned} \quad (10)$$

where the second equality holds because $N_t, Z_t, (X_t, Y_t)$ are independent.

It also follows from (7) that for all $j \in [2]$:

$$\begin{aligned} \mathbb{P}(\hat{X}_t = 0, \hat{Y}_t = j) &= \mathbb{P}(X_t = Y_t = 0, \hat{Y}_t = j) \\ &\overset{(7)}{=} \mathbb{P}(X_t = Y_t = 0, N_t = 1, Z_t = j) \\ &= \mathbb{P}(X_t = Y_t = 0)\mathbb{P}(N_t = 1)\mathbb{P}(Z_t = j) \\ &= \frac{1}{37} \cdot \frac{36}{37} \cdot \frac{1}{2} = \frac{1}{37} \cdot \frac{18}{37} \\ &= \mathbb{P}(\hat{X}_t = 0) \cdot \mathbb{P}(\hat{Y}_t = j), \end{aligned} \quad (11)$$

where the third equality holds because $N_t, Z_t, (X_t, Y_t)$ are independent.

We can conclude from (6), (10), (11) that $\hat{Y}_t$ is independent of $\hat{X}_t$.

Put

$$f(x, y, z) := x(y\mathbb{1}(y \neq 0) + z\mathbb{1}(y = 0)).$$

Notice that (X_t, Y_t, N_t, Z_t), $t \in \mathbb{N}_1$, are i.i.d. Therefore, $\hat{X}_t = X_t$, $t \in \mathbb{N}_1$, are i.i.d., $\hat{Y}_t = f(N_t, Y_t, Z_t)$, $t \in \mathbb{N}_1$, are i.i.d., and $(\hat{X}_t, \hat{Y}_t)$, $t \in \mathbb{N}_1$, are i.i.d. Now, we have for all $x = (x_t)_{t=1}^n \in ([2]_0)^n$ and all $y = (y_t)_{t=1}^n \in ([2]_0)^n$,

$$\begin{aligned} \mathbb{P}(\hat{X} = x, \hat{Y} = y) &= \prod_{t=1}^{n} \mathbb{P}(\hat{X}_t = x_t, \hat{Y}_t = y_t) \\ &= \prod_{t=1}^{n} \mathbb{P}(\hat{X}_t = x_t)\mathbb{P}(\hat{Y}_t = y_t) \\ &= \prod_{t=1}^{n} \mathbb{P}(\hat{X}_t = x_t) \cdot \prod_{t=1}^{n} \mathbb{P}(\hat{Y}_t = y_t) \\ &= \mathbb{P}(\hat{X} = x) \cdot \mathbb{P}(\hat{Y} = y), \end{aligned} \tag{12}$$

where the first equality holds because $(\hat{X}_t, \hat{Y}_t)$, $t \in \mathbb{N}_1$, are i.i.d., the second equality holds because $\hat{X}_t, \hat{Y}_t$ are independent, and the last equality holds because $\hat{X}_t$, $t \in \mathbb{N}_1$, are i.i.d. and $\hat{Y}_t$, $t \in \mathbb{N}_1$, are i.i.d. We can conclude from (12) that $\hat{X}$, $\hat{Y}$ are independent.

We can conclude from (9) and (8) that $\hat{X}_t, \hat{Y}_t$ have the same marginal distributions as X_t, Y_t. Since $X, \hat{X}, Y, \hat{Y}$ are sequences of i.i.d. random variables, it follows that $\hat{X}, \hat{Y}$ have the same marginal distributions as X, Y, because,

$$\begin{aligned} \mathbb{P}_{\hat{X}} &= \bigotimes_{t=1}^{n} \mathbb{P}_{\hat{X}_t} = \bigotimes_{t=1}^{n} \mathbb{P}_{X_t} = \mathbb{P}_X, \\ \mathbb{P}_{\hat{Y}} &= \bigotimes_{t=1}^{n} \mathbb{P}_{\hat{Y}_t} = \bigotimes_{t=1}^{n} \mathbb{P}_{Y_t} = \mathbb{P}_Y. \end{aligned} \tag{13}$$

□

3.2.3 Quantification of differences between original and coupling

Now we will try to quantify the difference between (X, Y) and its independent coupling $(\hat{X}, \hat{Y})$.

Notice that

$$Y_t = \hat{Y}_t \iff (N_t = 1 \wedge Y_t \neq 0) \vee (N_t = 0 \wedge Y_t = 0)$$

It follows that

$$\begin{aligned}\mathbb{P}(Y_t = \hat{Y}_t) &= \mathbb{P}((N_t = 1 \wedge Y_t \neq 0) \vee (N_t = 0 \wedge Y_t = 0)) \\ &= \mathbb{P}(N_t = 1 \wedge Y_t \neq 0) + \mathbb{P}(N_t = 0 \wedge Y_t = 0) \\ &= \left(\frac{36}{37}\right)^2 + \left(\frac{1}{37}\right)^2 \geq \left(\frac{36}{37}\right)^2 > 0.94.\end{aligned}$$

Theorem 3.2. *Suppose $n < \infty$. Put*

$$\alpha := 1 - \left(\left(\frac{36}{37}\right)^2 + \left(\frac{1}{37}\right)^2\right). \tag{14}$$

Then: $\mathcal{D}_n := \#\{t \in [n] \ : \ Y_t \neq \hat{Y}_t\} \sim Bin(n, \alpha)$.

Proof. Notice that

$$\mathcal{D}_n = \#\{t \in [n] \ : \ Y_t \neq \hat{Y}_t\} = \sum_{t=1}^{n} \mathbb{1}(Y_t \neq \hat{Y}_t) \tag{15}$$

We have already seen that $(Y_t, \hat{Y}_t), t \in \mathbb{N}_1$, are i.i.d., so

$$\mathbb{1}(Y_1 \neq \hat{Y}_1),\ \mathbb{1}(Y_2 \neq \hat{Y}_2), \ldots \overset{\text{iid}}{\sim} \text{Bern}(\mathbb{P}(Y_1 \neq \hat{Y}_1)) = \text{Bern}(\alpha). \tag{16}$$

Combining (15) and (16) yields

$$\mathcal{D}_n = \#\{t \in [n] \ : \ Y_t \neq \hat{Y}_t\} \sim \text{Bin}(n, \alpha).$$

□

Theorem 3.2 tells us that after n rounds the number $\mathcal{D}_n$ of differences between the original simultaneous process (X, Y) and our independent coupling $(\hat{X}, \hat{Y}) = (X, \hat{Y})$ is binomially distributed with parameters n, α, where $\alpha \approx 0.0526$ (see (14) for the precise definition of α). Our legal client plays a maximum number of 21 rounds, so in our case the number of differences is distributed as described in the table below:

	p.m.f.	c.d.f.
0	0.322	0.322
1	0.374	0.696
2	0.208	0.905
3	0.073	0.978
4	0.018	0.996
5	0.003	0.999
6	0.001	1.000
≥ 7	0.000	1.000

Table 2: This table describes the pmf and cdf of $\mathcal{D}_n$.

3.2.4 Differences in capital between original and coupling

What does Theorem 3.2 tell us about the differences in capital gains between the original simultaneous processes (X, Y) and the coupling $(\hat{X}, \hat{Y})$ that we have just constructed? Notice that by construction $X = \hat{X}$, so there are no capital differences between X and $\hat{X}$, and we will be only looking at capital differences in Y and $\hat{Y}$. We start with the same capital in both Y and $\hat{Y}$. Moreover, let us assume for a moment that we can keep playing forever, regardless of our capital levels. This assumption is made for simplifying our analysis.

Denote the capital level for game Y in round n by C_n, and similarly use the notation $\hat{C}_n$ for the process $\hat{Y}$. Denote the number of wins minus the number of losses for game Y in round n by W_n, and similarly use the notation $\hat{W}_n$ for $\hat{Y}$ likewise .

It is immediate that

$$|W_n - \hat{W}_n| \leq 2\mathcal{D}_n. \tag{17}$$

By using formula (1), and by using the fact that we start with the same capital in both Y and $\hat{Y}$, we find:

$$C_n - \hat{C}_n = W_n - \hat{W}_n + \frac{1}{2}\left(\hat{W}_n^2 - W_n^2\right). \tag{18}$$

Now, it follows from (17) that

$$
\begin{aligned}
|C_n - \hat{C}_n| &\leq |W_n - \hat{W}_n| + \frac{1}{2}\left|\hat{W}_n^2 - W_n^2\right| \\
&\leq 2\mathcal{D}_n + \frac{1}{2}\left|\hat{W}_n^2 - W_n^2\right| \\
&= 2\mathcal{D}_n + \frac{1}{2}\left|\hat{W}_n - W_n\right|\left|\hat{W}_n + W_n\right| \\
&\leq 2\mathcal{D}_n + \mathcal{D}_n\left|W_n + \hat{W}_n\right| \\
&\leq 2\mathcal{D}_n + 2n\mathcal{D}_n \\
&= 2(1+n)\mathcal{D}_n
\end{aligned}
\tag{19}
$$

This shows us that the differences in capital gains between the original simultaneous processes and the coupling can be upper bounded by a multiple of a binomially distributed random variable.

Our legal client plays a maximum number of 21 rounds. We derived the upper bound (19) for the capital difference on the assumption that we can play forever. This seems to be a problem since sometimes the client stops before the maximum number of rounds is reached. However, for all $t \in [n]$, we have

$$|C_t - \hat{C}_t| \overset{(19)}{\leq} 2(1+t)\mathcal{D}_t \leq 2(1+n)\mathcal{D}_n,$$

where the last inequality holds because $0 \leq \mathcal{D}_1 \leq \mathcal{D}_2 \leq \ldots$. Thus, even when the player stops before the maximum number of rounds is reached, we can still use our derived upper bound.

By using the upper bound we find the following probabilities:

x	$\mathbb{P}(\lvert C_{21} - \hat{C}_{21}\rvert \leq x)$
0	0.322
44	≥ 0.696
88	≥ 0.905
132	≥ 0.978
176	≥ 0.996
220	≥ 0.999
264 and higher	1.000

Table 3: This table describes probabilities based on our theoretical upper bound for $|C_{21} - \hat{C}_{21}|$

However, in §3.3 we will prove a lemma (Lemma 3.3) that tells us that the capital levels never go beyond 49 units. This, and table 3, suggest that our theoretical upper bound is not very useful for practical purposes. The fact that our theoretical upper bound may not work for practical purposes is likely to be caused by the crude upper bound $\mathcal{D}_n \left| W_n + \hat{W}_n \right| \leq 2n\mathcal{D}_n$ in our derivation.

We could try to find a better upper bound for $\mathcal{D}_n \left| W_n + \hat{W}_n \right|$ in order to improve the upper bound found in (19). However, this is technically tricky because $W_n, \hat{W}_n, \mathcal{D}_n$ are dependent. We will demonstrate below why W_n and $\mathcal{D}_n$ are dependent.

Dependence of W_n and $\mathcal{D}_n$ can be shown in the following way. Suppose $W_n = -n$. This means that we lose n times in a row. Each time there is a probability of 1/19 that this loss is caused by a zero. When this loss is caused by a zero, there is a probability of 36/37 that there is a different outcome in the coupled game $\hat{Y}$. When this loss is not caused by a zero, there is a probability of 1/37 that the outcome in the coupled game is different. Hence, $\mathbb{E}[\mathcal{D}_n \mid W_n = -n] = n \cdot \frac{1}{19} \cdot \frac{36}{37} + n \cdot \frac{18}{19} \cdot \frac{1}{37} = \frac{54}{703} n$. However, we win all the time when $W_n = n$, so there are no zero outcomes conditional on the event $\{W_n = n\}$. Hence, $\mathbb{E}[\mathcal{D}_n \mid W_n = n] = \frac{1}{37} n \neq \mathbb{E}[\mathcal{D}_n \mid W_n = -n]$. This shows us that $\mathcal{D}_n$ and W_n are indeed dependent.

Finding a better upper bound for $\mathcal{D}_n \left| W_n + \hat{W}_n \right|$ is very tricky due to the dependence of $W_n, \hat{W}_n, \mathcal{D}_n$. That is the reason why will not endeavour this.

We can use our upper bound found in (19) in order to bound the expectation of the capital differences. Since $\mathcal{D}_n \sim \text{Bin}(n, \alpha)$, we have $\mathbb{E}\mathcal{D}_n = n\alpha$. Now it follows from (19) that

$$\begin{aligned} \mathbb{E}|C_n - \hat{C}_n| &\leq 2(1+n)n\alpha = 2\alpha n^2 + 2\alpha n \\ &\approx 0.1052 n^2 + 0.1052 n \end{aligned} \tag{20}$$

In the case of the client, this means that the mean (final) capital difference can be bounded from above by 48.6. This upper bound is enormous, so it does not provide us any useful information on the mean of the (final) capital difference.

3.2.5 Simulations of capital differences

In order to gain a deeper understanding of the distribution of $|C_{21} - \hat{C}_{21}|$ we will run computer simulations. For this purpose we adapt the .m -files of

§3.4. We run 100000 simulations with the MATLAB-function `dalembertstatistical012adapted.m`. We use the function `counting.m` to do the counting for Table 4.

We find the following sample mean

$$\overline{|C_{21} - \hat{C}_{21}|} \approx 1.87,$$

which is of course much better than the theoretical upper bound of 48.5961. We also find the following sample standard deviation

$$s_{|C_{21} - \hat{C}_{21}|} \approx 4.66.$$

In order to get an insight into the distribution of the capital difference $|C_{21} - \hat{C}_{21}|$ we plot a histogram based on our simulations.

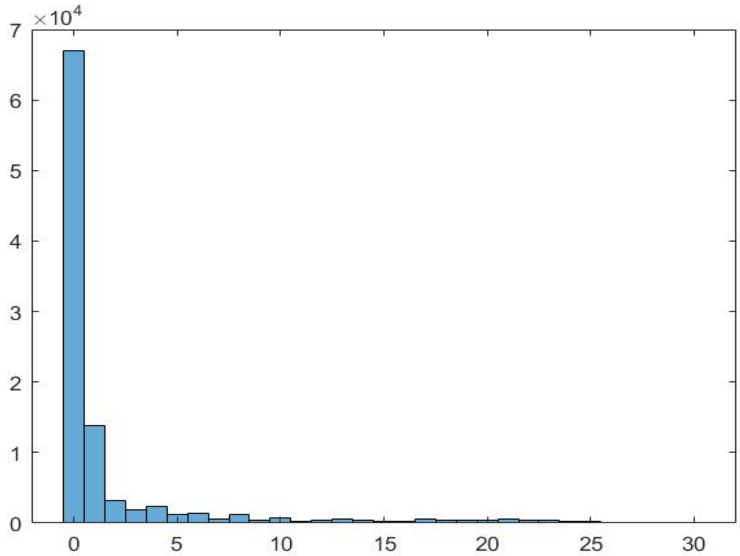

Figure 2: Histogram based on simulations of capital difference $|C_{21} - \hat{C}_{21}|$

By counting how many times a certain value of $|C_{21} - \hat{C}_{21}|$ comes up in our simulations, we get the following table

The histogram and table show us that in two of out three cases there is no capital difference between the original processes and the coupling. In more than 80% of the cases, the capital difference does not exceed a single unit, and in more than 90% the capital difference does not exceed six units. However,

x	$\lvert C_{21} - \hat{C}_{21} \rvert = x$	$\lvert C_{21} - \hat{C}_{21} \rvert \leq x$
0	66967	66967
1	13804	80771
2	3112	83883
3	1924	85807
4	2363	88170
5	1289	89459
6	1461	90920
10	793	93934
20	400	97813

Table 4: Numbers of observations. Based on 100000 simulations.

sometimes the capital difference gets terribly out of hand. In more than 2% of the cases the difference is even larger than 20 units.

Our coupling makes it much easier to do all kinds of calculations with regards to the money laundering case and the use of the d'Alembert system in this case. Based on our simulations we can conclude that the coupling can be used to approximate the original processes, but that we should do that with some caution, since sometimes the capital differences can get terribly out of hand.

3.3 The roulette games as Markov process

The capital/stake levels of the simultaneous red/black and low/high games can be thought of as a Markov process. First we look at the two games separately.

3.3.1 One game as Markov chain

We consider the capital/stake levels $(c_t, s_t)_{t=0}^{21}$ in one of the games, namely the red/black game or the low/high game (these two games are interchangeable when you look at them separately), where $s_t = 0$ denotes a stopped game. Our goal is to model this as a discrete-time time-homogeneous Markov chain with finite state space. This is possible because the transition from one capital/stake-pair (c, s) to the next $(c\prime, s\prime)$ only depends on the current pair (c, s) and the outcome of the roulette wheel, which is independent for each round, and because there are only finitely many possibilities for these pairs, as we will see in the next lemma.

Lemma 3.3. *In each game (red/black or low/high) we have in every round t that capital c and stake s satisfy*

$$0 \leq c \leq 49 \quad \textit{and} \quad 0 \leq s \leq 9$$

when we follow the rules of the previously described d'Alembert strategy.

Proof. Suppose we lost $n - 1$ times in a row and we are still playing. Then we lost $\sum_{k=1}^{n-1} k = \frac{1}{2}n^2 - \frac{1}{2}n$ units of capital, and in the next round we play a stake of $s = n$. Notice that $\frac{1}{2}n^2 - \frac{1}{2}n \geq n$ for all $n \geq 3$. Thus, when we already lost ≥ 2 times in a row, we cannot offset our loss by winning the next round. This leads to the conclusion that losing ≥ 2 in a row is always unfavourable in comparison with winning all the time. Suppose we lose only one time and then win the next round. Then our capital gain is $-1+2 = 1$. However, winning two times leads to a capital gain of $1+1 = 2$. Thus, the conclusion is again that in terms of capital, losing is unfavourable in comparison with winning all the time. Since winning all the time is the most favourable outcome of a game and since we play at most 21 rounds, we have $c \leq 25 + 21 = 46 \leq 49$.

It is obvious that $s \geq 0$ in every round. Suppose we have a stake of $s = 10$. Then we lost the previous 9 rounds, so we already lost $\sum_{k=1}^{8} k = 36$ units of capital in the first eight losing rounds. After these eight rounds, our capital is at most $46 - 36 = 10$ units, so we must stop there. Hence, a stake of 10 is impossible. In fact, a stake of 9 is impossible as well. Hence, $0 \leq s \leq 8 \leq 9$.

Since $s \leq 8$ in every round, and since we must stop when capital falls below the threshold of 16, we have $c \geq 16 - 8 = 8$. Thus, we can conclude that capital c always satisfies $0 \leq 8 \leq c \leq 46 \leq 49$. □

Because of Lemma 3.3, we can choose a finite state space for our Markov chain:

$$S = [49]_0 \times [9]_0 = \{0, 1, \dots, 49\} \times \{0, 1, \dots, 9\}. \tag{21}$$

Let us assume for simplicity that a zero outcome at the roulette table means a regular loss for the player. Then, in each game, the player loses with probability 19/37 and the player wins with probability 18/37. Moreover, the player must stop when she hits the maximum number of rounds, which is 21, or when the capital falls below the threshold of 16 units. The latter is modelled by setting the next stake equal to zero and by setting $P((c, 0), (c\prime, s\prime)) = \mathbb{1}((c\prime, s\prime) = (c, 0))$, that is, nothing changes once our stake is zero. This leads to a transition matrix $P = [P((c, s), (c\prime, s\prime))]_{(c,s),(c\prime,s\prime)\in S}$ given by

$$\begin{aligned} P((c, s), ((c + s) \wedge 49, (s - 1) \vee 1)) &= \frac{18}{37} \quad \text{if } s \neq 0, \\ P((c, s), ((c - s) \vee 0, (s + 1) \wedge 9)) &= \frac{19}{37} \quad \text{if } s \neq 0 \text{ and } c - s \geq 16, \\ P((c, s), ((c - s) \vee 0, 0)) &= \frac{19}{37} \quad \text{if } s \neq 0 \text{ and } c - s \leq 15, \\ P((c, s), (c, s)) &= 1 \quad \text{if } s = 0, \end{aligned} \tag{22}$$

where all the other entries of P are equal to zero. We can use the MATLAB function `Pmatrix.m` to compute the entries of P. See the appendix for the code.

We can use the transition matrix P to compute the distribution of the final capital level. Since we start with initial capital 25 and initial stake 1, we should take as initial distribution the probability row vector $\lambda^{(0)} = (\lambda^{(0)}(c, s))_{(c,s)\in S}$ given by

$$\lambda^{(0)}((25, 1)) = 1 \quad \text{and} \quad \lambda^{(0)}((c, s)) = 0,\ (c, s) \neq (25, 1). \tag{23}$$

Then the distribution $\lambda^{(21)} = (\lambda^{(21)}(c, s))_{(c,s)\in S}$ at time 21 is given by

$$\lambda^{(21)} = \lambda^{(0)} P^{21}. \tag{24}$$

Finally, the distribution of the final capital is given by the pmf f given by

$$f(c) = \sum_{s=0}^{9} \lambda^{(21)}(c, s) \quad \text{for} \quad c \in [49]_0, \quad \text{and zero elsewhere.} \tag{25}$$

We can use MATLAB to do all these computations. We can use the MATLAB function `initialdistribution.m` to define $\lambda^{(0)}$ in the MATLAB environment and we can use the MATLAB function `Pmatrix.m` to define P in the MATLAB environment. We can use the build-in function `reshape` to let these MATLAB objects have the appropriate dimensions, and we can use `Distr.m` to aggregate probabilities in accordance to (25). See the appendix for the code. In the MATLAB console, we type in:

```
>> P=Pmatrix(50,10);
>> Q=reshape(P,[500,500]);
>> v=initialdistribution(50,10,25,1);
>> w=reshape(v,[1,500]);
>> distr=w*Q^21;
>> D=Distr(distr)
```

Notice that the entries $f(0), f(1), \ldots, f(49)$ of the vector `D` together with (25) give us the pmf of the final capital. We get:

$$(f(k))_{k=0}^{16} \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0001 \\ 0.0255 \\ 0.0743 \\ 0.0821 \\ 0.0730 \\ 0.0625 \\ 0.1246 \\ 0.0003 \end{bmatrix}, \quad (f(k))_{k=17}^{33} \approx \begin{bmatrix} 0.0001 \\ 0.0074 \\ 0.0122 \\ 0.0092 \\ 0.0045 \\ 0.0015 \\ 0.0004 \\ 0.0204 \\ 0.0214 \\ 0.0133 \\ 0.0058 \\ 0.0183 \\ 0.0306 \\ 0.0255 \\ 0.0142 \\ 0.0337 \\ 0.0366 \end{bmatrix} \quad \text{and} \quad (f(k))_{k=34}^{49} \approx \begin{bmatrix} 0.0396 \\ 0.0467 \\ 0.0582 \\ 0.0598 \\ 0.0474 \\ 0.0292 \\ 0.0141 \\ 0.0054 \\ 0.0016 \\ 0.0004 \\ 0.0001 \\ 0.0000 \\ 0.0000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding bar chart can be plotted by typing

```
>> x=0:1:49;
>> bar(x,D)
```

We get the following bar chart:

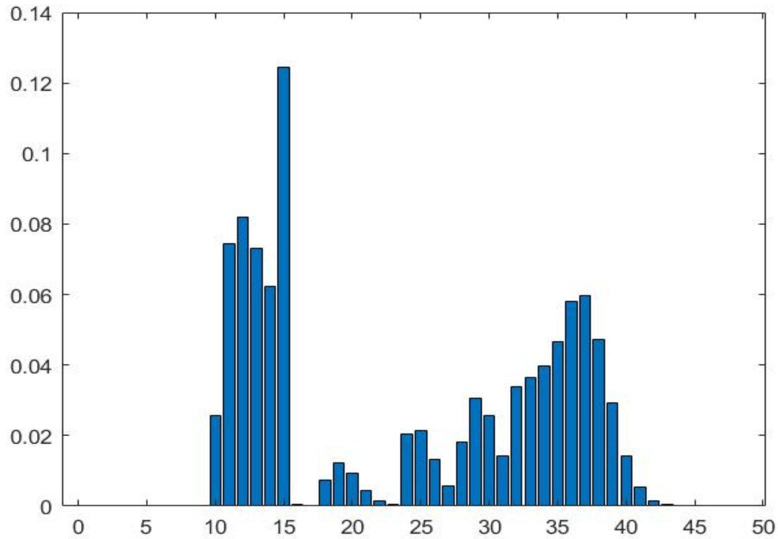

Figure 3: Bar chart of pmf of final capital in one of the games

We type the following in the MATLAB console

```
>> firstmoment = x * transpose(D)
>> secondmoment = (x .^2 ) * transpose(D)
>> var = secondmoment - (firstmoment)^2
>> sd = (var)^0.5
```

to find that the mean of the final capital is approximately 24.0, which means that on average we lose around 4.15% to the house, and that the standard deviation of the final capital is around 10.6.

3.3.2 Simultaneous games as Markov chain

For the two games simultaneously we still have that in each round the two capital levels are bounded between 0 en 49, and the two stake levels are bounded between 0 en 9. Thus, for the two games simultaneously, we can choose as state space

$$\mathcal{S} = S \times S. \tag{26}$$

Let $(u_{X,0}, u_{Y,0}), (u_{X,1}, u_{Y,1}) \in \mathcal{S}$. Let $E_{X,0}$ denote the event that the current state in the X-game is $u_{X,0}$, let $E_{X,1}$ denote the event that the next state in the X-game is $u_{X,1}$, let $E_{Y,0}$ denote the event that the current state in the Y-game is $u_{Y,0}$, and let $E_{Y,1}$ denote the event that the next state in the X-game is $u_{X,1}$. Because the two processes X, Y (red/black and low/high) are almost independent, we can approximate the transition probability

$$\mathcal{P}((u_{X,0}, u_{Y,0}), (u_{X,1}, u_{Y,1})),$$

by

$$\begin{aligned}
\mathcal{P}((u_{X,0}, u_{Y,0}), (u_{X,1}, u_{Y,1})) &= \mathbb{P}(E_{X,1} \cap E_{Y,1} \mid (E_{X,0} \cap E_{Y,0}) \\
&= \frac{\mathbb{P}(E_{X,1} \cap E_{X,0} \cap E_{Y,1} \cap E_{Y,0})}{\mathbb{P}(E_{X,0} \cap E_{Y,0})} \\
&\approx \frac{\mathbb{P}(E_{X,1} \cap E_{X,0})}{\mathbb{P}(E_{X,0})} \cdot \frac{\mathbb{P}(E_{Y,1} \cap E_{Y,0})}{\mathbb{P}(E_{Y,0})} \\
&= \mathbb{P}(E_{X,1} \mid E_{X,0}) \cdot \mathbb{P}(E_{Y,1} \mid E_{Y,0}) \\
&= P(u_{X,0}, u_{X,1}) \cdot P(u_{Y,0}, u_{Y,1}).
\end{aligned} \tag{27}$$

We would like to learn more about the accuracy of this kind of approximations.

Suppose that currently the capital/stake positions of the X- and Y-games are $u_{X,0} = (c_{X,0}, s_{X,0})$ and $u_{Y,0} = (c_{Y,0}, s_{Y,0})$ respectively. Denote the winning outcome of each game by 2, the losing non-zero outcome by 1, and the losing zero outcome by 0. Denote the new outcomes of the X- and Y-games by X_{new} and Y_{new} respectively.

Winning in the X-game means that the capital/stake position moves from $u_{X,0} = (c_{X,0}, s_{X,0})$ to $\hat{u}_{X,1} = (\hat{c}_{X,1}, \hat{s}_{X,1})$ with $\hat{c}_{X,1} = c_{X,0} + s_{X,0}$ and $\hat{s}_{X,1}$ given by a more complicated update rule. See (22) for a description of this update rule. Losing in the X-game means that the capital/stake position moves from $u_{X,0} = (c_{X,0}, s_{X,0})$ to $\tilde{u}_{X,1} = (\tilde{c}_{X,1}, \tilde{s}_{X,1})$ with $\tilde{c}_{X,1} = c_{X,0} - s_{X,0}$ and $\tilde{s}_{X,1}$ given by the update rule. The consequences of winning and losing in the Y-game can be described in a similar manner and we use similar notations and symbols for this.

This shows us that given current positions $(u_{X,0}, u_{Y,0})$, there are only four possibilities, with positive conditional probability, for the next positions $(u_{X,1}, u_{Y,1})$ of the X- and Y-games.

The analysis above gives us the tools to exactly compute the transition probabilities and compare them with the approximated transitions probabilities.

The transition probabilities are given by

$$\begin{aligned}\mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\hat{u}_{Y,1})\right)&=\mathbb{P}(X_{\text{new}}=Y_{\text{new}}=2)=\frac{9}{37}\approx 0.243,\\ \mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\tilde{u}_{Y,1})\right)&=\mathbb{P}(X_{\text{new}}=2,Y_{\text{new}}=1)=\frac{9}{37}\approx 0.243,\\ \mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\hat{u}_{Y,1})\right)&=\mathbb{P}(X_{\text{new}}=1,Y_{\text{new}}=2)=\frac{9}{37}\approx 0.243,\end{aligned}\tag{28}$$

and

$$\begin{aligned}\mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\tilde{u}_{Y,1})\right)&=\mathbb{P}(X_{\text{new}}=Y_{\text{new}}=1)+\mathbb{P}(X_{\text{new}}=Y_{\text{new}}=0)\\ &=\frac{9}{37}+\frac{1}{37}=\frac{10}{37}\approx 0.270,\end{aligned}\tag{29}$$

and all the other entries equal to zero.

Approximations of the transition probabilities, based on (27) would be

$$\begin{aligned}\mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\hat{u}_{Y,1})\right)&\approx\left(\frac{18}{37}\right)^2\approx 0.237\\ \mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\tilde{u}_{Y,1})\right)&\approx\frac{18}{37}\cdot\frac{19}{37}\approx 0.250\\ \mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\hat{u}_{Y,1})\right)&\approx\frac{19}{37}\cdot\frac{18}{37}\approx 0.250,\\ \mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\tilde{u}_{Y,1})\right)&\approx\left(\frac{19}{37}\right)^2\approx 0.264.\end{aligned}\tag{30}$$

By comparing the actual transition probabilities in (28), (29) with the approximated transition probabilities in (30), we see that the approximations based on (27) are pretty accurate. However, small differences in transition probabilities can possibly accumulate into huge differences in capital when we let the Markov process run 21 rounds.

Moreover, we also see that the actual transition probabilities are quite easy to compute, so it seems that there is no need for approximating the transition probabilities. The only complication in computing the actual transition probabilities is the fact that we have to keep track of so many indices and that we have to reshape arrays in an appropriate way in order to do actual calculations with them.

The initial distribution

$$p^{(0)}=(p^{(0)}(u_X,u_Y))_{(u_X,u_Y)\in\mathcal{S}}=(p^{(0)}((c_X,s_X),(c_Y,s_Y)))_{c_X,c_Y\in[49]_0,\ s_X,s_Y\in[9]_0}$$

of the capital/stake levels in the two simultaneous is given by

$$\begin{aligned} p^{(0)}((25,1),(25,1)) &= 1 \quad \text{and} \\ p^{(0)}((c_X,s_X),(c_Y,s_Y)) &= 0, \quad ((c_X,s_X),(c_Y,s_Y)) \neq ((25,1),(25,1)). \end{aligned} \tag{31}$$

3.3.3 The total capital level in the two simultaneous games

Let $\mathcal{P}$ be the transition matrix for the capital/stake levels of the two simultaneous games (X,Y). The entries of this matrix are given by (28) and (29). We consider the initial distribution $p^{(0)}$ as a row vector and $\mathcal{P}$ as a two-dimensional matrix. The distribution

$$\begin{aligned} p^{(21)} &= (p^{(21)}(u_X,u_Y))_{(u_X,u_Y)\in\mathcal{S}} \\ &= (p^{(21)}((c_X,s_X),(c_Y,s_Y))_{c_X,c_Y\in[49]_0,\ s_X,s_Y\in[9]_0} \end{aligned}$$

at time 21 can be computed by

$$p^{(21)} = p^{(0)}\mathcal{P}^{21}. \tag{32}$$

The distribution $p_{CT}^{(21)} = (p_{CT}^{(21)}(c))_{c=0}^{98}$ of the total final capital level at time 21 can be computed by

$$p_{CT}^{(21)}(c) = \sum_{k\in\mathcal{I}_c}\sum_{s_Y=0}^{9}\sum_{s_X=0}^{9} p^{(21)}((k,s_X),(c-k,s_Y)), \quad c\in[98]_0, \tag{33}$$

where

$$\mathcal{I}_c = \{j\in[49]_0 \mid 0\le j, c-j\le 49\} = [49]_0\cap[c-49,c]. \tag{34}$$

It is possible to model the total final capital level through a Markov model, but we have to keep track of the stakes and the capital level in at least one of the games (X or Y), because the stakes do not only depend on the previous stakes and the outcomes of the games, but also on current capital levels in each of the games, since we set the stake to zero when capital falls below a certain threshold. Thus, states must be of the form (c_X,c_T,s_X,s_Y), (c_Y,c_T,s_X,s_Y) or (c_X,c_Y,c_T,s_X,s_Y), where c stands for capital, s stand for stake, X stands for the process X, Y stands for the process Y, and C_T denotes the total capital.

3.3.4 Markov chain description of a single game under the Holland Casino policy for zero outcomes

Let us assume that the zero outcome is treated in accordance to the policy of Holland Casino. Let us further assume that the player chooses to share half of her stake with the house when zero comes up. See §2.2.2. The policy on allowed stakes may differ from one casino to another, but let us not be bothered too much with minimum and maximum stakes.

Let us reformulate the d'Alembert strategy for the situation in which we follow the Holland Casino rules. As described in §2.2.2, the player loses half of her stake when zero comes up. However, this is still a loss, so we will label this as a loss. Thus, a losing non-zero outcome and a zero outcome have the same label "loss." A winning outcome is of course still labeled as a win.

Based on our labels "loss" and "win" we follow the d'Alembert strategy, which we will describe now. After every loss, the player increases the stake by one unit, unless her capital has already fallen below the threshold of 16 units, in which case she will bet nothing (stake = zero). After every win, the player decreases the stake by one unit, unless the size of the stake is already one unit. Moreover, we have a time horizon of 21 rounds.

The main difference between treating a zero outcome as a regular loss and treating a zero outcome in the Holland Casino way is that in the latter case a zero outcome leads to a capital level of $capital - \frac{stake}{2}$ in the next round instead of $capital - stake$. This requires us to modify the state space S as defined in (21). For one game, red/black or low/high, the modified state space becomes

$$S = \left(\frac{1}{2}[98]_0\right) \times [9]_0 \qquad (35)$$

and the modified transition matrix $P = [P((c,s),(c\prime,s\prime))]_{(c,s),(c\prime,s\prime)\in S}$ becomes given by

$$
\begin{aligned}
P((c,s),((c+s)\wedge 49,(s-1)\vee 1)) &= \frac{18}{37} \quad \text{if } s\neq 0,\\
P((c,s),((c-s)\vee 0,(s+1)\wedge 9)) &= \frac{18}{37} \quad \text{if } s\neq 0 \text{ and } c-s\geq 16,\\
P((c,s),((c-s)\vee 0,0)) &= \frac{18}{37} \quad \text{if } s\neq 0 \text{ and } c-s< 16,\\
P((c,s),((c-s/2)\vee 0,(s+1)\wedge 9)) &= \frac{1}{37} \quad \text{if } s\neq 0 \text{ and } c-\frac{s}{2}\geq 16,\\
P((c,s),((c-s/2)\vee 0,0)) &= \frac{1}{37} \quad \text{if } s\neq 0 \text{ and } c-\frac{s}{2}< 16,\\
P((c,s),(c,s)) &= 1 \quad \text{if } s=0,
\end{aligned}
\tag{36}
$$

where all the other entries of P are equal to zero. For computer computations it is convenient to subdivide each monetary unit into two sub units and to do all the calculations with sub units. Expressed in sub units, (35) and (36) become

$$
S=[98]_0\times(2[9]_0) \tag{37}
$$

and

$$
\begin{aligned}
P((c,s),((c+s)\wedge 98,(s-2)\vee 2)) &= \frac{18}{37} \quad \text{if } s\neq 0,\\
P((c,s),((c-s)\vee 0,(s+2)\wedge 18)) &= \frac{18}{37} \quad \text{if } s\neq 0 \text{ and } c-s\geq 32,\\
P((c,s),((c-s)\vee 0,0)) &= \frac{18}{37} \quad \text{if } s\neq 0 \text{ and } c-s< 32,\\
P((c,s),((c-s/2)\vee 0,(s+2)\wedge 18)) &= \frac{1}{37} \quad \text{if } s\neq 0 \text{ and } c-\frac{s}{2}\geq 32,\\
P((c,s),((c-s/2)\vee 0,0)) &= \frac{1}{37} \quad \text{if } s\neq 0 \text{ and } c-\frac{s}{2}< 32,\\
P((c,s),(c,s)) &= 1 \quad \text{if } s=0,
\end{aligned}
\tag{38}
$$

where all the other entries of P are equal to zero. Expressing the initial distribution $\lambda^{(0)}=(\lambda^{(0)}(c,s))_{(c,s)\in S}$, as given in (23), in terms of sub units yields

$$
\lambda^{(0)}((50,2))=1 \quad \text{and} \quad \lambda^{(0)}((c,s))=0,\ (c,s)\neq(50,2). \tag{39}
$$

Of course, when everything is expressed in terms of sub units, the distribution $\lambda^{(21)}$ at time 21 can still be calculated by using (24). However, the formula

(25) for the pmf of the final capital needs a slight change:

$$f(c) = \sum_{s \in 2[9]_0} \lambda^{(21)}(c,s) \quad \text{for} \quad c \in [98]_0, \quad \text{and zero elsewhere.} \tag{40}$$

In order to compute this pmf for the Holland Casino scenario we adapt `Pmatrix.m` and `Distr.m` and call these adapted functions `PmatrixHC.m` and `DistrHC.m` respectively. See the appendix. Notice that we allow stakes to have an odd number of sub units. However, this will not affect the computations since we start with an initial stake of two sub units and since the stakes make only even jumps (in terms of sub units).

The bar chart corresponding to the pmf can be plotted by typing

```
>> P=PmatrixHC(99,19);
>> d=99*19;
>> Q=reshape(P,[d,d]);
>> v=initialdistribution(99,19,50,2);
>> w=reshape(v,[1,d]);
>> distr=w*Q^21;
>> D=DistrHC(distr);
>> x=0:1:98;
>> bar(x,D)
```

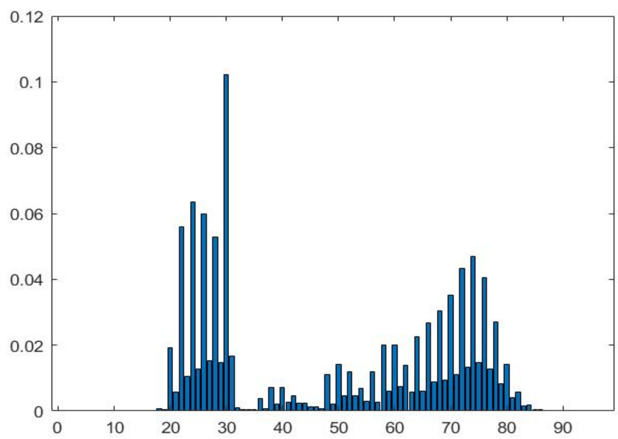

Figure 4: Bar chart of pmf of final capital in one of the games. Holland Casino scenario. Final capital is expressed in sub units of 25 euro each. (1 unit = 50 euro, 1 unit = 2 sub units)

It is striking that the bar chart for the Holland Casino scenario (figure 4) and the bar chart for the 'zero-treated-as-regular-loss' scenario (figure 3) look very similar. This indicates that the game under the Holland Casino rules can be well approximated by a simplification in which we treat a zero outcome as regular loss.

It is also striking that an even final capital (in terms of sub units) seems to be more probable than an odd one (in figure 4 the bars for even capital levels are larger than for odd ones). This can be explained by the fact that an odd final capital can only occur when there are zero outcomes, but that zero outcomes do not necessarily imply that the final capital is odd. For example, a zero outcome does not lead to an odd decrease in capital when the stake equals four. Another example, two zero outcomes can lead to two odd decreases in capital, which amount to a an even total decrease in capital. We have:

$$\mathbb{P}(\text{even final capital}) \geq \mathbb{P}(\text{no zero outcomes}) = \left(\frac{36}{37}\right)^{21} > \frac{1}{2}. \qquad (41)$$

This shows us that an even final capital is more probable than an odd one.

We type the following in the MATLAB console

```
>> firstmoment = x * transpose(D)
>> secondmoment = (x .^2 ) * transpose(D)
>> var = secondmoment - (firstmoment)^2
>> sd = (var)^0.5
```

to find that the mean of the final capital is approximately 48.9, which means that on average we lose around 2.11% to the house, and that the standard deviation of the final capital is around 21.5.

3.3.5 Markov chain description of simultaneous games under the Holland Casino policy for zero outcomes

Let us now mathematically describe the capital/stake-levels of the simultaneous games (X, Y) under the Holland Casino policy, again under the same assumptions (use of the d'Alembert strategy as described above, choice for sharing half of the stake with the house when zero comes up). We will do this in terms of Markov processes and their transition probabilities. This description will be very similar to the description for the simultaneous games under the "zero-treated-as-regular-loss rule". See §3.3.2.

Of course we can approximate the transition probabilities of the simultaneous processes (X, Y) by multiplying the transition probabilities of the marginal processes, since these processes are almost independent. However, we will endeavor to exactly compute the transition probabilities.

The state space of the simultaneous games will again be $\mathcal{S} = S \times S$, but this time S refers to (39) instead of (21). By this choice of S we implicitly choose to express capital and stake levels in terms of sub units (1 unit = 2 sub units). We will hugely adopt the same analysis and same notations as for the case in which a zero outcome is treated as a regular loss, but there will be small differences.

Suppose that currently the capital/stake positions of the X- and Y-games are $u_{X,0} = (c_{X,0}, s_{X,0})$ and $u_{Y,0} = (c_{Y,0}, s_{Y,0})$ respectively. Denote the winning outcome of each game by 2, the losing non-zero outcome by 1, and the losing zero outcome by 0. Denote the new outcomes of the X- and Y-games by X_{new} and Y_{new} respectively.

Winning in the X-game means that the capital/stake position moves from $u_{X,0} = (c_{X,0}, s_{X,0})$ to $\hat{u}_{X,1} = (\hat{c}_{X,1}, \hat{s}_{X,1})$ with $\hat{c}_{X,1} = c_{X,0} + s_{X,0}$ and $\hat{s}_{X,1}$ given by a more complicated update rule. See (38) for a description of this update rule. A loss caused by a non-zero outcome in the X-game means that the capital/stake position moves from $u_{X,0} = (c_{X,0}, s_{X,0})$ to $\tilde{u}_{X,1} = (\tilde{c}_{X,1}, \tilde{s}_{X,1})$ with $\tilde{c}_{X,1} = c_{X,0} - s_{X,0}$ and $\tilde{s}_{X,1}$ given by the update rule. A loss caused by a zero outcome in the X-game means that the capital/stake position moves from $u_{X,0} = (c_{X,0}, s_{X,0})$ to $\check{u}_{X,1} = (\check{c}_{X,1}, \check{s}_{X,1})$ with $\check{c}_{X,1} = c_{X,0} - s_{X,0}/2$ and $\check{s}_{X,1}$ given by the update rule. The consequences of winning and losing in the Y-game can be described in a similar manner and we use similar notations and symbols for this.

Based on the analysis above, and table 1, we find the following transition probabilities:

$$\begin{aligned}
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\hat{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=2,Y_{\text{new}}=2\right)=\frac{9}{37}\approx 0.243,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\tilde{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=2,Y_{\text{new}}=1\right)=\frac{9}{37}\approx 0.243,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\hat{u}_{X,1},\check{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=2,Y_{\text{new}}=0\right)=0,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\hat{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=1,Y_{\text{new}}=2\right)=\frac{9}{37}\approx 0.243,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\tilde{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=1,Y_{\text{new}}=1\right)=\frac{9}{37}\approx 0.243,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\tilde{u}_{X,1},\check{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=1,Y_{\text{new}}=0\right)=0,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\check{u}_{X,1},\hat{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=0,Y_{\text{new}}=2\right)=0,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\check{u}_{X,1},\tilde{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=0,Y_{\text{new}}=1\right)=0,\\
\mathcal{P}\left((u_{X,0},u_{Y,0}),(\check{u}_{X,1},\check{u}_{Y,1})\right) &= \mathbb{P}\left(X_{\text{new}}=0,Y_{\text{new}}=0\right)=\frac{1}{37}\approx 0.027,
\end{aligned}\tag{42}$$

and all other transition probabilities equal to zero.

In a way the transition matrix $\mathcal{P}$ for the Holland Casino scenario is simpler than the transition matrix for the scenario in which zero outcomes are treated as regular losses, since the transition probabilities in (42) seem to just resemble table 1.

The initial distribution

$$p^{(0)}=(p^{(0)}(u_X,u_Y))_{(u_X,u_Y)\in\mathcal{S}}=(p^{(0)}((c_X,s_X),(c_Y,s_Y)))_{c_X,c_Y\in[98]_0,\ s_X,s_Y\in 2[9]_0},$$

in terms of sub units, is given by

$$\begin{aligned}
p^{(0)}((50,2),(50,2)) &= 1 \quad \text{and}\\
p^{(0)}((c_X,s_X),(c_Y,s_Y)) &= 0, \quad ((c_X,s_X),(c_Y,s_Y))\neq((50,2),(50,2)).
\end{aligned}\tag{43}$$

The distribution $p^{(21)}$ at time 21 can be computed through $p^{(21)}=p^{(0)}\mathcal{P}^{21}$, where we write $p^{(0)},p^{(21)}$ as row vectors and $\mathcal{P}$ as matrix. The distribution $p^{(21)}_{CT}=(p^{(21)}_{CT}(c))_{c=0}^{196}$ of the total final capital level at time 21 can be computed through

$$p^{(21)}_{CT}(c)=\sum_{k\in\mathcal{I}_c}\sum_{s_Y\in 2[9]_0}\sum_{s_X\in 2[9]_0}p^{(21)}((k,s_X),(c-k,s_Y)),\quad c\in[196]_0,$$

with $\mathcal{I}_c=[98]_0\cap[c-98,c]$.

3.3.6 Doing computations with the Markov model for the simultaneous games

Consider the Markov chain model for the capital/stake-levels of the two simultaneous games (X, Y). This model is specified by the transition matrix $\mathcal{P}$ given by (42) and the initial distribution $p^{(0)}$ given by (43).

We have tried to write a MATLAB function `PsimHC.m` (see appendix) for constructing the transition matrix $\mathcal{P}$ in the MATLAB environment, both for the Holland Casino case and for the simplified case. However, we encountered some memory issues. We have tried to solve these issues by using sparse arrays. For this purpose we have tried to use `ndSparse`, a class written by Matt Jocobson, a research scientist at Xoran Technologies, which is a company for CT scanners in Michigan, US. This software can be found on `https://nl.mathworks.com/matlabcentral/fileexchange/29832-n-dimensional-sparse-arrays` (including its license). Unfortunately, we did not succeed in making this solution work.

This opens up new research questions. From a computer scientific point of view, we could ask ourselves how to store and process enormous arrays. How should we deal with memory allocation and what kind of objects do we need to use? Could we construct our own classes for doing computations with large sparse arrays or could we use someone else's classes? Which programming languages and statistical packages are best suited for doing this? From a probabilistic point of view, we could ask ourselves how to refine our coupling idea, and how to use that refinement for getting fairly accurate approximations of probabilistic properties of the simultaneous games and their corresponding capital- and stake-levels.

3.4 Simulations

3.4.1 Simulations when zero is treated as regular loss

For simplicity we assume in our simulations that a zero outcome at the roulette table means a regular loss for the player. Thus, our simulations provide lower bounds for the capital gains of the player.

We use the MATLAB function `simultaneous.m` for simulating the outcomes of the two simultaneous roulette games (red/black and low/high). With the function `dalembert012.m` we simulate the d'Alembert strategy for the two games and the corresponding capitals and stakes. The function `dalembertstatistical012.m` is written in order to collect the final capitals for each game. This enables us to do statistical analysis on the d'Alembert system applied to these two simultaneous games. See the appendix for more details on the code of the m-files.

Performing the simulations (with $n = 10^6$ games) give rise to the following summary statistics:

	Sample average of final capital	Sample standard deviation of final capital
Red/black game	24.0	10.6
Low/high game	24.0	10.6
The two games combined	47.9	15.2

Table 5: Summary statistics based on 10^6 simulations.

These figures are completely in line with our theoretical findings about a single game (red/black or low/high) in §3.3.1. Moreover, we see that on average we lose approximately 4.16% to the house.

Denote the sample standard deviation of the final capital of the red/black game by s_X, the sample standard deviation of the final capital of the low/high game by s_Y and the sample standard deviation of the combined final capital by s_T and notice that

$$\begin{aligned} s_X^2 + s_Y^2 &\approx 226, \\ s_T^2 &\approx 230. \end{aligned}$$

This suggests that the two final capitals (one for red/black and one for low/high) are weakly correlated. The sample correlation r_{XY} turns out to be approximately 0.017. Under the null hypothesis of zero correlation, we have that asymptotically the test statistic $t = r_{XY}\sqrt{\frac{n-2}{1-r_{XY}^2}}$ follows a standard normal distribution. (See the appendix.) We have $t \approx 17$, so the sample correlation r_{XY} differs significantly from zero. This makes it reasonable to think that the two final capitals (one for red/black and one for low/high) are weakly correlated.

We will plot a number of histograms, based on the simulations, in order to get an idea of the distributions of the final capitals, and in order to check the findings in §3.3.

Figures 5, 6, 7 show us histograms for the final capital of the red/black game, the low/high game, and the two games combined:

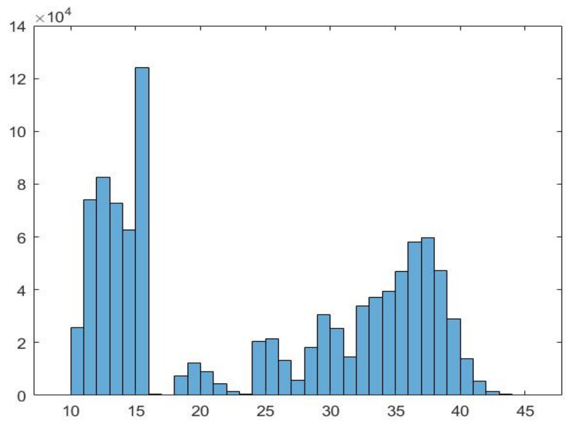

Figure 5: Final capital in red/black game

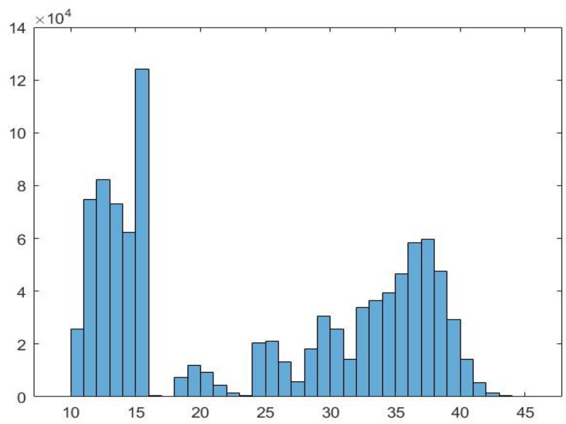

Figure 6: Final capital in low/high game

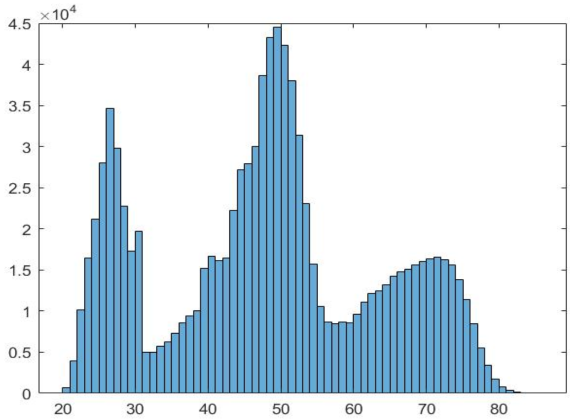

Figure 7: Final capital for the two simultaneous games combined

Figure (5) and (6) show us that the final capitals in the two games have (at least almost) the same distribution. Moreover, these figures confirm the theoretical results for a single game (black/red or low/high), since the histograms look very similar to the bar chart of the actual pmf of the final capital of a single game. See figure (3) for this bar chart.

The histogram in figure 7 gives us an idea of the distribution of the total final capital. Let us look at a player who plays on red/black and low/high in roulette, under the policy of treating zero outcomes as regular losses, and using the d'Alembert strategy with the parameters from §3.1. By using the MATLAB-function `counting.m` we find that the player loses with an approximate probability of around 0.560. By using conditionalaverage.m we find that a losing player loses on average 25.6% and a not-losing player wins on average 23.2%.

3.4.2 Simulations under the rules of Holland Casino

We adapt the files `dalemebert012.m`, `dalembertstatistical012.m` in order to do simulations for the case in which zero outcomes are treated in accordance to the Holland Casino policy. See the appendix for these adapted MATLAB-functions `dalembert012HC.m`, `dalembertstatistical012HC.m`.

Performing the simulations (with $n = 10^6$ games) give rise to the following summary statistics:

These numbers are completely in line with our theory about a single game (red/black or low/high) under the Holland Casino policy. See §3.3.4. On average we lose (in total) approximately 2.10% to the house.

	Sample average of final capital	Sample standard deviation of final capital
Red/black game	49.0	21.5
Low/high game	48.9	21.5
The two games combined	97.9	30.5

Table 6: Summary statistics, based on 10^6 simulations, for the Holland Casino policy on zero outcomes. All quantities are expressed in terms of sub units (1 unit = 2 sub units = 50 euros).

We will plot a number of histograms, based on the simulations, in order to get an idea of the distributions of the final capitals under the Holland Casino policy, and in order to check the theoretical findings about this in §3.3.

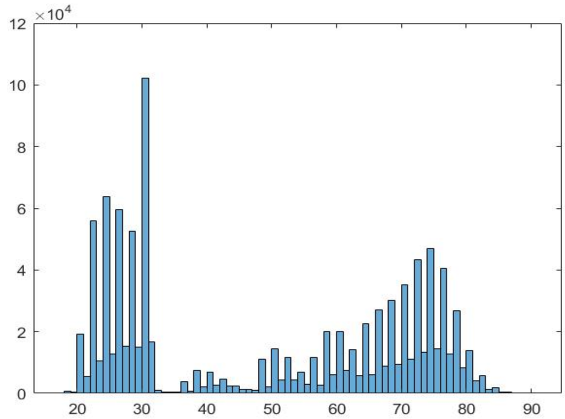

Figure 8: Final capital in red/black game under the Holland Casino policy. In sub units of 25 euros each.

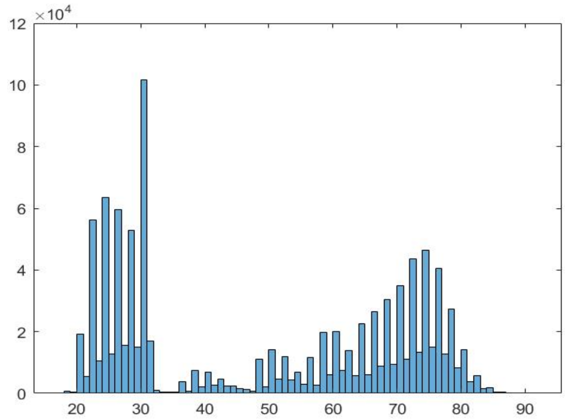

Figure 9: Final capital in low/high game under the Holland Casino policy. In sub units of 25 euros each.

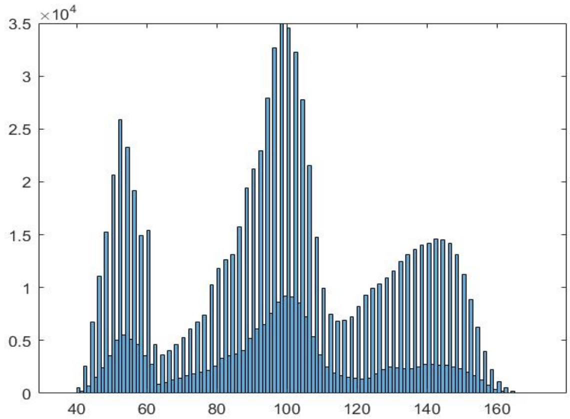

Figure 10: Final capital for the two simultaneous games combined, under the Holland Casino policy. In sub units of 25 euros each.

Figure (8) and (9) show us that under the Holland Casino policy the final capitals in the two games have (at least almost) the same distribution. Moreover, these figures confirm the theoretical results for a single game (red/black or low/high), since the histograms look very similar to the bar chart for the actual pmf of the final capital of a single game under the Holland Casino policy. See figure 4 for this bar chart.

The histogram in figure 10 gives us an idea of the distribution of the total final capital. Even levels of total final capital are more prevalent as a consequence of even levels of "marginal" final capitals being more prevalent (it takes at least one odd “marginal” to have an odd total final capital). The reason for more prevalent even “marginal” final capital levels is explained in §3.3.4. All of this is of course in terms of sub units.

Let us look at a player who plays on red/black and low/high in roulette, under the Holland Casino rules, and using the d’Alembert strategy with the parameters from §3.1.By using the MATLAB-function `counting.m` we find that the player loses with an approximate probability of around 0.527. By using conditionalaverage.m we find that a losing player loses on average 25.1% and a not-losing player wins on average 23.6%.

These numbers are better under the Holland Casino policy than under the policy of treating zero outcomes as regular losses since Holland Casino penalizes zero outcomes less severely.

3.5 Forensic statistical analysis

The client told his lawyer and the court that he had a net profit of about 10,000 euro a year for five years in a row by simultaneously playing red/black and low/high in roulette at Holland Casino. He used the d'Almbert strategy with the parameters from §3.1. The client supported his claims by showing a booklet of notes from the casino.

This booklet shows us that in each year he attended the casino 25 evenings and he had a net profit of 10,000 euros in total. Moreover, the booklet shows us that there is no evening in which he lost more than 10% of his start capital.

However, the booklet is undated and it is not sure whether this booklet tells us the complete truth. We will try to statistically investigate how convincing his story is by comparing his hypothesis with a relevant competing hypothesis:

$$
\begin{aligned}
H_1 :\ & \text{The client tells the truth.} \\
& \text{The booklet tells us the complete truth.} \\
\\
H_2 :\ & \text{The client does not tell the truth.} \\
& \text{The booklet is not complete.} \\
& \text{He attended the casino 40 times each year,} \\
& \text{and he also lost a lot of money.}
\end{aligned} \tag{44}
$$

Before we do that we will first have a look at the likelihood ratio, a very important forensic concept, which we will use in our analysis.

3.5.1 The likelihood ratio

This part is based on [8], [9], and [10].

In forensic science the likelihood ratio is often used in evaluating evidence. According to [8] the likelihood ratio can even be used when there is no possibility to express likelihoods and/or likelihood ratios numerically.

The likelihood ratio shows up when writing Bayes' theorem in the so called odds-form. See (45). There are many formulations of Bayes' rule. Let us first mention a simple version of Bayes' rule.

Theorem 3.4 (Bayes' rule)**.** *Let A and B be events such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then:*

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Let us now formulate a definition of the likelihood ratio.

Definition 3.5. *Let H_1 and H_2 be two competing (mutually exclusive) hypotheses. Let E be the observed data. Then the* likelihood ratio *LR of H_1 versus H_2 is defined as*

$$LR := \frac{\mathbb{P}(E \mid H_1)}{\mathbb{P}(E \mid H_2)}.$$

The likelihood ratio is a measure for the evidence provided by the data E for the first hypothesis H_1 relative to the second hypothesis H_2. Notice that

$$\frac{\mathbb{P}(H_1 \mid E)}{\mathbb{P}(H_2 \mid E)} \overset{\text{Thrm.3.4}}{=} \frac{\mathbb{P}(E \mid H_1)\mathbb{P}(H_1)/\mathbb{P}(E)}{\mathbb{P}(E \mid H_2)\mathbb{P}(H_2)/\mathbb{P}(E)} = LR \cdot \frac{\mathbb{P}(H_1)}{\mathbb{P}(H_2)}. \tag{45}$$

This is the *odds-form* of Bayes' theorem. The ratio $\mathbb{P}(H_1)/\mathbb{P}(H_2)$ tells us something about our judgement of the probability of H_1 relative to that of H_2 before we have seen the data E. This ratio is called the *prior odds* of H_1 and H_2. The fraction $P(H_1 \mid E)/\mathbb{P}(H_2 \mid E)$ tells us the same thing after we have seen the data. This fraction is called the *posterior odds*. The likelihood ratio tells us how the evidence changes our judgement of the probabilities of H_1 versus H_2. Thus the likelihood ratio can be seen as a measure of evidential strength.

3.5.2 Analysis of the client's story

Let us use the likelihood ratio in order to compare the two hypotheses from (44). The data E are given by the possibly incomplete booklet. By using `counting.m` we find that there is an approximate probability of 0.650 that you do not lose more than 10% of your start capital in one evening. The probability that you can play 25 evenings without losing more than 10% in any evening is thus extremely low, and the probability that you can do that five years in a row is even much smaller. Therefore,

$$\mathbb{P}(E \mid H_1) \leq (0.65^{25})^5 \approx 4.11 \cdot 10^{-24}. \tag{46}$$

Notice that when you play 40 evenings in a row, the number of evenings in which you do not lose more than 10% follows an $\text{Bin}(40, 0.650)$-distribution. With probability 0.695 this number of evenings is at least 25 so that you can just choose to record the right 25 evenings. By using the MATLAB function `conditionalaverage.m` we find that if you do not lose more than 10% in an evening, then the average net profit is about 15.7%, so for the recorded evenings the expected profit can be approximated by

$$25 \cdot 15.7 = 392.5 \text{ sub units} \approx 9,800 \text{ euros.}$$

If you just record 25 evenings in which you do not lose more than 10% of your start capital, then with a approximate probability of 0.869 your total net profit (of all these evenings combined) is at least 7,000 euros (which is close enough to 10,000 euros). We used `dalembertstatistical012HC.m`, `sumfinalcapital.m` and `counting.m` for approximating these probabilities. See the appendix. Now we have

$$\mathbb{P}(E \mid H_2) \approx (0.695 \cdot 0.869)^5 \approx 0.0804. \tag{47}$$

Combining (46) and (47) yields

$$LR = \frac{\mathbb{P}(E \mid H_1)}{\mathbb{P}(E \mid H_2)} \leq 5.12 \cdot 10^{-23}.$$

This shows us that the data (the booklet) provides strong evidence in favor of the second hypothesis (client played 40 evenings) in comparison to the first hypothesis (client only played the evenings that are recorded in the booklet). Without any prior knowledge on the client, we may assume that the prior odds are $\mathbb{P}(H_1)/\mathbb{P}(H_2) \approx 20$. Then the posterior odds become

$$\frac{\mathbb{P}(H_1 \mid E)}{\mathbb{P}(H_2 \mid E)} = LR \cdot \frac{\mathbb{P}(H_1)}{\mathbb{P}(H_2)} \leq 1.02 \cdot 10^{-21}.$$

Thus, given the evidence, the first hypothesis (booklet is everything) is very unlikely in comparison to the second hypothesis (client played 40 evenings).

4 Conclusion

A crime suspect is accused of money laundering in casinos, but he told his criminal defence lawyer that he just won 10,000 euros each year for five years in a row by playing simultaneously on red/black and low/high in roulette at Holland Casino, and that this was possible due to the use of the d'Alembert. The lawyer's client used this strategy with parameters mentioned in §3.1: a start capital of 25 units, at most 21 rounds, and a stopping rule for when the capital drops too low. The suspect supported his claims by showing a booklet of notes from the casino. The question is of course how convincing his story actually is.

Holland Casino is a state-owned gambling company. It is the only company in the Netherlands with a casino license. When you bet on red/black, low/high or odd/even in French roulette at Holland Casino, and when zero comes up, you have to choose between sharing half of your stake with the house or putting your stake 'en prison.' We assume that the first choice is always made. The d'Alembert system is a gambling strategy in which the player starts with a unit bet, increases the stake by one unit after every loss, and decreases the stake by one unit after every win, unless the stake is already one unit. By following this strategy, the net gain equals the number of wins when the numbers of wins and losses equalize. However, the problem with this strategy is that there is a positive probability that these numbers will never equalize.

The red/black-outcomes $X = (X_t)_{t=1}^n$ and low/high-outcomes $Y = (Y_t)_{t=1}^n$ are dependent, which makes it more difficult to do computations with these processes. That is why we construct an independent coupling $(X, \hat{Y})$ to approximate (X, Y). It turns out that $(X, \hat{Y})$ approximates (X, Y) pretty well. The number $\mathcal{D}_n$ of differences between Y and $\hat{Y}$ follows an $\mathrm{Bin}(n, \alpha)$-distribution with $\alpha \approx 0.0526$. The difference in capital gains can be bounded from above by $2(1 + n)\mathcal{D}_n$, but this upper bound is not sharp enough for practical purposes. It would be nice if further research could improve this upper bound. Simulations show that for $n = 21$ rounds the average difference in capital gains is about 1.87 units. However, the sample standard deviation of about 4.66 units is pretty large and there is a small (approximated) probability that the difference in capital gains can get completely out of hand.

It is possible to model the capital/stake-levels of the two games X, Y as a Markov chain with transition matrix P and initial distribution $\lambda^{(0)}$. This model enables us to compute exactly the distribution of the final capitals and to use that distribution to compute all kinds of probabilistic properties of the final capitals. This results in some beautiful concrete results for a single game. However, we did not succeed in doing the computations for the two games X, Y simultaneously, because the arrays turn out to be too large to work with. Luckily, by performing simulations we can still get some insights into the distribution of the total final capital for the two simultaneous games. See §3.4 for our simulation results.

The transition probabilities for the simultaneous games can be approximated accurately by the products of the transition probabilities for the single games. However, small differences in transition probabilities can possibly accumulate into large differences in capital and distribution when we let the Markov process run 21 rounds. Moreover, we still need to store and book-keep these products. The memory issues with large arrays may motivate future research into refining the coupling idea and into the computational use of sparse arrays for the topics of this thesis.

We did a forensic statistical analysis to investigate how convincing the story of the suspect is. We used the likelihood ratio to compare the hypothesis of the suspect (he tells the truth and the booklet is complete) with an alternative hypothesis in which the booklet is not complete. The booklet forms the evidence. We found a very low likelihood ratio ($< 10^{-22}$) and very low posterior odds ($< 10^{-20}$). Therefore we come to the conclusion that the story of the suspect is not convincing.

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A Preliminary material

A.1 Discrete-time Markov chains

This section follows chapter 12 of [2]. For more information on Markov chains, see also [11].

A.1.1 Distributions and transition probabilities

Let $X = (X_k)_{k=0}^{\infty}$ be random sequence with countable state space S. We call X a *Markov chain* if it satisfies the *Markov property*, that is,

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)$$

for every non-negative integer n and all states $x_0, \ldots, x_{n+1}$. We say that X is *homogeneous* if in addition

$$\mathbb{P}(X_{n+1} = x_1 | X_n = x_0) = \mathbb{P}(X_1 = x_1 | X_0 = x_0)$$

for every non-negative integer n and all states x_0 and x_1.

In the sequel, we will assume that X is a homogeneous Markov chain.

The *transition matrix* P of X is given by the *1-step transition probabilities* $p_{i,j} := \mathbb{P}(X_1 = j | X_0 = i)$. This matrix is a *stochastic matrix*, i.e.

$$p_{i,j} \geq 0, \text{ and } \sum_{j \in S} p_{i,j} = 1.$$

The *n-step transition probabilities* are given by $p_{i,j}(n) := \mathbb{P}(X_n = j | X_0 = i)$, and they form the *n-step transition matrix* $P(n)$.

Theorem A.1 (Chapman-Kolmogorov Equations)**.** *For all $i, j \in S$ and $m, n \in \mathbb{N}_{\geq 0}$,*

$$p_{i,j}(m+n) = \sum_{k \in S} p_{i,k}(m) \cdot p_{k,j}(n),$$

and $P(m+n) = P(m)P(n)$.

The Chapman-Kolmogorov equations provide us with an easy way to compute n-step transition matrices, namely $P(n) = P^n$.

The *distribution at time n* is a probability row vector $\lambda^{(n)}$ given by $\lambda_i^{(n)} = \mathbb{P}(X_n = i)$. We call $\lambda^{(0)}$ the *initial distribution*. We have $\lambda^{(n)} = \lambda^{(0)} P^n$.

A.1.2 Communicating classes and their class properties

For $i, j \in S$, we say that i *leads to* j, written $i \longrightarrow j$, if $p_{i,j}(n) > 0$ for some $n \geq 0$. If i leads to j, we may also say that j is *accessible from* i. We say that i and j *communicate*, written $i \longleftrightarrow j$, if $i \longrightarrow j$ and $j \longrightarrow i$.

Proposition A.2. *The relation* $\longleftrightarrow$ *is an* equivalence relation.

The equivalence relation $\longleftrightarrow$ defines *equivalence classes* of the following form $C_i = \{j \in S : i \longleftrightarrow j\}$, $i \in S$. These equivalence classes are called *communicating classes*. The chain X or the state space S is called *irreducible* if there is a single communicating class, i.e. $i \longleftrightarrow j$ for all $i, j \in S$. A subset $C \subseteq S$ is called *closed* if for all $i \in C$, $i \longrightarrow j \Rightarrow j \in C$. If a singleton set $\{i\} \subseteq S$ is closed, we call i an *absorbing state*.

Proposition A.3. *A subset $C \subseteq S$ is closed if and only if*

$$p_{i,j} = 0 \;\; \text{for} \;\; i \in C, \; j \notin C.$$

Define the *first-passage time to state* j by $T_j := \min\{n \geq 1 : X_n = j\}$. Define the *first-passage probabilities* by $f_{i,j}(n) := \mathbb{P}_i(T_j = n) := \mathbb{P}(T_j = n | X_0 = i)$. Define $f_i := \mathbb{P}(\exists n \geq 1 : X_n = i | X_0 = i) = \mathbb{P}_i(T_i < \infty) = \sum_{n=1}^{\infty} f_{i,i}(n)$. A state i is called *recurrent* if $f_i = 1$ and it is called *transient* if it is not recurrent. The *mean recurrence time* is $\mu_i = \mathbb{E}_i(T_i)$. If i is recurrent, we call it *null* if $\mu_i = \infty$ and *positive* or *non-null* if $\mu_i < \infty$.

Theorem A.4. *The state i is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_{i,i}(n) = \infty.$$

Theorem A.5 (Pólya's Theorem)**.** *The* symmetric random walk *on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$ and transient if $d \geq 3$.*

Theorem A.6. *Suppose $X_0 = i$ and let $V_i := \#\{n \in \mathbb{N}_{\geq 1} : X_n = i\}$. Then,*

$$\mathbb{P}_i(V_i = r) = (1 - f_i) f_i^r \quad \text{for} \quad r \in \mathbb{N}_{\geq 0},$$

with f_i being the return probability $f_i = \mathbb{P}_i(T_i < \infty)$.

The *period* d_i of the state i is given by $d_i = \gcd\{n \in \mathbb{N}_{\geq 1} : p_{i,i}(n) > 0\}$. The state i is called *aperiodic* if $d_i = 1$, and *periodic* if $d_i > 1$. State i is called *ergodic* if it is aperiodic and positive recurrent.

If a communicating class contains a recurrent state, then the class is closed. Periodicity, (positive/null) recurrence, transience, and ergodicity are class properties. See Theorems 12.37, 12.39, 12.75 of [2] for more information on recurrence and transience.

A.1.3 Stationary distributions and limiting behavior

The probability row vector $\pi = (\pi_i)_{i \in S}$ is called an *invariant distribution* of X if $\pi = \pi P$.

Theorem A.7. *Consider an irreducible Markov chain X.*

There exists an invariant distribution π if and only if X is positive recurrent. If there exists an invariant distribution π, then

$$\pi_i = \frac{1}{\mu_i}, \quad i \in S,$$

where $\mu_i = \mathbb{E}_i(T_i)$ (mean recurrence time).

Theorem A.8. *Consider an irreducible, ergodic (aperiodic, positive recurrent) Markov chain X, and let $\pi = (\pi_i)_{i \in S}$ be the unique invariant distribution of X. Then, for all $i, j \in S$,*

$$\lim_{n \to \infty} p_{i,j}(n) = \pi_j.$$

Theorem A.9. *Let X be an irreducible, recurrent Markov chain. Then, the following are equivalent.*

$$\begin{aligned} &(1) \quad \exists i \in S : \lim_{n \to \infty} p_{i,i}(n) = 0 \\ &(2) \quad \textit{Every state is null recurrent.} \end{aligned}$$

Theorem A.10. *Let X be an irreducible, positive recurrent Markov chain and let $i \in S$. Then, irrespective of the initial distribution of the chain,*

$$\frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{X_k = i\}} \quad \textit{coverges in distribution to} \quad \frac{1}{\mu_i} \ (= \pi_i),$$

where $\mu_i = \mathbb{E}_i(T_i)$.

Theorem A.11. *Let X be an irreducible Markov chain. Suppose $\pi = (\pi_i)_{i \in S}$ is a probability row vector satisfying the* detailed balance equations*:*

$$\forall i, j \in S : \quad \pi_i p_{i,j} = \pi_j p_{j,i}.$$

Then, π is the unique invariant distribution of X. Furthermore, X is reversible in equilibrium*.*

See §12.11 of [2] for a more in-depth explanation of *time reversal*.

B Testing significance of sample correlation

Let $(X_i, Y_i), i \in [n]$ be a random sample drawn from an uncorrelated bivariate normal distribution. Let r_{XY} be the sample correlation based on this random sample. According to Wikipedia, which cites [12], the test statistic $t = r_{XY}\sqrt{\frac{n-2}{1-r_{XY}^2}}$ follows a t-distribution with $n-2$ degrees of freedom. Wikipedia also mentions that $t \approx t(n-2)$ in case of non-normal observations when n is sufficiently large, and references to [13]. The fact that $t \approx t(n-2)$ suggests that t converges in distribution to a standard normal random variable.

However, the proofs in [12] and [13] seem to be daunting and tedious. Therefore, we will do an effort in constructing an understandable proof of $t \xrightarrow{d} \mathcal{N}(0,1)$ based on standard statistical techniques: the (Strong) Law of Large Numbers ((S)LLN), the Central Limit Theorem (CLT), the continuous mapping theorem, and Slutsky's theorem. These techniques are extensively described in [14], so we mention them only shortly. We will construct such a proof of $t \xrightarrow{d} \mathcal{N}(0,1)$ in order to gain a better understanding of the statistical asymptotic properties of the test statistic t.

Theorem B.1 (*SLLN*)**.** *Let $\overline{X_n}$ be the sample mean based on the first n observations of a random sample drawn from a (multivariate) distribution with finite mean μ. Then $\overline{X_n} \xrightarrow{a.s.} \mu$ as $n \to \infty$. [14]*

Theorem B.2 (*CLT*)**.** *Let X_n ($n \in \mathbb{N}_1$) be i.i.d. $\mathbb{R}^k$-valued random vectors with finite mean μ and finite covariance matrix Σ. Then $\sqrt{n}(\overline{X_n} - \mu) \xrightarrow{d} \mathcal{N}_k(0, \Sigma)$ as $n \to \infty$. [14]*

Theorem B.3 (*Continuous Mapping Theorem*)**.** *Let X, X_n ($n \in \mathbb{N}_1$) be random vectors and let g be a continuous map. If X_n converges to X, then $g(X_n)$ converges to $g(X)$ for the following modes of stochastic convergence: a.s.-convergence, convergence in probability and weak convergence. [14]*

Theorem B.4 (*Slutsky*)**.** *Let X, X_n, Y_n ($n \in \mathbb{N}_1$) be random variables and let c be a number s.t. $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$. Then $X_n + Y_n \xrightarrow{d} X + c$, $X_n Y_n \xrightarrow{d} cX$, and $X_n Y_n^{-1} \xrightarrow{d} c^{-1}X$ (provided $c \neq 0$). [14]*

From now on, let $(X_i, Y_i), i \in \mathbb{N}_{\geq 1}$, be a random sample drawn from a joint distribution $\mathbb{P} = \mathbb{P}_X \otimes \mathbb{P}_Y$ with independent components, finite mean (μ_X, μ_Y), and finite variances σ_X^2, σ_Y^2. Let r_{XY} be the sample correlation based on the sample $(X_i, Y_i), i \in [n]$, with sample size n.

Lemma B.5.

$$\frac{1}{\sqrt{1-r_{XY}^2}} \xrightarrow{d} 1.$$

Proof. Notice that $r_{XY} = \frac{s_{XY}}{s_{XX}^{1/2} S_{YY}^{1/2}}$ with $s_{XY} = n^{-1}\sum(X_i - \overline{X})(Y_i - \overline{Y})$ (similarly for s_{XX}, s_{YY}). By taking a careful look, we see that

$$\begin{aligned} s_{XY} &= n^{-1}\sum X_iY_i - \overline{Y}n^{-1}\sum X_i - \overline{X}n^{-1}\sum Y_i + \overline{X}\cdot\overline{Y} \\ &= \overline{X\cdot Y} - \overline{X}\cdot\overline{Y}. \end{aligned}$$

Since we have finite first two moments, we can apply the (S)LLN. Applying (S)LLN and Slutsky yields:

$$s_{XY} \xrightarrow{d} \mathbb{E}(X_1Y_1) - \mathbb{E}X_1\mathbb{E}Y_1 = \mathrm{cov}(X_1, Y_1) \quad \text{(similarly for } s_{XX}, s_{YY}).$$

Since X_1, Y_1 are independent and since the first two moments are finite, we have $s_{XY} \xrightarrow{d} 0$ and $s_{XX} \xrightarrow{d} \sigma_{XX} < \infty$ (similarly for Y). By continuous mapping and Slutsky, we have $r_{XY} \xrightarrow{d} 0$. Again by continuous mapping and Slutsky we have $\frac{1}{\sqrt{1-r_{XY}^2}} \xrightarrow{d} 1$. □

Proposition B.6.

$$r_{XY}\sqrt{n} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. By the proof of the previous lemma and the continuous mapping theorem, we have

$$s_{XX}^{1/2}s_{YY}^{1/2} \xrightarrow{d} \sigma_X\sigma_Y,$$

where σ_X, σ_Y are the standard deviations of X_1, Y_1 respectively. By Slutsky the only thing left to prove is thus

$$s_{XY}\sqrt{n} \xrightarrow{d} \mathcal{N}(0, \sigma_X^2\sigma_Y^2). \tag{48}$$

By the CLT we have asymptotic normality of $s_{XY}\sqrt{n}$. Since X_i are independent of Y_i, we have $\mathbb{E}(s_{XY}\sqrt{n}) = \sqrt{n}\mathbb{E}(s_{XY}) = 0$. By using the fact that the random vectors $(X_i, Y_i), i \in \mathbb{N}_{\geq 1}$, form a random sample with independent components, the variance of $s_{XY}\sqrt{n}$ can be calculated as

$$\begin{aligned}
\mathrm{var}[s_{XY}\sqrt{n}] =& n\mathrm{var}[s_{XY}] \\
=& n\mathrm{var}\left[\overline{XY} - \overline{X}\cdot\overline{Y}\right] \\
=& n\mathbb{E}\left[\left(\overline{XY} - \overline{X}\cdot\overline{Y} - \mathbb{E}\left[\overline{XY} - \overline{X}\cdot\overline{Y}\right]\right)^2\right] \\
=& n\mathbb{E}\left[\left(\overline{XY} - \overline{X}\cdot\overline{Y}\right)^2\right] \\
=& n\mathbb{E}\left[\overline{XY}^2 - 2\overline{X}\cdot\overline{Y}\cdot\overline{XY} + \overline{X}^2\overline{Y}^2\right] \\
=& n\mathbb{E}\left[n^{-2}\sum_i\left(X_i^2Y_i^2 + \sum_{j\neq i}X_iX_jY_iY_j\right)\right. \\
& - 2n^{-3}\sum_k\left(X_k^2Y_k^2 + \sum_{j\neq k}X_k^2Y_jY_k + \sum_{i\neq k}\left(X_iX_kY_k^2 + \sum_{j\neq k}X_iX_kY_jY_k\right)\right) \\
& \left. + n^{-4}\left(\sum_i\left(X_i^2 + \sum_{j\neq i}X_iX_j\right)\sum_i\left(Y_i^2 + \sum_{j\neq i}Y_iY_j\right)\right)\right] \\
=& n\left(\left(\frac{1}{n} - \frac{1}{n^2}\right)\mathbb{E}\left[X_1^2\right]\mathbb{E}\left[Y_1^2\right] - \frac{n-1}{n^2}\mathbb{E}\left[X_1^2\right](\mathbb{E}Y_1)^2\right. \\
& \left. - \frac{n-1}{n^2}\mathbb{E}\left[Y_1^2\right](\mathbb{E}X_1)^2 + \frac{n-1}{n^2}(\mathbb{E}X_1)^2(\mathbb{E}Y_1)^2\right) \\
\xrightarrow{n\uparrow\infty}& \mathbb{E}\left[X_1^2\right]\mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[X_1^2\right](\mathbb{E}Y_1)^2 - \mathbb{E}\left[Y_1^2\right](\mathbb{E}X_1)^2 + (\mathbb{E}X_1)^2(\mathbb{E}Y_1)^2 \\
=& \left(\mathbb{E}\left[X_1^2\right] - (\mathbb{E}X_1)^2\right)\left(\mathbb{E}\left[Y_1^2\right] - (\mathbb{E}Y_1)^2\right) \\
=& \sigma_X^2\sigma_Y^2.
\end{aligned}$$

Hence, we conclude that the convergence (48) indeed holds. This completes the proof. □

Proposition B.7.

$$t := r_{XY}\sqrt{\frac{n-2}{1-r_{XY}^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Notice that $t = r_{XY}\sqrt{n}\cdot\frac{\sqrt{n-2}}{\sqrt{n}}\cdot\frac{1}{\sqrt{1-r_{XY}}}$. By Proposition B.6 we have $r_{XY}\sqrt{n} \xrightarrow{d} \mathcal{N}(0,1)$, obviously $\frac{\sqrt{n-2}}{\sqrt{n}} \longrightarrow 1$, and by Lemma B.5 we have $\frac{1}{\sqrt{1-r_{XY}^2}} \xrightarrow{d} 1$. Applying Slutsky yields the desired result. □

C m-files

C.1 simultaneous.m

```
function [X,Y] = simultaneous(N)

X=zeros(N,1);
%declaration and initialization of vector X

Y=zeros(N,1);
%declaration and initialization of vector Y
%X represents the red/black-game,
%Y represents the low/high-game

D=binornd(1,36/37,N,1);
%vector of i.i.d. bernoulli(36/37) variables
%with probability 1/37 ZERO will come up

for i=1:N
    if D(i)==0
        X(i)=D(i);
        Y(i)=D(i);
        %If ZERO comes up at the roulette wheel,
        %both games have ZERO as outcome
    else
        U=binornd(1,1/2);
        V=binornd(1,1/2);
        X(i)=D(i)+U;
        Y(i)=D(i)+V;
        %Conditional on "not ZERO":
        %w.p. 1/2 you win the red/back-game,
        %w.p. 1/2 you win the low/high game,
        %both games are (cond) indepdendent.
    end
end
```

C.2 dalembert012.m

```
function [CX, SX, CY, SY]=dalembert012(tstop, Ctreshold, Cstart)
%Two games are played:
%red/black (X) and low/high (Y).
%Cstart is start capital.
%Each game has start capital Cstart.
%Player always stops at/before time tstop.
%Player always stops when cap. falls below Ctreshold.

[X,Y] = simultaneous(tstop);
%the two games are simulated

CX = zeros(1+tstop,1);
%declaration and initialization of vector CX
%vector of capital in game X

SX = zeros(1+tstop,1);
%declaration and initialization of vector SX
%vector of stakes in game X

CY = zeros(1+tstop,1);
%declaration and initialization of vector CY
%vector of capital in game Y

SY = zeros(1+tstop,1);
%declaration and initialization of vector SY
%vector of stakes in game Y

CX(1) = Cstart;
%at beginning, capital game X = start capital

CY(1) = Cstart;
%at beginning, capital game Y = start capital

SX(1) = 1;
%first stake equals one unit in game X

SY(1) = 1;
```

```
%first stake equals one unit in game Y

for i=1:tstop
    %Game X
    if X(i) == 2 %Winning outcome in round i
                CX(i+1) = CX(i) + SX(i);
        %new capital = capital + stake
                if SX(i)>1
                        SX(i+1) = SX(i) - 1;
            %if stake>1, next stake = stake - 1
                else
                        SX(i+1) = SX(i);
            %if stake<=1, next stake = stake
                end
    else %Losing outcome in round i
                CX(i+1) = CX(i) - SX(i);
        %new capital = capital - stake
                if CX(i+1) <Ctreshold
                        SX(i+1) = 0;
            %if new capital below level Ctreshold
            %the player stops (so next stake = 0)
        else
                        SX(i+1) = SX(i) + 1;
            %if new capital => Ctreshold
            %new stake = stake + 1
        end
    end

    %Game Y
    %Game Y is treated in the same way as X
        if Y(i) == 2
                CY(i+1) = CY(i) + SY(i);
                if SY(i)>1
                        SY(i+1) = SY(i) - 1;
                else
                        SY(i+1) = SY(i);
                end
        else
                CY(i+1) = CY(i) - SY(i);
                if CY(i+1) <Ctreshold
                        SY(i+1) = 0;
```

```
                else
                        SY(i+1) = SY(i) + 1;
                end
        end
end
```

C.3 dalembert012HC.m

```
function [CX, SX, CY, SY]=dalembert012HC(tstop, Ctreshold, Cstart)
%Two games are played:
%red/black (X) and low/high (Y).
%Cstart is start capital.
%Each game has start capital Cstart.
%Player always stops at/before time tstop.
%Player always stops when cap. falls below Ctreshold.

[X,Y] = simultaneous(tstop);
%the two games are simulated

CX = zeros(1+tstop,1);
%declaration and initialization of vector CX
%vector of capital in game X

SX = zeros(1+tstop,1);
%declaration and initialization of vector SX
%vector of stakes in game X

CY = zeros(1+tstop,1);
%declaration and initialization of vector CY
%vector of capital in game Y

SY = zeros(1+tstop,1);
%declaration and initialization of vector SY
%vector of stakes in game Y

CX(1) = 2*Cstart; %THIS HAS BEEN CHANGED, IN SUB UNITS
%at beginning, capital game X = start capital

CY(1) = 2*Cstart; %THIS HAS BEEN CHANGED, IN SUB UNITS
%at beginning, capital game Y = start capital

SX(1) = 2; %THIS HAS BEEN CHANGED, IN SUB UNITS
%first stake equals one unit in game X
```

```
SY(1) = 2; %THIS HAS BEEN CHANGED, IN SUB UNITS
%first stake equals one unit in game Y

for i=1:tstop
    %Game X
    if X(i) == 2 %Winning outcome in round i
                CX(i+1) = CX(i) + SX(i);
        %new capital = capital + stake
                if SX(i)>2 %THIS HAS BEEN CHANGED, IN SUB UNITS
                        SX(i+1) = SX(i) - 2; %THIS HAS BEEN CHANGED, IN SUB UNITS
            %if stake>2, next stake = stake - 2 THIS HAS BEEN CHANGED, IN S
        else
                        SX(i+1) = SX(i);
            %if stake<=2, next stake = stake %THIS HAS BEEN CHANGED, IN SUB
        end
    %THIS PART HAS BEEN ADDED/CHANGED
    elseif X(i) == 1 %Loss due to non-zero outcome in round i
        CX(i+1) = CX(i) - SX(i);
        %new capital = capital - stake
        if CX(i+1) <2*Ctreshold
            SX(i+1) = 0;
            %if new capital below level 2*Ctreshold
            %the player stops (so next stake = 0)
        else
            SX(i+1) = SX(i) + 2;
            %if new capital => 2*Ctreshold
            %new stake = stake + 2
        end
    else %Loss due to zero outcome in round i
        CX(i+1) = CX(i) - (SX(i))/2;
        %new capital = capital - stake/2
        if CX(i+1) <2*Ctreshold
            SX(i+1) = 0;
            %if new capital below level 2*Ctreshold
```

```
            %the player stops (so next stake = 0)
        else
            SX(i+1) = SX(i) + 2;
            %if new capital => 2*Ctreshold
            %new stake = stake + 2
        end
    end
    %END OF 'THIS PART HAS BEEN ADDED/CHANGED'

    %Game Y
    %Game Y is treated in the same way as X
    %THE SAME CHANGED HAVE BEEN APPLIED TO THE Y-PART OF THIS CODE
    if Y(i) == 2
                CY(i+1) = CY(i) + SY(i);
        if SY(i)>2
                        SY(i+1) = SY(i) - 2;
        else
                        SY(i+1) = SY(i);
        end
    elseif Y(i) == 1
        CY(i+1) = CY(i) - SY(i);
        if CY(i+1) <2*Ctreshold
                        SY(i+1) = 0;
        else
                        SY(i+1) = SY(i) + 2;
        end
    else
                CY(i+1) = CY(i) - (SY(i))/2;
        if CY(i+1) <2*Ctreshold
                        SY(i+1) = 0;
        else
                        SY(i+1) = SY(i) + 2;
        end
    end
end
```

C.4 dalembertstatistical012.m

```
function [CXfinal, CYfinal, CTfinal] = dalembertstatistical012(tstop, Ctreshold, Cstart, n)

%CXfinal: final capital levels w.r.t X,
%CYfinal: final capital levels w.r.t Y,
%each entry represents one game of (at most) tstop rounds,
%CTfinal = CXfinal + CYfinal.
%When the player plays X and Y simultaneously,
%then CTfinal = total final capital levels

CXfinal=zeros(n,1);
%declaration and initialization of vector CXfinal

CYfinal=zeros(n,1);
%declaration and initialization of vector CYfinal

for i=1:n
    [CX, SX, CY, SY]=dalembert012(tstop, Ctreshold, Cstart);
    CXfinal(i)=CX(1+tstop);
    %CX(1+tstop) is final capital in "i-th X-game"
    CYfinal(i)=CY(1+tstop);
    %CY(1+tstop) is final capital in "i-th Y-game"
end
CTfinal=CXfinal+CYfinal;
%total capital = capital X + capital Y
```

C.5 dalembertstatistical012HC.m

```
function [CXfinal, CYfinal, CTfinal] =
   dalembertstatistical012HC(tstop, Ctreshold,
   Cstart, n)

%CXfinal: final capital levels w.r.t X,
%CYfinal: final capital levels w.r.t Y,
%each entry represents one game of (at most) tstop
   rounds,
%CTfinal = CXfinal + CYfinal.
%When the player plays X and Y simultaneously,
%then CTfinal = total final capital levels

CXfinal=zeros(n,1);
%declaration and initialization of vector CXfinal

CYfinal=zeros(n,1);
%declaration and initialization of vector CYfinal

for i=1:n
    [CX, SX, CY, SY]=dalembert012HC(tstop, Ctreshold
        , Cstart);
    %For the Holland Casino scenario we use
    %dalembert012HC.m instead of dalembert012.m
    CXfinal(i)=CX(1+tstop);
    %CX(1+tstop) is final capital in "i-th X-game"
    CYfinal(i)=CY(1+tstop);
    %CY(1+tstop) is final capital in "i-th Y-game"
end
CTfinal=CXfinal+CYfinal;
%total capital = capital X + capital Y
```

C.6 coupling.m

```
function [hatX, hatY] = coupling(X,Y)

%This function constructs a coupling of the X- and Y-games.

N=length(Y);

M=binornd(1,36/37,N,1);
%Random vector for deciding when there is a zero in the coupled Y-game.

Z=binornd(1,1/2,N,1);
%Random vector for constructing new non-zero outcomes to replace old
%zero outcomes in the original Y-game.

hatX=X;
%The coupled X-game is by construction equal the original X-game.

hatY=zeros(N,1);
%Declaration and initialisation of coupled Y-game.

for t=1:N
    if M(t,1)==0
        %Vector M tells us that there is a zero in the coupled Y-game.
        hatY(t,1)=0;
    else
        %Vector M tells us there is no zero in the coupled Y-game.
        if Y(t,1)==0
            %When there is a zero in the original Y-game,
            %replace that old zero by a non-zero outcome.
            hatY(t,1)=1+Z(t,1);
        else
```

```
            %When there is a non-zero outcome in the
                original Y-game,
            %retain that non-zero outcome in the
               coupled Y-game.
            hatY(t,1)=Y(t,1);
        end
    end
end
```

C.7 countingdifferences.m

```
function D = countingdifferences(X1,X2)

%Function for counting the number of difference
   components between two
%vectors of equal dimension.

D=0;
%Counting starts with 0.

N=length(X1);
%Dimension of the vectors.

for i=1:N
    if X1(i,1)==X2(i,1)
        D=D;
        %We do not count a difference when there is
           no difference.
    else
        D=D+1;
        %We do count a difference when there is a
           difference.
    end
end
```

C.8 dalembert012adapted.m

```
function [X, CX, SX, WX, hatX, hatCX, hatSX, hatWX, Y, CY, SY, WY, hatY, hatCY, hatSY, hatWY]= dalembert012adapted(tstop, Ctreshold, Cstart)
%
% This function can be used for
% simulating random walks related to X,Y, coupled Y.
%
% Random walk: #wins - #losses.
%
% This function can be used for
% simulating capital/stake-levels of X, Y, coupled Y.
%
% This function is an adaptation of dalembert012.m
%
% Two games are played: red/black (X) and low/high (Y).
% Cstart is the start capital. Each game has start capital Cstart.
% Player always stops at/before time tstop.
% Player always stops when capital falls below Ctreshold.
% These rules also apply to the coupling (hatX, hatY).
%

[X,Y] = simultaneous(tstop);
%the two games are simulated

[hatX, hatY] = coupling(X,Y);
%independent coupling of X,Y

hatWX=zeros(1+tstop, 1);
WX=zeros(1+tstop, 1);
%declaration and initialization of random walk for (coupled) X-game
%declaration and initialization of random walk for X-game

```

```
hatCX = zeros(1+tstop,1);
CX = zeros(1+tstop,1);
%declaration and initialization of vectors hatCX, CX
%vectors of capital in (coupled) X-game and X-game resp.

hatSX = zeros(1+tstop,1);
SX = zeros(1+tstop,1);
%declaration and initialization of vectors hatSX, SX
%vectors of stakes in (coupled) X-game and X-game resp.

hatWY=zeros(1+tstop,1);
WY=zeros(1+tstop, 1);
%declaration and initialization of random walk for coupled Y-game
%declaration and initialization of random walk for Y-game

hatCY = zeros(1+tstop,1);
CY = zeros(1+tstop,1);
%declaration and initialization of vectors hatCY, CY
%vectors of capital in coupled Y-game and Y-game resp.

hatSY = zeros(1+tstop,1);
SY = zeros(1+tstop,1);
%declaration and initialization of vectors hatSY, SY
%vectors of stakes in coupled Y-game and Y-game resp.

hatCX(1) = Cstart;
CX(1) = Cstart;
%at beginning, capital (coupled) X-game = start capital
%at beginning, capital X-game = start capital

hatCY(1) = Cstart;
CY(1) = Cstart;
%at beginning, capital coupled Y-game = start capital
```

```
%at beginning, capital Y-game = start capital

hatSX(1) = 1;
SX(1) = 1;
%first stake equals one unit in (coupled) X-game
%first stake equals one unit in X-game

hatSY(1) = 1;
SY(1) = 1;
%first stake equals one unit in coupled Y-game
%first stake equals one unit in Y-game

for i=1:tstop
    %Game X
    if X(i) == 2 %Winning outcome in round i
                CX(i+1) = CX(i) + SX(i);
        WX(i+1) = WX(i) + 1;
        %new capital = capital + stake
        %random walk makes a move upwards
                if SX(i)>1
                        SX(i+1) = SX(i) - 1;
            %if stake>1, next stake = stake - 1
                else
                        SX(i+1) = SX(i);
            %if stake<=1, next stake = stake
                end
    else %Losing outcome in round i
                CX(i+1) = CX(i) - SX(i);
        WX(i+1) = WX(i) - 1;
        %new capital = capital - stake
        %random walk makes a move downwards
                if CX(i+1) <Ctreshold
                        SX(i+1) = 0;
            %if new capital below level Ctreshold
            %the player stops (so next stake = 0)
        else
                        SX(i+1) = SX(i) + 1;
            %if new capital => Ctreshold
            %new stake = stake + 1
        end
    end
```

```

    %Game Y
    %Game Y is treated in the same way as X
        if Y(i) == 2
                CY(i+1) = CY(i) + SY(i);
        WY(i+1) = WY(i) + 1;
                if SY(i)>1
                        SY(i+1) = SY(i) - 1;
                else
                        SY(i+1) = SY(i);
                end
        else
                CY(i+1) = CY(i) - SY(i);
        WY(i+1) = WY(i) - 1;
                if CY(i+1) <Ctreshold
                        SY(i+1) = 0;
                else
                        SY(i+1) = SY(i) + 1;
                end
        end
end

% Same iterations for the coupling (hatX, hatY).
% Read "(coupled) X-game" when you read "X-game".
% Read "coupled Y-game" when you read "Y-game".
for i=1:tstop
    %Game X
    if hatX(i) == 2 %Winning outcome in round i
                hatCX(i+1) = hatCX(i) + hatSX(i);
        hatWX(i+1) = hatWX(i) + 1;
        %new capital = capital + stake
        %random walk makes a move upwards
                if hatSX(i)>1
                        hatSX(i+1) = hatSX(i) - 1;
            %if stake>1, next stake = stake - 1
                else
                        hatSX(i+1) = hatSX(i);
            %if stake<=1, next stake = stake
                end
    else %Losing outcome in round i
                hatCX(i+1) = hatCX(i) - hatSX(i);
```

```
            hatWX(i+1) = hatWX(i) - 1;
            %new capital = capital - stake
            %random walk makes a move downwards
                    if hatCX(i+1) <Ctreshold
                            hatSX(i+1) = 0;
                %if new capital below level Ctreshold
                %the player stops (so next stake = 0)
            else
                            hatSX(i+1) = hatSX(i) + 1;
                %if new capital => Ctreshold
                %new stake = stake + 1
            end
        end

        %Game Y
        %Game Y is treated in the same way as X
            if hatY(i) == 2
                    hatCY(i+1) = hatCY(i) + hatSY(i);
            hatWY(i+1) = hatWY(i) + 1;
                    if hatSY(i)>1
                            hatSY(i+1) = hatSY(i) - 1;
                    else
                            hatSY(i+1) = hatSY(i);
                    end
            else
                    hatCY(i+1) = hatCY(i) - hatSY(i);
            hatWY(i+1) = hatWY(i) - 1;
                    if hatCY(i+1) <Ctreshold
                            hatSY(i+1) = 0;
                    else
                            hatSY(i+1) = hatSY(i) + 1;
                    end
            end
end
```

C.9 dalembertstatistical012adapted.m

```
function [WX21absolute, WY21absolute, CXfinal,
   CYfinal, CTfinal, hatWX21absolute,
   hatWY21absolute, hatCXfinal, hatCYfinal,
   hatCTfinal,D] = dalembertstatistical012adapted(
   tstop, Ctreshold, Cstart, n)
%
% This function is meant for doing multiple
   simulations of the X-,Y-, and
% coupled-Y-games, and to collect a data set (based
   on simulations) of
% associated features of these games:
% - final capital levels of (X,Y) and their sum (the
    total final capital);
% - final capital levels of the coupling, and their
   sum;
% - number of difference between original (X,Y) and
   coupling;
% - 'final distance' from origin of corresponding
   random walks,
% - position random walk = net number of wins.
% CXfinal: vector of final capital levels w.r.t X;
% CYfinal: vector of final capital levels w.r.t Y;
% CTfinal = CXfinal + CYfinal = vector of total
   final capital levels;
% hatCXfinal, hatCYfinal, hatCTfinal are defined
   similarily, but based on
% coupling of (X,Y);
% each entry of such a vector represents
% one game of (at most) tstop rounds.
%
hatCXfinal=zeros(n,1);
CXfinal=zeros(n,1);
%declaration and initialization of vectors
   hatCXfinal, CXfinal.

hatCYfinal=zeros(n,1);
CYfinal=zeros(n,1);
%declaration and initialization of vectors
   hatCYfinal, CYfinal.
```

```
hatWX21absolute=zeros(n,1);
WX21absolute=zeros(n,1);
%declaration and initialization of vectors hatWX21absolute, WX21absolute.

hatWY21absolute=zeros(n,1);
WY21absolute=zeros(n,1);
%declaration and initialization of vectors hatWY21absolute, WY21absolute.

D=zeros(n,1);
%declaration and initialization of vector D

for i=1:n
    [X, CX, SX, WX, hatX, hatCX, hatSX, hatWX, Y, CY, SY, WY, hatY, hatCY, hatSY, hatWY]=dalembert012adapted(tstop, Ctreshold, Cstart);
    %[hatX, hatY] is coupling of (X,Y);
    CXfinal(i)=CX(1+tstop);
    hatCXfinal(i)=hatCX(1+tstop);
    WX21absolute(i)=abs(WX(1+tstop));
    hatWX21absolute(i)=abs(hatWX(1+tstop));
    %CX(1+tstop) is final capital in "i-th X-game"
    %abs(WX(1+tstop)) is distance from origin at the end of "i-th X-game"
    CYfinal(i)=CY(1+tstop);
    hatCYfinal(i)=hatCY(1+tstop);
    WY21absolute(i)=abs(WY(1+tstop));
    hatWY21absolute(i)=abs(hatWY(1+tstop));
    D(i)=countingdifferences(Y,hatY);
    %CY(1+tstop) is final capital in "i-th Y-game"
    %abs(WY(1+tstop)) is distance from origin at the end of "i-th Y-game"
end
CTfinal=CXfinal+CYfinal;
hatCTfinal=hatCXfinal+hatCYfinal;
%total capital = capital X + capital Y
```

C.10 Pmatrix.m

```
function P = Pmatrix(m,n)
% m = maximal capital + 1
% n = maximal stake + 1
%indices start by one
%so index 1 stands for capital/stake 0,
%   index 2 stands for capital/stake 1,
%and so on.
P=zeros(m,n,m,n);
for i=1:m
    for j=1:n
        c=i-1;
        %capital = capital index - 1
        s=j-1;
        %stake = stake index -1
        if s==0
            P(i,j,i,j)=1;
            %zero stake means that nothing changes
        else
            %when stake is not zero,
            %so when you are still playing,
            %cpositive = new capital after winning
            %ipositive = new capital index after winning
            %cnegative = new capital after losing
            %inegative = new capital index after losing
            %spositive = new stake after winning
            %jpositive = new stake index after winning
            %snegative = new stake after losing
            %jnegative = new stake index after losing
            cpositive = c+s;
            cpositive = min(cpositive, m-1);
            spositive = s-1;
            spositive = max(spositive, 1);
            ipositive = cpositive + 1;
            jpositive = spositive + 1;
            P(i,j,ipositive, jpositive)=18/37;
```

```
            %probability that you win is 18/37
            cnegative = c-s;
            cnegative = max(cnegative, 0);
            if cnegative < 16
                snegative = 0;
                %stop playing when new capital falls
                %below treshold of 16,
                %that is, choose zero as new stake.
            else
                snegative = s+1;
                snegative = min(snegative, n-1);
                %when you lose and you still have
                %capital of at least 16 (our treshold),
                %increase stake with 1
            end
            inegative = cnegative + 1;
            jnegative = snegative + 1;
            P(i,j,inegative, jnegative)=19/37;
            %probability that you lose is 19/37
        end
    end
end
```

C.11 PmatrixHC.m

```
function P = PmatrixHC(m,n)
% m = maximal capital + 1
% n = maximal stake + 1
%indices start by one
%so index 1 stands for capital/stake 0,
%   index 2 stands for capital/stake 1,
%and so on.
P=zeros(m,n,m,n);
for i=1:m
    for j=1:n
        c=i-1;
        %capital = capital index - 1
        s=j-1;
        %stake = stake index -1
        if s==0
            P(i,j,i,j)=1;
            %zero stake means that nothing changes
        else
            %when stake is not zero,
            %so when you are still playing,
            %cpositive = new capital after winning
            %ipositive = new capital index after winning
            %cnegative = new capital after non-zero loss %HAS BEEN CHANGED
            %inegative = new capital index non-zero loss %HAS BEEN CHANGED
            %    czero = new capital after zero outcome %HAS BEEN ADDED
            %    izero = new capital index after zero outcome %HAS BEEN ADD
            %spositive = new stake after winning
            %jpositive = new stake index after winning
            %snegative = new stake after non-zero loss %HAS BEEN CHANGED
            %jnegative = new stake index after non-zero loss %HAS BEEN CHAN
```

```
%    szero = new stake after zero outcome %HAS BEEN ADDED
%    jzero = new stake index after zero outcome %HAS BEEN ADDED
cpositive = c+s;
cpositive = min(cpositive, m-1);
spositive = s-2; %THIS HAS BEEN CHANGED.
spositive = max(spositive, 2); %THIS HAS BEEN CHANGED.
ipositive = cpositive + 1;
jpositive = spositive + 1;
P(i,j,ipositive, jpositive)=18/37;
%probability that you win is 18/37
cnegative = c-s;
cnegative = max(cnegative, 0);
if cnegative < 32 %THIS HAS BEEN CHANGED
    snegative = 0;
    %stop playing when new capital falls
    %below treshold of 16 (32 sub units), %THIS HAS BEEN ...
    %that is, choose zero as new stake.
else
    snegative = s+2; %THIS HAS BEEN CHANGED
    snegative = min(snegative, n-1);
    %when you lose and you still have
    %capital of at least 16 (32 sub units) (our treshold),%THIS
    %increase stake with 1 (2 sub units) %THIS HAS BEEN CHANGED
end
inegative = cnegative + 1;
jnegative = snegative + 1;
P(i,j,inegative, jnegative)=18/37; %THIS HAS BEEN CHANGED
%probability of non-zero loss is 18/37 %THIS HAS BEEN CHANGED

%THIS HAS BEEN ADDED%
czero = c-ceil(s/2);
czero = max(czero, 0);
```

```
            if czero < 32
                szero = 0;
                %stop playing when new capital falls
                %below treshold of 16 (32 sub units)
                    ,
                %that is, choose zero as new stake.
            else
                szero = s+2;
                szero = min(szero, n-1);
                %when you lose and you still have
                %capital of at least 16 (32 sub
                   units) (our treshold),
                %increase stake with 1 (2 sub units)
            end
            izero = czero + 1;
            jzero = szero + 1;
            P(i,j,izero, jzero)=1/37;
            %probability of zero outcome is 1/37
        end
    end
end
```

C.12 PsimHC.m

```
function P = PsimHC(h)
n = 99*19*99*19;
P = sparse(n, n);
P = reshape(ndSparse(P),[99,19,99,19,99,19,99,19]);
for i=1:99
    for j=1:19
        for k=1:99
            for l=1:19
                cX = i-1;
                sX = j-1;
                cY = k-1;
                sY = l-1;
                sXwin = max(sX-2,2);
                sYwin = max(sY-2,2);
                sXloss= min(sX+2,18);
                sYloss= min(sY+2,18);
                cXwin = max(cX+sX,98);
                cYwin = max(cY+sY,98);
                cXloss= min(cX-sX,0);
                cYloss= min(cY-sY,0);
                if h==1   %zero outcome treated as regular loss
                    P(i,j,k,l,cXwin+1,sXwin+1,cYwin+1,sYwin+1) = 9/37; %++
                    if cX-sX<32
                       P(i,j,k,l,cXloss+1,1,cYwin+1,sYwin+1)= 9/37; %-+
                       if cY-sY<32
                           P(i,j,k,l,cXloss+1,1,cYloss+1,1) = 10/37; %--
                           P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,1) = 9/37; %+-
                       else
                           P(i,j,k,l,cXloss+1,1,cYloss+1,sYloss+1)=10/37; %--
```

```
                    P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,sYloss+1)=9/37; %+-
                end
            else
                if cY-sY<32
                    P(i,j,k,l,cXloss+1,sXloss+1,cYloss+1,1) = 10/37; %--
                    P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,1) = 9/37; %+-
                else
                    P(i,j,k,l,cXloss+1,sXloss+1,cYloss+1,sYloss+1)=10/37; %--
                    P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,sYloss+1)=9/37; %+-
                end
            end
        else %h=2, Holland Casino policy
            cZ=min(cX-ceil(sX/h),0);
            P(i,j,k,l,cXwin+1,sXwin+1,cYwin+1,sYwin+1) = 9/37; %++
            if cX-sX<32
                P(i,j,k,l,cXloss+1,1,cYwin+1,sYwin+1)= 9/37; %-+
                if cY-sY<32
                    P(i,j,k,l,cXloss+1,1,cYloss+1,1) = 9/37; %--
                    P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,1) = 9/37; %+-
                else
                    P(i,j,k,l,cXloss+1,1,cYloss+1,sYloss+1)=9/37; %--
                    P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,sYloss+1)
```

```
                        =9/37; %+-
                    end
                else
                    if cY-sY<32
                        P(i,j,k,l,cXloss+1,sXloss+1,cYloss+1,1) = 9/37; %--
                        P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,1) = 9/37; %+-
                    else
                        P(i,j,k,l,cXloss+1,sXloss+1,cYloss+1,sYloss+1)=9/37; %--
                        P(i,j,k,l,cXwin+1,sXwin+1,cYloss+1,sYloss+1)=9/37; %+-
                    end
                end
                if cX-sX/h<32
                    if cY-sY/h<32
                         P(i,j,k,l,cZ+1,1,cZ+1,1)=1/37; %zz
                    else
                         P(i,j,k,l,cZ+1,1,cZ+1,sYloss+1)=1/37; %zz
                    end
                else
                    if cY-sY/h<32
                         P(i,j,k,l,cZ+1,sXloss+1,cZ+1,1)=1/37; %zz
                    else
                         P(i,j,k,l,cZ+1,sXloss+1,cZ+1,sYloss+1)=1/37; %zz
                    end
                end
            end
        end
    end
end
```

```
end
```

C.13 initialdistribution.m

```
function v = initialdistribution(m,n,startC,startS)
% m = maximal capital + 1
% n = maximal stake + 1
%indices start by one
%so index 1 stands for capital/stake 0,
%   index 2 stands for capital/stake 1,
%and so on.
%startC = start capital,
%startS = start stake.
v=zeros(m,n);
for i=1:m
    for j=1:n
        c=i-1;
        %capital = capital index - 1
        s=j-1;
        %stake = stake index -1
        if c==startC
            if s==startS
                v(i,j)=1;
                %W.p. 1 we start in (StartC, StartS)
            end
        end
    end
end
```

C.14 Distr.m

```
function D = Distr(k)

%k must be a row vector with 500 components, giving the stationary
%distribution of the possible capital/stake-pairs.
%k must be organised in such a way that the first block of 50 components
%stands for stake 0, the second block for stake 1 etcetera, and that the
%first component of a block stands for capital 0, the second component for
%capital 1 and so on.

D=zeros(1,50);
%Declaration and initialization of row vector with 50 components, each
%component for each possible value of our capital level.

for j=1:50 %This loop runs through all possible values of capital
    for i=1:10
        D(1,j)=D(1,j)+k(1, j+50*(i-1));
    end
end

%We end up with a row vector D giving the stationary distribution of the
%capital levels.
```

C.15 DistrHC.m

```
function D = DistrHC(k)
%THE NUMBERS IN THIS PROGRAM HAVE BEEN MODIFIED
%k must be a row vector with 99*19 components, giving the stationary
%distribution of the possible capital/stake-pairs.
%k must be organised in such a way that the first block of 99 components
%stands for stake 0, the second block for stake 1 etcetera, and that the
%first component of a block stands for capital 0, the second component for
%capital 1 and so on.

D=zeros(1,99);
%Declaration and initialization of row vector with 00 components, each
%component for each possible value of our capital level.

for j=1:99 %This loop runs through all possible values of capital
    for i=1:19
        D(1,j)=D(1,j)+k(1, j+99*(i-1));
    end
end

%We end up with a row vector D giving the stationary distribution of the
%capital levels.
```

C.16 counting.m

```
function [count , countcum] = counting(x,y)

%x must be a column vector.
%y must be a number.
%count is the number of times that y appears in the
   components of x.
%countcum is the number of components that are less
   or equal to y.

l=length(x);
%Length/dimension of vector x.

count=0;
%Counting starts with 0.

countcum=0;
%Counting start with 0.

%Loop for counting the number of times that y
   appears in x.
for i=1:l
    if x(i,1) == y
        count = count + 1;
        %Count when component of x equals y.
    else
        count = count;
        %Do NOT count component of x does NOT equal
           y.
    end
end

%Loop for counting the number of components that are
    less or equal to y
for i=1:l
    if x(i,1) <= y
        countcum = countcum + 1;
        %Count when component of x is less or equal
           to y.
    else
```

```
        countcum = countcum;
        %Do NOT count when component of x is larger
            than y.
    end
end
```

C.17 conditionalaverage.m

```
function [ccL, ccl, cch, ccH] = conditionalaverage(x,y)
% N is the number of observations
% NL is the number of observations below y
% Nl is the number of observations below or equal to y
% Nh is the number of observations above or equal to y
% NH is the number of observations above y
% sumL is the sum of all observations below y
% suml is the sum of all observations below or equal to y
% sumh is the sum of all observations above or equal to y
% sumH is the sum of all observations above y
% ccL is the average of all observations below y
% ccl is the average of all observations below or equal to y
% cch is the average of all observations above or equal to y
% ccH is the average of all observations above y
N=length(x);
NL = 0;
sumL=0;
Nl = 0;
suml=0;
Nh = 0;
sumh=0;
NH = 0;
sumH=0;
for i=1:N
    if x(i,1) < y
        NL=NL+1;
        Nl=Nl+1;
        sumL=sumL+x(i,1);
        suml=suml+x(i,1);
    elseif x(i,1) == y
        Nl=Nl+1;
        Nh=Nh+1;
```

```
        suml=suml+x(i,1);
        sumh=sumh+x(i,1);
    else
        NH=NH+1;
        Nh=Nh+1;
        sumH=sumH+x(i,1);
        sumh=sumh+x(i,1);
    end
end

if sumL==0
    ccL=-10^12;
else
    ccL=sumL/NL;
end

if suml==0
    ccl=-10^12;
else
    ccl=suml/Nl;
end

if sumh==0
    cch=-10^12;
else
    cch=sumh/Nh;
end

if sumH==0
    ccH=-10^12;
else
    ccH=sumH/NH;
end
```

C.18 sumfinalcapital.m

```
function CTT = sumfinalcapital(CTfinal, y,
   nrsummands, n)
CTT=zeros(n,1);
j=1;
for i=1:n
    t=0;
    while t<nrsummands
        if CTfinal(j,1) > y
            CTT(i,1)=CTT(i,1)+CTfinal(j,1);
            j=j+1;
            t=t+1;
            %If final capital is large enough,
               record evening
        else
            j=j+1;
            %Else, move on to next evening
        end
    end
end
```