Gerrymandering

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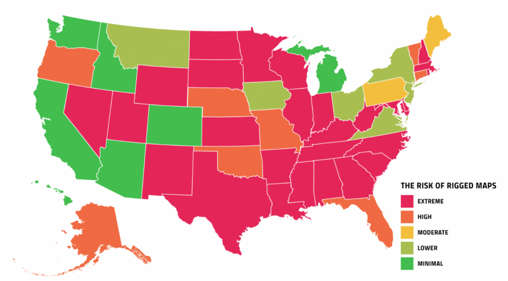

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Contents

1 Introduction

Sometimes, political elections working with a district system are not as democratic as they seem. While it is true that each voter has its own choice to vote for one or more political contenders and each vote counts, some politicians with the right authority have the means to control the election more or less. This can be done by designing the borders of political districts in such a way to benefit a political contender. This type of manipulation is called gerrymandering. In this thesis we will investigate how gerrymandering can be performed, detected and prevented. We will not thoroughly discuss the judicial aspects of this "unfair" procedure, namely who exactly are able to perform this and whether and to what extent they have the right to do so. We will make some comments about this throughout.

Chapter 2 introduces a complete definition of gerrymandering and gives some of its history. Here, we will also discuss how gerrymandering can be performed, i.e. how these new borders can be designed. Chapter 2 additionally introduces some other mathematical topics that serve as relevant background for the rest of this thesis.

Chapters 3 to 7 are dedicated to detection and prevention of gerrymandering. Part I of this thesis includes Chapters 3, 4 and 5 and discusses three ways to help determine whether gerrymandering has in fact occurred to alter the outcome of a certain election. Chapter 3 discusses the efficiency gap, a tool that measures the vote distribution in districts. A high efficiency gap corresponds to an unfair distribution of voters among districts and can hint towards gerrymandering. Chapter 4 talks about the unlikeliness of the outcome of the election, were it not rigged by gerrymandering. This will be done with the help of Monte Carlo simulation, which will be introduced. The literature in Chapter 5 comprises the shape of the districts. One or more districts with a really irregular shape will immediately give people the idea that this shape is purposely designed to alter the outcome. Most suspicions of gerrymandering namely start by looking at the map. Chapter 3 and 4 have a common theme: extending a method or definition that exists for two political contenders to more contenders.

Part II of this thesis includes Chapters 6 and 7 and discusses two ways to prevent gerrymandering in the future. Chapter 6 talks about adapting a new voting system. Here we propose some other voting systems and discuss their advantages and disadvantages. The theory about voting systems will be addressed in Chapter 2. Chapter 7 discusses a solution that gets to the root of the problem: redraw the districts in a fair way.

2 Relevant Background

Since gerrymandering will be the core of this thesis, we will introduce the definition and main aspects of this phenomenon. We will however not give a rigorous and mathematical definition for gerrymandering, unlike what is to be expected in a mathematics thesis, because gerrymandering is not a purely mathematical, but rather a political or natural phenomenon. Alongside this, there are a few other relevant matters to be addressed as well. They serve as mathematical background, which readers that are familiar with multiple basic subjects in mathematics possibly already know. These will aid in understanding some sections that will be encountered later on. For now, this chapter can be perceived as an incoherent chapter discussing all relevant knowledge that is needed to read the rest of the thesis, including a fairly recent research area called *voting theory*.

2.1 Gerrymandering

The setting for the rest of the thesis will be as follows: a country is divided into multiple electoral districts.[1] We consider political elections with at least two contenders, where at the end of the election one contender wins, based on votes submitted by residents (that fulfill voting requirements, for instance in The Netherlands this is to be at least 18 years old) in a certain country or region. We generally refer to those contenders as *political parties*, such as the Republicans and the Democrats in the presidential elections in the United States, but also people that are running for mayor in a town. We make no distinction between the different type of contenders, unless this is specifically noted.

After the election, in each electoral district there is a single winning political party. Based on the winners of each electoral district in a country, a single political party will be declared as the winner of the nationwide election and thus is the winning contender in the whole country. In most occasions we will assume that the political party who won most districts, is the nationwide winner. However, in general, each district represents a seat (which is a unit of control) and a party benefits from having as much seats as possible. For the election of the new prime minister in The Netherlands there is only one electoral district and this is the whole country. An interesting example and also an example that we will reference throughout this thesis is the setting of the elections in the United States of America.

2.1.1 American Politics

For the presidential elections in the United States, the usage of the word “electoral districts” might cause confusion. There are 435 electoral districts[2], but

[1] Sometimes we will be talking about a “county” instead of a country, or more generally a “nation”.

[2] All data regarding the number of electoral districts in the United States is valid in the decade 2010–2020.

it only matters which presidential candidate is the winner in each of the 50 states (with the exception of Maine and Nebraska). An American state contains a number of electoral districts, where this number depends on the amount of residents in that state. Highly populated states contain a lot more electoral districts than states with lower population, for example California contains 53 districts and Alaska only 1. In each state, residents vote for their preferred presidential candidate, and the candidate with the largest number of votes wins in that state (this is called plurality voting or a first-past-the-post system. This will be clarified later on). If a state has X districts, the candidate who won in that state receives $X + 2$ *electoral votes*, where the 2 stems from the pair of senators each state receives, regardless of their population size. See Figure 1.

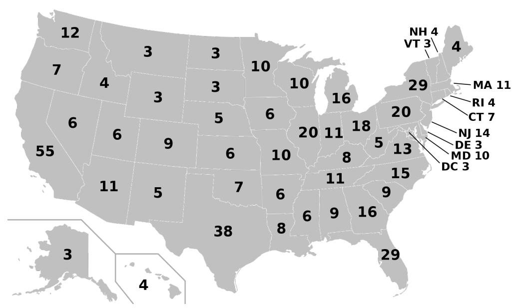


Figure 1: Electoral votes allocated to each state in the decade 2010–2020.

In this way, there are 538 electoral votes cast for presidential candidates (435 districts + 2×50 senators + 3 remaining votes stemming from the District of Columbia). A candidate gets to be elected as the president and is hence declared the winner, if he or she receives the majority of the electoral votes. In this case this amounts to at least 270 (in the case of a tie, the House of Representatives breaks that tie by giving each state one vote).
For these presidential elections, which are quadrennial, the shape of the electoral districts does not influence the outcome, but only the number of districts in each state. However, for elections held in a specific state, these shapes turn out to have a major influence. Figure 1 shows that North Carolina has 13 congressional districts (which is essentially the same as electoral districts in the US). In the main setting described before, North Carolina is divided into 13 electoral districts. A pre-2016 map of North Carolina that depicts the shape of each district looks as follows, see Figure 2 [1].

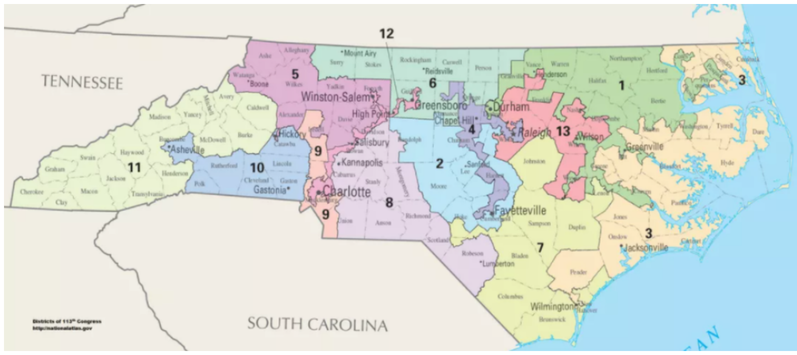


Figure 2: North Carolina divided into 13 congressional districts.

There are some oddly shaped districts visible here, but by far the most irregular shaped district is district 12. An explanation for the weird shape that is displayed here can be deduced from Figure 3, which shows the percentage of African American people in each neighborhood in North Carolina.

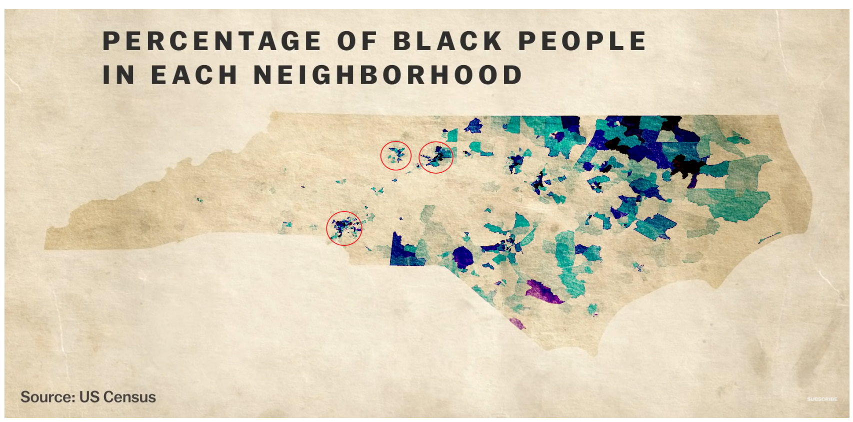


Figure 3: A darker shade of blue displays a higher percentage of African American people (source: Vox).

It seems that the shape of district 12 is designed to capture three cities with predominantly black neighborhoods, which are Charlotte, Greensboro and Winston-Salem and are circled in red on Figure 3. This is a tactic, which will be later defined as *packing*, where the regional influence of black voting behaviour is reduced by packing as many neighborhoods of a certain demographic (in this case black) as possible in a single district. This will prevent these neighborhoods to influence the voting behaviour in other districts. Knowing that Thomas B. Hoffeler, a Republican, was in control of creating the map in Figure 2 [2] makes this even more suspicious. It is generally known that African American people do not, in general, support the Republican party[3]. This was Hoffelers way to establish a political advantage for the Republican party in North Carolina and is one of the most famous cases of *gerrymandering*.

2.1.2 Gerrymandering Tactics

Gerrymandering is a practice where the boundaries of electoral districts are (re)drawn in such a way to benefit a specific political party or group. The name is a combination of "Gerry" and "salamander"; one of the districts in Massachusetts with Governor Elbridge Gerry resembled the shape of a mythological salamander [3].
As seen before, data regarding the distribution of certain demographic groups or voting behaviour, based on previous elections, can be very useful in this manipulation process. They allow people in control of redistricting to apply strategies like *packing* and *cracking*. Packing is to concentrate as many voters of a certain (demographic, political or religious) group in as few districts as possible. A few districts will be sacrificed (the districts where these groups of voters are packed in), but the rest will, with a high certainty, have a majority of voters of the group or party that is trying to be aided. When each district represents a unit of control, a party strives to win as many districts as possible. When winning the majority of districts guarantees a party to win, it is simply enough to pack certain voters in districts in such a way that a majority of districts will be won. Packing is more effective in combination with cracking. Cracking is to divide as many voters of a certain group among as many districts as possible to dilute their votes and hence deny them to have a majority, or sufficiently large enough voting bloc, in those districts. An example of cracking done in practice is a college campus in North Carolina split into two congressional districts [4].
To cite Thomas B. Hoffeler, who was accused of gerrymandering:

> *Usually the voters get to pick the politicians. In redistricting, the politicians get to pick the voters.*

For the presidential elections in the United States it is only important what the shapes of the states are. A notable example of gerrymandering in these type of elections is the split of the Dakota Territory into North Dakota and South Dakota, ensuring that each newly formed state now at least has two electoral votes instead of one (due to each state receiving two electoral votes, regardless of the population). Another notable result of gerrymandering is the election of the members of congress in North Carolina in 2012. As mentioned before, the pre-2016 map of North Carolina and its 13 congressional districts (see Figure 2) has been gerrymandered and the results of the election are depicted in Table 1 [5].

[3] A famous exception is African American rapper Kanye West.

Party	Total # votes in percentages	# districts or seats won
Democrats	50.60 %	4
Republicans	48.75 %	9

Table 1: Results of 2012 election in North Carolina .

Even though the Democrats won the majority of the votes, they only won 4 out of 13 seats, which is not proportional to the total vote count. Another more famous, but less extreme situation where a similar thing has occurred, is the 2016 presidential elections in the United States. Nominee Donald Trump received 46.1% of the total votes and nominee Hillary Clinton received 48.2% [6], but Trump received 304 electoral votes whereas Clinton received only 227, making Trump the president of the United States. Whether this was in fact a result of gerrymandering is disputable and will not be in the scope of this thesis. These types of results, however, can raise a few eyebrows and can lead to the start of an investigation where the use of gerrymandering will be proved (or disproved).

2.2 Voting Theory

A *voting system* is a rule that determines how elections are executed and how their results are decided. More specific, a *voting rule* (or *social choice function*) decides, on the basis of ballots that voters declare, which contenders win the election. In these ballots, each voter chooses from a set of contenders (in voting theory these are generally called *alternatives*) and states their preference in the form of a linear order over the alternatives. After all the ballots are declared, the voting rule decides which alternatives are the winners of the election.
In order to give a formal definition of a voting rule, we need to construct a mathematical model related to voting theory. Let $N = \{1, 2, \ldots, n\}$ be the set of voters in an election, thus we have $|N| = n$ voters. Let $\mathcal{P}$ be the set of alternatives, which we will in this thesis often refer to as the set of political parties, with $|\mathcal{P}| = m$. Let $\mathcal{L}(\mathcal{P})$ denote the set of all linear orders over $\mathcal{P}$. That is, an element of $\mathcal{L}(\mathcal{P})$ is an ordering over all m alternatives. Each voter i, where $1 \leq i \leq n$, declares a ballot $\succ_i \in \mathcal{L}(\mathcal{P})$ and this gives rise to a profile $\succ = (\succ_1, \ldots, \succ_n) \in \mathcal{L}(\mathcal{P})^n$.

Definition 1. A voting rule (or social choice function) F is defined by

$$F : (\mathcal{L}(\mathcal{P}))^n \rightarrow \wp(\mathcal{P}) \backslash \{\emptyset\}$$

where $\wp(X)$ denotes the power set of X.

So $F(\succ) \subseteq \mathcal{P}$ denotes the set of alternatives that won the election according to profile $\succ$. When $|F(\succ)| = 1$ for all profiles $\succ$, we say that F is *resolute.* For resolute F, we have that

$$F : (\mathcal{L}(\mathcal{P}))^n \rightarrow \mathcal{P}.$$

Resoluteness can often be achieved by using tie-breaking rules, for example random tie-breaking or lexicographic tie-breaking.
One of the most trivial voting rules is the *plurality rule*, often referred to as the first-past-the-post system. Here, the alternative that is chosen as top alternative the most is the winner and gets elected. Another way of saying this is to assign one point to each alternative that is ranked first on a ballot and the alternative with the largest number of points wins. For this voting rule, the only relevant information on a ballot is the alternative ranked first. Therefore, a ballot can contain one alternative in this case. As mentioned before, the presidential elections in the United States use a first-past-the-post system and the candidate with most votes in a state receives all those electoral votes. See Figure 4 for a general election ballot in 2016.

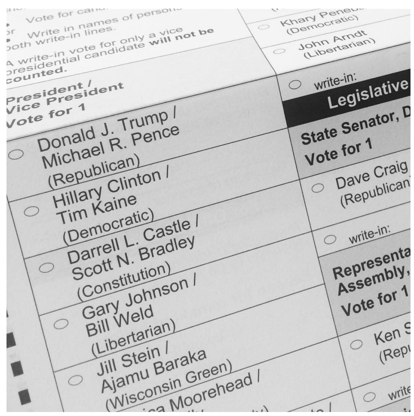


Figure 4: General ballot for the US presidential elections in 2016.

Formally, the plurality rule is the voting rule F with

$$F(\succ) = \{P \in \mathcal{P} : \sum_{i=1}^{n} \mathbb{1}_{\text{top}(\succ_i)=P} \geq \sum_{i=1}^{n} \mathbb{1}_{\text{top}(\succ_i)=H} \text{ for all } H \in \mathcal{P} \backslash \{P\}\}$$

where $\text{top}(\succ_i)$ is the top alternative of ballot $\succ_i$.
Another well-known voting rule is the *Borda rule*. Here, each voter gives $m-1$ points to the alternative she ranks first, $m-2$ points to the alternative she ranks second, and so on. The alternatives with most points win. Now a declared ballot is a full linear ordering of the alternatives instead of picking a single alternative. For $1 \leq k < m$, *k-approval voting* assigns a point to each alternative placed in the top k alternatives in a ballot and the alternatives with most points win. *Approval voting* is similar, but each voter decides which and how many alternatives she gives a point to. It can be perceived as k-approval voting, but k is not fixed and each voter decides for herself what k is.
The plurality rule, Borda rule and k-approval are all an example of a *positional scoring rule*.

Definition 2. A positional scoring rule is a voting rule that can be defined by a scoring vector $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m$ where $s_1 \geq s_2 \geq \ldots \geq s_m$ and $s_1 > s_m$.

In a declared ballot $\succ$, the alternative positioned at the i-th place according to that ballot gets s_i points. The alternatives with most points win. The plurality rule has scoring vector $\mathbf{s} = (1, 0, \ldots, 0)$, the Borda rule has scoring vector $\mathbf{s} = (m-1, m-2, \ldots, 0)$ and k-approving has scoring vector $\mathbf{s} = (1, 1, \ldots, 1, 0, \ldots, 0)$ where there is a total of k 1-s. Another example of a positional scoring rule is the *veto rule*, which has scoring vector $\mathbf{s} = (1, 1, \ldots, 1, 0)$. A few other (slightly more complicated) voting rules will be defined in later chapters.

Example 1. Consider a household of 11 people that has to decide what to eat for a common dinner. They can choose between chicken, salmon or spinach (those are the only things they supplied in their fridge in large amounts). Since a combination of these dishes sounds like too much effort on this lazy sunday, they decide to run a little election to select one single dish. Each of them declares a ballot where the dishes are placed in descending order.

- Chicken $\succ$ Spinach $\succ$ Salmon ($4\times$)
- Salmon $\succ$ Chicken $\succ$ Spinach ($2\times$)
- Salmon $\succ$ Spinach $\succ$ Chicken ($3\times$)
- Spinach $\succ$ Chicken $\succ$ Salmon ($2\times$)

The first line displays that four people prefer chicken the most, then spinach and salmon is the least preferred among them. Using the plurality rule, salmon wins with a total of 5 votes. However, using the Borda rule, chicken has 12 points, salmon 10 and spinach 11, so chicken is being served. But using 2-approval, they enjoy a vegetarian meal of spinach.

Voting rules can satisfy certain desirable properties, called *axioms*. A voting rule is considered fair if it at least fulfills the requirements of the anonymity and neutrality axioms.

Definition 3. A voting rule F is *anonymous* if $F(\succ) = F(\pi(\succ))$ for any profile $\succ$ and permutation $\pi : N \to N$.

π is a permutation over the set of voters, so $\pi(\succ)$ is defined as $\pi(\succ_1, \ldots, \succ_n) = (\succ_{\pi(1)}, \ldots, \succ_{\pi(n)})$. Anonymity means that the social choice function F does not look at which voter declares which ballot, so that any permutation over the set of voters gives the same result. Each voter is "anonymous".

Definition 4. A voting rule F is *neutral* if $F(\pi(\succ)) = \pi(F(\succ))$ for any profile $\succ$ and permutation $\pi : \mathcal{P} \to \mathcal{P}$.

This time, π is a permutation over the set of alternatives, so $\pi(\succ)$ is defined as $\pi(\succ_1, \ldots, \succ_n) = (\pi(\succ_1), \ldots, \pi(\succ_n))$, where

$$\succ_i = P_1 \succ P_2 \succ \ldots \succ P_m \Rightarrow \pi(\succ_i) = \pi(P_1) \succ \pi(P_2) \succ \ldots \succ \pi(P_m),$$

for $P_1, \ldots, P_m \in \mathcal{P}$. Neutrality means that the social choice function F does not look at the "name" of each alternative, only at its respective order in the ballots. The rule is symmetric with respect to alternatives.
All the aforementioned voting rules are anonymous and neutral. An example of a voting rule that is not anonymous is the *dictatorship*. F is a dictatorship if there exists $i \in N$, such that $F(\succ) = \text{top}(\succ_i)$ for every profile $\succ$. Voter i is called the dictator and gets to decide which alternative wins.
Another useful property for a voting rule is satisfying the *Pareto Principle*.

Definition 5. A voting rule F is *weakly Paretian* if $P \succ_i H$ for all $1 \leq i \leq n$ implies that $F(\succ) \neq H$ for any profile $\succ = (\succ_1, \ldots, \succ_n)$ and alternatives $P, H \in \mathcal{P}$.

Being weakly Paretian means that, whenever all voters rank alternative P above alternative H, then H will not be a winning alternative.
The last axiom that we will define is called Independence of irrelevant alternatives.

Definition 6. A voting rule F is *independent* if, for any two profiles $\succ = (\succ_1, \ldots, \succ_n)$ and $\succ^2 = (\succ_1^2, \ldots, \succ_n^2)$ and any two alternatives $P, H \in \mathcal{P}$, such that

$$P \succ_i H \Leftrightarrow P \succ_i^2 H$$

for all $1 \leq i \leq n$ we have that

$$F(\succ) = P \Rightarrow F(\succ^2) \neq H.$$

Being independent means that, if the relative rankings of P and H are the same in two profiles $\succ$ and $\succ^2$, and P keeps H from winning in $\succ$, then it also keeps H from winning in $\succ^2$. Another way of phrasing this is to say that if P is preferred to H out of the choice set $\{P, H\}$, introducing a third option K, expanding the choice set to $\{P, H, K\}$, must not make H preferable to P. K is in this case an irrelevant alternative when only considering P and H.
We will state one impossibility theorem that tells us that a resolute voting rule that satisfies some of the above axioms can only be of a certain type. This theorem was officially designed for *social welfare functions*.

Definition 7. A social welfare function F is defined by

$$F : \mathcal{L}(\mathcal{P})^n \rightarrow \mathcal{L}(\mathcal{P}).$$

A social welfare function distinguishes from a social choice function (or voting rule) in the sense that it ouputs all alternatives in order, instead of outputting a list of alternatives that wins the election. It outputs a collective ballot that represents the voting behaviour of each voter that declared a ballot.

Theorem 1. Any resolute social choice function F with at least three alternatives that is weakly Paretian and independent must be a dictatorship.

Theorem 1 is called Arrows impossibility theorem and is named after economist and mathematician Kenneth Arrow [7]. This theorem tells us that no resolute voting rule can satisfy three fairness criteria at the same time, which are the Pareto Principle, being independent and non-dictatorship. This is a rather surprising result with quite a long proof. A proof can be found in [7], where the social welfare function variant has been stated and proven.

2.3 Jordan Curves

A *Jordan curve*, or a simple closed curve in the plane, is a non-self-intersecting continuous loop in the plane [8]. More formally, see Definition 8.

Definition 8. A Jordan curve C is the image of a continuous map $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = \gamma(1)$ and $\gamma|_{[0,1)}$ is injective.

The injectivity is needed to stipulate that C has no self-intersecting points. It can also be defined as the image of a non-self-intersecting *path* from x to y in $\mathbb{R}^2$ where $x = y$.

Theorem 2. (*Jordan curve theorem*) Let C be a Jordan curve. The complement $\mathbb{R}^2 \backslash C$ is the union of two disjoint nonempty open connected sets.

One of the sets is called the *interior* and is bounded, the other is called the *exterior* and is unbounded, see Figure 5.

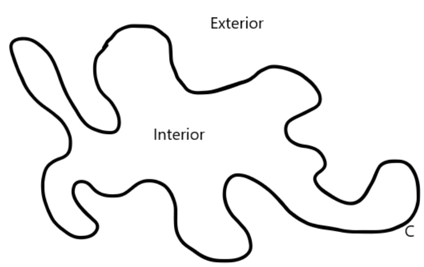


Figure 5: Example of a Jordan curve C.

The Jordan curve C is the boundary of the interior and the exterior, hence separates these components. The result seems intuitive and perhaps trivial, but the proof is rather technical and we refer to [9] for a proof of the Jordan curve theorem that imitates the original proof designed by Camille Jordan himself. The most important thing to remember about this theorem is that the interior is the bounded region inside C and we will refer to this as the *interior of C*, denoted by $\text{int}(C)$. Since by the Jordan curve theorem, $\text{int}(C)$ is an open set, it is a Borel set, which implies that $\text{int}(C)$ is Lebesgue-measurable. The Lebesgue measure of $\text{int}(C)$ will be denoted by $\lambda(\text{int}(C))$ and this coincides with the *area* of the interior of C, denoted by $A(\text{int}(C))$.[4]

2.4 Statistical Tests

In statistical testing we try to give an affirmative answer to certain questions based on the result of one or more experiments. The type of questions we strive to answer reduces to the decision between two conflicting hypotheses. One of these hypotheses is the *null hypothesis* and is typically used for indicating the default position where there is no significant relationship between certain phenomena or a significant difference. The other hypothesis, the *alternative hypothesis*, usually indicates that there is a significant difference or relation.
For a more formal definition of these hypotheses, we must also define what we mean by a *statistical model*. This is a collection of probability distributions on a given outcome space. When all these distributions are similar but differ only in a certain parameter, the outcome space can be defined by Θ, which is a space of parameters θ and will be called the *parameter space*. The statistical model is then defined by $\{p_\theta : \theta \in \Theta\}$, where p_θ indicates a probability distribution with parameter θ. Let $\mathbf{X}$ be the outcome of an experiment, or rather an *observation*, where $\mathbf{X}$ can also be a collection of observations: $\mathbf{X} = (X_1, \ldots, X_n)$. On the basis of $\mathbf{X}$, one decides between the two conflicting hypotheses, which are formally defined as follows.

Definition 9. The null-hypothesis H_0 is defined as the event $\{\theta \in \Theta_0\}$ and the alternative hypothesis is defined as the event $\{\theta \in \Theta_1\}$, where the parameter space $\Theta = \Theta_0 \cup \Theta_1$ is a disjoint union of Θ_0 and Θ_1.

So, on the basis of $\mathbf{X}$, one decides whether the true parameter θ lies in Θ_0 or Θ_1. When $|\Theta_0| = 1$, we call the null-hypothesis a *simple hypothesis*. The same holds for the alternative hypothesis.

Example 2. A typical and simple example of statistical testing is that of throwing a coin with two sides called head and tails. A coin throw can be represented by the random variable X. X can take the values 0 and 1, where 0 represents throwing the coin and tails comes up and 1 represents throwing the coin and heads come up. Denote $\mathbb{P}(X = 1)$ by θ, this means that throwing heads with a single coin has probability θ, where $\theta \in [0, 1]$. In other words, the statistical model of throwing a coin is defined by $\{p_\theta : \theta \in [0, 1]\}$, where p_θ is Bernoulli distributed with parameter θ, i.e. $p_\theta(X = 1) = \theta$. In this sense, each p_θ is a different distribution for different values of θ and θ represents the probability of throwing heads. Assume that one would like to test whether a specific coin is fair, which means that the probability of throwing heads is equal to the probability of throwing tails (as is the case for most coins we encounter in our daily life). The null-hypothesis is that the coin is fair (the default position) and the

[4] We will from now on abbreviate $A(\text{int}(C))$ by $A(C)$ when it is stated that C is a Jordan curve.

alternative hypothesis that it is unfair (a more "interesting" position). Then H_0 and H_1 are defined as follows:

$$H_0 : \theta = \frac{1}{2} \text{ (coin is fair)}$$
$$H_1 : \theta \neq \frac{1}{2} \text{ (coin is unfair).}$$

Put differently, the outcome space $\Theta = [0, 1]$ is a disjoint union of $\Theta_0 = \{\frac{1}{2}\}$ and $\Theta_1 = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. The null-hypothesis is here a simple hypothesis. $\theta = \frac{1}{2}$ is indeed equivalent to saying that the coin is fair, since the probability of heads is $\theta = \frac{1}{2}$ and of tails is $1 - \theta = \frac{1}{2}$. One might use a binary vector of length n, where n is the amount of throws of a coin, as the observation vector $\mathbf{X}$, in order to conclude whether H_0 is more likely, or H_1. For example, $\mathbf{X} = (1, 1, 0, 1)$ tells us that the first two throws and the last throw resulted in heads and the third throw resulted in tails.

The two hypotheses can also be stated verbally instead of being indicated by different parameter values, as in the following example.

Example 3. In a household of five people, a mysterious disappearance has happened during one night in October. One member of the household, Alizé de Varmuennel, noticed that, from the plate in the fridge with five pieces of fried chicken, only two pieces were left the following morning. She immediately suspects her brother, Odin de Varmuennel, to have eaten the three pieces of fried chicken, as the two pieces of burned chicken were left over and she knows that he dislikes burned meat. She defines the hypotheses as follows:

H_0 : Odin did not eat the fried chicken.

H_1 : Odin ate some of the fried chicken.

In these kind of (legal) accusations, the observation $\mathbf{X}$ is usually called *evidence* (hence, sometimes noted by E). The evidence is stated as follows:

E : The pieces of burned chicken were still on the plate.

E can also consist of more pieces of evidence, in which case it can be interpreted as a vector of evidence.

It is common in statistical testing to try to accept H_1, by rejecting the null-hypothesis H_0. In Example 2, one would like to presume that the coin is unfair by rejecting the hypothesis that it is fair. In Example 3, Alizé would like to confirm her accusation that Odin ate the chicken. So statistical analysis can lead to two conclusions:

- Reject H_0 and thus accept H_1.
- Do not reject H_0, but do not accept H_1.

This procedure can however lead to false conclusions, since we do not know the true nature of θ, or more generally, we do not know with certainty which statement is true. There are two types of false conclusions:

1. Reject H_0, while H_0 is correct.
2. Do not reject H_0, while H_0 is incorrect.

The first false conclusion is called a *type I error* and the second false conclusion is called a *type II error*. A type I error is usually much worse, as it can lead to false accusations (such as in the context of Example 3 or more serious legal contexts), whereas a type II error usually leads to no further actions, but only a continuation of the investigation. Call the probabilities of making these types of errors respectively $\alpha := \mathbb{P}(\text{type I error})$ and $\beta := \mathbb{P}(\text{type II error})$.
It is interesting to compute the probability of an observation (or evidence) $\mathbf{X}$, given a parameter value θ. This probability is called the *likelihood function*.

Definition 10. Let $\mathbf{X}$ be a discrete random variable that depends on the parameter θ, then the likelihood function is $\mathcal{L}(\theta|\mathbf{x}) = \mathbb{P}(\mathbf{X} = \mathbf{x}|\theta)$.

This function of θ indicates how likely a certain observation is given the parameter θ. To compare the two hypotheses $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$ based on observation $\mathbf{x}$, we use the so-called *likelihood ratio*.

Definition 11. The likelihood ratio for observation $\mathbf{x}$ is

$$LR_{H_0,H_1}(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta|\mathbf{x})}{\sup_{\theta \in \Theta} \mathcal{L}(\theta|\mathbf{x})}.$$

A high likelihood ratio indicates a bigger support for H_0, given $\mathbf{x}$, and a low ratio indicates a bigger support for H_1. We chose to take the supremum over $\theta \in \Theta$ in the denominator instead of $\theta \in \Theta_1$ to avoid dividing by zero.
When H_0 and H_1 are both simple hypotheses, i.e. $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$, we define the likelihood ratio for observation $\mathbf{x}$ a bit different.

Definition 12. Let H_0 and H_1 be simple hypotheses, then the likelihood ratio for observation $\mathbf{x}$ is

$$LR_{H_0,H_1}(\mathbf{x}) = \frac{\mathcal{L}(\theta_0|\mathbf{x})}{\mathcal{L}(\theta_1|\mathbf{x})}.$$

The difference between Definitions 11 and 12 is in the denominator.
When the two hypotheses are verbal statements, as in Example 3, and the observation is called evidence and denoted by E, the likelihood ratio as defined in Definition 12 takes a special form:

$$LR_{H_0,H_1}(E) = \frac{\mathbb{P}(E|H_0)}{\mathbb{P}(E|H_1)}. \tag{1}$$

$\mathbb{P}(E|H_0)$ is just the likelihood of the statement from the null-hypothesis H_0 given the evidence E. Again, a high value of $LR_{H_0,H_1}(E)$ indicates a bigger support for H_0 and vice versa.
We would like to decide, based on the value of the likelihood ratio, whether to reject H_0 or to not yet reject it. The Neyman–Pearson test [10] is such a procedure that works for simple hypotheses and we will formulate this test with a likelihood in the form of Equation 1 [11] . Therefore, let H_0 and H_1 be simple competing hypotheses and E be the evidence. It is always the case that H_0 is true or H_1 is true. If H_i is true, we want to take some action A_i, for $i = 0, 1$. We decide on the basis of evidence E whether to take action A_0 or A_1. We do so by defining a region $R \subseteq \mathcal{E}$, where $\mathcal{E}$ is the set of all the evidence we might obtain. The test is as follows: if $E \in R$, we choose action A_1, otherwise if $E \notin R$, we choose action A_0. Then the probabilities of type I and type II error are equal to $\alpha = \mathbb{P}(E \in R|H_0)$ and $\beta = \mathbb{P}(E \notin R|H_1)$. Define the region R_t by

$$R_t = \{E | LR_{H_0,H_1}(E) \leq t\}. \tag{2}$$

This means that, when the evidence in favour of H_0 is strong enough (likelihood ratio is strictly bigger than t) we undertake action A_0, otherwise we undertake action A_1 (which is similar to rejecting H_0). The following lemma tells us that, among all such tests with region R, the Neyman–Pearson test, which is the above test with region R_t for a fixed t, is the most “powerful”.

Lemma 1. (*Neyman–Pearson Lemma.*) Define $\alpha_t = \mathbb{P}(E \in R_t|H_0)$ and $\beta_t = \mathbb{P}(E \notin R_t|H_1)$ for a fixed t. There can be no other test with region R (where $R \neq R_t$ for any t) and corresponding α, β, such that $\alpha < \alpha_t$ and $\beta < \beta_t$.

The most powerful test means that it ensures minimal α and β, i.e. minimal probabilities of type I and type II error. This can also be read as

$$\alpha < \alpha_t \Rightarrow \beta > \beta_t, \tag{3}$$

i.e., if another test gives a smaller probability of type I error, it gives a bigger probability of type II error. So the likelihood ratio threshold is an optimal way (in the sense as in Equation 3) to decide whether to take action A_0 or A_1, i.e. to reject H_0 or to not yet reject H_0.

Part I
Detection

3 Efficiency Gap

The efficiency gap is the first tool we will discuss that helps determining whether gerrymandering has occurred. It measures the amount of wasted votes for a party, which are "redundant" votes, i.e. votes for a party that lost or would still have won without those votes. By applying gerrymandering tactics like packing and cracking, as discussed in Chapter 2, the party being "packed" or "cracked" has many wasted votes. Packing ensures that this party receives many votes more than necessary in some districts, to dilute their votes in other districts. Cracking ensures that this party receives many votes from districts it loses in. Hence, a party receiving many wasted votes is a good indication for gerrymandering.

3.1 Efficiency Gap for Two Parties

We will first discuss the general definition of the efficiency gap. The efficiency gap is designed as a measure for the case that there are only two (political) parties competing against each other to win some election for a particular cause, e.g. rule the nation. In this election, the most reasonable voting rule for two parties will be considered: the aforementioned *plurality rule*. To write down the mathematical definition of the efficiency gap, we will first have to present a handful of definitions and notions.

Let the two competing parties be called A and B and let there be S legislative districts in the country. Let $\mathcal{D}$ be the set of (legislative) districts, denoted by $\mathcal{D} = \{d_1, \ldots, d_S\}$. We denote by $\mathcal{D}^P \subseteq \mathcal{D}$ the set of districts won by party P (where P is A or B). Furthermore, define

$$S_i^P = \begin{cases} 1, & \text{if party } P \text{ won in district } d_i \\ 0, & \text{if party } P \text{ lost in district } d_i \end{cases}$$

Then $S^P := \sum_{i=1}^{S} S_i^P = |\mathcal{D}^P|$ is the number of districts won by party P in the country. Let T_i^P be the number of votes for party P in district d_i. Then $T^P := \sum_{i=1}^{S} T_i^P$ is the number of votes for party P in the country and $T_i := T_i^A + T_i^B$ is the total number of votes in district d_i.

As mentioned before, a wasted vote is a vote for a losing party or a useless vote for a winning party such that the party would still have won in that district without that vote. Alternatively, one might consider wasted districts. This is however not a useful notion, since a party sometimes benefits from having as many controlled districts as possible. Here we will mostly consider the case where a party wins if it controls the most districts in a region, i.e. party A wins if $S^A > S^B$. Still, considering wasted districts is not a helpful tool to tackle gerrymandering.

We compute the number of wasted votes W_i^P for party P in district d_i as follows

$$W_i^P = \begin{cases} T_i^P - \lceil \frac{T_i}{2} \rceil, & \text{if } d_i \in \mathcal{D}^P \\ T_i^P, & \text{if } d_i \notin \mathcal{D}^P \end{cases} \tag{4}$$

$$= T_i^P - S_i^P \lceil \frac{T_i}{2} \rceil, \tag{5}$$

where $d_i \in \mathcal{D}^P$ is equivalent to party P winning in district d_i.[5] We always have that the total number of wasted votes in d_i is

$$W_i := W_i^A + W_i^B = \lfloor \frac{T_i}{2} \rfloor = \lfloor \frac{T_i^A + T_i^B}{2} \rfloor.$$

The principle of a wasted vote for party A is that, even if all of these wasted votes would be allocated as regular votes to party B, party A would still have the majority of the votes.[6] Since a wasted vote remains a vote, this vote will turn into a vote for the other party, ensuring both parties to have $\frac{T_i}{2}$ votes after allocation (if T_i is odd, the winning party will still have slightly more). For example, take a district d_1 with $T_1 = 10$ voters, suppose that 7 of these vote for party A and 3 vote for party B. Then A wins in district d_1 and $W_1^A = 2$, so that $T_1^B + W_1^A = 5 = \frac{T_1}{2}$.
Let $W^P = \sum_{i=1}^S W_i^P$ be the number of wasted votes for party P in the country and let $T = T^A + T^B$ be the total number of votes in the country. We have now all the means to mathematically define the efficiency gap [12].

Definition 13. For two parties A and B competing in a country where $\mathcal{D}$ is the set of S districts, the efficiency gap EG is

$$EG = \sum_{i=1}^{S} \frac{W_i^A - W_i^B}{T} = \frac{1}{T}(W^A - W^B). \tag{6}$$

Following from this, we have that $EG \in [-1, 1]$.
EG being high is an unfair situation for A, as there are relatively many more wasted votes for A than there are for B. This is possibly a result of packing, a gerrymandering strategy. Equivalently, EG being very low (as in very negative) is an unfair situation for B in the sense that B has much more wasted votes. $EG \approx 0$ is a situation that is fair to both parties with respect to the wasted votes. Note that

$$EG = 0 \Leftrightarrow W^A = W^B.$$

[5] When T_i is even, we have $\lceil \frac{T_i}{2} \rceil = \frac{T_i}{2}$. When T_i is odd, we round to the first integer above, i.e. $\lceil \frac{T_i}{2} \rceil = \frac{T_i}{2} + \frac{1}{2}$.

[6] Perhaps a better choice for W_i^P would be $T_i^P - \frac{T_i}{2} - 1$ when $d_i \in \mathcal{D}^P$ and T_i is even, so that $T_i^P - W_i^P = \frac{T_i}{2} + 1$ when party P wins in district d_i. This ensures that P has indeed the majority, not half of the votes. However, the wasted votes are in the literature defined as in (5), since in applications generally the numbers are high enough to justify neglecting the +1 for simplicity reasons.

If there is only one district, this is the case when one party has 75% of the votes and the other 25%. Also notice that, to work with the efficiency gap, one has to specify party A and party B, since it is defined associated to party A.
When is EG considered high, i.e. from what score is EG considered problematic and could hint towards possible gerrymandering? Stephanopoulos and McGhee, who invented the notion of efficiency gap, argued that the right number for gerrymandering detection is $|EG| > 0.08$ [5]. Hence, the situation is suspicious whenever $|EG|$ exceeds this threshold 0.08.

3.2 Efficiency Gap for Finitely Many Parties

As mentioned before, we are interested in tactics to tackle the problem of gerrymandering in more cases than just the trivial case of two competing parties. The first generalization that comes to mind, is to include more than two parties into the equation. In this section we will consider possible generalizations of the efficiency gap defined in Definition 13 to three competing parties and we will discuss some (dis)advantages. Afterwards, an easy generalization can be made to any finite number of parties. For now, we will consider the plurality rule applied to three alternatives.
Let the set of parties be denoted by $\mathcal{P} = \{A, B, C\}$. For two parties, the efficiency gap has the anti-symmetric property, meaning that $EG = -\widetilde{EG}$, where $\widetilde{EG}$ is the efficiency gap with parties A and B swapped. As a consequence, knowing exclusively EG or $\widetilde{EG}$, one can easily calculate the other. This anti-symmetric property gets lost for three parties. In this case, it makes sense to define a measure EG^P to infer something about party $P \in \mathcal{P}$ and not necessarily about the other parties. We will discuss three possible extensions[7] of the definition of wasted votes per district and in the end define EG^P.

1. The total number of wasted votes for a party P, say A, is defined with respect to the two remaining parties, B and C. Let $d_i \in \mathcal{D}$ be a district. If A lost in d_i, then each vote in d_i for party A is a wasted vote for A. If A won in d_i, then a vote in d_i is a wasted vote for A if A would still have a majority of the votes without that vote. Thus, the number of wasted votes by A-voters in district d_i is given by

$$\begin{aligned} W_i^A &= \begin{cases} T_i^A, & \text{if } d_i \notin \mathcal{D}^A \\ T_i^A - \max\{T_i^B, T_i^C\}, & \text{if } d_i \in \mathcal{D}^A \end{cases} \\ &= T_i^A - S_i^A \max\{T_i^B, T_i^C\}. \end{aligned}$$

When $d_i \in \mathcal{D}^A$, then

$$\begin{aligned} T_i^A - W_i^A &= T_i^A - (T_i^A - \max\{T_i^B, T_i^C\} \\ &= \max\{T_i^B, T_i^C\}, \end{aligned}$$

[7] The attentive reader will notice that one of them is actually not an extension, we will come back to this.

so A has as many votes as the runner-up.
Next, we define EG^A as follows

$$EG^A = \sum_{i=1}^{S} \frac{W_i^A - \frac{1}{2}(W_i^B + W_i^C)}{T} = \frac{W^A - \frac{1}{2}(W^B + W^C)}{T}.$$

This is the proportional difference between wasted votes for A and the average number of wasted votes for the other parties.
More generally, if $P \in \mathcal{P}$ with $|\mathcal{P}| \geq 3$,

$$W_i^P = \begin{cases} T_i^P, & \text{if } d_i \notin \mathcal{D}^P \\ T_i^A - \max\limits_{P \in \mathcal{P}}\{T_i^P\}, & \text{if } d_i \in \mathcal{D}^P \end{cases}$$

and

$$EG^P = \sum_{i=1}^{S} \frac{W_i^P - \frac{1}{|\mathcal{P}|-1}\sum_{K \in \mathcal{P}\setminus P} W_i^K}{T} = \frac{W^P - \frac{1}{|\mathcal{P}|-1}\sum_{K \in \mathcal{P}\setminus P} W^K}{T}.$$

We can still conclude that $EG^A > 0$, large, implies an unfair situation for A and $EG^A < 0$, with $|EG^A|$ large, imples an unfair situation for another party. However, there are several problems rising with this extension. Firstly, as mentioned before, the value of EG^A can be used to infer something about the fairness of the outcome to party A, but solely to this party. It would be nice, however, if we can make a conclusion for all the parties based on this one value EG^A, or even based on the computation of EG^A. Secondly, we assume here that a wasted vote in a district just vanishes, but what if it turned into a vote for another party, just like it does in the case with two parties? These problems (or questions) imply the essence of the following proposed extension.

2. The total number of wasted votes for a party P, say A, is defined with respect to each party separately. Hence, a vote is wasted by A-voters *against* B-voters if we restrict a district to only A and B. Thus we neglect C, and consider the original two-party notion of wasted votes. The number of wasted votes by A-voters in district d_i, with respect to party B, is then given by

$$\begin{aligned} W_{i\,B}^{\,A} &= \begin{cases} T_i^A & \text{if } d_i \notin \mathcal{D}^A \\ T_i^A - \lceil \frac{T_i - T_i^C}{2} \rceil & \text{if } d_i \in \mathcal{D}^A \end{cases} \\ &= T_i^A - S_i^A \lceil \frac{T_i - T_i^C}{2} \rceil \\ &= T_i^A - S_i^A \lceil \frac{T_i^A + T_i^B}{2} \rceil. \end{aligned}$$

We simply compute EG^A as follows[8]

$$\begin{aligned}EG^A &= \frac{1}{2}(EG^A_B + EG^A_C) = \frac{1}{2}(\sum_{i=1}^{S} \frac{W_{i}{}^{A}_{B} - W_{i}{}^{B}_{A}}{T - T^C} + \sum_{i=1}^{S} \frac{W_{i}{}^{A}_{C} - W_{i}{}^{C}_{A}}{T - T^B}) \\ &= \frac{1}{2}(\frac{W^A_B - W^B_A}{T - T^C} + \frac{W^A_C - W^C_A}{T - T^B}).\end{aligned}$$

More generally, if $P, K \in \mathcal{P}$ with $|\mathcal{P}| = n \geq 3$, then

$$W_{i}{}^{P}_{K} = \begin{cases} T_i^P & \text{if } d_i \notin \mathcal{D}^P \\ T_i^P - \lceil \frac{T_i^P + T_i^K}{2} \rceil & \text{if } d_i \in \mathcal{D}^P \end{cases}$$

and

$$\begin{aligned}EG^P &= \frac{1}{n-1} \sum_{K \in \mathcal{P} \setminus \{P\}} EG^P_K = \frac{1}{n-1} \sum_{K \in \mathcal{P} \setminus \{P\}} \frac{W^P_K - W^K_P}{T - \sum_{L \in \mathcal{P} \setminus \{P,K\}} T^L} \\ &= \frac{1}{n-1} \sum_{K \in \mathcal{P} \setminus \{P\}} \frac{W^P_K - W^K_P}{T^P + T^K}.\end{aligned}$$

We still have the same results about the fairness of the outcome to party A being dependent on the sign of EG^A and the size of that number, but we now have more information concerning the fairness of A relative to another party. To give a better description of how (un)fair the outcome is to A in the case that $|\mathcal{P}| = 3$, consider the components EG^B and EG^C. Then $EG^A < 0$ and $|EG^A|$ large gives a probable indication that the outcome is really favorable for party A, but observing the signs and magnitude of EG^A_B and EG^A_C tells us at the expense of which party this favorable outcome is. Furthermore, we can now assume that, just as in the original model with $|\mathcal{P}| = 2$, for the computation of e.g. EG^A_B, a wasted vote for A in a district where A won turns into a vote for B. This extension sounds reasonable, especially by the two arguments above, but may not be the most intuitive one.

3. The total number of wasted votes is defined as the maximum number of votes that can be distributed over the other parties without ending in a losing way, where we distinguish between two cases.

(a) Suppose that a party P, say A, wins in district d_i (hence $T_i^A > \frac{T_i}{3}$) and the other two parties, B and C, lose with less than $\frac{1}{3}$ of the total votes (hence $T_i^B, T_i^C < \frac{T_i}{3}$). Then the number of wasted votes for party A in district d_i is defined as the difference between T_i^A and $\lceil \frac{T_i}{3} \rceil$. Divide these wasted votes amongst the other parties such that they both have $\frac{T_i}{3}$ votes[9]. Calculate the

[8] The factor $\frac{1}{2}$ is to ensure that $EG^P \in [-1, 1]$.

[9] Or, in the case where $\frac{T_i}{3}$ is not an integer, spread these wasted votes such that no other party exceeds $\lceil \frac{T_i}{3} \rceil$.

wasted votes for B and C as their total votes. The number of wasted votes by A-voters in district d_i is then given by

$$W_i^A = \begin{cases} T_i^A - \lceil \frac{T_i}{3} \rceil & \text{if } d_i \in \mathcal{D}^A \\ T_i^A & \text{if } d_i \notin \mathcal{D}^A \end{cases} \tag{7}$$

$$= T_i^A - S_i^A \lceil \frac{T_i}{3} \rceil. \tag{8}$$

This may seem like the most intuitive way to extend (5) to three parties.
For example, if we fix district d_i and we have $T_i = 900$ with $T_i^A = 600, T_i^B = 100$ and $T_i^C = 200$, then $W_i^A = 600 - \frac{900}{3} = 300$ and we allocate these wasted votes as follows: $200 \to B$ and $100 \to C$. The wasted votes for B and C are respectively $W_i^B = 100$ and $W_i^C = 200$, since A won in d_i. Remember that the allocation of wasted votes is only a procedure to ensure that these votes do not disappear and still preserves the same winner, or at least does not result in another winning party (this will be formally proven).
(b) Suppose that a party P, say A, wins in district d_i (hence $T_i^A > \frac{T_i}{3}$) and one of the other two parties, say B, has more than $\frac{T_i}{3}$ votes, while the third party does not. Hence, $T_i^B > \frac{T_i}{3}$ and $T_i^C < \frac{T_i}{3}$. Then the number of wasted votes for party A is defined as the difference between the votes for A and the second winner B. Allocate the wasted votes for A to the losing party C to preserve the winner in the district. The number of wasted votes for B and C are defined as their total number of votes. The number of wasted votes by A-voters in district d_i is then given by

$$W_i^A = \begin{cases} T_i^A - T_i^B, & \text{if } d_i \in \mathcal{D}^A \cap \mathcal{D}_2^B \\ T_i^A - T_i^C, & \text{if } d_i \in \mathcal{D}^A \cap \mathcal{D}_2^C \\ T_i^A, & \text{if } d_i \notin \mathcal{D}^A \end{cases} \tag{9}$$

$$= T_i^A - S_i^A (X_i^B T_i^B + X_i^C T_i^C), \tag{10}$$

where we denote by $\mathcal{D}_2^P \subseteq \mathcal{D}$ the set of districts where party P has the second most votes (is second winner or runner-up) and

$$X_i^P = \begin{cases} 1 & \text{if party } P \text{ is second winner in } d_i \\ 0 & \text{if party } P \text{ is (at most) third winner in } d_i. \end{cases} \tag{11}$$

For example, if we fix district d_i and we have $T_i = 900$ with $T_i^A = 400, T_i^B = 350$ and $T_i^C = 150$, then $W_i^A = 400 - 350 = 50$ and $50 \to C$ (A still wins). $W_i^B = 350$ and $W_i^C = 150$, since A won.

In general, W_i^A is given by the following formula,

$$W_i^A = T_i^A - S_i^A \max\{\lceil \frac{T_i}{3} \rceil, T_i^B, T_i^C\}. \tag{12}$$

We then only need to know whether we are in case (a) or case (b) to allocate the wasted votes via the mentioned procedures.
For both cases[10] we compute EG^A as follows

$$EG^A = \sum_{i=1}^{S} \frac{W_i^A - \frac{1}{2}(W_i^B + W_i^C)}{T} = \frac{W^A - \frac{1}{2}(W^B + W^C)}{T},$$

where in each district d_i, we compute W_i^P according to Equation 12.
More generally, let $|\mathcal{P}| = n \geq 3$. Write $\mathcal{P} = \{P_1, \ldots, P_n\}$, we again consider two cases.

(a) Party P_1 wins in district d_i ($T_i^{P_1} > \frac{T_i}{n}$) and all other parties have less than $\frac{T_i}{n}$ votes ($T_i^{P_j} < \frac{T_i}{n}$ for $2 \leq j \leq n$). Then we divide the wasted votes for party P_1 such that $T_i^{P_j} \leq \lceil \frac{T_i}{n} \rceil$ for $2 \leq j \leq n$. The number of wasted votes by P-voters in district d_i is then given by

$$W_i^P = \begin{cases} T_i^P - \lceil \frac{T_i}{n} \rceil, & \text{if } d_i \in \mathcal{D}^P \\ T_i^P, & \text{if } d_i \notin \mathcal{D}^P \end{cases} \tag{13}$$

$$= T_i^P - S_i^P \lceil \frac{T_i}{n} \rceil. \tag{14}$$

(b) Party P_1 wins in district d_i ($T_i^{P_1} > \frac{T_i}{n}$) and some other (definitely not all) parties have more than $\frac{T_i}{n}$ votes, say $T_i^{P_j} > \frac{T_i}{n}$ for $2 \leq j \leq m$ and $m < n$. Then the number of wasted votes for P_1 is defined as the difference between $T_i^{P_1}$ and $T_i^{P_2}$ if P_2 has the second most votes. These votes are then allocated to the party with fewest votes. We then have

$$W_i^P = \begin{cases} T_i^P - T_i^K, & \text{if } d_i \in \mathcal{D}^P \cap \mathcal{D}_2^K \\ T_i^P, & \text{if } d_i \notin \mathcal{D}^P \end{cases}$$

$$= T_i^P - S_i^P \sum_{K \in \mathcal{P} \setminus \{P\}} X_i^K T_i^K,$$

where we use the same definition of $\mathcal{D}_2^P$ and X_i^P as in Equation 11.

Again, in both cases, W_i^P is given by

$$W_i^P = T_i^P - S_i^P \max_{K \in \mathcal{P} \setminus P} \{\lceil \frac{T_i}{n} \rceil, T_i^K\}, \tag{15}$$

and we compute EG^P as follows

$$EG^P = \sum_{i=1}^{S} \frac{W_i^P - \frac{1}{n-1} \sum_{K \in \mathcal{P} \setminus \{P\}} W_i^K}{T} = \frac{W^P - \frac{1}{n-1} \sum_{K \in \mathcal{P} \setminus \{P\}} W^K}{T}.$$

[10] There are no other cases: if A won, hence $T_i^A > \frac{T_i}{3}$, we can not have that $T_i^B, T_i^C > \frac{T_i}{3}$, since then we would have $T_i = T_i^A + T_i^B + T_i^C > \frac{T_i}{3} + \frac{T_i}{3} + \frac{T_i}{3} > T_i$. We exclude the case $T_i^A = T_i^B = T_i^C = \frac{T_i}{3}$, since there is no winner here.

Remark 1. It is easily verifiable that for $|\mathcal{P}| = 2$ Extensions 2 and 3 coincide with the original definition of EG. However, Extension 1 is not really an extension, as can be concluded from Footnote 7 and the fact that a wasted vote vanishes instead of going to another party.

Remark 2. One may wonder whether Extension 3(b) is well-defined. That means, does the winning party still win after the wasted votes for that party have been assigned to another party? The following lemma tells us that this is indeed the case.

Lemma 2. The allocation will not result in a new winner.

Proof. Say, w.l.o.g., that in district d_i we have $T_i^{P_1} > T_i^{P_2} > \ldots > T_i^{P_{n-1}} > T_i^{P_n}$, so P_1 has won. Assume that the allocation in Extension 3(b) results in a new winner. This winner can only be P_n, since that party receives the wasted votes of P_1. So $T_i^{P_n} + W_i^{P_1} > T_i^{P_1}$, i.e. $T_i^{P_n} + (T_i^{P_1} - T_i^{P_2}) > T_i^{P_1}$. This would mean $T_i^{P_n} - T_i^{P_2} > 0$, but $T_i^{P_2} > T_i^{P_n}$, hence a contradiction. □

Remark 3. Other allocations of the wasted votes for the winning party do not necessarily have this winner-preserving property. For example, in a district with 100 voters, there are 40 votes for party A, 35 for party B and 25 for party C. According to Extension 3, we are in case b, hence party A has 5 wasted votes. If these wasted votes were to be allocated to party B, party B would have a total of 40 votes and party A would have 35, resulting in a new winner.

Evaluation. We have proposed three different ways to define the efficiency gap for any finite number of parties. There are some important aspects to regard when using these different definitions in practice. Firstly, it will be useful that these generalizations give rise to a well-defined extension, in the sense that the restriction to $n = 2$ parties coincides with the original definition of the efficiency gap (see Definition 13). This is a property that the first generalization lacks, as discussed in Remark 1. Secondly, the procedure of allocating the wasted votes of the winning party must preserve the winning party. Lemma 2 implies that this is the case for Extension 3b. Extensions 2 and 3a also fulfill this property. Finally, a straightforward generalization is mostly one that is intuitive. As intuition is usually open to discussion and not as dogmatic as the two properties discussed above, there is still a small preference for Extension 3 when it comes to this point. Although Extension 2 has its own benefits as explained before, the disadvantages would be the complexity of computing each EG^P and the lack of simplicity and intuition, in contrary to Extension 3.
We will show two examples using Extension 2: one where the distribution of votes is proportional to the distribution of districts won by a party (the proportional case) and one where there is full disproportionality between these two distributions (the extreme case). We will see that in particular in these extreme cases, the efficiency gap is a good indication.

Example 4. The Proportional Case: Say there is a country with $150,000$ voters ($T = 150,000$), 50 districts ($S = 50$) and three parties that can be voted on ($\mathcal{P} = \{A, B, C\}$). The distribution of votes in districts is as follows:

- in districts d_1 to d_{15}: $T_i^A = 2,000$ and $T_i^B = 1,000$
- in districts d_{16} to d_{20}: $T_i^A = 2,000$ and $T_i^C = 1,000$
- in districts d_{21} to d_{35}: $T_i^A = 1,000$ and $T_i^B = 2,000$
- in districts d_{36} to d_{40}: $T_i^A = 1,000$ and $T_i^C = 2,000$
- in districts d_{41} to d_{50}: $T_i^C = 3,000$.

We made the simplifying assumption that votes are evenly distributed among districts: $T_i = \frac{T}{S} = 3,000$, for all $1 \leq i \leq 50$. Simple math shows that $T^A = 60,000$, $T^B = 45,000 = T^C$ and $|\mathcal{D}^A| = 20$, $|\mathcal{D}^B| = 15 = |\mathcal{D}^C|$, i.e. there is full proportionality between the number of votes and number of districts won by each party. To compute EG^P for each $P \in \mathcal{P}$, we need to compute ${W_i}_K^P$ for each $P, K \in \mathcal{P}$, $P \neq K$ and $i \in \{1, \ldots, 50\}$. For districts d_1 to d_{15} we have ${W_i}_B^A = 500$, ${W_i}_C^A = 1,000$, ${W_i}_A^B = 1,000$, ${W_i}_C^B = 1,000$, ${W_i}_A^C = {W_i}_B^C = 0$. Hence

$$\begin{aligned} EG_B^A &= \sum_{i=1}^{50} \frac{{W_i}_B^A - {W_i}_A^B}{105,000} \\ &= \frac{1}{105,000}(15(500 - 1,000) + 5(1,000 - 0) + 15(1,000 - 500) \\ &+ 5(1,000 - 0) + 10(0 - 0)) \\ &= \frac{10,000}{105,000} = \frac{10}{105} \approx 0.10. \end{aligned}$$

Analogously,

$$EG_C^A = \sum_{i=1}^{50} \frac{{W_i}_C^A - {W_i}_A^C}{105,000} = \frac{15,000}{105,000} \approx 0.14$$

and

$$EG_C^B = \sum_{i=1}^{50} \frac{{W_i}_C^B - {W_i}_B^C}{90,000} = \frac{5,000}{90,000} \approx 0.06.$$

Since $EG_K^P = -EG_P^K$ for $P, K \in \mathcal{P}$, we find that $EG^A = \frac{1}{2}(EG_B^A + EG_C^A) \approx 0.12$, $EG^B \approx -0.02$ and $EG^C \approx -0.10$.
The threshold for the efficiency gap in the original (two-party) case is 0.08, but here $|EG^A|, |EG^C| > 0.08$.[11] So even proportionality, the fairest distribution of (wasted) votes, will not always result in a low efficiency gap. However, this is not due to this choice of the extension, since the same problem occurs with the original notion for the efficiency gap with two parties [5].

[11] A question is whether this threshold makes sense in the case with three (or even more) parties. This is a relevant question, although we do not have the means to verify this.

Example 5. The Extreme Case: Our fictional country with three parties $\mathcal{P} = \{A, B, C\}$ that can be voted on, has the following properties: $T = 20,000$, $S = 20$, $T_i = \frac{T}{S} = 1,000$ for all $1 \leq i \leq 20$ and the following vote distribution:

- in districts d_1 to d_{10}: $T_i^A = 334$, $T_i^B = 333$, $T_i^C = 333$
- in districts d_{11} to d_{20}: $T_i^A = 0$, $T_i^B = 499$, $T_i^C = 501$,

so that $T^A = 3,340$, $T^B = 8,320$, $T^C = 8,340$, $|\mathcal{D}^A| = 10$, $|\mathcal{D}^B| = 0$ and $|\mathcal{D}^C| = 10$, i.e. this is a really unfair seat/district distribution. We compute W_{iK}^P again for $P \neq K$, $1 \leq i \leq 20$, and from this we find $EG^A \approx -0.61$, $EG^B \approx 0.51$ and $EG^C \approx 0.10$. These are relatively large values, certainly exceeding the threshold. The efficiency gap indicates that this outcome is really unfair to party B and advantageous to party A, as can be observed from the disproportionality between T^P and $|\mathcal{D}^P|$.

Examples 4 and 5 confirm a relevant comment from [14] :

"The (generalized) efficiency gap is a measure, not a test."

By this we mean that the efficiency gap is one of the tools that can be used to indicate potential gerrymandering. We can not solely use the efficiency gap as a test, as we saw that in the proportional case the efficiency gap still exceeds the threshold (this was already the case for $|\mathcal{P}| = 2$). For extreme cases, like in Example 5, the efficiency gap tells us that this voting distribution is extremely biased towards favoring one particular party, which refers directly to potential gerrymandering. Hence, the efficiency gap should be used as one of the measures for gerrymandering detection and, together with other measures that we will discuss, will serve as methods to make conjectures regarding the presence or absence of political manipulation in the form of gerrymandering.

3.3 Efficiency Gap and Proportionality

Example 4 in the case of Extension 2 with $|\mathcal{P}| = 3$ and in the case of $|\mathcal{P}| = 2$ indicated that proportionality will not always result in an efficiency gap that is close to 0. It would be convenient to formalize this relation between the efficiency gap and proportional representation. Therefore, we first strive to find a formula that represents the direct relation between the efficiency gap and the difference between the vote share and district share in the case of two parties. We will work with $\mathcal{P} = \{A, B\}$ and EG as in Definition 13. We define $\mathcal{T}^B = \frac{T^B}{T}$ and $\mathcal{S}^B = \frac{S^B}{S}$, where we denote by S^B the number of districts won by party B. Furthermore, define $\mathcal{T}_d^B = \mathcal{T}^B - \frac{1}{2}$ and $\mathcal{S}_d^B = \mathcal{S}^B - \frac{1}{2}$. $\mathcal{T}^B$ can be seen as the fraction of votes for B and $\mathcal{T}_d^B$ as the *deviation of* $\frac{1}{2}$ of the fraction of votes for B, and analogously for $\mathcal{S}^B$ and $\mathcal{S}_d^B$, with votes being replaced by districts.

Lemma 3. Assume $T_i = \frac{T}{S}$, $1 \leq i \leq S$, and $T >> S$, then $EG \approx \mathcal{S}_d^B - 2\mathcal{T}_d^B$.

Proof. Define $\epsilon := \lceil \frac{T}{2S} \rceil - \frac{T}{2S}$.

$$\begin{aligned}
EG &= \frac{W^A - W^B}{T} \\
&= \sum_{i=1}^{S} \frac{W_i^A - W_i^B}{T} \\
&= \sum_{i=1}^{S} \frac{(T_i^A - S_i^A \lceil \frac{T_i}{2} \rceil) - (T_i^B - S_i^B \lceil \frac{T_i}{2} \rceil)}{T} \\
&= \sum_{i=1}^{S} \frac{(T_i^A - S_i^A \lceil \frac{T}{2S} \rceil) - (T_i^B - S_i^B \lceil \frac{T}{2S} \rceil)}{T} \\
&= \sum_{i=1}^{S} \frac{T_i^A - T_i^B}{T} - \sum_{i=1}^{S} \frac{S_i^A - S_i^B}{T} (\frac{T}{2S} + \epsilon) \\
&= \sum_{i=1}^{S} \frac{T_i^A - T_i^B}{T} - \frac{1}{2} \sum_{i=1}^{S} \frac{S_i^A - S_i^B}{S} - \sum_{i=1}^{S} \frac{(S_i^A - S_i^B)\epsilon}{T} \\
&= \frac{T^A - T^B}{T} - \frac{1}{2} \frac{S^A - S^B}{S} - (S^A - S^B) \frac{\epsilon}{T} \\
&\approx \frac{T^A - T^B}{T} - \frac{1}{2} \frac{S^A - S^B}{S} \\
&= \frac{1}{2} \frac{S^B - S^A}{S} - \frac{T^B - T^A}{T} \\
&= \frac{1}{2} (\frac{2S^B}{S} - 1) - (\frac{2T^B}{T} - 1) \\
&= \frac{S^B}{S} - \frac{1}{2} - 2(\frac{T^B}{T} - \frac{1}{2}) \\
&= \mathcal{S}_d^B - 2\mathcal{T}_d^B.
\end{aligned}$$

where the fourth equality follows from $T_i = \frac{T}{S}$ and the ninth equality from $S^A + S^B = S$ and $T^A + T^B = T$. The approximation follows from the assumption[12] $S >> T$, hence $\frac{S}{T} \approx 0$ and thus $(S^A - S^B)\frac{\epsilon}{T} \approx 0$, since $0 < \epsilon < 1$. □

$\mathcal{S}_d^B \approx \mathcal{T}_d^B$ means that parties are represented proportionally to their vote share, but from Lemma 3 we infer the important relation

$$EG \approx 0 \Leftrightarrow \mathcal{S}_d^B \approx 2\mathcal{T}_d^B.$$

This relation tells us that, as long as both parties do not come close to having half of the votes and half of the districts ($\mathcal{S}_d^B = \mathcal{T}_d^B = 0$), a low efficiency gap implies a disproportional outcome, and vice versa. Another relation that can

[12] This is a realistic assumption; in most real life scenarios the number of voters per districts is quite large, so $\frac{S}{T} = \frac{1}{T_i}$ is quite small.

be inferred from Lemma 3 is

$$\mathcal{S}_d^B \approx \mathcal{T}_d^B \Leftrightarrow EG \approx -\mathcal{T}_d^B,$$

which means that, at proportionality, the efficiency gap is close to $-\mathcal{T}_d^B$, i.e. the deviation of $\frac{1}{2}$ of the fraction of votes for party A.
For $|\mathcal{P}| = n$, let $\mathcal{P} = \{P_1, \ldots, P_n\}$ and define $\mathcal{T}_d^{P_1} = \frac{T^{P_1}}{T} - \frac{1}{n}$ and $\mathcal{S}_d^{P_1} = \frac{S^{P_1}}{S} - \frac{1}{n}$.[13]

Lemma 4. Consider Extension 3. Suppose that $|\mathcal{P}| = n$ and assume again that $T_i = \frac{T}{S}$, where $T >> S$. Assume furthermore that the outcome is as in case (a). Then $EG^{P_1} \approx \frac{1}{n-1}(n\mathcal{T}_d^{P_1} - \mathcal{S}_d^{P_1})$.

Proof. Define $\epsilon := \lceil \frac{T}{nS} \rceil - \frac{T}{nS}$. By Equation 14

$$\begin{aligned}
EG^{P_1} &= \sum_{i=1}^{S} \frac{1}{T}[(T_i^{P_1} - S_i^{P_1}\lceil\frac{T_i}{n}\rceil) - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}}(T_i^K - S_i^K\lceil\frac{T_i}{n}\rceil)] \\
&= \sum_{i=1}^{S} \frac{T_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} T_i^K}{T} - \sum_{i=1}^{S} \frac{(S_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S_i^K)\lceil\frac{T}{nS}\rceil}{T} \\
&= \sum_{i=1}^{S} \frac{T_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} T_i^K}{T} - \sum_{i=1}^{S} \frac{(S_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S_i^K)}{T}(\frac{T}{nS} + \epsilon) \\
&= \sum_{i=1}^{S} \frac{T_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} T_i^K}{T} - \frac{1}{n}\sum_{i=1}^{S} \frac{S_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S_i^K}{S} \\
&- \sum_{i=1}^{S} \frac{(S_i^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S_i^K)\epsilon}{T} \\
&= \frac{T^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} T^K}{T} - \frac{1}{n}\frac{S^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S^K}{S} - (S^{P_1} \\
&- \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S^K)\frac{\epsilon}{T} \\
&\approx \frac{T^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} T^K}{T} - \frac{1}{n}\frac{S^{P_1} - \frac{1}{n-1}\sum_{K\in\mathcal{P}\setminus\{P_1\}} S^K}{S}
\end{aligned}$$

[13] Note that we define these notions now in terms of P_1, our first party in the list, instead of B, the second party of $\{A, B\}$.

$$
\begin{aligned}
&= \frac{T^{P_1}}{T} - \frac{1}{n-1}(1 - \frac{T^{P_1}}{T}) - \frac{1}{n}(\frac{S^{P_1}}{S} - \frac{1}{n-1}(1 - \frac{S^{P_1}}{S})) \\
&= \frac{n}{n-1}\frac{T^{P_1}}{T} - \frac{1}{n-1} - \frac{1}{n}(\frac{n}{n-1}\frac{S^{P_1}}{S} - \frac{1}{n-1}) \\
&= \frac{n}{n-1}(\frac{T^{P_1}}{T} - \frac{1}{n}) - \frac{1}{n-1}(\frac{S^{P_1}}{S} - \frac{1}{n}) \\
&= \frac{n}{n-1}\mathcal{T}_d^{P_1} - \frac{1}{n-1}\mathcal{S}_d^{P_1} \\
&= \frac{1}{n-1}(n\mathcal{T}_d^{P_1} - \mathcal{S}_d^{P_1}).
\end{aligned}
$$

□

Corollary 1. For Extension 3, case (a), we have that $EG^{P_1} \approx 0 \Leftrightarrow \mathcal{S}_d^{P_1} \approx n\mathcal{T}_d^{P_1}$. Hence, if there are more parties, an efficiency gap close to 0 results in less proportionality.

3.4 Efficiency Gap for More Voting Rules

For two parties or alternatives $\{A, B\}$, the only deterministic voting rule that is neutral and anonymous is the plurality rule. Hence, this is a naturally voting rule when talking about an election with two competing parties. When we have more alternatives however, there are many more deterministic voting rules that are neutral and anonymous as well (two properties that we want a voting rule to have for it being considered fair). We have seen some examples before. The most trivial one is still the plurality rule applied to any finite number of alternatives. All results and definitions for the efficiency gap we have seen so far, apply to this rule. It is also interesting to define the (generalized) efficiency gap when the voting rule in dispute is something other than the plurality rule. The reasoning behind this will be clarified later. Since many deterministic, neutral and anonymous voting rules defined for any finite number of parties reduce to the plurality rule when we consider only two parties, we will straightaway consider three parties and from this extend our definition to any finite number of parties.

Let $\mathcal{P} = \{A, B, C\}$. Recall that a ballot is an ordering over the alternatives, for instance $A \succ B \succ C$, and that the Borda rule is a positional scoring rule that assigns points to the ballot just mentioned as follows: A gets 2 points, B gets 1 point and C gets 0 points. Suppose that we are in district d_i. Let T_i be the number of votes (ballots) in district d_i. Each alternative receives a number of points according to the Borda rule and the alternative with the most points among all the votes in district d_i is the winner in district d_i. Let H_i^P be the total number of points for party P in district d_i and $H_i = H_i^A + H_i^B + H_i^C$ the total number of points (not ballots or votes) assigned in district d_i. We have that $T_i = \frac{H_i}{3}$, since each vote gives three points in total.

We can define the (generalized) efficiency gap equivalently for the Borda rule by simply considering each *point* for a party as a *vote* for that party. With this convention, every Borda winner is a plurality winner (a party wins in a district if it has the largest number of votes among all parties in that district). Hence, the generalizations of the efficiency gap when the applied voting rule is the Borda rule are the generalizations as seen before (1, 2 or 3), but with the substitution $T_i^P \to H_i^P$ and $T_i \to H_i$. For example, Extension 3 translates to

(a)

$$\begin{aligned}\overline{W_i^A} &= \begin{cases} H_i^A - \frac{H_i}{3}, & \text{if } d_i \in \mathcal{D}^A \\ H_i^A, & \text{if } d_i \notin \mathcal{D}^A \end{cases} \\ &= H_i^A - S_i^A \frac{H_i}{3},\end{aligned}$$

where $\overline{W_i^A}$ is the number of wasted points for party P.

(b)

$$\begin{aligned}\overline{W_i^A} &= \begin{cases} H_i^A - H_i^B, & \text{if } d_i \in \mathcal{D}^A \cap \overline{\mathcal{D}^B} \\ H_i^A - H_i^C, & \text{if } d_i \in \mathcal{D}^A \cap \overline{\mathcal{D}^C} \\ H_i^A, & \text{if } d_i \notin \mathcal{D}^A \end{cases} \\ &= H_i^A - S_i^A(X_i^B T_i^B + X_i^C T_i^C),\end{aligned}$$

where X_i^P is defined analogously to (11).

More generally,

$$\overline{W_i^A} = H_i^A - S_i^A \max\{\lceil \frac{H_i}{3} \rceil, H_i^B, H_i^C\}. \tag{16}$$

We then have

$$\overline{EG^A} = \sum_{i=1}^{S} \frac{\overline{W_i^A} - \frac{1}{2}(\overline{W_i^B} + \overline{W_i^C})}{H} = \frac{\overline{W^A} - \frac{1}{2}(\overline{W^B} + \overline{W^C})}{H},$$

where we compute $\overline{W_i^A}$ using Equation 16 and $H = \sum_{i=1}^{S} H_i$ is the total number of points.

Example 6. In a surprisingly small country with one district (which we denote by district 1) and 8 voters we have the following ballots (with multiplicity indicated)

- $A \succ B \succ C$ (2×)
- $C \succ B \succ A$ (3×)
- $B \succ A \succ C$ (1×)
- $B \succ C \succ A$ (2×)

Then $H_1 = 24$, $T_1 = 8$ and furthermore A has 5 points, B has 11 points and C has 8 points, so party B wins the district (hence wins the election).
If we consider Extension 3, it is easy to see that we are in case (a), hence $W_1^A = 5$, $W_1^C = 8$ and $W_1^B = 11 - \frac{24}{3} = 3$ wasted points. Then

$$\overline{EG^A} = \frac{5 - \frac{1}{2}(3+8)}{24} = -\frac{1}{48} \approx -0.02$$

and $\overline{EG^B} = -\frac{7}{48} \approx -0.15$ and $\overline{EG^C} = \frac{1}{6} \approx 0.17$.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$, so that $|\mathcal{P}| = n$. It is straightforward to generalize the above construction to n parties, where now the total number of points assigned to parties in one ballot is

$$(n-1) + (n-2) + \ldots + 1 + 0 = \frac{n^2 - n}{2} = \binom{n}{2},$$

and $T_i = \frac{H_i}{\binom{n}{2}} = \frac{2H_i}{n^2-n}, 1 \leq i \leq n$.
We make the important observation that we can define the (generalized) efficiency gap for every positional scoring rule by applying the same idea as we did for the Borda rule: consider each point for a party as a vote for that party. Recall that $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ is the scoring vector, i.e. $s = (n-1, n-2, \ldots, 0)$ for the Borda rule. Fix a positional scoring rule with a scoring vector s, e.g. Borda rule or (anti)plurality rule. In a ballot $\succ$, the alternative positioned at the i-th place according to that ballot gets s_i points. Denote by $\succ_P$ the number of points party P gets from a ballot $\succ$, i.e. $\succ_P = s_i$ if party P is positioned at the i-th place according to ballot $\succ$. Denote furthermore by $\succ^{i,j}$ the j-th ballot in district d_i for $j = 1, \ldots, T_i$ and $i = 1, \ldots, S$. Then $H_i^P = \sum_{j=1}^{T_i} \succ_P^{i,j}$ is the number of points for party P in district d_i and we have the following relation between T_i and H_i: $T_i = \frac{H_i}{\overline{s}}$ where $\overline{s} = \sum_{i=1}^{n} s_i$.
Obviously, the plurality rule is a positional scoring rule with $s = (1, 0, \ldots, 0)$. It is not hard to see that the proposed generalization to any arbitrary positional scoring rule will reduce to the efficiency gap of the plurality rule using this scoring vector, as $T_i = H_i$.

3.5 Statistical Tests using Efficiency Gap

A typical way in (statistical) science to prove the presence or absence of a certain phenomenon is by using statistical hypotheses, as explained in Chapter 2. Here, we would like to infer whether gerrymandering took place in a certain election, using the outcome of that election[14]. This outcome includes each vote in a country and whether that vote is for party A or B.
The null hypothesis is typically used for indicating the default position where nothing significantly different is happening. On the contrary, the alternative

[14]By 'gerrymandering in an election' we mean that the district borders have been gerrymandered before the election to alter the outcome of that election.

hypothesis usually indicates that something is happening. Hence, in this case it makes sense to construct the null hypothesis H_0 and the alternative hypothesis H_1 as follows:

H_0 : The outcome of the election has not been affected by gerrymandering.

H_1 : The election took place in a gerrymandered map.

We would like to find a test that tells us exactly when to reject H_0 and thereby strengthen the conjecture of gerrymandering, based on the results of the election. For such a test it is beneficial to have small probabilities of type I and type II errors. Recall from Chapter 2 that $\alpha := \mathbb{P}(\text{type I error})$ and $\beta := \mathbb{P}(\text{type II error})$. The Neyman–Pearson Lemma, Lemma 1 from Chapter 2, ensures α and β that are minimal in the NP sense[15]. It tells us to construct a region $\mathcal{R}$, so that, if the evidence E (which are the election results) lies in the region $\mathcal{R}$, we take action A_1, otherwise action A_0. A_0 is the action that corresponds to a true null hypothesis and is "leaving the districts intact". A_1 corresponds to a rejection of the null hypothesis and is "rearranging the districts" (in later chapters we will go deeper into this topic). Recall that the rejection region $\mathcal{R}$ proposed by Neyman and Pearson is

$$\mathcal{R}_t = \{E \in \mathcal{E} | LR_{H_0,H_1}(E) \leq t\}$$

for a fixed value t, where

$$LR_{H_0,H_1}(E) = \frac{\mathbb{P}(E|H_0)}{\mathbb{P}(E|H_1)}.$$

Both probabilities are quite problematic to compute, certainly because of the "vague" formulation of H_0 and H_1. Another useful way in Bayesian statistics to compute the likelihood ratio is to divide the likelihood ratio in *posterior odds* and *prior odds* as follows:

$$LR_{H_0,H_1}(E) = \frac{\mathbb{P}(E|H_0)}{\mathbb{P}(E|H_1)} = \frac{\mathbb{P}(H_0|E)}{\mathbb{P}(H_1|E)} \times \frac{\mathbb{P}(H_1)}{\mathbb{P}(H_0)}$$

so that we have LR = posterior odds × prior odds. To compute the posterior odds, we would need a more explicit definition of H_0 and H_1. We did however not succeed in constructing H_i, $i = 0, 1$, so that $\mathbb{P}(H_i|E)$ is computable. One way it may be computable is to let H_0 and H_1 only depend on the outcome E, but that would give $\mathbb{P}(H_i|E) \in \{0, 1\}$ for $i = 0, 1$. This can result in dividing by zero, but even if we would use the convention $\frac{a}{0} = \infty$ for any $a \in \mathbb{R}$, we would have that for every t

$$\begin{aligned} LR_{H_0,H_1}(E) \leq t &\Leftrightarrow LR_{H_0,H_1}(E) = 0 \\ &\Leftrightarrow \mathbb{P}(H_0|E) = 0 \\ &\Leftrightarrow \mathbb{P}(H_1|E) = 1, \end{aligned}$$

[15]Minimal in the Neyman–Pearson (NP) sense means $\alpha' < \alpha \Rightarrow \beta' > \beta$ for a pair (α', β') from another test.

so that

$$\begin{aligned}\mathcal{R}_t &= \{E \in \mathcal{E} | \mathbb{P}(H_1|E) = 1\} \\ &= \{H_1\},\end{aligned}$$

where H_1 is a hypothesis that depended solely on the outcome. This is a useless region.

We found out that computing the likelihood ratio to define a suitable region $\mathcal{R}_t$ did not help for finding a procedure to decide on the basis of evidence which of the two actions A_0 and A_1 to take. We can however use the results in this chapter to form another test. Recall that the threshold for gerrymandering detection for the efficiency gap is 0.08 (in the case with two parties). That means that $|EG| > 0.08$ implies caution regarding potential gerrymandering and $|EG| \leq 0.08$ implies that no gerrymandering is assumed to have taken place. From Lemma 3 we deduce the important relation $EG \approx \mathcal{S}^B - 2\mathcal{T}^B + \frac{1}{2}$, with $\mathcal{T}^B = \frac{T^B}{T}$ and $\mathcal{S}^B = \frac{S^B}{S}$. This allows us to conclude that

$$|EG| \lesssim 0.08 \Longleftrightarrow 0.21 + \frac{\mathcal{S}^B}{2} \lesssim \mathcal{T}^B \lesssim 0.29 + \frac{\mathcal{S}^B}{2}, \tag{17}$$

where $\lesssim$ denotes "smaller, equal to or almost equal to", i.e.

$$A \lesssim B \Leftrightarrow (A < B) \vee (A \approx B).$$

This is equivalent to

$$|EG| \lesssim 0.08 \Longleftrightarrow -0.58 + 2\mathcal{T}^B \lesssim \mathcal{S}^B \lesssim -0.42 + 2\mathcal{T}^B. \tag{18}$$

As argued before, the null hypothesis is rejected if the data suggests that it is unlikely for the statement in the null hypothesis to hold. For simplicity, we will fully rely on the aforementioned threshold for the efficiency gap, namely 0.08. This means that we work with the following inference:

$$\begin{aligned}\text{Reject } H_0 &\Leftrightarrow |EG| > 0.08 \\ &\Leftrightarrow \mathcal{T}^{\mathcal{B}} \notin [0.21 + \frac{\mathcal{S}^{\mathcal{B}}}{2}, 0.29 + \frac{\mathcal{S}^{\mathcal{B}}}{2}] \\ &\Leftrightarrow \mathcal{S}^{\mathcal{B}} \notin [-0.58 + 2\mathcal{T}^{\mathcal{B}}, -0.42 + 2\mathcal{T}^{\mathcal{B}}].\end{aligned}$$

This coincides with the following rejection region $\mathcal{R} \subset \mathcal{E}$,

$$\mathcal{R} = \{(s,t) \in \mathcal{E} : 0.29 < t - \frac{s}{2} < 0.21\}, \tag{19}$$

where the evidence $E = (s,t)$ has $s = \mathcal{S}^B$ as first component and $t = \mathcal{T}^B$ as second component. This region tells us that, if $E \in \mathcal{R}$ (which means that $|EG| > 0.08$), we take action A_1 and if $E \notin \mathcal{R}$ (which means that $|EG| \lesssim 0.08$), we take action A_0.

It is clear that there is a strictly positive probability of a type I error , i.e. a false

positive: in some proportional instances there is no gerrymandering (H_0 is true), while we reject H_0 ($|EG| > 0.08$). For example, if $\mathcal{S}^B = 0.8$, which means that party B won in 80% of the total districts, then $|EG| \lesssim 0.08 \Leftrightarrow 0.61 \leq \mathcal{T}^B \lesssim 0.69$. However, full proportionality ($\mathcal{T}^B = 0.08$) yields that $|EG| > 0.08$. We saw a similar scenario in Example 4. There is also a positive probability of a type II error, i.e. a false negative.
The Neyman–Pearson test did not help for finding a suitable rejection region $\mathcal{R}$, but the test based on the threshold for the efficiency gap and Equation 17 gave us a suitable region $\mathcal{R}$ as defined in 19. We only need evidence consisting of the fraction of the votes a party has and the fraction of the districts a party won. We do however still have that $\alpha, \beta > 0$ and those probabilities are not minimal in the NP sense. Therefore, we conclude again that the efficiency gap can not be solely used to indicate or prove gerrymandering. It is mere a helpful tool, along with other measures that will be discussed in the following chapters.

4 Probabilistic Methods

A different approach to indicate potential gerrymandering is to highlight the "unlikeliness" of the outcome. A highly unlikely outcome of an election, meaning that the probability of the outcome *given* the vote distribution is very low, will give rise to the suspicion of authorities having deliberately designed the borders to achieve this outcome. For calculations, or rather approximations, of this probability, we will make use of the Monte Carlo method: an algorithmic method that uses random sampling for numerical results.

4.1 Monte Carlo for Two Parties

In this section we assume that there are two (political) parties competing against each other. The slight difference with the previous section is, that we now consider (legislative) districts as before and *Voting Tabulation Districts* (VTD's). Each voting tabulation district is *fixed*. In this sense, VTD's are never prone to change, whereas districts are. Let $\mathcal{V} = \{x|x \text{ is a VTD}\}$ be the set of VTD's and $\mathcal{D} = \{d_1, \ldots . d_S\}$ be the set of districts, such that we have S legislative districts in the country. The number of VTD's $|\mathcal{V}|$, denoted by V, is at least S, since each district consists of at least one VTD.

Definition 14. A *redistricting plan* is a surjective function $\xi : V \to \mathcal{D}$ that assigns each VTD to a district $d_i \in \mathcal{D}$, such that $\xi^{-1}(d_i) \neq \emptyset$ for all $1 \leq i \leq S$.

See Figure 6 for an illustrative image of a redistricting plan where $|\mathcal{V}| = 7$ and $|\mathcal{D}| = 3$ (this is not by any means a geographic representation).

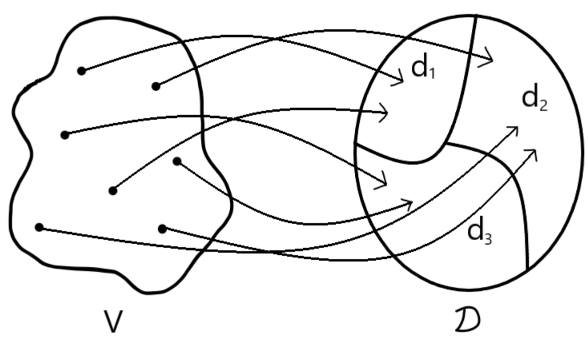


Figure 6: Example of a redistricting function ξ.

$\xi^{-1}(d_i) \subset \mathcal{V}$ is the set of VTD's contained in district d_i for redistricting plan ξ. We denote the set of districts that are now fixed according to redistricting plan ξ by $\mathcal{D}_\xi := \{\xi^{-1}(d_1), \ldots, \xi^{-1}(d_S)\}$. For each $1 \leq i \leq S$, d_i can be seen as the i-th district and $\xi^{-1}(d_i)$ as the set of all VTD's that are assigned to district d_i.

The following lemma tells us how many different redistricting plans there are.[16]

Lemma 5. The number of redistricting plans ξ is

$$\sum_{n=0}^{S}(-1)^n\binom{S}{n}[(S-n)^V]. \tag{20}$$

Proof. Without the requirement that $\xi^{-1}(d_i) \neq \emptyset$ for every $1 \leq i \leq S$, this number is simply S^V: for each VTD (of which there are V), you can choose either of the districts (of which there are S). With the requirement however, there are less such redistricting plans. In all S^V redistricting plans (without the requirement), we counted every function where each district is empty. Hence, we have to subtract all functions where a single district is empty. The number of redistricting plans where d_i is empty, for a $1 \leq i \leq S$, is $(S-1)^V$. Thus the total number of such redistricting plans with the requirement is $S^V - S[(S-1)^V]$. But now we have to add every occasion where a combination of two districts is empty, of which there are $\binom{S}{2}$. The number of redistricting plans where d_i and d_j are empty, for some $1 \leq i < j \leq S$ is $(S-2)^V$. From this total number, we have to subtract again every occasion where a combination of three districts is empty, and so forth. Continuing this alternating process, which is known as the *inclusion-exclusion principle* [15] , we get a total number of

$$\begin{aligned}
&S^V - S[(S-1)^V] + \binom{S}{2}[(S-2)^V] - \binom{S}{3}[(S-3)^V] + \ldots + (-1)^S \cdot 0\\
=&\binom{S}{0}[(S-0)^V] - \binom{S}{1}[(S-1)^V] + \binom{S}{2}[(S-2)^V] - \ldots + (-1)^S\binom{S}{S}[(S-S)^V]\\
=&\sum_{n=0}^{S}(-1)^n\binom{S}{n}[(S-n)^V].
\end{aligned}$$

□

Note that the cardinality of the set of redistricting plans expressed in Lemma 5 is a very large number, as V is usually quite large (in 2012, the American state North Carolina counted a total of 2,500 VTD's [5]). Obviously, not every such assignment of VTD's to the set of districts makes sense. Hence, we want to reduce the set of redistricting plans to a much smaller but more meaningful set. To this end, we need the concept of a *legal redistricting plan*.

Definition 15. A legal (or fair) redistricting plan is a redistricting plan that is geocompact, contiguous and balanced.

By these notions, we mean the following:

[16]In practice, it makes a difference in which district the VTD's are placed, i.e. the permutations of the districts differ from each other. For example, the redistricting plan $\{\{VTD_1, VTD_2\}, \{VTD_3, VTD_4\}$ is essentially different than $\{\{VTD_3, VTD_4\}, \{VTD_1, VTD_2\}\}$. This is because each district has already a fixed "name".

- Geocompactness: this includes the shape of each district. If there is at least one district with a really odd or irregular shape, this would be suspicious and could hint towards potential gerrymandering. We will dedicate a whole section to this aspect and investigate it more mathematically and rigorously. Briefly speaking, a district d_i is *geocompact*, short for geometrically compact, when $\text{gcomp}(d_i) \geq C$ for a threshold $C \in (0,1)$.[17]

- Contiguity: or the connectedness of districts. A legal redistricting plan ensures that each district is coherent, which means that you can travel from each point in that district to any other point in that district without leaving the district. From a graph-theoretical point of view, this means that the subgraph of the whole network corresponding to a district is connected. More precisely, let $G_\xi = (\mathcal{V}, E_\xi)$ be the graph representing the network of the country after applying redistricting plan ξ. $\mathcal{V}$, the set of nodes, is the set of VTD's as before. E_ξ is the edge set consisting of edges that connect two VTD's when they are adjacent in geographical sense and belong to the same district in $\mathcal{D}_\xi$. It means that, for two VTD's j and k, $(j,k) \in E_\xi$ if and only if j and k are adjacent and $\exists i$ such that $j, k \in \xi^{-1}(d_i)$. G_ξ then consists of at least S connected components. A district is coherent if all the nodes corresponding to the VTD's assigned to that district belong to the same connected component. If every district is coherent, i.e. full contiguity in the country, G_ξ consists of exactly S connected components.When there is at least one VTD belonging to a district that is not connected to the other VTD's belonging to that district, one of the obvious reasons will be political manipulation, which we want to manifest.

- Balancedness: meaning that the population distribution among the districts is balanced. This is a property that we assumed multiple times last section to make calculations easier. But most importantly, this is also fair from a political point of view, since we usually do not want to distinguish between the importance of different districts. Each district results in one seat and therefore each seat should represent the same amount of voters. Allowing unequal populations is unfair for people voting in a larger district, as they have relatively less influence on the voting outcome. Hence, we will call a redistricting plan fair or legal if each district has roughly the same population count. More mathematically, let C_i^ξ denote the population count of district $\xi^{-1}(d_i)$. We do not want the population counts from each district to deviate too much from each other. Therefore, we define a threshold for the largest difference. Let $K > 0$ be an integer threshold, then whenever

$$\max_{1 \leq i \leq S} C_i^\xi - \min_{1 \leq i \leq S} C_i^\xi \leq K \tag{21}$$

we say that redistricting plan ξ is balanced.

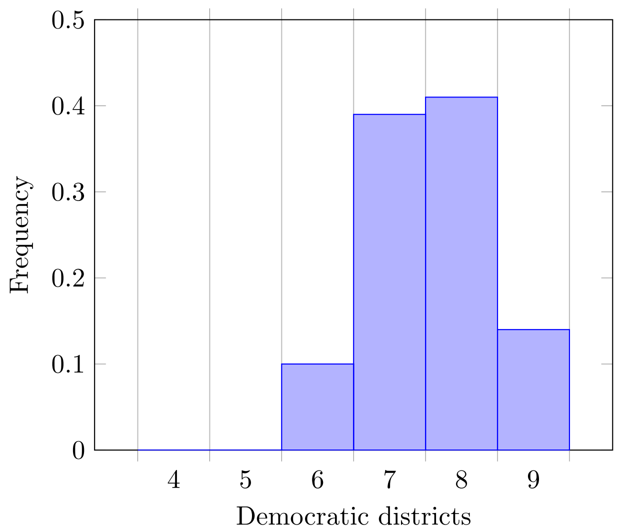
[17] gcomp will be defined later. For now, regard it as a measure of geocompactness.

We are interested in the probability that there is a certain outcome (i.e. number of districts won by parties A and B), given a legal redistricting plan and given a distribution of the votes in the VTD's. It is important that we know the given distribution of votes beforehand, but that we do not know which (legal) redistricting plan was used. This probability is equal to the number of legal redistricting plans that result in the corresponding outcome given the distribution of votes in VTD's, divided by the total number of legal redistricting plans. As mentioned before, the number of redistricting plans expressed in Lemma 5 is usually very large ($C \cdot 10^k$ where $C < 10$ and $k > 100$ is no exception). The aforementioned restrictions reduce the set of redistricting plans, but the set of legal redistricting plans is still too large to do computations with. To compute the numerator in the previously mentioned probability would be already infeasible. However, by simulations we can estimate the probability in dispute when the outcome space is unreasonably large. Here we will make use of a *Monte Carlo method*: take a large sample from the space of feasible outcomes (legal redistricting plans) and count how many times you get a certain (desirable) result. The estimated probability would then be the frequency of the outcome that is investigated. The idea behind this simulation is the following [16]

Bizarre outcomes will be very unlikely.

This could strengthen the conjecture one has regarding the unlikeliness of a certain outcome.

Example 7. An election in North Carolina in 2012, a state with 13 congressional (legislative) districts, between the Democrats and the Republicans resulted in $\pm 51\%$ of the votes for the Democrats (party D) and $\pm 49\%$ of the votes for the Republicans (party R) [5]. This means that $\mathcal{T}^D = \frac{T^D}{T} \approx \frac{1}{2}$ and likewise $\mathcal{T}^R = \frac{T^R}{T} \approx \frac{1}{2}$. A proportional seat distribution, $\mathcal{S}^D \approx \mathcal{S}^R \approx \frac{1}{2}$, would suggest that either R or D won 6 districts and the other one 7. In reality, R won 9 districts and D won 4 districts. This is suspicious, certainly given the fact that the Republicans drew the district boundaries after the 2010 Census.
100 samples drawn uniformly from the space of legal redistricting plans give the result as seen in Histogram 7 [17]. In these simulations, the distributions of votes per VTD are fixed and known beforehand.



In none of the 100 simulated redistricting plans the Democrats have won only 4 districts. Thus, such an event could be regarded as highly unlikely. In more than 80% of the cases, the Democrats have won 7 districts or more.

This Monte Carlo simulation is based on (a corollary of) the Law of Large Numbers: the frequency of an outcome converges to the probability of that outcome, given that the outcome is in the space of legal redistricting plans. More precisely, denote by Ξ the set of redistricting plans and by $\Xi_0 \subset \Xi$ the set of *legal* redistricting plans. Furthermore, let $\mathcal{D}_X^A$ denote the number of districts won by party A using a redistricting plan from set $X \subseteq \Xi$ ($\mathcal{D}_X^A$ is a random variable for a fixed X). Then

$$\frac{\#\text{simulations where } A \text{ won } x \text{ districts}}{\#\text{simulations}} \to \mathbb{P}(\mathcal{D}_{\Xi_0}^A = x),$$

as #simulations $\to \infty$. So, another strategy for gerrymandering detection for a certain election would be:

- Simulate a large number of legal redistricting plans.
- Calculate the frequency of the *real outcome* (hence $\mathcal{D}^A = x$ for a $0 \leq x \leq S$) among all simulations. This is close to the probability of the real outcome.
- Verify whether this frequency is lower than a certain threshold $t \in (0, 1)$.

This verification can also be done by plotting a histogram (as in Histogram 7) and checking whether the bar corresponding to the real outcome has a value below t. The lower the threshold, the stronger the conjecture that this very unlikely outcome is a result of gerrymandering. As for the efficiency gap, this does not *prove* gerrymandering, but could be used as one of the measures for gerrymandering detection.

4.1.1 The Simulation

There are multiple ways to simulate a legal redistricting plan. The trivial way would suggest to generate a function ξ by randomly assigning a $v \in \mathcal{V}$ to a $d \in \mathcal{D}$ for all $v \in \mathcal{V}$. That means that, for any $v \in \mathcal{V}$, we choose a random $d \in \mathcal{D}$ and let $\xi(v) = d$. After doing this, $\xi(v)$ is defined for every $v \in \mathcal{V}$ and takes value in the set of districts $\mathcal{D}$. Hence, we get a function ξ that is not yet surjective. Then, a legal redistricting plan can be simulated by generating functions ξ as above until a function is generated that is surjective, geocompact, contiguous and balanced. The downside is that this is computationally very heavy, certainly if we wish to do a large number of simulations. Since almost every random assignment of VTD's to districts does not create a map in which each district is geocompact, connected and sufficiently large, one would have to generate many such functions ξ in order to simulate a legal redistricting plan. Therefore, we look for another way of assigning VTD's to districts.
We propose Algorithm 1 as a more efficient way to simulate legal redistricting plans.

Algorithm 1. *Construction of* ξ

for $d \in \mathcal{D}$

  choose $v \in V$ randomly;

  count $= 0$;

  $\xi^{-1}(d) = \{v\}$;

  while $\text{pop}(\xi^{-1}(d)) < \frac{T}{S} - \frac{K}{2}$

    for $w \in \delta(\xi^{-1}(d)) \cap V$

      count $\to 0$;

      if $\text{gcomp}(\{\xi^{-1}(d), w\}) \geq C$ and $\text{pop}(\{\xi^{-1}(d), w\}) < \frac{T}{S} + \frac{K}{2}$

        $\xi^{-1}(d) \to \{\xi^{-1}(d), w\}$;

        $\delta(\xi^{-1}(d)) \to \delta(\xi^{-1}(d)) \backslash \{w\}$;

        $V \to V \backslash \{w\}$;

        count $\to 1$;

    if $\delta(\xi^{-1}(d)) == \emptyset$ and $\text{pop}(\xi^{-1}(d)) < \frac{T}{S} - \frac{K}{2}$

      STOP, NO REDISTRICTING PLAN;

    if count $== 0$

      STOP, NO REDISTRICTING PLAN;

A couple of things need to be clarified about this pseudocode.

- $\text{pop}(U)$, where $U \subseteq V$, denotes the population count of a subset of VTD's

U. More formally, $\text{pop}(\{v_1,\ldots,v_n\}) = \sum_{i=1}^n \text{pop}(v_i)$ where $\text{pop}(v_i)$ is the population count of VTD $v_i \in V$.

- $\delta(U)$, where $U \subseteq V$, denotes the set of VTD's that are adjacent to a VTD $v \in U$. More formally, $\delta(\{v_1,\ldots,v_n\}) = \bigcup_{i=1}^n \{w \in V : w \text{ is adjacent to } v_i\}$, where we consider adjacency in the geographical sense.

- $\text{gcomp}(U)$, where $U \subseteq V$, denotes the *ratio of geocompactness* of a subset of VTD's U. This is a value between 0 and 1 and tells us how geocompact a certain district or set of VTD's is. There are multiple ways to define this ratio, we will discuss this in the next chapter.

- $C \in (0,1)$ is a threshold value for the geocompactness. When $\text{gcomp}(U) \geq C$, we can consider U as a geocompact set of VTD's, i.e. the area enclosed by the closed curve that represents the border of U is geocompact. The constant C is dependent of the choice of the function gcomp: $2^V \to [0,1]$.

- $K > 0$ is the integer threshold for the largest difference in population size as encountered in Equation 21.

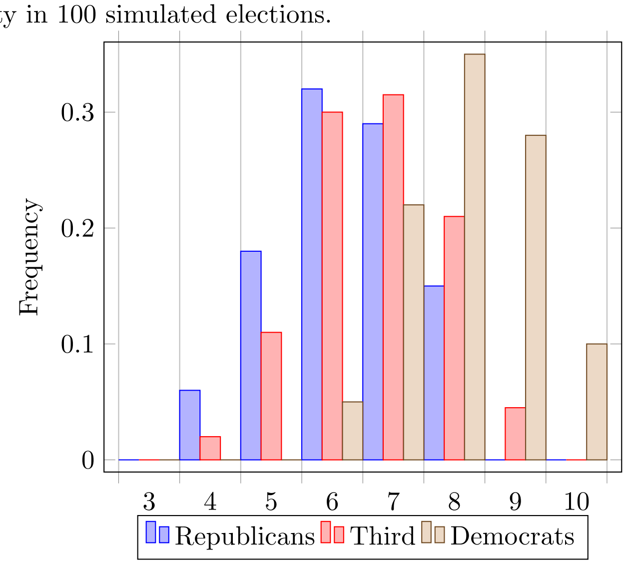
- T is the total number of people (not voters!) in the country and S is the number of districts.

The construction of each district d starts from a random VTD that is not yet assigned to another district. The while loop ensures that Equation 21 always holds for each district. Then we check for each VTD that is adjacent to the set of VTD's that we have assigned so far, whether the addition of this VTD preserves geocompactness and still satisfies Equation 21. If this is the case, we add this VTD and remove it from V and $\delta(\xi^{-1}(d))$, so that it will not be picked again. *count* is a Boolean variable that verifies if such a VTD can be added. When this is not the case, the algorithm stops (by the command STOP, NO REDISTRICTING PLAN) and fails to terminate.
If Algorithm 1 terminates, it does so with $\xi^{-1}(d)$ for each $d \in D$, thus a set of districts $\mathcal{D}_\xi$. By construction, this ξ is a legal redistricting plan. However, in most such constructions it will not terminate, due to *count* staying at the value 0. This happens when, at the construction of $\xi^{-1}(d)$ for a particular $d \in \mathcal{D}$, there is no other VTD left that can be added preserving geocompactness and connectedness. It also fails to terminate when the set of VTD's to be added is empty and the population count in the district is still too low. The amount of times the algorithm fails to terminate is expected to be lower than the amount of times a random assignment of VTD's (as seen before) fails to be a legal redistricting plan. Hence, this method is computationally less heavy than the method described before. The price we have to pay for that is a fairly complicated construction of ξ compared to the ξ that is generated by randomly assigning VTD's to districts.

4.2 Monte Carlo for Finitely Many Parties

This concept can be easily extended to any finite number of parties (while, for now, still using the plurality rule). First, let $\mathcal{P} = \{A, B, C\}$. Again, each VTD is fixed and the definition of a (legal) redistricting plan is analogous. An outcome now consists of the number of districts won by A, B and C. Similarly, sample a large number of elections, using legal redistricting plans, and plot the frequency of the number of districts won by each party in the same histogram. Distinguish different parties by bars of different colors, to keep an overview.[18] When either of the bars corresponding to a real outcome is below a certain threshold, this could again indicate possible gerrymandering.

Example 8. The following results are not from a real election. Imagine that, besides the Democrats (D) and the Republicans (R), there is a third party (T). There is an election in a state with 20 districts and each of these districts contain many VTD's. The outcome from a certain election is such that $\pm 40\%$ from the votes are for D, $\pm 30\%$ are for R and $\pm 30\%$ are for T. Also, following the current redistricting plan, 11 districts are won by R, 6 districts are won by D and 3 districts are won by T. This seems unfair and not proportional at first sight. The following histogram illustrates the frequencies of the districts won by each party in 100 simulated elections.



Again, there are no cases registered where T wins 3 districts and R wins 11 districts (the latter falls outside the scope of the histogram). There are just a few cases, approximately 5, where D wins 6 districts. Hence, such an event can be regarded as highly unlikely. With even more simulations and the same

[18] In this case, we have three bars in each column. For the case $\mathcal{P} = \{A, B\}$, there were not two bars, but only one in each column. This is because the other bar would be redundant, as, for instance, the frequency of the outcome with x Republican districts is the frequency of the outcome with $S - x$ Democratic districts. Thus the frequency of x Republican districts can be deduced from the histogram.

result (or a result that is even less similar to the real outcome), the conjecture of gerrymandering would be even stronger.

The generalization to any number of parties, $|\mathcal{P}| = n \geq 3$, is straightforward. Keep in mind that a histogram will be hectic and cluttered when many parties are considered, thus another form of data representation could be useful in that case.

4.3 Monte Carlo for More Voting Rules

This can also easily be extended to any number of parties *and* any (resolute) voting rule that is neutral.[19] A VTD contains many people that all declare their own ballot, i.e. an ordering over the parties. Using any voting rule, the national result of the election can be decided in two ways.

1. This is when we still want to preserve one winning party for each district. The district winner is the output of the social choice function, with as input the ballots from each voter in that district. More precisely, we say that person j is a voter in district d_i if $j \in d_i$. Then the winning party in district d_i is $F((\succ_j)_{j \in d_i})$, where F is the social choice function and $\succ_j$ the ballot of voter j. The winner in the country can be chosen as the party that won most districts, or via a more advanced system as in the presidential elections in the United States.

2. This is when we want every district to end with a *collective ballot* after there has been voted, that represents the ballots from each voter in that district. Recall from Definition 7 that a *social welfare function* takes a profile of ballots as input, but outputs a full ordering over the alternatives, whereas a social choice function (a voting rule) outputs a set of alternatives. The following lemma guarantees the extension to a social welfare function, so that each district can be represented by a fully ordered ballot.

Lemma 6. Let $F : \mathcal{L}(\mathcal{P})^m \to \mathcal{P}$ be a *neutral* voting rule for any integer m and any set of parties $\mathcal{P}$, then F can be extended to a *social welfare function* $F^* : \mathcal{L}(\mathcal{P})^m \to \mathcal{L}(\mathcal{P})$ such that $F(\succ) = top(F^*(\succ))$ for any $\succ \in \mathcal{L}(\mathcal{P})^m$.

Proof. Let $\succ = (\succ_1, \ldots, \succ_m) \in \mathcal{L}(\mathcal{P})^m$ be a profile of m different ballots, i.e. orderings over $\mathcal{P}$, where $\mathcal{P}$ is the set of parties. Use the voting rule F to select a single winner, i.e. $F(\succ) = P$ for a $P \in \mathcal{P}$. This will be the highest ranked party in the ballot $F^*(\succ)$, hence we have $F(\succ) = P = top(F^*(\succ))$. Define a new profile $\succ^{\mathbf{1}} = (\succ_1^1, \ldots, \succ_m^1)$, where $\succ_i^1$ is the same as ballot $\succ_i$ but with party P removed, for $1 \leq i \leq m$. Hence, $\succ_i^1$ is a ballot over the set of parties $\mathcal{P} \backslash \{P\}$, i.e. $\succ_i^1 \in \mathcal{L}(\mathcal{P} \backslash \{P\})$. Use F again to select a winner from profile $\succ^{\mathbf{1}}$, i.e. $F(\succ^{\mathbf{1}}) = K$ for a $K \in \mathcal{P} \backslash \{P\}$. Note that neutrality of F is required in order to do this,

[19] Recall, that resolute means that the outcome of the voting rule is always exactly one party. However, the applications are usually with such high numbers that with very low probability there will be two or more winners. When not convinced by this argument, use a random tie-breaking rule to always select a unique winner.

since F must work similarly when picking a winner from $\succ^{\mathbf{1}} \in \mathcal{L}(\mathcal{P}\backslash\{P\})^m$ and thus can not have a 'preference' for party P, which is the case when F is neutral. Place K below P in the ballot $F^*(\succ)$. Now define profile $\succ^{\mathbf{2}}$ similarly with ballots ordered over $\mathcal{P}\backslash\{P, K\}$ and proceed iteratively until there is a full ballot $F^*(\succ)$. Thus we have constructed a F^* such that $F^*(\succ) \in \mathcal{L}(\mathcal{P})$ and $F(\succ) = top(F^*(\succ))$, so F^* is a social welfare function and naturally extends F. □

Example 9. In this simplistic scenario there are 4 voters and $\mathcal{P} = \{A, B, C, D\}$ is the set of parties with the following ballots

- $A \succ B \succ C \succ D$
- $D \succ B \succ C \succ A$
- $A \succ C \succ D \succ B$
- $B \succ A \succ C \succ D$.

The Borda winner is party A with 8 points, so A will be the top party in the collective ballot. Removing A from each ballot results in the following four ballots

- $B \succ C \succ D$
- $D \succ B \succ C$
- $C \succ D \succ B$
- $B \succ C \succ D$.

The Borda winner is now party B with 5 points. Removing B from each ballot results in four ballots in which three have the order $C \succ D$ and one $D \succ C$, so C wins. The collective ballot is then $A \succ B \succ C \succ D$.

m voters in a district give rise to a profile $(\succ_j)_{j\in d_i} = (\succ_1, \ldots, \succ_m) \in \mathcal{L}(\mathcal{P})^m$ of m different ballots (if the voters in district d_i are denoted by $1, \ldots, m$). Extend the neutral voting rule F that is applied in the country to a social welfare function F^* as constructed in Lemma 6. The collective ballot that district d_i represents, is $\succ^{d_i} = F^*((\succ_j)_{j\in d_i})$. Then we apply the voting rule F to each ballot represented by a district, i.e. the winner is $P = F((\succ^{d_i})_{1\leq i\leq S})$ for a party $P \in \mathcal{P}$.
In the first case, applying the Monte Carlo method works analogously, since the result is that each district is won by a single party. However, this is infeasible in the second case, as each district is represented by a full ordering over the alternatives instead of a single alternative.

5 Geometry

Designing the shapes of districts specifically to include or exclude certain neighborhoods can result in irregular shaped districts. We have seen in Figure 2 that North Carolina consisted of 13 districts in 2012, where there was one district with a particularly odd shape. This was proven to be indeed the result of gerrymandering. In this chapter we will investigate and quantify the concept of shapes of districts. We will use this as last resource to help determine whether gerrymandering has occurred.

5.1 Geocompactness of a District

Roughly speaking, the geocompactness of a district tells us how similar the shape of a district is to a perfectly geocompact shape, namely a circle. Examples of reasonably geocompact shapes in $\mathbb{R}^2$ are ellipses and regular convex n-gons for n large enough, e.g. $n \geq 5$. An example of a barely geocompact shape in $\mathbb{R}^2$ is a regular pentagram, which is just the shape of a star. We will quantify these shapes below. Note that (usually) the shape of a district has non-intersecting edges, so we consider shapes that are not self-intersecting.
As mentioned in the previous chapter, $\text{gcomp}(U)$ denotes the ratio of geocompactness of a subset of VTDs $U \subseteq V$. There are multiple ways to define this ratio, hence multiple ways to measure the geocompactness of a district.

5.1.1 Polsby-Popper Test

The Polsby-Popper test was developed by lawyers Daniel D. Polsby and Robert Popper [18] and is based on Theorem 3, which is called the *isoperimetric inequality.* In the proof, we make the following assumption.

Assumption 1. The straight line connecting every two points on Jordan curve C has finitely many intersections with C, i.e.

$$\{l(p,q) \cup C : p, q \in C\} < \infty,$$

where $l(p,q)$ is the straight line connecting p and q.

Theorem 3. (*Isoperimetric inequality.*) Let C be a Jordan curve of finite length[20] that satisfies Assumption 1, $L(C)$ be the length of C and $A(C)$ be the area of the interior of C. Then

$$\frac{4\pi A(C)}{L(C)^2} \leq 1 \tag{22}$$

and equality holds whenever C is a circle.

[20]In Chapter 2 we defined what a Jordan curve C is and that we denote the Lebesgue measure $\lambda(\text{int}(C))$ of the interior of C, $\text{int}(C)$, by $A(C)$.

Proof. We will prove the slight reformulation of (22), namely $A(C) \leq \frac{L(C)^2}{4\pi}$ with equality when C is a circle [19].
First of all, let C be a circle. Then C has a radius $R(C)$ and we know that the circumference $L(C)$ is equal to $2\pi R(C)$ and the area $A(C)$ enclosed by C is equal to $\pi R(C)^2$. Then

$$A(C) = \pi R(C)^2 = \frac{4\pi^2 R(C)^2}{4\pi} = \frac{L(C)^2}{4\pi},$$

so equality holds.
We will now prove that, for any closed curve C with length $L(C)$, the closed curve with maximum area is the circle C^* with circumference $L(C^*) = L(C)$. Then

$$A(C) \leq A(C^*) = \frac{L(C^*)^2}{4\pi} = \frac{L(C)^2}{4\pi}$$

for any such C and Theorem 3 is proved.

Lemma 7. $C^* = \underset{C:L(C)=L}{\operatorname{argsup}} A(C)$ is a circle with circumference L.

Proof. This will be proven on the basis of three small claims. Let $A^* = \sup_{C:L(C)=L} A(C)$ be the area of the interior of $C^* = \underset{C:L(C)=L}{\operatorname{argsup}} A(C)$.

Claim 1. $\operatorname{int}(C^*)$ is convex.

Proof. Assume $\operatorname{int}(C^*)$ is not convex. Then there are infinitely many pairs of points p and q on C^* such that the straight line $l(p,q)$ connecting p and q belongs to $\mathbb{R}^2 \backslash \operatorname{int}(C^*)$ (see Figure 7).

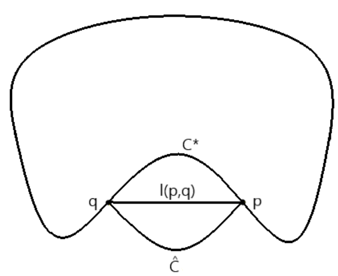


Figure 7: Example of a non-convex C^*.

Pick such a pair p and q such that the piece of the curve C^* connecting p and q, $C^*(p,q)$, reflected with respect to $l(p,q)$ creates another curve $\hat{C}$ that does not intersect C^* itself (this is possible by Assumption 1). Note that $\hat{C}$ also has a length of L. But then

$$A(\hat{C}) > A(C^*) = A^* = \sup_{C:L(C)=L} A(C).$$

□

Claim 2. $\text{int}(C^*)$ can be divided into two equally sized parts with equal perimeters.

Proof. Pick an arbitrary point p on C^*. Now let q be the (unique) point on C^* such that C^* is bisected into C_1^* and C_2^*, where C_1^* is a path over C^* from p to q and C_2^* is the other path over C^* from q to p, and $L(C_1^*) = L(C_2^*)$. Since $L(C_1^*) + L(C_2^*) = L(C^*) = L$, we know that $L(C_1^*) = L(C_2^*) = \frac{L}{2}$. Let $l(p,q)$ be the straight line connecting p and q. Notice that $l(p,q)$ is contained in $\text{int}(C^*)$ by the convexity of $\text{int}(C^*)$. We obtain two new areas A_1^* and A_2^*, where A_i^* is the area of the interior of the closed (Jordan) curve $C_i^* \odot l(p,q)$, for $i = 1, 2$, and $\odot$ denotes the concatenation of two connected curves.

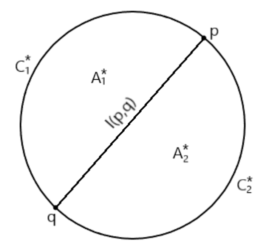


Figure 8: Bisection of $\text{int}(C^*)$.

Then we have that $A_1^* + A_2^* = A^*$ with $A_1^* = A_2^*$. Assume that this is not the case, then without loss of generality $A_1^* > A_2^*$. We can then reflect C_1^* with respect to $l(p,q)$, call the resulting curve $\hat{C}_1$. $C_1^* \odot \hat{C}_1$ is then a closed curve with length L and the area $\hat{A}$ enclosed by $C_1^* \odot \hat{C}_1$ is $2A_1^*$. But then

$$\hat{A} = 2A_1^* > A_1^* + A_2^* = A^* = \sup_{C:L(C)=L} A(C),$$

so this can not happen. □

A *semicircle* is a closed shape consisting of half a circle and a diameter of that circle.

Claim 3. C_1^* and C_2^* as defined in the proof of Claim 2 are semicircles.

Proof. $A_1^* = A_2^*$ by Claim 2, so without loss of generality we will prove that C_1^* is a semi-circle. Assume therefore that C_1^* is not a semi-circle. Then according to Thales Theorem there is a point r on C^* such that the angle $\angle prq$, with p and q as in the proof of Claim 2, is not 90°. We fix the lengths of $l(q,r)$ and $l(p,r)$, but allow $\angle prq$ to change, thereby changing $l(p,q)$ too, in order to maximise the area of $\triangle pqr$. By doing this, the length $L(C_1^*)$ stays the same, but the area of $\triangle pqr$ changes. But $\text{area}(\triangle pqr) = \frac{1}{2} \times \text{length}(l(p,r)) \times \text{length}(l(q,r)) \times \sin(\angle prq)$ is maximised for $\angle pqr = 90°$, contradicting our assumption of allowing $\angle pqr$ to change in order to maximise the area of $\triangle pqr$. So C_1^* is a semi-circle. □

By Claim 3, C^* is a circle with circumference $L(C_1^*) + L(C_2^*) = \frac{L}{2} + \frac{L}{2} = L$, proving Lemma 7. $\square$

$\square$

Definition 16. Let d_i be a district. Then the *Polsby-Popper score* of d_i is

$$PP(d_i) = \frac{4\pi A(d_i)}{L(d_i)^2},$$

where $A(d_i)$ is the area of district d_i and $L(d_i)$ is the perimeter of district d_i.[21]

If we regard the border of a district as a Jordan curve, then the district itself is the interior of the Jordan curve. By the area of the district and perimeter of the border being non-negative and by Theorem 3 we have that $PP(d_i) \in [0, 1]$. This score is a ratio and tells us how geocompact district d_i is: a score close to 1 (looks like the shape of a circle) means that d_i is a geocompact district and a score close to 0 (large perimeter, small area) means that d_i is not a geocompact district. The higher the score, the more the shape looks like a circle and the more geocompact the district is.

5.1.2 Reock Test

Another way to measure geocompactness of a district is to compare the area of the district to the area of the smallest circle containing that district. Denote by S_C^i the set of circles in $\mathbb{R}^2$ containing district d_i and $C_{min}^i = \underset{C \in S_C^i}{\operatorname{argmin}}\, A(C)$ is the smallest circle containing d_i. In Appendix A will be proven that C_{min}^i indeed exists.

Definition 17. Let d_i be a district. Then the *Reock score* [20] of d_i is

$$Reock(d_i) = \frac{A(d_i)}{A(C_{min}^i)},$$

where $A(d_i)$ is the area of district d_i and $A(C_{min}^i)$ is the area of C_{min}^i.

Of course, $A(C_{min}^i) \geq A(d_i)$, so $Reock(d_i) \in [0, 1]$ and is a ratio denoting geocompactness of district d_i as well. A score close to 1 means that the smallest circle containing d_i is approximately as large as d_i itself, hence that implies that d_i is a geocompact district. A score close to 0 indicates a less geocompact district.

[21] The interpretation of $PP(C)$ is the ratio between the area and the perimeter of C. $L(C)$ is squared to compare with the two-dimensional notion of area $A(C)$. 4π is a normalization such that $PP(C) = 1$ for a circle C.

5.1.3 Reverse Reock Test

The method applied for the Reock test can be naturally reversed to create another method for measuring the geocompactness of a district. Instead of comparing the area of a district to the area of the smallest circle containing that district, one can compare it to area of the largest circle contained in that district. Denote by S_i^C the set of circles in $\mathbb{R}^2$ contained in district d_i (note that this is different from S_C^i) and $C_{max}^i = \underset{C \in S_i^C}{\text{argmax}}\, A(C)$ is the largest circle contained in d_i. In Appendix B will be proven that C_{max}^i indeed exists.

Definition 18. Let d_i be a district. Then the *Reverse Reock score* of d_i is

$$RR(d_i) = \frac{A(C_{max}^i)}{A(d_i)},$$

where $A(d_i)$ is the area of district d_i and $A(C_{max}^i)$ is the area of C_{max}^i.

Since $A(d_i) \geq A(C_{max}^i)$, $RR(d_i) \in [0, 1]$ and analogously to Definitions 16 and 17, it is a ratio denoting geocompactness of district d_i.

5.1.4 Examples of the Three Tests

Example 10. Let the Jordan curve C be an ellipse with width $2a$ and height $2b$ (see Figure 9). If the ellipse is centered at the origin, then we have

$$C = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}. \tag{23}$$

The area of the interior of C, which we denote by $A(C)$, can be found formally, but can also be deduced informally from the area of a circle. A circle with radius b has area πb^2, if we scale this by a factor $\frac{a}{b}$ in the direction of the x-axis, we get ellipse C, hence the area is $A(C) = \pi b^2 \frac{a}{b} = \pi ab$. The perimeter $L(C)$ of the ellipse, or in this case the circumference of C, has a slightly more complicated formula and we will give this without deduction:

$$L(C) = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin(\theta)} d\theta$$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$ is the eccentricity of C, assuming $a > b$, that measures the elongation of the ellipse. Then

$$PP(d_i) = \frac{4\pi A(d_i)}{L(d_i)^2} = \frac{4\pi^2 ab}{16a^2[\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin(\theta)} d\theta]^2},$$

if d_i is a district with the shape of the ellipse given by Equation 23. When $a = b$, the ellipse described in Equation 23 is a circle, and indeed $PP(d_i) = 1$. Take now, for example, the values $a = 4$ and $b = 2$ (the unit of these distances can be regarded as kilometers or miles in the context of real world districts). Then a district d_i with the shape of an ellipse with the above values has area and perimeter $A(d_i) = 8\pi$ and $L(d_i) \approx 19.38$, respectively. So the Polsby-Popper score of d_i is

$$PP(d_i) \approx \frac{32\pi^2}{(19.38)^2} \approx 0.84,$$

which is reasonably close to 1.

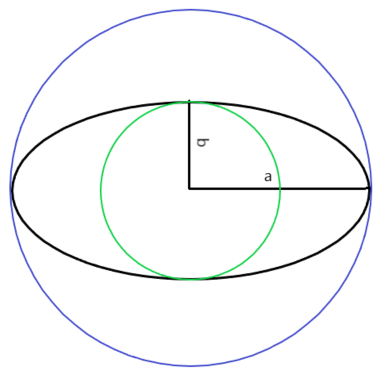


Figure 9: Smallest circle outside and largest circle contained in an ellipse.

The smallest circle containing an ellipse given by Equation 23 is the circle with radius $\max\{a, b\}$ also centered at the origin. If d_i is a district with the shape of the ellipse given by Equation 23, then the Reock score of d_i is

$$Reock(d_i) = \frac{A(d_i)}{A(C^i_{min})} = \frac{\pi ab}{\pi \max\{a, b\}^2} = \frac{\min\{a, b\}}{\max\{a, b\}}$$

In case $a = 4$ and $b = 2$, we have $\text{Reock}(d_i) = \frac{1}{2}$.
Conversely, the biggest circle contained in the ellipse given by Equation 23 is the circle with radius $\min\{a, b\}$ also centered at the origin. If d_i is a district with the shape of the ellipse given by Equation 23, then the Reverse Reock score of d_i is

$$RR(d_i) = \frac{A(C^i_{max})}{A(d_i)} = \frac{\pi \min\{a, b\}^2}{\pi ab} = \frac{\min\{a, b\}}{\max\{a, b\}}.$$

We see that in the case of an ellipse, the Reock and Reverse Reock scores coincide. In case $a = 4$ and $b = 2$, we have $RR(d_i) = \frac{1}{2}$.

Example 11. Let the Jordan curve C be a concave decagon in the shape of a *star*, i.e. a regular pentagram. All the metric properties of the star, including the area of the interior and perimeter of C, can be deduced from the single parameter a, which is the distance between the two points D and F (see Figure 10).

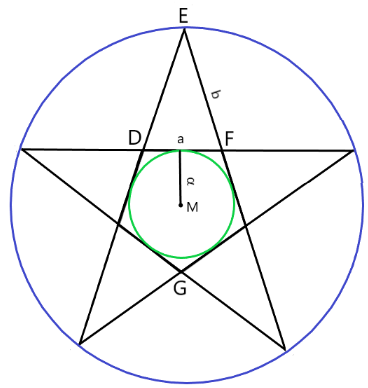


Figure 10: Smallest circle outside and largest circle contained in a star.

The area of the interior of a regular pentagram with parameter a can be found as follows

$$A(C) = A(\text{regular pentagon}) + 5 \times A(\triangle DEF)$$

The area of a regular pentagon with side length a is $5a^2/(4\tan(36))$. Using some elementary geometric computations, it can be shown that b, the length from E to F, is also the length from F to G. According to Ptolemy's Theorem, a and b are *in the golden ratio*, which means that $b/a = \varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The area of $\triangle DEF$ is twice the area of $\triangle DEH$, where H is the midpoint of line DF, see Figure 11.

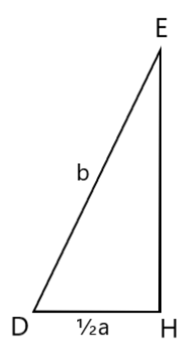


Figure 11: $\triangle DEH$

Using the Pythagorean Theorem and using $b = a(1+\sqrt{5})/2$, we find that EH has length $\frac{1}{2}a\sqrt{(1+\sqrt{5})^2-1}$. Therefore,

$$\begin{aligned}A(\triangle DEF) &= 2A(\triangle DEH) = 2(\frac{1}{2}DH \times EH)\\ &= \frac{1}{2}a \times \frac{1}{2}a\sqrt{(1+\sqrt{5})^2-1} = \frac{1}{4}a^2\sqrt{(1+\sqrt{5})^2-1},\end{aligned}$$

where the second equality follows from Herons formula for the computation of the area of a triangle. Hence, the area of the interior of C is

$$A(C) = \frac{5a^2}{4\tan(36)} + \frac{5}{4}a^2\sqrt{(1+\sqrt{5})^2-1}.$$

Further, since $L(C)$ equals ten times the length of EF, we get $L(C) = 10b = 5(1+\sqrt{5})a$.

A district d_i with the shape of a star C with parameter $a = 5$ has area and perimeter $A(d_i) \approx 139.19$ and $L(d_i) \approx 80.9$, respectively. So the Polsby-Popper score of d_i is

$$PP(d_i) = \frac{4\pi A(d_i)}{L(d_i)^2} \approx \frac{556\pi}{(80.9)^2} \approx 0.27, \tag{24}$$

which is much closer to 0 than to 1.

Writing out the exact formula of the Polsby-Popper score for any a leads us to an important observation:

$$\begin{aligned}PP(d_i) &= \frac{5a^2\pi(\frac{1}{\tan(36)} + \sqrt{(1+\sqrt{5})^2-1})}{(5(1+\sqrt{5}))^2a^2}\\ &= \frac{5\pi(\frac{1}{\tan(36)} + \sqrt{(1+\sqrt{5})^2-1})}{(5(1+\sqrt{5}))^2}.\end{aligned}$$

This outcome is independent of a, hence equal to the outcome in (24). We see that the Polsby-Popper score of a star shaped district is approximately 0.27, for any choice of a.

For convenient calculations, we first start with the computation of the Reverse Reock score of a star-shaped district. The largest circle contained in the interior of C is the inscribed circle of the pentagon inside the star polygon. The radius of this circle is the *apothem* of the pentagon, which is the distance from the center to the midpoint of one of its sides (see Figure 10). The formula for the apothem α of a regular pentagon with side length a is $\alpha = \frac{a}{2}\tan(\frac{3}{10}\pi)$. Hence, the area of the largest circle containing C, the shape of district d_i, is

$$A(C^i_{max}) = \pi\alpha^2 = \pi\frac{a^2}{4}\tan(\frac{3}{10}\pi)^2.$$

A district d_i with the shape of a star C has Reverse Reock score

$$RR(d_i) = \frac{A(C^i_{max})}{A(d_i)} = \frac{\pi\tan(\frac{3}{10}\pi)^2}{5\tan(36)^{-1} + 5\sqrt{(1+\sqrt{5})^2-1}} \approx 0.37.$$

where, again, the outcome is not dependent on the parameter a.
To compute the Reock score, we need to find the area of the smallest circle which circumscribes the star polygon C. The circumscribing circle intersects each of the five endpoints of the star, hence the radius is the distance from the center point to any endpoint. The distance from the center to the top point is equal to the apothem α, as described before, plus the height of the upper triangle. This height is equal to the length EH as found before and equals $\frac{a}{2}\sqrt{(1+\sqrt{5})^2-1}$. Thus the area of the smallest circle with C contained in, where C is the shape of district d_i, is

$$A(C^i_{min}) = \pi(\frac{a}{2}\tan(\frac{3}{10}\pi) + \frac{a}{2}\sqrt{(1+\sqrt{5})^2-1})^2.$$

A district d_i with the shape of a star C has Reock score

$$Reock(d_i) = \frac{A(d_i)}{A(C^i_{min})} = \frac{5\tan(36)^{-1} + 5\sqrt{(1+\sqrt{5})^2-1}}{\pi(\tan(\frac{3}{10}\pi) + \sqrt{(1+\sqrt{5})^2-1})^2} \approx 0.26.$$

An important result is that, for *any* parameter $a > 0$, the Polsby-Popper, Reock and Reverse Reock scores are constants. This is not the case for ellipse shaped districts, see Example 10. We will investigate the dependency on a single parameter, when the shape is defined by two parameters, in the following example.

Example 12. Let the Jordan curve C be a rectangle with height a and width b.[22] We will find out what happens when we keep the height fixed and vary the width of the rectangle. The area of this rectangle is $A(C) = ab$ and the length of C is $L(C) = 2(a+b)$. A district d_i with the shape of a rectangle with parameters a and b has Polsby-Popper score

$$PP(d_i) = \frac{4\pi A(d_i)}{L(d_i)^2} = \frac{4\pi ab}{(2(a+b))^2} = \frac{\pi ab}{(a+b)^2}.$$

The smallest circle containing a rectangle with parameters a and b has radius equal to the distance from the midpoint of the rectangle to one of the four corners. More formally, the radius r is $d(M, H_i)$, where M is the midpoint and H_i is the i-th corner of the rectangle, for $i = 1, 2, 3, 4$ (see Figure 12).

[22] The original notions for the measurements of a rectangle are length and width. However, this can cause ambiguity between this length and the length of the perimeter, so we will use height and width just like for the ellipse in Example 1.

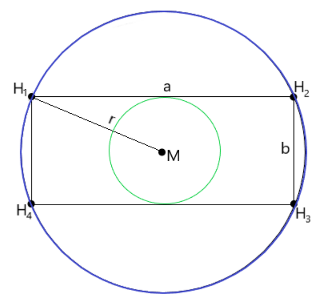


Figure 12: Smallest circle outside and largest circle contained in a rectangle.

This distance is equal to $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}$. Hence, the area of the smallest circle that contains district d_i, where d_i has the shape of rectangle C, is

$$A(C^i_{min}) = \pi r^2 = \pi \frac{a^2+b^2}{4}.$$

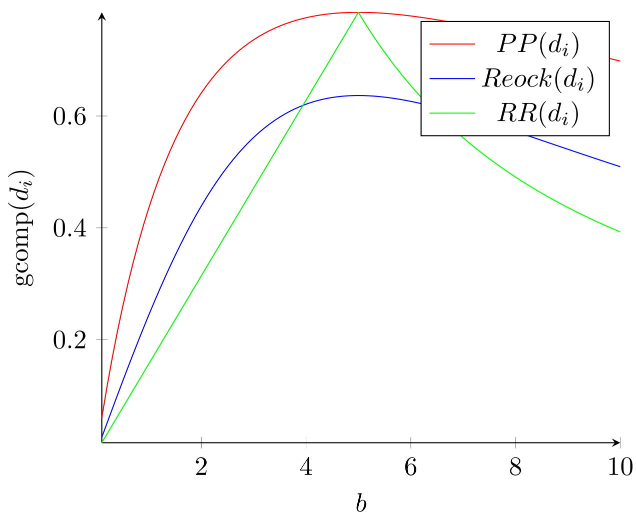
Therefore, district d_i has Reock score

$$Reock(d_i) = \frac{A(d_i)}{A(C^i_{min})} = \frac{ab}{\pi\frac{a^2+b^2}{4}} = \frac{4ab}{\pi(a^2+b^2)}.$$

Conversely, the biggest circle contained in such a rectangle has radius equal to $\frac{1}{2}\min\{a,b\}$, see Figure 12. Hence, the area of the largest circle that is contained in district d_i is $A(C^i_{max}) = \frac{\pi}{4}\min\{a,b\}^2$. Therefore, district d_i has a Reverse Reock score of

$$RR(d_i) = \frac{A(C^i_{max})}{A(d_i)} = \frac{\frac{\pi}{4}\min\{a,b\}^2}{ab} = \frac{\pi}{4} \times \begin{cases} \frac{b}{a}, & \text{if } b \leq a \\ \frac{a}{b}, & \text{if } b > a \end{cases}.$$

As mentioned in the begin of this example, we want to find out how the different scores behave for a different parameter.



The above plot showcases this for a rectangle with fixed height $a = 5$ and varying width b. We immediately see that all scores have a maximum for $b = 5$. This makes sense intuitively, the rectangle with the best geocompact shape is the square ($a = b$). All three scores have different ways of attending the maximum $b = 5$. The kink we observe in the green line is caused by the difference in the RR score for $b \leq a$ and $b > a$. By comparing the red and blue line, we see that C always has a higher Polsby-Popper score than the Reock score (this will be the case for all realisations of a, hence for all rectangles). Another observation is that the Polsby-Popper score and Reverse Reock score coincide at $b = 5$. A square thus always has the same PP and RR score and this score is relatively high (in this case $\pi/4 \approx 0.79$).

Example 13. This example includes no calculations, but indicates the importance of using different measures for geocompactness of a district with an odd or irregular shape. Consider a district which is shaped as a circle, but with a long, narrow cove in each "side", such that it is similar to the shape of a clover, see Figure 13.

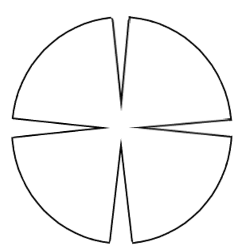


Figure 13: Clover-like shape.

The Reock score is then close to 1, as those coves themselves do not have a large area and the difference between the area of the district and the smallest circumscribing circle is the total area of the four coves. However, the Reverse Reock score is rather low, as there are five possible locations for the largest circle contained in the district: either in one of the four "leafs" of the shape or right in the middle. The area of this circle is significantly smaller than the area of the district itself, therefore having a small Reverse Reock score.
Conversely, consider a district that is shaped as a circle, but now has long, narrow protrusions on each "side". By similar arguments, it follows that the Reverse Reock score is close to 1 and the Reock score is small and this will be smaller for longer protrusions.

5.1.5 The Weighted Average Test

In the figure in Example 12 it is shown that the three scores behave quite differently for the rectangle. In Example 11, the scores were all constant, but the three scores differed from each other. In Example 10, for specific choice of the parameters a and b, the Polsby-Popper score of an ellipse shaped district is quite large, while the Reock and Reverse Reock scores are both exactly a half. The different outcomes between these tests indicate that it is, in most instances, quite important to consider all three scores when constructing a statement or conjecture about the geocompactness of a district. Especially Example 13 showcases this, as the Reock score and Reverse Reock score differ very much for those two particular shapes. However, those three tests have the following desired property in common: they are typically low for non-geocompact shapes (for example a star shape) and typically high for geocompact shapes (for example an ellipse or a rectangle with the right parameters). That is why it may be useful to define the following score when we can only consider one score, i.e. one choice for $\text{gcomp}(d_i)$ (as is the case in the algorithm in the previous chapter).

Definition 19. Let d_i be a district. Then the *Weighted Average Geocompactness score* of d_i is

$$WAG(d_i) = \frac{1}{3}(PP(d_i) + Reock(d_i) + RR(d_i)),$$

where $PP(d_i)$, $Reock(d_i)$ and $RR(d_i)$ are the Polsby-Popper score, Reock score and Reverse Reock score of d_i respectively.

The WAG score serves as an average of the three aforementioned geocompactness scores and has therefore the desired property explained before: the score will be higher for a more geocompact district. We also still have that $WAG(d_i) \in [0, 1]$ and $WAG(d_i) = 1$ if and only if d_i has the shape of a circle. As mentioned in the previous chapter, we will regard a district d_i as a *geocompact district* when $WAG(d_i) \geq C$ for a suitable $C \in (0, 1)$. The geocompactness of a district will be the last measure of gerrymandering detection we will discuss. The key concept behind this is that a district with a "normally structured" shape should be geocompact and a district with a shape created as a result from gerrymandering is typically not geocompact.

5.1.6 Regional Score of Geocompactness

We have found several ways to quantify the measure of geocompactness for each congressional district, after the districts are drawn according to a Census. After fixing a geocompactness score and fixing a suitable $C \in (0, 1)$, each district is classified as geocompact or non-geocompact. It is still disputable how many non-geocompact districts need to be located in a certain nation or region, so that it might suggest that gerrymandering has taken place in that region. A possible way to infer that partisan gerrymandering has been taken place in a certain region, would be to give the region as a whole a geocompactness score. Therefore, we might take the average of the geocompactness scores of all the districts in that region. More mathematically, if $\mathcal{D}$ is the region in dispute, consisting of districts $d_1, \dots, d_S$, then the geocompactness score of region $\mathcal{D}$ is

$$\text{gcomp}(\mathcal{D}) = \frac{1}{S} \sum_{i=1}^{S} \text{gcomp}(d_i), \tag{25}$$

where $\text{gcomp}(d_i)$ is the geocompactness score in consideration of district d_i, quite possibly the WAG score. The region $\mathcal{D}$ would then be a *geocompact region* if and only if $\text{gcomp}(\mathcal{D}) \geq \hat{C}$, in which $\hat{C} \in (0, 1)$ and can be chosen equal to C. Although this is an average and hence takes into account the geocompactness score of each relevant district, this number may not be as representative as it sounds. If $\text{gcomp}(\mathcal{D})$ is (slightly) above $\hat{C}$, it might still be the case that there is one or more district with a very low geocompactness score, but enough geocompact districts to elevate the average above the threshold. However, those districts with a very low score are probably the ones that are constructed by gerrymandering (keep in mind that this will not always be the case). Hence, it may be more meaningful to consider the district with the lowest geocompactness score and associate the geocompactness of the region with this score. More mathematically, if $\mathcal{D} = \{d_1, \dots, d_S\}$, then the geocompactness of region $\mathcal{D}$ is

$$\text{gcomp}(\mathcal{D}) = \min_{i=1,\dots,S} \text{gcomp}(d_i). \tag{26}$$

The region $\mathcal{D}$ would again be called a geocompact region if $\text{gcomp}(\mathcal{D}) \geq \hat{C}$ for a suitable $\hat{C} \in (0, 1)$. When $\hat{C}$ is chosen equal to C, this means that a region $\mathcal{D}$ is geocompact if and only if all the districts in $\mathcal{D}$ are geocompact. Hence, by violation of one of the districts, we would consider that region as not being geocompact. Therefore, it is useful to pick a low value for C, as usually very "weird-looking" districts are non-geocompact and can make the whole region look suspicious. For example, in Figure 14 the American state Pennsylvania is depicted that consists of 18 congressional districts, where these districts were drawn according to the 2010 Census [21]. This redistricting was valid from 2011 to 2018. The districts differ in size because of the equal population regulation: even though district 5 is much larger than district 2, they have roughly the same population count, because district 2 is much more densely populated than district 5 (mainly because of the big city Philadelphia located in district 2, with currently over 1.5 million inhabitants).

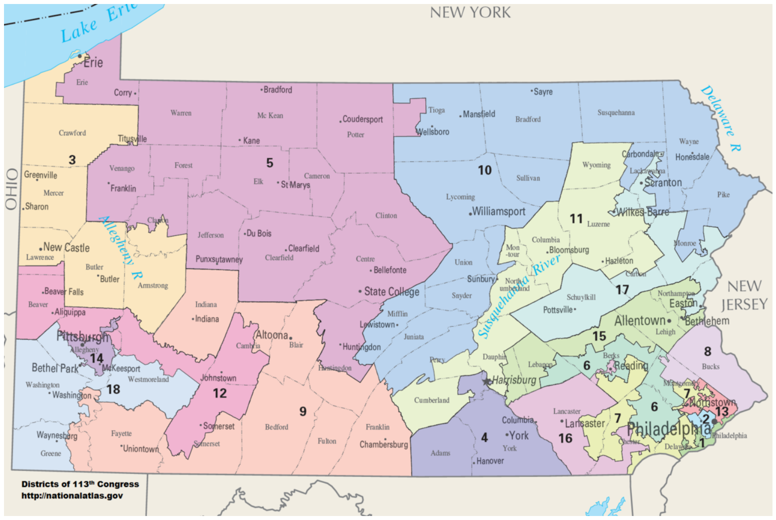


Figure 14: The 18 congressional districts of Pennsylvania

It is immediately visible that none of these districts have a geocompactness score of close to 1. This indicates that a relatively low choice for C, e.g. $C = 0.1$, is not ruled out. However, by a quick scan one could conclude that by far the most "interestingly shaped" district is district 7, at the bottom right. The shape is similar to a cartoon in which Goofy kicks Scrooge McDuck[23] at the buttocks, such a description alone would make for a suspiciously shaped district. Hence, a strategic choice for defining the geocompactness of a region would be the definition in (26) with $C = \hat{C}$ relatively low.

[23]Goofy and Scrooge McDuck are both fictional characters that live in the cartoonish town Duckburg, known by the famous cartoon Donald Duck.

Part II
Prevention

6 Prevention by Voting System

Part II of this thesis discusses two ways that aid in preventing gerrymandering in the future. In this chapter we will look at the benefits of adapting a voting system other than the plurality rule. In Part I we have generalized a few notions to more voting systems, here we will discuss a few of them and their ability to help prevention of gerrymandering.

6.1 Changing the Voting System

If a party can somehow predict the preferences of certain demographic groups, such as political, ethnic or religious groups, it is relatively easy to establish a political advantage by manipulating district boundaries. Gerrymandering occurs when this manipulation of the district boundaries can actually be realised. One of the possible strategies for gerrymandering prevention is a nationwide change in the electoral system that has been applied to select a winning party. The voting system for two or more alternatives, that we have mainly assumed to be the one applied, is the plurality rule. Applying other voting rules makes it harder to predict the voting behaviour of certain groups, certainly when requiring each voter to fill in a whole ballot instead of only picking a favourite party (as is the case for the plurality rule). Therefore, in this section we will consider a number of other voting rules and look at the trade-off between effectiveness and applicability. By effectiveness we mean the extent to which the voting rule affects potential gerrymandering. By applicability we mean the possibility to apply the voting rule in the nation for the election in dispute and whether this is realistic to implement.
We will still assume that a nation consists of different districts. Each district still gives rise to a single winning party. These are called *single-member districts.* The voting rule decides which party wins in each district, hence we are only considering resolute voting rules[24]. A district still consists of multiple VTD's. In some simple cases each VTD is just a voter.

6.1.1 Borda Rule

This voting rule probably needs no introduction by now, but here is still a quick recap of how it works. Assuming there are m parties, each voter gives $m-1$ points to the party she ranks first, $m-2$ points to the party she ranks second, and so on. The party with the most points wins.
The most notable difference with the plurality rule is, that voters need to declare a whole ranked ballot in which they rank all parties instead of picking one party. This serves as an advantage for effectiveness, but a disadvantage for applicability. The former will be clarified later. The impracticalities from using this voting system are especially present, when there is a large amount of parties to choose from. Declaring an ordering over all parties requires much more

[24]Again, when a voting rule is naturally irresolute, use random tie breaking.

work than picking one favourite and this will likely affect the voter turnout, the amount of people who turn up to vote, in a negative way. Implementing the Borda rule has some nice properties, besides gerrymandering reduction, from a social choice theoretic point of view. Each ranked ballot represents the voters true preferences better than picking only their favourite, as most voters already implicitly have a partial linear ordering over the parties.[25] For example, most opinionated voters have a favourite party (or more), but also one or more least favourite parties. Those are mostly the parties with opinions opposite to their favourite party. This can be expressed in a ranked ballot by ordering their least favourite at the bottom of their ranking and their most favourite ones at the top. It is still a debate whether those advantages, possible gerrymandering reduction and better representation of voters' opinions, are worth the difficulties of declaring much more advanced ballots. When there are, for instance, approximately 15 political parties and each of these parties has at average 20 representatives, it is already infeasible to construct a full linear ordering over all representatives. This is the case for the second parliamentary elections ("tweede kamerverkiezingen" in Dutch) in the Netherlands [22]. See Figure 15.

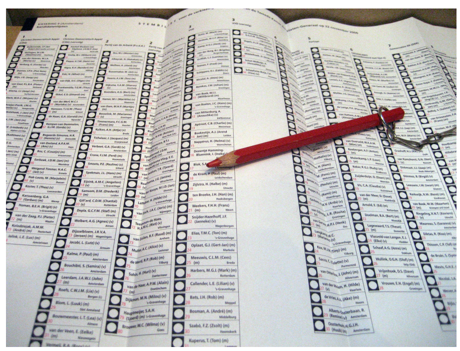


Figure 15: Voting list for the second parliamentary elections in the Netherlands on March 15, 2017.

To see why this is an advantage regarding effectiveness, notice that it is harder for parties to predict the voting behaviour of certain demographic groups now. Predicting the favourite party of a demographic group can be slightly problematic by itself, but predicting the preference of an ordering of a certain demographic group is of course a more difficult task. We will come back to this in more detail.

[25] Assuming the voter is truthful, i.e. the ballot a voter reports coincides with her actual preference order. In some cases, the voter has a incentive to misrepresent her preferences, this is called strategic voting.

6.1.2 Two-Round Rule

The two-round rule, also called runoff voting, is a voting rule for two or more alternatives which requires, as the name suggests, two voting rounds for a single election. In the first round, each voter votes for one alternative (hence, they do not declare a ranked ballot, but a single vote). If after this round, there is an alternative with the majority of votes, this alternative is the winner. Otherwise, in the second round, each voter votes for one of the two top alternatives as a result from the first round. The second round is a simple majority contest between two alternatives, hence the winner is the alternative that receives the most votes in this contest. It is commonly used to select a president, for example in France, or to select a mayor, for example in Italy [23]. It may be the case that the top alternative from the first round loses from the second most voted alternative, see Example 14.

Example 14. Suppose a small village decides to select a mayor by running a two-round rule among the candidates Anderson, Bennett and Charles. The 450 voters vote in the first round as follows: 200 voters pick Anderson, 150 voters pick Bennett and the remaining 100 voters pick Charles. Anderson has most votes, but not an absolute majority (which would be over 225 votes), so we require a second round. This is a majority contest between Anderson and Bennett. Of course the Anderson voters and Bennett voters will not change their vote, but the Charles voters need to alter their vote. Say, 90 of these voters prefer Bennett and the remaining 10 prefer Anderson. Hence, in the second round, 210 voters pick Anderson and 240 voters pick Bennett, thus electing Bennett as the mayor of the village. Such a 90/10 split seems odd, but it could be the case that Charles and Bennett have similar opinions or characteristics, so that Charles voters prefer voting for Bennett instead of for Anderson.
Note, that this two-round system involves at most two rounds (a single round in the case that in the first round an alternative has an absolute majority). But it can also be done in one round by requiring each voter to declare a full ranked ballot. Here, the following ballots will imply the same winner, with a second round being held implicitly.

- Anderson $\succ$ Bennett $\succ$ Charles (120$\times$)
- Anderson $\succ$ Charles $\succ$ Bennett (80$\times$)
- Bennett $\succ$ Anderson $\succ$ Charles (65$\times$)
- Bennett $\succ$ Charles $\succ$ Anderson (85$\times$)
- Charles $\succ$ Anderson $\succ$ Bennett (10$\times$)
- Charles $\succ$ Bennett $\succ$ Anderson (90$\times$)

It follows that Anderson and Bennett will make it to the second round. Also the result of the second round can be implied by these ballots, as we see that 10 Charles voters pick Anderson as second favourite mayor (ballot 5) and 90 Charles voters pick Bennett (ballot 6).

The following important observation thus follows from Example 14: declaring full ranked ballots will ensure that only one round is required in order to have the same result as holding two rounds. However, in practice, the two-round system with actually two rounds is preferred, certainly when considering a large amount of alternatives. The reasoning is the same as for the Borda rule: constructing ranked ballots is much more advanced than picking a favourite alternative, even when picking a favourite must be done twice. Therefore, the two-round approach is even more desirable in practice than the Borda rule.
Also, it is more effective for gerrymandering prevention compared to the plurality rule, as predicting the voting behaviour becomes harder: the two winners from the first round can be predicted, however this is more difficult than predicting only the most voted alternative, but predicting the outcome of the second round will be harder. This may be easier to predict than the Borda rule, as the Borda rule requires *all information* from a ballot and this rule does not. As in Example 14, it does not utilize the fact that from the 200 Anderson voters, 120 prefer Bennett and 80 prefer Charles. This property is also desirable from the same social choice theoretic point of view as explained before, as the Borda rule utilizes all information from a declared ballot. Therefore, it represents the opinions better and also takes into account every party in a ballot, whereas this rule does not. See the following table for a comparison of different properties from the plurality rule, Borda rule and the two-round system.

Property	Plurality rule	Two-round system	Borda rule
Utilization	Uses very little information	Uses partial information	Uses all information
Effectiveness	Not very effective	More effective	Most effective
Applicability	Very easy to implement	Not difficult to implement	Difficult to implement

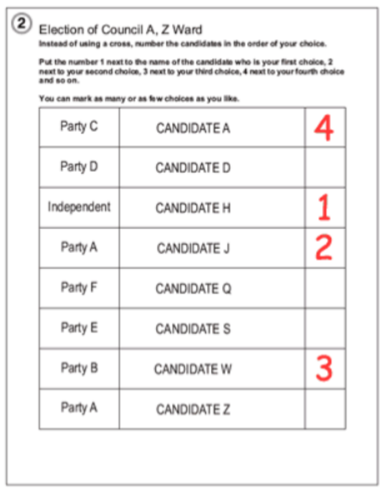
Table 2: Comparison of voting rules.

Hence, the choice of the voting rule to apply is a trade-off between utilization of information from ballots, effectiveness for gerrymandering prevention and applicability in the nation. However, there are more voting rules to consider.

6.1.3 Cumulative and Preference Voting

In *cumulative voting*, also called range voting, each voter divides a fixed amount of points among all alternatives. For instance, when a voter gets 100 points to assign and there are five alternatives, she can assign all 100 points to her favourite alternative or spread those points more uniformly, respecting a certain (partial) ordering over the alternatives. A fully uniform division, 20 points to each alternative, can also be submitted by an indifferent voter, but this does not have any influence on the outcome. This method gives each voter usually more "freedom" than the plurality rule or even the Borda rule. That is, if the number of points to be assigned is large enough. If there are m alternatives, the number of points needs to be at least 2 to be more flexible than the plurality rule and at least $\binom{m}{2}$ to be more flexible than the Borda rule. This is because $(m-1)+(m-2)+\ldots+1 = \binom{m}{2}$ is the total number of points assigned to parties from each ballot by using the Borda rule. This method is commonly used to select multiple winners. However, since we work with single-member districts, the district winner can be chosen as the alternative that receives most points.

Preference voting, also called single transferable voting, is a voting rule for two or more alternatives that requires a series of rounds. Each voter submits a ranked ballot, but only for as many candidates as she wishes. E.g., out of a total of 8 candidates, she may only select 4, see Figure 16 [24].


② Election of Council A, Z Ward

Instead of using a cross, number the candidates in the order of your choice.

Put the number 1 next to the name of the candidate who is your first choice, 2 next to your second choice, 3 next to your third choice, 4 next to your fourth choice and so on.

You can mark as many or as few choices as you like.

Party C	CANDIDATE A	4
Party D	CANDIDATE D	
Independent	CANDIDATE H	1
Party A	CANDIDATE J	2
Party F	CANDIDATE Q	
Party E	CANDIDATE S	
Party B	CANDIDATE W	3
Party A	CANDIDATE Z	

Figure 16: Voting list for the Scottish elections in 2007 using the STV system.

Let T be the number of voters. The preference voting algorithm is as follows:

1. The alternative that has at least $\lfloor \frac{T}{2} \rfloor + 1$ votes[26] is the winner. If there is no such alternative, go to step 2.

2. The alternative with the fewest votes is eliminated. A vote for this losing alternative is allocated to the next choice on the ballot (that has not been eliminated). If there is no next choice, the vote is lost. If only one alternative is left, this is the winner. Otherwise, go to step 1.

The number $\lfloor \frac{T}{2} \rfloor + 1$ is called the *Droop quota*. Clearly, there can be at most one alternative with at least $\lfloor \frac{T}{2} \rfloor + 1$ votes.

[26]The number of votes is the number of first-choice votes plus the number of votes that are allocated from eliminated alternatives in step 2.

The STV system is mainly used in multi-member districts, i.e. it is mainly used to select a team of representatives instead of only one representative [25]. The description above only applies to one winning alternative. This can be modified to any number of winning alternatives, say k, by iterating as many rounds as necessary to elect k alternatives that exceed the Droop quota or are remaining. Whenever an alternative exceeds the Droop quota and is elected, the excess votes (the number of votes they receive above the quota) are reassigned to other alternatives in a proportional way based on their next votes. However, in this case

$$\text{Droop quota} = \lfloor \frac{T}{k+1} \rfloor + 1,$$

where T is the total number of voters and k the number of seats/winners. In the case of single-member districts, this reduces to $\lfloor \frac{T}{2} \rfloor + 1$, as in the algorithm.

Example 15. The set of parties is $\mathcal{P} = \{A, B, C, D\}$. There is a total of 100 voters that declare the following ballots:

- $A \succ B \succ D$ (6×)
- $A \succ C$ (11×)
- $A \succ C \succ D$ (2×)
- $B \succ A$ (15×)
- $B \succ C \succ A \succ D$ (15×)
- C (6×)
- $C \succ B \succ A$ (20×)
- $D \succ A \succ C$ (8×)
- $D \succ B$ (16×)
- A (1×)

The Droop quota is 51 votes. No party has at least 51 first-choice votes, so there is no direct winner in round 1. In round 2 we eliminate A, as this party has the fewest first-choice votes with a total of 20. Then, 6 of these votes go to party B, 13 go to party C and one vote gets lost, which brings B and C at a total of 36 and 39 votes, respectively. In round 3, party D is eliminated with a total of 24 votes. Party B gets 16 of these votes and party C the remaining 8. In round 4, party B and C have a total of 52 and 47 votes respectively. Party B is the winning party, as 52 votes exceeds the Droop quota.

Evaluation. We will discuss some pros and cons of preference voting and cumulative voting.

1. Minority vote dilution. In his article [26], Steven J. Mulroy proposed

these two voting systems as a remedy for *minority vote dilution*, which is closely related to racial gerrymandering. Minority vote dilution occurs when, in the current electoral system used, no minority group has any real influence in the electoral outcome. Think of a racial or ethnic group forming a minority of the total population, e.g. blacks or Mexicans living in the United States. By gerrymandering the districts in such a way that a certain demographic group forms a minority in any district, the voting behaviour of that group can be expelled in those districts, if a voting rule is being used that allows for this. However, minority vote dilution is mainly an issue in multi-member districts, where there are multiple winning alternatives. By any real influence we mean then the ability to assure the election of at least one winner.
The *threshold of exclusion* is defined to be the minimum percentage a minority group needs to be in order to have any real influence. For preference voting and cumulative voting, the threshold of exclusion is $\frac{1}{k+1}$, where k is the number of alternatives to be elected [26]. E.g., when preference voting or cumulative voting is used in a district where they want to select a committee of 5 alternatives, a politically cohesive group needs to consist of at least 17% of the district population to have the ability to elect one candidate in the committee. When $k = 1$, as is the case in single-member districts, the threshold of exclusion is $\frac{1}{2}$. In other words, only a majority has the ability to select the winning alternative and hence regulate the whole outcome. This is not the case for the Borda rule for example, where a minority (which is still a relatively large minority, such as 40%) can counteract the voting behavior of a majority (which is a not too large majority, such as 60%). To see this, let $n = 100$ and $\mathcal{P} = \{A, B, C\}$. 60 voters (i.e. 60%) vote as follows: $A \succ B \succ C$, so party A is expected to win the election. However, when the remaining 40 voters (i.e. 40%) have the preferences $B \succ C \succ A$, party A will get 120 points and party B 140. Hence, party B wins. Furthermore, the article by Mulroy mentions a few other advantages of preference voting and cumulative voting.
2. Implementation and flexibility. Firstly, as opposed to the Borda rule where each voter has to declare a full ordering over all alternatives, here each voter declares a less advanced ballot. For cumulative voting, each voter can construct a ballot as advanced or as simple as she wishes. This is also the case for preference voting, although a linear ordering over the selected alternatives is required, as opposed to cumulative voting. Intuitively, the simpler the construction of a ballot, the less flexibility and freedom a voter has to express her preferences. The rationale behind this, is that giving more information on a ballot allows for better representing your opinion. However, for these two systems each voter can decide how much information she gives, giving her even more freedom at constructing a ballot. Cumulative voting has a strong appeal in this case; it provides voters the flexibility to express the intensity of their preferences by regulating the number of points one can give to each alternative. Preference voting in some sense has the same advantage, as it lets one pick alternatives for ranking, however with less flexibility than in cumulative voting. Although preference voting might sound complicated at first sight, each voter only needs to know how to fill in such a ballot, as in Figure 16. Both systems are thus

not problematic to implement and still gives the voter enough flexibility and freedom.
3. Strategic voting. An advantage of preference voting only is that filling in your true preferences is also strategically the smartest thing to do, i.e. no voter has an incentive to misrepresent her preferences [27]. For cumulative voting, strategic voting can be eliminated by adapting the rule as follows: each voter marks one or more alternatives and each marked alternative gets an equal share of the total number of points given. E.g., when 100 points are to be distributed and a voter marks 5 alternatives, each alternative receives 20 points. This may result in a non-integer amount of points cast to an alternative, but this does not need to be problematic. This method is called *equal and even cumulative voting.*
4. Reduction of wasted votes. Another important advantage of these two systems is that they are designed to reduce wasted votes [27]. Initially, they are designed for multi-member districts, where the reduction of wasted votes is even "better" than for single-member districts. This is because these two voting systems ensure a more "proportional" representation, meaning that if $n\%$ of the voters support a set of candidates, then roughly $n\%$ of the committee (candidates to be elected) will consist of these candidates. For most voting systems, including plurality and Borda, a vote cast for an alternative that is less well-known or non-mainstream is one less vote available for a more electable candidate and can be interpreted as a wasted vote. In preference voting, the voter can also vote for such a long-shot candidate, while the rest of the ballot stays invariant, and thus support this candidate. Whenever this candidate is not elected, which is to be expected, this vote is less wasted in a sense. The same holds for equal and even cumulative voting.
5. Less predictability. Lastly, one can argue that, by implementing either of these two systems, it makes it harder to predict the voting behaviour than for the plurality rule. As mentioned before, cumulative voting gives each voter almost a maximum amount of flexibility, certainly when the number of points to be assigned is very large. Preference voting requires multiple rounds and ballots where the amount of votes to be assigned is not fixed beforehand and can be different per person. It is disputable whether these systems make it harder to predict the voting behaviour than for, e.g., the Borda rule or the two-round rule. In the next subsection we give a more formal discussion on this type of predictability.

6.1.4 Predictability of Different Voting Systems

By predictability we mean the ability to predict the voting behaviour of certain (demographic) groups and therefore to predict the outcome of an election. When using plurality rule, for example, it comes down to computing the individual probabilities of a party to win the election when voting behaviour can be predicted by sampling some voters from each specific, politically cohesive group (representative sampling). We will then use the relative frequency that a group voted for party P as the probability that a person from this group will vote for

P in a future election.

Example 16. In a district with three electoral candidates, $\mathcal{P} = \{A, B, C\}$, and 5 politically cohesive groups, the plurality rule is used to select a winner. By sampling a set of voters from each group, the following results are obtained:

%	1	2	3	4	5
A	70	30	0	50	20
B	20	30	10	50	70
C	10	40	90	0	10

Table 3: Relative frequencies for each group and party.

From this we see that, e.g., 70% of the voters in the sample selection from group 1 voted for party A. When each group has roughly the same amount of voters, an educated guess would be that party B wins the election, as it has an average frequency of $180/5 = 36\%$. But what would be the actual probability that a party wins, given the results from the sample? In what follows we strive to find explicit bounds for this probability, when we drop the requirement that each group has the same population count.

Let there be M politically cohesive groups and let n be the number of voters in a district. Denote by T_g the number of voters in group g and by $\mathbb{P}_g^P$ the frequency of P-voters in group g, according to the sample, where $P \in \mathcal{P}$ and $g \in \{1, \ldots, M\}$. For example, $\mathbb{P}_1^A = 0.7$ in Table 3. Let X_g^P be the random variable denoting the number of P-voters in group g. Then we have that $X_1^P, \ldots, X_M^P$ are independent and distributed as follows: $X_g^P \sim B(T_g, \mathbb{P}_g^P)$, where $B(n, p)$ is the Binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$. This is because a voter from group g votes for P with probability $\mathbb{P}_g^P$. Furthermore, $\sum_{g=1}^M X_g^P = T^P$, where T^P is the number of votes for party P in the district. The following theorem provides a lower bound for the probability that a party wins, using the plurality rule, in the case of two parties.

Theorem 4. When $|\mathcal{P}| = 2$,

$$\mathbb{P}(\text{party } P \text{ wins}) \geq \begin{cases} 0, & \text{if } \sum_{g=1}^M \frac{T_g}{n} \mathbb{P}_g^P < \frac{1}{2} \\ 1 - \frac{\prod_{g=1}^M [(1-\mathbb{P}_g^P)\mathcal{C}^P + \mathbb{P}_g^P]^{T_g}}{(\mathcal{C}^P)^{\frac{n}{2}}}, & \text{if } \sum_{g=1}^M \frac{T_g}{n} \mathbb{P}_g^P \geq \frac{1}{2} \end{cases} \tag{27}$$

$$\text{where } \mathcal{C}^P = \frac{\sum_{g=1}^M \frac{T_g}{n} \mathbb{P}_g^P}{1 - \sum_{g=1}^M \frac{T_g}{n} \mathbb{P}_g^P}. \tag{28}$$

Proof.

$$\begin{aligned}
\mathbb{P}(\text{party } P \text{ wins}) &\geq \mathbb{P}(\text{at least } \frac{n}{2} \text{ people vote for } P) \\
&= \mathbb{P}(T^P \geq \frac{n}{2}) \\
&= \mathbb{P}(n - T^P < \frac{n}{2}) \\
&= \mathbb{P}(\sum_{g=1}^{M}(T_g - X_g^P) < \frac{n}{2}) \\
&= 1 - \mathbb{P}(\sum_{g=1}^{M}(T_g - X_g^P) \geq \frac{n}{2}) \\
&= 1 - \mathbb{P}(\sum_{g=1}^{M} Y_g^P \geq \frac{n}{2}) \\
&= 1 - \mathbb{P}(Y^P \geq \frac{n}{2}), \qquad (29)
\end{aligned}$$

where $Y_g^P := T_g - X_g^P$ is the number of non P-voters in group g and $Y^P := \sum_{g=1}^{M} Y_g^P$. $Y_1^P, \ldots, Y_M^P$ are independent and distributed as follows: $Y_g^P \sim B(T_g, 1 - \mathbb{P}_g^P)$.
Chebyshev's inequality [28] states that

$$\mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}[f(Y)]}{f(a)} \qquad (30)$$

for a random variable Y, constant a and non-decreasing and non-negative function f. We will use Equation 30 to find an upper bound for $\mathbb{P}(Y^P \geq \frac{n}{2})$, hence a lower bound for $\mathbb{P}$(party P wins). We will consider multiple options for the function f.

Case 1: $f(x)$=x.

$$\begin{aligned}
\mathbb{P}(Y^P \geq \frac{n}{2}) &\leq \frac{\mathbb{E}[Y^P]}{\frac{n}{2}} \\
&= \frac{\sum_{g=1}^{M} \mathbb{E}[Y_g^P]}{\frac{n}{2}} \\
&= \frac{\sum_{g=1}^{M} [T_g(1 - \mathbb{P}_g^P)]}{\frac{n}{2}} \\
&= \frac{n - \sum_{g=1}^{M} T_g \mathbb{P}_g^P}{\frac{n}{2}} \\
&= 2(1 - \frac{1}{n}\sum_{g=1}^{M} T_g \mathbb{P}_g^P),
\end{aligned}$$

where the second equality follows because $\mathbb{E}[X] = np$, when $X \sim B(n, p)$.

Case 2: $f(x)$=x^2.

$$\begin{aligned}
\mathbb{P}(Y^P \geq \frac{n}{2}) &\leq \frac{\mathbb{E}[(Y^P)^2]}{(\frac{n}{2})^2} \\
&= \frac{\text{Var}[Y^P] + (\mathbb{E}[Y^P])^2}{(\frac{n}{2})^2} \\
&= \frac{\sum_{g=1}^{M} \text{Var}[Y_g^P] + (\sum_{g=1}^{M} \mathbb{E}[Y_g^P])^2}{(\frac{n}{2})^2} \\
&= \frac{\sum_{g=1}^{M} T_g \mathbb{P}_g^P (1 - \mathbb{P}_g^P) + (n - \sum_{g=1}^{M} T_g \mathbb{P}_g^P)^2}{(\frac{n}{2})^2} \\
&= \frac{4}{n^2} (\sum_{g=1}^{M} T_g \mathbb{P}_g^P (1 - \mathbb{P}_g^P) + (n - \sum_{g=1}^{M} T_g \mathbb{P}_g^P)^2),
\end{aligned}$$

where the first equality follows because $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ for a random variable X and the third equality follows because $\text{Var}[X] = np(1-p)$ when $X \sim B(n, p)$.

Case 3: $f(x)$=e^{sx}.

$$\begin{aligned}
\mathbb{P}(Y^P \geq \frac{n}{2}) &\leq \frac{\mathbb{E}[e^{sY^P}]}{e^{s\frac{n}{2}}} \\
&= \frac{\mathbb{E}[e^{s \sum_{g=1}^{M} Y_g^P}]}{e^{s\frac{n}{2}}} \\
&= \frac{\prod_{g=1}^{M} \mathbb{E}[e^{sY_g^P}]}{e^{s\frac{n}{2}}},
\end{aligned}$$

where $s \geq 0$.
Write $Y_g^P = \sum_{i=1}^{T_g} U_i$, where

$$U_i = \begin{cases} 1, & \text{with probability } 1 - \mathbb{P}_g^P \\ 0, & \text{with probability } \mathbb{P}_g^P. \end{cases}$$

This is true, since every voter in group g votes not for P with probability $1 - \mathbb{P}_g^P$ (success) and votes for P with probability $\mathbb{P}_g^P$ (failure). We have

$$\begin{aligned}
\mathbb{E}[e^{sU_i}] &= \mathbb{P}(U_i = 1) \times e^s + \mathbb{P}(U_i = 0) \times e^0 \\
&= (1 - \mathbb{P}_g^P) e^s + \mathbb{P}_g^P.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[e^{sY_g^P}] &= \prod_{i=1}^{T_g} \mathbb{E}[e^{sU_i}] \\ &= \prod_{i=1}^{T_g} [(1-\mathbb{P}_g^P)e^s + \mathbb{P}_g^P] \\ &= [(1-\mathbb{P}_g^P)e^s + \mathbb{P}_g^P]^{T_g}.\end{aligned}$$

Thus we conclude

$$\mathbb{P}(Y^P \geq \frac{n}{2}) \leq \frac{\prod_{g=1}^{M}[(1-\mathbb{P}_g^P)e^s + \mathbb{P}_g^P]^{T_g}}{e^{\frac{sn}{2}}}. \tag{31}$$

We will be looking for the value of s that minimizes this upper bound, as a tighter upper bound for $\mathbb{P}(Y^P \geq \frac{n}{2})$ gives a tighter lower bound for $\mathbb{P}(\text{party } P \text{ wins})$. Therefore, write

$$h_g(s) := [(1-\mathbb{P}_g^P)e^s + \mathbb{P}_g^P]^{T_g}.$$

The derivative of this upper bound to s is equal to

$$\frac{\frac{d}{ds}[\prod_{g=1}^{M} h_g(s)]e^{\frac{sn}{2}} - \prod_{g=1}^{M} h_g(s)[\frac{n}{2}e^{\frac{sn}{2}}]}{e^{sn}}. \tag{32}$$

Using

$$\frac{d}{ds}[\prod_{g=1}^{M} h_g(s)] = (\prod_{g=1}^{M} h_g(s))(\sum_{g=1}^{M} \frac{h_g'(s)}{h_g(s)})$$

and setting (32) equal to 0, we find

$$\sum_{g=1}^{M} \frac{(1-\mathbb{P}_g^P)T_g e^s}{((1-\mathbb{P}_g^P)e^s + \mathbb{P}_g^P)} = \frac{n}{2}.$$

The latter equation for s is difficult to solve, unless we use the simplifying condition $T_g = \frac{n}{M}$, $\mathbb{P}_g^P = \mathbb{P}^P$ for all $g = 1, \ldots, M$ (which means that there is only one group). Then, we infer the following

$$\begin{aligned}&\frac{(1-\mathbb{P}^P)e^s n}{(1-\mathbb{P}^P)e^s + \mathbb{P}^P} = \frac{n}{2} \\ &\Rightarrow 2(1-\mathbb{P}^P)e^s = (1-\mathbb{P}^P)e^s + \mathbb{P}^P \\ &\Rightarrow e^s = \frac{\mathbb{P}^P}{1-\mathbb{P}^P} \\ &\Rightarrow s = \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P}).\end{aligned}$$

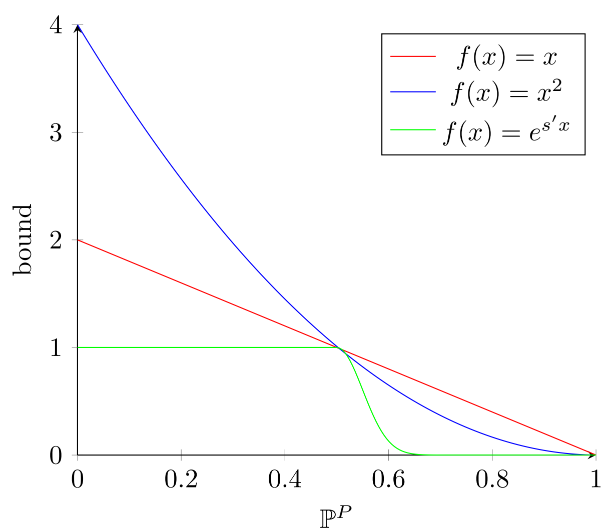
The problem with this choice of s is that $s < 0$ whenever $\mathbb{P}^P < \frac{1}{2}$. Indeed, $f(x) = e^{sx}$, $x \in \mathbb{R}$, is decreasing for $s < 0$, so Chebyshev's inequality to find an upper bound can not be used. We fix this by defining

$$\begin{aligned} s' &= \max\{0, s\} \\ &= \max\{0, \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P})\} \\ &= \begin{cases} 0, & \text{if } \mathbb{P}^P < \frac{1}{2} \\ \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P}), & \text{if } \mathbb{P}^P \geq \frac{1}{2}, \end{cases} \end{aligned}$$

so that s' coincides with s on positive values.
Using this simplifying condition again, we find the following bounds for the different choices of f:

- $f(x) = x$ gives $2(1 - \mathbb{P}^P)$.
- $f(x) = x^2$ gives $\frac{4}{n}\mathbb{P}^P(1-\mathbb{P}^P) + 4(1-\mathbb{P}^P)^2$.
- $f(x) = e^{sx}$ gives $(\frac{(1-\mathbb{P}^P)e^s + \mathbb{P}^P}{e^{\frac{s}{2}}})^n$.
- $f(x) = e^{s'x}$ with $s' = \max\{0, \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P})\}$ gives $\begin{cases} 1 & \text{if } \mathbb{P}^P < \frac{1}{2} \\ 2^n(\mathbb{P}^P)^{\frac{n}{2}}(1-\mathbb{P}^P)^{\frac{n}{2}} & \text{if } \mathbb{P}^P \geq \frac{1}{2} \end{cases}$

We will be looking for the function that is generally (which means for most realizations of n) the smallest. A plot for $n = 100$ looks as follows



For $n = 100$, the upper bound for $f(x) = e^{s'x}$ is everywhere (except for $\mathbb{P}^P = \frac{1}{2}$) lower than for $f(x) = x$ and $f(x) = x^2$. This turns out to generally be the case, hence we will choose (31) to be the best upper bound. However, we chose $s' = \max\{0, \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P})\}$, where each group has frequency $\mathbb{P}^P$. In the general case, groups do not have the same frequency, but we can incorporate $\mathbb{P}_1^P, \ldots, \mathbb{P}_M^P$ into one "collective" $\mathbb{P}^P$ by simply putting

$$\mathbb{P}^P = \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P. \tag{33}$$

This coincides with $\mathbb{P}^P$ by using the simplifying conditions and also incorporates the number of voters, T_g, in each group g, hence using (33) for $\mathbb{P}^P$ is still a good choice. Hence,

$$\mathbb{P}(Y^P \geq \frac{n}{2}) \leq \frac{\prod_{g=1}^{M}[(1-\mathbb{P}_g^P)C^P + \mathbb{P}_g^P]^{T_g}}{(C^P)^{\frac{n}{2}}},$$

where

$$C^P = e^{s'} = e^{\max\{0, \ln(\frac{\mathbb{P}^P}{1-\mathbb{P}^P})\}} = \begin{cases} 1 & \text{if } \mathbb{P}^P < \frac{1}{2} \\ \frac{\mathbb{P}^P}{1-\mathbb{P}^P} & \text{if } \mathbb{P}^P \geq \frac{1}{2} \end{cases}$$

and we use the $\mathbb{P}^P$ as in (33). Equivalently,

$$\mathbb{P}(Y^P \geq \frac{n}{2}) \leq \begin{cases} 1 & \text{if } \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P < \frac{1}{2} \\ \frac{\prod_{g=1}^{M}[(1-\mathbb{P}_g^P)\mathcal{C}^P + \mathbb{P}_g^P]^{T_g}}{(\mathcal{C}^P)^{\frac{n}{2}}} & \text{if } \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P \geq \frac{1}{2} \end{cases}$$

where

$$\mathcal{C}^P = \frac{\sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P}{1 - \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P}.$$

The result then follows from Equation 29. □

Theorem 5. When $|\mathcal{P}| = m$,

$$\mathbb{P}(\text{party } P \text{ wins}) \leq \begin{cases} \frac{\prod_{g=1}^{M}[\mathbb{P}_g^P \tilde{\mathcal{C}}^P + (1-\mathbb{P}_g^P)]^{T_g}}{(\tilde{\mathcal{C}}^P)^{\frac{n}{m}}}, & \text{if } \frac{1-\mathbb{P}^P}{\mathbb{P}^P} > m-1 \\ 1, & \text{if } \frac{1-\mathbb{P}^P}{\mathbb{P}^P} \leq m-1 \end{cases} \tag{34}$$

where

$$\tilde{\mathcal{C}}^P = \frac{1}{m-1} \frac{1 - \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P}{\sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P}.$$

Proof. The proof is analogous to the proof of Theorem 4. □

Example 17. We will try to find upper and lower bounds for the probability that a party wins in the district from Example 16. In Table 3 the frequencies are specified, but not the respective size of the groups, which are $T_1 = 10, T_2 = 20, T_3 = 20, T_4 = 30$ and $T_5 = 20$, so that $n = 100$. According to Equation 33, we find that $\mathbb{P}^A = 0.32$, $\mathbb{P}^B = 0.39$ and $\mathbb{P}^C = 0.29$. However, none of theme exceed $\frac{1}{2}$, such that the lower bound is 0. Of course, this lower bound does not give us helpful information on the probability that a party wins. Nevertheless, we are able to find some helpful upper bounds. Since $\frac{1-\mathbb{P}^A}{\mathbb{P}^A} = 2.125$, $\frac{1-\mathbb{P}^B}{\mathbb{P}^B} \approx 1.564$ and $\frac{1-\mathbb{P}^C}{\mathbb{P}^C} \approx 2.448$, we can find non-trivial upper bounds for parties A and C. Since $\tilde{\mathcal{C}}^A = 1.0625$, we find

$$\begin{aligned} \mathbb{P}(\text{party } A \text{ wins}) &\leq \frac{\prod_{g=1}^{M} [\mathbb{P}_g^A \times 1.0625 + (1 - \mathbb{P}_g^A)]^{T_g}}{(1.0625)^{\frac{100}{3}}} \\ &\approx 0.95 \end{aligned}$$

and likewise we find

$$\begin{aligned} \mathbb{P}(\text{party } C \text{ wins}) &\leq \frac{\prod_{g=1}^{M} [\mathbb{P}_g^C \times 1.224 + (1 - \mathbb{P}_g^C)]^{T_g}}{(1.224)^{\frac{100}{3}}} \\ &\approx 0.50 \end{aligned}$$

This tells us that the odds of party C winning the election, using plurality, is not bigger than 50%.

Example 17 giving us no essential information on the lower bound of the win probabilities is not really coincidental. Since

$$\begin{aligned} \sum_{P \in \mathcal{P}} \mathbb{P}^P &= \sum_{P \in \mathcal{P}} \sum_{g=1}^{M} \frac{T_g}{n} \mathbb{P}_g^P \\ &= \frac{1}{n} \sum_{g=1}^{M} \sum_{P \in \mathcal{P}} T_g \mathbb{P}_g^P \\ &= \frac{1}{n} \sum_{g=1}^{M} T_g \sum_{P \in \mathcal{P}} \mathbb{P}_g^P \\ &= \frac{1}{n} \sum_{g=1}^{M} T_g \\ &= \frac{1}{n} \times n \\ &= 1, \end{aligned}$$

$\mathbb{P}^P > \frac{1}{2}$ can occur for at most one party. Also, when $m > 3$, it will sporadically be the case that there is one party P with $\mathbb{P}^P > \frac{1}{2}$. When there is actually such a party, we can already expect it beforehand to have a high lower bound. This

tells us that we typically can extract more information from the upper bounds, as they are more often non-trivial (as Example 17 also tells us).
Since the outcome of an election in single-member districts is solely the winning of a single party, the ability to predict the outcome of an election, given the relative frequencies of voters for each party in each group (as in Table 16), is proportional to the probability that the party wins that is most likely to win, i.e.

$$\text{Predictability} \sim \max_{P \in \mathcal{P}} \mathbb{P}(\text{party } P \text{ wins}).$$

The logic behind this is that, when there is a party with a high percentage chance of winning, the prediction that this party wins is quite likely to happen.
Gerrymandering partly comes down to the addition or deletion of groups (columns from a frequency table as in Table 16) to a certain district to increase the probability that a specific party wins. A straightforward "strategy" for gerrymandering (note that we are *not* talking about a strategy for gerrymandering *prevention* now, as is the topic of this chapter) would be either of the following options, when using plurality rule and striving to benefit party A in district d_i:

1. Replace (if possible) group $g \in d_i$ with group $h \notin d_i$ if $\mathbb{P}_h^A > \mathbb{P}_g^A$.
2. Delete (if possible) group $g \in d_i$ from d_i if $\mathbb{P}_g^A < \frac{1}{m}$.
3. Add (if possible) group $g \notin d_i$ to d_i if $\mathbb{P}_g^A > 0.5$.

Of course, these options are dealing with geographic restrictions. For example, the first option is only possible when groups g and h are at the border of the district, to retain connectedness and geocompactness of the district. For further discussion on this, we refer to the chapter "Probabilistic Methods", where we defined a legal redistricting plan to be any redistricting of groups or VTD's that satisfy criteria such as contiguity and geocompactness in an appropriate way.
This strategy will indeed aid in manipulating district boundaries for a specific cause, in this case to increase the probability of a party winning in a specific district d_i. This may already be clear by intuition: including a demographic group that frequently votes for party A, or is known to have many supporters for party A, in a district increases the odds of party A winning in that district. This is also clear by implementing the probability $\mathbb{P}_g^A$ in the equations corresponding to the lower and upper bound for the probability that party A wins, see Theorem 4 and Equation 34. By following the options above it follows that the lower and upper bounds for $\mathbb{P}(\text{party } A \text{ wins})$ indeed do not decrease. Option 2 does not directly have this result, but states that a group g does certainly not "help" in electing party A when the frequency of A-voters in that group is less than one divided by the number of parties.
We have no proof that this strategy is indeed followed by those accused of gerrymandering, but implicitly this must be the train of thought when trying to perform gerrymandering.[27] In order to do this, one must have good or at least

[27] That is, when the goal of gerrymandering is to increase the chance for a party to win in specific district(s).

reasonable predictions of the relative frequencies that a party is voted on by voters from different regions or groups, such as in Table 16. When we extend this reasoning to the Borda rule for example, we have other (possibly more complicated-looking) bounds for the probability that a party wins in a district, but similar results as how to strategically perform gerrymandering. A frequency table as in Table 16 is not sufficient as basis for decisions regarding the three options of the aforementioned strategy plan. Rather, we need a table with *more information* as in the following example.

Example 18. In the same district as in Example 16, with $\mathcal{P} = \{A, B, C\}$ and 5 politically cohesive groups, the Borda rule is now used to select a winner. By sampling a set of voters from each group, the following results are obtained:

%	1	2	3	4	5
$A \succ B \succ C$	30	10	0	20	15
$A \succ C \succ B$	40	20	0	30	5
$B \succ A \succ C$	15	10	5	30	50
$B \succ C \succ A$	5	20	5	20	20
$C \succ A \succ B$	0	20	60	0	0
$C \succ B \succ A$	10	20	30	0	10

Table 4: Relative frequencies for each group and party.

It is easy to check that those results are consistent with the results in Table 3.

When there are $|\mathcal{P}| = m$ parties and M demographic groups, such a frequency table consists of $m!\,M$ entries. This is because we have to include every possible voting ballot, of which there are $m!$ in total. Hence, for the plurality rule it is easier to sample, i.e. make a table with all corresponding $\mathbb{P}_g^P$, than for any other voting rule, since each other voting rule requires at least mM entries (which is the number of entries for the plurality rule). A second observation is, that for the plurality rule sampling is less necessary to predict frequencies. When one has some impression of the voting behaviour of a specific group, a frequency table as Table 3 can be predicted much easier than a table as Table 4. Of course, sampling results in better predictions, but this is not always realisable. All in all, we can conclude that predicting the voting behaviour and utilizing that to benefit a specific party is easier to perform for the plurality rule than for the Borda rule. But what if we compare the other aforementioned voting rules?
As implicitly argued before, the number of entries in a frequency table is M times the number of different ballots. The more different ballots are possible, the harder it is to construct a frequency table. Even without using a frequency table, it is still true that the number of different ballots is proportional to the "difficulty" of predicting the voting behaviour and therefore predicting the outcome. We therefore look at the number of different ballots for each of the five voting rules mentioned in this chapter, in terms of the number of parties m

(in the case of cumulative voting, there is an additional parameter).
As mentioned before, for the plurality rule and the Borda rule, this number is respectively m and $m!$. For the two-round rule, there are (at least) two ways to submit your vote. We have seen before that it is possible to deduce the winner, using the two-round rule, when each voter declares a full ranked ballot as for the Borda rule. In this case there are $m!$ different ballots, but this method is not desirable in practice when m is large. In the original case, each voter declares her favourite choice twice, where at the second time she chooses between two parties. A possible voting ballot for the two-round rule, where $\mathcal{P} = \{P_1, \ldots, P_m\}$, is

$$P_1, P_2 \vee P_3 \rightarrow P_2$$

meaning that the voters first choice is P_1 and, when in the second round she has to choose between P_2 and P_3, she chooses P_2. Her declared ballots will then only be P_1 and P_2, but it is important to specify which two parties "survive" the first round in order to distinguish between declared ballots. Since there are m options in the first round, $\binom{m}{2}$ possible contenders to survive to the second round and 2 options in the second round, the number of different ballots[28] is $m \times \binom{m}{2} \times 2 = m^3 - m^2$. The cumulative voting rule involves a more complicated explanation, therefore we state the number in the following lemma.

Lemma 8. For the cumulative voting rule, where each voter divides Y points among m parties, the number of different ballots is $\binom{Y+m-1}{Y}$.

Proof. First assume that every party must have at least one point in each ballot, later we will drop this assumption. The number of different ballots is equal to the number of different allocations of points to the parties. The number of ways these points can be assigned is equal to the number of m-*compositions* of the integer Y: a m-composition of Y is a way of writing Y as the sum of m positive integers, i.e. $Y = a_1 + \ldots + a_m$ where $a_j > 0$ for $j = 1, \ldots, m$. This is because of the following interpretation of the composition problem: a_j can be seen as the number of points party P_j gets in a ballot, where $\mathcal{P} = \{P_1, \ldots, P_m\}$. The number of m-compositions of Y is $\binom{Y-1}{m-1}$. A visual explanation for this is as follows: draw Y identical stripes next to each other, where in the gap between two consecutive stripes a bar can be placed. Fill $m-1$ of these $Y-1$ gaps with bars to indicate what each number a_j is: a_j is the number of stripes between the $j-1$-th and j-th bar, where a_1 is the number of stripes before the first bar and a_m the number of stripes after the last bar. See Figure 17 for an example where $m = 5$, $\mathcal{P} = \{A, B, C, D, E\}$ and $Y = 10$. Here, $a_1 = 3$ for example, so A gets 3 points.

[28] In this computation, we allow ballots like $P_1, P_1 \vee P_2 \rightarrow P_2$. This can be the result of strategic voting (the voter voted for P_1 in the first round to "bother" another party, say P_3) or a voter that changed her mind between the first and second round.

Figure 17: A 5-composition of 10.

In this way, each allocation of bars to the gaps is a m-composition of Y, and there are $\binom{Y-1}{m-1}$ ways to do this.
We are in fact looking for the number of *weak* m-compositions of Y: that is, the number of m-compositions of Y, but we drop the requirement that $a_j > 0$, i.e. a way of writing Y as the sum of m non-negative integers. This is equal to the number of different ballots without the assumption that every party must have at least one point, hence the number we are looking for. Because $a_1+\ldots+a_m = Y+m$ implies that $(a_1-1)+\ldots+(a_m-1) = Y$, we have that every m-composition of $Y+m$ corresponds to a weak m-composition of Y. Hence, the number of weak m-compositions of Y equals $\binom{Y+m-1}{m-1} = \binom{Y+m-1}{Y}$. □

For preference voting, a ranked ballot contains as many parties as the voter wishes to include. When a voter wants to declare a full ordering over all m parties, the number of different orderings is $m!$, as for the Borda rule. When a voter wants to declare an ordering over $m-j$ parties, where $0 \leq j \leq m-1$, the number of different orderings is $(m-j)!\binom{m}{m-j}$. For $j = 0, \ldots, m-1$, each of these orderings of $m-j$ parties is possible, hence the total number of possible ballots is

$$\begin{aligned}\sum_{j=0}^{m-1}(m-j)!\binom{m}{m-j} &= \sum_{j=0}^{m-1}(m-j)!\,\frac{m!}{(m-j)!\,j!} \\ &= \sum_{j=0}^{m-1}\frac{m!}{j!} \\ &= m!\sum_{j=0}^{m-1}\frac{1}{j!} \\ &\sim m!\,e.\end{aligned}$$

For a comparison of this property for the five voting rules, see Table 5.

Rule	# ballots in m	$m = 5$	$m = 10$
Plurality	m	5	10
Borda	$m!$	120	3,628,800
Two-round	$m^3 - m^2$	100	900
Cumulative	$\binom{Y+m-1}{Y}$	$\frac{1}{24}\prod_{i=1}^{4}(Y+i)$	$\frac{1}{9!}\prod_{i=1}^{9}(Y+i)$
Preference	$m!\sum_{j=0}^{m-1}\frac{1}{j!}$	325	9,864,100

Table 5: Comparison of the number of different ballots for five voting rules.

There is no direct comparison between cumulative voting and each of the other four voting rules, because of the parameter Y. When $Y = 1$, then $\binom{Y+m-1}{Y} = \binom{m}{1} = m$. Hence, it has the same number of different ballots as the plurality rule. This makes sense, since cumulative voting coincides with plurality voting for $Y = 1$.

It can be verified from Table 5, that cumulative and preference voting are the voting rules that are hardest to predict in terms of the number of possible ballots, if Y is not too small. The question remains which of these two numbers of different ballots is larger for Y fixed. In the case of $m = 5$, one can deduce that cumulative voting has a bigger number than preference voting for $Y > 7$. For $m = 10$, this is the case for $Y > 20$. For general m, we have that the number for preference voting is bigger than for cumulative voting when

$$\begin{aligned}
&\binom{Y+m-1}{Y} < m! \sum_{j=0}^{m-1} \frac{1}{j!} \\
\Leftrightarrow &\frac{1}{(m-1)!} \prod_{i=1}^{m-1} (Y+i) < m! \sum_{j=0}^{m-1} \frac{1}{j!} \\
\Leftrightarrow &\prod_{i=1}^{m-1} (Y+i) < m!\,(m-1)! \sum_{j=0}^{m-1} \frac{1}{j!} \\
\Leftrightarrow &Y^{m-1} + C_1 Y^{m-2} + \ldots + C_{m-1} < m!\,(m-1)! \sum_{j=0}^{m-1} \frac{1}{j!},
\end{aligned}$$

where $C_1, \ldots, C_{m-1}$ are the corresponding coefficients. This implies that

$$\begin{aligned}
&Y^{m-1} < m!\,(m-1)! \sum_{j=0}^{m-1} \frac{1}{j!} \\
\Leftrightarrow &Y < \sqrt[m-1]{m!\,(m-1)! \sum_{j=0}^{m-1} \frac{1}{j!}}.
\end{aligned}$$

Hence, by using the logical relation $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$, we find that the number for cumulative voting is higher than the number for preference voting when

$$Y > \sqrt[m-1]{m!\,(m-1)!\sum_{j=0}^{m-1}\frac{1}{j!}}.$$

This is of course not the best bound, by the omission of $\sum_{j=1}^{m-1} C_j Y^{m-j-1}$ in the fourth line, but we at least have a bound.
We can find a more precise bound by noting that

$$\prod_{i=1}^{m-1}(Y+i) = \sum_{k=0}^{m-1} Y^k (m-1)! \sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k}.$$

So we have that

$$\begin{aligned}
& Y < \sqrt[k]{\frac{m!}{k!}\Big(\sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k}\Big)^{-1}},\ \text{for all } k = 1,\ldots,m-1 \\
\Leftrightarrow & Y^k < \frac{m!}{k!}\Big(\sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k}\Big)^{-1},\ \text{for all } k = 1,\ldots,m-1 \\
\Leftrightarrow & Y^k \sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k} < \frac{m!}{k!},\ \text{for all } k = 1,\ldots,m-1
\end{aligned}$$

implies that

$$\begin{aligned}
& \sum_{k=0}^{m-1} Y^k \sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k} < \sum_{k=0}^{m-1}\frac{m!}{k!} \\
\Leftrightarrow & \frac{1}{(m-1)!}\sum_{k=0}^{m-1} Y^k (m-1)! \sum_{1\leq i_1<i_2<\ldots<i_k\leq m-1} \frac{1}{i_1 i_2 \cdots i_k} < m!\sum_{j=0}^{m-1}\frac{1}{j!} \\
\Leftrightarrow & \frac{1}{(m-1)!}\prod_{i=1}^{m-1}(Y+i) < m!\sum_{j=0}^{m-1}\frac{1}{j!} \\
\Leftrightarrow & \binom{Y+m-1}{Y} < m!\sum_{j=0}^{m-1}\frac{1}{j!}.
\end{aligned}$$

Together with the earlier bound for Y, we have two bounds which imply a strict ordering between the number for cumulative voting and the number for preference voting.
Even when sampling a group of voters is not possible or realisable, the numbers in the second row of Table 5 indicate how easy or complicated it can be to predict frequencies of certain groups or, more in the context of gerrymandering in general, certain VTD's. For voting rules with a low number of different ballots, such as the plurality rule, sampling is not even necessary to predict frequencies, although it of course gives a better prediction to do so. We conclude that voting rules such as cumulative voting and preference voting are, in this sense, the hardest to predict.

6.1.5 Conclusion

When comparing the Borda rule, two-round rule, cumulative voting and preference voting as potential substitutes of the plurality system, we find that the last two stand out as contenders. This is not even because of the reason that Steven J. Mulroy explained in his article, namely that they serve as a remedy for minority vote dilution. We find that they offer advantages from a social-choice theoretic point of view and, because of the reduction of wasted votes and difficult predictability, also serve as a remedy for gerrymandering with single-member districts. Preference voting even has a slight favour between the two. This voting rule is naturally strategyproof, a problem that can only be avoided for cumulative voting by adapting it to equal and even cumulative voting. Also, the reduction of wasted votes in single-member districts only makes sense for cumulative voting when adapting it to this special case. Equal and even cumulative voting, however, gives the voters less flexibility than for cumulative voting and for preference voting. As mentioned before, a possible disadvantage of preference voting is, that it sounds rather complicated and not every voter will immediately understand this procedure. However, each voter only needs to know how to construct a ballot and this is not as troublesome as understanding how preference voting operates.

With this solution in the form of alternative electoral systems extensively discussed, there is one more remedy for potential gerrymandering to be addressed. This is a district remedy and focuses on adjusting the redistricting map instead of adjusting the electoral system. In the next chapter we will give some insight about what remedy plan, i.e. district remedy or electoral system remedy, will be practical to use and will thoroughly discuss what a possible district remedy is. We will introduce an algorithm that composes a “fair” redistricting map.

7 Prevention by Redistricting

In the last chapter we addressed a way to reduce gerrymandering by altering the voting system. In this chapter we present another way that gets to the root of the problem, namely by redrawing the districts. First we discuss which of the two solutions to use in which case. Then we look at two methods concerning the redistricting of the map.

7.1 Choosing Between Solutions

We will briefly address the question when courts should impose a district solution, or when to impose an alternative system solution, as discussed in the last chapter, without discussing the judicial aspects of this question too much in detail. For a more extensive discussion on this, we refer again to the article of Steven Mulroy [26].

In the United States, the Decennial Census is the once-a-decade population count of all the 50 states and the results are used to (re)draw legislative districts [29].[29] Implementing an alternative system, instead of redrawing the districts, is a potentially more attractive approach in the sense that it circumvents the burden of a redistricting every decade after the Decennial Census. However, an alternative system solution can be troublesome in its own way, depending on the voting system being applied. In the previous chapter we concluded that preference voting would serve as the best alternative voting system among the four described. For pure gerrymandering prevention, changing the nationwide voting system from plurality voting to preference voting might seem like a big leap. Proponents of district solutions will probably state that redistricting every ten years is the more practical solution.

Not only this practical reasoning can be decisive for the choice between these two solutions, there are also legal considerations. For example, a district solution must be scrutinized to verify whether it fully complies with the Voting Rights Act, an act in the United States that prohibits racial discrimination in voting [30]. Another remark is, that the court should choose the solution that conflicts with the state law the least, i.e. does the least violence to state law [26].

In the case that the court chooses to go for a prevention (or reduction) tactic in the form of redistricting, we will present a way to draw districts that is supposed to be fair. The key point is to introduce precise criteria with which any district map should comply.

7.2 Changing the Map

Firstly, we introduce the Shortest Splitline Algorithm, an algorithm that outputs a fair redistricting map without using election results from the past, but

[29] In the rest of this discussion, we solely consider the situation in the United States. Therefore, names like the Decennial Sensus or U.S. Supreme Court only apply to this scenario, but the reasoning is the same for (perhaps hypothetical) countries other than the United States.

incorporating the population distribution. Then we make the (trivial) observation that an independent commission would eliminate the partisanism from redrawing districts. Related to this is the competitive district method, where the commission prioritizes districts that satisfy a certain criterion (competitiveness) and chooses the best among those suitable redistricting plans.

7.2.1 Shortest Splitline Algorithm

We will describe a simple algorithm, designed by the Center of Range Voting [31], that outputs a district map with simple-looking districts, such that each of these districts have roughly the same population count. The input required is the shape of the county with the distribution of residents (not necessarily voters) and the number of districts S. For a more formal algorithm, we will use the mathematical convention of district shapes as used in Chapter 5. Therefore, let C denote the Jordan curve representing the border of a region. The interior of C, denoted by $\text{int}(C)$, now incorporates additional data, namely the distribution of residents. Let there be n residents in the county.
Algorithm 5 provides such a construction. We denote by $S(C)$ the number of districts that has to be contained in $\text{int}(C)$ and by $N(C)$ the number of residents living in $\text{int}(C)$.

Algorithm 2. *Shortest Splitline Algorithm*

1. Let C be the Jordan curve representing the county border, so that $\text{int}(C)$ is the county. Initialize $S(C) = S$, $N(C) = n$ and $\mathcal{C} = \{C\}$.
2. Define $A(C) = \lceil \frac{S(C)}{2} \rceil$ and $B(C) = \lfloor \frac{S(C)}{2} \rfloor$.
3. For $C \in \mathcal{C}$: From *all pairs of points* p and q on C such that
   - $N(C_1 \odot l(p, q)) = N(C) \times \frac{A(C)}{S(C)}$,
   - $S(C_1 \odot l(p, q)) = A(C)$,
   - $N(C_2 \odot l(p, q)) = N(C) \times \frac{B(C)}{S(C)}$ and
   - $S(C_2 \odot l(p, q)) = B(C)$

   where $l(p, q)$ is the straight line segment from p to q that lies in $\text{int}(C)$, C_1 is a path over C from p to q and C_2 is the other path over C from p to q, choose the pair p and q with *shortest length* $l(p, q)$. If $S(C)$ is even, $l(p, q)$ divides the region into two parts with same population count. If there is an exact length-tie for "shortest" then break that tie by using the line closest to North-South orientation (explained why below).
4. $\mathcal{C} = \mathcal{C} \cup \{C_1 \odot l(p, q), C_2 \odot l(p, q)\} \backslash \{C\}$.
5. Continue from step 3 until $|\mathcal{C}| = S$. Each element of $\mathcal{C}$ is now a district in the county.[30]

[30] If anybody's residence is split in two by one of the splitlines, then they are automatically declared to lie in the most-western of the two districts.

In the case of multiple shortest lines, the tie-breaking rule in step 3 tells us to choose the line that is closest to a fully vertical line. In other words, choose the line that has the smallest positive (counterclockwise) angle with a vertical, straight line. If there is still a tie between multiple lines, then they must all be parallel. Because they split the population in exactly the same way, there are no inhabitants between these parallel lines, hence the choice between these is indifferent for the outcome.
This algorithm comes down to splitting each region approximately in half, so that we end up with exactly S districts which all roughly have the same size. Note that each district is not built up from VTD's as encountered in Chapter 4. See Figure 18 for a sketch of the state Missouri, if it has to be divided into 13 districts. After one iteration of Algorithm 5, the state has been divided by the green line into two regions, where the region above will contain 7 districts and the other region 6 districts. Six highly populated cities in Missouri are depicted, that clarify the placement of the green line a little.

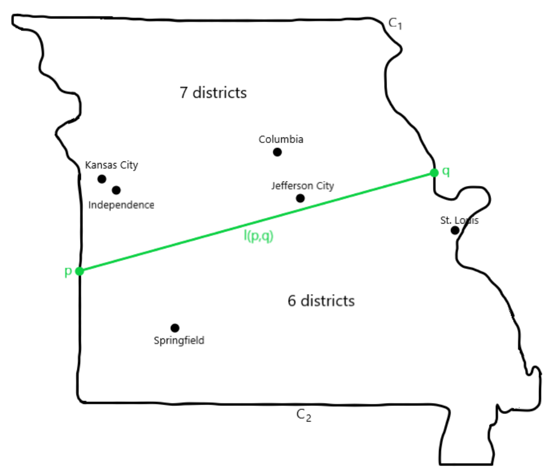


Figure 18: A sketch of the American state Missouri.

The reasoning behind the choice to pick the shortest straight line with smallest positive angle, and not just any straight line that divides the region in the right way, is to ensure that this algorithm generates a unique map[31] and stays non-partisan. To clarify this, look at Figure 19, where a fictional state is sketched. There are six towns in this state (the rest of the state is uninhabited), where each town has a population count that is roughly a thousand times the number depicted on the map. In this state, there is a plurality contest between the red and the blue parties. The color of the circle corresponding to the town represents the voting behaviour of a large majority in that town, e.g. it is likely

[31] Unique up to parallelity as discussed before.

that a large group of residents living in the town in the upper left corner is going to vote for the red party. If the state has to be divided into two districts, the straight line bisects the state such that each district has roughly 10,000 residents. Without the requirement of choosing the shortest such line, there are multiple placements of this line. The vertical green line is the shortest, hence will be picked by the algorithm. However, the blue party benefits from the horizontal green line, as this increases the probability of winning at least one district (the lower district in this case), while the vertical line almost ensures that the red party wins in both districts. This example shows that dropping the requirement of choosing the shortest line (with a certain angle)[32] can result in multiple options, leaving the algorithm open for bias.

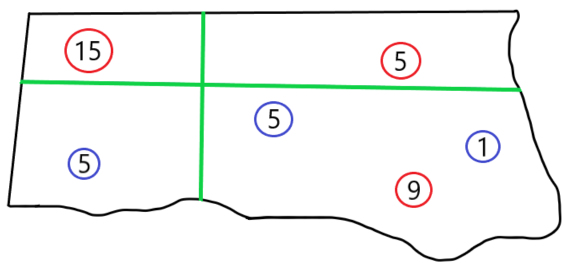


Figure 19: A fictional state with six towns. Population count $\times 1,000$.

The advantages of this algorithm and the map it produces, are i) the simplicity of the shapes of the districts, ii) the uniqueness of the map (partly due to the tie-breaking rule in point 3 of Algorithm 5) and iii) the fact that it is created following an objective and non-partisan procedure. See Figures 20 and 21 for the difference between the map of Tennessee in 2004 without using the Shortest Spitline Algorithm and using it [31].

[32]To see that it also differs to pick the (shortest) line with smallest positive angle, picture a state with the shape of a square. Among two straight lines with even length, a horizontal and a vertical one, the algorithm picks the vertical one.

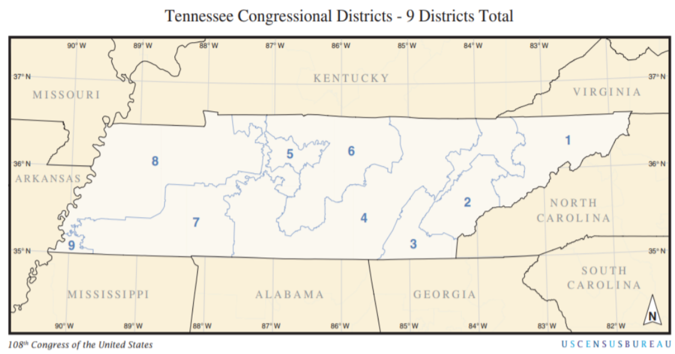


Figure 20: Tennessee divided into 9 congressional districts.

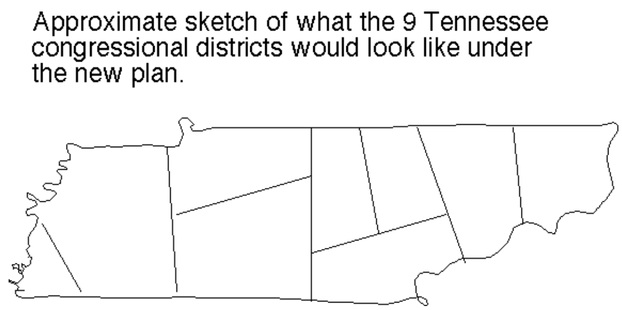


Figure 21: A sketch of Tennessee after applying SSA.

However, the simplicity is also due to ignoring geographical features, such as rivers, cliffs or lakes, which is a drawback of this algorithm. Another (minor) drawback is that the line sometimes cuts through a large metropolis, dividing residents from this metropolis into different districts. We nonetheless think that the reduction of gerrymandering, by creating a map that prevents bias and corruption[33] in the process, preponderates the disadvantages.

7.2.2 Redistricting by Neutral Commission

A straightforward solution of gerrymandering is to create an independent and non-partisan (boundary) commission that has the authority to redistricting, hence determining the boundaries. This is a way to remove the bias from the decision of adopting a certain redistricting plan. The U.S. state of Ohio proposed a constitutional amendment, where, with a majority of yes-voters, a state redistricting commission would be accepted [33]. One of the requirements is that *the new commission would be composed of five members, two of whom would be*

[33] In 2001, California State Democrats paid political consultant Michael Berman 1.36 million USD to draw the US House district map for California [31]

chosen by sitting judges, and the remaining members appointed by the first two or chosen by lot, as cited from the second bullet point in [33]. Another example is Australia, where the independent Australian Electoral Commission is in charge of determining electoral boundaries [34].

The amendment of Ohio gives us another method of altering the map, by creating criteria that the map should satisfy. Those criteria relate to the so-called *competitiveness* of a district.

7.2.3 Competitive District Method

The independent commission, created by the amendment of Ohio, has a priority to accept a redistricting plan where each district is competitive. Broadly speaking, a district is competitive, when there is a close competition between the party that won and the party that finished second, the runner-up, based on results from the previous election. More formally, a district is competitive if the ratio of competitiveness in a district is not too large[34]. Blais and Lago argued in their article [32], that a general measure of district competitiveness in an SMP system (Single Member Plurality) is proportional to the *margin of victory*. The margin of victory in an SMP system indicates the additional number of votes the runner-up would have needed to win the election[35]. However, the more competitive a district (in the sense that there is a close competition between the winner and the runner-up), the less the margin of victory. Therefore, a general measure of district competitiveness will be called Lack of Competitiveness of a district, say d_i, denoted by LoC(d_i)[36].

LoC(d_i) is equal to the margin of victory divided by the number of votes cast in district d_i. More mathematically, recall that

$$S_i^P = \begin{cases} 1 & \text{if party } P \text{ won in district } d_i \\ 0 & \text{if party } P \text{ lost in district } d_i, \end{cases}$$

and

$$X_i^P = \begin{cases} 1 & \text{if party } P \text{ is runner-up in } d_i \\ 0 & \text{if party } P \text{ is (at most) third winner in } d_i, \end{cases}$$

based on previous election results. Define $\mathcal{P}$ again to be the set of parties, T_i^P to be the number of votes for party P in district d_i and $T_i = \sum_{P\in\mathcal{P}} T_i^P$ to be the total number of votes in district d_i, where $P \in \mathcal{P}$ and $1 \leq i \leq S$.

[34] One can already sense a contradiction between the terms "competitive" and "ratio of competitiveness". We will come back to this.

[35] Unlike the calculation of wasted votes, these additional votes are not from voters who change party. The margin of victory can be interpreted as the number of additional voters, who vote for the runner-up, needed to be added to the district to guarantee the runner-up to win.

[36] The "reverse logic" encountered earlier now makes sense: the less competitive a district, the more the Lack of Competitiveness is.

Definition 20.

$$LoC(d_i) = \frac{\sum_{P\in\mathcal{P}} S_i^P T_i^P - \sum_{P\in\mathcal{P}} X_i^P T_i^P}{T_i}$$

The numerator can be rewritten as $\sum_{P\in\mathcal{P}} T_i^P[S_i^P - X_i^P]$. This is just the difference between the number of votes of the winner and runner-up.
We consider a district d_i *competitive* if $LoC(d_i) \leq R$ for a suitable $R \in (0,1)$, based on the results of the previous election. Thus, when the runner-up needed at most RT_i additional votes to win district d_i, the district is called competitive, due to a close competition. A *qualifying plan* is a redistricting plan where every district is competitive.
We consider a district d_i *balanced* when $LoC(d_i) \leq Q$ for a suitable $Q \in (0,1)$, where $Q < R$. In this sense, a district being balanced is a stronger requirement than its being competitive. In a balanced district d_i, the runner-up needed at most QT_i points to win, which is smaller than RT_i, resulting in an even closer competition.
Referring back to the amendment of Ohio, we find a new method in which a (preferably independent) commission can accept or construct a certain redistricting plan. This method is to find the redistricting plan with the highest *competitiveness number*. According to [33] the competitiveness number of a plan is defined as follows: *The Amendment defines the "competitiveness number" of a plan by a mathematical formula, that is the product of the number of balanced districts multiplied by two, plus the total number of other remaining competitive districts, minus the total number of unbalanced uncompetitive districts multiplied by two.* To phrase this more mathematically, recall from Chapter 3 that $\mathcal{D} = \{d_1, \ldots, d_S\}$ is a set of S districts according to a given redistricting plan[37]. Define $\mathcal{D}_C$ to be the subset of $\mathcal{D}$ consisting of all competitive districts and $\mathcal{D}_B$ to be the subset consisting of all balanced districts, then $\mathcal{D}_B \subseteq \mathcal{D}_C \subseteq \mathcal{D}$. Then the verbal phrase converts to defining the competitiveness number of a (redistricting) plan as $2|\mathcal{D}_B| + (|\mathcal{D}_C| - |\mathcal{D}_B|) - 2|\mathcal{D}\backslash\mathcal{D}_C|$, where $|A|$ denotes the cardinality of set A. Rewriting this, we get the following definition.

Definition 21. The competitiveness number $CN(\mathcal{D})$ of a redistricting plan $\mathcal{D}$ is

$$CN(\mathcal{D}) = 3|\mathcal{D}_C| + |\mathcal{D}_B| - 2|\mathcal{D}|.$$

Hence, among all redistricting plans $\mathcal{D}$, choose the $\mathcal{D}$ with highest competitiveness number $CN(\mathcal{D})$. In other words, choose

$$\mathcal{D}^* = \operatorname*{argmax}_{\mathcal{D}} CN(\mathcal{D}). \tag{35}$$

[37]Moreover, $\mathcal{D}$ itself can be seen as a redistricting plan, since each district in $\mathcal{D}$ is uniquely determined by its shape resulting from the redistricting plan.

Additionally, this method picks a qualifying plan, i.e. a plan $\mathcal{D}$ where every district is competitive, if it exists. Among all qualifying plans, it will then look for the one that has the highest competitiveness number, which then reduces to $CN(\mathcal{D}) = |\mathcal{D}| + |\mathcal{D}_B|$, since $\mathcal{D} = \mathcal{D}_C$. So, if there is at least one qualifying plan, (35) reduces to

$$\mathcal{D}^* = \underset{\text{qualifying } \mathcal{D}}{\text{argmax}} |\mathcal{D}_B|.$$

If there are more choices for $\mathcal{D}^*$, the commission has to decide which plan to adopt. Here it is of course preferred to let an independent commission do this, to prevent any bias when picking between different alternatives. However, even though there may be a choice between different redistricting plans at the end, the commission will never have as much freedom as without using this method. Thus this method has a certain objectivity to it, but also ensures that each district does not have too much dispersion with respect to voters for different parties.

7.2.3.1 Constructing a Redistricting Plan

Consider each district as a collection of VTD's as explained in Chapter 4. A VTD, or voting district, is fixed and can be interpreted as a municipality that can not be divided into several districts. Let V be the set of VTD's and $\mathcal{D} = \{d_1, \dots, d_S\}$ denote the set of districts. A redistricting plan is formally a surjective function $\xi : V \to \mathcal{D}$. Recall from Chapter 4 that $\mathcal{D}_\xi$ is the set of districts after applying redistricting function ξ. In this chapter however, we considered $\mathcal{D}$ as a redistricting plan. In the case that a district is built up of multiple VTD's, we actually mean the redistricting plan ξ that has been implicitly applied to V to get a set of districts $\mathcal{D}$, where $\mathcal{D} = \{\xi^{-1}(d_1), \dots, \xi^{-1}(d_S)\}$. As an extra requirement, we can look for all *legal* redistricting plans and pick the one with the highest competitiveness number, while still being a qualifying plan if possible. This will reduce the total number of qualifying plans we have to choose from drastically and leaves us with a set of redistricting plans that have purposeful properties, such as contiguity and (geo)compactness of the districts. Therefore, run Algorithm 1 for the construction of a redistricting plan ξ a large amount of times. The algorithm does not always terminate, but when it does, it does so with a legal redistricting plan ξ. We denote by Ξ the set of legal redistricting plans as output from the algorithm, i.e. $\Xi = \{\xi_1, \dots, \xi_k\}$, where k is the number of outputted plans. Define $\mathcal{D}_i$ such that $\mathcal{D}_i = \{\xi_i^{-1}(d_1), \dots, \xi_i^{-1}(d_S)\}$ for $1 \leq i \leq k$ and let $\mathbb{D} = \{\mathcal{D}_1, \dots, \mathcal{D}_k\}$. Then define $\mathbb{D}_C \subseteq \mathbb{D}$ to be the set of qualifying plans. For the construction of $\mathbb{D}_C$, check Algorithm 3.[38]

[38] Algorithm 3 always terminates, but can terminate with $\mathbb{D}_C$ being an empty set.

Algorithm 3. Construction of $\mathbb{D}_C$

$\mathbb{D}_C = \{\}$;

for $\mathcal{D} \in \mathbb{D}$

    comp = 1;

    for $d \in \mathcal{D}$

        if $LoC(d) \leq R$

            comp $\rightarrow$ 1;

        else

            comp $\rightarrow$ 0;

    if comp == 1

        $\mathbb{D}_C \rightarrow \{\mathbb{D}_C, \mathcal{D}\}$;

Then construct $\mathcal{D}^*$ according to Algorithm 4 (which is in pseudocode).

Algorithm 4. Construction of $\mathcal{D}^*$.

if $\mathbb{D}_C \neq \emptyset$

    $\mathcal{D}^* \rightarrow \underset{\mathcal{D}\in\mathbb{D}_C}{\operatorname{argmax}} |\mathcal{D}_B|$;

else

    $\mathcal{D}^* \rightarrow \underset{\mathcal{D}\in\mathbb{D}}{\operatorname{argmax}}\, 3|\mathcal{D}_C| + |\mathcal{D}_B|$;

The dichotomy in Algorithm 4 comes from the fact that $CN(\mathcal{D}) = |\mathcal{D}| + |\mathcal{D}_B|$ when $\mathcal{D} \in \mathbb{D}_C$. When Algorithm 3 leaves $\mathbb{D}_C$ empty, there are no qualifying plans in the simulated set $\mathbb{D}$ and we choose $\mathcal{D}^*$ as in Equation 35. $\mathcal{D}^*$ is the "best choice" among all k simulated legal redistricting plans.

8 Appendix

8.1 Appendix A

We will formally prove here that $C^i_{min} = \underset{C \in S^i_C}{\operatorname{argmin}} A(C)$ indeed exists, as encountered in Section 5.1.2. Recall that we denote by S^i_C the set of circles in $\mathbb{R}^2$ containing district d_i.
Let $A^* = \inf_{C \in S^i_C} A(C)$, say $A^* = \pi(r^*)^2$. This exists, since surfaces are bounded below by 0.
A circle C in $\mathbb{R}^2$ can be defined by the pair (r, m), where $r \in \mathbb{R}_+$ is the radius of C and $m \in \mathbb{R}^2$ is the centre. Choose a sequence of circles $(C_n)_{n \geq 0} \subset S^i_C$ where $C_n = (r_n, m_n)$, with the property that $A(C_n) \downarrow A^*$ as $n \to \infty$. Then, since $r^* = \sqrt{\frac{A^*}{\pi}}$, we have that $r_n \downarrow r^*$. In other words, $\lim_{n\to\infty} r_n = r^*$.
Since each district is bounded and the centre of a circle in S^i_C must be contained in district d_i, we know that $(m_n)_{n \geq 0}$ is a bounded sequence. By the Bolzano-Weierstrass theorem $(m_n)_{n \geq 0}$ has a convergent *subsequence*. Denote this subsequence by $(m_{n_k})_{k \geq 0}$. For example, the sequence $(m_n)_{n \geq 0} = (1, 2, 3, 1, 2, 3, 1, \ldots)$ is bounded (it contains only three elements in $\mathbb{R}$). It has thus a convergent subsequence, e.g. $(m_{n_k})_{k \geq 0} = (1, 1, 1, \ldots)$, where $n_0 = 0, n_1 = 3, n_2 = 6$ and so forth. The subsequence $(m_{n_k})_{k \geq 0}$ has a limit, i.e. $\lim_{k\to\infty} m_{n_k} = m^*$. We also have that $\lim_{k\to\infty} r_{n_k} = r^*$, by picking any subsequence $(r_{n_k})_{k \geq 0}$.
We now have that $(r_{n_k}, m_{n_k}) \to (r^*, m^*)$ for $k \to \infty$. Does this also mean that $C_{n_k} \to C^*$ for $k \to \infty$, where $C^* = (r^*, m^*)$? The answer is yes; every point $p^\alpha_{n_k}$ on a circle C_{n_k} can be described by

$$p^\alpha_{n_k} = m_{n_k} + r_{n_k} \times (\cos(\alpha), \sin(\alpha)), 0 \leq \alpha \leq 2\pi.$$

So, for $k \to \infty$, every such point converges to

$$p^\alpha = m^* + r^* \times (\cos(\alpha), \sin(\alpha)), 0 \leq \alpha \leq 2\pi,$$

which is a point on C^*. Furthermore, C^* is a circle containing d_i, i.e. $C^* \in S^i_C$. To see this, note that every point $p^\alpha_{n_k}$ on C_{n_k} lies in $\overline{\mathbb{R}^2 \backslash d_i}$, the *closure* of the complement of d_i. This is a closed set, so every limit point $p^\alpha = \lim_{k\to\infty} p^\alpha_{n_k}$ lies in $\overline{\mathbb{R}^2 \backslash d_i}$ as well, concluding that C^* contains d_i (see Figure 22).
Hence $(C_{n_k})_{k \geq 0}$ is a sequence with $\lim_{k\to\infty} C_{n_k} = C^*$. Therefore

$$\inf_{C \in S^i_C} A(C) = \inf_{C \in S^i_C} \pi r(C)^2 = \pi r(C^*)^2 = \pi(r^*)^2,$$

where $r(C)$ is the radius of circle C. Hence, $\underset{C \in S^i_C}{\operatorname{argmin}} A(C)$ exists, since the minimum and infimum coincide as $C^* \in S^i_C$.

8.2 Appendix B

The proof of the existence of $C^i_{max} = \underset{C \in S_i^C}{\text{argmax}}\, A(C)$ has a similar approach as Appendix A. Here we choose a sequence of circles $(C_n)_{n \geq 0} \subset S_i^C$ where $C_n = (r_n, m_n)$, but with the property that $A(C_n) \uparrow A^*$ as $n \to \infty$, where $A^* = \sup_{C \in S_i^C} A(C)$. Via similar arguments we conclude that $C^* = (r^*, m^*)$, where $r^* = \lim_{k \to \infty} r_{n_k}$ and $m^* = \lim_{k \to \infty} m_{n_k}$, has the maximum area of all circles in S_i^C.

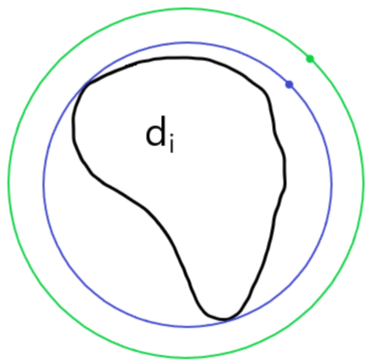


Figure 22: District d_i with two circles in S^i_C. The green circle is C_{n_7} and the blue circle is C^*. The green dot is point $p_{n_7}^{\frac{5}{18}\pi}$ and the blue dot is point $p^{\frac{5}{18}\pi}$. Both circles lie in $\overline{\mathbb{R}^2 \backslash d_i}$, which is the region outside d_i (but including the border).

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