Synchronized Push and Shove
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Abstract

Combinatorial games where both players choose a move simultaneously are called synchronized games. If we consider games where any pair of Left and Rights moves can always be performed in some order, then a synchronized version of the game comes naturally. As it turns out, such games are always numbers as combinatorial games. However, games which are equal to each other as combinatorial games may behave differently as synchronized games. We may still assign each game a numerical value. One approach is to view synchronized games as zero-sum matrix games, and we call the value of the zero-sum game associated to a game its Nash value. Push and Shove are examples of rule sets which have natural synchronized versions, and for synchronized Push and Shove we show a result which is similar to the number avoidance theorem for combinatorial games. We also show that, for certain positions of Push and Shove, the difference in Nash value of n copies of the position and $n-1$ copies of the position tends to the combinatorial value of the position as n tends to infinity.

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1 Introduction

In *combinatorial game theory* we typically study two-player sequential games, that is, games where two players take turns moving in the game. Combinatorial game theory started out by only considering impartial games, which are games in which any move available to one player must also be available to the other player. This requirement is relaxed in partisan games, which Elwyn R. Berlekamp, John H. Conway and Richard K. Guy jointly introduced in the books *Winning Ways for your Mathematical Plays* [3] and *On Numbers and Games* [6].

Alessandro Cincotti and Hiroyuki Iida where the first to study a rule set with the stipulation that games are played sequentially also relaxed [5], leading to so called *synchronized games*. Here, the two players move at the same time. Most of the results of combinatorial game theory break down under synchronized play. Among those is the equivalence relation on the group of short games, which also includes the subgroup of numbers. In [5], games are assigned a different type of value. However, this definition is problematic for certain synchronized games. A different approach is to assign end positions integer values based on the winner and the number of moves (s)he has left, and non-end positions are recursively assigned the value of the zero-sum game of the values of its options.

For synchronized games, we will introduce the concept of separability, which relates to the ability to perform a synchronized move by letting the players move sequentially. Whether or not games are separable depends on the rule set. A key result is that all separable games are numbers in combinatorial sense. Furthermore, we will observe that the difference in Nash value between n copies and $n-1$ copies of certain separable games tends to the combinatorial value of the game, as n tends to infinity.

This thesis is structured as follows. Section 2 introduces the relevant background in combinatorial game theory, as well as some useful theorems on zero-sum games and a brief explanation on a data storage method that we used to efficiently compute the value of large synchronized games. In Section 3 we talk about the games Push and Shove, which are the main rule sets that we study. In Section 4 we introduce synchronized play and define the Nash value for synchronized games, as well as discuss separable games. We finish by studying several positions of synchronized Push and synchronized Shove in Section 5.

2 Preliminaries

2.1 Combinatorial game theory

This section introduces the topics in combinatorial game theory that are used throughout this thesis. It has largely been adapted from *Combinatorial game theory* by Siegel [9] and *Lessons in Play* by Albert, Nowakowski and Wolfe [2]. The world of combinatorial game theory is much larger than what we discuss here. To read more, as well as to find proofs of the theorems we present in this section, we refer to the introductory chapters of these works.

2.1.1 Core definitions and terminology

In combinatorial game theory, we study games played between two players. Traditionally, these players are called *Left* and *Right*, although various other names are used in the literature, as well. In this thesis, we will always use Left and Right. In games with colored pieces, Left will always use the color bLue, while Right will use the color Red.

Left and Right take turns playing a game. When it is a player's turn to move, (s)he can consider all possible game states that result from each respective move. These are called the *options* of a game. When an option is chosen, the respective move is made and the process repeats with the other player choosing one of the options from the current game state. The winner of the game is the player that makes the last move. This is called *normal play*, as opposed to *misère play*, in which the last player to move loses the game. Because a game is characterised by the options and those options themselves are games as well, we arrive at the following formal recursive definition of game.

Definition 2.1. A *game* (or *position*) G is defined as the set of its options,

$$G = \left\{G_1^L, \dots, G_m^L \,\middle|\, G_1^R, \dots, G_n^R\right\},$$

where the games $G_1^L, \dots, G_m^L$ are the *Left options* of G and the games $G_1^R, \dots, G_n^R$ are the *Right options* of G. More commonly, this is written as

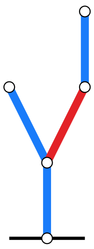
$$G = \left\{\mathcal{G}^L \,\middle|\, \mathcal{G}^R\right\},$$

where $\mathcal{G}^L$ and $\mathcal{G}^R$ denote the sets of Left and Right options, respectively.

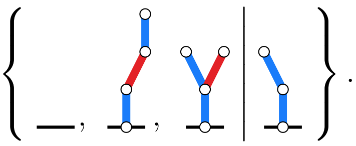
The smallest game is $\{\,|\,\}$, in which neither player can make a move. We denote this game by 0.

Remark 2.2. For a game G, we use G^L resp. G^R as typical representatives of arbitrary Left resp. Right options of G. By our recursive definition, options are themselves games and thus have their own set of options. The set of Left options of a Left option G^L is denoted by $\mathcal{G}^{LL}$ and its set of Right options by $\mathcal{G}^{LR}$. Typical representatives of these sets are G^{LL} and G^{LR}, respectively. For Right, an analogous definition holds, as well as for higher order options and sets of options.

Example 2.3. Consider the game of *Red Blue-Hackenbush G*, given by

$G =$ 

For the rule set of Hackenbush, see [2, p. 309]. By looking at the options available to each player, we find that G is formally defined as the game

$G =$ 

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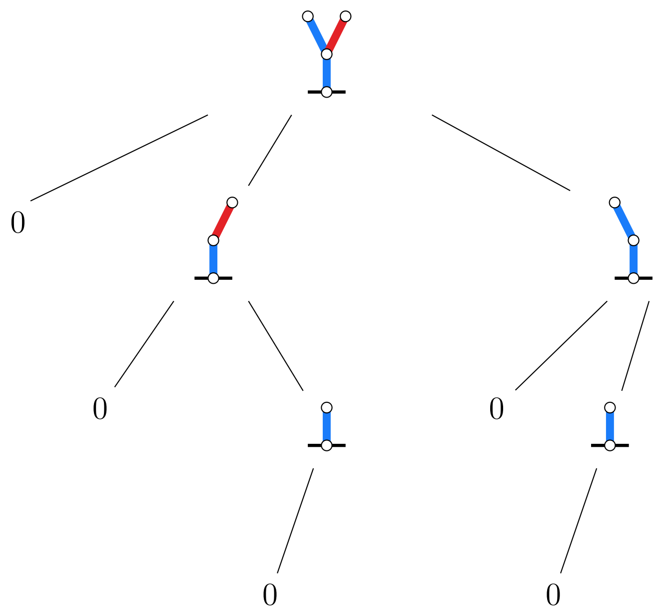
By definition, each option of a game is a game as well. If one writes each option as the set of its options, and those options as the sets of their respective options, etc., it leads to multiple levels of brackets. As games grow, this quickly does not lend itself well to readability. A common way to better visualize the structure of games is by drawing them as a tree graph, known as the *game tree*. The game tree of a game G is a directed tree graph with root G. From G we draw directed edges to its options, and we draw directed edges from each option to their respective options, etc. We draw trees downwards and, as a visual aid, we draw Left options by branching to the left and Right options by branching to the Right. The leaves of the tree are the games in which both players have no options, i.e., the game 0.

Example 2.4. Consider the game tree of one of the Left options from the game from Example 2.3.



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We say that two games are *isomorphic* if they have the same structure, or in other words, if their game trees are isomorphic as graphs. A trivial example of this would be the Hackenbush position  and the Push position [P]. In both games, Left can move to 0 and Right has no legal move. Therefore, they have isomorphic game trees and we say that the two games are isomorphic, denoted by  $\cong$ [P].

The set of *positions* of a game are all those games which appear in its game tree, including the game itself. We will only consider so-called *short* games. Short games are games which have a finite game tree. Games where we drop this assumption are called *loopy* games, as they can loop around back to themselves. Additionally, we require that every position of a short game has only finitely many distinct options. Short games will always end in a finite number of moves, while loopy games may go on forever. This leads to the following useful definition.

Definition 2.5. The *birthday* of game G, denoted by $b(G)$, is defined as

$$b(G) = 1 + \max\left\{b(H)\colon H \in \mathcal{G}^L \cup \mathcal{G}^R\right\},$$

and $b(0) = 0$.

Intuitively, the birthday of a game is the height of its game tree. Short games will always have a finite birthday, while loopy games may have infinite birthday.

We say that a game is born by day n if its birthday is at most n. The only game born by day 0 is the game 0. Using the definition of a game, we can easily see that there are only four games born by day 1. These games are very common and are given their own names:

$$\begin{aligned} 0 &\stackrel{\text{def}}{=} \{\,|\,\}, \\ 1 &\stackrel{\text{def}}{=} \{0\,|\,\}, \\ -1 &\stackrel{\text{def}}{=} \{\,|\,0\}, \\ * &\stackrel{\text{def}}{=} \{0\,|\,0\}. \end{aligned}$$

We can define the set of all games born by day n,

$$\mathbb{G}_n \stackrel{\text{def}}{=} \{G\colon b(G) \leq n\}.$$

The set of all short games is then given by

$$\mathbb{G} \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \mathbb{G}_n.$$

One of the core principles in combinatorial game theory is that games can be broken down into their components, which can be analyzed individually. The reverse of this is that we must define what it means to add games together.

Definition 2.6. Let G, H be combinatorial games. The *(disjunctive) sum* $G + H$ is recursively defined as

$$G + H \stackrel{\text{def}}{=} \left\{ \mathcal{G}^L + H, G + \mathcal{H}^L \,\middle|\, \mathcal{G}^R + H, G + \mathcal{H}^R \right\},$$

where $\mathcal{G}^L + H = \{G^L + H\}_{G^L \in \mathcal{G}^L}$, etc.

Intuitively, this means that the sum of two games is the same as laying the games next to each other and allowing a move to be made on any of the components. This is why in the game tree, we do not omit the paths where one player makes consecutive moves; it could be the case that this game is part of a sum of games, and the other player moved elsewhere, allowing for consecutive moves on components of the game.

Another important definition is that of the negative of a game.

Definition 2.7. Let G be a combinatorial game. The *negative* of G, written as $-G$, is recursively defined as

$$-G \stackrel{\text{def}}{=} \left\{ -\mathcal{G}^R \,\middle|\, -\mathcal{G}^L \right\}.$$

The negative of a game can be thought of as switching the players. For instance, in Hackenbush, this constitutes to swapping the colors of the edges. With respect to the disjunctive sum, we have

$$G - H \stackrel{\text{def}}{=} G + (-H).$$

2.1.2 Outcome classes and the partial order on $\mathbb{G}$

Theorem 2.8 (Fundamental Theorem of Combinatorial Game Theory)**.** *Let G be a short combinatorial game, and assume normal play. Then, either Left can force a win playing first on G, or Right can force a win playing second, but not both.*

Proof. Consider a Left option G^L. By induction on birthday, on G^L, either Right can force a win playing first, or Left can force a win playing second, but not both. If Right can win *all* such G^L by playing first, then he wins G playing second, since Left must move to any G^L. However, if there is a G^L on which Left wins playing second, then she wins G by moving to such a G^L. Exactly one of these possibilities must hold. □

Remark 2.9. Note that in the proof of Theorem 2.8, we used induction without specifying the base case. This type of induction is called *top-down induction* and is commonly used in combinatorial game theory. With top-down induction, the base case if often handled implicitly within the proof. For instance, in the proof above, there is an implicit base case in which Left has no moves at all. Since Left now loses playing first, this also satisfies the inductive hypothesis. Other times the base case may simply be trivial. This usually happens when induction on the birthday of a game is used. The game 0 is the only game with birthday 0, and most statements hold trivially for 0.

A consequence of the Fundamental Theorem is that $\mathbb{G}$ can be partitioned into four distinct classes that we refer to as the *outcome classes*. We denote the outcome class (or simply the *outcome*) of a game G by $o(G)$. As the name suggests, the outcome class of a game determines the winner(s) of the game, provided that both players play perfectly. The four classes are the following:

Class	Name	Definition
$\mathcal{N}$	Fuzzy	The $\mathcal{N}$ext player to move can force a win.
$\mathcal{P}$	Zero	The $\mathcal{P}$revious player (or *second* player) to move can force a win.
$\mathcal{L}$	Positive	$\mathcal{L}$eft can force a win regardless of who moves first.
$\mathcal{R}$	Negative	$\mathcal{R}$ight can force a win regardless of who moves first.

Definition 2.10. Let G and H be short games. We write $G = H$ if

$$o(G + X) = o(H + X) \text{ for all } X \in \mathbb{G}.$$

Note that $G \cong H$ implies $G = H$. The converse is usually not true.

Lemma 2.11. $=$ *is an equivalence relation on* $\mathbb{G}$.

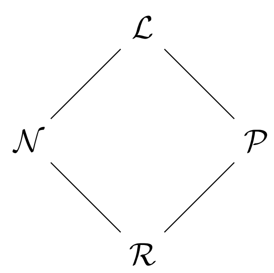
Theorem 2.12. $(\mathbb{G}, +)$ *is an Abelian group.*

Theorem 2.13. *Let G be a short game. Then, $G = 0$ if and only if $o(G) = \mathcal{P}$.*

Corollary 2.14. *For short games G and H, we have $G = H$ if and only if $G - H = 0$.*

Theorem 2.13 and Corollary 2.14 are the main tools we use to show that two games are equal, as working directly from the definition is often infeasible. Instead, we only have to show that the difference game is won by the second player to move, which we do by considering every opening move and showing that the other player can force a win.

The outcome classes can be ordered by favorability to a particular player. As mentioned previously, the classes $\mathcal{L}$ and $\mathcal{R}$ are called Positive and Negative, respectively. Therefore, it makes sense to say that $\mathcal{L} \geq \mathcal{R}$. The classes $\mathcal{N}$ and $\mathcal{P}$ fall in between $\mathcal{L}$ and $\mathcal{R}$, as in those classes, both Left and Right can win depending on who moves first. It turns out that $\mathcal{N}$ and $\mathcal{P}$ are incomparable, which we denote by $\mathcal{N} \parallel \mathcal{P}$. This gives us the following partial order on the outcome classes:



We can extend this to $\mathbb{G}$.

Definition 2.15. Let G and H be short games. We write $G \geq H$ if

$$o(G + X) \geq o(H + X) \text{ for all } X \in \mathbb{G}.$$

Lemma 2.16. *$\geq$ is a partial order on $\mathbb{G}$:*
For all $G, H, I \in \mathbb{G}$, we have

(i) $G \geq G$.

(ii) If $G \geq H$ and $H \geq G$, then $G = H$.

(iii) If $G \geq H$ and $H \geq I$, then $G \geq I$.

To fully describe the partial-order relation between any two short games we introduce some more notation.

Definition 2.17. Let G and H be short games. We say that G is *confused with* H ($G \parallel H$) if $G \ngeq H$ and $H \ngeq G$. We say that G is *greater than or confused with* H ($G \mathrel{|\!\!\triangleright} H$) if $H \ngeq G$. Lastly, we say that G is *less than or confused with* H ($G \mathrel{\triangleleft\!\!|} H$) if $G \ngeq H$.

Theorem 2.18. *Let G be a short game. The outcome class of G is determined by its partial-order relationship to 0:*

$$\begin{array}{lcllcl}
G = 0 & \Leftrightarrow & G \in \mathcal{P}, & G \geq 0 & \Leftrightarrow & G \in \mathcal{P} \cup \mathcal{L}, \\
G > 0 & \Leftrightarrow & G \in \mathcal{L}, & G \leq 0 & \Leftrightarrow & G \in \mathcal{P} \cup \mathcal{R}, \\
G < 0 & \Leftrightarrow & G \in \mathcal{R}, & G \mathrel{|\!\!\triangleright} 0 & \Leftrightarrow & G \in \mathcal{N} \cup \mathcal{L}, \\
G \parallel 0 & \Leftrightarrow & G \in \mathcal{N}, & G \mathrel{\triangleleft\!\!|} 0 & \Leftrightarrow & G \in \mathcal{N} \cup \mathcal{R}.
\end{array}$$

Since $\mathbb{G}$ is a partially ordered group, this can be extended to the partial-order relation between any two short games. For example, one can show that $G \mathrel{\triangleleft\!\!|} H$ by showing that the game $G - H$ is in $\mathcal{N} \cup \mathcal{R}$, or in other words, by showing that Right wins $G - H$ playing first.

2.1.3 Numbers

We begin with two methods that can be used to "simplify" games.

Theorem 2.19 (Domination)**.** *Let*

$$G = \{A, B, C, \cdots \mid H, I, J, \ldots\}.$$

If $B \geq A$, then $G = G'$, where

$$G' = \{B, C, \cdots \mid H, I, J, \ldots\}.$$

Similarly for Right, if $I \leq H$, then G is equal to G with option H removed. Here, we say that B resp. I dominates *A resp. H, as B is at least as favorable to Left as A and I is at least as favorable to Right as H.*

Theorem 2.20 (Reversibility)**.** *Let*

$$G = \{A, B, C, \cdots \mid H, I, J, \dots\},$$

and suppose that A *has some Right option* A^R *such that* $G \geq A^R$*. Suppose* A^R *has Left options* $\mathcal{A}^{RL} = \{W, X, Y, \dots\}$*. Then,* $G = G'$*, where*

$$G' = \{W, X, Y, \dots, B, C, \cdots \mid H, I, J, \dots\}.$$

In Theorem 2.20, we say that A is a *reversible option*, A^R is the the *reversing option* and $\mathcal{A}^{RL}$ is the *replacement set*. The intuition behind reversing is that if A^R is just as good or better than G for Right, he will move there is he gets the chance. This means that Left should consider moving to A only if she intends to immediately follow up Right's move to A^R with her own move to an option A^{RL}, and therefore she might as well consider those Left options as the Left options of G, instead of A being a Left option of G. An analogous statement holds for Right.

Before we show an example of these techniques in action, let us first define a few more special games. Previously, we defined the games $0, 1$ and -1. Likewise, we can define every integer as a game.

Definition 2.21. We define the game 0 as $0 \stackrel{\text{def}}{=} \{\,|\,\}$. For a positive integer n we define the games n and $-n$ recursively as

$$n \stackrel{\text{def}}{=} \{n-1 \mid \}, \qquad\qquad -n \stackrel{\text{def}}{=} \{ \mid -(n-1)\}.$$

In the integer game n, Left has n moves, while Right has no moves, and vice versa if n were a negative integer. The partial-order relation between the integers is exactly as expected.

Example 2.22. Consider $G = \{-3, 1 \mid \{4 \mid 2, 5\}\}$. First, we use domination to find $G = \{1 \mid \{4 \mid 2\}\}$. Next, we verify that $4 - G \geq 0$. Indeed, Right can only move to $4 - 1 > 0$, so Left wins $4 - G$ playing second. Since $G \leq 4$, we know that $\{4 \mid 2\}$ is a reversible option. The reversing option is $4 = \{3 \mid \}$, so the replacement set is empty. As a result, $G = \{1 \mid \}$, hence G is equal to the game 2. ◁

Definition 2.23. Let G be a short game. We say that G is in *canonical form* if no position of G has dominated or reversible options. The canonical form of G is denoted by $\text{can}(G)$.

The canonical form of a game G can be thought of as the "simplest" game such that $\text{can}(G) = G$. The canonical form is unique.

Theorem 2.24. *Let* G *and* H *be short games in canonical form and assume* $G = H$*. Then,* $G \cong H$*.*

Integers are the simplest games of their value. Hence, if a game is equal to an integer, than that integer is its canonical form.

Consider the game $G = \{0 \mid 1\}$. One can easily verify that $G + G - 1 \in \mathcal{P}$, hence $G + G = 1$. This suggests that we should call this game $\frac{1}{2}$, leading to the following definition.

Definition 2.25. For $n \geq 1$ we define the game $\frac{1}{2^n}$ recursively as

$$\frac{1}{2^n} \stackrel{\text{def}}{=} \left\{ 0 \,\middle|\, \frac{1}{2^{n-1}} \right\},$$

with $\frac{1}{2^0} \stackrel{\text{def}}{=} 1$. For m odd we also define

$$\frac{m}{2^n} \stackrel{\text{def}}{=} \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{m \text{ times}}.$$

Once again, these fractional games behave exactly as their rational counterparts.

Theorem 2.26. *Let $A, B, C \in \mathbb{G}$, and suppose that $A = a$, $B = b$ and $C = c$ for some $a, b, c \in \mathbb{Z} \cup \left\{ \frac{m}{2^n} : m \in \mathbb{Z} \text{ odd}, n \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}$. Then,*

(i) $A + B = C$ in $\mathbb{G}$ if and only if $a + b = c$ in $\mathbb{Q}$.

(ii) $A \geq B$ in $\mathbb{G}$ if and only if $a \geq b$ in $\mathbb{Q}$.

We defined the game $\frac{m}{2^n}$ for m odd as m copies of the game $\frac{1}{2^n}$. If we have an even number of copies, then by Theorem 2.26, it is equal to the game with the simplified fraction. The canonical form of any fraction with odd numerator is given by the following theorem.

Theorem 2.27. *For $n \geq 1$ and m odd, consider the game $G = \frac{m}{2^n}$. Then,*

$$\operatorname{can}(G) = \left\{ \frac{m-1}{2^n} \,\middle|\, \frac{m+1}{2^n} \right\}.$$

We call the set $\mathbb{D} = \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}$ the *dyadic rationals.* $\mathbb{D}$ is a subgroup of $\mathbb{Q}$, and we have an injective group homomorphism $\mathbb{D} \to \mathbb{G}$ mapping a dyadic rational to its corresponding game. As a result, $\mathbb{G}$ contains a natural subgroup isomorphic to $\mathbb{D}$. We will identify this subgroup with $\mathbb{D}$ itself, and refer to this subgroup and its elements as *numbers.*

Consider two numbers x and y and assume $x > y$. The *simplest number* z such that $x < z < y$ is the smallest (in absolute value) integer, if it exists. Otherwise, $z = \frac{i}{2^n}$ for some i and with n as small as possible. The following theorem tells us how to easily spot numbers.

Theorem 2.28 (Simplest Number Theorem)**.** *Let G be a short game and suppose that $G^L < G^R$ for all $G^L \in \mathcal{G}^L, G^R \in \mathcal{G}^R$. Then, $G = x$, with $x \in \mathbb{D}$ the simplest number such that $G^L < x < G^R$ for all $G^L \in \mathcal{G}^L, G^R \in \mathcal{G}^R$.*

Definition 2.29. Let G be a short game. The *Left incentive* of a move G^L is $G^L - G$. Similarly, the *Right incentive* of a move G^R is $G - G^R$. The *incentives* of G is the union of all Left and Right incentives of G.

Incentives can be thought of as the "amount gained" by a particular move. In the case of numbers, all incentives are negative, and the best move (by domination) has the greatest incentive. For games that are equal to integers, the best incentive is -1, meaning that moving on an integer "costs" exactly one move. For games that are not integers (or numbers), we always have a move that costs less than one move.

Theorem 2.30. *Let G be a short game, and assume G is equal to an integer. Then, G has both a Left incentive $G^L - G > -1$ and a Right incentive $G - G^R > -1$.*

As a result, it is always best to save your integers and to move on other components of a game. it turns out that one should avoid playing on any number, if possible.

Theorem 2.31 (Weak Number Avoidance)**.** *Let x, G be short games and assume that x is a number and G is not. If Left can win moving first on $x + G$, then Left can do so with a move on G.*

An analogous statement holds for Right. A stronger version of this statement also exists.

We end this section with another way to spot number games, in particular, integers.

Theorem 2.32 ([9] Exercise 3.5)**.** *Let G be a short game and suppose G has no Right options. Then, G is an integer.*

Proof. Let G be a short game with no Right options. The exact integer that G is equal to depends on the outcome class of its Left options.

If all $G^L \in \mathcal{P}$, then $G = \{0 \mid \} = 1$. If all $G^L \in \mathcal{N}$, then Left loses G playing second. Right also loses G playing second, since he has no moves to begin with, so $G = 0$. The same hold true if all $G^L \in \mathcal{R}$. If all $G^L \in \mathcal{L}$, then we set

$$n = \min\{n' \in \mathbb{Z}_{\geq 0} \colon n' \mathrel{\rhd} G^L \text{ for all } G^L \in \mathcal{G}^L\}.$$

We have $n < \infty$, since G is a short game and thus G must have finitely many Left options, and each Left option must have a finite birthday. Additionally, $n > 0$, as each $G^L > 0$. Now, consider the game $G - n$. If Left plays first, she must move to $G^L - n \mathrel{\lhd} 0$, which Right wins playing first, hence Left loses G playing first. If Right plays first on G, he must move to $G - (n-1)$. Left responds by moving to $G^L - (n-1)$, where G^L is such that $n - 1 \not\mathrel{\rhd} G^L$. Hence, $n - 1 \leq G^L$ and thus $G^L - (n-1) \geq 0$ and Left wins G playing second. We find that $G - n \in \mathcal{P}$, hence $G = n$.

Lastly, if the Left options of G are a mixture of outcome classes, then G us equal to the "maximum" of the cases we discussed. If $\mathcal{G}^L$ is a mixture of outcome classes that includes games in $\mathcal{L}$, then we still have $G = n$. The only extra thing we have to check is that Left loses by moving to $G^L - n$ for $G^L \in \mathcal{R} \cup \mathcal{P} \cup \mathcal{N}$. If $G^L \in \mathcal{P}$, then Right wins. If $G^L \in \mathcal{R}$ is a number, then $G^L - n < 0$. If $G^L \in \mathcal{R} \cup \mathcal{N}$ and G^L not a number, then by weak number avoidance, players can only win $G^L - n$ by moving on G^L, which Right wins playing first. If $\mathcal{G}^L$ doesn't have options in $\mathcal{L}$, but does have options in $\mathcal{P}$, then $G = 1$, as $G^L - 1$ is won by Right playing second if $G^L \in \mathcal{R} \cup \mathcal{N}$ by the same argument. Finally, if $\mathcal{G}^L$ is only a mixture of games in $\mathcal{R}$ and $\mathcal{N}$, then $G = 0$ since both players lose playing first. □

2.2 Two person zero-sum games

In this section, we take a step out of the world of combinatorial game theory and dive into economic game theory. We will consider the two person zero-sum game, which is a type of non-cooperative game. This is going to be the foundation of Nash-synchronization for combinatorial games, which is the main topic of the thesis and which will be introduced in Section 4.

Definition 2.33. Consider an $m \times n$ matrix A with real coefficients. A *two-person zero-sum game* is a non-cooperative two-player game in which two players, say *Left* and *Right*, each independently of one another choose a row resp. column of A. If Left chooses row i and Right chooses column j, then Left receives a payout of a_{ij} from Right.

Zero-sum refers to the sum of the payouts being zero (Right "gains" a payout of $-a_{ij}$). Left wants to maximise the money she will receive, while Right wants to minimize the money he will spend. One can make the distinction between deterministic strategies and mixed strategies. With deterministic strategies, we only allow each player to choose an option with probability 1, while with mixed strategies we allow players to make their choice based on a probability distribution on their options. For our purposes, we will allow mixed strategies. Left chooses a strategy $x \in \Delta^m$ and Right chooses a strategy $y \in \Delta^n$, where Δ^k is the k-dimensional unit simplex:

$$\Delta^k = \left\{ (x_1, x_2, \dots, x_k)^\top \colon x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{k} x_i = 1 \right\}.$$

Since Left and Right choose their option based on some chosen probability distribution, it makes sense to talk about the *expected* payout of the zero-sum game, which is given by $\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = x^\top A y$. Let us consider what the minimal and maximal expected payouts are when that we allow mixed strategies. Left can force an expected payout of at least

$$\underline{v}(A) = \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^\top A y,$$

by simply considering Right's best response to any $x \in \Delta^m$. Similarly, Right loses at most

$$\underline{v}(A) = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^\top A y,$$

by considering Left's best response to any $y \in \Delta^n$.

Theorem 2.34 (Fundamental Theorem of Matrix Games)**.** *For any real matrix A, we have $\bar{v}(A) = \underline{v}(A)$.*

Proof. See [7]. □

We call $v(A) \stackrel{\text{def}}{=} \bar{v}(A) = \underline{v}(A)$ the *value* of the game.

Definition 2.35. A pair of strategies $(x^*, y^*) \in \Delta^m \times \Delta^n$ is called a *Nash equilibrium* if

$$(x^*)^\top A y^* \geq x^\top A y^* \text{ and } (x^*)^\top A y^* \leq (x^*)^\top A y \text{ for all } x \in \Delta^m, y \in \Delta^n.$$

Theorem 2.36 (Nash [8]). *Every matrix game has at least one Nash equilibrium.*

Proof. See [8]. □

Consider the following linear program, given by

$$\min \left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 \geq \sum_{j=1}^{n} a_{ij} y_j, & 1 \leq i \leq m, \\ \sum_{j=1}^{n} y_j = 1, & \\ y_j \geq 0, & 1 \leq j \leq n. \end{array} \right\},$$

with its dual linear program given by

$$\max \left\{ x_0 \;\middle|\; \begin{array}{ll} x_0 \leq \sum_{i=1}^{m} a_{ij} x_i, & 1 \leq j \leq n, \\ \sum_{i=1}^{m} x_i = 1, & \\ x_i \geq 0, & 1 \leq i \leq m. \end{array} \right\}.$$

Suppose (y_0^*, y^*) and (x_0^*, x^*) are solutions to the LP and dual LP respectively. For all $x \in \Delta^m$ and $y \in \Delta^n$, we have

$$x^\top A y^* \leq y_0^* = x_0^* \leq (x^*)^\top A y.$$

Hence, $v(A) = y_0^* = x_0^*$ and (x^*, y^*) is a Nash equilibrium. Moreover, we now know that every zero-sum game will have at least one Nash equilibrium. Since the set solution set of a LP is convex, we find that the set of Nash-strategies is convex as well. In Section 4, we will compute the the Nash equilibrium of zero-sum games by solving the associated LP. For the remainder of this section, we will show a few ways to more easily find a Nash equilibrium.

Definition 2.37. Let A be a real matrix. We call an entry a_{ij} a *saddle point* if

$$a_{ij} \geq a_{kj} \text{ for } k = 1, \ldots, m,$$

and

$$a_{ij} \leq a_{ik} \text{ for } k = 1, \ldots, n.$$

I.e., a_{ij} is larger resp. smaller than or equal to any other entry in its column resp. row.

Lemma 2.38. *If a matrix A has a saddle point a_{ij}, then Left choosing row i with probability 1 and Right choosing column j with probability 1 is a Nash equilibrium.*

Proof. Let x^* resp. y^* be the strategies of choosing row i resp. column j with probability 1. For all $x \in \Delta^m$ we have

$$(x^*)^\top A y^* = a_{ij} \geq \sum_{k=1}^{m} x_k a_{kj} = x^\top A y^*.$$

Analogously for y^*. □

Another useful result is presented in [4].

Theorem 2.39. *Let A be a $n \times n$ square matrix of the form*

$$A = \begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix},$$

for some $a, b \in \mathbb{R}$. Then, for Left and Right, choosing any row or column, respectively, uniformly at random is a Nash equilibrium. Moreover, the value of the game is $\frac{1}{n}a + \frac{n-1}{n}b$.

So, if all rows and columns are permutable, we know that simply choosing any uniformly at random is a Nash equilibrium. Now, consider a more general case.

Theorem 2.40. *Let G be a block matrix of the form*

$$G = \left(\begin{array}{c|c} A & W \\ \hline V & B \end{array} \right),$$

where A is as in Theorem 2.39, V is a $m_1 \times n$ matrix with identical columns $v = (v_1, \ldots, v_{m_1})^\top$, W is a $n \times m_2$ matrix with identical rows $w = (w_1, \ldots, w_{m_2})$ and B is any $m_1 \times m_2$ matrix. Consider the zero-sum game G', given by

$$G' = \left(\begin{array}{c|ccc} \frac{1}{n}a + \frac{n-1}{n}b & w_1 & \cdots & w_{m_2} \\ \hline v_1 & & & \\ \vdots & & B & \\ v_{m_1} & & & \end{array} \right),$$

If $(x, y) \in \Delta^{n+m_1} \times \Delta^{n+m_2}$ is Nash equilibrium for G with value $v(G)$, then $(x', y') \in \Delta^{1+m_1} \times \Delta^{1+m_2}$ is a Nash equilibrium for G' with value $v(G') = v(G)$, where $x'_1 = nx_1$ and $x'_i = x_{n-1+i}$ for $i = 2, \ldots, m_1$, analogously for y'.

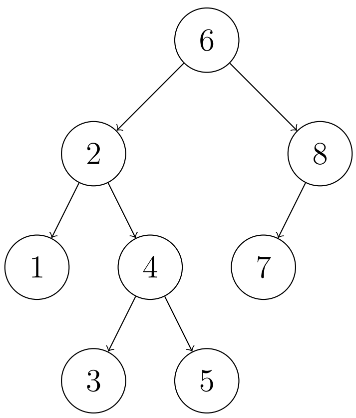
In other words, the rows resp. columns containing A are all identical, so we may consider them as one row resp. column, replacing A with its expected payout. Any Nash equilibrium would have the same probability for the rows resp. columns containing A, so we can take those together as the probability to play on any row resp. column of A. For a proof of Theorem 2.39 and Theorem 2.40, see [4].

2.3 Binary search trees

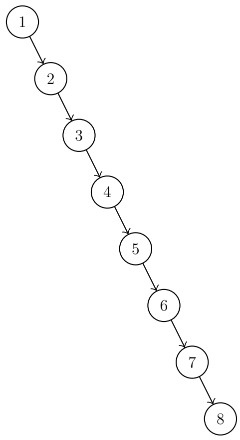
In Section 5, we will do a large number of recursive computations, and to greatly reduce computation time we can store each computed value and simply recall it later when needed. We will use *self-balancing binary search trees* to accomplish this. In this section we will briefly talk about how they work.

A *binary search tree (BST)* is a rooted binary tree. Each node stores a key and possibly other data as well. Each node has two sub-trees, which we denote by the *left sub-tree* and the *right sub-tree*. The key of a node is always greater than any key in its left sub-tree and less than any key in its right sub-tree. This is referred to as the *binary search property.* As the name implies, this is what allows us to find or add a node with a given key. To find a node, we start at the root of the tree. If this node has the key we are looking for, then we are done. If not, we check whether our key is greater or less than the key of the current node. If our key is less than the node key, we traverse to the root node of its left sub-tree, and if our key was greater, then we traverse to the root node of its right sub-tree. We repeat until we either find a node with our key or until we get sent into an empty sub tree. If this happens, we know that there is no node with our given key present in the BST, and it should be added here if we wished to do that.

The shape of a binary search tree depends entirely on the order in which the nodes are added. The shape greatly impacts the lookup efficiency. Consider the following two binary search trees:



(A)



(B)

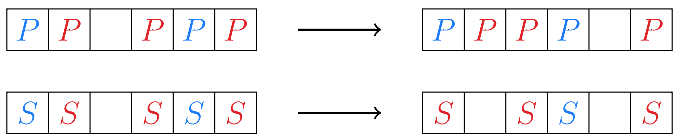
Both trees contain the keys 1 up to and including 8. Tree A is the result of adding the nodes in the order 6,2,1,4,3,5,8,7 (this is not the only order that produces this tree), while tree B is the result of adding nodes in the order 1,2,3,4,5,6,7,8. A has a depth of 3, while B has a depth of 7. This means that we need 3 resp. 7 comparisons to find the deepest node(s). As a result, the worst-case time complexity is $O(n)$ for both searching and inserting nodes.

For our use, we will recursively compute values of synchronized combinatorial games. When a value is computed, we store it in a node whose key is a number that is based on the structure of the game. Because we work recursively, the values will be computed more or less in the order of their keys, leading to an unbalanced tree similar to tree B we saw earlier. A solution to this problem is to use a modification of the BST, the *self-balancing binary search tree.* Several data structures exist that implement self-balancing binary search trees, one of them being the AVL tree [1] (named after inventors Adelson-Velsky and Landis), which is the particular variant that we used. In an AVL tree, we try to keep the height of the tree at a minimum using transformations known as tree rotations. For the details, see [1]. Important is that, for as tree with n nodes, the depth of a BST will be at least $\lfloor \log_2(n) \rfloor$. Since self-balancing binary search trees transform the tree to have this depth for every inserted node, we find that AVL trees have a worst case time complexity of $O(\log n)$ for search and insertion.

3 Push and Shove

Push and Shove are two games with similar rule sets. In both games, a position is a finite board or strip of squares, where each square is empty or occupied by a blue or red piece. No more than one piece is permitted per square. We denote the pieces by colored letters P and S for Push and Shove, respectively. A move consists of taking a piece of your color and moving it one space to the left. The games differ in how the other pieces on the strip behave after a move. In Push, other pieces move one square to the left if and only if another piece threatens to occupy the same space, i.e., all pieces directly left of the piece that intends to move, up to the first empty space, all move one space to the left. In Shove, all pieces left of the piece intended to move get moved one space to the left, regardless of whether they would interfere with other pieces. In both games, pieces may be pushed off the end of the strip and are then removed from the game.

Example 3.1. Example of a move in Push and a move in Shove. In both games, Left moved its rightmost piece.



◁

3.1 Push

We first compute the value of a few simple games. For this we use the notation

$$\square^n \boxed{P} = \underbrace{\boxed{}\boxed{}\dots\boxed{}}_{n}\boxed{P},$$

where we use a superscript to denote repeated blank squares. Note that the negative of a game is the same game with the colors reversed. The following lemma, taken from an exercise in Lessons in Play [2], provides us with the values of a few basic Push positions.

Lemma 3.2. *The values of a few simple games of Push.*

(i) $\square^n \boxed{P} = n + 1$,

(ii) $\square^n \boxed{P}\boxed{P} = 2 - \frac{1}{2^{n+1}}$,

(iii) $\square^n \boxed{P}\square^m \boxed{P} = m + 1$ *(for* $m > 0$*).*

Proof. (i) The canonical form of $\square^n \boxed{P}$ is

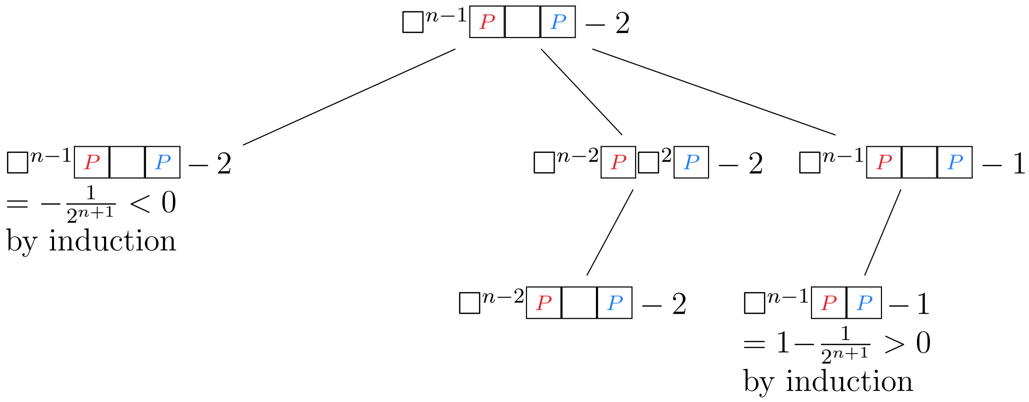
$$\square^n \boxed{P} = \left\{ \square^{n-1} \boxed{P} \;\middle|\; \right\}, n > 0$$

and for $n = 0$ we have $\square^0 \boxed{P} = \boxed{P} = \{0 \mid \}$, so the claim follows by induction.

(ii) We proceed by induction. $\boxed{P}\boxed{P} = \left\{ \boxed{P} \,\middle|\, \boxed{}\boxed{P} \right\} = \{1 \mid 2\} = \frac{3}{2}$. For $n > 1$ we have

$$\square^n \boxed{P}\boxed{P} = \left\{ \square^{n-1}\boxed{P}\boxed{P} \,\middle|\, \square^{n-1}\boxed{P}\boxed{}\boxed{P} \right\}$$
$$= \left\{ 2 - \frac{1}{2^n} \,\middle|\, \square^{n-1}\boxed{P}\boxed{}\boxed{P} \right\}.$$

It remains to show that $\square^{n-1}\boxed{P}\boxed{}\boxed{P} = 2$. We will show that the game $\square^{n-1}\boxed{P}\boxed{}\boxed{P} - 2$ is a second player win.

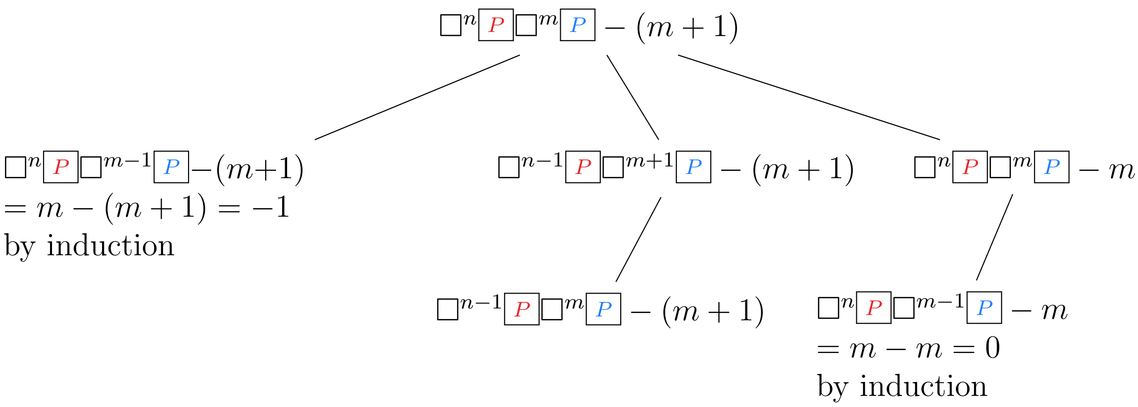


By a separate induction argument, $\square^{n-2}\boxed{P}\boxed{}\boxed{P} - 2 = 0$, since for $n = 0$ we have

$$\boxed{P}\boxed{}\boxed{P} = \left\{ \boxed{P}\boxed{P} \,\middle|\, \square^2\boxed{P} \right\} = \left\{ \frac{3}{2} \,\middle|\, 3 \right\} = 2,$$

completing the proof of (ii).

(iii) Induction on m. We proved the base case in the proof of (ii). For $m > 1$ we consider the game $\square^n\boxed{P}\square^m\boxed{P} - (m+1)$.



We now do induction on n to show that $\square^{n-1}\boxed{P}\square^m\boxed{P} - (m+1) = 0$, since for $n = 0$ we have

$$\boxed{P}\square^m\boxed{P} = \left\{ \boxed{P}\square^{m-1}\boxed{P} \,\middle|\, \square^{m+1}\boxed{P} \right\} = \{m \mid m+2\} = m+1,$$

completing the proof.

□

Lemma 3.2 tells us that any Push position consisting of two pieces somewhere on the board is a number. It is possible to show that any Push position is a number.

Theorem 3.3. *All Push games are numbers.*

Proof. Consider an arbitrary Push game G. We will show that $G^L < G^R$ for any arbitrary Left and Right options of G. By induction, all G^L and all G^R are numbers. We distinguish three cases.

First, suppose that the piece that Left intended to move in G^L gets pushed by Right's move to G^R. In other words, $G^R \in \mathcal{G}^{LR}$. G^L is a number by induction, so $G^R > G^L$.

Secondly, suppose Right's piece gets pushed by Left, or $G^L \in \mathcal{G}^{RL}$. By the same argument, $G^R > G^L$.

Lastly, suppose both piece do not interfere with each other. Then, Left can move from G^R to a position G^{RL} by moving the same piece that she moves to get from G to G^L, which in this case should have remained in the same square. However, Right can do the same with his piece, moving from G^L to G^{LR}, and we find that $G^{LR} = G^{RL}$. By induction,

$$G^L < G^{LR} = G^{RL} < G^R.$$

Thus, $G^L < G^R$ for all $G^L \in \mathcal{G}^L$ and all $G^R \in \mathcal{G}^R$. By the simplest number theorem, G is a number as well. □

3.2 Value of a Shove position

In [2] a formula is given to calculate the value of any Shove position.

Theorem 3.4. *Let G be a Shove position consisting of n pieces on a single strip. For piece $i = 1, \ldots, n$ let $p(i)$ be the position of i on the strip (counting from the left, with the leftmost square being position 1) and let $r(i)$ be the number of pieces strictly to the right of i up to and including the last color alteration. Furthermore, we define $c(i)$ by*

$$c(i) = \begin{cases} 1 & \text{if } i \text{ is blue,} \\ -1 & \text{if } i \text{ is red.} \end{cases}$$

Then the value of G is given by

$$G = \sum_{i=1}^{n} c(i) \frac{p(i)}{2^{r(i)}}$$

Example 3.5. Using Theorem 3.4, we find that the following game of Shove has value $12\frac{15}{16}$.

S	S	S		S	S			S	
$-\frac{1}{16}$	$-\frac{2}{8}$	$+\frac{3}{4}$		$-\frac{5}{2}$	$+6$			$+9$	$= 12\frac{15}{16}$

◁

For synchronized play, it would be nice to find positions of certain values that are contained on a single strip. However, Theorem 3.4 shows that certain values are impossible to attain. Namely, we would like to study powers of $\frac{1}{2}$, but for Shove this is impossible on a single strip. To see that this is true, we consider the following position:

$$M_m = \underbrace{\boxed{S}\boxed{S}\ldots\boxed{S}\boxed{S}}_{m}$$

The value of M_m can be computed recursively. By Theorem 3.4 we have

$$\begin{aligned} M_m &= m - \frac{m-1}{2} - \frac{m-2}{4} - \cdots - \frac{1}{2^{m-1}} \\ &= m - 1 - \frac{m-2}{2} - \frac{m-3}{4} - \cdots - \frac{1}{2^{m-2}} \\ &+ 1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^{m-1}} \\ &= M_{m-1} + \frac{1}{2^{m-1}} \ . \end{aligned}$$

With $M_1 = \boxed{S} = 1$ this gives us the following recurrence relation:

$$\begin{cases} M_m = M_{m-1} + \frac{1}{2^{m-1}}, & m > 1, \\ M_1 = 1. \end{cases}$$

The solution of this equation is given by

$$M_m = 2 - \frac{1}{2^{m-1}}.$$

M_m is the smallest positive (in value) position containing m pieces. To see that this is true, suppose we have a game with m pieces occupying the first m squares of the strip. There has to be at least one blue piece, otherwise the game would be negative and there must be a blue piece occupying the last square, as otherwise the game would be bounded from above by $-M_m < 0$. For the smallest positive value, the remaining pieces should be red. Finally, if M'_m is a game with a blue piece in position $m' > m$ and $m - 1$ red pieces placed to the left of it, then $M'_m \geq M_{m'}$, as the latter game has more red pieces, which makes the game more favorable for Right, while nothing changes for Left. As $M_{m'} > M_m$, this game in not smaller than M_m.

We can conclude that any game with m pieces that has positive value smaller than M_m does not exist on a single strip. In particular this implies that, for Shove, no single strip game G can have value $G \in (0, \inf_m\{M_m\}) = (0, 1)$.

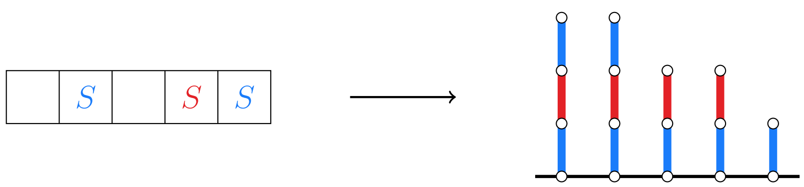
3.3 Equal games and (non-)isomorphic games

Theorem 3.3 and Theorem 3.4 tell us that all Push games and all Shove games are numbers. The same holds true for RB-Hackenbush games [3]. For all three rule sets, we can find a position with the value of any power of $\frac{1}{2}$. Consequently, by taking sums, we can find a position with any numerical value. This means that for any two of these rule sets we can find two games, one of either rule set, such that they are equal in the combinatorial sense. However, it turns out that for some positions, an isomorphic position in one of the other rule sets does *not* exist. In this chapter, we will show examples of such games.

Work on the behavior of synchronized Hackenbush has already been done [10]. One might assume that all statements about synchronized Hackenbush hold for the corresponding synchronized Push or Shove positions as well, since after all they are *equal* in the combinatorial sense. However, it turns out that two synchronized games, which may be equal in canonical form, can behave differently when played synchronized. This will be further discussed in Section 4. While not strictly necessary, isomorphic games would behave the same under synchronized play. Since an isomorphic game might not exist, finding a Hackenbush position that is equal to a Shove position may not be useful, but we will still demonstrate it as a fun exercise. In this section we identify the colors blue and red with the signs + and –, respectively.

Suppose S is a Shove position. We assume S consists of a single strip with n pieces. If S is a sum of Shove games, we may construct a Hackenbush position for each component and then take the sum of all these positions. We construct a Hackenbush position H as follows: We connect $p(n)$ edges of color $c(n)$ to the ground. Then, for $i = 1, \dots, n-1$, we connect an edge of color $c(n-i)$ to $p(n-i)$ out of the $p(n-(i-1))$ edges of color $c(n-(i-1))$. This gives us a forest of Hackenbush stalks.

Example 3.6. The Shove position below consists of three pieces. The third piece is blue and $p(3) = 5$, so we start with 5 blue stalks connected to the ground. The second piece is red, so on top of the blue stalks we place $p(2) = 4$ red stalks. The last piece is blue, so we place $p(1) = 2$ blue stalks on top of the previously placed red stalks.



◁

We see that is we started with n pieces in a Shove game, we obtain n unique Hackenbush strings $x_1, \dots, x_n$, and there are $p(1)$ copies of x_1 and $p(i) - p(i-1)$ copies of $x_i, i = 2, \dots n$. We can use Thea van Roode's method [11] to compute the value of each string. We find that, for $i = 1, \ldots, n$:

$$x_i = \sum_{k=i}^{n} c(k) \frac{1}{2^{r(k)}}.$$

This gives us the following expression for H (putting $p(0) = 0$):

$$\begin{aligned}
H &= \sum_{i=1}^{n} (p(i) - p(i-1)) x_i \\
&= \sum_{i=1}^{n} p(i) x_i - \sum_{i=1}^{n} p(i-1) x_i \\
&= \sum_{i=1}^{n} p(i) \left(\sum_{k=i}^{n} c(k) \frac{1}{2^{r(k)}} \right) - \sum_{i=1}^{n} p(i-1) x_i \\
&= \sum_{i=1}^{n-1} p(i) \left(\sum_{k=i+1}^{n} c(k) \frac{1}{2^{r(k)}} \right) - \sum_{i=1}^{n} p(i-1) x_i + \sum_{i=1}^{n} c(i) \frac{p(i)}{2^{r(i)}} \\
&= \sum_{i=1}^{n-1} p(i) x_{i+1} - \sum_{i=1}^{n} p(i-1) x_i + \sum_{i=1}^{n} c(i) \frac{p(i)}{2^{r(i)}} \\
&= \sum_{i=1}^{n} c(i) \frac{p(i)}{2^{r(i)}}.
\end{aligned}$$

Thus, by Theorem 3.4, $H = S$.

In this Hackenbush representation, each piece of our Shove game is represented by a layer of edges in our forest of strings, where the "thickness" of the layer represents the position of the piece on the strip. Moving a piece then corresponds to cutting one of the edges, and since in Shove all pieces to left of a moving piece move as well, one must cut the edge that is part of the largest string. It is, however, immediately clear that the games are not isomorphic, since H has many more options which are invalid as Shove positions.

Going from Hackenbush to Shove is more difficult, given that there does not exist a formula for the value of a general RB-Hackenbush game, so one must first compute the canonical form. We know that the canonical form of H is some dyadic rational x. It turns out for any dyadic rational there exists a game of Shove of equal value. First, we need to introduce the concept of sign expansion.

Definition 3.7. Consider a string of signs $s = s_1 s_2 \ldots s_n$, where $n \in \mathbb{N}, s_i \in \{-, +\}, i = 1, \ldots, n$. Let k be the smallest index for which $s_k \neq s_{k+1}$. We say that s is the *sign expansion* of the dyadic rational $x \in \mathbb{D} = \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \right\}$ if x can be written as

$$x = s_k k + \sum_{i=k+1}^{n} s_i \frac{1}{2^{i-k}}.$$

The sign expansion of a number can be used to find games of this value. For Push, Shove and Hackenbush, we know how to construct any power of $\frac{1}{2}$ as a game, and can use the sign expansion to construct any number as a sum of powers of $\frac{1}{2}$. For Hackenbush, an alternative method is to use Van Roode's method, which can give any number as a Hackenbush string. For Shove, we also found a better method, allowing us to construct any number as a sum of up to three games.

Example 3.8. Let $x \in \mathbb{D}$, with sign expansion $s = s_1 s_2 \ldots s_n$, with k the smallest index for which $s_k \neq s_{k+1}$. We then construct the following Shove positions:

$S_1 =$

s_n	s_{n-1}	s_{n-2}	$\cdots$	s_{k+1}	s_k

$S_2 =$

s_{n-1}	s_{n-2}	$\cdots$	s_{k+1}	s_k

By definition of sign expansion, we know that the last color alteration in both S_1 and S_2 are at s_{k+1} and s_k. Using Theorem 3.4 we then find the values for both positions,

$$S_1 = \sum_{i=k}^{n} s_i \frac{n-i+1}{2^{i-k}}, \qquad S_2 = \sum_{i=k}^{n-1} s_i \frac{n-i}{2^{i-k}}.$$

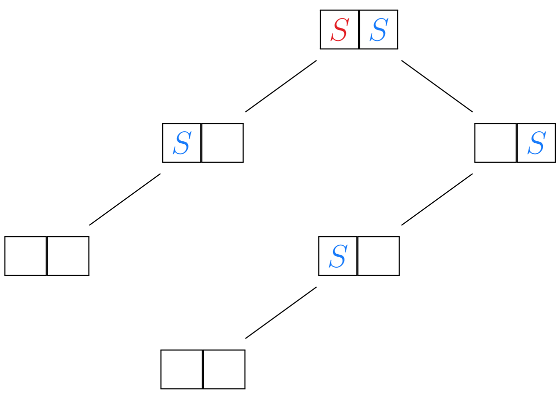
Subtracting S_2 from S_1, we find

$$\begin{aligned} S_1 - S_2 &= \sum_{i=k}^{n} s_i \frac{1}{2^{i-k}} \\ &= s_k + \sum_{i=k+1}^{n} s_i \frac{1}{2^{i-k}}. \end{aligned}$$

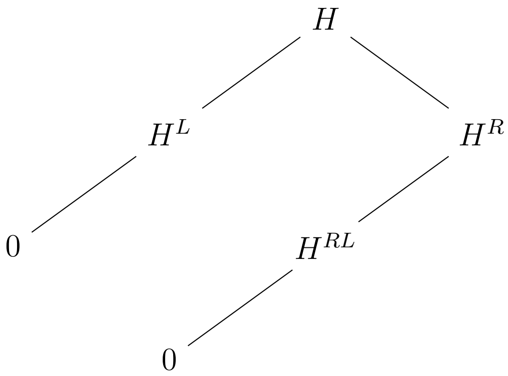
If $k = 1$, then we have found a Shove game of equal value to H. For $k > 1$, we need to add a third game $S_3 = \square^{k-2}\boxed{s_k} = s_k(k-1)$. ◁

As mentioned before, we have only constructed S based on the combinatorial value of H, so in general they would not be isomorphic and there is no relation between a move on S and a move on H. A valid question now would be if it is even possible for any Shove game to find an isomorphic Hackenbush game and vice versa. As we will now demonstrate by example, there do exist games for which this is impossible.

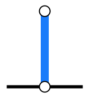
Example 3.9. Consider $S = \boxed{S}\boxed{S}$. The game tree of S is as follows,



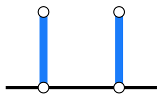
Our goal is to find a Hackenbush position H with a game tree of the same shape,



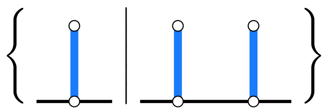
Working our way up from the leaves we can try to fill this tree. We find

$$H^L = H^{RL} =$$  ,

since this is the only position in which Left can *only* play to 0 and Right has no move *at all* (remember that we look at the game tree as is, i.e., without dominating or reversing any options). Then, H^R must be a position in which Left can only play to H^{RL} and Right again has no move, so the only possibility is

$$H^R =$$  .

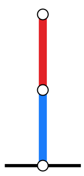
So, H must be a Hackenbush position with the following options,

$$H =$$  .

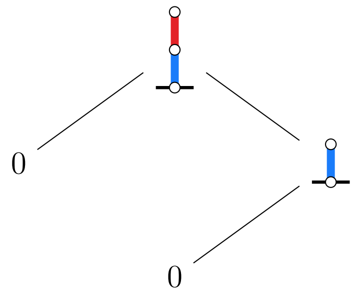
Such a position H does not exist: it would need to have a red edge somewhere, which when cut by Right should lead to H^R. But, the red edge should also disappear as a result of Left moving to H^L, so it should be connected to one of the two blue edges and not to the ground. Since Left has one unique Left option, it should be connected to both edges, but such a position does not have H^L as Left option. ◁

Now, we might try to do it the other way around.

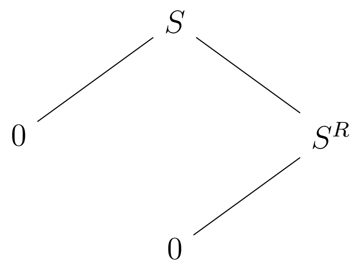
Example 3.10. Consider

$$H =$$  .

The corresponding game tree is



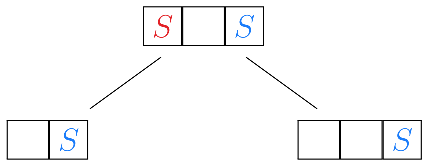
We are looking for a Shove game S with the following game tree,



The only possible game for S^R is $S^R = \boxed{\color{blue}S}$. The only game is which Right can only move to S^R is $S = \boxed{\color{blue}S} + \boxed{\color{red}S}$. However, his gives us $S^L = \boxed{\color{red}S} \neq 0$, hence there is no Shove game isomorphic to H. ◁

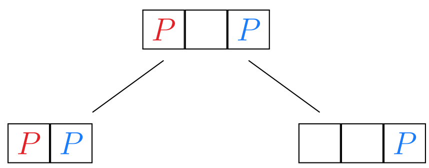
Example 3.9 and Example 3.10 also show that finding an isomorphic Hackenbush game cannot be done for certain Push games, as all the Shove positions used are isomorphic to their Push counterparts. Lastly, we will show that there also exist Shove games which do not have an isomorphic Push game and vice versa.

Example 3.11. Consider $S = \boxed{\color{red}S}\boxed{}\boxed{\color{blue}S}$. The two upper layers of the game tree of S are given by



Since $\boxed{}\boxed{\color{blue}S} \cong \boxed{}\boxed{\color{blue}P}$ and $\boxed{}\boxed{}\boxed{\color{blue}S} \cong \boxed{}\boxed{}\boxed{\color{blue}P}$, we are looking for the Push game $P = \left\{ \boxed{}\boxed{\color{blue}P} \,\middle|\, \boxed{}\boxed{}\boxed{\color{blue}P} \right\}$, which does not exist: Right needs a piece somewhere and Left's move to $\boxed{}\boxed{\color{blue}P}$ must push this red piece off the board, meaning that Left needs to have a piece on the leftmost square of the strip, which she does not have.

Vice versa, for $P = \boxed{\color{red}P}\boxed{}\boxed{\color{blue}P}$, we have the game tree



and we are looking for the Shove game $S = \left\{ \boxed{\color{red}S}\boxed{\color{blue}S} \,\middle|\, \boxed{}\boxed{}\boxed{\color{blue}S} \right\}$, since again the Left and Right options are isomorphic to their Shove counterparts. We see that S is impossible as a Shove position, since the only Shove games with $\boxed{}\boxed{}\boxed{\color{blue}S}$ as Right option are $\boxed{\color{red}S}\boxed{}\boxed{\color{blue}S}$ and $\boxed{}\boxed{}\boxed{\color{blue}S} + \boxed{\color{red}S}$, neither of which have $\boxed{\color{red}S}\boxed{\color{blue}S}$ as a Left option. $\triangleleft$

4 Synchronized play

In [5], the concept of synchronized play is introduced. Synchronized games differ from regular combinatorial games in that we consider both players to move simultaneously. This requires us to define what a synchronized move does, which for some games might be tricky. However, for Push and Shove it turns out we can do it in a fairly natural way. When presented with a game of either Push or Shove, both players will have to decide which piece to move. The synchronized move will then consist of moving both pieces. If the pieces are located on different strips, then they can be moved in any order. If the two pieces to move lie on the same strip, then we move the leftmost piece first, ensuring that each player may move their intended piece from their original position.

4.1 General definition of a synchronized game

In Section 1 we introduced combinatorial games, which are defined as its sets of Left and Right options,

$$G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}.$$

We now define the concept of synchronized games in a similar fashion.

Definition 4.1. A *synchronized game* G is a triple $\{\mathcal{G}^L \mid \mathcal{G}^S \mid \mathcal{G}^R\}$, where $\mathcal{G}^L = (G_1^L, \ldots, G_m^L)$ is a sequence of m synchronized games (the *Left options* of G), $\mathcal{G}^R = (G_1^R, \ldots, G_n^R)$ is a sequence of n synchronized games (the *Right options* of G) and $\mathcal{G}^S = (G_{ij}^S)_{ij}$ is a $m \times n$ matrix of synchronized games (the *synchronized options* of G).

Once again we denote the smallest synchronized game by $0 \stackrel{\text{def}}{=} \{\,|\,|\,\}$. Sometimes, synchronized moves are denoted by G^{L+R}, instead of G^S, to emphasize that it is the synchronized move of a particular Left move G^L and a particular Right move G^R. We may also represent a synchronized game using a matrix representation. In matrix representation, a game G is given by

$$G = \left(\begin{array}{c|c} & \mathcal{G}^R \\ \hline \mathcal{G}^L & \mathcal{G}^S \end{array}\right).$$

Here, each row corresponds to a move by Left and each column corresponds to a move by Right, with the first row and column corresponding to the respective player passing to account for solo moves. Later, we will introduce the concept of synchronized value, for which we use $\mathcal{G}^S$ as a matrix with its rows and columns defined in this manner.

Definition 4.2. Let G and H be synchronized games, and set $|\mathcal{G}^L| = m$ and $|\mathcal{G}^R| = n$. We define the *(disjunctive) sum* $K = G + H$ as follows: $\mathcal{K}^L$ is the concatenation of $\mathcal{G}^L$ and $\mathcal{H}^L$; $\mathcal{K}^R$ is the concatenation of $\mathcal{G}^R$ and $\mathcal{H}^R$; and

$$\mathcal{K}_{ij}^S = \begin{cases} \mathcal{G}_{ij}^S + H & \text{if } i \leq m, j \leq n, \\ G + \mathcal{H}_{i-m,j-n}^S & \text{if } i > m, j > n, \\ \mathcal{G}_i^L + \mathcal{H}_{j-m}^R & \text{if } i \leq m, j > n, \\ \mathcal{G}_j^R + \mathcal{H}_{i-n}^L & \text{if } i > m, j \leq n. \end{cases}$$

In matrix notation, this is expressed as

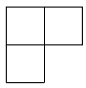
$$G+H=\left(\begin{array}{c|cc} & \mathcal{G}^R+H & G+\mathcal{H}^R \\ \hline \mathcal{G}^L+H & \mathcal{G}^S+H & \mathcal{G}^L+\mathcal{H}^R \\ G+\mathcal{H}^L & \mathcal{G}^R+\mathcal{H}^L & G+\mathcal{H}^S \end{array}\right).$$

Definition 4.3. Let G be a synchronized game. We define the *negative* of G by

$$-G=\left\{-\mathcal{G}^R \,\middle|\, -\left(\mathcal{G}^S\right)^\top \,\middle|\, -\mathcal{G}^L\right\}.$$

Definition 4.4. Let G be a combinatorial game. We call G *separable* if, for every position H of G, for every H^L and H^R, it holds that $H^L \in \mathcal{H}^{RL}$ or $H^R \in \mathcal{H}^{LR}$ or $\mathcal{H}^{LR} \cap \mathcal{H}^{RL} \neq \emptyset$. If, for all H^L and H^R, it holds that $\mathcal{H}^{LR} \cap \mathcal{H}^{RL} \neq \emptyset$, then we call G *strongly separable.*

Example 4.5. Consider the following position of *Domineering* (see [2, p. 307]):

$G =$ 

Under non-synchronized play, both players have a move which places a domino on the centre square, after which the other player cannot place any more dominoes. As a combinatorial game it is equal to $* = \{0 \mid 0\}$. As a synchronized game, it is unclear what the synchronized move should be. We could define the synchronized move to have both players overlap their dominoes, in which case $G^S = 0$. By this definition, G is not separable. ◁

Definition 4.6. Let G be a separable combinatorial game. We inductively construct a *synchronized version of* G, say $\hat{G} = \{\widehat{\mathcal{G}^L} \mid \widehat{\mathcal{G}^S} \mid \widehat{\mathcal{G}^R}\}$, as follows:

- $\widehat{\mathcal{G}^L} = \hat{\mathcal{G}}^L$;
- $\widehat{\mathcal{G}^R} = \hat{\mathcal{G}}^R$;
- For every $G_i^L \in \mathcal{G}^L$, $G_j^R \in \mathcal{G}^R$, if $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} \neq \emptyset$, pick $G_{ij}^S \in \mathcal{G}^{LR} \cap \mathcal{G}^{RL}$ and set $\widehat{G_{ij}^S} = \hat{G}_{ij}^S$. Otherwise, if $G_i^L \in \mathcal{G}^{RL}$, set $\widehat{G_{ij}^S} = \hat{G}_i^L$. Otherwise, $G_j^R \in \mathcal{G}^{LR}$ and set $\widehat{G_{ij}^S} = \hat{G}_j^R$.

Push and Shove are both rule sets in which every position is separable. For Push, if the two pieces to move by Left and Right do not interact with each other, meaning moving one of them does not push the other, then we have a unique $G^{LR} = G^{RL}$, which will be our synchronized option for this pair of options. If one of the pieces to move does get pushed by the other, then we have either $G^L \in \mathcal{G}^{RL}$ or $G^R \in \mathcal{G}^{LR}$ and we set the synchronized option to this option. We do the same for Shove, except here he also have $G^{LR} = G^{RL}$ if one piece to move gets shoved by the other piece. $G^L \in \mathcal{G}^{RL}$ or $G^R \in \mathcal{G}^{LR}$ only happens when a piece gets moved off the board.

Definition 4.7. We call a synchronized game $G = \{\mathcal{G}^L \mid \mathcal{G}^S \mid \mathcal{G}^R\}$ *decided* if $\mathcal{G}^S$ is the empty matrix.

In combinatorial games, we defined the outcome classes $\mathcal{N}, \mathcal{P}, \mathcal{L}$ and $\mathcal{R}$. We will now define a similar concept for synchronized games. Similarly to combinatorial games, a synchronized game belongs to class $\mathcal{L}$ resp. $\mathcal{R}$, if Left resp. Right has a winning strategy, regardless of the other player's strategy. The classes $\mathcal{N}$ and $\mathcal{P}$ do not make sense for synchronized play, since both players move at the same time. This means that a game is either won by Left or Right, *or* both players run out of moves at the same time: a draw. Games that always result in a draw if both players play optimally belong to the class $\mathcal{D}$.

While the classes $\mathcal{L}$ and $\mathcal{R}$ appear both in regular combinatorial game theory and in synchronized game theory, games that belong to either of those categories as combinatorial games do not necessarily belong to them as synchronized games.

Example 4.8. Consider the synchronized version of the Push game

$$\boxed{P}\boxed{P}.$$

By Lemma 3.2, this game has combinatorial value $\frac{3}{2}$, hence it is in $\mathcal{L}$ (as a combinatorial game). As a synchronized game it is in $\mathcal{L}$ as well: Both players only have one legal move, resulting in the game $\boxed{P}\boxed{}$, in which only Left can move. Now consider the game

$$\boxed{P} + \boxed{P}\boxed{P},$$

which has combinatorial value $\frac{1}{2}$, hence it is in $\mathcal{L}$ (as a combinatorial game) as well. Right now has two legal moves, however Right knows that Left must move the only piece she has. Left's move will cause one of Right's pieces to fall off the board, so Right's optimal move is to move that piece first, since it is guaranteed to move anyway. Indeed, the resulting game is $\boxed{P} + \boxed{P}\boxed{}$ (a draw) if Right moves that piece, while moving the other piece results in the game $\boxed{P}\boxed{}$ (a Left-player win). Right can at best force a draw, hence this game is in $\mathcal{D}$ as a synchronized game. $\triangleleft$

In the game $\frac{1}{2}$ in Example 4.8, Rights knows exactly what Left will do and can plan accordingly. In general, this is not the case. Optimal play may therefore involve mixed strategies. As a result, we must introduce more classes. Games that, under optimal play, have a positive chance to result in a draw and a positive chance to result in a Left-player win belong the the class $\mathcal{LD}$. The class $\mathcal{RD}$ is similarly defined, but with either a draw or a Right-player win. Lastly, there exists the class $\mathcal{LR}$ for games that result in a Left-player or Right-player win and the class $\mathcal{LRD}$ for games that have a positive chance to end with any result.

Example 4.9. Consider two copies of the game from Example 4.8,

$$\boxed{P} + \boxed{P}\boxed{P} + \boxed{P} + \boxed{P}\boxed{P}.$$

Both players now have multiple options. The matrix if synchronized moves for this game is as follows,

$$\begin{pmatrix} \boxed{P}\boxed{} + \boxed{P} + \boxed{P}\boxed{P} & \boxed{P} + \boxed{P}\boxed{} + \boxed{P} + \boxed{P}\boxed{P} & \boxed{P} + \boxed{P}\boxed{} + \boxed{P} + \boxed{}\boxed{P} \\ \boxed{P}\boxed{P} + \boxed{P} + \boxed{P}\boxed{} & \boxed{P} + \boxed{}\boxed{P} + \boxed{P} + \boxed{P}\boxed{} & \boxed{P} + \boxed{P}\boxed{P} + \boxed{P} + \boxed{P}\boxed{} \end{pmatrix},$$

where we have omitted the column belonging to Right's third piece, as the resulting synchronized moves are identical to those of his first piece. For Right, two of his pieces are in danger of being pushed off the board by Left. If Right moves the piece that would be pushed by Left, the resulting game is $\boxed{\color{red}P} + \boxed{\color{blue}P}\boxed{} + \boxed{\color{red}P} + \boxed{\color{red}P}\boxed{\color{blue}P}$. Since $\boxed{\color{red}P} + \boxed{\color{blue}P}$ adds exactly one move for both players, the outcome of the game is the same as the outcome of $\boxed{\color{red}P} + \boxed{\color{red}P}\boxed{\color{blue}P}$, i.e., a draw. However, in Left's case, it is optimal to move on a strip on which Right does not move, since in this case two red pieces will move, costing Right an additional move. This results in the game $\boxed{\color{red}P} + \boxed{\color{blue}P}\boxed{} + \boxed{\color{red}P} + \boxed{}\boxed{\color{blue}P}$, which is guaranteed to be won by Left. We see that Right wants to move on the same strip as Left, while Left wants to avoid moving on the same strip as Right. It follows that choosing either of the strips $\boxed{\color{red}P}\boxed{\color{blue}P}$ with probability $\frac{1}{2}$ for both players is a Nash equilibrium. As the outcome of the game is either a Left-player win or a draw, the game is in $\mathcal{LD}$. ◁

Lemma 4.10. *Let G be a separable game in canonical form. Then, G is a number.*

Proof. Induction on the birthday of G. The only game with birthday 0 is 0 itself, which is a number in canonical form. Now let G be a separable game in canonical form of birthday larger than 0. If $\mathcal{G}^L = \emptyset$ or $\mathcal{G}^R = \emptyset$, then we know G is an integer (Theorem 2.32) and we are done. Now suppose G has both Left and Right options. By the definitions of separability and canonical form, all options of G are also separable games in canonical form. Then, by induction, all options of G are numbers. Since G is in canonical form, by domination, we have a unique Left option G^L and a unique Right option G^R. Using the simplest number theorem, G is a number if $G^L < G^R$. Using the separability of G there are three possible ways in which G^L and G^R are related. If $G^R \in \mathcal{G}^{LR}$ or $G^L \in \mathcal{G}^{RL}$, then we immediately have $G^L < G^R$ by induction. If $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} \neq \emptyset$, then we have $G^L < G^{LR} = G^{RL} < G^R$. Thus, in any case we find $G^L < G^R$, hence G is a number. □

The last part of this proof is very similar to our reasoning in the proof of Theorem 3.3. Indeed, every Push game is separable, so the upcoming Theorem 4.12 can be used as an alternative argument to show that all Push games are numbers.

Lemma 4.11. *Let G be a separable game. Then,* $\operatorname{can}(G)$ *is separable.*

Proof. We need to show that G remains separable when options are removed by domination, or when a reversible option in replaced by the options in its replacing set. Because G is separable, so too is every position of G, so it is enough to check that for every Left option G^L and every Right option G^R of $\operatorname{can}(G)$ we have $G^L \in \mathcal{G}^{RL}$ or $G^R \in \mathcal{G}^{LR}$ or $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} \neq \emptyset$.

Domination is trivial, as we know that every pair of left and right options of G can be performed in any order, so removing options still means that any pair of Left and Right options can be performed in some order. By induction, all options of G can be brought into canonical form and remain separable, so by Lemma 4.10, all options of G are numbers. Using domination we then find a unique left option G^L and a unique right option G^R and, since *every* position of G is separable, by induction all positions of G (except G itself) are numbers, either in canonical form or not. So, we can assume every position of G to have at most one option for Left and at most one option for Right.

Now, suppose that G^L is reversible. Then $G^{LR} \leq G$ and we replace G^L with G^{LRL}, and we need to check the aforementioned condition for G^{LRL} and G^R. If $\mathcal{G}^{LRL} = \emptyset$, then we are done. If it's not, then we consider the three possible ways in which G^R can relate to G^L.

If $G^R \in \mathcal{G}^{LR}$, then $G^R = G^{LR}$. As a result, $G^{LRL} \in \mathcal{G}^{LRL} = \mathcal{G}^{RL}$.

If $G^L \in \mathcal{G}^{RL}$, then $G^L = G^{RL}$, so $G^L < G^R$. Then, we must also have $G^L < G < G^R$ and, since $G^{LR} \leq G$, G^L is also a reversible option of G^R and by reversing it we find $G^{LRL} \in \mathcal{G}^{RL}$.

Finally, if $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} \neq \emptyset$, then $G^{LR} = G^{RL}$. We first show that in this case, we must have $G^{LR} = G$, by showing that the game $G^{LR} - G$ is a second player win. If Left begins, she can play to $G^{LRL} - G < 0$, since $G^{LRL} < G^{LR} \leq G$, or she could play to $G^{LR} - G^R$, from which Right can play to $G^{LR} - G^{RL} = 0$. If Right begins, he can either make the same mistake, moving to $G^{LR} - G^L$, where Left can answer with $G^{LR} - G^{LR} = 0$. Alternatively, Right can play to $G^{LRR} - G$, from which Left can respond by moving to $G^{LRR} - G^R = G^{RLR} - G^R$. By induction, we can assume that G^R has no reversible options. Hence, $G^{RLR} > G^R$ resulting in a win for Left. So, $G = G^{LR} = G^{RL}$, meaning that G^R is a reversible option for Right and G^R gets replaced by G^{RLR}. The separability of G then follows by the separability of $G^{LR} = G^{RL}$. □

Theorem 4.12. *If G is a separable game, then G is a number.*

Proof. This follows from Lemma 4.10 and Lemma 4.11. □

While every separable game is a number, the converse is not true.

Example 4.13. Consider the game $G = \{-2 \mid 2\}$. By the simplest number theorem, $G = 0$ as a combinatorial game, hence G is a number. As a synchronized game, however, G is not separable; we have $G^L = \{\,\mid -1\}$ and $G^R = \{1 \mid\,\}$, however, $1 \notin \mathcal{G}^{LR}$, $-1 \notin \mathcal{G}^{RL}$ and $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} = \emptyset$.

◁

4.2 Nash synchronization

We have seen that, with synchronized play, numbers behave differently depending on their game tree. This means that numbers do not form a ring or even a subgroup, like they do for regular short games. One might still wish to assign some sort of value to synchronized games. [5] proposes a method where values are determined by considering sums of moves. However, this method turns out to be problematic for certain synchronized rule sets. In particular, the Push position

| P | + | P | P |

is a draw and should be given a synchronized value of 0. However, as we will see later, adding this game to itself gives us a game that is in $\mathcal{LD}$, meaning that it should be given a value greater than 0. A different approach is to consider a synchronized game as a zero-sum game.

Definition 4.14. Let G be a synchronized version of a combinatorial rule set, with decided positions D. We call $f\colon D \to \mathbb{R}$ a *value function* if it has the following properties.

(i) For $H \in D$ with $H \in \mathcal{L}$, we have $f(H) > 0$. Moreover, if every position of H is a decided win for Left, we have $f(H) = \mathrm{can}(H)$.

(ii) For $H \in D$ with $H \in \mathcal{R}$, we have $f(H) < 0$. Moreover, if every position of H is a decided win for Right, we have $f(H) = \mathrm{can}(H)$.

(iii) For $H \in D$ with $\mathcal{H}^L = \mathcal{H}^R = \mathcal{H}^S = \emptyset$, we have $f(H) = 0$. In other words, $f(0) = 0$.

(iv) For $H \in D$, we have $f(-H) = -f(H)$.

Definition 4.15. Let G be a synchronized game and let f be a value function for the game. For every position H of G, we define its *Nash value* $v(H)$ to be $v(H) = f(H)$ if H is decided, or the Nash value of H as a zero-sum game otherwise.

Example 4.16. We compute the synchronized value of the games in Example 4.8. In the game $\boxed{P}\boxed{P}$ both players have only one option, so $\mathcal{G}^S$ is a 1×1 matrix and the value of the game is equal to the value of the only possible resulting game,

$$v\left(\boxed{P}\boxed{P}\right) = v\left(\boxed{P}\right) = 1.$$

For the game $\boxed{P} + \boxed{P}\boxed{P}$ we have that $\mathcal{G}^S$ is a 1×2 matrix, since Right now has two options. However, we already determined that this game is in $\mathcal{D}$, hence

$$v\left(\boxed{P} + \boxed{P}\boxed{P}\right) = 0.$$

$\triangleleft$

Theorem 4.17. *Let G be a separable synchronized game. Then $v(G^L) \leq v(G) \leq v(G^R)$ for all Left options G^L and Right options G^R of G.*

Proof. Consider an arbitrary Left option G^L. Knowing G^L, we can determine G^R such that the synchronized move G^{L+R} is the best-case outcome for Right. This implies that $v(G^{L+R}) \leq v(G)$. Now we consider the two ways in which we can perform the move G^{L+R}. First, suppose that $G^L \in \mathcal{G}^{RL}$, in which case $G^{L+R} = G^L$ by definition and we are done. If this is not the case, then we must have either $G^R \in \mathcal{G}^{LR}$ or $\mathcal{G}^{LR} \cap \mathcal{G}^{RL} \neq \emptyset$ by separability, and then $G^{L+R} = G^{LR}$ is a legal way to perform the synchronized move. From G^L we now consider an arbitrary Left move and assume it is synced with Right's move, i.e., we consider $(G^L)^{L+R}$. Then again, either $(G^L)^{L+R} = G^{LRL}$ or $(G^L)^{L+R} = G^{LLR}$ (or both). If G^{LRL} is legal, we note that $G^{LRL} = (G^{LR})^L$ and, by induction, $v((G^{LR})^L) \leq v(G^{LR})$. If G^{LRL} is not legal, then $G^{LR} \in \mathcal{G}^{LLR}$, hence $(G^L)^{L+R} = G^{LR}$. Thus, in any case we have $v((G^L)^{L+R}) \leq v(G^{LR}) \leq v(G)$ for any Left move from G^L, which implies that $v(G^L) \leq v(G)$. A symmetric argument holds for G^R. $\square$

Definition 4.18. Let G be a decided synchronized game. We say that G is *terminal* if for all positions H of G, we have that H is decided and $o(H) = o(G)$.

For Push and Shove, the synchronized versions of the integer games are all terminal. In fact, these are the only terminal positions, since any non-integer game has at least one piece of each color, meaning that a synchronized move exists and, as a result, the game is not decided.

Definition 4.19. Let G be a synchronized game. If every decided position of G is terminal, we call G *rebound-free.*

Since for Push and Shove, all terminal positions are integers, we know that all synchronized Push and Shove games are rebound-free. An example of a synchronized rule set which has non-rebound-free positions is *Cherries* [2, p. 305]. In Cherries, there exists positions where one player cannot make a legal move, but every option of the other player does allow the first player to move. This means that the original position is decided, while it has options that are not decided, hence the position is not terminal.

Say that we have a separable synchronized game $G + H$, where G is not decided and H is terminal for Left. Then, every position of H is decided for Left. Since all separable games are numbers, we find that H must be an integer. For regular combinatorial games, we have seen that Left has a higher incentive to play on G than on H. This leads to the following conjecture for separable synchronized games.

Conjecture 4.20. *Let G, H be separable synchronized games, and assume H is terminal. Then,*

$$v(G + H) = v(G) + v(H).$$

In isolation, moving on H decreases its Nash value by 1. Similar to Theorem 2.30, we would expect G to have a synchronized option which decreases the Nash value by less than 1. If this is the case for every non-decided game, than Left will ignore moving on H, from which the above statements follows. However, it turns out difficult to show the existence of such a move for a general synchronized game. For Push and Shove, it can be shown.

Theorem 4.21. *Let G, H be synchronized versions of Push (or Shove) games, and assume H is terminal. Then,*

$$v(G + H) = v(G) + v(H)$$

Proof. We will only show the case for Push. All arguments also hold for Shove.

If G is also terminal, then the statement is trivial. Assume G to be non-terminal and assume H to be won by Left. We will first show that Left has a move on G which is at least as good as a move on H, resulting in the move on H getting dominated.

Then, we will show that the Nash value of $G + H$ is indeed the sum if the Nash values of G and H.

Consider the non-terminal game G. Since G is non-terminal, there must be some move for Left. Consider Left moving her leftmost blue piece on any particular strip of G. For any arbitrary Right move, the resulting synchronized move will result in

$$G_1 = G^{L+R} + H,$$

while moving on H will result in

$$G_2 = G^R + H^L.$$

By induction, G_1, G_2 have Nash values

$$\begin{aligned} v(G_1) &= v\left(G^{L+R}\right) + v(H), \\ v(G_2) &= v\left(G^R\right) + v\left(H^L\right) \\ &= v\left(G^R\right) + v\left(H\right) - 1. \end{aligned}$$

There are several cases we need to consider. If the blue piece Left intended to move would already be moved as a result of Right's move, i.e., $G^{L+R} = G^R$, then we have

$$v(G_1) = v\left(G^{L+R}\right) + v(H) = v\left(G^R\right) + v\left(H\right) > v\left(G^R\right) + v\left(H\right) - 1 = v(G_2).$$

Next, consider the case if Right's move does not push Left's piece to move and Left's move will not push any other pieces. The resulting games G_1 and G_2 are then identical, with the only difference being one blue piece having moved one square to the left on G in G_1 and on H in G_2:

$$\begin{aligned} G_1 &= \boxed{\quad\dots\quad}\boxed{P}\boxed{}\boxed{\quad\dots\quad} + \square^n\boxed{P}, \\ G_2 &= \boxed{\quad\dots\quad}\boxed{}\boxed{P}\boxed{\quad\dots\quad} + \square^{n-1}\boxed{P}, \end{aligned}$$

where, without loss of generality, we have assumed G to be a single strip and have written $H = \square^n\boxed{P}$, for some $n \in \mathbb{Z}_{\geq 0}$, $n = \text{can}(H) - 1$. First, we need to introduce some notation. Let A be some game with an $m \times n$ matrix $\mathcal{A}^S$ of synchronized options. Let $p^L \in \Delta^m$, $p^R \in \Delta^n$ be arbitrary strategies for Left and Right, respectively. Then, we define the value corresponding to these strategies as

$$v_{p^L,p^R}(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i^L p_j^R v(A_{ij}^S)$$

Note that for the Nash equilibrium (π^L, π^R) we have

$$v_{\pi^L,\pi^R}(A) = v(A).$$

Since p^L and p^R possibly non-Nash strategies for each respective player, we have

$$v_{p^L,\pi^R}(A) \leq v(A) \leq v_{\pi^L,p^R}(A).$$

Now, we couple G_1 and G_2; we compute the Nash equilibrium for both games. Let $\pi_1 = (\pi_1^L, \pi_1^R)$ be the Nash equilibrium of G_1 and let $\pi_2 = (\pi_2^L, \pi_2^R)$ be the Nash equilibrium of G_2. Then, Left will play the Nash strategy of G_2 on both games and Right will play the Nash strategy of G_1 on both games. This means that Left will select the "best" piece to move on G_2 and also move the corresponding piece on G_1, and vice versa for Right. By induction, both players will only play on G. We repeat for each pair of resulting games and stop when the blue piece moved by Left on G_1 has fallen off the board, meaning that, eventually, G_1 and G_2 will be played to

$$G_1' = \boxed{}\boxed{\quad\dots\quad} + \square^n\boxed{P},$$
$$G_2' = \boxed{P}\boxed{\quad\dots\quad} + \square^{n-1}\boxed{P},$$

if this blue piece stays ahead by one square on G_1 and, as a result of that, will fall off the board first, or they will be played to

$$G_1' = \boxed{\quad\dots\quad} + \square^n\boxed{P},$$
$$G_2' = \boxed{\quad\dots\quad} + \square^{n-1}\boxed{P},$$

if some move pushed the corresponding piece on G_2 to catch up with the blue piece on G_1, in which case they leave their respective boards in the same turn. In the second case, it is easy to see that $v(G_1') \geq v(G_2')$. In the first case, this is also true; we couple G_1' and G_2' the same way as before, except we say that Left also moves on $\square^n\boxed{P}$ in G_1' if Left moves on the leftover blue piece in G_2'. If we continue playing until the leftover blue piece falls off, then we we either arrive at a situation like the second case or we arrive in a situation where $G_1' = G_2'$. As $v(G_1') \geq v(G_2')$ holds for all couples, we have $v_{\pi_2^L, \pi_1^R}(G_1) \geq v_{\pi_2^L, \pi_1^R}(G_2)$ and thus

$$v(G_1) \geq v_{\pi_2^L, \pi_1^R}(G_1) \geq v_{\pi_2^L, \pi_1^R}(G_2) \geq v(G_2).$$

Lastly, consider the case where, again, Right's move does not push Left's piece to move, but Left's move now also pushes some other pieces. Because we assumed Left to move its leftmost blue piece, these pieces can only be red:

$$G_1 = \boxed{\quad\dots\quad}\boxed{P}\dots\boxed{P}\boxed{P}\boxed{}\boxed{\quad\dots\quad} + \square^n\boxed{P},$$
$$G_2 = \boxed{\quad\dots\quad}\boxed{}\boxed{P}\dots\boxed{P}\boxed{P}\boxed{\quad\dots\quad} + \square^{n-1}\boxed{P}.$$

We apply the same coupling argument. However, because on G_1 some red pieces have moved one square to the left compared to G_2, it is possible that one of such pieces will fall off the board on G_1 first. This does not pose a problem, since Right plays the Nash equilibrium strategy of G_1, therefore he can always move a piece in both games. We can then continue until we reach one of the pairs G_1', G_2' from above and we are done.

Now that we have shown that a move on H is dominated by moving on G, we will show that $v(G+H) = v(G) + v(H)$. By definition of Nash value, $v(G+H)$ is given by the following linear program:

$$v(G+H) = \min\left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 \geq \sum_{j=1}^{|\mathcal{G}^R|} v\left(\mathcal{K}_{ij}^S\right) y_j, & 1 \leq i \leq |\mathcal{G}^L| + |\mathcal{H}^L|, \\ \sum_{j=1}^{|\mathcal{G}^R|} y_j = 1, & \\ y_j \geq 0, & 1 \leq j \leq |\mathcal{G}^R|. \end{array} \right\},$$

where $\mathcal{K}^S_{ij}$ is as defined in Definition 4.2. Dominating moves on H means that we may remove the restrictions corresponding to the rows in the zero-sum game that correspond to moves on H, giving us the equivalent linear program

$$v(G+H) = \min \left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 \geq \sum_{j=1}^{|\mathcal{G}^R|} v\left(\mathcal{G}^S_{ij} + H\right) y_j, & 1 \leq i \leq |\mathcal{G}^L|, \\ \sum_{j=1}^{|\mathcal{G}^R|} y_j = 1, & \\ y_j \geq 0, & 1 \leq j \leq |\mathcal{G}^R|. \end{array} \right\}.$$

By induction, we have $v(\mathcal{G}^S_{ij} + H) = v(\mathcal{G}^S_{ij}) + v(H)$, so rewriting the restrictions gives us

$$\begin{aligned} v(G+H) &= \min \left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 \geq v(H) + \sum_{j=1}^{|\mathcal{G}^R|} v\left(\mathcal{G}^S_{ij}\right) y_j, & 1 \leq i \leq |\mathcal{G}^L|, \\ \sum_{j=1}^{|\mathcal{G}^R|} y_j = 1, & \\ y_j \geq 0, & 1 \leq j \leq |\mathcal{G}^R|. \end{array} \right\} \\ &= v(H) + \min \left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 \geq \sum_{j=1}^{|\mathcal{G}^R|} v\left(\mathcal{G}^S_{ij}\right) y_j, & 1 \leq i \leq |\mathcal{G}^L|, \\ \sum_{j=1}^{|\mathcal{G}^R|} y_j = 1, & \\ y_j \geq 0, & 1 \leq j \leq |\mathcal{G}^R|. \end{array} \right\} \\ &= v(H) + v(G), \end{aligned}$$

completing the proof. □

Example 4.22. We use Theorem 4.21 to simplify the computation of the synchronized value of the game from Example 4.9. We find that

$$v\left(2 \cdot \left(\boxed{P} + \boxed{P}\boxed{P}\right)\right) = v\left(2 \cdot \boxed{P}\boxed{P}\right) - 2,$$

and we now only have to consider the game $G = 2 \cdot \boxed{P}\boxed{P} = \boxed{P}\boxed{P} + \boxed{P}\boxed{P}$. $\mathcal{G}^S$ is now a 2×2 matrix, with the diagonal being given by the option of both players move on the same game and the anti-diagonal being given by the option of moving on different games,

$$\mathcal{G}^S = \begin{pmatrix} \boxed{P}\boxed{} + \boxed{P}\boxed{P} & \boxed{P}\boxed{} + \boxed{}\boxed{P} \\ \boxed{}\boxed{P} + \boxed{P}\boxed{} & \boxed{P}\boxed{P} + \boxed{P}\boxed{} \end{pmatrix}.$$

Using Example 4.16 and Theorem 4.21, we replace each game by its synchronized value to obtain the zero-sum game associated to G,

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

By Theorem 2.39, for both players, choosing either game with probability $\frac{1}{2}$ is a Nash-equilibrium, and the value of the Nash-equilibrium is $\frac{5}{2}$. Thus, we find that

$$\begin{aligned} v\left(2 \cdot \left(\boxed{P} + \boxed{P}\boxed{P}\right)\right) &= \frac{5}{2} - 2 \\ &= \frac{1}{2}. \end{aligned}$$

Note that $2v\left(\boxed{P} + \boxed{P}\boxed{P}\right) \neq v\left(2 \cdot \left(\boxed{P} + \boxed{P}\boxed{P}\right)\right)$. ◁

Remark 4.23. Because of Theorem 4.21, from now on we will write the game $\boxed{P}$ as 1. In general, for $n > 0$, we define the synchronized Push games

$$n \overset{\text{def}}{=} \square^{n-1}\boxed{P}, \qquad\qquad -n \overset{\text{def}}{=} \square^{n-1}\boxed{P}.$$

Because every such game n is terminal with Nash value n, we may "add" the games together like we would regular integers. For example, we write $\boxed{}\boxed{P} + \boxed{P} = 2 - 1 = 1$. While these games are not technically isomorphic, this has no impact on Nash value and Nash equilibria by Theorem 4.21.

5 Synchronized Push

5.1 Halves

We will consider synchronized Push games of combinatorial value $\frac{1}{2}$. To this end, consider the games

$$H' = -1 + \boxed{P}\boxed{P} \text{ and } H = 2 + \boxed{P}\boxed{P},$$

where the games -1 and 2 are as defined in Remark 4.23. From H', Left can move to $H'^L = -1 + \boxed{P}\boxed{} = -1 + 1 = 0$. Right technically has two options, but by Theorem 4.21, we know Right will not move on -1, hence we only consider $H'^R - 1 + \boxed{}\boxed{P} = -1 + 2 = 1$. This also leaves us with the synchronized option $H'^S = -1 + \boxed{P}\boxed{} = 0$. In matrix form,

$$H' = \left(\begin{array}{c|c} & 1 \\ \hline 0 & 0 \end{array}\right).$$

Similarly, for H we find

$$H = \left(\begin{array}{c|c} & 1 \\ \hline 0 & 1 \end{array}\right).$$

Using Lemma 3.2 we see that that $H' = H = \frac{1}{2}$ as combinatorial games. However, we see that they behave differently as synchronized games; A synchronized turn on H' results in the game $-1+1 = 0$, a draw, while a synchronized turn on H results in the game $2 + 1 = 1$, a Left-player win. Moreover, we find $v(H') = 0$ and $v(H) = 1$.

We will now consider $n \in \mathbb{N}$ copies of either game. It turns out the synchronized values of n copies satisfy a recurrence relation. The exact recurrence relations differ between both games, but we will see that the difference between n and $n-1$ copies tends to $\frac{1}{2}$, for both games. In the case of H', the recurrence relation of $v(nH')$ is the same as for n copies of a Hackenbush position consisting of a red edge on top of a blue edge [10], a position which also has combinatorial value $\frac{1}{2}$.

Theorem 5.1. *Let $u_n = v(nH')$. Then u_n satisfies the following recurrence relation:*

$$\begin{cases} u_n = \frac{1}{n}u_{n-1} + \frac{n-1}{n}(u_{n-2} + 1), & n \geq 3, \\ u_1 = 0, u_2 = \frac{1}{2}. \end{cases}$$

Proof. u_1, u_2 follow from Example 4.16 and Example 4.22, respectively. Now, consider $n \geq 3$ copies of H'. By Theorem 2.39, picking any copy of H' with equal probability is a Nash equilibrium for both players. If both players move on the same copy of H', then the resulting game is

$$(n-1)H' + H'^S = (n-1)H' + 0 = (n-1)H'.$$

If both players play on different copies of H', the resulting game is

$$(n-2)H' + H'^L + H'^R = (n-2)H' + 0 + 1 = (n-2)H' + 1.$$

As the former happens with probability $\frac{1}{n}$ and the latter with probability $\frac{n-1}{n}$, we arrive at the desired recurrence relation. $\square$

This recurrence relation is rather ill-behaved. However, it is possible to find a direct solution for the difference $u_n - u_{n-1}$. Before we examine the difference sequence, we will first find a recurrence relation for the Nash value of nH.

Theorem 5.2. *Let $v_n = v(nH)$. Then, v_n satisfies the following recurrence relation.*

$$\begin{cases} v_n = 1 + \frac{1}{n} v_{n-1} + \frac{n-1}{n} v_{n-2}, & n \geq 3, \\ v_1 = 1, v_2 = \frac{3}{2}. \end{cases}$$

Proof. We have already seen that $v_1 = 1$. For v_2, we know that choosing either copy of H with probability $\frac{1}{2}$ is a Nash equilibrium, so

$$\begin{aligned} v_2 &= \tfrac{1}{2} v \left(H + H^S\right) + \tfrac{1}{2} v \left(H^L + H^R\right) \\ &= \tfrac{1}{2} v \left(H + 1\right) + \tfrac{1}{2} v \left(0 + 1\right) \\ &= \tfrac{1}{2} v(H) + \tfrac{1}{2} + \tfrac{1}{2} \\ &= \tfrac{3}{2}. \end{aligned}$$

Now, consider $n \geq 3$ copies of H. Again, choosing any copy with equal probability is a Nash equilibrium for both players. Thus, we arrive with probability $\frac{1}{n}$ at the game

$$(n-1)H + H^S = (n-1)H + 1,$$

and with probability $\frac{n-1}{n}$ at the game

$$(n-2)H + H^L + H^R = (n-2)H + 0 + 1 = (n-2)H + 1,$$

giving us the desired recurrence relation. □

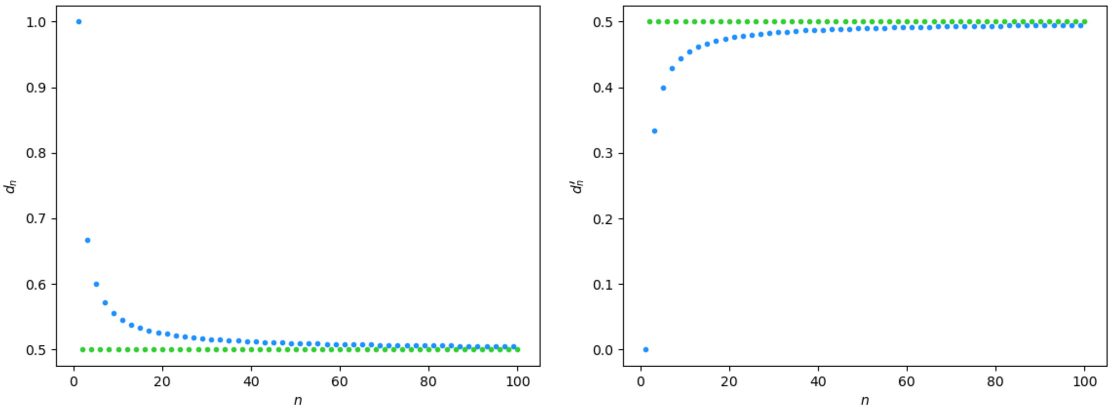
This recurrence relation is difficult to solve as well. We see that, even though the games H and H' are equal in the combinatorial sense, the Nash value of n copies of the games behaves differently.

We will now analyze the difference sequences $d_n = v_n - v_{n-1}$ and $d'_n = u_n - u_{n-1}$, $n \geq 1$. We write

$$\begin{aligned} d'_n &= u_n - u_{n-1} \\ &= \tfrac{n-1}{n}(1 + u_{n-2}) + \tfrac{1}{n} u_{n-1} - u_{n-1} \\ &= \tfrac{n-1}{n} + \tfrac{n-1}{n} u_{n-2} - \tfrac{n-1}{n} u_{n-1} \\ &= \tfrac{n-1}{n} - \tfrac{n-1}{n} d'_{n-1} \\ &= \tfrac{n-1}{n}(1 - d'_{n-1}). \end{aligned}$$

This gives us the following recurrence relation for d'_n,

$$\begin{cases} d'_n = \frac{n-1}{n}(1 - d'_{n-1}), & n \geq 2, \\ d_1 = 0. \end{cases}$$



(a) (b)

Figure 1: Plots of d_n (Figure 1a) and d'_n (Figure 1b) for $n \in \{1, \dots, 100\}$. Points are colored green for n even and blue for n odd.

Similarly for d_n we find

$$\begin{aligned} d_n &= v_n - v_{n-1} \\ &= 1 + \tfrac{1}{n} v_{n-1} + \tfrac{n-1}{n} v_{n-2} - v_{n-1} \\ &= 1 - \tfrac{n-1}{n} \left(v_{n-1} - v_{n-2} \right) \\ &= 1 - \tfrac{n-1}{n} d_{n-1}, \end{aligned}$$

giving us the recurrence relation

$$\begin{cases} d_n = 1 - \frac{n-1}{n} d_{n-1}, & n \geq 2, \\ d_1 = 1. \end{cases}$$

The recurrence relations for d_n, d'_n do behave nicely, and we can find a direct formula for both. As it turns out, both d_n, d'_n will tend to $\frac{1}{2}$, the combinatorial value of both H and H', as n tends to infinity.

Theorem 5.3. *For the Push games H and H' as defined above, we have* $\lim_{n\to\infty} v(nH) - v((n-1)H) = \lim_{n\to\infty} v(nH') - v((n-1)H') = \frac{1}{2}$.

Proof. Solutions to the recurrence relations of d_n and d'_n are given by

$$d_n = \frac{2n + (-1)^{n-1} + 1}{4n}, \qquad d'_n = \frac{2n + (-1)^n - 1}{4n},$$

which can be verified by substituting the solutions into their respective recurrence relation. One can easily see that $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d'_n = \frac{1}{2}$. In fact, for n even we have $d_n = d'_n = \frac{1}{2}$ exactly. $\square$

In Figure 1 we have plotted d_n and d'_n. In Figure 1a, we see that the subsequence $(d_{2k+1})_{k\in\mathbb{Z}_0}$ converges to $\frac{1}{2}$ from above, while in Figure 1b, we see that the subsequence $(d'_{2k+1})_{k\in\mathbb{Z}_0}$ converges to $\frac{1}{2}$ from below. This is ultimately caused by the difference in behaviour of H and H' when we play synchronized. For both games, $v(H'^L) = v(H^L) = 0$ and $v(H'^R) = v(H^R) = 1$. However, a synchronized move on H' results in the game 0, while a synchronized move on H results in the game 1. Consequently, when n is small and the probability to play on the same copy is large, nH' will be more favorable to Right and nH will be more favorable to Left.

We end this section with a small corollary that will be useful in sections to come.

Corollary 5.4. *Consider $v_n = v(nH)$, $n \in \mathbb{Z}_{\geq 0}$. Then,*

$$v_n + \tfrac{1}{2} \leq v_{n+1} \leq v_n + 1.$$

Proof. Using the solution to the recurrence relation of d_n from the proof of Theorem 5.3, we find that

$$v_{n+1} - v_n = \frac{2(n+1) + (-1)^n + 1}{4(n+1)} \geq \frac{2(n+1)}{4(n+1)} = \frac{1}{2}.$$

For the second inequality, we note that from $(n+1)H$, Right can move to

$$nH + 1,$$

which, by Theorem 4.21, has Nash value $v_n + 1$. By Theorem 4.17, $v_{n+1} \leq v_n + 1$. ☐

5.2 Quarters

Now that we have seen how Push games of combinatorial value $\frac{1}{2}$ behave, we will move on to higher powers of $\frac{1}{2}$. To do this, we first define for $i, j \in \mathbb{N}_0$ the game $H_{i,j}$, given by

$$H_{i,j} = 2 + \square^i \boxed{P} \square^j \boxed{P}.$$

Using Lemma 3.2, we see that $H_{i,0}$ has combinatorial value $\frac{1}{2^i}$, while $H_{i,j}, j \geq 1$ has combinatorial value $1 - j$. The $2 = \boxed{}\boxed{P}$ component of each $H_{i,j}$ only serves to "even the playing field" for Left and to give us the powers of $\frac{1}{2}$. By Theorem 4.21, Left will always ignore moving on this component until no other possible moves remain, and it has no impact on the Nash equilibrium, only raising the Nash value by 2 for each copy of $H_{i,j}$. When we mention Left "moving on" a game $H_{i,j}$ it is therefore unambiguous what this entails.

In this section we will start by considering the game

$$H_{1,0} = 2 + \boxed{}\boxed{P}\boxed{P},$$

which has combinatorial value $\frac{1}{4}$. Possible followers of $H_{1,0}$ are

$$H_{0,0} = 2 + \boxed{P}\boxed{P}\boxed{},$$

which is the result of Right's solo move or *the* synchronized move, and

$$H_{0,1} = 2 + \boxed{P}\boxed{}\boxed{P},$$

the result of a Left solo move. In matrix form,

$$H_{1,0} = \left(\begin{array}{c|c} & H_{0,0} \\ \hline H_{0,1} & H_{0,0} \end{array}\right).$$

In order to compute the Nash value of any number of copies of $H_{1,0}$, we need to know the Nash value games consisting of copies of $H_{1,0}$ and copies of its followers. Note that $H_{0,0}$ is the same game as H from Section 5.1, and we have seen how multiples of this game behave (Theorem 5.2). The game $H_{0,1}$ is a draw and therefore has Nash value 0. In fact, any number of copies of $H_{0,1}$ is a draw, which we prove in the following lemma.

Lemma 5.5. *For all $k \in \mathbb{N}$, we have $v(kH_{0,1}) = 0$.*

Proof. We will show that $kH_{0,1} \in \mathcal{D}$ for all $k \in \mathbb{N}$ by induction on k.

The game $H_{0,1}$ is a draw: Left will not play on her terminal position, so the resulting game will be

$$2 + \begin{array}{|c|c|c|}\hline & P & \\ \hline\end{array} = 2 - 2 = 0 \in \mathcal{D}.$$

If we have $k > 1$ copies of $H_{0,1}$, then the game is still a draw, as both players pick a copy uniformly at random. If they play on the same copy, then the resulting game is

$$(k-1)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline & P & \\ \hline\end{array} = (k-1)H_{0,1} + 2 - 2 = (k-1)H_{0,1},$$

which is a draw by induction. If they play on different copies, then the resulting game is

$$(k-2)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline & & P \\ \hline\end{array} + 2 + \begin{array}{|c|c|c|}\hline P & P & \\ \hline\end{array} = (k-2)H_{0,1} + H_{0,0} - 1.$$

Here, Left can force the game to end in a draw at worst by moving on $H_{0,0}$. If Right follows suit by moving on $H_{0,0}$ as well, then the resulting game is

$$(k-2)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline P & & \\ \hline\end{array} - 1 = (k-2)H_{0,1},$$

which is a draw by induction. Finally, if Right moves on a copy of $H_{0,1}$, then the resulting game is

$$(k-3)H_{0,1} + H_{0,0} + 2 + \begin{array}{|c|c|c|}\hline & P & \\ \hline\end{array} - 1 = (k-3)H_{0,1} + H_{0,0} - 1,$$

once again, by induction, ending in a draw. □

Having calculated the Nash value of games of the form $kH_{0,1}, k \in \mathbb{N}$, the next step will be to consider games of the form $nH_{0,0} + kH_{0,1}, n, k \in \mathbb{N}$. Since $H_{0,1}$ is a draw, one might expect that adding copies of $H_{0,1}$ has no impact on the Nash value. We will see that this is indeed the case.

Lemma 5.6. *For $n, k \in \mathbb{Z}_{\geq 0}$, we have $v(nH_{0,0} + kH_{0,1}) = v(nH_{0,0})$.*

Proof. We write $v(nH_{0,0}) = v_n$. Using induction on the birthday, we will compute the zero-sum game a associated with $nH_{0,0} + kH_{0,1}$. We will use Theorem 2.40 to decrease the size of our zero-sum game. Instead of having having, say, n rows and n column, one for each copy of $H_{0,0}$, we will only have one row and column for the event where a player chooses to play on any copy of $H_{0,0}$. Since all copies are identical, we know that choosing any copy with equal probability is a Nash equilibrium. By Theorem 2.40, we replace the matrix block corresponding to both players moving on $H_{0,0}$ by the average over all coefficients of this block. We do the same for $H_{0,1}$ and will do so for any game we might encounter in the future, of which we have multiple copies.

So, for $n, k \in \mathbb{Z}_{\geq 0}$, consider

$$nH_{0,0} + kH_{0,1} = n \cdot \left(2 + \begin{array}{|c|c|}\hline P & P \\ \hline\end{array}\right) + k \cdot \left(2 + \begin{array}{|c|c|c|}\hline P & & P \\ \hline\end{array}\right).$$

First, consider the event where both players play on a copy of $H_{0,0}$. If both players play on the same copy of $H_{0,0}$, the resulting game will be

$$G_1 = (n-1)H_{0,0} + 1 + kH_{0,1},$$

while moving on different copies results in the game

$$G_2 = (n-2)H_{0,0} + 0 + 1 + kH_{0,1},$$

Since each player chooses any copy uniformly at random, the expected Nash value is, by induction,

$$\begin{aligned}\tfrac{1}{n}v\left(G_1\right) + \tfrac{n-1}{n}v\left(G_2\right) &= \tfrac{1}{n}\left(v_{n-1}+1\right) + \tfrac{n-1}{n}\left(v_{n-2}+1\right) \\ &= 1 + \tfrac{1}{n}v_{n-1} + \tfrac{n-1}{n}v_{n-2} \\ &= v_n,\end{aligned}$$

where the first equality follows Theorem 4.21 and the second equality follows from Theorem 5.2.

If Left moves on a copy of $H_{0,0}$, while Right moves on a copy of $H_{0,1}$, then the resulting game is

$$(n-1)H_{0,0} + 0 + (k-1)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline P & P & \\ \hline\end{array} = nH_{0,0} + (k-1)H_{0,1},$$

which has Nash value v_n by induction. If Right moves on $H_{0,0}$, while Left moves on $H_{0,1}$, then the resulting game is

$$(n-1)H_{0,0} + 1 + (k-1)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline & & P \\ \hline\end{array} = (n-1)H_{0,0} + (k-1)H_{0,1},$$

which has Nash value v_{n-1} by induction.

Lastly, suppose both players play on a copy of $H_{0,1}$. If they play on the same copy, the resulting game is

$$\begin{aligned}G_3 &= nH_{0,0} + (k-1)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline & P & \\ \hline\end{array} \\ &= nH_{0,0} + (k-1)H_{0,1},\end{aligned}$$

while moving on different copies results in the game

$$\begin{aligned} G_4 &= nH_{0,0} + (k-2)H_{0,1} + 2 + \begin{array}{|c|c|c|}\hline & & P \\ \hline \end{array} + 2 + \begin{array}{|c|c|c|}\hline P & P & \\ \hline \end{array} \\ &= (n+1)H_{0,0} + (k-2)H_{0,1} - 1. \end{aligned}$$

By induction, the expected Nash value is

$$\tfrac{1}{k} v\left(G_3\right) + \tfrac{k-1}{k} v\left(G_4\right) = \tfrac{1}{k} v_n + \tfrac{k-1}{k}\left(v_{n+1} - 1\right).$$

With these four values, we find the following zero-sum game corresponding to $nH_{0,0}+kH_{0,1}$:

$$\begin{pmatrix} v_n & v_n \\ v_{n-1} & \frac{1}{k} v_n + \frac{k-1}{k}\left(v_{n+1}-1\right) \end{pmatrix},$$

where the first row and column correspond to moving on any copy of $H_{0,0}$ uniformly at random and the second row and column correspond to moving on any copy of $H_{0,1}$ uniformly at random. By Corollary 5.4, $v_n \geq v_{n-1}$ and $v_n \geq v_{n+1} - 1$, so for Left, moving on a copy of $H_{0,0}$ dominates moving on a copy of $H_{0,1}$. As a result, this zero-sum has a Nash equilibrium with a value of v_n. □

Now, we are almost ready to examine how copies of the game $H_{1,0}$ behave. First, we must consider games consisting of copies of $H_{1,0}$, $H_{0,0}$ and $H_{0,1}$. Once again, the copies of $H_{0,1}$ do not contribute to the Nash value of this game.

Lemma 5.7. *For $m, n, k \in \mathbb{Z}_{\geq 0}$, we have $v(mH_{1,0} + nH_{0,0} + kH_{0,1}) = v(mH_{1,0} + nH_{0,0})$.*

Proof. Consider $G = mH_{1,0} + nH_{0,0} + kH_{0,1}$. This proof is similar to that of Lemma 5.6; using induction on the birthday of G, we will compute the Nash values of the 3×3 zero-sum game corresponding to G. Then, we will show that, for both players, moving on $H_{1,0}$ dominates moving on $H_{0,1}$.

We have the following table of synchronized options:

	$H_{1,0}$	$H_{0,0}$	$H_{0,1}$
$H_{1,0}$	Same copy: $(m-1)H_{1,0} + (n+1)H_{0,0} + kH_{0,1}$ Different copy: $(m-2)H_{1,0} + (n+1)H_{0,0} + (k+1)H_{0,1}$	$(m-1)H_{1,0} + (n-1)H_{0,0} + (k+1)H_{0,1} + 1$	$(m-1)H_{1,0} + (n+1)H_{0,0} + kH_{0,1}$
$H_{0,0}$	$(m-1)H_{1,0} + nH_{0,0} + kH_{0,1} + 0$	Same copy: $mH_{1,0} + (n-1)H_{0,0} + kH_{0,1} + 1$ Different copy: $mH_{1,0} + (n-2)H_{0,0} + kH_{0,1} + 0 + 1$	$mH_{1,0} + nH_{0,0} + (k-1)H_{0,1} + 0$
$H_{0,1}$	$(m-1)H_{1,0} + (n+1)H_{0,0} + (k-1)H_{0,1} - 1$	$mH_{1,0} + (n-1)H_{0,0} + (k-1)H_{0,1} + 1 - 1$	Same copy: $mH_{1,0} + nH_{0,0} + (k-1)H_{0,1} + 0$ Different copy: $mH_{1,0} + (n+1)H_{0,0} + (k-2)H_{0,1} - 1$

Writing $v_{m,n} = v(mH_{1,0} + nH_{0,0})$, we can compute the Nash values of the above games. The zero-sum game associated with G is then given by

$$\begin{pmatrix} \begin{array}{c}\frac{1}{m}v_{m-1,n+1}\\ +\frac{m-1}{m}v_{m-2,n+1}\end{array} & v_{m-1,n-1}+1 & v_{m-1,n+1} \\ v_{m-1,n} & \begin{array}{c}\frac{1}{n}(v_{m,n-1}+1)\\ +\frac{n-1}{n}(v_{m,n-2}+1)\end{array} & v_{m,n} \\ v_{m-1,n+1}-1 & v_{m,n-1} & \begin{array}{c}\frac{1}{k}v_{m,n}\\ +\frac{k-1}{k}(v_{m,n+1}-1)\end{array} \end{pmatrix}.$$

We will start by showing that the first column dominates the third column, which corresponds to moving on $H_{1,0}$ dominating moving on $H_{0,1}$ for Right. To do this, we will first show that $v_{m,n}$ is non-decreasing in m.

For arbitrary $m', n' \in \mathbb{Z}_{\geq 0}$, we have that from $(m'+1)H_{1,0} + n'H_{0,0}$, Left can move to $m'H_{1,0} + n'H_{0,0} + H_{0,1}$. By induction, this game has Nash value $v_{m',n'}$ and, by Theorem 4.17, we find $v_{m',n'} \leq v_{m'+1,n'}$.

Now, by the above statement, for the first two rows we find

$$\tfrac{1}{m}v_{m-1,n+1} + \tfrac{m-1}{m}v_{m-2,n+1} \leq v_{m-1,n+1},$$

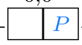
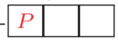
and

$$v_{m-1,n} \leq v_{m,n}.$$

For the third row, we only have $v_{m-1,n+1}-1 \leq v_{m,n+1}-1$. To see that also $v_{m-1,n+1} - 1 \leq v_{m,n}$, we use that $v_{m,n+1} \leq v_{m,n}+1$. To see that this is true, consider a game with m copies of $H_{1,0}$ and $n+1$ copies of $H_{0,0}$, and consider Right moving on a copy of $H_{0,0}$. As a result,

$$\begin{aligned} v_{m-1,n+1} - 1 &\leq v_{m,n+1} - 1 \\ &\leq v_{m,n}, \end{aligned}$$

hence the third column gets dominated by the first column. We will now show domination for Left. Similarly to the columns, the first row dominates the third row. Because the third column has been removed, we only have to check the first two columns. To this end, we first show the following.

For arbitrary $m', n' \in \mathbb{Z}_{\geq 0}$, consider $(m'+1)H_{1,0}+n'H_{0,0}$. Using two Right moves on a copy of $H_{1,0}$, we arrive at the game $m'H_{1,0}+n'H_{0,0}+$$+$. Consequently, $v_{m'+1,n'} \leq v_{m',n'}+1$ for any $m', n' \geq 0$.

As a result of the above, we find

$$\begin{aligned} \tfrac{1}{m}v_{m-1,n+1} + \tfrac{m-1}{m}v_{m-2,n+1} &\geq \tfrac{1}{m}v_{m-1,n+1} + \tfrac{m-1}{m}(v_{m-1,n+1}-1) \\ &= v_{m-1,n+1} - \tfrac{m-1}{m} \\ &> v_{m-1,n+1} - 1, \end{aligned}$$

and

$$v_{m-1,n-1}+1 \geq v_{m,n-1},$$

showing that for Left, moving on $H_{1,0}$ dominates moving on $H_{0,1}$ as well.

Now, we must show that the Nash value of G is indeed $v_{m,n}$. Both player will ignore all copies of $H_{0,1}$ and only play strategies involving $H_{1,0}$ and $H_{0,0}$. We have not computed the Nash equilibrium of G yet, however, we know that any resulting game will again be of the form $m'H_{1,0} + n'H_{0,0} + k'H_{0,1}$, $m', n', k' \in \mathbb{Z}_{\geq 0}$, with possibly some additional terminal values. Terminal values are ignored and inductively, both players will continue to ignore $H_{0,1}$ until no copies of $H_{1,0}$ are left. At this point, only copies of $H_{0,0}$ and $H_{0,1}$ and terminal positions are left, and Lemma 5.6 tells us that the copies of $H_{0,1}$ are irrelevant for the Nash value of this game. Since this holds for any game that can result from the Nash equilibrium, we have that the number of copies of $H_{0,1}$ in G is irrelevant for its Nash value, hence $v(G) = v_{m,n}$. □

In the proof of Lemma 5.7 we also showed some properties of $v_{m,n}$ that are similar to Corollary 5.4. These properties are as follows.

Lemma 5.8. *Consider* $v_{m,n} = v(mH_{1,0} + nH_{0,0})$, $m, n \in \mathbb{Z}_{\geq 0}$. *Then,*

(i) $v_{m,n} \leq v_{m+1,n} \leq v_{m,n} + 1$,

(ii) $v_{m,n} \leq v_{m,n+1} \leq v_{m,n} + 1$,

(iii) $v_{m+1,n} \leq v_{m,n+1}$.

Proof. Let $m, n \in \mathbb{Z}_{\geq 0}$ and consider $G = (m+1)H_{1,0} + nH_{0,0}$. Right can move to $G^R = mH_{1,0} + (n+1)H_{0,0}$. From G^R, Left can move to $G^{RL} = mH_{1,0} + nH_{0,0} + 0$. By Theorem 4.17, $v_{m+1,n} \leq v_{m,n+1}$ and $v_{m,n} \leq v_{m,n+1}$. For the three remaining inequalities, see the proof of Lemma 5.7. □

In Lemma 5.7 we have shown that, in a game consisting of copies of $H_{1,0}$, copies of $H_{0,0}$ and copies of $H_{0,1}$, both players will avoid moving on $H_{0,1}$. We can use this to more easily compute the Nash equilibrium for games of this form. In Table 1, we have computed $v_{m,n}$ for different values of m, n by solving the associated linear program. In the games for which we computed the Nash value we found that, if n is large enough, the Nash equilibrium consists of a deterministic strategy. If $m \geq 2$, then the Nash equilibrium is to move on any copy of $H_{1,0}$ uniformly at random for both players, while if $m = 1$, the Nash equilibrium is for Left to move on $H_{1,0}$, while Right moves on any copy of $H_{0,0}$ uniformly at random. In the following theorems we will show that this is indeed the case.

Theorem 5.9. *Let* $G = H_{1,0} + nH_{0,0}$, *for* $n \geq 4$. *Then, Left moving on* $H_{1,0}$ *and Right moving on any copy of* $H_{0,0}$ *uniformly at random is a Nash equilibrium. Consequently,* $v(G) = v_{n-1} + 1$.

Proof. We use induction on n. Consider $G = H_{1,0} + nH_{0,0}$. For $n = 4$ and $n = 5$, we see in Table 1 that the statement holds. For $n \geq 6$, we have the following table of synchronized options:

$m \setminus n$	0	1	2	3	4	5	6	7
0	0	1	1.5	2.1667	2.6667	3.2667	3.7667	4.3381
1	1	1.3333	2.0333	2.5196	3.1667	3.6667	4.2667	4.7667
2	1.1667	1.7667	2.3363	2.9167	3.4667	4.0167	4.5524	5.0881
3	1.4778	2.1343	2.6520	3.2667	3.7833	4.3619	4.8738	5.4307
4	1.8586	2.4152	3.0042	3.5458	4.1030	4.6327	5.1737	5.6969
5	2.1905	2.7224	3.3225	3.8473	4.4161	4.9338	5.4839	5.9977
6	2.4664	3.0572	3.5961	4.1552	4.6829	5.2254	5.7470	6.2803
7	2.7702	3.3616	3.8912	4.4542	4.9755	5.5215	6.0381	6.5745

Table 1: The Nash value $v_{m,n} = v(mH_{1,0}+nH_{0,0})$ for different values of m, n. Values have been rounded to four decimal places. For the values marked in red, the Nash equilibrium consists of mixed strategies for both players. For the unmarked values, the Nash equilibrium consists of deterministic strategies for both players.

	$H_{1,0}$	$H_{0,0}$
$H_{1,0}$	$(n+1)H_{0,0}$	$(n-1)H_{0,0} + H_{0,1} + 1$
$H_{0,0}$	$nH_{0,0} + 0$	Same copy: $H_{1,0} + (n-1)H_{0,0} + 1$ Different copy: $H_{1,0} + (n-2)H_{0,0} + 0 + 1$

Using Lemma 5.7 to compute the Nash values of the synchronized options, we find that the zero-sum game associated to G is

$$\begin{pmatrix} v_{n+1} & v_{n-1}+1 \\ v_n & 1 + \frac{1}{n}v_{1,n-1} + \frac{n-1}{n}v_{1,n-2} \end{pmatrix}.$$

We will show that $v_{n-1}+1$ is a saddle point. By Corollary 5.4,

$$\begin{aligned} v_{n+1} &\geq v_n + \tfrac{1}{2} \\ &\geq \left(v_{n-1} + \tfrac{1}{2}\right) + \tfrac{1}{2} \\ &= v_{n-1} + 1, \end{aligned}$$

showing that $v_{n-1}+1$ is the smallest value of its row. Now, we show that it is the largest value of its column. By induction, we know that on the games $H_{1,0} + (n-1)H_{0,0}$ and $H_{1,0} + (n-2)H_{0,0}$, Left moving on $H_{1,0}$ and Right moving on $H_{0,0}$ is a Nash equilibrium. As a result,

$$\begin{aligned} v_{1,n-1} &= v_{n-2} + 1, \\ v_{1,n-2} &= v_{n-3} + 1. \end{aligned}$$

This gives us

$$\begin{aligned} 1 + \tfrac{1}{n}v_{1,n-1} + \tfrac{n-1}{n}v_{1,n-2} &= 1 + \tfrac{1}{n}(v_{n-2}+1) + \tfrac{n-1}{n}(v_{n-3}+1) \\ &= 2 + \tfrac{1}{n}v_{n-2} + \tfrac{n-1}{n}v_{n-3} \\ &\leq 2 + \tfrac{1}{n-1}v_{n-2} + \tfrac{n-2}{n-1}v_{n-3} \\ &= 1 + v_{n-1}, \end{aligned}$$

where the inequality comes from the fact that v_n is increasing in n (Corollary 5.4) and the last equality uses the recurrence relation of v_n (Theorem 5.2).

We have

$$v_{n+1} \geq v_{n-1} + 1 \geq 1 + \tfrac{1}{n} v_{1,n-1} + \tfrac{n-1}{n} v_{1,n-2},$$

so $v_{n-1}+1$ is a saddle point, hence the corresponding strategy is a Nash equilibrium, and $v(G) = v_{n-1} + 1$. ☐

Theorem 5.10. *Let $G = mH_{1,0}+nH_{0,0}$, for $m \geq 1$. Then, there exists an $N(m) \geq 1$ such that for all $n \geq N(m)$, choosing any copy of $H_{1,0}$ uniformly at random is a Nash equilibrium for both players.*

Proof. Consider $G = mH_{1,0}+nH_{0,0}$, for $m \geq 2$ and $n \geq 1$. The synchronized options of G are

	$H_{1,0}$	$H_{0,0}$
$H_{1,0}$	Same copy: $(m-1)H_{1,0} + (n+1)H_{0,0}$ Different copy: $(m-2)H_{1,0} + (n+1)H_{0,0} + H_{0,1}$	$(m-1)H_{1,0} + (n-1)H_{0,0} + H_{0,1} + 1$
$H_{0,0}$	$(m-1)H_{1,0} + nH_{0,0} + 0$	Same copy: $mH_{1,0} + (n-1)H_{0,0} + 1$ Different copy: $mH_{1,0} + (n-2)H_{0,0} + 0 + 1$

This gives us the following zero-sum game:

$$\begin{pmatrix} \frac{1}{m} v_{m-1,n+1} + \frac{m-1}{m} v_{m-2,n+1} & v_{m-1,n-1} + 1 \\ v_{m-1,n} & 1 + \frac{1}{n} v_{m,n-1} + \frac{n-1}{n} v_{m,n-2} \end{pmatrix}.$$

First, by Lemma 5.8, $v_{m-1,n} \leq v_{m-2,n+1}$ and $v_{m-2,n+1} \leq v_{m-1,n+1}$, so we have

$$\begin{aligned} v_{m-1,n} &\leq v_{m-2,n+1} \\ &\leq \tfrac{1}{m} v_{m-1,n+1} + \tfrac{m-1}{m} v_{m-2,n+1}. \end{aligned}$$

Next, we will find an $N \geq 1$ and show that for all $n \geq N$: $\frac{1}{m} v_{m-1,n+1} + \frac{m-1}{m} v_{m-2,n+1} \leq v_{m-1,n-1} + 1$. This is considerably more work. Consider

$$\begin{aligned} G_1 &= (m-1)H_{1,0} + (n-1)H_{0,0} + 1, \\ G_2 &= (m-2)H_{1,0} + (n+1)H_{0,0}, \\ G_3 &= (m-1)H_{1,0} + (n+1)H_{0,0}, \end{aligned}$$

which have Nash value $v_{m-1,n-1}+1$, $v_{m-2,n+1}$ and $v_{m-1,n+1}$, respectively, and we will show that $\frac{1}{m} v(G_3) + \frac{m-1}{m} v(G_2) \leq v(G_1)$. We couple G_1, G_2 and G_3. By induction on m, both players will play on copies of $H_{1,0}$ until only one is left or none are left. Let us define the stochastic process X given by $X(0) = 0$, $X(1) = -1$, and

$$X(k) = \begin{cases} 1 + X(k-1), & \text{w.p. } \frac{1}{k}, \\ 1 + X(k-2), & \text{w.p. } \frac{k-1}{k}, \end{cases}$$

for $1 < k < m$. $X(k)$ models the number of copies of $H_{0,0}$ that we gain when we start with a game $kH_{1,0}$ and play according to the Nash equilibrium strategy. Note that when the process lands on $X(1)$, we must play according to Theorem 5.9 and a copy of $H_{0,0}$ becomes $\boxed{}\boxed{P} + \boxed{P} = 1$. We also introduce a sequence of random variables Z_k, given by

$$Z_k = \begin{cases} 1, & \text{w.p. } \frac{1}{k}, \\ 0, & \text{w.p. } \frac{k-1}{k}. \end{cases}$$

We consider three copies of the process X, say X_1, X_2 and X_3. For $k = 2, \ldots, m-1$ we couple them by setting

$$X_i(k) = 1 + Z_k X_i(k-1) + Z_k X_i(k-2), \qquad i = 1, 2, 3.$$

This gives us $X_1(k) = X_2(k) = X_3(k)$ for $k = 0, 1, \ldots, m-1$. The games G_1, G_2 and G_3 then result in the games

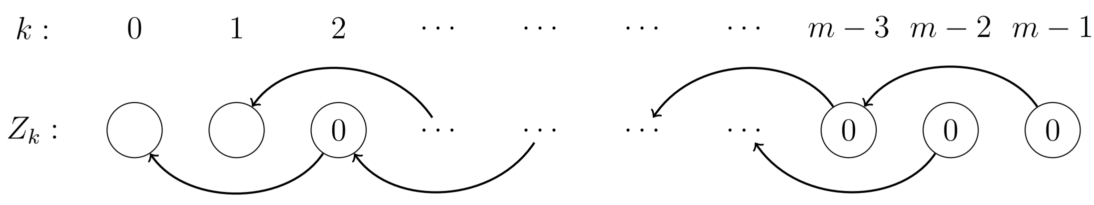
$$\begin{aligned} G_1' &= (n - 1 + X_1(m-1))H_{0,0} + 1 + 1 \cdot \mathbb{1}_{\{X_1(m-1) \text{ lands on } 1\}}, \\ G_2' &= (n + 1 + X_2(m-2))H_{0,0}, +1 \cdot \mathbb{1}_{\{X_2(m-2) \text{ lands on } 1\}}, \\ G_3' &= (n + 1 + X_3(m-1))H_{0,0}, +1 \cdot \mathbb{1}_{\{X_3(m-1) \text{ lands on } 1\}}, \end{aligned}$$

and we have $v(G_i) = \mathbb{E}_{Z_2,\ldots,Z_{m-1}}[v(G_i')]$ for $i = 1, 2, 3$. We have omitted the copies of $H_{0,1}$ that are created every time the players play on different copies of $H_{1,0}$, since removing them does not change the Nash equilibrium or the Nash value. Now, let

$$\ell = \begin{cases} \max\{2 \le k \le m-1 : Z_k = 1\}, & \text{if } Z_k = 1 \text{ for some } 2 \le k \le m-1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\ell = 0$ denotes the event $\{Z_2 = Z_3 = \cdots = Z_{m-1} = 0\}$, while $\ell \geq 2$ tells us the the first time both players play on the same copy of $H_{1,0}$.

First, consider the case $\ell = 0$. Then, a copy of X starting at $m-1$ and a copy of X starting at $m-2$ will never meet. If m is even, then both use the same amount of "jumps" before they reach 0 or 1:



Because the total number of jumps is $\frac{m-2}{2}$, we have

$$\begin{aligned} X(m-2) &= \tfrac{m-2}{2} + X(0) = \tfrac{m-2}{2}, \\ X(m-1) &= \tfrac{m-2}{2} + X(1) = \tfrac{m-2}{2} - 1. \end{aligned}$$

If we set $n' = n + X(m-1) = n + \frac{m-2}{2} - 1$, then Nash values of G_1', G_2', G_3' given $\ell = 0$ are

$$\begin{aligned} v(G_1') &= v_{n'-1} + 2, \\ v(G_2') &= v_{n'+2}, \\ v(G_3') &= v_{n'+1} + 1. \end{aligned}$$

Since $v(G_2') \leq v(G_3')$, we have

$$\begin{aligned} \tfrac{1}{m} v(G_3') + \tfrac{m-1}{m} v(G_2') &\leq \tfrac{1}{2} v(G_3') + \tfrac{1}{2} v(G_2') \\ &= \tfrac{1}{2}(v_{n'+1} + 1) + \tfrac{1}{2} v_{n'+2} \\ &= \tfrac{1}{2}(v_{n'-1} + d_{n'} + d_{n'+1} + 1) + \tfrac{1}{2}(v_{n'-1} + d_{n'} + d_{n'+1} + d_{n'+2}) \\ &= v_{n'-1} + d_{n'} + d_{n'+1} + \tfrac{1}{2} d_{n'+2} + \tfrac{1}{2}. \end{aligned}$$

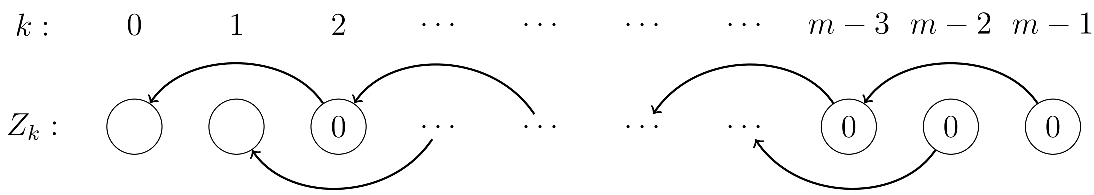
We know that $d_i = \frac{1}{2}$ for even values of i, while the subsequence d_{2i+1} is strictly decreasing with limit $\frac{1}{2}$. For $n' = 2$ and $n' = 3$ we have

$$\begin{aligned} d_2 + d_3 + \tfrac{1}{2} d_4 + \tfrac{1}{2} &= \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{23}{12} \leq 2, \\ d_3 + d_4 + \tfrac{1}{2} d_5 + \tfrac{1}{2} &= \tfrac{2}{3} + \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{3}{5} + \tfrac{1}{2} = \tfrac{59}{30} \leq 2. \end{aligned}$$

Hence, for $n' \geq 2$ we have $d_{n'} + d_{n'+1} + \frac{1}{2} d_{n'+2} + \frac{1}{2} \leq 2$, and thus

$$\tfrac{1}{m} v(G_3') + \tfrac{m-1}{m} v(G_2') \leq v_{n'-1} + 2 = v(G_1').$$

If $\ell = 0$, but m is odd, then the process starting at $m-1$ makes an additional jump compared to the process starting at $m-2$:



Now, we have

$$\begin{aligned} X(m-2) &= \tfrac{m-1}{2} + X(1) - 1 = \tfrac{m-1}{2} - 2, \\ X(m-1) &= \tfrac{m-1}{2} + X(0) = \tfrac{m-1}{2}. \end{aligned}$$

Setting $n' = n + X(m-1) = n + \frac{m-1}{2}$, then the Nash values of G_1', G_2', G_3' are

$$\begin{aligned} v(G_1') &= v_{n'-1} + 1, \\ v(G_2') &= v_{n'-1} + 1, \\ v(G_3') &= v_{n'+1}. \end{aligned}$$

Again, $v(G_2') \leq v(G_3')$, so we have

$$\begin{aligned} \tfrac{1}{m} v(G_3') + \tfrac{m-1}{m} v(G_2') &\leq \tfrac{1}{3} v(G_3') + \tfrac{2}{3} v(G_2') \\ &= \tfrac{1}{3} v_{n'+1} + \tfrac{2}{3}(v_{n'-1} + 1) \\ &= \tfrac{1}{3}(v_{n'-1} + d_{n'} + d_{n'+1}) + \tfrac{2}{3}(v_{n'-1} + 1). \end{aligned}$$

Since d_i converges to $\frac{1}{2}$ from above, we have $d_{n'}+d_{n'+1} > 1$ for all n', hence $\frac{1}{m}v(G_3')+\frac{m-1}{m}v(G_2') \not\leq v_{n'-1}+1 = v(G_1')$. Instead, for $n' \geq 2$, we have

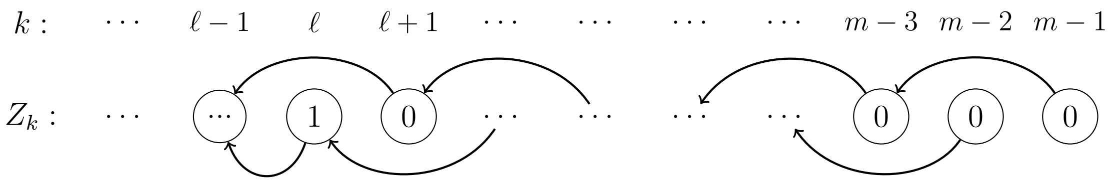
$$\tfrac{1}{3}(d_{n'}+d_{n'+1}-1) \leq \tfrac{1}{3}\left(\tfrac{1}{2}+\tfrac{2}{3}-1\right) = \tfrac{1}{18} < \tfrac{1}{10},$$

giving us

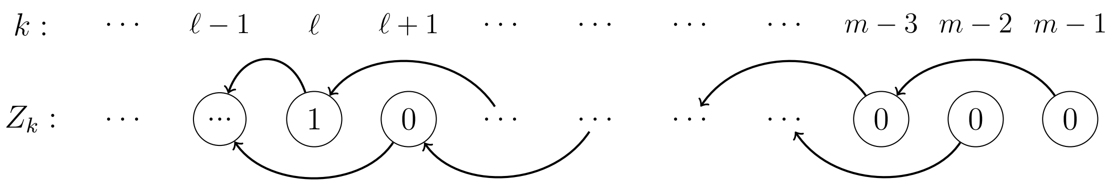
$$\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') \leq v(G_1') + \tfrac{1}{10}.$$

When the time comes to put everything together, we will see that this is enough.

We continue with the case $\ell \geq 2$. If $\ell \equiv m \bmod 2$, then a copy of X starting from $m-1$ will reach $\ell-1$ with the same number of "jumps" as a coupled copy starting from $m-2$:



In contrast, if $\ell \equiv m-1 \bmod 2$, then a process starting form $m-1$ needs an additional jump to get to the same point as a coupled copy that starts at $m-2$:



Since from ℓ onward they follow the same path, we find

$$X(m-1) = \begin{cases} X(m-2)+1, & \text{if } \ell \equiv m-1 \bmod 2, \\ X(m-2), & \text{if } \ell \equiv m \bmod 2. \end{cases}$$

and we know that either $X_1(m-1), X_2(m-2), X_3(m-1)$ all land on 1 or they all land on 0. We once again set $n' = n + X(m-1)$. This time, n' is still a stochastic value. By slight abuse of notation, we can say that the Nash values of G_1', G_2', G_3' are

$$\begin{aligned} v(G_1') &= v_{n'-1} + 1 + \mathbb{1}_{\{X_1(m-1) \text{ lands on } 1\}}, \\ v(G_2') &= \begin{cases} v_{n'} + \mathbb{1}_{\{X_1(m-1) \text{ lands on } 1\}}, & \text{if } \ell \equiv m-1 \bmod 2, \\ v_{n'+1} + \mathbb{1}_{\{X_1(m-1) \text{ lands on } 1\}}, & \text{if } \ell \equiv m \bmod 2, \end{cases} \\ v(G_3') &= v_{n'+1} + \mathbb{1}_{\{X_1(m-1) \text{ lands on } 1\}}. \end{aligned}$$

Note that $v(G_2') \leq v(G_3')$. For $\ell \equiv m-1 \bmod 2$, we get

$$\begin{aligned}\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1') &\leq \tfrac{1}{2}v(G_3') + \tfrac{1}{2}v(G_2') - v(G_1') \\ &= \tfrac{1}{2}v_{n'+1} + \tfrac{1}{2}v_{n'} - v_{n'-1} - 1 \\ &= \tfrac{1}{2}(v_{n'-1} + d_{n'} + d_{n'+1}) + \tfrac{1}{2}(v_{n'-1} + d_{n'}) - v_{n'-1} - 1 \\ &= d_{n'} + \tfrac{1}{2}d_{n'+1} - 1.\end{aligned}$$

Since $d_{n'} + \frac{1}{2}d_{n'+1}$ converges to $\frac{3}{4}$, we can choose a stricter upper bound to solve the problem we had earlier. We have

$$\begin{aligned}d_4 + \tfrac{1}{2}d_5 - 1 &= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{3}{5} - 1 = -\tfrac{1}{5} \leq -\tfrac{1}{10}, \\ d_5 + \tfrac{1}{2}d_6 - 1 &= \tfrac{3}{5} + \tfrac{1}{2}\cdot\tfrac{1}{2} - 1 = -\tfrac{3}{20} \leq -\tfrac{1}{10}.\end{aligned}$$

As a result, we have

$$\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1') \leq -\tfrac{1}{10},$$

for $n' \geq 4$. For $\ell \equiv m \bmod 2$, we run into the same problem:

$$\begin{aligned}\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1') &= v_{n'+1} - v_{n'-1} - 1 \\ &= d_{n'} + d_{n'+1} - 1.\end{aligned}$$

Now, $d_4 + d_5 - 1 = \frac{1}{2} + \frac{3}{5} - 1 = \frac{1}{10}$, hence

$$\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1') \leq \tfrac{1}{10},$$

for $n' \geq 4$.

We now have multiple inequalities, all of which hold for $n' \geq 4$, where $n' = n+X(m-1)$. In the worst case, only half of the $m-1$ copies of $H_{1,0}$ we began with become copies of $H_{0,0}$ and we land on 1, losing a copy. In other words, $X(m-1) = \left\lceil \frac{m-1}{2} \right\rceil - 1$. So, at worst we need to have at least

$$\begin{aligned}n + \left\lceil \tfrac{m-1}{2} \right\rceil - 1 &\geq 4 \\ n &\geq 5 - \left\lceil \tfrac{m-1}{2} \right\rceil.\end{aligned}$$

Thus, we set

$$N(m) = \max\left\{1, 5 - \left\lceil \tfrac{m-1}{2} \right\rceil\right\}.$$

Now to compute the relevant probabilities. Since the Z_k are independent, we have, for $2 \leq k \leq m-1$,

$$\begin{aligned}\mathbb{P}(\ell = k) &= \mathbb{P}(Z_k = 1)\prod_{j=k+1}^{m-1}\mathbb{P}(Z_j = 0) \\ &= \tfrac{1}{k}\cdot\left(\tfrac{k}{k+1}\cdot\tfrac{k+1}{k+2}\cdot\ldots\cdot\tfrac{m-2}{m-1}\right) \\ &= \tfrac{1}{m-1}.\end{aligned}$$

For $\ell = 0$, we have

$$\begin{aligned}\mathbb{P}(\ell = 0) &= 1 - \mathbb{P}(\ell \neq 0) \\ &= 1 - \sum_{k=2}^{m-1} \mathbb{P}(\ell = k) \\ &= 1 - \tfrac{m-2}{m-1} \\ &= \tfrac{1}{m-1}.\end{aligned}$$

Furthermore, for $\ell \geq 2$, we have

$$\mathbb{P}(\ell \geq 2, \ell \equiv m-1 \bmod 2) = \begin{cases} \frac{m-2}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ even,} \\ \frac{m-1}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ odd,} \end{cases}$$

$$\mathbb{P}(\ell \geq 2, \ell \equiv m \bmod 2) = \begin{cases} \frac{m-2}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ even,} \\ \left(\frac{m-1}{2} - 1\right) \cdot \frac{1}{m-1}, & \text{if } m \text{ odd.} \end{cases}$$

Putting this all together, for m even, we have

$$\begin{aligned}\mathbb{E}_{Z_2,\ldots,Z_{m-1}}\left[\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1')\right] &\leq \mathbb{P}(\ell = 0) \cdot 0 \\ &+ \mathbb{P}(\ell \geq 2, \ell \equiv m-1 \bmod 2) \cdot -\tfrac{1}{10} \\ &+ \mathbb{P}(\ell \geq 2, \ell \equiv m \bmod 2) \cdot \tfrac{1}{10} \\ &= -\tfrac{m-2}{2} \cdot \tfrac{1}{m-1} \cdot \tfrac{1}{10} + \tfrac{m-2}{2} \cdot \tfrac{1}{m-1} \cdot \tfrac{1}{10} \\ &= 0.\end{aligned}$$

Similarly, for m odd we have

$$\begin{aligned}\mathbb{E}_{Z_2,\ldots,Z_{m-1}}\left[\tfrac{1}{m}v(G_3') + \tfrac{m-1}{m}v(G_2') - v(G_1')\right] &\leq \mathbb{P}(\ell = 0) \cdot \tfrac{1}{10} \\ &+ \mathbb{P}(\ell \geq 2, \ell \equiv m-1 \bmod 2) \cdot -\tfrac{1}{10} \\ &+ \mathbb{P}(\ell \geq 2, \ell \equiv m \bmod 2) \cdot \tfrac{1}{10} \\ &= \tfrac{1}{m-1} \cdot \tfrac{1}{10} - \tfrac{m-1}{2} \cdot \tfrac{1}{m-1} \cdot \tfrac{1}{10} \\ &+ \left(\tfrac{m-1}{2} - 1\right) \cdot \tfrac{1}{m-1} \cdot \tfrac{1}{10} \\ &= 0.\end{aligned}$$

Both hold for all $n \geq N(m)$, completing the proof. □

Remark 5.11. In the proof of Theorem 5.10 we found

$$N(m) = \max\left\{1, 5 - \left\lceil \tfrac{m-1}{2} \right\rceil\right\}.$$

which is non-increasing in m. This was a conservative estimate and as we saw in Table 1, for lower values of m we have overestimated the true value of $N(m)$. However, this proof does show that $N(m) = 1$ for all $m \geq 8$, which will be important later.

Theorem 5.9 and Theorem 5.10 tell us exactly how a game starting with m copies of $H_{1,0}$ plays out, provided that m and n satisfy $n \geq N(m)$. Let us first assume that this holds for all m, n to build a two-dimensional recurrence relation for $v_{m,n}$ and for the difference $d_{m,n} = v_{m,n} - v_{m-1,n}$. Then, we will argue for which values of m, n our recurrence relations follow the true Nash value of $mH_{1,0} + nH_{0,0}$.

Assuming Theorem 5.9 and Theorem 5.10 hold for all m, n, we know that moving on any copy of $H_{1,0}$ uniformly at random for both players is a Nash equilibrium, if $m \geq 2$. If $m = 1$, then Left moving on $H_{1,0}$ and Right moving on a copy of $H_{0,0}$ is a Nash equilibrium. This gives us the following two-dimensional recurrence relation for $v_{n,m}$:

$$\begin{cases} v_{m,n} = \frac{1}{m} v_{m-1,n+1} + \frac{m-1}{m} v_{m-2,n+1}, & m \geq 2 \\ v_{1,n} = v_{n-1} + 1, \\ v_{0,n} = v_n. \end{cases}$$

Similar to the recurrence relation of v_n, this too is difficult to analyze directly. We once again turn to the difference $d_{m,n} = v_{m,n} - v_{m-1,n}$. Using the above recurrence of $v_{m,n}$, we find, for $m \geq 3$:

$$\begin{aligned} d_{m,n} &= \left(\tfrac{1}{m} v_{m-1,n+1} + \tfrac{m-1}{m} v_{m-2,n+1}\right) - \left(\tfrac{1}{m-1} v_{m-2,n+1} + \tfrac{m-2}{m-1} v_{m-3,n+1}\right) \\ &= \tfrac{1}{m} v_{m-1,n+1} + \tfrac{(m-1)^2 - m}{m(m-1)} v_{m-2,n+1} - \tfrac{m-2}{m-1} v_{m-3,n+1} \\ &= \tfrac{1}{m} v_{m-1,n+1} - \tfrac{1}{m} v_{m-2,n+1} + \tfrac{m-2}{m-1} v_{m-2,n+1} - \tfrac{m-2}{m-1} v_{m-3,n+1} \\ &= \tfrac{1}{m} d_{m-1,n+1} + \tfrac{m-2}{m-1} d_{m-2,n+1}. \end{aligned}$$

For $m = 2$ we have

$$\begin{aligned} d_{2,n} &= v_{2,n} - v_{1,n} \\ &= \tfrac{1}{2} v_{1,n+1} - \tfrac{1}{2} v_{0,n+1} - v_{n-1} - 1 \\ &= \tfrac{1}{2} v_n + \tfrac{1}{2} + \tfrac{1}{2} v_{n+1} - v_{n-1} - 1 \\ &= \tfrac{1}{2} v_{n+1} - \tfrac{1}{2} v_n + v_n - v_{n-1} - \tfrac{1}{2} \\ &= \tfrac{1}{2} d_{n+1} + d_n - \tfrac{1}{2}, \end{aligned}$$

and lastly,

$$\begin{aligned} d_{1,n} &= v_{1,n} - v_{0,n} \\ &= v_{n-1} + 1 - v_n \\ &= 1 - d_n. \end{aligned}$$

This gives us the following two-dimensional recurrence relation for $d_{m,n}$:

$$\begin{cases} d_{m,n} = \frac{1}{m} d_{m-1,n+1} + \frac{m-2}{m-1} d_{m-2,n+1}, & m \geq 3 \\ d_{2,n} = \frac{1}{2} d_{n+1} + d_n - \frac{1}{2}, \\ d_{1,n} = 1 - d_n. \end{cases}$$

By Theorem 5.9, we know that $v_{1,n} = v_{n-1} + 1$ holds for $n \geq 4$. Hence, $d_{1,n} = 1 - d_n$ also holds for $n \geq 4$. For $m \geq 2$, we see that $v_{m,n}$ is a function of $v_{m-1,n+1}$ and

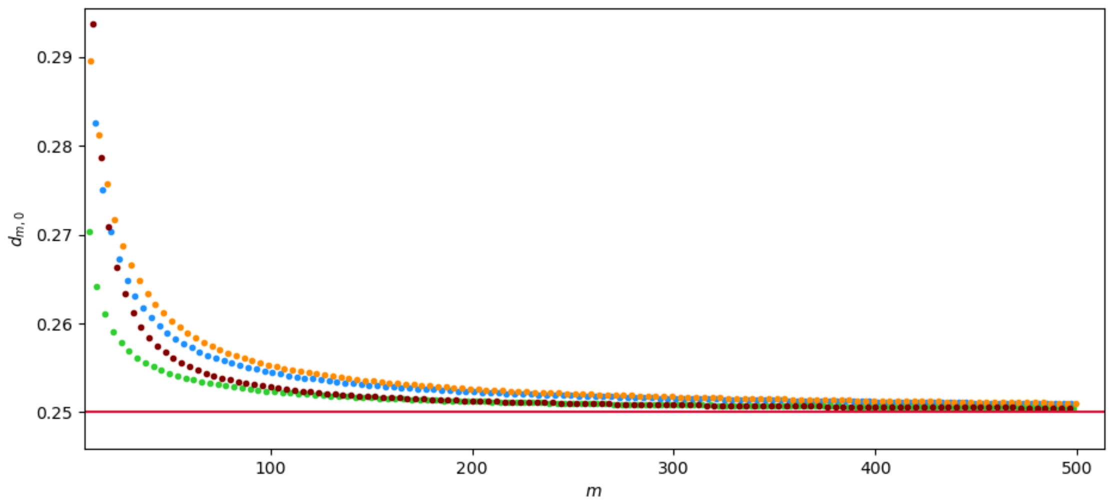


Figure 2: $d_{m,0}$ for $m \in \{10, \ldots, 500\}$. The line $y = \frac{1}{4}$ is drawn in red. Points are given a color based on the value of their index modulo 4.

$v_{m-2,n+1}$. Likewise, $d_{m,n}$ is a function of $d_{m-1,n+1}$ and $d_{m-2,n+1}$. The function $N(m)$ we found in the proof of Theorem 5.10 has $N(2) = 4$ and decreases by 1 if m increases by 2. We can work out that, for $m \geq 10$, $v_{m,0}$ depends only on games $v_{m',n'}$ such that $n' \geq N(m')$. As a result, for $m \geq 10$, we find that $v_{m,0}$ and $d_{m,0}$ coincide with the true values for $mH_{1,0}$. Finding an analytical solution to this recurrence relation is difficult. Calculating the values directly, we see that $d_{m,0}$ appears to converge to $\frac{1}{4}$ (Figure 2), which is the combinatorial value of $H_{1,0}$. We can prove that this is indeed the case.

First, we will point out another interesting phenomenon we see in Figure 2, which is that the points appear to converge to $\frac{1}{4}$ with four distinct subsequences. In Section 5.1, we saw the same thing happen with d_n with two subsequences $(d_{2n})_{n\in\mathbb{Z}_{\geq 0}}$ and $(d_{2n})_{n\in\mathbb{Z}_{\geq 0}}$, and one of them happened to be a constant sequence. Here, we have the four subsequences $(d_{km,0})_{m\in\mathbb{Z}_{\geq 0}}$, $k = 1, 2, 3, 4$. It is not immediately clear from the recurrence relation of $d_{m,n}$ why this pattern emerges.

Now, let us provide a proof that $d_{m,n}$ does indeed converge to $\frac{1}{4}$ as m tends to infinity.

Theorem 5.12. $\lim_{m\to\infty} v(mH_{1,0} - (m-1)H_{1,0}) = \frac{1}{4}$.

Proof. For $n \in \mathbb{Z}_{\geq 0}$, we define the stochastic process $X(n)$ by $X(0) = 0$, $X(1) = 1$ with probability 1 and

$$X(n) = \begin{cases} 1 + X(n-1), & \text{w.p. } \frac{1}{n}, \\ 1 + X(n-2), & \text{w.p. } \frac{n-1}{n}. \end{cases}$$

If we start with n copies of $H_{0,0}$, then $X(n)$ models the number of copies of [ ][P] + [P] = 1 that are generated. Hence, $\mathbb{E}X(n) = v(nH_{0,0})$ and, by Theorem 5.3, we have $\mathbb{E}[X(n) - \tilde{X}(n-1)] \to \frac{1}{2}$ for $n \to \infty$ for any two copies $X, \tilde{X}$ of the process. Likewise, we define the process $Y(m)$ by $Y(0) = 0$, $Y(1) = -1$ with probability 1 and

$$Y(m) = \begin{cases} 1 + Y(m-1), & \text{w.p. } \frac{1}{m}, \\ 1 + Y(m-2), & \text{w.p. } \frac{m-1}{m}, \end{cases}$$

By Theorem 5.9 and Theorem 5.10, for $m \geq 10$, $Y(m)$ models the number of copies of $H_{0,0}$ that are generated when we start with m copies of $H_{1,0}$. If we define the indicator random variable $W(m) = \mathbb{1}_{\{Y(m) \text{ lands on } 1\}}$, then we have $\mathbb{E}[X(Y(m)) + W(m)] = v(mH_{1,0})$.

Now, suppose that we have two copies of the process X, say X and $\tilde{X}$, and two copies of the process Y, say Y and $\tilde{Y}$, with their respective W and $\tilde{W}$. For $k = 2, \ldots, m$, we define the random variables Z_k by

$$Z_k = \begin{cases} 1, & \text{w.p. } \frac{1}{k}, \\ 0, & \text{w.p. } \frac{k-1}{k}. \end{cases}$$

We couple X and $\tilde{X}$ for all $k = 2, \ldots, m$ by setting

$$X(k) = 1 + Z_k X(k-1) + (1 - Z_k) X(k-2),$$

and

$$\tilde{X}(k) = 1 + Z_k \tilde{X}(k-1) + (1 - Z_k) \tilde{X}(k-2).$$

We couple Y and $\tilde{Y}$ for all $k = 2, \ldots, m$ analogously. By coupling the stochastic processes, we have $X(k) = \tilde{X}(k), Y(k) = \tilde{Y}(k)$ for all $k = 2, \ldots, m$. Once again, let

$$\ell = \begin{cases} \max\{2 \leq k \leq m-1 : Z_k = 1\}, & \text{if } Z_k = 1 \text{ for some } 2 \leq k \leq m-1, \\ 0, & \text{otherwise.} \end{cases}$$

We know that $\mathbb{P}(\ell = 0) = \frac{1}{m} \to 0$ as $m \to \infty$ (proof Theorem 5.10), so we only have to consider the case of $\ell \neq 0$. For $k = 2, \ldots, m$, if $k \equiv \ell \bmod 2$, then

$$X(k) = 1 + X(k-2) = 2 + X(k-4) = \cdots = \tfrac{k-\ell}{2} + X(\ell) = \tfrac{k-\ell}{2} + 1 + X(\ell-1),$$

and

$$\tilde{X}(k-1) = 1 + \tilde{X}(k-3) = 2 + \tilde{X}(k-5) = \cdots = \tfrac{k-\ell}{2} + X(\ell-1).$$

Similarly, if $k \equiv \ell - 1 \bmod 2$, then

$$X(k) = \tfrac{k-\ell-1}{2} + X(\ell+1) = \tfrac{k-\ell-1}{2} + 1 + X(\ell-1),$$

and

$$\tilde{X}(k-1) = \tfrac{k-\ell-1}{2} + X(\ell) = \tfrac{k-\ell-1}{2} + 1 + X(\ell-1),$$

In both cases, we find $X(k) - \tilde{X}(k-1) \in \{0, 1\}$. It follows that

$$\mathbb{P}(X(n) - \tilde{X}(n-1) = 1) = \mathbb{E}[X(n) - \tilde{X}(n-1)] \to \frac{1}{2}.$$

These arguments also hold for Y and $\tilde{Y}$. Moreover, we have $\mathbb{E}[W(m)-\tilde{W}(m-1)] \to 0$ as $m \to \infty$, since $\mathbb{P}(W(m) - \tilde{W}(m-1) \neq 0) = \mathbb{P}(\ell = 0) \to 0$. Now, we compute

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{E}[X(Y(m)) + W(m) - \tilde{X}(\tilde{Y}(m-1)) - \tilde{W}(m-1)] \\
&= \lim_{m\to\infty} \mathbb{E}[X(Y(m)) - \tilde{X}(\tilde{Y}(m-1))] \\
&= \lim_{m\to\infty} \Big(\mathbb{E}[X(Y(m)) - \tilde{X}(\tilde{Y}(m-1)) \mid Y(m) - \tilde{Y}(m-1) = 1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{P}(Y(m) - \tilde{Y}(m-1)) = 1) \\
&\qquad + \mathbb{E}[X(Y(m)) - \tilde{X}(\tilde{Y}(m-1)) \mid Y(m) - \tilde{Y}(m-1) = 0] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{P}(Y(m) - \tilde{Y}(m-1)) = 0) \Big) \\
&= \tfrac{1}{2} \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{2} \\
&= \tfrac{1}{4}.
\end{aligned}$$

Note that $\lim_{m\to\infty} \mathbb{E}[X(Y(m)) - \tilde{X}(\tilde{Y}(m-1)) \mid Y(m) - \tilde{Y}(m-1) = 1] = \frac{1}{2}$ only holds if $Y(m), \tilde{Y}(m-1) \to \infty$ as $m \to \infty$, but this is true by definition. □

5.3 Eighths

We have now seen that for two Push games with combinatorial value $\frac{1}{2}$, when we have n copies of such a game G, then the difference in Nash value $v(nG)-v((n-1)G)$ converges to $\frac{1}{2}$. We have seen the same happen with a Push game with combinatorial value $\frac{1}{4}$. We arrive at the following conjecture.

Conjecture 5.13. *Let G be a separable combinatorial game. Then,*

$$\lim_{n\to\infty} \big(v(nG) - v((n-1)G)\big) = \operatorname{can}(G).$$

In [10], this has been shown to hold true for various Hackenbush positions, including Hackenbush "flowers", which are positions with combinatorial value $\frac{1}{2^n}$ for $n \geq 1$. The proof for Hackenbush flowers is analogous to the proof of Theorem 5.12. With Hackenbush flowers, it is a Nash equilibrium for both players to move on the flowers that are the highest power of $\frac{1}{2}$ in combinatorial value, similar to how, with Push, both players first move on $H_{1,0}$ until one or none remain(s). With Hackenbush flowers, however, a Right solo move on a flower $\frac{1}{2^n}$ turns it into a flower $\frac{1}{2^{n-1}}$ and a Left solo move or a synchronized move removes the flower entirely. As a result, a game consisting of any number of Hackenbush flowers will always remain a game of only flowers, which lends itself well to an induction argument. In contrast, we saw that a game consisting of copies of the Push position $H_{1,0}$ eventually contains copies of the position $H_{0,1}$, which has combinatorial value 0. In the case of $H_{0,1}$, we showed that this game can be ignored (Lemma 5.6 and Lemma 5.7). However, it turns out that when we examine Push games of higher powers of $\frac{1}{2}$, then these types of non-fractional games will result in entirely different Nash equilibria from what we have seen before, preventing us from using a similar induction argument.

Consider the Push game

$$H_{2,0} = 2 + \boxed{}\boxed{}\boxed{P}\boxed{P},$$

which has combinatorial value $\frac{1}{8}$. In matrix form,

$$H_{2,0} = \left(\begin{array}{c|c} & H_{1,0} \\ \hline H_{1,1} & H_{1,0} \end{array}\right).$$

When we compute the Nash value of n copies of $H_{2,0}$, it does appear as if Conjecture 5.13 holds true in this case as well (Figure 3). We also see the same subsequence pattern as with $H_{0,0}$ and $H_{1,0}$. However, we are unable to provide a proof that

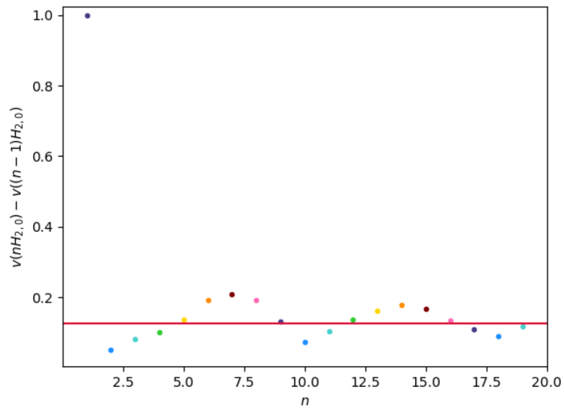


Figure 3: Plot of $v(nH_{2,0}) - v((n-1)H_{2,0})$ for $n \in \{1, \ldots, 19\}$. The line $y = \frac{1}{8}$ is drawn in red. Points are given a color based on the value of their index modulo 8.

$\lim_{n\to\infty}(v(nH_{2,0}) - v((n-1)H_{2,0})) = \frac{1}{8}$. We will discuss some of our findings.

Consider a game consisting of some number of copies of $H_{2,0}$. As usual, both players will choose to move on any of the copies uniformly at random, leading to a game of the form

$$mH_{2,0} + nH_{1,0} + kH_{1,1}.$$

Similar to what we saw with $\frac{1}{4}$, both players will continue to move on a copy of $H_{2,0}$, as long as $m \geq 2$. If $m = 1$, then Left will move on $H_{2,0}$ and Right will move on a copy of $H_{1,0}$. When all copies of $H_{2,0}$ are "gone", our game only consists of copies of $H_{1,0}$, $H_{1,1}$ and possibly a copy of $H_{0,0}$. We have $H_{1,1} \in \mathcal{D}$; however, $H_{1,1}$ *does* have an impact on the Nash value and Nash equilibrium.

Example 5.14. Consider the Push game

$$G_1 = H_{0,0} + H_{1,1}.$$

The synchronized options of G_1 are

$$\mathcal{G}_1^S = \begin{pmatrix} 1 + H_{1,1} & 0 + H_{1,0} \\ 1 + H_{0,2} & H_{0,0} + H_{0,1} \end{pmatrix},$$

where the first row and column correspond to moving on $H_{0,0}$ and the second row and column correspond to moving on $H_{1,1}$. Since $v(H_{0,0} + H_{0,1}) = v(H_{0,0})$ (Lemma 5.6), we find the following zero-sum game associated to G_1:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We see that moving on $H_{0,0}$ dominates moving on $H_{1,1}$ for Left, hence $v(G_1) = 1$. Now, consider the game

$$G_2 = 2H_{0,0} + H_{1,1}.$$

G_2 has the following table of synchronized options:

	$H_{0,0}$	$H_{1,1}$
$H_{0,0}$	Same copy: $1 + H_{0,0} + H_{1,1}$ Different copy: $0 + 1 + H_{1,1}$	$0 + H_{0,0} + H_{1,0}$
$H_{1,1}$	$1 + H_{0,0} + H_{0,2}$	$2H_{0,0} + H_{0,1}$

The game $H_{0,0} + H_{0,2}$ is a draw: Both players move on $H_{0,0}$, resulting in the game $1 + H_{0,2} \in \mathcal{D}$. This gives us the zero-sum game

$$\begin{pmatrix} \frac{3}{2} & \frac{4}{3} \\ 1 & \frac{3}{2} \end{pmatrix}.$$

Here, we find that $\left(\left(\frac{1}{4}, \frac{3}{4}\right)^\top, \left(\frac{3}{4}, \frac{1}{4}\right)^\top \right)$ is a Nash equilibrium, and

$$v(G_2) = \tfrac{11}{8} \neq \tfrac{3}{2} = v(2H_{0,0}).$$

$\triangleleft$

Going back to our game consisting of copies of $H_{1,0}$ and $H_{1,1}$, we find that $H_{1,1}$ gets ignored by both players. They will play the same strategy as with a game only consisting of copies of $H_{1,0}$, i.e., they play on copies of $H_{1,0}$ until one or none remain(s). The resulting game is of the form

$$mH_{1,0} + nH_{0,0} + kH_{0,1} + \ell H_{1,1},$$

with $m \in \{0, 1\}$. Here, we diverge from the strategies presented in Theorem 5.9 and Theorem 5.10. If $m = 1$, then a Nash equilibrium is for both players to play some mixed strategy which moves on $H_{1,0}$ and $H_{0,0}$ with positive probability, where Left's probability to move on $H_{1,0}$ seems to approach 1 as the total number of games increases. If $m = 0$, then Left moving on $H_{0,0}$ with probability 1 and Right moving on $H_{1,0}$ with probability 1 is a Nash equilibrium.

Assuming that all of theses strategies hold, a game n copies of $H_{2,0}$ would play out roughly as follows. As $n \to \infty$, on average, half of the copies of $H_{2,0}$ will be be moved to $H_{1,0}$, while the other half will be moved to $H_{1,1}$. Then, of the games $H_{1,0}$, on average, half will become $H_{0,0}$ and the other half will become $H_{0,1}$. Now, we will alternate between the following strategies. Right moves a copy of $H_{1,1}$ to $H_{1,0}$ and

Left moves a copy of $H_{0,0}$ to 0. Then, Right moves a copy of $H_{0,0}$ to 1 and Left moves the only copy of $H_{1,0}$ to $H_{0,1}$. This last strategy is played with probability 1 as $n \to \infty$, but becomes increasingly mixed as more games get played to 0 or 1. The ratio between copies of $H_{1,1}$ and copies of $H_{0,0}$ was approximately $2:1$ and we remove a copy of $H_{0,0}$ (almost) every turn, while copies of $H_{1,1}$ are only removed once every two turns. So, we eventually run out of copies of $H_{0,0}$ and are left with a game consisting only of copies of $H_{1,1}$, $H_{0,1}$ and terminal positions 0 and 1, which appear approximately in equal proportions. A mixture of $H_{1,1}$ and $H_{0,1}$ is a draw, so the Nash value is determined by the expected number of the game 1 that are left. As $n \to \infty$, this number would converge to half of the copies of $H_{0,0}$ that were created, which is one eighth of the number of copies of $H_{2,0}$ that we started with, which explains why we see that $\lim_{n\to\infty}(v(nH_{2,0}) - v((n-1)H_{2,0})) = \frac{1}{8}$.

5.4 Negatives

In regular combinatorial game theory, we can show that $G - G = 0$ for any short game G, using a concept known as *mirroring*. For any move that the starting player makes, the second player can respond by making the same move but on the negative side (or vice versa), ensuring that (s)he always has a responding move and will win the game. With Nash synchronization, we find something similar; it turns out that if a game G is equal to 0 in combinatorial sense, then it must have Nash value 0. Moreover, both players will play the same strategy.

Theorem 5.15. *Let G be a synchronized game and assume that $G = -G$. Then, $v(G) = 0$ and G will have a Nash equilibrium of the form (x^*, x^*) for some $x \in \Delta^{|\mathcal{G}^L|}$.*

Proof. If $G = -G$, then $\mathcal{G}^S = -(\mathcal{G}^S)^\top$ and $\mathcal{G}^L = -\mathcal{G}^R$. Let $n = |\mathcal{G}^L| = |\mathcal{G}^R|$. Since $\mathcal{G}^S$ is skew-symmetric, viewing G as a zero-sum games gives us $x^\top Gy = -y^\top Gx$ for all $x, y \in \Delta^n$. By symmetry, (x^*, y^*) is a Nash equilibrium if and only if (y^*, x^*) is a Nash equilibrium. It follows that the value of G must be 0. Moreover, $x^\top Gx = 0$. □

We will now consider synchronized Push games of the form $mH_{0,0} - nH_{0,0}$. We write $s_{m,n} = v(mH_{0,0} - nH_{0,0})$. In Table 2, we see that $s_{m,m} = 0$, as expected by Theorem 5.15. We also see that $s_{m,n} = -s_{n,m}$, which follows from symmetry. For $s_{m,n}$, we can show a few simple inequalities.

Lemma 5.16. *Consider $s_{m,n} = v(mH_{0,0} - nH_{0,0})$, $m, n \in \mathbb{Z}_{\geq 0}$. Then,*

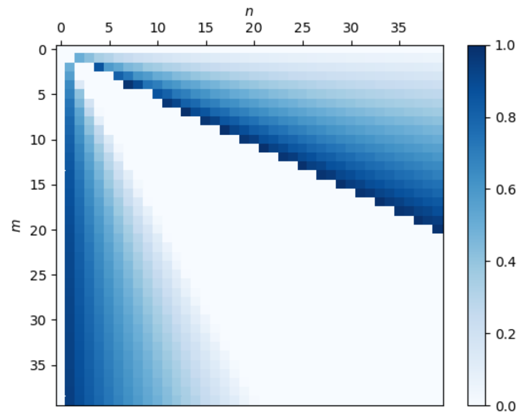
(i) $s_{m,n} \leq s_{m+1,n} \leq s_{m,n} + 1$,

(ii) $s_{m,n} - 1 \leq s_{m,n+1} \leq s_{m,n}$.

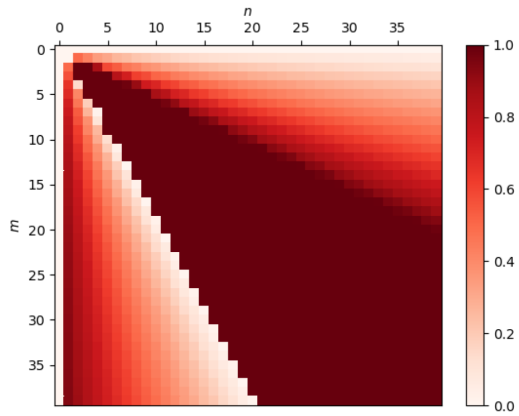
Proof. From $G_1 = (m+1)H_{0,0} - nH_{0,0}$, Left can move to $G_1^L = mH_{0,0} - nH_{0,0}$, while Right can move to $G_1^R = 1 + mH_{0,0} - nH_{0,0}$. From $G_2 = mH_{0,0} - nH_{0,0}$, Left can move to $G_2^L = -1 + mH_{0,0} - nH_{0,0}$, while Right can move to $G_2^R = mH_{0,0} - nH_{0,0}$. The inequalities follow from Theorem 4.17. □

$m \setminus n$	0	1	2	3	4	5	6	7
0	0	-1	-1.5	-2.1667	-2.6667	-3.2667	-3.7667	-4.3381
1	1	0	-0.75	-1.3571	-1.9935	-2.5415	-3.1333	-3.6608
2	1.5	0.75	0	-0.75	-1.3338	-1.9411	-2.4923	-3.0719
3	2.1667	1.3571	0.75	0	-0.75	-1.3338	-1.9236	-2.4721
4	2.6667	1.9935	1.3338	0.75	0	-0.75	-1.3338	-1.9220
5	3.2667	2.5415	1.9411	1.3338	0.75	0	-0.75	-1.3338
6	3.7667	3.1333	2.4923	1.9236	1.3338	0.75	0	-0.75
7	4.3381	3.6608	3.0719	2.4721	1.9220	1.3338	0.75	0

Table 2: The Nash value $s_{m,n} = v(mH_{0,0} - nH_{0,0})$ for different values of m, n. Values have been rounded to four decimal places.



(a)



(b)

Figure 4: We compute the Nash equilibrium of the game $mH_{0,0} - nH_{0,0}$. Figure 4a resp. Figure 4b shows the Nash probability to play on a copy of $+H_{0,0}$ for Left resp. Right.

The data suggests that, for all $n \in \mathbb{Z}_{\geq 0}$, we have $\lim_{m\to\infty}(s_{m,n} - s_{m-1,n}) = \frac{1}{2}$. Likewise, for all $m \in \mathbb{Z}_{\geq 0}$, we have $\lim_{n\to\infty}(s_{m,n} - s_{m,n-1}) = -\frac{1}{2}$. This is all in agreement with Conjecture 5.13. We also spot the following pattern.

Conjecture 5.17. *Let $s_{m,n} = v(mH_{0,0} - nH_{0,0})$. Then,*

(i) For all $m \geq 1$, we have $s_{m+1,m} = \frac{3}{4}$.

(ii) For all k, the limit $\lim_{m\to\infty} s_{m+k,m}$ exists.

If we consider the Nash equilibrium of each game, we find that there exist three cases, which we have visualized in Figure 4. If $m \geq 2n$, then the Nash equilibrium consists of the same strategy for Left and Right, and the probability to move on a copy of $+H_{0,0}$ seems to converge to 1 as $m \to \infty$. If $n \geq 2m$, by symmetry, we find the opposite, i.e., the probability to move on a copy of $-H_{0,0}$ seems to converge to 1 as $m \to \infty$, and both players play the same strategy. Between these two lines, Left moves on $-H_{0,0}$ with probability 1 and Right moves on $+H_{0,0}$ with probability 1.

5.5 Synchronized Shove

In this section, we will briefly discuss how synchronized versions of powers of $\frac{1}{2}$ behave in Shove. For $n \in \mathbb{Z}_{\geq 0}$, let us define the Shove position

$$I_j = \boxed{}\boxed{S} + \underbrace{\boxed{S}\boxed{S}\dots\boxed{S}\boxed{S}}_{j+1}.$$

From Section 3.2, we know that $I_j = \frac{1}{2^{j+1}}$ as combinatorial games. For $j = 0$, we have $\boxed{}\boxed{S} + \boxed{S}\boxed{S} \cong \boxed{}\boxed{P} + \boxed{P}\boxed{P}$. Consequently, for the synchronized version of I_0, for all $n \in \mathbb{Z}_{\geq 0}$ we have

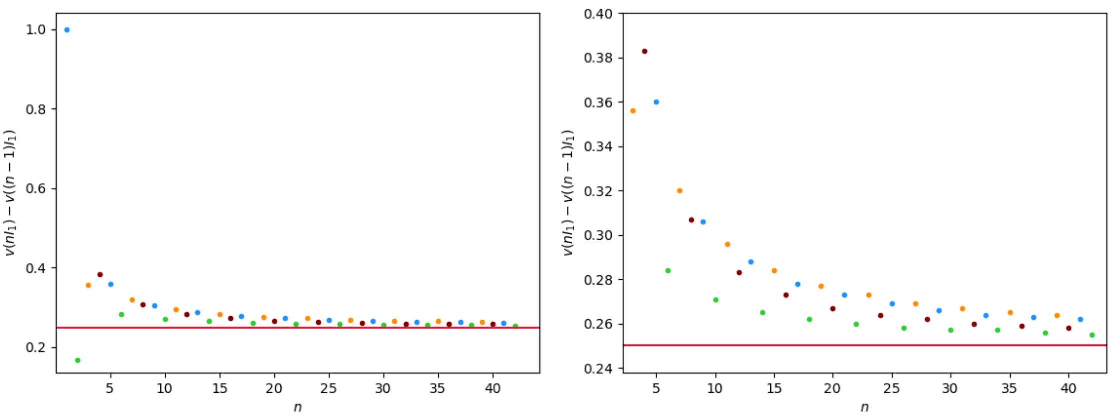
$$v(nI_0) = v(nH_{0,0}).$$

It follows that $\lim_{n\to\infty} v(nI_0) - v((n-1)I_0) = \lim_{n\to\infty} v(nH_{0,0}) - v((n-1)H_{0,0}) = \frac{1}{2}$. Moreover, $mI_0 - nI_0 \cong mH_{0,0} - nH_{0,0}$, hence we also have $v(mI_0 - nI_0) = v(mH_{0,0} - nH_{0,0})$, which we discussed in Section 5.4.

For $j > 1$, things get more interesting. Consider

$$I_1 = \boxed{}\boxed{S} + \boxed{S}\boxed{S}\boxed{S},$$

which has combinatorial value $\frac{1}{4}$. In Figure 5, we have plotted $v(nI_1) - v((n-1)I_1)$ for $n \in \{1, \dots, 42\}$ and we see that it converges to $\frac{1}{4}$, as expected. Similar to the plot of $d_{m,0} = v(mH_{1,0}) - v((m-1)H_{1,0})$, we see that there appear to be four distinct converging subsequences. In the Shove position I_j, Left has many more options than



(a) (b)

Figure 5: Figure 5a shows $v(nI_1) - v((n-1)I_1)$ for $n \in \{1, \dots, 42\}$. In Figure 5b, we have zoomed in and cut off the first two points. The line $y = \frac{1}{4}$ is drawn in red. Points are given a color based on the value of their index modulo 4.

in the corresponding Push position $H_{j,0}$. In the case of I_1, we found that Left would always move its leftmost piece. For the games that we encounter when we compute the value of nI_1, we find different Nash equilibria than for the corresponding games from $nH_{1,0}$. We find that most Nash equilibria consist of mixed strategies. For I_2 we also find similar results to what we found with $H_{2,0}$. See Figure 6. Note that, to cut down on computation time, we assume that Left will always want to move its leftmost piece, since any moved piece will move all pieces to the left of it as well. We observed that this was the case for I_1, but we do not have a proof that it holds for I_j, $j \geq 2$. Interestingly, we see that $v(nI_2) - v((n-1)I_2)$ converges to $\frac{1}{8}$ from above, whereas for Shove it oscillates around $\frac{1}{8}$. This could be due to Left having two more pieces in I_2 than she has in $H_{2,0}$.

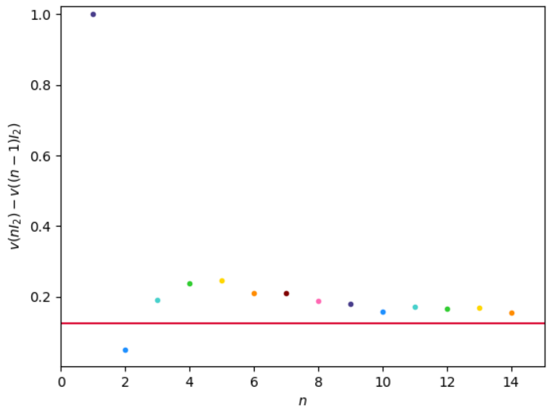


Figure 6: Plot of $v(nI_2) - v((n-1)I_2)$ for $n \in \{1, \dots, 14\}$. The line $y = \frac{1}{8}$ is drawn in red. Points are given a color based on the value of their index modulo 8.

6 Summary and future work

We have seen the definition of a separable synchronized combinatorial game, and we showed that any such game must be a number (as a combinatorial game). Equivalently, any combinatorial game that is not a number (which is the vast majority of combinatorial games) must have a synchronized version that is not separable. This is rather disappointing, as separable games have a natural way of defining its synchronized version, although this definition may not lead to a unique synchronized version. For non-separable games, we can still define a synchronized version; however, such a definition will depend on the rule set of the game.

For synchronized games, its synchronized options form a matrix of options, where one player chooses the row and the other player chooses the column. This means that synchronized games can be thought of as two-player zero-sum matrix games if we can assign some sort of value to each option. This gives rise to the Nash value, which is defined recursively by assigning decided games an integer value and assigning non-decided games the value equal to the value of their zero-sum game.

With regards to the Nash value, we formulate two conjectures, which we have only been able to prove in a few specific cases. First, assuming a non-terminal game, we conjecture that a move on a terminal component of a game is dominated by a move on the non-terminal component, and the Nash value of the game is the sum of the Nash values of its non-terminal component and its terminal component. We have been able to show that this is true for synchronized versions of any Push or Shove game. The proof hinges on the fact that, in these rule sets, we can explicitly point to a move which dominates the move on the terminal component. Showing that this is true for arbitrary (separable) synchronized games is still an open problem. In particular, we know that a move on a terminal components "costs" exactly 1 move (hence, also increases resp. decreases the Nash value by 1) and we need to find a move on the non-terminal component that costs at most 1 move. For regular combinatorial games, a similar result using incentives is stated in [9, p. 80].

Next, we found a few instances of separable games for which we see that the difference in the Nash value of n copies and the Nash value of $n-1$ copies of said game converges to the synchronised value of the game as n tends to infinity. We show that this is true for a few Push and Shove positions, as well as show data which suggests this holds for even more Push and Shove positions. [10] shows the same for several Hackenbush games. Since any separable game is a number, the next step would be to find a proof that this holds for any separable game. Future work may also study non-separable synchronized games. As we have discussed in Section 4, non-separable games require us to explicitly define the synchronized move in some cases. For example, with Domineering, we have to decide what happens when two dominoes are placed on the same space. One solution is to allow two dominoes that are placed in the same turn to overlap. Another solution would be to let both players repeat their move until no pieces occupy the same space, and to declare the game a draw if the Nash equilibrium has both players move on the same space with probability 1. As non-separable games are not numbers, it would be interesting to see if a analogous statement to Conjecture 5.13 can be formulated.

Lastly, we found that the difference in Nash value shows a pattern based on combinatorial value of the game, which could also be explored further. In the case of Push and Shove positions with combinatorial value $\frac{1}{2}$, assuming we have n copies of the game H, we know that the probability to play on the same copy is $\frac{1}{n}$. For the Nash value, this means that $v(nH) \approx 1 + v((n-2)H)$ as $n \to \infty$. Consequently, $d_n \approx d_{n-2}$, which explains the modulo 2 pattern. For $\frac{1}{4}$, we also have that the probability to play on the same copy of $H_{1,0}$ decreases. Here, for every two copies of $H_{1,0}$ that gets removed, we gain a copy of $H_{0,0}$ and a copy of 1. This gives us a modulo 2 pattern embedded in a modulo 2 pattern, which should result in a modulo 4 pattern. However, this explanation is very vague and requires further examination.

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