# Subproduct systems from SU(2)-representations

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# Subproduct systems from $SU(2)$-representations

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## Abstract

This thesis is motivated by the study of multivariate operator theory as well as the study of $SU(2)$-symmetries of $C^*$-algebras. We partially extend the results in [1]. In particular, we extend the fusion rules of the $SU(2)$-equivariant subproduct system from irreducible representations to the case of multiple copies; we extend the commutation relations of the resulting Toeplitz algebra of $SU(2)$-equivariant subproduct system from irreducible representations to the case of multiple copies. As applications, we prove the commutation relations in the resulting Cuntz–Pimsner algebra and obtain that the resulting Cuntz–Pimsner algebra is the closed span of noncommutative polynomials of a specific form.

*The impression that I have after many years is that each human being is unique and could well be a hero of some kind depending on the circumstances.*

*—- Alain Connes*

# Contents

Chapter 1

# Introduction

## Background

The theory of operators on Hilbert spaces is at the basis of the mathematical foundation of quantum mechanics, thanks to the works of Werner Heisenberg [9] on matrix mechanics, and later by von Neumann on rings of operators [14]. Later work by Gelfand and Naimark in [8] formally established the theory of $C^*$-algebras as an independent research field within mathematics.

Following results by Gelfand, there exists an equivalence relation between the category of commutative $C^*$-algebras over $\mathbb{C}$ and the category of locally compact topological spaces. This analogy gives birth to non-commutative topology as the study of not necessarily commutative $C^*$-algebras.

On the other hand, operator theory on Hilbert spaces led to the prosper of the theory of partial differential equations, for instance, Riesz representation theory implies the existence of weak solutions for elliptic PDEs [7].

The development of operator theory and operator algebras is getting more and more prosperous nowadays. This project focuses on the theory of subproduct systems and their associated $C^*$-algebras, a relative new branch within operator algebras.

## Motivations

Dilation theory is a powerful technique in operator theory. The general philosophy is to study a complicated class of operators (e.g. contractive operators) by viewing them as compressions of a well-studied class of operators (e.g. unitary operators). For example, let $T \in B(H)$ be a contractive operator on some Hilbert space $H$ i.e. $\|T\| \leq 1$, then we have $1 - T^*T$ is positive. The von Neumann inequality can be proved easily using the tools of dilation theory: by using the continuous functional calculus and a limit argument, we have that

$$U := \begin{bmatrix} T & \sqrt{1 - TT^*} \\ \sqrt{1 - T^*T} & -T^* \end{bmatrix}.$$

is a self-adjoint operator on $H \oplus H$. Moreover, we have $T = P_H U$ for $P_H : H \oplus H \to H \oplus 0 \cong H$ an orthogonal projection. Then we claim that for every polynomial $p$, we have $\|p(T)\| \leq \sup_{|z|=1} |p(z)|$ whenever $T$ is contractive: let $p \in \mathbb{C}[z]$, we have

$$\|p(T)\| = \|P_H p(U)|_H\| \leq \|p(U)\| = \sup_{z \in \sigma(U)} |p(z)| \leq \sup_{|z|=1} |p(z)|, \tag{1.1}$$

where we use the spectral theorem for unitary operators. The inequality $\|p(T)\| \leq \sup_{|z|=1} |p(z)|$ is the famous von Neumann inequality [16, Theorem 1.1].

An important recent research direction in dilation theory is the study of subproduct systems. Those consist of a family of $C*$-correspondences indexed by a semigroup, typically $\mathbb{N}_0$, and subject to certain compatibility conditions. Subproduct systems were first formally described by Shalit and Solel in [15] and were independently studied by Bhat and Mukherjee in the Hilbert space setting [3], under the name of inclusion systems.

# Outline

Inspired by dilation theory, multivariate operator theory [2] as well as the recent paper [1], in this thesis, we focus on $SU(2)$-equivariant subproduct systems of finite-dimensional Hilbert spaces. We recall the results from [1] on $SU(2)$-equivariant subproduct systems induced by an irreducible $SU(2)$-representation. Then we show how to generalize those results to some reducible cases. Finally, we describe the commutation relations in the resulting Toeplitz and Cuntz–Pimsner algebras in both irreducible and reducible cases. We end the project with a corollary of how the resulting Cuntz–Pimsner algebras can be described as completion of algebras of polynomials, and with an outlook on open problems.

The outline of the thesis is as follows.

Chapter 2 is devoted to preliminaries on the classical theory of $C^*$-algebras and their representation theory.

In Chapter 3, we introduce the theory of subproduct systems of Hilbert spaces and the resulting Toeplitz and Cuntz–Pimsner algebras, which are our main objects of study.

In Chapter 4, we describe the structure of an $SU(2)$-equivariant subproduct system induced by an irreducible representations, following [1], and then generalize our treatment to some reducible cases.

Chapter 5 we start by recalling the commutation relations in the Toeplitz algebras of an irreducible $SU(2)$-subproduct system. Then we prove the commutation relations in the Toeplitz algebras arising from reducible representations. Finally, we describe the commutation relations in the Cuntz–Pimsner algebras and end the chapter by showing that the our Cuntz–Pimsner algebras are the closed span of noncommutative polynomials of some form. We conclude with an outlook section concerning possible further research questions.

This thesis contains three appendices: one on the theory of Lie groups and their representations, one on subproduct systems of $C^*$-correspondences, and the last one on some integer sequences appearing in this work.

Chapter 2

# Preliminaries: $C^*$-algebras and Their Representation Theory

*Mathematics is the art of giving the same name to different things.*

*Henri Poincaré*

In this Chapter, we will firstly define $C^*$-algebras and recall some important results. Then we will turn to representation theory of $C^*$-algebras in which the famous *GNS* Theorem will be proved.

## 2.1 $C^*$-algebras

The main references for this chapter is [12].

### 2.1.1 $C^*$-algebras and their properties

**Definition 2.1** (Normed algebra)**.** *Let $A$ be an algebra over some field $\mathbb{F}$, we call $A$ a normed algebra if $A$ is endowed with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\| \cdot \|b\|$. We call $A$ unital if there exists $1_A \in A$ such that $1_A a = a 1_A = a, \forall a \in A$. If $A$ is unital, we also assume $\|1_A\| = 1$.*

From now on, the ground field $\mathbb{F}$ of all vector spaces and algebras will be fixed as $\mathbb{C}$. Once we have norm, the most interesting spaces are the complete ones.

**Definition 2.2** (Banach algebras). *A Banach algebra $A$ is a normed algebra that is complete in the topology induced by the norm, that is, all Cauchy sequences in $A$ converge in $A$.*

One of the standard examples is the vector space $\mathbb{C}^n$ with point-wise multiplication.

**Example 2.3.** *Let $A$ denote the vector space $\mathbb{C}^n$. We claim that $A$ is a Banach algebra with the Euclidean norm and point-wise multiplication. The Euclidean norm is complete since $\mathbb{C}^n$ is finite dimensional. The remaining thing to check is the inequality $\|xy\| \leq \|x\| \cdot \|y\|$. Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ we have $xy = (x_1y_1, x_2y_2, \ldots, x_ny_n)$ and*

$$\begin{aligned}
\|xy\|^2 &= \sum_{i=1}^{n} \|x_iy_i\|^2 \\
&= \sum_{i=1}^{n} \|x_i\|^2 \cdot \|y_i\|^2 \\
&\leq (\sum_{i=1}^{n} \|x_i\|^2) \cdot (\sum_{i=1}^{n} \|y_i\|^2) \\
&= \|x\|^2\|y\|^2.
\end{aligned}$$

*Then $A$ is a Banach algebra.*

Let us look at another example of Banach algebras.

**Example 2.4.** *Consider the set $C(X)$ of continuous complex valued functions on $X$, where $X$ is a compact space. We endow $C(X)$ with the supremum norm (which is well-defined as $X$ is assumed to be compact)*

$$\| \cdot \|_\infty : C(X) \to \sup_{x \in X} |f(x)| < \infty.$$

*One can check that $C(X)$ is an algebra with respect to pointwise addition, multiplication, scalar multiplication, and with the function $1_X(x) = 1, \forall x \in X$ as unit. Furthermore, we have $\|fg\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ and $(C(X), \| \cdot \|_\infty)$ is complete. Therefore $C(X)$ is a unital Banach algebra.*

**Definition 2.5** (Involution). *An involution on an algebra $A$ (over $\mathbb{C}$) is a conjugate-linear map $^* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. If an algebra $A$ is endowed with an involution, we call $A$ a $*$-algebra. If $B$ is a subalgebra of $A$ with $x^* \in B, \forall x \in B$, we call $B$ a $*$-subalgebra of $A$.*

Roughly speaking, the $C^*$-algebras are Banach algebras with an isometric involution, that is compatible with the norm in a way we shall describe below.

**Definition 2.6** (Banach $*$-algebra). *A Banach $*$-algebra $A$ is a $*$-algebra, together with a complete norm such that $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$. If $A$ is unital with $\|1_A\| = 1$, we call $A$ a unital Banach $*$-algebra.*

**Remark 2.7.** *Some authors define the involution $^*$ of a Banach $*$-algebra to be isometric e.g. [12], however, we do not follow this convention in this thesis.*

We are now ready to formally define $C^*$-algebra.

**Definition 2.8** ($C^*$-algebra). *A $C^*$-algebra $A$ is a Banach $*$-algebra with an involution $^*$ such that*

$$\|x^*x\| = \|x\|^2. \tag{2.1}$$

*The identity 2.1 is called $C^*$-identity.*

An element $a \in A$ is self-adjoint (or Hermitian) if $a^* = a$. We call $a$ normal if $aa^* = a^*a$. A projection in a $C^*$-algebra $A$ is an element $p \in A$ such that $p^* = p = p^2$. For a unital $C^*$-algebra $A$, we say an element $a \in A$ is invertible if there exists $b \in A$ such that $ab = ba = 1$.

**Remark 2.9.** *If $B$ is a closed (with respect to the norm topology) $*$-subalgebra of a $C^*$-algebra $A$, then $B$ itself is again a $C^*$-algebra.*

**Definition 2.10.** *Let $A, B$ be two $C^*$-algebras, a $*$-homomorphism $\varphi : A \to B$ is an algebra homomorphism such that*

$$\varphi(a^*) = \varphi(a)^*, \quad \forall a \in A.$$

We list some important properties of $C^*$-algebras below.

**Lemma 2.11.** *Let $A$ be a $C^*$-algebra then we have*

1. *For any $x \in A$, if $x^*x = 0$, then $x = 0$;*

2. *The involution is bijective and isometric;*

3. *If $x \in A$ is invertible, then $(x^*)^{-1} = (x^{-1})^*$.*

*Proof.* Those properties follow from the $C^*$-identity:

1. By the $C^*$-identity we have $x^*x = 0$ implies $\|x\|^2 = 0$ thus $x = 0$;

2. The fact that $(x^*)^* = x$ implies $^*$ is surjective and $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ implies $\|x\| \leq \|x^*\|$. If we replace $x$ by $x^*$, we have $\|x^*\| \leq \|x\|$, which implies that $\|x\| = \|x^*\|$. Hence the involution is bijective and isometric.

3. If $x$ is invertible, then there exists $x^{-1} \in A$ such that $x^{-1}x = 1$, which implies that $1 = 1^* = x^*(x^{-1})^*$ thus we have $(x^*)^{-1} = (x^{-1})^*$.

□

We will now consider some examples of $C^*$-algebras.

**Example 2.12.** *The simplest example of $C^*$-algebra is the complex numbers with the standard norm ($\|a + bi\| = \sqrt{a^2 + b^2}$), and with complex conjugate as involution. Indeed, write $x = a + bi$ then we have$\|x^*x\| = \|\overline{x}x\| = \|a^2 + b^2\| = a^2 + b^2 = \|x\|^2$.*

**Example 2.13.** *Let $H$ be a Hilbert space and $B(H)$ be the algebra of bounded operators on $H$ with norm the operator norm. Then we have that $B(H)$ is a $C^*$-algebra with the adjoint as the involution. The only non-trivial thing is the $C^*$ identity: take $A \in B(H)$*

$$\|A^*x\|^2 = \langle A^*x, A^*x\rangle = \langle AA^*x, x\rangle \leq \|A\|\|A^*x\|\|x\|.$$

*Thus we have $\|A^*x\| \leq \|A\|\|x\| = \|A\|$ when $\|x\| = 1$. It implies that $\|A^*\| \leq \|A\|$ and thus $\|A^*\| = \|A\|$. On the other hand, we have $\|Ax\|^2 \leq \|A^*A\|\|x\|^2 = \|A^*A\|$ when $\|x\| = 1$, thus $\|A\|^2 \leq \|A^*A\|$. Therefore the $C^*$ identity holds:*

$$\|A\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2 \quad \textit{implies} \quad \|A^*A\| = \|A\|^2.$$

*We conclude that $B(H)$ is a $C^*$-algebra.*

### 2.1.2 Spectral theory

In finite-dimensional linear algebra, eigenvectors as well as their eigenvalues play an important role. Here we will define the notion of spectrum which plays the role of eigenvalues in the infinite dimensional case.

We first state some results on unital Banach algebras, then turn to the case of $C^*$-algebras.

**Definition 2.14** (Spectrum)**.** *Let $A$ be a unital Banach algebra. The spectrum of $a \in A$ is defined as $\sigma(a) := \{\lambda \in \mathbb{C} : \lambda \cdot 1 - a \text{ is not invertible}\}$.*

**Definition 2.15** (Positive element)**.** *Let $A$ be a unital $C^*$-algebra. We call $a \in A$ positive if $a$ is self-adjoint and $\sigma(c) \in [0, \infty)$.*

The following theorem describes the geometric property of the spectrum $\sigma(a)$ for all $a \in A$.

**Theorem 2.16.** *Let $A$ be a unital Banach algebra then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$, for all $a \in A$.*

*Proof.* We first show $\sigma(a)$ is bounded. We claim that for all $x \in A$, $\|x - 1\| < 1$ implies that $x$ is invertible. Indeed, let $y := 1 - x$. Then $\sum_{n=0}^{\infty} y^n$ converges as $\|y\| < 1$ and

$$\| \sum_{n=0}^{\infty} y^n \| \leq \sum_{n=0}^{\infty} \|y^n\| \leq \sum_{n=0}^{\infty} \|y\|^n \to \frac{1}{1 - \|y\|},$$

implies that $x^{-1} = \sum_n y^n$, since

$$x(\sum_{n=0}^{\infty} y^n) = \lim_{n \to \infty} x(1 + y + \cdots + y^n) = \lim_{n \to \infty} 1 - y^{n+1} = 0.$$

Moreover, $\lambda \in \sigma(a)$ implies that $\lambda - a = \lambda(1 - a/\lambda)$ is not invertible. Thus we have $\|1 - (1 - a/\lambda)\| = \|a/\lambda\| = \|a\|/|\lambda| \geq 1$ otherwise $1 - a/\lambda$ would be invertible. Therefore we have $|\lambda| \leq \|a\|$.

To show the compactness of $\sigma(a)$, it is sufficient to show $\sigma(a)$ is closed. However, we have that $O(A)$, the set of invertible elements of $A$, is open and $f(\lambda) := \lambda - a$ is clearly continuous. Then we have $\sigma(a) = \mathbb{C} \setminus f^{-1}(O(A))$ is closed.

The proof of the fact that $\sigma(a)$ is non-empty for all $a \in A$ can be found in [12, Theorem 1.2.5]. □

We can now state the famous Gelfand–Mazur's theorem.

**Theorem 2.17** (Gelfand–Mazur)**.** *Let $A$ be a unital Banach algebra. If all non-zero elements of $A$ are invertible, then $A = \mathbb{C}1$.*

*Proof.* Let $a \in A$. Then by Theorem 2.16 we have that $\sigma(a)$ is non-empty. Then there exists $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. By assumption we have $\lambda - a = 0$, that is $a = \lambda$. □

Using Theorem 2.16 we can define the concept of spectral radius.

**Definition 2.18.** *Let $A$ be a unital Banach algebra. We call*

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|$$

*the spectral radius of $a \in A$.*

It is not hard to compute the spectral radius by Beurling's Theorem below.

**Theorem 2.19** (Beurling)**.** *Let $A$ be a unital Banach algebra then we have $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ for all $a \in A$.*

*Proof.* The proof can be found in [12, Theorem 1.2.7]. □

As a quick consequence, it is easy to determine the spectral radius of normal elements.

**Corollary 2.20.** *Let $A$ be a $C^*$-algebra and $a \in A$ be a normal element i.e. $aa^* = a^*a$, then $r(a) = \|a\|$.*

*Proof.* By using $C^*$ identity we have $\|a^{2^n}\| = \|a^{2^n}(a^{2^n})^*\|^{1/2}$. Then we have $\|a^{2^n}(a^{2^n})^*\|^{1/2} = \|(aa^*)^{2^n}\|^{1/2} = \|a\|^{2^n}$ as $a$ is normal. That leads to $\|a\| = \|a^{2^n}\|^{1/2^n}$ thus $r(a) = \lim_n \|a^n\|^{1/n} = \lim_n \|a^{2^n}\|^{1/2^n} = \|a\|$ by Beurling's Theorem. □

Now we turn our attention to Abelian $C^*$-algebras. Actually, the results presented here hold for general Abelian Banach algebras.

**Definition 2.21.** *A character $\tau$ of an Abelian Banach algebra $A$ is a homomorphism from $A$ to $\mathbb{C}$. The set of characters is denoted by $\Omega(A)$.*

We shall list some properties of characters without proof, and then prove the Gelfand representation theorem.

**Lemma 2.22** ([12, Theorem 1.3.3])**.** *Let $A$ be a unital Abelian Banach algebra. Then we have:*

1. *If $\tau \in \Omega(A)$ then we have $\|\tau\| = 1$;*
2. *The set $\Omega(A)$ is nonempty;*
3. *If $A$ is unital then $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$.*

In fact, $\Omega(A)$ is even compact when endowed with the weak-$*$ topology.

**Theorem 2.23.** *If $A$ is a unital Abelian Banach algebra, then $\Omega(A)$ is a compact Hausdorff space.*

*Proof.* We have that $\Omega(A)$ is weak $*$ closed in the unit ball of $A^*$ as if $\tau_\lambda \in \Omega(A)$ converges weakly to $\tau$, then we have $\tau_\lambda(a) \to \tau(a)$ for all $a \in A$. That gives $\tau_\lambda(a \star b) = \tau_\lambda(a) \star \tau_\lambda(b) \to \tau(a) \star \tau(b)$ where $\star$ stands for addition and multiplication. Then by Banach–Alaoglu's theorem, we have that $\Omega(A)$ is compact. □

Now we are ready for the main theorem of this section.

**Theorem 2.24** (Gelfand)**.** *Let $A$ be a unital Abelian Banach algebra. Let the Gelfand transform be defined by:*

$$\varphi : A \to C(\Omega(A)), a \mapsto (\hat{a} : \tau \mapsto \tau(a)).$$

*Then the following hold:*

1. *The Gelfand transform is a norm-decreasing homomorphism and $r(a) = \|\hat{a}\|_\infty$.*
2. *If $A$ is a unital Abelian $C^*$-algebra then the Gelfand transform is an isometric isomorphism.*

*Proof.* It is easy to check that $\varphi$ is a homomorphism. Indeed, we have $\varphi(a \star b)(\tau) = \tau(a \star b) = \tau(a) \star \tau(b) = \varphi(a)(\tau) \star \varphi(b)(\tau)$ where $\star$ stands for addition and multiplication since $\tau$ is additive and multiplicative on $A$. Then, to show that $\varphi$ is a homomorphism, it is sufficient to show that $\varphi(a^*) = \overline{\varphi(a)}$. That follows, as

$$\varphi(a^*)(\tau) = \tau(a^*) = \overline{\tau(a)} = \overline{\varphi(a)}.$$

By Lemma 2.22, we have that the range of $\hat{a}$ coincides with the spectrum of $a$ and $\|\tau\| = 1$, thus $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\hat{a}\|_\infty$.

The fact that the Gelfand transform is an isometry is due to the $C^*$-identity:

$$\|\varphi(a)\|_\infty^2 = \|\varphi(a)\varphi(a)^*\|_\infty = \|\varphi(a^*a)\|_\infty = r(a^*a) = \|a^*a\| = \|a\|^2.$$

To show that we have an isomorphism, it is sufficient to show that $\varphi(A)$ is a closed $*$-subalgebra of $C(\Omega(A))$ that separates points. The closedness of $\varphi(A)$ is clear and for any $\tau \in \Omega(A)$ we have $a \in A$ such that $\tau(a)$ as $\|\tau\| = 1$ which means that $\varphi(A)$ separates points. By the Stone–Weierstrass theorem [5, Theorem 8.1], we have $\varphi(A) = C(\Omega(A))$, as desired. □

## 2.2 Representation theory of $C^*$-algebras

### 2.2.1 Ideals and positive linear functionals

For non-unital $C^*$-algebras, approximate units play an important role by extending the notion of unit to this setting.

**Definition 2.25** (Approximate unit)**.** *An approximate unit for a $C^*$-algebra $A$ is an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ of positive elements within the closed unit ball of $A$ such that* $\lim_{\lambda \to \infty} au_\lambda = \lim_{\lambda \to \infty} u_\lambda a = a$ *for all $a \in A$.*

The standard example of approximate unit is the net of projections in $K(H)$, where $K(H)$ denotes the $C^*$-algebra of compact operators from a separable infinite-dimensional Hilbert space $H$ to itself.

Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $H$. Let $\{p_k\}$ be the projection onto span$\{e_1, \ldots, e_k\}$ then we have that $\{p_k\}$ is an approximate unit for $K(H)$. Indeed, as the set of finite rank operators which we denote by $F(H)$, is dense in $K(H)$, it is sufficient to show $\lim_{k \to \infty} p_\lambda T = T$ under the operator norm, for all $T \in F(H)$.

The following Theorem ensures the existence of approximate units in arbitary $C^*$-algebras.

**Theorem 2.26** ([12, Theorem 3.1.1])**.** *Every $C^*$-algebra admits an approximate unit.*

Indeed, an approximate unit can be constructed as follows: let $\Lambda$ denote the set of positive elements with norm less than one in $A$, which has a natural order. Set $u_\lambda = \lambda \in \Lambda$. Then we have that $\{u_\lambda\}$ is an approximate unit. This is called standard approximate unit.

Using Theorem 2.26, we can easily show the existence of approximate unit for closed ideals.

**Corollary 2.27.** *Let $L$ be a closed left ideal of a $C^*$-algebra $A$. Then $L$ admits an approximate unit $\{u_\lambda\} \in I$.*

*Proof.* Set $B = L \cap L^*$. Since $B$ is a $C^*$-algebra, $B$ admits an approximate unit by Theorem 2.26. Let $a \in L$. We have $a^*a \in B$, which implies that

$$\lim_{\lambda \to \infty} a^*a(1 - u_\lambda) = 0.$$

Then we have

$$\begin{aligned}\lim_{\lambda \to \infty} \|a(1 - u_\lambda)\|^2 &= \lim_{\lambda \to \infty} \|(1 - u_\lambda)a^*a(1 - u_\lambda)\|^2 \\ &\leq \lim_{\lambda \to \infty} \|a^*a(1 - u_\lambda)\|^2 = 0.\end{aligned}$$

Hence we have $\lim_{\lambda \to \infty} au_\lambda = \lim_{\lambda \to \infty} u_\lambda a = a$ for all $a \in L$. □

**Corollary 2.28.** *If $I$ is a closed ideal in a $C^*$-algebra $A$, then $I$ is self-adjoint and therefore a $C^*$-subalgebra of $A$. If $\{u_\lambda\}$ is an approximate unit for $I$, then for all $a \in A$:*

$$\|a + I\| = \lim_\lambda \|a - au_\lambda\| = \lim_\lambda \|a - u_\lambda a\|.$$

*Proof.* The proof uses Corollary 2.27 and the details can be found in [12]. □

By using Corollary 2.28, we can prove that the quotient of a $C^*$-algebra by a closed ideal is a $C^*$-algebra.

**Corollary 2.29.** *If $I$ is a closed ideal of a $C^*$-algebra $A$, then the quotient $A/I$ is a $C^*$-algebra with the operations defined on the quotient, and the quotient norm.*

*Proof.* Let $\{u_\lambda\}$ be an approximate unit of $I$, which exists by Corollary 2.27. By Corollary 2.28, for each $a \in A, b \in I$ we have

$$\begin{aligned}
\|a + I\|^2 &= \lim_\lambda \|a - au_\lambda\|^2 \\
&= \lim_\lambda \|(1 - u_\lambda)a^*a(1 - u_\lambda)\|^2 \\
&= \lim_\lambda \|(1 - u_\lambda)(a^*a + b - b)(1 - u_\lambda)\|^2 \\
&\leq \|a^*a + b\| + \lim_\lambda \|b - bu_\lambda\| \\
&= \|a^*a + b\|,
\end{aligned}$$

which gives $\|a + I\|^2 \leq \|a^*a + I\| \leq \|a + I\|^2$. Thus $A/I$ is a $C^*$-algebra. □

We are now ready to prove the main theorem of this section.

**Theorem 2.30.** *If $\varphi : A \to B$ is an injective $*$-homomorphism between $C^*$-algebras, then $\varphi$ is isometric.*

*Proof.* It is sufficient to show $\|\varphi(a^*a)\| = \|a^*a\|$, thus we can restrict to the case $A$ is generated by 1 and $a^*a$ and $B = \overline{\varphi(A)}$ which are Abelian.

Let $\tau$ be a character on $B$, thus $\tau \circ \varphi$ is a character on $A$. Clearly we have that the $*$-homomorphism $\varphi$ induces a continuous map $\varphi^\#$ from $\Omega(B)$ to $\Omega(A)$ which sends $\tau$ to $\tau \circ \varphi$. As $\Omega(B)$ is compact, we then have $\varphi^\#(\Omega(B))$ is closed in $\Omega(A)$. By Urysohn's Lemma, there exists $f : \Omega(A) \to \mathbb{C}$ such that $f|_{\varphi^\#} = 0$. However, via Theorem 2.24, we have $f = \hat{x}$ for some $x \in A$. Then we have $0 = \hat{x}(\tau \circ \varphi) = \tau \circ \varphi(x)$ for all $\tau \in \Omega(B)$ which shows that

$x = 0$, a contradiction! Therefore we have $\Omega(A) = \varphi^\#(\Omega(B))$, from which it follows that

$$\|x\| = \|\hat{x}\|_\infty = \sup_{\substack{\tau\in\Omega(A)\\ \|\tau\|=1}} \|\tau(x)\| = \sup_{\substack{\tau'\in\Omega(B)\\ \|\tau\|=1}} \|\tau(\varphi(x))\| = \|\varphi(x)\|.$$

□

The last essential concept in the representation theory of $C^*$-algebras is that of a positive linear functional.

**Definition 2.31.** *A linear map $\varphi$ between $C^*$-algebras $A$ and $B$ is said to be positive if it maps positive elements to positive elements. A linear functional is said to be positive if it is positive as linear map from $A$ to $\mathbb{C}$. A state is a positive linear functional with norm* 1.

Let us look at some properties of positive linear functionals.

**Lemma 2.32.** *Positive linear functionals are bounded.*

*Proof.* Let $\tau$ be a positive linear functional. Suppose $\tau$ is not bounded, then there exists $\{s_n : \|s_n\| \leq 1\}$ such that $\tau(s_n) \geq 2^n$. Then we have the series $s := \sum_n \frac{s_n}{2^n}$ converges:

$$\|\sum_n \frac{s_n}{2^n}\| \leq \sum_n \|\frac{s_n}{2^n}.\| \leq \sum_n 2^{-n} \leq 1.$$

On the other hand, we have that $s \geq s_n$ for all $s_n$ by construction. Therefore we have $\tau(s) \geq \tau(s_n) \geq 2^n$ for all $n \in \mathbb{N}$, which is a contradiction. Hence we conclude that $\tau$ is bounded. □

**Lemma 2.33.** *If $\tau$ is a positive linear functional on a $C^*$-algebra $A$ , then $\tau(a^*) = \overline{\tau(a)}$ and $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$.*

*Proof.* Let $\{u_\lambda\}$ be an approximate unit of $A$. Then we have

$$\tau(a^*) = \lim_\lambda \tau(a^* u_\lambda) = \lim_\lambda \overline{\tau(u_\lambda a)} = \overline{\tau(a)},$$

where the second equality is due to the fact that $\langle x, y\rangle := \tau(y^*x)$ defines a sesquilinear linear form. The inequality $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$ is a consequence of

$$|\tau(a)|^2 = \lim_\lambda |\tau(u_\lambda a)|^2 \leq \lim_\lambda |\tau(u_\lambda^2)|\tau(a^*a) = \|\tau\|\tau(a^*a),$$

which in turn follows from the Cauchy–Schwarz inequality. □

The following Lemma provides a characterization of the positive linear functionals.

**Lemma 2.34** ([12, Theorem 3.3.2]). *Let $\tau$ be a linear functional. The following facts are equivalent:*

1. *$\tau$ is positive;*
2. *For every approximate unit $\{u_\lambda\}$ we have $\lim_\lambda \tau(u_\lambda) = \|\tau\|$;*
3. *For some approximate unit $\{u_\lambda\}$ we have $\lim_\lambda \tau(u_\lambda) = \|\tau\|$.*

**Corollary 2.35.** *Let $A$ be a unital $C^*$-algebra, then we have*

1. *If $\tau$ is a bounded linear functional on $A$, then $\tau$ is positive if and only if $\tau(1) = \|\tau\|$.*
2. *If $\tau$ and $\tau^{'}$ are positive linear functionals on $A$, then we have $\|\tau + \tau^{'}\| = \tau(1) + \tau^{'}(1) = \|\tau\| + \|\tau^{'}\|$*

*Proof.* Let $\tau$ be a bounded linear functional on $A$, then by Lemma 2.34, we have that $\tau$ is positive if and only if $\lim_\lambda \tau(u_\lambda) = \|\tau\|$ for every approximate unit. Since $A$ is unital, we can take $u_\lambda = 1$ for all $\lambda$, which proves the first statement. The second statement follows from the first one. □

**Theorem 2.36.** *Suppose $\tau$ is a positive linear functional on a $C^*$-algebra $A$. Then we have*

$$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b), \quad \forall a, b \in A.$$

*Proof.* We may suppose $\tau(b^*b) > 0$ otherwise $\tau(b^*a^*ab) = 0$ by the Cauchy–Schwarz inequality. Then define the positive linear functional

$$\rho : A \to \mathbb{C}, c \mapsto \frac{\tau(b^*cb)}{\tau(b^*b)}.$$

Let $\{u_\lambda\}$ be an approximate unit for $A$, the above functional satisfies

$$\|\rho\| = \lim_\lambda \rho(u_\lambda) = \lim_\lambda \tau(b^*u_\lambda b)/\tau(b^*b) = 1,$$

which implies $\rho(a^*a) \leq \|a^*a\|$, thus $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$. □

### 2.2.2 Gelfand–Naimark–Segal Theorem

**Definition 2.37.** *A representation of a $C^*$-algebra is a pair $(H, \varphi)$ where $H$ is a Hilbert space and $\varphi$ is a $*$- homomorphism from $A$ to $B(H)$. We call $(H, \varphi)$ faithful if $\varphi$ is injective.*

Given a representation $(H, \varphi)$, we call $H$ the representation space. If $(H_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ is a family of representations, the direct sum of $H_\lambda$ is automatically a representation space, and the $*$- homomorphism is defined as

$$\varphi(a)(x_\lambda) := (\varphi_\lambda(a)(x_\lambda))_\lambda, x_\lambda \in H_\lambda.$$

Moreover, if all $(H_\lambda, \varphi_\lambda)$ are faithful, $(\oplus_\lambda H_\lambda, \oplus_\lambda \varphi_\lambda)$ is faithful.

We shall prove that for any positive linear functional on $C^*$-algebra $A$, there exists an associated representation. This result is known as the GNS construction.

**Lemma 2.38.** *Let $\tau$ be any positive linear functional on $A$. The set*

$$N_\tau = \{a \in A : \tau(a^*a) = 0\}.$$

*is a closed left ideal of $A$.*

*Proof.* Let $b \in A$, $a \in N_\tau$, we have

$$0 \leq \tau((ba)^*ba) = \tau(a^*b^*ba) \leq \|b^*b\|\tau(a^*a) = 0 \quad \text{implies} \quad ba \in N_\tau.$$

Let $\{a_\lambda\}$ be a net in $N_\tau$,

$$\tau(a^*a) = \tau(\lim_\lambda a_\lambda^* a_\lambda) = \lim_\lambda \tau(a_\lambda^* a_\lambda) = \lim_\tau 0 = 0.$$

Therefore, $N_\tau$ is a closed left ideal of $A$. □

On $A/N_\tau$, define the inner product:

$$(a + N_\tau, b + N_\tau) \mapsto \tau(b^*a).$$

To see that the inner product is well-defined, take $a_1 + N_\tau = a_2 + N_\tau$ i.e. $a_1 - a_2 = a \in N_\tau$ then

$$\begin{aligned}
(a_1 + N_\tau, b + N_\tau) &= (a_2 + a + N_\tau, b + N_\tau) && (2.2)\\
&= \tau(b^*a_2) + \tau(b^*a) && (2.3)\\
&= \tau(b^*a_2) && (2.4)\\
&= (a_2 + N_\tau, b + N_\tau) && (2.5)
\end{aligned}$$

where equality from (2.3) to (2.4) is due to the Cauchy–Schwartz inequality by considering the sesqui-linear form described in the proof of Theorem 2.33.

Hence the space $(A/N_\tau, (\cdot, \cdot))$ is a pre-Hilbert space. Denote by $H_\tau$ the Hilbert space completion of $A/N_\tau$.

Now, let us construct the representation $\varphi$. Define a $*$-homomorphism $\varphi : A \to B(A/N_\tau)$ by

$$\varphi(a)(b + N_\tau) = ab + N_\tau.$$

Then we have

$$\begin{aligned}\|\varphi(a)\|^2 &= \sup_{\|b+N_\tau\| \leq 1} \|\varphi(a)(b + N_\tau)\|^2 \\ &= \sup_{\|b+N_\tau\| \leq 1} \tau(b^* a^* ab) \\ &\leq \sup_{\tau(b^*b) \leq 1} \|a\|^2 \tau(b^* b) \leq \|a\|,\end{aligned}$$

where the first inequality follows from Theorem 2.36 .

The the operator $\varphi(a)$ can be uniquely extended to a bounded operator $\varphi_\tau(a)$ on $H_\tau$. The map $\varphi_\tau : a \mapsto \varphi_\tau(a)$ is clearly a $*$-homomorphism.

The representation $(H_\tau, \varphi_\tau)$ constructed above is called the *Gelfand–Naimark-Segal (GNS) representation* associated to the positive linear functional $\tau$.

**Definition 2.39** (Universal representation). *Let $A$ be a $C^*$-algebra, the universal representation of $A$ is defined as the direct sum of $(\varphi_\tau, H_\tau)$ for all states $\tau$ on $A$.*

This leads us to the famous **Gelfand–Naimark–Segal Theorem**:

**Theorem 2.40** (Gelfand–Naimark–Segal). *If $A$ is a $C^*$-algebra, then it has a faithful representation. Specifically, its universal representation if faithful.*

*Proof.* Let $(H, \varphi)$ be the universal representation of $A$. Then suppose $(H, \varphi)$ is not faithful i.e. there exists $a \in A$ such that $\varphi(a) = 0$. Since $a^*a$ is normal, there exists a state $\tau$ such that $\|a^*a\| = \tau(a^*a)$. Set $b = (a^*a)^{1/4}$, using functional calculus, we have

$$\|a\|^2 = \|a^*a\| = \tau(a^*a) = \tau(b^4) = \|\varphi_\tau(b)(b + N_\tau)\|^2.$$

On the other hand, we have $\varphi_\tau(b^4) = \varphi_\tau(a^*a) = 0$ hence we have

$$\varphi_\tau(b) = 0 \text{ implies } \|a\| = 0 \text{ if and only if } a = 0,$$

from which we conclude that $(H, \varphi)$ is faithful. □

We end this section with an application of the $GNS$ theorem.

**Example 2.41.** *Consider the matrix algebra $M_n(A)$ with entries in a $*$-algebra $A$. The involution operation on $M_n(A)$ is defined as $(a_{ij})^*_{ij} = (a^*_{ji})_{ij}$. If there is a $*$-homomorphism between $A$ and another $*$-algebra $B$, then there is a corresponding a $*$- homomorphism between matrix algebras, by sending $(a_{ij})$ to $\varphi(a_{ij})$.*

*Let $H$ be a Hilbert space, and $H^{(n)}$ be the direct sum of $n$ copies of $H$. Choose $u \in M_n(B(H))$, we define $\varphi(u) \in B(H^{(n)})$ by*

$$\varphi(u) : (x_1, \ldots, x_n)^T \mapsto u(x_1, \ldots, x_n)^T = (\sum_{j=1}^{n} u_{1j}x_j), \ldots, \sum_{j=1}^{n} u_{nj}x_j).$$

*Clearly, the homomorphism $\varphi$ is a $*$-isomorphism. We call it the canonical $*$-isomorphism. Thus the matrix algebra of $B(H)$ can be identified with $B(H^{(n)})$. Note that $M_n(B(H))$ has no norms right now. We define the norm by setting $\|u\|_M := \|\varphi(u)\|$. As $B(H^{(n)})$ is clearly a $C^*$-algebra, it is not hard to show $M_n(B(H))$ is a $C^*$-algebra with the norm $\|\cdot\|_M$. Moreover, by using the GNS Theorem we can prove that $\|\cdot\|_M$ is the only norm with which $M_n(B(H))$ becomes a $C^*$-algebra. This fact holds in much bigger generality, as the next theorem shows.*

**Theorem 2.42** ([12, Theorem 3.4.2])**.** *If $A$ is a $C^*$-algebra, then there is a unique norm on $M_n(A)$ making it a $C^*$-algebra.*

Chapter 3

# Subproduct Systems from $SU(2)$-representations

## 3.1 Subproduct systems of Hilbert spaces

Subproduct systems were originally defined in the Hilbert space setting in [3]. Later, the concept of subproduct system was extended to the more general setting of $C^*$-correspondences in [15]. The interested reader can refer to the Appendix C for a detailed introduction to subproduct systems of $C^*$-correspondences. In this section, we only focus on subproduct systems of Hilbert spaces.

**Definition 3.1.** *Suppose that $H = \{H_m\}, m \in \mathbb{N}_0$ is a sequence of Hilbert spaces and that $\iota_{k,m} : H_{k+m} \to H_k \otimes_{\mathbb{C}} H_m$ is a bounded isometry for every $k, m \in \mathbb{N}_0$. We say that $(H, \iota)$ is a subproduct system over $\mathbb{C}$ when the following holds for all $k, l, m \in \mathbb{N}_0$:*

1. *$H_0 = \mathbb{C}$;*
2. *The structure maps $\iota_{0,m} : H_m \to H_0 \otimes_{\mathbb{C}} H_m$ and $\iota_{m,0} : H_m \to H_m \otimes_{\mathbb{C}} H_0$ are the canonical identifications and;*
3. *The two bounded isometries $(1_k \otimes \iota_{l,m}) \circ \iota_{k,l+m}$ and $(\iota_{k,l} \otimes 1_m) \circ \iota_{k+l,m}$ : $H_{k+l+m} \otimes H_k \otimes_{\mathbb{C}} H_l \otimes_{\mathbb{C}} H_m$ agree, where $1_k$ and $1_m$ denote the identity operators on $H_k$ and $H_m$, respectively.*

Let us look at an example.

**Example 3.2.** *Let $H$ be a finite-dimensional Hilbert space over $\mathbb{C}$. Then we have that $\{H^{\otimes n}\}_{n=0}^{\infty}$ is a subproduct system with structure maps given by the canonical identifications $H^{\otimes n+m} \cong H^{\otimes n} \otimes H^{\otimes m}$. This subproduct system is called the full product system.*

**Example 3.3.** *Let $H$ be a fixed finite-dimensional Hilbert space. Define the projection $p_n$ from $H^{\otimes n}$ onto the symmetric subspace of $H^{\otimes n}$ by*

$$p_n(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma^{-1}(1)} \otimes \xi_{\sigma^{-1}(2)} \otimes \cdots \otimes \xi_{\sigma^{-1}(n)},$$

*where $S_n$ denotes the $n$-degree permutation group.*

*We define $E_n := p_n(H^{\otimes n})$ and the structure maps $\iota_{k,l}$ as the adjoints of the maps $u_{k,l} : E_k \otimes E_l \to E_{k+l}$ such that $u_{k,l}(x \otimes y) = p_{k+l}(x \otimes y)$. The only non-trivial thing to check is the third identity in the Definition 3.2. It is sufficient to prove that*

$$u_{k+l,m} \circ (u_{k,l} \otimes 1_m) = u_{k,l+m} \circ (1_k \otimes u_{l,m}).$$

*Indeed, we have*

$$\begin{aligned}
&u_{k+l,m} \circ (u_{k,l} \otimes 1_m)(x \otimes y \otimes z) \\
&= u_{k+l,m}(p_{k+l}(x \otimes y) \otimes z) \\
&= p_{k+l+m}(x \otimes y \otimes z) \\
&= u_{k,l+m} \circ (x \otimes p_{l+m}(y \otimes z)) \\
&= u_{k,l+m} \circ (1_k \otimes u_{l,m})(x \otimes y \otimes z).
\end{aligned}$$

*Therefore, $(E_n, \iota)$ is a subproduct system, which we call the symmetric subproduct system.*

### 3.1.1 The Toeplitz and Cuntz–Pimsner algebras of a subproduct system

Once we have a subproduct system, we can associate to it two $C^*$ algebras, namely, the Toeplitz and Cuntz–Pimsner algebras. They were studied by Viselter in [17]. Before introducing the concepts, we shall firstly define the Fock space on which we will represent the Toeplitz $C^*$-algebra.

**Definition 3.4** (Fock space). *Given a subproduct system $(E, \iota)$, its Fock space is defined as the infinite direct sum of Hilbert spaces $F := \oplus_{i=0}^{\infty} E_i$.*

For every $\xi \in E_k$, we define the creation operator $T_\xi \in \mathcal{L}(F)$ by

$$T_\xi : F \to F, \quad T_\xi(\zeta) := \iota_{k,m}^*(\xi \otimes \zeta), \quad \zeta \in E_k \subset F$$

Via the creation operators, we can define the Toeplitz $C^*$-algebra of a subproduct system.

**Definition 3.5** (Toeplitz algebra)**.** *Let $(E, \iota)$ be a subproduct system. The Toeplitz algebra of $(E, \iota)$, denoted by $\mathbb{T}_E$, is defined as the smallest unital $C^*$-subalgebra of $\mathcal{L}(F)$ containing all creation operators.*

The following definition gives the *Cuntz–Pimsner algebra* of a subproduct system.

**Definition 3.6** (Cuntz–Pimsner algebra)**.** *Given a subproduct system $(E, \iota)$ consisting of finite-dimensional Hilbert spaces over $\mathbb{N}_0$. The Cuntz–Pimsner algebra $\mathbb{O}_E$ of $(E, \iota)$ is the unital $C^*$-algebra obtained as the quotient of the Toeplitz algebra $\mathbb{T}_E$ by the ideal $\mathbb{K}_F$ of compact operators over the Fock space $F$.*

**Remark 3.7.** *In the setting of a subproduct system of $C^*$-correspondences, the Cuntz–Pimsner algebra is defined by quotienting the Toeplitz algebra by a suitable ideal. Interested readers may refer to Appendix C for more details.*

### 3.1.2 $G$-subproduct systems

In this project, we are interested in subproduct systems of Hilbert spaces with some additional structure in the form of a Lie group action.

**Definition 3.8** ($G$-action)**.** *Let $G$ be a locally compact topological group and $H$ be a Hilbert space. Then a $G$-action on $H$ is a pair $(G, \rho)$ where $\rho : G \to B(H)$ is a strongly continuous homomorphism i.e. the orbit map $\xi_x : G \to H$ sending $g$ to $\rho(g)(x)$ is continuous for all $x \in H$.*

Note that we will abbreviate the action on some $x \in H$ to $g(x)$ in the rest of this thesis.

**Definition 3.9** ($G$-subproduct systems)**.** *Let $G$ be a locally compact topological group and let $(E, \iota)$ be a subproduct system of Hilbert spaces $E_m, m \in N_0$. We say that $(E, \iota)$ is a $G$-subproduct system when there is a $G$-action $\rho$ on each fiber $E_m, m \in \mathbb{N}_0$, such that*

$$\rho(g) \circ \iota_{k,m} = \iota_{k,m} \circ \rho(g), \quad k, m \in \mathbb{N}. \tag{3.1}$$

*The $G$-action on $E_m \otimes E_n$ is given by $g(\xi \otimes \eta) := g(\xi) \otimes g(\eta)$.*

Property (3.1) is called $G$-equivariance.

Once we have a $G$-subproduct system, then its Fock space inherits the group action by $g(\{\xi_m\}_{m=0}^{\infty}) := \{g(\xi_m)\}_{m=0}^{\infty}$.

Furthermore, thanks to the following Lemma, the group action on a subproduct system induces an action on its Toeplitz algebra.

**Lemma 3.10** ([1, Lemma 1.6])**.** *Let $G$ be a locally compact group and suppose that $(E, \iota)$ is a $G$-subproduct system. Then the assignment $g(T_\xi) := T_g(\xi)$ defines a strongly continuous action of $G$ on the Toeplitz algebra $\mathbb{T}_E$.*

As Cuntz–Pimsner algebra is defined as a quotient algebra of the Toeplitz $\mathbb{T}_E$, the action on $\mathbb{T}_E$ descends to an action of $G$ on $\mathbb{O}_E$. Indeed, this follows immediately, since $g(\mathbb{K}_F) \subset \mathbb{K}_F$ for all $g \in G$.

**Example 3.11.** *Consider the Lie group $U(1)$ and the subproduct system defined in Example 3.2. We define a $U(1)$-action $\rho$ on $\{H^{\otimes m} : m \in N\}$ by multiplication:*

$$\rho(z)(\xi_m) := z \cdot \xi_m, \quad \forall \xi_m \in H^{\otimes m}.$$

*Clearly, this action is $U(1)$-equivariant as structure maps are linear.*

## 3.2 Subproduct system from $SU(2)$-representations

Let $\tau : SU(2) \to B(H)$ be a strongly continuous representation of the Lie group $SU(2)$ on a finite-dimensional Hilbert space $H$. Since $SU(2)$ is compact, then we shall furthermore assume that $\tau$ is unitary. In this section, we will construct a subproduct system starting from a representation of $SU(2)$.

We first define the *determinant* of an $SU(2)$-representation.

**Definition 3.12.** *Let $(\tau, H)$ be a representation of $SU(2)$. The determinant of $\tau$ is the subspace of $H \otimes H$ whose elements are fixed under the diagonal action:*

$$\det(\tau) := \{\xi \in H \otimes H : \tau(g) \otimes \tau(g)(\xi) = \xi\}.$$

For each $m \in \{2, 3, \dots\}$ and each $i \in \{1, 2, \dots, m-1\}$ we define a strongly continuous unitary representation $\Delta_m(i) := 1^{\otimes(i-1)} \otimes (\tau^{\otimes 2}) \otimes 1^{\otimes(m-i-1)} : SU(2) \to U(H^{\otimes m})$. We denote the subspace of $H^{\otimes m}$ invariants under $\Delta_m(i)$ by $K_m(i)$. Finally we define $K_m := \sum_{i=1}^{m-1} K_m(i)$.

**Remark 3.13.** *Note that we have $K_m = K_2 \otimes H^{\otimes(m-2)} + H \otimes K_2 \otimes H^{\otimes(m-3)} + \cdots + H^{\otimes(m-2)} \otimes K_2$.*

We are ready to construct a subproduct system associated to the $SU(2)$-representation $(H, \tau)$. Set

$$E_m(\tau, H) = \begin{cases} K_m^{\perp} & , m \geq 2 \\ H & , m = 1 \\ \mathbb{C} & , m = 0 \end{cases}. \tag{3.2}$$

The following Lemma shows that the diagonal representation on $H^{\otimes m}$ restricted to $E_m$ is still strongly continuous.

**Lemma 3.14** ([1, Lemma 2.2])**.** *Let $m \in \{2, 3, \dots\}$. The diagonal representation $\tau^{\otimes m} : SU(2) \to U(H^{\otimes m})$ restricts to a strongly continuous unitary representation of $SU(2)$ on the subspace $E_m \subset H^{\otimes}$.*

Finally we let $\iota_{k,m} : E_{k+m} \to E_k \otimes E_m$ be the map obtained from identification $H^{\otimes(k+m)} \cong H^{\otimes k} \otimes H^{\otimes m}$.

**Proposition 3.15.** *The pair $(E, \iota)$ defined above is a subproduct system.*

*Proof.* Firstly, the case $k = 0$ holds as $E_0 = \mathbb{C}$ and $\mathbb{C} \otimes E_m \cong E_m$. Now let $k = 1$, we have $E_k \otimes E_m = H \otimes E_m$ which is trivially a subspace of $H \otimes E_m$ itself.

Let $k, m \in \mathbb{N}$ with $k, m \geq 2$, it is sufficient to show that $E_{k+m} \subset E_k \otimes E_m$ which is equivalent to showing that $K_{k+m}^{\perp} \subset K_k^{\perp} \otimes K_m^{\perp}$. First assume that $k, m \geq 2$. By definition, we have on the one hand

$$K_k \otimes H^{\otimes m} + H^{\otimes k} \otimes K_m \subset K_{k+m}$$

On the other hand, we have

$$(K_k^{\perp} \otimes K_m^{\perp})^{\perp} = K_k \otimes H^{\otimes m} + H^{\otimes k} \otimes K_m$$

which implies that $K_{k+m}^{\perp} \subset K_k^{\perp} \otimes K_m^{\perp}$, as desired.

□

### A Case Study: the standard Representation

**Example 3.16.** *We are now going to describe the subproduct system induced by the standard representation (or fundemental representation) $\rho : SU(2) \to U(\mathbb{C}^2)$.*

*Let $f_0, f_1$ denote the standard orthonormal basis for $\mathbb{C}^2$, we claim that*

$$\det\Big(\rho, \mathbb{C}^2\Big) = span_{\mathbb{C}}\{f_0 \otimes f_1 - f_1 \otimes f_0.\}$$

*Assume that $v = a_{00} f_0 \otimes f_0 + a_{01} f_0 \otimes f_1 + a_{10} f_1 \otimes f_0 + a_{11} f_1 \otimes f_1 \in \det(\rho, \mathbb{C}^2)$. Then, by the definition of determinant, we have*

$$(\rho(X) \otimes \rho(X))(v) = v, \forall X \in SU(2),$$

*which holds if and only*

$$\begin{cases} a_{00} = a_{00}x^2 + a_{01}xy + a_{10}xy + a_{11}y^2 \\ a_{01} = -a_{00}x\overline{y} + a_{01}|x|^2 - a_{10}y^2 + a_{11}\overline{x}y \\ a_{10} = -a_{00}x\overline{y} + a_{01}|y|^2 + a_{10}|x|^2 + a_{11}\overline{x}y \\ a_{11} = a_{00}\overline{y}^2 - a_{01}\overline{xy} - a_{10}\overline{xy} + a_{11}|x|^2 \end{cases},$$

*where we write* $\rho(X) = \begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}$ *with* $|x|^2 + |y|^2 = 1$. *By using the fact that* $\det(\rho(X)) = 1$, *we have that the solution space is one dimensional and it is spanned by* $f_0 \otimes f_1 - f_1 \otimes f_0$.

*Therefore we have that* $K_m(i) = (\mathbb{C}^2)^{\otimes(i-1)} \otimes \text{span}_{\mathbb{C}}\{f_0 \otimes f_1 - f_1 \otimes f_0\} \otimes (\mathbb{C}^2)^{\otimes(m-i-1)}$ *for all* $m \in \{2, 3, \dots\}$ *and* $i \in \{1, 2, \dots, m-1\}$.

*Recall from Theorem A.18 and Example A.29 that every n-dimensional irreducible representation* $(\phi, V_n)$ *with* $\dim(V_n) = n + 1$ *is isomorphic to*

$$(\rho^{\otimes n+1}, (\mathbb{C}^2)^{\otimes_s n}).$$

*For simplicity, we denote* $(\rho^{\otimes n+1}, (\mathbb{C}^2)^{\otimes_s n})$ *by* $(\rho_n, (\mathbb{C}^2)^{\otimes_s n})$. *As a consequence of Clebsch–Gordan theory and properties of symmetric subproduct system (Example 3.3), we have*

$$E_m(\rho, \mathbb{C}^2) = (\mathbb{C}^2)^{\otimes_s m}.$$

*Let* $p_m$ *be the orthogonal projection from* $(\mathbb{C}^2)^{\otimes m}$ *onto the symmetric tensor product* $(\mathbb{C}^2)^{\otimes_s n} \subset (\mathbb{C}^2)^{\otimes m}$. *We define the vectors*

$$f_0^k f_1^{m-k} := p_m(f_0^{\otimes k} \otimes f_1^{\otimes m-k}).$$

*Then the set* $\{f_0^k f_1^{m-k} : k = 0, 1, \dots, m\}$ *forms an orthogonal basis for* $E_m(\rho, \mathbb{C}^2)$ *with norm satisfying*

$$\|f_0^k f_1^{m-k}\|^2 = \frac{k!(m-k)!}{m!}.$$

Define the number operator $N : \text{Dom}(N) \to F(\rho, \mathbb{C}^2)$ by sending $\xi$ to $m \cdot \xi$ whenever $\xi \in E_m$. It is clear that the number operator is unbounded and self-adjoint. Then we have the following famous result.

**Theorem 3.17** ([2, Proposition 5.3]). *The Toeplitz algebra* $\mathbb{T}(\rho, \mathbb{C}^2)$ *induced by the standard representation is the universal* $C^*$*-algebra generated by two operators* $T_0 := T_{f_0}$ *and* $T_1 := T_{f_1}$ *which satisfy the following commutation relations:*

1. $T_0 T_1 = T_1 T_0$,

2. $T_0^*T_0 + T_1^*T_1 = (2+N)(1+N)^{-1}$,

3. $T_i^*T_j^* - T_jT_i = (1+N)^{-1}(\delta_{i,j}1 - T_jT_i^*)$

In this thesis, we plan to extend this result to representations other than the fundamental one.

Moreover, as proved in Theorem A.30, we can identify the symmetric tensors and homogeneous polynomials. This yields an isomorphism between the Fock space $F(\rho, \mathbb{C}^2)$ with the Drury–Arveson space $H_2^2$.

**Theorem 3.18** ([2, Theorem 5.7]). *The Toeplitz algebra $\mathbb{T}(\rho, \mathbb{C}^2)$ contains the algebra of compact operators on the Drury–Arveson space $H_2^2$, and we have an exact sequence of $C^*$-algebras*

$$0 \longrightarrow \mathbb{K}(H_2^2) \longrightarrow \mathbb{T}(\rho, \mathbb{C}^2) \longrightarrow C(S^3) \longrightarrow 0,$$

*where $C(S^3)$ is the commutative $C^*$-algebra of continuous functions on the 3-sphere $S^3 \subset \mathbb{C}^2$. In particular, we have that the Cuntz–Pimsner algebra $\mathbb{O}(\rho, \mathbb{C}^2)$ is isomorphic to $C(S^3)$.*

### 3.2.1 The structure of the determinant

In this section we shall study the structure of determinant. Let $L_n = (\mathbb{C}^2)^{\otimes_S n}$ be the representation space of an irreducible $SU(2)$ representation. Consider the orthonormal basis defined by

$$e_k := \frac{n!}{k!(n-k)!} \cdot f_0^k f_1^{n-k} \in L_n, \quad k = 0, 1, \ldots, n. \tag{3.3}$$

**Proposition 3.19** ([1, Lemma 2.8]). *Suppose that $\tau : SU(2) \to U(H)$ is irreducible and let $V : L_n \to H$ be a unitary operator intertwining $\tau$ with $\rho_n$. Then the determinant* $\det(\tau, H) \subset H \otimes H$ *is an one-dimensional vector space spanned by the vector*

$$(V \otimes V)(n+1)^{\frac{1}{2}} \sum_{k=0}^{n} (-1)^{n-k} e_k \otimes e_{n-k}.$$

Furthermore, we have the following Proposition describing the determinant in the case of a reducible representation.

**Proposition 3.20** ([1, Proposition 2.10]). *Let $H \cong \sum_{m=0}^{\infty} L_m^{\oplus k_m}$. Then we have* $\det(\tau, H) \subset H \otimes H$ *has dimension $\sum_{m=0}^{\infty} k_m^2$ and it is unitarily isomorphic to the Hilbert space*

$$\bigoplus_{m=0}^{\infty} \det(\rho_m, L_m)^{\oplus k_m^2} \subset \bigoplus_{m=0}^{\infty} (L_m \otimes L_m)^{\oplus k_m^2}$$

*Proof.* Let us compute the tensor product $H \otimes H$ using the isomorphism $H \cong \sum_{m=0}^{\infty} L_m^{\oplus k_m}$:

$$H \otimes H \cong (\sum_{m=0}^{\infty} L_m^{\oplus k_m}) \otimes (\sum_{m=0}^{\infty} L_m^{\oplus k_m}) \cong \bigoplus_{s,l=0}^{m} (L_s \otimes L_l)^{\oplus k_s \cdot k_l}.$$

Then by Theorem A.33 we have that $L_s \otimes L_l$ contains some copies of the trivial representation if and only if $s = l$. This leads to

$$\det(\tau, H) \cong \det\left( \oplus_{m=0}^{\infty} \rho_m^{\oplus k_m}, \sum_{m=0}^{\infty} L_m^{\oplus k_m} \right) \cong \bigoplus_{m=0}^{\infty} \det(\rho_m, L_m)^{\oplus k_m^2}.$$

as desired. □

Chapter 4

# Fusion Rules for $SU(2)$-subproduct Systems

*I used to say: "Everything is Representation Theory". Now I say: "Nothing is Representation Theory".*

*Israel Gelfand*

## 4.1 Irreducible case

In this section, we shall recall and describe the fusion rules for the fibers of the subproduct system induced by an irreducible $SU(2)$-representation. Our main reference for this section is [1]. Our main goal is to prove the following:

**Theorem 4.1** (Fusion rules, [1, Theorem 3.1])**.** *Let $k, m \in \mathbb{N}_0$ and put $l := \min\{k, m\}$. We have an $SU(2)$-equivariant unitary isomorphism*

$$W_{k,m} : \bigoplus_{j=0}^{l} E_{k+m-2j} \to E_k \otimes E_m.$$

Let $(\rho, H)$ be an irreducible unitary representation of $SU(2)$ on a finite-dimensional Hilbert space. We define two linear operators $G_m : E_{m-1} \to$

$K_{m+1}$ and $G_m^{'} : E_{m-1} \to K_{m+1}$ recursively:

$$G_1(1) := \delta, \quad G_m := G_{m-1} \otimes 1 + (-1)^{(n+1)(m-1)} d_{m-1} \cdot 1_{m-1} \otimes G_1,$$
$$G_1^{'}(1) := \delta, \quad G_m^{'} := 1 \otimes G_{m-1}^{'} + (-1)^{(n+1)(m-1)} d_{m-1} \cdot G_1^{'} \otimes 1_{m-1},$$

where $\delta$ is such that $\mathbb{C} \cdot \delta = \det(\rho, H)$. The reasons behind this definition will become apparent later.

**Lemma 4.2.** *Let $m \in \mathbb{N}$. The linear maps $G_m$ and $G_m^{'}$ are equivariant i.e.*

$$\rho_n^{\otimes(m+1)}(g) G_m = G_m \rho_n^{\otimes(m-1)}(g), \quad \forall g \in SU(2), \tag{4.1}$$
$$\rho_n^{\otimes(m+1)}(g) G_m^{'} = G_m^{'} \rho_n^{\otimes(m-1)}(g), \quad \forall g \in SU(2). \tag{4.2}$$

*Proof.* We only give the proof of (4.1) since the proof of (4.2) works similarly. We prove it by induction on $m \in \mathbb{N}$. For $m = 1$, we have

$$\rho_n^{\otimes(2)}(g) G_1(1) = \rho_n^{\otimes(2)}(g)(\delta) = \delta = G_1(1) \rho_n^{\otimes 0}(g), \quad \forall g \in SU(2),$$

where the second equality is due to the definition of $\delta$. Now assume the (4.1) holds for $m \leq k$. Then we have for $\xi \in E_k$

$$\begin{aligned} &\rho_n^{\otimes(k+2)}(g) G_{k+1}(\xi) \\ &= \rho_n^{\otimes(k+2)}(g)((G_k \otimes 1)(\xi) + (-1)^{(n+1)k} d_k \cdot \xi \otimes \delta) \\ &= (G_k \otimes 1) \rho_n^{\otimes k}(g)(\xi) + (-1)^{(n+1)k} d_k \cdot \rho_n^{\otimes k}(g)(\xi) \otimes \delta \\ &= G_{k+1} \rho_n^{\otimes k}(g)(\xi), \end{aligned}$$

which completes the proof. □

**Lemma 4.3** ([1, Lemma 3.6, Lemma 3.12])**.** *Let $m \in \mathbb{N}$. The linear map $G_m$ satisfies the following identities.*

1. $\langle (G_m \otimes 1)(\xi), \eta \otimes \delta \rangle = (-1)^{(n+1)m+1} d_{m-1}/d_1 \cdot \langle \xi, \eta \rangle, \quad \forall \xi \in E_{m-1} \otimes E_1, \eta \in E_m$ ;
2. $\langle G_m(\xi), G_m(\eta) \rangle = \mu_m \cdot \langle \xi, \eta \rangle, \quad \forall \xi, \eta \in E_{m-1}$ ;
3. $\langle (G_m \otimes 1)(\xi), G_{m+1}(\eta) \rangle = 0, \quad \forall \in \xi \in E_{m-1} \otimes E_1, \eta \in E_m$ .

*Similiarly, we have the following three identities for $G_m^{'}$:*

1. $\langle (1 \otimes G_m^{'})(\xi), \eta \otimes \delta \rangle = (-1)^{(n+1)m+1} d_{m-1}/d_1 \cdot \langle \xi, \eta \rangle, \quad \forall \xi \in E_{m-1} \otimes E_1, \eta \in E_m$ ;

2. $\langle G_m^{'}(\xi), G_m^{'}(\eta)\rangle = \mu_m \cdot \langle \xi, \eta\rangle, \quad \forall \xi, \eta \in E_{m-1}$ ;

3. $\langle (1 \otimes G_m^{'})(\xi), G_{m+1}^{;}(\eta)\rangle = 0, \quad \forall \in \xi \in E_{m-1} \otimes E_1, \eta \in E_m$ .

**Lemma 4.4** ([1, Lemma 3.7, Lemma 3.13]). *There are unitary isomorphisms of Hilbert spaces*

$$(K_m \otimes E_1) \oplus G_m(E_{m-1}) \cong K_{m+1}, \quad \forall m \geq 1,$$
$$(E_1 \otimes K_m) \oplus G_m^{'}(E_{m-1}) \cong K_{m+1}, \quad \forall m \geq 1.$$

Note that the image of $G_m$ is equal to $K_{m+1} \cap (E_m \otimes E_1)$ and the image of $G_m^{'}$ is equal to $K_{m+1} \cap (E_1 \otimes E_m)$. Combining this observation with Lemma 4.3, we obtain that the following two maps are isometries

$$V_m := \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} \cdot G_m : E_{m-1} \to E_m \otimes E_1 \tag{4.3}$$
$$V_m^{'} := \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} \cdot G_m^{'} : E_{m-1} \to E_1 \otimes E_m. \tag{4.4}$$

**Lemma 4.5.** *Let $\{d_m\}$ be the sequence of positive integers defined recursively by the following:*

$$d_{-1} := 0, \quad d_0 := 1, \quad d_1 := n+1, \quad d_m := d_1 d_{m-1} - d_{m-2},\ m \geq 2.$$

*Then the sequence of fractions $\{d_{m-1}/d_m\}_{m=0}^{\infty}$ is strictly increasing and converges to the limit $\gamma_n := (n+1-\sqrt{(n+1)^2-4})/2$.*

*Proof.* We first note that $d_m > d_{m-1}$ for $m \in \mathbb{N}$ since $d_m - d_{m-1} = (d_1 - 1)d_{m-1} - d_{m-2} = (n-1)d_{m-1} + d_{m-1} - d_{m-2}/d_{m-1}$. Indeed, we have $d_0 > d_{-1}$ and $d_m \geq 0, \forall m \in \mathbb{N}$ then $d_m > d_{m-1}$ follows from an induction argument.

By using Lemma B.2 we have that

$$\frac{d_{m-1}}{d_m} = \sum_{j=1}^{m}(\frac{d_{j-1}}{d_j} - \frac{d_{j-2}}{d_{j-1}}) = \sum_{j=1}^{m} \frac{d_{j-1}^2 - d_{j-2}d_{j-1}}{d_j d_{j-2}} = \sum_{j=1}^{m} \frac{1}{d_j d_{j-1}}.$$

which yields that $d_m/d_{m+1} - d_{m-1}/d_m = 1/d_m d_{m-1} > 0$, thus $\{d_m/d_{m+1}\}$ is strictly increasing.

To compute the limit, we observe that if we take the limit of $d_m/d_{m-1} = n+1-d_{m-2}/d_{m-1}$ on both sides, we have

$$\lim_{m\to\infty} d_m/d_{m-1} = n+1-\lim_{m\to\infty} d_{m-2}/d_{m-1} \implies \frac{1}{\gamma_n} = n+1-\gamma_n.$$

which gives that $\gamma_n = (n+1-\sqrt{(n+1)^2-4})/2 \in (0,1]$. □

We claim that the sequence $\{d_m\}$ agrees with the sequence of dimensions of the fibers $E_m$.

**Corollary 4.6.** *It holds that* $\dim(E_m) = d_m$ *for all* $m \in \mathbb{N}_0$.

*Proof.* By Lemma 4.4 we have that

$$\begin{aligned}\dim(E_{m+1}) &= (n+1)^{m+1} - \dim(K_{m+1})\\ &= (n+1)^{m+1} - \dim(K_m \otimes E_1) - \dim(G_m(E_{m-1}))\\ &= (n+1)^{m+1} - (n+1)\dim(K_m) - \dim(E_{m-1})\\ &= (n+1)\dim(E_m) - \dim(E_{m-1}).\end{aligned}$$

where the last equality uses the fact that $\dim(E_m) = (n+1)^m - \dim(K_m)$. Finally the fact that $\dim(E_0) = 1$ and $\dim(E_1) = n+1$ completes the proof. □

**Proposition 4.7.** *Let* $m \in \mathbb{N}$. *The linear maps*

$$\begin{aligned}(\iota_{m,1}, V_m) &: E_{m+1} \oplus E_{m-1} \to E_m \otimes E_1\\ (\iota_{1,m}, V_m') &: E_{m+1} \oplus E_{m-1} \to E_1 \otimes E_m,\end{aligned}$$

*are $SU(2)$-equivariant unitary isomorphisms.*

*Proof.* We only prove the first isomorphism, the second one can be proved similarly. The map $(\iota_{m,1}, V_m)$ is clearly a linear isometry. By a dimension-counting argument, we have that $(\iota_{m,1}, V_m)$ is an isomorphism. Indeed, we have

$$\begin{aligned}\dim(E_{m+1} \oplus E_{m-1}) &= \dim(E_{m+1}) + \dim(E_{m-1})\\ &= d_{m+1} + d_{m-1}\\ &= d_1 d_m\\ &= \dim(E_1) \cdot \dim(E_m)\\ &= \dim(E_1 \otimes E_m)\end{aligned}$$

which completes the proof. □

Since the image of $G_m$ and $G_m^{'}$ lie in the intersections $K_{m+1} \cap (E_m \otimes E_1)$ and $K_{m+1} \cap (E_1 \otimes E_m)$, respectively, we may define the following maps

$$\widehat{G}_m := (J_m^* \otimes 1)G_m \quad \text{and} \quad \widehat{G}_m^{'} := (1 \otimes J_m^*)G_m, \tag{4.5}$$

where $J_m$ is the inclusion from $E_m$ into $E_1^{\otimes m}$, given by

$$J_m = (\iota_{1,1} \otimes 1^{\otimes m-2}) \dots (\iota_{m-2,1} \otimes 1)(\iota_{m-1,1}) : E_m \hookrightarrow E_1^{\otimes m},$$

which step by step isometrically sends

$$E_{m-k} \otimes E_1^{\otimes k} \quad \text{to} \quad E_{m-k-1} \otimes E_1^{\otimes k+1},$$

for $k = 0, 1, \dots, m$.

**Lemma 4.8.** *Let $m \in \mathbb{N}$. We have that the maps $\widehat{G}_m$ and $\widehat{G}_m^{'}$ defined in* (4.5) *satsify*

$$(\iota_{m-1,1} \otimes 1)\widehat{G}_m = (\widehat{G}_{m-1} \otimes 1)\iota_{m-2,1} + (-1)^{(n+1)(m-1)} d_{m-1} 1_{m-1} \otimes \widehat{G}_1,$$
$$(1 \otimes \iota_{1,m-1})\widehat{G}_m^{'} = (1 \otimes \widehat{G}_{m-1}^{'})\iota_{1,m-2} + (-1)^{(n+1)(m-1)} d_{m-1} \widehat{G}_1^{'} \otimes 1_{m-1}.$$

*Proof.* We shall prove the claims via induction. For $m = 1$ we have that

$$\begin{aligned}(\iota_{0,1} \otimes 1)\widehat{G}_1 &= (\iota_{0,1} \otimes 1)(J_1^* \otimes 1)G_1 \\ &= \delta \\ &= (\widehat{G}_0 \otimes 1)\iota_{m-2,1} + (-1)^{(n+1)(1-1)} d_{1-1} 1_{1-1} \otimes \widehat{G}_1,\end{aligned}$$

holds trivially. Then suppose the claim holds for $m \leq l$. For $m = l + 1$ we have

$$\begin{aligned}&(\iota_{l,1} \otimes 1)\widehat{G}_{l+1} \\ &= \iota_{l,1}^* \dots (\iota_{1,1}^* \otimes 1^{\otimes l-1})G_{l+1} \\ &= \iota_{l,1}^* \dots (\iota_{1,1}^* \otimes 1^{\otimes l-1})(G_l \otimes 1 + (-1)^{(n+1)l} d_l \cdot 1_l \otimes G_1) \\ &= (\widehat{G}_l \otimes 1)\iota_{l-1,1} + (-1)^{(n+1)l} d_l \cdot 1_l \otimes \widehat{G}_1.\end{aligned}$$

where the last equality uses the definition of $G_m$ and the induction hypothesis. The proof for $\widehat{G}_m^{'}$ works similarly. □

Before proving our main theorem, we need some further lemmas.

**Lemma 4.9** ([1, Lemma 3.15]). *Let $m \in \mathbb{N}$. It holds that*

$$\widehat{G}_m = (-1)^{(n+1)(m-1)} d_{m-1} \cdot (\iota^*_{m-1,1} \otimes 1)(1_{m-1} \otimes \widehat{G}_1),$$
$$\widehat{G}'_m = (-1)^{(n+1)(m-1)} d_{m-1} \cdot (1 \otimes \iota^*_{1,m-1})(\widehat{G}_1 \otimes 1_{m-1}).$$

**Lemma 4.10** ([1, Lemma 3.16]). *Let $m \in \mathbb{N}$. It holds that*

$$\iota^*_{m-1,1} = (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (1_m \otimes \widehat{G}^*_1)(\widehat{G}_m \otimes 1) : E_{m-1} \otimes E_1 \to E_m,$$
$$\iota^*_{1,m-1} = (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot ((\widehat{G}'_1)^* \otimes 1_m)(1 \otimes \widehat{G}'_m) : E_1 \otimes E_{m-1} \to E_m.$$

**Lemma 4.11** ([1, Lemma 3.17]). *Let $m \in \mathbb{N}$. It holds that*

$$p_{m-1,1} = 1_{m-1} \otimes 1 + (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} \cdot (\widehat{G}_{m-1} \otimes \widehat{G}^*_1)(\iota_{m-2,1} \otimes 1),$$
$$p_{1,m-1} = 1 \otimes 1_{m-1} + (-1)^{(n+1)m+1} \frac{d_1}{d_{m-1}} (\widehat{G}^*_1 \otimes \widehat{G}'_{m-1})(1 \otimes \iota_{1,m-2}).$$

*where $p_{m-1,1} : E_{m-1} \otimes E_1 \to E_{m-1} \otimes E_1$ and $p_{1,m-1} : E_1 \otimes E_{m-1} \to E_1 \otimes E_{m-1}$.*

Now we shall introduce an essential $SU(2)$-equivariant linear map which plays an important role in the proof of the fusion rules. For $k, m \in \mathbb{N}_0$, we define the map by: $\sigma_{k,m} : E_k \otimes E_m \to E_{k+1} \otimes E_{m+1}$

$$\sigma_{k,m} := (1_{k+1} \otimes \iota^*_{1,m})(\widehat{G}_{k+1} \otimes 1_m).$$

Combining Lemma 4.9, Lemma 4.10, and Lemma 4.11, we obtain the following proposition. For the sake of simplicity, we omit the subscripts when no confusion could arise.

**Proposition 4.12** ([1, Proposition 3.18,Proposition 3.19,Proposition 3.20]). *Let $k, m \in \mathbb{N}_0$ and and $j \in \mathbb{N}$. We have the following identities*

1. $\sigma^*_{k,m}\sigma_{k,m} = \frac{d_k d_{k+m+1}}{d_1 d_m} 1_k \otimes 1_m + \frac{d_k d_{m-1}}{d_{k-1} d_m} \sigma_{k-1,m-1}\sigma^*_{k-1,m-1} :$ $E_k \otimes E_m \to E_k \otimes E_m$;
2. $\sigma^*\sigma^j = \mu_{k+j} \cdot (1 - \frac{d_k d_{m-1}}{d_{k+j} d_{m+j+1}} \sigma^{j-1}) + \frac{d_{m-1} d_{k+j-1}}{d_{k-1} d_{m+j-1}} \sigma^j \sigma^* :$ $E_k \otimes E_m \to E_{k+j-1} \otimes E_{m+j-1}$;

3. $\sigma^* \iota_{k,m} = 0$;

4. $(\sigma^*)^j \sigma^j \iota_{k,m} = \prod_{i=1}^{j} \mu_{k+i}(1 - \frac{d_k d_{m-1}}{d_{k+i} d_{m+i-1}})\iota_{k,m}$.

*where we use the notation* $\sigma^j := \sigma_{k+j,m+j} \dots \sigma_{k+1,m+1}\sigma_{k,m}$, *and* $\sigma^* \iota_{k,m} : E_{k+m} \to E_{k-1} \otimes E_{m-1}$.

Finally, we can prove the main Theorem of this section.

**Theorem 4.13.** *Let* $k, m \in \mathbb{N}_0$ *and put* $l := \min\{k, m\}$. *We have an* $SU(2)$-*equivariant unitary isomorphism*

$$W_{k,m} = (W^0_{k,m} W^1_{k,m} \dots W^l_{k,m}) : \bigoplus_{j=0}^{l} E_{k+m-2j} \to E_k \otimes E_m.$$

*where* $W^j_{k,m} : E_{k+m-2j} \to E_k \otimes E_m$ *is defined by*

$$W^j_{k,m} = \prod_{i=1}^{j} \frac{1}{\sqrt{\mu_{k-j+i}}} \left(1 - \frac{d_{k-j} d_{m-j-1}}{d_{k-j+i} d_{m-j+i-1}}\right)^{-\frac{1}{2}} \cdot \sigma^j \iota_{k-j,m-j},$$

*for all* $j \in \{1, \dots, l\}$, *and* $W^0_{k,m} := \iota_{k,m} : E_{k+m} \to E_k \otimes E_m$.

*Proof.* (of Theorem 4.1) Firstly, the third identity in Lemma 4.12 implies that $(W^i_{k,m})^* W^j_{k,m} = 0$ whenever $i \neq j$. Secondly, the last identity in Lemma 4.12 implies that

$$(W^j_{k,m})^* W^j_{k,m} = \iota^*_{k,m} (\sigma^*)^j \sigma^j \iota_{k,m} = \prod_{i=1}^{j} \mu_{k+i}(1 - \frac{d_k d_{m-1}}{d_{k+i} d_{m+i-1}}) \iota^*_{k,m} \iota_{k,m}.$$

where $\iota_{k,m}$ is an isometry thus $W^j_{k,m}$ is an isometry. Therefore we have $W^j_{k,m}$ is an isometry and thus $W_{k,m} : \oplus_{j=0}^{l} E_{k+m-2j} \to E_k \otimes E_m$ is an isometry.

Surjectivity of $W_{k,m}$ follows from a dimension counting argument by (B.3).

Finally, $SU(2)$-equivariance follows from the definition of $W_{k,m}$ in which all factors are $SU(2)$-equivariant. □

## 4.2 Several copies of the same irreducible representations

By Theorem A.25, we know that the matrix Lie Group of $SU(2)$ is completely reducible, thus any representation of $SU(2)$ is a direct sum of irreducible ones. In this section, we will study the fusion rules for reducible case.

Let us consider the case when the representation space $H$ is isomorphic to the $t$ copies of $L_n$ i.e., $H \cong L_n^{\oplus t}$ and $\tau = \rho_n^{\oplus t}$.

Let $\widetilde{K}_2$ denote the $\det(\rho_n^{\oplus t}, L_n^{\oplus t})$. By Proposition 3.20, we have that the dimension of $\widetilde{K}_2$ is $t^2$. We set

$$\widetilde{K}_m = \widetilde{K}_2 \otimes H^{\otimes m-2} + H \otimes \widetilde{K_2} \otimes H^{\otimes m-3} + \cdots + H^{\otimes m-2} \otimes \widetilde{K}_2.$$

We define each $\widetilde{E}_m$ as in (3.2):

$$\widetilde{E}_m(\tau, H) = \widetilde{E}_m = \begin{cases} \widetilde{K}_m^{\perp} & , m \geq 2 \\ H & , m = 1 \\ \mathbb{C} & , m = 0 \end{cases}.$$

We have the following lemma.

**Lemma 4.14.** *For the subproduct system we constructed above, we have the recurrence formula for $\widetilde{E}_m$ when $m \geq 3$:*

$$\widetilde{E}_m = \widetilde{E}_1 \otimes \widetilde{E}_{m-1} \cap \widetilde{E}_{m-1} \otimes \widetilde{E}_1 \tag{4.6}$$

$$= H \otimes \widetilde{E}_{m-1} \cap \widetilde{E}_{m-1} \otimes H \tag{4.7}$$

*Proof.* Observe that $K_m = H \otimes K_{m-1} + K_{m-1} \otimes H$ then we have

$$\begin{aligned} \widetilde{E}_m &= \widetilde{K}_m^{\perp} \\ &= (H \otimes \widetilde{K}_{m-1} + \widetilde{K}_{m-1} \otimes H)^{\perp} \\ &= (\widetilde{E}_1 \otimes \widetilde{K}_{m-1})^{\perp} \cap (\widetilde{K}_{m-1} \otimes \widetilde{E}_1)^{\perp} \\ &= \widetilde{E}_1 \otimes \widetilde{E}_{m-1} \cap \widetilde{E}_{m-1} \otimes \widetilde{E}_1. \end{aligned}$$

where the third equality is due to the fact that $(V_1 + V_2)^{\perp} = V_1^{\perp} \cap V_2^{\perp}$. □

We claim:

$$\begin{aligned}\widetilde{K}_m &= \widetilde{K}_2 \otimes H^{\otimes m-2} + H \otimes \widetilde{K_2} \otimes H^{\otimes m-3} + \cdots + H^{\otimes m-2} \otimes \widetilde{K}_2 \\ &= K_2^{\oplus t^2} \otimes H^{\otimes m-2} + \cdots + H^{\otimes m-2} \otimes K_2^{\oplus t^2} \\ &= K_2^{\oplus t^2} \otimes (L_n^{\otimes m-2})^{\oplus t^{m-2}} + \cdots + (L_n^{\otimes m-2})^{\oplus t^{m-2}} \otimes K_2^{\oplus t^2} \\ &= K_m^{\oplus t^m}.\end{aligned}$$

Therefore we have

$$\widetilde{E}_m = \widetilde{K}_m^{\perp} = (K_m^{\oplus t^m})^{\perp} = (K_m^{\perp})^{\oplus t^m} = E_m^{\oplus t^m}. \tag{4.8}$$

Indeed, we have $\widetilde{E}_1 = L_n^{\oplus t} = E_1^{\oplus t}$. Then assume (4.8) holds for $\widetilde{E}_k$ for all $k \leq m-1$, we have for $\widetilde{E}_m$:

$$\begin{aligned}\widetilde{E}_m &= \widetilde{E}_1 \otimes \widetilde{E}_{m-1} \cap \widetilde{E}_{m-1} \otimes \widetilde{E}_1 \\ &\cong (E_1^{\oplus t} \otimes E_{m-1}^{\oplus t^{m-1}}) \cap (E_{m-1}^{\oplus t^{m-1}} \otimes E_1^{\oplus t}) \\ &\cong (E_1 \otimes E_{m-1})^{\oplus t^m} \cap (E_{m-1} \otimes E_1)^{\oplus t^m} \\ &\cong E_m^{\oplus t^m}.\end{aligned}$$

which also proves the isomorphism (4.8).

By using the fusion rules for the irreducible case (Theorem 4.1), we have that

$$\widetilde{E}_k \otimes \widetilde{E}_l \cong E_k^{\oplus t^k} \otimes E_l^{\oplus t^l} \cong (E_{k+l} \oplus E_{k+l-2} \oplus \cdots \oplus E_{|k-l|})^{\oplus t^{k+l}}. \tag{4.9}$$

To sum up, we have the fusion rules for the $t$ copies case.

**Theorem 4.15** (Fusion rules for $H \cong L_n^{\oplus t}$)**.** *Let $k, l \in \mathbb{N}$ we have $\widetilde{E}_m \cong E_m^{\oplus t^m}$, and there exists an $SU(2)$-equivariant unitary isomorphism*

$$\widetilde{E}_k \otimes \widetilde{E}_l \cong \widetilde{E}_{k+l} \oplus \widetilde{E}_{k+l-2}^{\oplus t^2} \oplus \cdots \oplus \widetilde{E}_{|k-l|}^{\oplus t^{k+l-|k-l|}} \tag{4.10}$$

*Proof.* By (4.9) we have $\widetilde{E}_k \otimes \widetilde{E}_l \cong (E_{k+l} \oplus E_{k+l-2} \oplus \cdots \oplus E_{|k-l|})^{\oplus t^{k+l}}$. By definition, we have

$$\begin{aligned}&(E_{k+l} \oplus E_{k+l-2} \oplus \cdots \oplus E_{|k-l|})^{\oplus t^{k+l}} \\ &\cong E_{k+l}^{\oplus t^{k+l}} \oplus E_{k+l-2}^{\oplus t^{k+l}} \oplus \cdots \oplus E_{|k-l|}^{\oplus t^{k+l}} \\ &\cong E_{k+l}^{\oplus t^{k+l}} \oplus E_{k+l-2}^{\oplus t^{k+l-2} \cdot t^2} \oplus \cdots \oplus E_{|k-l|}^{\oplus t^{|k-l|} \cdot t^{k+l-|k-l|}} \\ &\cong \widetilde{E}_{k+l} \oplus \widetilde{E}_{k+l-2}^{\oplus t^2} \oplus \cdots \oplus \widetilde{E}_{|k-l|}^{\oplus t^{k+l-|k-l|}}.\end{aligned}$$

which completes the proof. □

Let us look at a concrete example.

**Example 4.16.** *Let us consider the case $H \cong L_1^{\oplus 2}$. We firstly compute the determinant:*

$$\det\{\xi \in L_1^{\oplus 2} \otimes L_1^{\oplus 2} | \rho(g) \otimes \rho(g)\xi = \xi\}.$$

*First note that we have*

$$\det(\rho, L_1) = span_{\mathbb{C}}\{f_0 \otimes f_1 - f_1 \otimes f_0\},$$

*which is one dimensional from Example 3.16.*

*Then by Proposition 3.20 we have the dimension of determinant of $(\rho^{\oplus 2}, L_1^{\oplus 2})$ is four. Let us define*

$$\begin{cases} \delta_1 := (f_0 \oplus 0) \otimes (f_1 \oplus 0) - (f_1 \oplus 0) \otimes (f_0 \oplus 0) \\ \delta_2 := (f_0 \oplus 0) \otimes (0 \oplus f_1) - (0 \oplus f_1) \otimes (f_0 \oplus 0) \\ \delta_3 := (0 \oplus f_0) \otimes (f_1 \oplus 0) - (f_1 \oplus 0) \otimes (0 \oplus f_0) \\ \delta_4 := (0 \oplus f_0) \otimes (0 \oplus f_1) - (0 \oplus f_1) \otimes (0 \oplus f_0) \end{cases}. \tag{4.11}$$

**Claim 1.** *The determinant of $(\rho^{\oplus 2}, L_1^{\oplus 2})$ is spanned by the elements $\{\delta_i\}_{i=1}^4$ i.e.*

$$\widetilde{K}_2 = \det\left(\rho^{\oplus 2}, L_1^{\oplus 2}\right) = span_{\mathbb{C}}\{\delta_1, \delta_2, \delta_3, \delta_4\}.$$

*where each $\delta_i$ is defined in* (4.11).

*We prove the claim for $\delta_1$. The other cases can be checked in the same way. Let $\rho(g)^{\oplus 2} \otimes \rho(g)^{\oplus 2}$ be the matrix*

$$\rho(g)^{\oplus 2} \otimes \rho(g)^{\oplus 2} = \begin{bmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix} \otimes \begin{bmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix},$$

*then we have*

$$\rho(g)^{\oplus 2} \otimes \rho(g)^{\oplus 2}((f_0 \oplus 0) \otimes (f_1 \oplus 0) - (f_1 \oplus 0) \otimes (f_0 \oplus 0)) \tag{4.12}$$

$$= \begin{bmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix}^{\otimes 2} ((f_0 \oplus 0) \otimes (f_1 \oplus 0) - (f_1 \oplus 0) \otimes (f_0 \oplus 0)). \tag{4.13}$$

*Then by the definition of tensor product of linear transformations, we obtain that* $(\rho(g)^{\oplus 2} \otimes \rho(g)^{\oplus 2})(\delta_1)$ *equals*

$$\begin{aligned} & ab(f_0 \oplus 0) \otimes (f_0 \oplus 0) + |a|^2 (f_0 \oplus 0) \otimes (f_1 \oplus 0) - |b|^2 (f_1 \oplus 0) \otimes (f_0 \oplus 0) \\ & - \overline{ab}(f_1 \oplus 0) \otimes (f_1 \oplus 0) - ab(f_0 \oplus 0) \otimes (f_0 \oplus 0) + \overline{ab}(f_1 \oplus 0) \otimes (f_1 \oplus 0) \\ & + |b|^2 (f_1 \oplus 0) \otimes (f_0 \oplus 0) - |a|^2 (f_0 \oplus 0) \otimes (f_1 \oplus 0), \end{aligned}$$

*where the last equality is due to the fact* $g \in SU(2)$.

*As many terms cancel, the above expression equals*

$$(f_0 \oplus 0) \otimes (f_1 \oplus 0) - (f_1 \oplus 0) \otimes (f_0 \oplus 0),$$

*as desired.*

*Then we have that*

$$\begin{aligned} \widetilde{K}_m &= \widetilde{K}_2 \otimes (L_2^{\oplus 2})^{\otimes m-2} + \cdots + (L_2^{\oplus 2})^{\otimes m-2} \otimes \widetilde{K}_2 \\ &= \widetilde{K}_2 \otimes ((\mathbb{C}^2)^{\oplus 2})^{\otimes m-2} + \cdots + ((\mathbb{C}^2)^{\oplus 2})^{\otimes m-2} \otimes \widetilde{K}_2 \\ &= span\{\delta_i\}_{i=1}^4 \otimes (\mathbb{C}^4)^{\otimes m-2} + \cdots + ((\mathbb{C}^4)^{\otimes m-2} \otimes span\{\delta_i\}_{i=1}^4. \end{aligned}$$

*By using the recurrence formula above (Lemma 4.14 ), we have* $\widetilde{E}_m = H \otimes \widetilde{E}_{m-1} \cap \widetilde{E}_{m-1} \otimes H$. *Then to compute the* $\widetilde{E}_m, m \in \mathbb{N}$, *it is sufficient to compute* $E_2 = \widetilde{K}_2^{\perp}$ *which is the orthonormal complement of the* $span_{\mathbb{C}}\{\delta_1, \delta_2, \delta_3, \delta_4\}$ *in* $\mathbb{C}^2 \otimes \mathbb{C}^2$.

*Moreover, for every* $m$ *we have* $dim(\widetilde{E}_m) = (m+1) \cdot 2^m$ *since* $\widetilde{E}_m \cong E_m^{\oplus 2^m}$ *and* $dim(E_m) = m + 1$.

Chapter 5

# Commutation Relations for the Resulting $C^*$-algebras

## 5.1 Toeplitz algebra

### 5.1.1 Irreducible case

In this section, we consider the commutation relations for the Toeplitz algebra induced by an irreducible $SU(2)$-representation. Fix $n \in \mathbb{N}$ and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$, we let $\{e_n\}$ denote the orthonormal basis for $L_n$ introduced in (3.3).

We define the Toeplitz operators $T_j$ as creation operators over the Fock space $F$:

$$T_j := T_{e_j} : F \to F, \quad j \in \{0, 1, \ldots, n\}.$$

Similarly, we define the bounded operators $T_j' := \iota^*_{m,1}(\xi \otimes e_j)$ for all $\xi \in E_m$. We call $T_j'$ the right creation operator associated to the basis vector $e_j \in E_1 = L_n$. Then we define the $SU(2)$-equivariant bounded operators $\iota_L := \iota_{1,m-1} : F \to E_1 \otimes F$ and $\iota_R := \iota_{m-1,1} : F \to F \otimes E_1$. We set $\iota_L(\xi) = \iota_R(\xi) = 0, \xi \in E_0 = \mathbb{C}$.

**Lemma 5.1.** *We have the identities*

$$\iota_L^* = \sum_{j=0}^{n} \langle e_j, \cdot \rangle \otimes T_j : E_1 \otimes F \to F,$$
$$\iota_R^* = \sum_{j=0}^{n} T_j' \otimes \langle e_j, \cdot \rangle : F \otimes E_1 \to F.$$

*Proof.* Let $\xi \in E_m$ and $e_i \in \{e_0, \dots, e_n\}$ we have

$$\iota_L^*(e_i \otimes \xi) = \iota_{1,m-1}^*(e_i \otimes \xi) = T_i(\xi) = \langle e_i, e_i \rangle T_i(\xi) = \sum_{j=0}^{n} (\langle e_j, \cdot \rangle \otimes T_j)(e_i \otimes \xi),$$

$$\iota_R^*(\xi \otimes e_i) = \iota_{m-1,1}^*(\xi \otimes e_i) = T_i^{'}(\xi) = T_i^{'}(\xi)\langle e_i, e_i \rangle = \sum_{j=0}^{n} (T_j \otimes \langle e_j, \cdot \rangle)(\xi \otimes e_i),$$

which completes the proof. □

Furthermore, recall the isometries $V_m$ and $V_m^{'}$ defined in (4.3) and (4.4) we shall relate those isometries with the Toeplitz operators.

**Lemma 5.2.** *Let $m \in \mathbb{N}$. For every $\xi \in E_{m-1}$, we have the identities*

$$V_m(\xi) = \sqrt{d_{m-1}/d_m} \cdot \sum_{j=0}^{n} (-1)^j T_{n-j}(\xi)^{'} \otimes e_j, \tag{5.1}$$

$$V_m^{'}(\xi) = \sqrt{d_{m-1}/d_m} \cdot \sum_{j=0}^{n} (-1)^j e_j \otimes T_{n-j}(\xi). \tag{5.2}$$

*Proof.* By definition of $V_m$ we have

$$\begin{aligned} V_m &= \frac{(-1)^{(n+1)(m-1)}}{\sqrt{\mu_m}} G_m \\ &= \frac{d_{m-1}}{\sqrt{\mu_m}} (\iota_{m-1,1}^* \otimes 1)(\delta \otimes \xi) \\ &= \frac{d_{m-1}}{\sqrt{\mu_m}} (\sum_{j=0}^{n} (-1)^{n-j} \cdot T_{n-j}^{'} \otimes e_j. \end{aligned}$$

The third equality follows from the expression of $\delta$ and Lemma 4.9.

The second identity can be proved similarly. □

We define the following $SU(2)$-equivariant positive bounded operator

$$\Phi : F \to F, \qquad \Phi(\xi) = \frac{d_m}{d_{m+1}} \xi, \quad \forall \xi \in E_m.$$

**Lemma 5.3.** *The invertible operator $\Phi$ belongs to the Toeplitz algebra $\mathbb{T}_E$.*

*Proof.* Let $\gamma_n \in (0, 1]$ be the limit of the quotient of the sequence of dimensions calculated in Lemma 4.5. Since $\Phi - \gamma_n \cdot 1$ is the limit of a finite rank operator, then it is compact on $F$. Then $\Phi \in \mathbb{T}$ follows from the fact that $\mathbb{K}_F \subset \mathbb{T}$ where $\mathbb{K}_F$ denotes the algebra of compact operators on $F$. □

**Lemma 5.4** ([1, Proposition 4.5]). *For $\xi \in E_{m-1} \subset F$, we have identities*

$$V_L(\xi) = V_m'(\xi) = \sum_{j=0}^{n} (-1)^j \cdot e_j \otimes T_{n-j}\Phi^{1/2}(\xi),$$
$$V_R(\xi) = V_m(\xi) = \sum_{j=0}^{n} (-1)^{n-j} \cdot T_{n-j}'\Phi^{1/2}(\xi) \otimes e_j.$$

Finally we can present the commutation relations for the Toeplitz algebra in the case of irreducible representation.

**Theorem 5.5** ([1, Theorem 4.6]). *Let $n \in \mathbb{N}$, and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. Then the Toeplitz operators $T_i$ with $i = 0, \dots, n$ satisfy the following commutation relations:*

1. $\sum_i T_i(T_i)^* = 1 - Q_0$
2. $\sum_{i=0}^n (-1)^i T_i T_{n-i} = 0$
3. $T_i^*(T_j) = \delta_{i,j} 1_F + (-1)^{i+j+1}((n+1) \cdot 1_F - \Phi^{-1}) T_{n-i} T_{n-j}^*$
4. $\sum_i (T_i)^* T_i = \Phi^{-1}$

### 5.1.2 Reducible cases

In this section, we shall focus on the case $H \cong L_n^{\oplus t}$. First of all, by Theorem 4.15 we then have $\widetilde{E}_m \cong E_m^{\oplus t^m}$.

By Proposition 3.20 we have that the dimension of $\widetilde{K}_2$ is $t^2$. Let $\delta = \frac{1}{\sqrt{n+1}} \sum_{i=0}^n (-1)^i e_i \otimes e_{n-i}$ be the generators of the determinant in the irreducible case. We have that $\widetilde{K}_2$ is spanned by the elements $\delta_i^j$, defined by

$$\delta_i^j = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} (-1)^i e_k^i \otimes e_{n-k}^j,$$

with the convention that $e_k^i = 0 \oplus 0 \oplus \cdots \oplus \underbrace{e_k}_{j\text{th}} \oplus \cdots \oplus 0$. The set

$$\{\delta_k^l : k, l = 1, \dots, t\},$$

forms an orthonormal basis for $\widetilde{K}_2$.

Then we can give recursive formula for $\widetilde{G}_m : \widetilde{E}_{m-1} \to \widetilde{K}_{m+1}$: Define

$$G_1^{i,j}(1) := \delta_i^j,$$
$$G_m^{i,j} := G_{m-1}^{i,j} \otimes 1 + (-1)^{(n+1)(m-1)} d_m \cdot 1_{m-1} \otimes G_1^{i,j}, m \geq 2,$$
$$\widetilde{G}_m := \bigoplus_{i,j=1}^{t} (G_m^{i,j})^{\oplus t^{m-1}}.$$

Note that for each $k, l$, $\mathrm{span}_{\mathbb{C}} \delta_{k,l}$ is isometrically isomorphic to the $K_2$ by the map $e_i^k \otimes e_{n-i}^l \mapsto e_i \otimes e_{n-i}$ where $k, l = 1, 2, \dots, t$, from which it follows that $G_m^{i,j}(E_{m-1}) \cong G_m(E_{m-1})$.

Then let us compute the inner product $\langle \widetilde{G}_m(\xi), \widetilde{G}_m(\eta) \rangle$ for $\xi = \oplus_{i=1}^{t^{m-1}} \xi_i \in \widetilde{E}_{m-1}, \eta = \oplus_{i=1}^{t^{m-1}} \eta_i \in \widetilde{E}_{m-1}$. By Lemma 3.6 in [1] we have

$$\langle \widetilde{G}_m(\xi), \widetilde{G}_m(\eta) \rangle = \sum_{i,j=1}^{t} \sum_{k=1}^{t^{m-1}} \langle G_m^{i,j}(\xi_k), G_m^{i,j}(\eta_k) \rangle = t^2 \mu_m \langle \xi, \eta \rangle.$$

Therefore, the map $\widetilde{V}_m := \frac{(-1)^{(n+1)(m-1)}}{t\sqrt{\mu_m}} \cdot \widetilde{G}_m$ is an isometry from $\widetilde{E}_{m-1}$ to $\widetilde{E}_m \otimes \widetilde{E}_1$. One thing to note that is, this map is not surjective as the dimension of $\widetilde{E}_m \otimes \widetilde{E}_1$ is $d_m d_1 t^{m+1}$ while the dimension of the image is $d_{m-1} t^{m-1}$.

**Lemma 5.6.** *We have a unitary isomorphism of Hilbert spaces*

$$(\widetilde{K}_m \otimes \widetilde{E}_1) \oplus \widetilde{G}_m(\widetilde{E}_{m-1}) \cong \widetilde{K}_{m+1}.$$

*Proof.* The proof follows from the fact that each component satisfies the isomorphism in Lemma 4.4. More precisely, we have

$$\begin{aligned}(\widetilde{K}_m \otimes \widetilde{E}_1) \oplus \widetilde{G}_m(\widetilde{E}_{m-1}) &\cong (K_m \otimes E_1)^{\oplus t^{m+1}} \oplus \bigoplus_{i,j=1}^{t} (G_m^{i,j})^{\oplus t^{m-1}}(E_{m-1}) \\ &\cong ((K_m \otimes E_1) \oplus G_m(E_{m-1}))^{t^{m+1}} \\ &\cong K_{m+1}^{\oplus t^{m+1}} \\ &\cong \widetilde{K}_{m+1}.\end{aligned}$$

□

**Proposition 5.7.** *Let $m \in \mathbb{N}$. The linear map*

$$(\widetilde{\iota}_{m,1}, \widetilde{V}_m^{\oplus t^2}) : \widetilde{E}_{m+1} \oplus \widetilde{E}_{m-1}^{\oplus t^2} \to \widetilde{E}_m \otimes \widetilde{E}_1,$$
$$(\widetilde{\iota}_{1,m}, (\widetilde{V}_m')^{\oplus t^2}) : \widetilde{E}_{m+1} \oplus \widetilde{E}_{m-1}^{\oplus t^2} \to \widetilde{E}_1 \otimes \widetilde{E}_m,$$

*are $SU(2)$-equivariant unitary isomorphisms.*

*Proof.* We prove the first identity, and the second one can be proved by a similar argument. The claim follows from a dimension-counting argument. By the results discussed in Chapter 4, we have $\dim(\widetilde{E}_m) = d_m \cdot t^m$. Thus we have

$$\begin{aligned}\dim(\widetilde{E}_{m-1} \oplus \widetilde{E}_{m-1}^{\oplus t^2}) &= d_{m+1} \cdot t^{m+1} + t^2 d_{m-1} \cdot t^{m-1} \\ &= t^{m+1} d_{m+1} d_{m-1} \\ &= \dim(\widetilde{E}_m \otimes \widetilde{E}_1).\end{aligned}$$

Since $(\widetilde{\iota}_{m,1}, \widetilde{V}_m^{\oplus t^2})$ is clearly injective, we finish the proof. □

Now let us turn to the Toeplitz algebra coming from $\rho_n^{\oplus t} : SU(2) \to U(L_n^{\oplus t})$. Let $\{e_i^k : i = 0, 1, \ldots, n; k = 1, 2, \ldots, t\}$ denote the basis of $L_n^{\oplus t}$. Then we have the associated Toeplitz operators $T_{i,j} := T_{e_i^j}$ and the right creation operator $(T_{i,k})' := \iota_{m,1}^*(\xi \otimes e_i^j), \xi \in E_m$. We define operators $\iota_L : F \to E_1 \otimes F$ and $\iota_R : F \otimes E_1$ by setting $\iota_L(\xi) = \iota_{1,m-1}(\xi)$ and $\iota_R(\xi) = \iota_{m-1,1}(\xi)$ for $\xi \in E_m$.

**Lemma 5.8.** *We have the identities*

$$\iota_L^* = \sum_{\substack{i=0,\ldots,n \\ j=1,\ldots,t}} \langle e_i^j, \cdot \rangle \otimes T_{i,j} : \widetilde{E}_1 \otimes F \to F,$$

$$\iota_R^* = \sum_{\substack{i=0,\ldots,n \\ j=1,\ldots,t}} (T_{i,j})' \otimes \langle e_i^j, \cdot \rangle : F \otimes \widetilde{E}_1 \to F.$$

*Proof.* Let $\xi \in \widetilde{E}_m$ and for each $(i, j)$ we have

$$\begin{aligned}\iota_L^*(e_i^j \otimes \xi) &= \iota_{1,m}^*(e_i^j \otimes \xi) \\ &= T_{i,j}(\xi) \\ &= \sum_{k,l} \langle e_k^l, e_i^j \rangle \otimes T_{l,k}(e_i^j \otimes \xi).\end{aligned}$$

A similar computation holds for the second identity. □

Now, we are going to analyze the isometries $\widetilde{V}_m$ and $\widetilde{V}_m'$.

**Lemma 5.9.** *Let $m \in \mathbb{N}$. For every $\xi \in \widetilde{E}_{m-1}$, we have the identities below*

$$\widetilde{V}_m = \bigoplus_{i,j=1}^{t} \widetilde{V}_m^{i,j}\ , \quad \textit{where } \widetilde{V}_m^{i,j} = \sum_{k=0}^{n} (-1)^{n-k} \frac{\sqrt{d_{m-1}}}{t\sqrt{d_m}} (T_{n-k,i})' \otimes e_k^j,$$

$$\widetilde{V}_m' = \bigoplus_{i,j=1}^{t} (\widetilde{V}_m^{i,j})'\ , \quad \textit{where} (\widetilde{V}_m^{1,1})' = \sum_{k=0}^{n} (-1)^{n-k} \frac{\sqrt{d_{m-1}}}{t\sqrt{d_m}} \cdot e_k^i \otimes T_{n-k,j}.$$

*Proof.* By using the Lemma 3.15 in [1] we have

$$\begin{aligned} \widetilde{V}_m^{i,j}(\xi) &= \frac{(-1)^{(n+1)(m-1)}}{2\sqrt{\mu_m}} \widetilde{G}_m, \\ &= \frac{(-1)^{(n+1)(m-1)}}{2\sqrt{\mu_m}} (G_m^{1,1}, \ldots, G_m^{t,t})(\xi). \end{aligned}$$

For simplicity, we only compute the map $G_m^{1,1}$ below. The other maps can be described using the same method.

Let $\xi \in E_{m-1} \hookrightarrow \widetilde{E}_{m-1}$ and $c := (-1)^{(n+1)(m-1)} d_{m-1}$ we then have

$$\begin{aligned} G_m^{1,1}(\xi) &= c \cdot (\iota_{m-1,1}^* \otimes 1)(1_{m-1} \otimes \delta_1^1)(\xi), \\ &= c \cdot (\iota_{m-1,1}^* \otimes 1)(1_{m-1} \otimes \frac{1}{\sqrt{n+1}} \sum_{i=0}^{n} (-1)^i e_i^1 \otimes e_{n-i}^1)(\xi), \\ &= \sum_{i=0}^{n} \frac{(-1)^i}{\sqrt{n+1}} \cdot c \cdot (\iota_{m-1,1}^* \otimes 1)(1_{m-1} \otimes e_i^1 \otimes e_{n-i}^1)(\xi), \\ &= \sum_{j=0}^{n} (-1)^{n-j} \frac{\sqrt{d_{m-1}}}{2\sqrt{d_m}} (T_{n-j,1})' \otimes e_j^1. \end{aligned}$$

Thus we complete the proof. □

Define the operator $\widetilde{\Phi} : \widetilde{F} \to \widetilde{F}$, sending $\xi \in \widetilde{E}_m$ to $\frac{d_m}{d_{m+1}} \xi$, where $\widetilde{F} := \oplus_{m=0}^{\infty} \widetilde{E}_m$ denotes the Fock space for reducible case. We have $\widetilde{\Phi}$ belongs to the Toeplitz algebra by Lemma 5.3.

Then we can reformulate Lemma 5.9.

**Lemma 5.10.** *For every $\xi \in \widetilde{F}$, we have identities:*

$$\widetilde{V}_m = \bigoplus_{i,j=1}^{t} \widetilde{V}_m^{i,j}\ , \textit{where } \widetilde{V}_m^{i,j} = \sum_{k=0}^{n} (-1)^{n-k} \widetilde{\Phi}^{1/2} (T_{n-k,i})' \otimes e_k^j, \tag{5.3}$$

$$\widetilde{V}_m' = \bigoplus_{i,j=1}^{t} (\widetilde{V}_m^{i,j})'\ , \textit{where} (\widetilde{V}_m^{i,j})' = \sum_{k=0}^{n} (-1)^{n-k} \cdot e_k^i \otimes T_{n-k,j} \widetilde{\Phi}^{1/2}. \tag{5.4}$$

**Theorem 5.11** (Commutation relations of the Toeplitz algebra $\mathbb{T}$). *Let $n \in \mathbb{N}$, and consider the reducible representation $\rho_n^{\oplus t} : SU(2) \to U(L_n^{\oplus t})$. Then the Toeplitz operators $T_{i,k}$ with $i = 0, \ldots, n; k = 1, 2, \ldots, t$ satisfy the following commutation relations:*

1. $\sum_{i,k} T_{i,k}(T_{i,k})^* = 1 - Q_0$
2. $\sum_{i=0}^{n}(-1)^i T_{i,k}T_{n-i,l} = 0$ *where* $k, l = 1, 2, \ldots, t$
3. $(T_{i,k})^*(T_{j,l}) = \delta_{i,j}\delta_{k,l}1_F + (-1)^{i+j+1}T_{n-i,k}\widetilde{\Phi}(T_{n-j,l})^*$
4. $\sum_{i,k}(T_{i,k})^*T_{i,k} = (t-1)(n+1)1_F + \widetilde{\Phi}^{-1}$

*Proof.* The first identity is from [2, Lemma 2.8], and the second one is from the expression of the determinant.

Let $i, j \in \{0, 1, \ldots, n\}$ and $k, l \in \{1, \ldots, t\}$. By definition of the Toeplitz operators, we have $T_{j,l} = \widetilde{\iota}_L^*(e_j^l \otimes 1_F)$ and $T_{i,k} = \widetilde{\iota}_L^*(e_i^k \otimes 1_F)$, thus

$$(T_{i,k})^*(T_{j,l}) = (\langle e_i^k, \cdot\rangle \otimes 1_F)\widetilde{\iota}_L\widetilde{\iota}_L^*(e_j^l \otimes 1_F).$$

Recall that $(\widetilde{\iota}_L, (\widetilde{V}_m')^{\oplus t^2})$ is a unitary isomorphism from $\widetilde{E}_{m+1} \oplus \widetilde{E}_{m-1}^{\oplus t^2}$ to $\widetilde{E}_1 \otimes \widetilde{E}_m$. Therefore we have $1_F \otimes 1 = \widetilde{\iota}_L\widetilde{\iota}_L^* + t^2(\sum_{r,s=1,\ldots,t}(\widetilde{V}_m')^{r,s})$. It implies that

$$\begin{aligned}
&(\langle e_i^k, \cdot\rangle \otimes 1_F)\widetilde{\iota}_L\widetilde{\iota}_L^*(e_j^l \otimes 1_F) \\
&= \delta_{i,j}\delta_{k,l} - t^2(\langle e_i^k, \cdot\rangle \otimes 1_F)\Big(\sum_{r,s=1,\ldots,t}(\widetilde{V}_m')^{r,s}\Big)(e_j^l \otimes 1_F) \\
&= \delta_{i,j}\delta_{k,l} + (-1)^{i+j+1}T_{n-i,k}\widetilde{\Phi}(T_{n-j,l})^*,
\end{aligned}$$

where the last equality is due to equation (5.4).

For the last one, we have

$$\begin{aligned}
\sum_{\substack{i=0,\ldots,n\\k=1,\ldots,t}}(T_{i,k})^*T_{i,k} &= t(n+1)1_F - t^2\sum_{\substack{i=0,\ldots,n\\k=1,\ldots,t}}T_{n-i,k}\widetilde{\Phi}(T_{n-i,k})^* \\
&= (t-1)(n+1)1_F + \widetilde{\Phi}^{-1}\sum_{\substack{i=0,\ldots,n\\k=1,\ldots,t}}T_{n-i,k}(T_{n-i,k})^* \\
&= (t-1)(n+1)1_F + \widetilde{\Phi}^{-1}(1 - Q_0) \\
&= (t-1)(n+1)1_F + \widetilde{\Phi}^{-1},
\end{aligned}$$

where the second equality holds for that

$$T_{i,k}(\widetilde{E}_m) \subset \widetilde{E}_{m+1} \text{ and } d_1 - d_{m+2}/d_{m+1} = d_m/d_{m+1},$$

which yields that

$$\begin{aligned} T_{n-i,k}\widetilde{\Phi}(\xi) &= \frac{d_m}{d_{m+1}} T_{n-i,k}(\xi) \\ &= ((n+1) - d_{m+2}/d_{m+1}) T_{n-i,k}(\xi) \\ &= ((n+1)1_F - \widetilde{\Phi}^{-1}/t^2) T_{n-i,k}(\xi). \end{aligned}$$

Then we complete the proof. □

We finish this section by showing that the Toeplitz algebra is even generated by the one-shift $T_i$ i.e. it is the closed span of non-commutative polynomials with variables $T_i$.

**Proposition 5.12.** *The Toeplitz algebra $\mathbb{T}_E$ of a $SU(2)$-equivariant subproduct system $(E, \iota)$ is generated by the identity operator and one-shift $T_i : F \to F$ which sends $\zeta \in E_m$ to $\iota^*_{1,m}(e_i \otimes \zeta)$ with $i = 0, \ldots, dim(L_n) - 1$.*

*Proof.* It is sufficient to show that for all creation operators $T_\xi \in E_k$, we can write it as linear combination of products of one-shifts. By the linearity of $\iota^*_{k,m}$ it is sufficient to show that for $T_\xi$ can be written as product of one-shift where $\xi$ is a basis vector of $E_k$. Since $\iota_{k,m}$ is an isometry, we have that $\iota^*_{k,m}$ is surjective. Therefore we have that $E_{k+m}$ is spanned by the $\iota^*_{k,m}(\xi \otimes \zeta)$ where $\xi$ and $\zeta$ are basis vectors of $E_k$ and $E_m$ respectively. Then for $x \in E_n$, by using the associativity of the structure maps, we have

$$\begin{aligned} T_\xi T_\zeta(x) &= T_\xi(\iota^*_{m,n}(\zeta \otimes x)) \\ &= \iota^*_{k,m+n}(\xi \otimes \iota^*_{m,n}(\zeta \otimes x)) \\ &= \iota^*_{k,m+n}(1_k \otimes \iota^*_{m,n})(\xi \otimes \zeta \otimes x) \\ &= \iota^*_{k+m,n}(\iota^*_{k,m}(\xi \otimes \zeta) \otimes x) \\ &= T_{\iota^*_{k,m}(\xi \otimes \zeta)}(x). \end{aligned}$$

which implies that $T_\gamma$ where $\gamma \in E_n$ is a linear combination of $T_\xi T_\zeta$ where $\xi \in E_k$, $\zeta \in E_m$ where $k + m = n$. Hence by an induction argument we conclude that the Toeplitz algebra $\mathbb{T}_E$ is generated by the one-shifts $T_j$. □

## 5.2 Cuntz–Pimsner algebras

Now we are ready to describe the resulting Cuntz–Pimsner algebra. Since the $SU(2)$-equivariant subproduct system consists of finite-dimensional Hilbert spaces, we have that the resulting Cuntz–Pimsner algebra $\mathbb{O}_E = \mathbb{T}_E/\mathbb{I}_E = \mathbb{T}_E/\mathbb{K}_E$ where $\mathbb{K}_E$ denotes the space of compact operators on the associated Fock space.

### 5.2.1 Irreducible case

By the definition of $\Phi$, we have the following lemma:

**Lemma 5.13.** *The operator $\Phi^{-1} - \frac{1}{\gamma_n}$ is compact.*

*Proof.* By definition we have $\gamma_n = \lim_{m\to\infty} d_m/d_{m+1}$. Then we have $\Phi^{-1} - \frac{1}{\gamma_n}$ is the limit of finite rank operators which implies that $\Phi^{-1} - \frac{1}{\gamma_n}$ is a compact operator. □

Let $S_i$ and $\overline{\Phi^{-1}}$ denote the equivalent classes of $T_i$ and $\Phi$ in the quotient algebra $\mathbb{T}_E/\mathbb{K}_E$. We can formulate the following theorem.

**Theorem 5.14** (Commutation relations for Cuntz–Pimsner algebra $\mathbb{O}_E$: irreducible case)**.** *Let $n \in \mathbb{N}$, and consider the reducible representation $\rho_n : SU(2) \to U(L_n)$. Then the Cuntz–Pimsner algebra $\mathbb{O}_E$ is generated by the one-shift Toeplitz operators $S_i$ with $i = 0, \dots, n$ which satisfy the following commutation relations:*

1. $\sum_i S_i S_i^* = 1$
2. $\sum_{i=0}^{n} (-1)^i S_i S_{n-i} = 0$
3. $S_i^* S_j = \delta_{i,j} 1_F + (-1)^{i+j+1} \gamma_n S_{n-i} S_{n-j}^*$
4. $\sum_i S_i^* S_i = \frac{1}{\gamma_n}$

*Proof.* To prove the theorem, it is sufficient to determine the compact operators in the relations.

For the first relation, recall that in the Toeplitz algebra we have $\sum_i T_i T_i^* = 1 - Q_0$ where $Q_0$ is a projection from the Fock space $F$ onto $E_0$ and it is clearly compact. Hence in the quotient algebra the first relation holds.

The second relation holds since zero operator remains zero in the quotient algebra.

For the third relation, we have that the third relation in Theorem 5.5 for $\mathcal{O}_E$ becomes the following

$$\begin{aligned} S_i^* S_j &= \delta_{i,j} 1_F + (-1)^{i+j+1}((n+1)1_F - \overline{\Phi^{-1}}) S_{n-i} S_{n-j}^* \\ &= \delta_{i,j} 1_F + (-1)^{i+j+1}((n+1-\frac{1}{\gamma_n})1_F + \frac{1}{\gamma_n}1_F - \overline{\Phi^{-1}}) S_{n-i} S_{n-j}^* \\ &= \delta_{i,j} 1_F + (-1)^{i+j+1}((n+1-\frac{1}{\gamma_n})1_F) S_{n-i} S_{n-j}^*. \end{aligned}$$

Recall the fact that $(n+1)\gamma_n = 1 + \gamma_n^2$. Then we have $n + 1 - \frac{1}{\gamma_n} = \gamma_n$ which yields that

$$S_i^* S_j = \delta_{i,j} 1_F + (-1)^{i+j+1} \gamma_n S_{n-i} S_{n-j}^*. \tag{5.5}$$

The forth relation holds since $\sum_i T_i^* T_i = \Phi^{-1} = \Phi^{-1} - \frac{1}{\gamma_n} + \frac{1}{\gamma_n}$, then in the quotient algebra we have $\sum_i S_i^* S_i = \frac{1}{\gamma_n}$. □

**Corollary 5.15.** *Let $n \in \mathbb{N}$, and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. Then every element in the Cuntz–Pimsner algebra $\mathcal{O}_E$ can be written as a polynomial of the form*

$$P(S_0, \ldots, S_n) = \sum_{p,q} a_{p,q} p(S_0^*, \ldots, S_n^*) \cdot q(S_0, \ldots, S_n).$$

*where $a_{p,q} \in \mathbb{C}$ and $p, q$ are non-commutative polynomials.*

*Proof.* By the Proposition (5.12) we have that $\mathcal{O}_E$ is the closed linear span of one-shifts. Theorem (5.17) shows that for the non-commutative polynomials, we have:

$$S_i S_j^* = \frac{(-1)^{i+j+1}}{\gamma_n} (S_{n-i}^* S_{n-j} - \delta_{i,j} 1_F).$$

Therefore we can write the non-commutative polynomials of variables $S_i^*$ and $S_j$ into the product of non-commutative polynomials $p(S_0^*, \ldots, S_n^*)$and $q(S_0, \ldots, S_n)$. □

For the case $n = 1, t = 1$, we have that the Cuntz–Pimsner algebra $\mathcal{O}$ is isomorphic to the algebra of complex continuous functions on $S^3$ i.e. $C(S^3)$ which has been studied in [2, Theorem 5.7].

**Example 5.16.** *Let $n = 2, t = 1$, we then have $\gamma_2 = \frac{3-\sqrt{5}}{2}$ and $\frac{1}{\gamma_2} = \frac{3+\sqrt{5}}{2}$. We then have the following commutation relations in the Cuntz–Pimsner algebra $\mathcal{O}$:*

1. $S_0S_0^* + S_1S_1^* + S_2S_2^* = 1$
2. $S_0S_2 + S_2S_0 = S_1S_1$
3. $S_0^*S_1 = (2-\phi)S_2S_1^*$, $S_0^*S_2 = -(2-\phi)S_2S_0^*$ *and* $S_1^*S_2 = (2-\phi)S_1S_0^*$
4. $S_0^*S_0 = -\varphi \cdot S_2S_2^*$, $S_1^*S_1 = -\varphi \cdot S_1S_1^*$ *and* $S_2^*S_2 = -\varphi \cdot S_0S_0^*$
5. $S_0^*S_0 + S_1^*S_1 + S_2^*S_2 = 1 + \varphi$

*where $\varphi$ is the golden ratio. Note that the the first relation shows the tuple $(S_0^*, S_1^*, S_2^*)$ is a row-contraction.*

### 5.2.2 Reducible cases

Recall that from commutation relations in the irreducible case we have

$$(T_{i,k})^*(T_{j,l}) = \delta_{i,j}\delta_{k,l}1_F + (-1)^{i+j+1}T_{n-i,k}\Phi(T_{n-j,l})^*.$$

Using Lemma 5.3, we have $\widetilde{\Phi}^{-1} - 1/\gamma_n$ is compact. Therefore we have

$$\begin{aligned}(T_{i,k})^*(T_{j,l}) &= \delta_{i,j}\delta_{k,l}1_F + (-1)^{i+j+1}T_{n-i,k}\Phi(T_{n-j,l})^* \\ &= \delta_{i,j}\delta_{k,l}1_F + (-1)^{i+j+1}T_{n-i}^k(\Phi - \gamma_n + \gamma_n)(T_{n-j,l})^*.\end{aligned}$$

which implies that in the $\mathcal{O}_E$, we have

$$(S_{i,k})^*(S_{j,l}) = \delta_{i,j}\delta_{k,l}1_F + \gamma_n(-1)^{i+j+1}S_{n-i,k}S_{n-j,l}^*. \tag{5.6}$$

Then we can formulate the commutation relations in $\mathcal{O}_E$ for the subproduct system induced by the reducible $SU(2)$ representation.

**Theorem 5.17** (Commutation relations for Cuntz–Pimsner algebra $\mathcal{O}_E$: reducible case)**.** *Let $n \in \mathbb{N}$, and consider the reducible representation $\rho_n^{\oplus t} : SU(2) \to U(L_n^{\oplus t})$. Then the Cuntz–Pimsner algebra $\mathcal{O}_E$ is generated by the one-shift Toeplitz operators $S_{i,j}$ with $i = 0, \ldots, n, j = 1, \ldots, t$, which satisfy the following commutation relations:*

1. $\sum_{i=0,\ldots,n;j=1,\ldots,t} S_{i,j}S_{i,j}^* = 1$
2. $\sum_{i=0,\ldots,n;j=1,\ldots,t}(-1)^i S_{i,j}S_{n-i,j} = 0$
3. $S_{i,k}^*S_{j,l} = \delta_{i,j}\delta_{k,l}1_F + \gamma_n(-1)^{i+j+1}S_{n-i,k}S_{n-j,l}^*$
4. $\sum_{i=0,\ldots,n;j=1,\ldots,t} S_{i,j}^*S_{i,j} = (t-1)(n+1)1_F + \frac{1}{\gamma_n}$

*Proof.* The first two relations hold for the same reason with irreducible case. The third one is due to (5.6). The last relation holds for that in the Toeplitz algebra we have

$$\begin{aligned}\sum_{i=0,\dots,n;j=1,\dots,t} T_{i,j}^* T_{i,j} &= (t-1)(n+1)1_F + \widetilde{\Phi}^{-1} \\ &= (t-1)(n+1)1_F + (\widetilde{\Phi}^{-1} - 1/\gamma_n + 1/\gamma_n).\end{aligned}$$

and the fact that $\widetilde{\Phi}^{-1} - 1/\gamma_n$ is compact implies that

$$\sum_{i=0,\dots,n;j=1,\dots,t} S_{i,j}^* S_{i,j} = (t-1)(n+1)1_F + \frac{1}{\gamma_n}$$

then we complete our proof. □

**Corollary 5.18.** *Let $n \in \mathbb{N}$, and consider the reducible representation $\rho_n^{\oplus t} : SU(2) \to U(L_n^{\oplus t})$. Then every element in the Cuntz–Pimsner algebra $\mathbb{O}_E$ is a polynomial of the form*

$$P(S_{i,j}) = \sum_{p,q} a_{p,q} p(S_{i,j}^*) \cdot q(S_{i,j}).$$

*where $i = 0, \dots, n; j = 1, \dots, t$; $a_{p,q} \in \mathbb{C}$ and $p, q$ are non-commutative polynomials.*

*Proof.* The proof is almost the same as the proof of Corollary 5.15. □

Chapter 6

# Outlook

There are at least three main questions remaining to be studied:

1. The first thing is that we still do not know whether the fusion rules exist for the disjoint direct sum case. If so, it is interesting to consider the resulting Toeplitz and Cuntz–Pimsner algebra and to investigate their commutation relations.

2. Secondly, we would like to know whether the Toeplitz and Cuntz–Pimsner algebras are universal in any sense with respect to the $SU(2)$-representations and Hilbert spaces. For Cuntz–Pimsner algebra, there is a $U(1)$-gauge invariant uniqueness theorem. But we do not know if there exists some $SU(2)$-gauge invariant uniqueness theorem.

3. Finally, the K-theory of Toeplitz algebra from the irreducible $SU(2)$-representation has been studied by Arici and Kaad. It is worthwhile to extend the arguments in [1] to the reducible case.

# Appendices

Appendix A

# Lie Groups and Their Representations

## A.1 Lie groups

In this section, we shall briefly introduce Lie groups and their representation theory focusing in particular on $SU(2)$ and its irreducible representations.

**Definition A.1** (Topological groups)**.** *A topological group $G$ is a topological space which is at the same time a group, and the group operations are continuous with respect to the topology $\tau$ of $G$. That means the following operations are continuous:*

$$m : G \times G \to G, (x, y) \mapsto xy \tag{A.1}$$

$$i : G \to G, x \mapsto x^{-1} \tag{A.2}$$

Lie groups are special cases of topological groups. Roughly speaking, a Lie group is a topological group with a smooth manifold structure, and the group operations are smooth.

**Definition A.2** (Lie groups)**.** *A Lie group $G$ is a smooth manifold endowed with a group structure such that the group operations* (A.1) *and* (A.2) *are smooth.*

In fact, Lie groups are almost everywhere in our daily mathematical study and research. In order to get accustomed with the notion of Lie group, let us look at two examples.

**Example A.3.** *The vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with the operations of addition and the zero vector as unit, are Lie groups.*

**Example A.4.** *The multiplicative groups* $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ *and* $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ *with the usual multiplication and one as unit, are Lie groups.*

we will now describe how we can construct new Lie groups from the old ones.

**Lemma A.5.** *Let* $G_1, G_2$ *be Lie groups then* $G := G_1 \times G_2$ *endowed with the product manifold structure and product group structure is a Lie group.*

*Proof.* The product manifold structure and product group structure are clear. The only nontrivial thing to check is the smoothness of the group operations. The multiplication map satisfies

$$\mu((x_1, y_1), (x_2, y_2)) = (x_1 x_2, y_1 y_2) = \mu_1 \times \mu_2((x_1, x_2), (y_1, y_2)),$$

which implies that $\mu = (\mu_1 \times \mu_2)(\mathrm{id}_{G_1} \times S \times \mathrm{id}_{G_2})$, where $S : G_1 \times G_2 \to G_2 \times G_1$ sending $(x, y)$ to $(y, x)$ . Then we have that $\mu$ is a composition of smooth maps, thus $\mu$ is smooth. Indeed, take point $(x, y) \in G_1 \times G_2$ pick any chart $((\varphi, \phi), U \times V)$ containing $(x, y)$ then we have

$$(\phi, \varphi) \circ S \circ (\varphi^{-1}, \phi^{-1}) = \mathrm{id}$$

thus $S$ is smooth. And the smoothness of the remaining maps is trivial. Finally, the inverse map $\iota = (\iota_1, \iota_2)$ is clearly smooth due to the smoothness of $\iota_1$ and $\iota_2$. □

Since a Lie group is not only a group but also a smooth manifold, a subgroup may not inherit smooth manifold structure thus may not be a Lie group anymore. However, under certain assumptions, this is still the case.

**Lemma A.6.** *Let* $G$ *be a Lie group and let* $H \subset G$ *be a subgroup. If* $H$ *is also a sub-manifold of* $G$ *then* $H$ *is a Lie group.*

*Proof.* Let $\mu$ be the multiplication on $G$, then consider the $\mu_H := \mu|_{H \times H}$ from $H \times H$ to $H$. Since $H$ is a submanifold of $G$ we have $H \times H$ is a submanifold of $G \times G$. Therefore, smoothness of $\mu$ implies smoothness of $\mu$ restricted to $H \times H$. Moreover since $H$ is a subgroup, then we have $H$ is closed under the multiplication $\mu_H$ (i.e. the image of $\mu_H$ is still in $H$). For the same reason, we have $\iota_H := \iota|_H : H \to H$ is smooth and closed. Hence $H$ is again a Lie group. □

**Remark A.7.** *Indeed, if* $H \subset G$ *is a subgroup that is closed in the sense of topology. Then* $H$ *is a sub-manifold thus a Lie group. The proof of this remark can be found in [4].*

Let us look at two examples about the new Lie groups constructed via above lemmas.

**Example A.8.** *The vector space* $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ *is again a Lie group.*

**Example A.9.** *Let* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^*$*. Clearly we have* $\mathbb{T}$ *is a submanifold and a subgroup. Therefore* $\mathbb{T}$ *is a Lie group.*

In order to be able to classify Lie groups, we will introduce the concept of isomorphism between Lie groups.

**Definition A.10.** *A Lie group homomorphism from a Lie group* $G$ *to a Lie group* $H$ *is a smooth map* $\varphi : G \to H$ *that is also a group homomorphism.*

*A Lie group isomorphism is a bijective Lie group homomorphism such that the inverse is also a Lie group homomorphism (and thus a diffeomorphism between smooth manifolds). Lie group isomorphisms from* $G$ *to itself are called automorphisms and the set of automorphisms is denoted by* $Aut(G)$*.* $Aut(G)$ *is a group with respect to composition of maps.*

Finally, we can talk about Lie subgroups.

**Definition A.11** (Lie subgroups)**.** *A Lie subgroup of a Lie group* $G$ *is a subgroup* $H$ *endowed with a Lie group structure such that the inclusion* $i : H \hookrightarrow G$ *is a Lie group homomorphism.*

Now, let us turn to matrix Lie groups.

### A.1.1 Matrix Lie groups

Since most interesting Lie groups are matrix groups and we mainly focus on $SU(2)$ in this thesis, in the rest of this appendix, we will restrict our attention to those that are matrix groups.

Firstly, we introduce the notion of matrix exponential.

**Proposition A.12.** *For all* $A \in M_n(\mathbb{R})$*. The matrix exponential*

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{A^n}{n!},$$

*is convergent.*

*Proof.* Firstly, since all norms on a finite-dimensional linear space are equivalent, to show the convergence of $\exp(A)$ it is sufficient to show convergence in the operator norm. We have

$$\|\exp(A)\| = \|\sum_{n=1}^{\infty} \frac{A^n}{n!}\| \leq \sum_{n=1}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|} < \infty,$$

which proves the convergence of $\exp(A)$. □

**Definition A.13** (Matrix exponential)**.** *Let $A$ be a $n \times n$ martix. The matrix exponential* $\exp(A)$ *is defined as the convergent series:*

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{A^n}{n!}.$$

The matrix exponential shares many nice properties with the usual exponential as summed up as the following lemma.

**Lemma A.14.** *The matrix exponential satisfies the following properties:*

1. $\frac{d}{dt}\exp(A) = A\exp(A)$
2. *If* $AB = BA$ *then* $\exp(A)\exp(B) = \exp(AB) = \exp(BA)$
3. $\exp(A)$ *is invertible with the inverse* $\exp(-A)$.

We cannot avoid Lie algebras when talking about Lie groups. Simply speaking, a Lie algebra is a tangent space of a Lie group at the identity with a Lie bracket operation $[\cdot,\cdot]$. And from now on, for the convention of Lie algebras, we denote $M_n(\mathbb{R})$ by $\mathfrak{gl}(n)$.

The matrix exponential exp gives a smooth map from $\mathfrak{gl}(n)$ to $GL(n)$, we shall prove that the local inverse log indeed exists.

**Lemma A.15.** *There exist neighborhoods* $0 \in U \subset \mathfrak{gl}(n)$ *and* $I \in V \subset GL(n)$ *such that the matrix exponential* exp *restricted on $U$ is bijective.*

*Proof.* The proof follows from the Inverse function theorem directly: we can identify $\mathfrak{gl}(n)$ by $\mathbb{R}^{n^2}$ and $GL(n) \subset \mathbb{R}^{n^2}$. Therefore we have the Jacobian matrix of $\exp(A)$ is defined by the following:

$$D\exp(A) = \begin{bmatrix} \frac{\partial(\exp(A)_{11})}{\partial a_{11}} & \frac{\partial(\exp(A)_{11})}{\partial a_{12}} & \cdots & \frac{\partial(\exp(A)_{11})}{\partial a_{nn}} \\ \frac{\partial(\exp(A)_{12})}{\partial a_{11}} & \frac{\partial(\exp(A)_{12})}{\partial a_{12}} & \cdots & \frac{\partial(\exp(A)_{12})}{\partial a_{nn}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\exp(A)_{nn})}{\partial a_{11}} & \frac{\partial(\exp(A)_{nn})}{\partial a_{12}} & \cdots & \frac{\partial(\exp(A)_{nn})}{\partial a_{nn}} \end{bmatrix}.$$

On the other hand, we have $\exp(A) = I + A + O(A^2)$ therefore we have $D\exp(0) = I \in M_{n^2\times n^2}(\mathbb{R})$ which is clearly invertible. Thus we have $\exp(A)$ is invertible near zero and we denote its inverse by log. □

**Remark A.16.** *The logarithm function is analytic, with convergent series*

$$\log(A) = A - \frac{A^2}{2} + \frac{A^3}{3} + \ldots.$$

Before the important theorem which reveals the relation of Lie groups and Lie algebras, we need the famous Baker–Campbell–Hausdorff formula.

**Theorem A.17** (Baker-Campbell-Hausdorff formula)**.** *For all $X, Y \in U(0) \subset\in M_n(\mathbb{F})$ where $U(0)$ is a sufficiently small neighborhood of zero, we have*

$$\exp(X)\exp(Y) = \exp\left(X + \int_0^1 g(\exp(ad_X)\exp(t \cdot ad_Y))(Y)\right).$$

*where $g(z) = \frac{\log z}{1-z^{-1}}$ and $ad_X(Y) := XY - YX$.*

The proof of Baker–Campbell–Hausdorff formula can be found in [6].

**Theorem A.18.** *Let $G$ be a topologically closed subgroup of $GL(n)$. Define*

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n) : \exp(tX) \in G, \forall t \in \mathbb{R}\}$$

*then we have*

1. *$\mathfrak{g}$ is a vector space;*
2. *For $X, Y \in \mathfrak{g}$ we have $[X, Y] = XY - YX \in \mathfrak{g}$;*
3. *The vector space $\mathfrak{g}$ defined above is the tangent space of $G$ at $I$.*

Before proving this theorem, we state the following version of Baker–Campbell–Hausdorff formula.

**Proposition A.19.** *We have that*

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{[X,Y]}{2} + \frac{[X,[X,Y]]}{12} - \frac{[Y,[X,Y]]}{12} + f(X,Y)\right)$$

*where $f(X, Y)$ consists of a linear combination of the Lie brackets of $X, Y$.*

Now we can prove Theorem A.18.

*Proof.* We firstly show that $\mathfrak{g}$ is a tangent space of $G$. To show $\mathfrak{g}$ is a tangent space of $G$ at the identity, it is sufficient to show that for any curve $\gamma(t) \in G$ with $\gamma(0) = I$ we have $\frac{d\gamma(t)}{dt}|_0 \in \mathfrak{g}$. That means, we need to prove $\exp\left(s\frac{d\gamma(t)}{dt}|_0\right) \in G$ for all $s \in \mathbb{R}$. We claim that it is enough to show $\exp\left(\frac{d\gamma(t)}{dt}|_0\right) \in G$. Indeed, define $\beta(t) := \gamma(ts)$, then we have $\frac{d\beta}{dt} = s\frac{d\gamma}{dt}$.

Let $h(t) = \log \gamma(t)$ for small $s$ (in order to make it well-defined), then we have

$$\frac{dh(t)}{dt}|_0 = \frac{d\log(s)}{ds}|_{s=0}\frac{d\gamma(t)}{dt}|_{t=0} = \frac{d\gamma(t)}{dt}|_{t=0},$$

so it is sufficient to show that $\exp(dh(0)/dt) \in G$. We have

$$\frac{dh(0)}{dt} = \lim_{n\to\infty}\frac{h(1/n)-h(0)}{1/n} = \lim_{n\to\infty} nh(1/n).$$

Since $\exp(h(t)) = \gamma(t) \in G$ then we have that $\exp(nh(t)) = \gamma(t)^n \in G$. As $G$ is topologically closed, we have

$$\lim_{n\to\infty}\exp\left(\frac{dh(0)}{dt}\right) = \lim_{n\to\infty}\exp(nh(1/n)) = \lim_{n\to\infty}\gamma(1/n)^n \in G.$$

Therefore $\mathfrak{g}$ is a tangent space of $G$ at $I$.

We now prove that $\mathfrak{g}$ is a vector space and it is closed under the bracket operation. We define the following curves

$$\begin{aligned}
\gamma_1(t) &:= \exp(sX)\exp(sY);\\
\gamma_2(t) &:= \exp(s\lambda X);\\
\gamma_3(t) &:= \exp\big(X\sqrt{s}\big)\exp\big(Y\sqrt{s}\big)\exp\big(-X\sqrt{s}\big)\exp\big(-Y\sqrt{s}\big).
\end{aligned}$$

Since $\mathfrak{g}$ is the tangent space of $G$ at the identity, then we have the derivatives of the above curves at zero should be in $\mathfrak{g}$. That is

$$\begin{aligned}
\gamma_1'(0) &= X\exp(0X)\exp(0Y) + \exp(0X)Y\exp(0Y) = X + Y \in G,\\
\gamma_2'(0) &= \lambda X\exp(0X) = \lambda X \in G.
\end{aligned}$$

which prove that $\mathfrak{g}$ is a vector space. To show the bracket operator is closed, we shall use the Baker–Campbell–Hausdorff formula above:

$$\begin{aligned}
\gamma_3(t) &= \exp\big(X\sqrt{s}\big)\exp\big(Y\sqrt{s}\big)\exp\big(-X\sqrt{s}\big)\exp\big(-Y\sqrt{s}\big)\\
&= \exp\Big(s[X,Y] - s[X+Y, X+Y] + O(s^{3/2})\Big)\\
&= \exp\Big(s[X,Y] + O(s^{3/2})\Big)
\end{aligned}$$

which implies $\gamma_3'(0) = ([X,Y] + O(\sqrt{s}))\exp\big(0[X,Y] + O(0^{3/2})\big) = [X,Y]$. Therefore we have $[X,Y] \in G$. □

The vector space $\mathfrak{g}$ we defined above is the so-called Lie algebra of the Lie group $G$. The formal definition is as follows

**Definition A.20.** *A vector space L is called a Lie algebra if it is endowed with a bilinear operator* $[\cdot,\cdot] : L \times L \to L$ *satisfying the following:*

1. $[x, x] = 0, \forall x \in L$;
2. *Jacobian identity:* $[x, [y, z]] + [y[z, x]] + [z, [z, y]] = 0$.

*A linear map* $\varphi : L \to M$ *between two Lie algebras is called a Lie algebra homomorphism if* $\varphi([x, y]) = [\varphi(x), \varphi(y)], \forall x, y \in L$.

As stated in the Theorem A.18, a Lie algebra of a (matrix) Lie group is the tangent space to the group at the identity element. For every abstract Lie group, the associated Lie algebra is defined likewise as the tangent space at the identity.

Homomorphism between Lie groups induces homomorphism between Lie algebras. Indeed, if $\varphi : G \to H$ is a Lie group homomorphism, then the induced Lie algebra homomorphism is given by the differential $D\varphi : \mathfrak{g} \to \mathfrak{h}$.

In some cases, the representation of Lie group (homomorphism from $G$ to the matrix Lie group $GL(n)$) is equivalent to the representation of Lie algebra (the corresponding differential). This is a consequence of the famous Lie's third theorem.

**Theorem A.21** (Lie's third theorem)**.** *Define the functor* Lie *between the categories of simply connected Lie groups and finite-dimensional Lie algebras which sends every Lie group to its Lie algebra and homomorphism of Lie groups to the corresponding Lie algebra homomorphism. Then* Lie *is an equivalence between category of simply connected Lie groups with Lie group homomorphisms and category of finite-dimensional Lie algebras with Lie algebra homomorphisms.*

The proof is a corollary of the theorem below whose proof is beyond the content of this thesis.

**Theorem A.22** ([11, Theorem 8.49] ). *Every finite-dimensional Lie algebra is a Lie subalgebra of* $\mathfrak{gl}(n, \mathbb{R})$ *for some* $n$.

## A.2 Representation theory of Lie groups

In this section, we will introduce the basics of representation theory of Lie groups. In particular, the representations of $SU(2)$ will be studied in detail.

### A.2.1 Representations of Lie groups

**Definition A.23** (Representations)**.** *A representation $\pi$ of a Lie group $G$ is a Lie group homomorphism $\pi : G \to GL(V)$ where $V$ is a (complex) vector space and it is called the representation space. A unitary representation $\pi$ of $G$ is the representation with the representation space being a (complex) Hilbert space $H$ and $\pi(x)$ being unitary for all $x \in G$;*

*Given two representations $(\pi, V)$ and $(\varphi, W)$, the linear map $\phi : V \to W$ is called equivalent (or intertwining) if the following diagram commutes:*

$$
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\big\downarrow{\scriptstyle \pi(x)} & & \big\downarrow{\scriptstyle \varphi(x)} \\
V & \xrightarrow{\phi} & W
\end{array}
$$

*if $\varphi$ is invertible we call $\varphi$ an isomorphism and two representations $(\pi, V)$ and $(\varphi, W)$ isomorphic. A sub-representation of $(\pi, V)$ is a subspace $W \subset V$ together with the representation $\pi|_W$ such that $\pi(x)|_W w := \pi(x)w \subset W, \forall w \in W$. A representation $(\pi, V)$ is irreducible if the only sub-representations are trivial ones.*

The following lemma allows us to classify unitary representations of a Lie group.

**Lemma A.24.** *If $G$ is a Lie group and $(\pi, H)$ is a finite-dimensional unitary representation of $G$, then $(\pi, H)$ decomposes into a finite direct sum of irreducibles.*

*Proof.* Suppose $(\pi, H)$ is a unitary representation of $G$ that is not irreducible. Let $H_1$ be a nontrivial invariant subspace. Since we have assumed that the dimension of $H$ is finite, $H_1$ as a subspace is closed, thus also a Hilbert space. We claim $H_1^\perp$ is also invariant: for any $x \in G$, take $w \in H_1$ and $v \in H_1^\perp$ we have

$$\langle \pi(x)v, w\rangle = \langle v, \pi(x^{-1})w\rangle = 0 \quad \text{implies} \quad \pi(x)v \in H_1^\perp, \forall x \in G.$$

Therefore we have $H = H_1 \oplus H_1^\perp$ i.e. $H$ decomposes into the direct sum of sub-representations. If $H_1$ and $H_1^\perp$ are irreducible, we complete the proof, otherwise we do the same step as above until $H$ decomposes into a direct sum of irreducibles. □

As one can see in the proof, the unitarity of $(\pi, H)$ is essential and it leads to the following theorem which helps us understand the representations of compact Lie groups.

**Theorem A.25.** *If G is a compact Lie group, then all finite-dimensional representations of G are completely reducible.*

*Proof.* The key of the proof is the so-called Weyl's unitary trick. To use the Lemma above, we need to construct an invariant inner product. Let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product on $G$, we claim that the following inner product is invariant.

$$\langle u, v \rangle_G := \int_G \langle \pi(x)u, \pi(x)v \rangle d\mu(x).$$

where $\mu$ denotes the Haar measure. It is not hard to check that $\langle \cdot, \cdot \rangle_G$ is indeed an inner product (which requires the compactness of $G$). Moreover, the representation $\pi$ is unitary with respect to this inner product:

$$\begin{aligned}
\langle \pi(y)u, \pi(y)v \rangle_G &= \int_G \langle \pi(y)\pi(x)u, \pi(y)\pi(x)v \rangle d\mu(x) \\
&= \int_G \langle \pi(yx)u, \pi(yx)v \rangle d\mu(x) \\
&= \int_G \langle \pi(yx)u, \pi(yx)v \rangle d\mu(yx) \\
&= \langle u, v \rangle_G,
\end{aligned}$$

where in the third equality we used the right invariance of the Haar measure. □

**Remark A.26.** *We have thus proved that for any compact Lie group G, every the finite-dimensional representation of it is unitarizable i.e. there exists an inner product such that the representation is unitary.*

Now we shall turn to the case of $SU(2)$ which plays a central role in this thesis. Since $SU(2)$ is compact, the representation of $SU(2)$ can be understood completely once we classify the irreducible representations of any fixed dimension.

### A.2.2 A case study: representations of $SU(2)$

**Definition A.27.** *The matrix Lie group $SU(2)$ is defined as*

$$SU(2) = \{A \in M_2(\mathbb{C}) : A^*A = I\}.$$

Clearly, we have

$$SU(2) = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} : |a|^2 + |b|^2 = 1, a, b \in \mathbb{C} \right\}.$$

To classify the representations of $SU(2)$ by Theorem A.25, it is sufficient to find all irreducible representations.

Let us start by looking at some examples of representations of $SU(2)$.

**Example A.28.** *The representation $(\rho, \mathbb{C}^2)$ defined by $\rho(A) = A, \forall A \in SU(2)$ is clearly a representation. We call $(\rho, \mathbb{C}^2)$ the standard representation of $SU(2)$.*

**Example A.29.** *Let $V_m$ denote the space of homogeneous polynomials of degree $m$ in two (complex) variables. Define a representation $(\pi_m, V_m)$ by:*

$$\pi_m(A)(p(z)) := p(A^{-1}z),$$

*where $z = (z_1, z_2)^T \in \mathbb{C}^2$ and $p(z)$ is a homogeneous polynomial.*

*Clearly $V_m$ is an $(m+1)$-dimension space with basis $\{p_k(z) = z_1^{m-k} z_2^k : k = 0, 1, \ldots, m\}$. It is not hard to see $(\pi_m, V_m)$ is indeed a representation:*

$$\begin{aligned}\pi_m(AB)(p(z)) &= p((AB)^{-1}z) \\ &= p(B^{-1}A^{-1}z) \\ &= \pi_m(B)p(A^{-1}z) \\ &= \pi_m(A)(\pi_m(B)p(z)).\end{aligned}$$

*We will show that $(\pi_m, V_m)$ is irreducible and indeed all irreducible representation of dimension $m+1$ is isomorphic to $(\pi_m, V_m)$. The representation of $SU(2)$ is built from the "bricks" $(\pi_m, V_m)$, that is, every representation of $SU(2)$ decomposes into the direct sum of representations of the $(\pi_m, V_m)$ and the decomposition is unique up to isomorphism.*

**Theorem A.30.** *The representation $(\pi_m, V_m)$ is irreducible and all irreducible representations of dimension $m+1$ are isomorphic to $(\pi_m, V_m)$.*

We first the Schur's lemma which implies that $(\pi_m, V_m)$ is irreducible.

**Lemma A.31** (Schur's lemma)**.** *Let $(\pi, V)$ be an irreducible representation of a group, then the intertwining map $\varphi : V \to V$ has the form $\lambda I$ for some scalar $\lambda \in \mathbb{C}$.*

*Conversely, if $(\pi, V)$ is unitary (or unitarizable) and the intertwining maps are of the form $\lambda I$ for some scalar $\lambda \in \mathbb{C}$. Then $(\pi, V)$ is irreducible.*

*Proof.* Suppose $\varphi : V \to V$ is a homomorphism between vector spaces, then there exists an eigenvalue $\lambda$ with its corresponding eigenspace $U$ (since $\mathbb{C}$ is algebraically closed). Since $\pi(g)\varphi = \varphi\pi(g)$ for all $g \in SU(2)$ then $U$ is indeed an invariant subspace of $V$. By irreducibility, we conclude that $V = U$ and thus $\varphi = \lambda I$.

Conversely, if $(\pi, V)$ is unitary and assume $W \subset V$ is a nontrivial invariant subspace (then so is $W^{\perp}$). Then consider the projection $P$ from $V$ onto $W$ which is intertwining. Thus we have $P = \lambda I$, which implies that $\lambda^2 I = P^2 = P = \lambda I$. Therefore we have $\lambda^2 = \lambda$. For $\lambda \neq 0$ we then have $\lambda = 1$ and $W = PV = V$. □

Now we can prove Theorem A.30.

*Proof.* Since $SU(2)$ is compact (thus unitarizable), to prove that a representation is irreducible, it is sufficient to show that the intertwining maps are all of form $\lambda I$. Let $\{p_k(z) = z_1^{m-k} z^k\}$ be a basis for the vector space $V_m$ and consider the following two closed subalgebras of $SU(2)$:

$$T := \left\{ t_\alpha = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} : \alpha \in \mathbb{R} \right\}, R := \{ r_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} : \alpha \in \mathbb{R} \}$$

Suppose $\varphi : V \to V$ is an intertwining map. We observe that

$$\pi(t_\alpha) p_k(z) = (e^{i\alpha} z_1)^{n-k} (e^{-i\alpha} z_2)^k = e^{i(n-2k)\alpha} p_k(z),$$

which shows that each $p_k$ is an eigenvector of $\pi(t_\alpha)$. Choose $\alpha$ such that each $e^{i(n-2k)\alpha}$ is different. Then, since $\varphi$ is intertwining, we have $\mathbb{C}p_k$ is an eigenspace of $\varphi$. Say, the eigenvalue of $\varphi$ with respect to $\mathbb{C}p_k$ is $\lambda_k$. Let now $E_0$ be the eigenspace with eigenvalue $\lambda_0$. What we will do is to prove that $\lambda_k = \lambda_0$ for each $k$. Since $\pi(a_\alpha)\varphi = \varphi\pi(a_\alpha)$ then we have

$$\varphi\pi(a_\alpha) p_0 = \varphi((\cos\alpha z_1 + \sin\alpha z_2)^n) \tag{A.3}$$

$$= \varphi(\sum_{m=0}^{n} \binom{n}{m} \cos^m \alpha z_1^m \sin^{n-m} \alpha z_2^{n-m}) \tag{A.4}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \cos^m \alpha \sin^{n-m} \alpha \lambda_{n-m} z_1^m z_2^{n-m} \tag{A.5}$$

On the other hand, we have

$$\pi(a_\alpha)\varphi p_0 = \lambda_0 (\sum_{m=0}^{n} \binom{n}{m} \cos^m \alpha z_1^m \sin^{n-m} \alpha z_2^{n-m}).$$

Combining it with (A.5), we obtain

$$\sum_{m=0}^{n} \binom{n}{m} \cos^m \alpha \sin^{n-m} \alpha (\lambda_{n-m} - \lambda_0) z_1^m z_2^{n-m} = 0.$$

Therefore we have $\lambda_0 = \lambda_k$ for all $0 \leq k \leq n$. Hence $\varphi = \lambda_0 I$ and then $(\pi_n, V_n)$ is irreducible.

The uniqueness is due to the uniqueness of irreducible representations of $\mathfrak{sl}(2)$ (the Lie algebra of $SL(2)$). Interested readers can refer to [6]. □

Let us look at another irreducible representation of $SU(2)$.

**Example A.32** (Tensor product of representations). *Let $(\phi, V)$ and $(\varphi, W)$ be two representations of $SU(2)$, then we define the tensor product of two representations $(\phi \otimes \varphi, V \otimes W)$. If $\{v_i\}$ and $\{w_j\}$ are the basis of $V$ and $W$ respectively, then $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$ and $\phi \otimes \varphi(g)(v \otimes w) := \phi(g)(v) \otimes \varphi(g)(w)$.*

*Now we can define the symmetric power of representations. Let $V$ be a vector space, consider the following map $S : V^{\otimes n} \to V^{\otimes n}$ such that*

$$S(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

*Then we call the image of $S$ the symmetric power of $V$ and denote it by $V^{\otimes_s n}$. We claim that for the standard representation $(\rho, \mathbb{C}^2)$ we have $(\rho^n, (\mathbb{C}^2)^{\otimes_s n})$ is an irreducible representation with dimension $n+1$ thus by the Theorem A.30, it is isomorphic to the representation in Example A.29.*

We end the appendix with the so called Clebsch-Gordan formula which reveals the decomposition of tensor product of $(\pi_m, V_m)$ and $(\pi_n, V_n)$. The proof can be found in [6].

**Theorem A.33** (Clebsch-Gordan formula). *Let $(\pi_m, V_m)$ and $(\pi_n, V_n)$ be two standard irreducible representations of $SU(2)$. Then we have the following formula of decomposition:*

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|} \tag{A.6}$$

**Remark A.34.** *The proof relies on the notions of character of a representation.*

Appendix B

# Integer Sequences Arising from $SU(2)$-subproduct Systems

In this chapter, we shall define an integer sequence which encodes the dimensions of the fibers in our subproduct system.

Let us recall the definition of dimension sequence firstly.

**Definition B.1.** *Let $\{d_m\}$ be the sequence of positive integers defined recursively by the following:*

$$d_{-1} := 0, d_0 := 1, d_1 := n+1, d_m := d_1 d_{m-1} - d_{m-2}, m \geq 2.$$

**Lemma B.2.** *Let $m, k, l \in \mathbb{N}_0$. We have the identities:*

$$d_m^2 - d_{m-1}d_{m+1} = 1, \tag{B.1}$$

$$\sum_{i=0}^{l} d_{k+m+2i} = d_{k+l}d_{m+l} - d_{k-1}d_{m-1}, \tag{B.2}$$

$$d_k d_m = \sum_{j=0}^{\min\{k,m\}} d_{k+m-2\min\{k,m\}-2j}. \tag{B.3}$$

*Proof.* We firstly prove the first identity by induction. For $m = 0$, the identity $d_0 = 1 = 1 + d_{-1}d_1 = 1$ holds clearly. By definition, we have

$$\begin{aligned} d_{m-1}d_{m+1} + 1 &= d_{m-1}(d_1 d_m - d_{m-1}) + 1 \\ &= d_1 d_{m-1} d_m - d_{m-1}^2 + 1 \\ &= d_1 d_{m-1} d_m - (d_{m-2}d_m + 1) + 1 \\ &= d_m(d_1 d_{m-1} - d_{m-2}) \\ &= d_m^2. \end{aligned}$$

We likewise prove the second identity by induction. For $l = 0$, we have to show that

$$d_{k+m} = d_k d_m - d_{k-1} d_{m-1} \tag{B.4}$$

For $m = 0, 1$ there is nothing to prove. So suppose (B.4) holds for $m \leq n$. Then we compute that

$$\begin{aligned} d_{k+n+1} &= d_{k+n} d_1 - d_{k+n-1} \\ &= (d_k d_n - d_{k-1} d_{n-1}) d_1 - d_k d_{n-1} + d_{k-1} d_{n-2} \\ &= d_k (d_n d_1 - d_{n-1}) - d_{k-1} (d_{n-1} d_1 - d_{n-2}) \\ &= d_k d_{n+1} - d_{k-1} d_n. \end{aligned}$$

Then suppose (B.2) holds for $l \leq n$. Then we have

$$\begin{aligned} \sum_{i=0}^{n+1} d_{k+m+2i} &= \sum_{i=0}^{n} d_{k+m+2i} + d_{k+m+2(n+1)} \\ &= d_{k+n} d_{m+n} - d_{k-1} d_{m-1} + d_{(k+n+1)+(m+n+1)} \\ &= d_{k+n} d_{m+n} - d_{k-1} d_{m-1} + d_{k+n+1} d_{m+n+1} - d_{k+n} d_{m+n} \\ &= d_{k+n+1} d_{m+n+1} - d_{k-1} d_{m-1}. \end{aligned}$$

Finally, combining (B.2) with the recursive definition, we obtain

$$\begin{aligned} d_k d_m &= d_{k+m} + d_{k-1} d_{m-1} \\ &= d_{k+m} + d_{k-1+m-1} + d_{k-2} d_{m-2} \\ &= \sum_{j=0}^{l} d_{k+m-2\min\{k,m\}+2j}. \end{aligned}$$

This proves the lemma. □

Finally we end this appendix by introducing sequence $\{\mu_m\}$.

**Lemma B.3.** *Let $\{\mu_m\}$ be the sequence defined as*

$$\mu_m := \frac{d_m d_{m-1}}{d_1}.$$

*We have*

$$\mu_m + \mu_{m+1} = d_m^2.$$

*Proof.* It follows from $d_m^2 = \mu_m + \mu_{m+1}$. Indeed, we have

$$\begin{aligned} \mu_m + \mu_{m+1} &= \frac{d_m d_{m-1}}{d_1} + \frac{d_{m+1} d_m}{d_1} \\ &= \frac{d_m(d_{m+1} + d_{m-1})}{d_1} \\ &= \frac{d_m(d_1 d_m - d_{m-1} + d_{m-1})}{d_1} \\ &= d_m^2. \end{aligned}$$

□

Furthermore, the sequence $\{\mu_m\}$ satisfies the recurrence relation:

$$\mu_1 = 1, \mu_2 = (n+1)^2 - 1, \mu_{m+1} = ((n+1)^2 - 2)\mu_m - \mu_{m-1} + 1$$

which one can check by the following:

$$\begin{aligned} &\mu_{m+1} = ((n+1)^2 - 2)\mu_m - \mu_{m-1} + 1 && \text{(B.5)} \\ \iff\ &\mu_{m+1} + \mu_m = d_1^2 \mu_m + 1 - (\mu_m + \mu_{m-1}) && \text{(B.6)} \\ \iff\ &d_m^2 = d_1 d_m d_{m-1} + 1 - d_{m-1}^2. && \text{(B.7)} \end{aligned}$$

by using (B.1) we have that (B.7) is equivalent to

$$1 + d_{m+1} d_{m-1} = d_1 d_m d_{m-1} + 1 - d_{m-1}^2 \iff d_{m+1} = d_1 d_m - d_{m-1}.$$

as $d_{m-1} \neq 0$ and the right hand is the definition of the sequence.

Appendix C

# Subproduct Systems of $C^*$-correspondence

## C.1 Hilbert modules and $C^*$-correspondences

The general definition of subproduct system consists of $C^*$ correspondences, a special case of Hilbert $C^*$-modules.

**Definition C.1** (Hilbert $C^*$-module)**.** *Let $A$ be a $C^*$-algebra, an inner product $A$-module is a vector space $E$ which is a right $A$-module endowed with a map $E \times E \to A, (x, y) \mapsto \langle x, y \rangle \in A$ satisfying the followings*

1. *for all $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ we have $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$;*

2. *for all $x, y \in E, a \in A$ we have $\langle x, ya \rangle = \langle x, y \rangle a$;*

3. *for all $x, y \in E$ we have $\langle y, z \rangle = \langle x, y \rangle^*$;*

4. *for all $x, y \in E$ we have $\langle x, x \rangle \geq 0$ and the equality holds if and only if $x = 0$.*

*This yields a well-defined norm $\| \cdot \|$ on $E$ defined by $\|x\| := \|\langle x, x \rangle\|^{1/2}$. We call $E$ a Hilbert $A$-module (or Hilbert $C^*$-module over $A$) if $E$ is complete with respect to this norm.*

Roughly speaking, an $(A, B)$-$C^*$ correspondence is Hilbert $C^*$-module with an additional left module structure given by endomorphisms. The formal definition is as follows:

**Definition C.2** (Adjointable $C^*$-honomorphism)**.** *Let $E$ and $F$ be two Hilbert $C^*$-modules over a $C^*$-algebra $A$. We define $\mathcal{L}_A(E,F)$ to be the set of all maps $\phi : E \to F$ for which there is a map $\phi^* : F \to E$ such that*

$$\langle \phi(x), y \rangle = \langle x, \phi^*(y) \rangle.$$

*We abbreviate $\mathcal{L}_A(E,F)$ to $\mathcal{L}_A(E)$ when $E = F$ and call such map adjointable.*

Furthermore, the maps in $\mathcal{L}(E,F)$ are bounded and $A$-linear [10].

**Definition C.3.** *Let $A$ and $B$ be $C^*$-algebras. An $(A,B)$-$C^*$-correspondence is a pair $(E,\varphi)$ consisting of a right Hilbert $B$ module $E$ and an injective nondegerate $C^*$ homomorphism $\varphi : A \to \mathcal{L}_B(E)$ which means that the image $\varphi(A)E$ is dense in $E$.*

**Remark C.4.** *Note that some authors define $C^*$-correspondences without the requirement of injectivity and density of $\varphi(A)E$, instead, if such conditions are satisfied, call the $C^*$-correspondence faithful and essential respectively. For the simplicity, we choose to put such requirements in our definition and abbreviate $\phi(a)x$ to $ax$ for all $x \in E$.*

From now on, we abbreviate $C^*$-corredpondence $(E,\phi)$ to $E$ when there is no confusion. Given two $C^*$-correspondences, one can take their product in a properly defined sense.

**Definition C.5** (Interior tensor product of $C^*$ correspondences)**.** *Let $X$ and $Y$ be $(A,B)$ and $(B,C)$ correspondences respectively. Then the interior tensor product $X \otimes_B Y$ is the completion of quotient of $X \otimes_{alg} Y$ by the subspace spanned by*

$$\{xb \otimes y - x \otimes by, x \in X, y \in Y, b \in B\}.$$

*Note that we get a right $C$-module by $(x \otimes y) \cdot c = x \otimes (y \cdot c)$ for all $c \in C$ with respect to the inner product defined as the following*

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle := \langle y_1, (\langle x_1, x_2 \rangle) y_2 \rangle.$$

The fact that the linear map $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle$ is an inner product is nontrivial, the detailed proof can be found in [10].

We are now ready to give the general definition of a subproduct system of $C^*$-correspondences.

**Definition C.6.** *Suppose that $E = \{E_m\}, m \in \mathbb{N}_0$ is a sequence $C^*$ correspondences over a $C^*$-algebra $B$ and that $\iota_{k,m} : E_{k+m} \to E_k \otimes_B E_m$ is a bounded adjointable isometry for every $k, m \in \mathbb{N}_0$. We say that $(E,\iota)$ is a subproduct system over $B$ when the following holds for all $k, l, m \in \mathbb{N}_0$:*

1. $E_0 = B$;
2. The structure maps $\iota_{0,m} : E_m \to E_0 \otimes_B E_m$ and $\iota_{m,0} : E_m \to E_m \otimes_B E_0$ are the canonical identifications and;
3. The two bounded adjointable isometries $(1_k \otimes \iota_{l,m}) \circ \iota_{k,l+m}$ and $(\iota_{k,l} \otimes 1_m) \circ \iota_{k+l,m} : E_{k+l+m} \otimes E_k \otimes_B E_l \otimes_B E_m$ agree, where $1_k$ and $1_m$ denote the identity operators on $E_k$ and $E_m$, respectively.

Let us look at an example.

**Example C.7.** *Let $X$ be a $B$-correspondence where $B$ is a $C^*$-algebra. Then we have $\{X^{\otimes_B n}\}_{n=0}^{\infty}$ is a subproduct system with the structure maps given by the canonical identification $X^{\otimes_B n+m} \cong X^{\otimes_B n} \otimes_B X^{\otimes_B m}$.*

**Definition C.8** (Fock correspondence). *Given a subproduct system of $C^*$-correspondence $(E, \iota)$ over $\mathbb{N}_0$, we define the Fock correspondence as the infinite Hilbert $C^*$-module direct sum $F := \oplus_{m=0}^{\infty} E_m$.*

Similarly, we have the Toeplitz algebra of subproduct system of $C^*$-corrrespondences. For each $\xi \in E_k$, we define the creation operator $T_\xi \in \mathcal{L}(F)$ as

$$T_\xi(\zeta) := \iota_{k,m}^*(\xi \otimes \zeta), \quad \forall \zeta \in E_m.$$

**Definition C.9** (Toeplitz algebra). *Let $(E, \iota)$ be a subproduct system over $\mathbb{N}_0$. We define the Toeplitz algebra of $(E, \iota)$ denoted by $\mathbb{T}_E$, as the smallest unital $C^*$-subalgebra of $\mathcal{L}(F)$ that contains all the creation operators.*

Finally, we define the Cuntz–Pimsner algebra as follows.

**Definition C.10** (Cuntz–Cuntz–Pimsner algebra). *Given a subproduct system $(E, \iota)$ of $C^*$ correspondences over $\mathbb{N}_0$. Let $Q_n : F \to F$ be the orthonormal projection onto $E_n$. The Cuntz–Pimsner algebra of $(E, \iota)$ is the unital $C^*$-algebra obtained as the quotient of the Toeplitz algebra $\mathbb{T}_E$ by the ideal*

$$\mathbb{I}_E := \{x \in \mathbb{T}_E : \lim_{n\to\infty} \|Q_n x\| = 0\},$$

*which is denoted by $\mathbb{O}_E := \mathbb{T}_E / \mathbb{I}_E$.*

**Remark C.11.** *Note that in the subproduct system of finite-dimensional Hilbert spaces, the ideal $\mathbb{I}_E$ is isomorphic to the ideal $\mathbb{K}_F$ of compact operators over the Fock space. Therefore the Definition C.10 coincides with the Definition 3.6.*

## C.2 $G$-subproduct systems

In this section, we consider subproduct systems with group actions. Before defining group actions on subproduct systems, we shall firstly define group actions on $C^*$-algebras.

**Definition C.12** ($G$-$C^*$-algebra)**.** *Let $G$ be a locally compact topological group, a $G$-$C^*$-algebra $A$ is a $C^*$-algebra endowed with a strongly continuous action of $G$ by $*$-automorphisms.*

We remind the reader that the strong continuity means that the orbit map $\xi_a$ defined by

$$\xi_a : G \to A, \quad g \mapsto g.a.$$

is continuous for all $a \in A$.

Actions on $C^*$-correspondence are defined in a similar way.

**Definition C.13** (Automorphisms of $C^*$-correspondences)**.** *Let $X$ be an $A$-correspondence. An automorphism from $X$ to $X$ is a pair $(\phi, \varphi)$ where $\phi : A \to A$ is a $*$-isomorphism and $\varphi$ is a bijective adjointable map from Hilbert $C^*$-module $X$ to itself such that*

$$\varphi(ax) = \phi(a)\varphi(x).$$

Now we are ready to define the $G$-$C^*$-correspondence.

**Definition C.14** ($G$-$C^*$-correspondence)**.** *Let $G$ be a locally compact topological group, a $G$-$C^*$-correspondence $E$ is a $C^*$-correspondence with a strongly continuous action of $G$ by automorphisms of $C^*$-correspondences.*

Using the above concepts, we can define $G$-subproduct systems.

**Definition C.15** ($G$-subproduct systems)**.** *Let $G$ be a locally compact topological group and let $(E, \iota)$ be a subproduct system over a $C^*$-algebra $B$. We say that $(E, \iota)$ is a $G$-subproduct system when $B$ is a $G$-$C^*$-algebra and $E_m$ is a $G$-$C^*$-correspondence for all $m \in \mathbb{N}$, such that the structure maps $\iota_{k,m} : E_{k+m} \to E_k \otimes_B E_m$ are $G$-equivariant for all $k, m \in \mathbb{N}_0$ i.e. $g \circ \iota_{k,m} = \iota_{k,m} \circ g$. The action on $E_m \otimes_B E_n$ is given by $g(\xi \otimes \eta) := g(\xi) \otimes g(\eta)$.*

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