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Motivic Homotopy Theory and the Classification of Vector Bundles

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Motivic Homotopy Theory and the Classification of Vector Bundles

THESIS

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Motivic Homotopy Theory and the Classification of Vector Bundles

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Abstract

We define a homotopy theory for schemes, namely \mathbb{A}^1 -homotopy theory using a model categorical approach. With this homotopy theory, we can classify vector bundles of rank r over affine smooth schemes. Furthermore, we show a K -theoretical method to compute the \mathbb{A}^1 -homotopy classes of morphisms and then discuss some examples.

Contents

1	Introduction	1
2	Model Category	5
2.1	Model Category	5
2.2	The Category of Simplicial Set	14
2.3	Simplicial Model Category	18
2.4	Homotopy Limit and Colimit	21
2.5	Pointed Construction	25
3	\mathbb{A}^1-Homotopy Theory	27
3.1	Category of Presheaves	27
3.2	Bousfield Localization	30
3.3	\mathbb{A}^1 -local Model Category	32
3.4	Singular Construction	38
3.5	cd-Structure	42
3.6	Pointed Theory	49
4	Application and K-Theory	55
4.1	Some Direct Computations	55
4.2	An Affine Representability	59
4.3	Classifying Spaces and K -Theory	62
4.3.1	Classifying Space	62
4.3.2	Algebraic Linear Groups	66
4.3.3	K -Theory	69
4.4	Examples	71
A	Purity	77
A.1	Purity Theorem	77

Introduction

Let k be a perfect field and \mathbf{Sm}_k be the category of finitely presented smooth k -schemes. Let \mathbf{sSets} be the category of simplicial sets, we denote $\mathbf{sPre}(\mathbf{Sm}_k)$ for the category of simplicial presheaves of sets on \mathbf{Sm}_k , i.e. the functors in form of $\mathbf{Sm}_k^{op} \times \Delta^{op} \rightarrow \mathbf{Sets}$. We will set the motivic homotopy theory or \mathbb{A}^1 -homotopy theory on a such foundation in approach of model categories. By this we mean assigning the injective model structure on the category of simplicial presheaves on \mathbf{Sm}_k . Then by (left) Bousfield localization, we modified the model structure to impose a Nisnevich-descent condition and the contractibility of the affine line \mathbb{A}^1 to the simplicial model category $\mathbf{sPre}(\mathbf{Sm}_k)$. Based on this, we can construct the \mathbb{A}^1 -homotopy sheaves of sets, which is an analogue of homotopy groups for topological spaces, and achieve certain computations with them. Furthermore, as working in the pointed category $\mathbf{Spc}_{\mathbb{A}^1, \bullet}$, we can set up the notation to \mathbb{A}^1 -homotopy fiber sequences and then, similar to topological homotopy theory, we obtain the canonical long exact sequences of \mathbb{A}^1 -homotopy sheaves.

In the thesis, the main goal is an affine motivic classification of vector bundles which is recently proven by Asok, Hoyois, and Wendt in [1]:

Theorem 1.0.1 (1). *Let $\mathcal{V}_r(X)$ be the isoclasses of vector bundles of rank r over affine scheme X and Gr_r be the infinite Grassmannian of rank r , then we have an isomorphism*

$$\mathcal{V}_r(X) \cong [X, Gr_r]_{\mathbb{A}^1}.$$

This classification leads to a connection to algebraic K -theory: as Morel and Voevodsky prove that in motivic homotopy theory, the infinite Grassmannian Gr_r represents the classifying space BGL_r of general linear group GL_r , the computation of the \mathbb{A}^1 -homotopy classes of morphisms to Gr_r is equivalent to compute \mathbb{A}^1 -homotopy sheaves of BGL_r in certain obstructional cases. It turns out that this computation is equivalent to the computation of K -theory:

Theorem 1.0.2 (2). For $n = 1$, $BGL_1 = BG_m$ is an \mathbb{A}^1 -Eilenberg-MacLane space $K(\mathbb{G}_m, 1)$.

For $n \geq 2$, we have isomorphisms of sheaves:

$$\begin{aligned}\pi_0^{\mathbb{A}^1} BGL_n &\cong * \\ \pi_1^{\mathbb{A}^1} BGL_n &\cong \mathbb{G}_m \\ \pi_i^{\mathbb{A}^1} BGL_n &\cong K_i\end{aligned}\tag{0-2}$$

where $2 \leq i < n$.

This thesis is organized as follows. In Chapter 1, we introduce the basic theory of model categories. Important concepts such as simplicial sets and homotopy (co)limits are defined and related properties are proved. In Chapter 2, we set up the \mathbb{A}^1 -homotopy starting from basics like simplicial presheaves. A technical construction of \mathbb{A}^1 -localization, which involves the singular construction and relations between different cd-structures on simplicial presheaves, is introduced. At last we define the \mathbb{A}^1 -homotopy fibers and connectedness of spaces in pointed theory. In Chapter 3, we start with some simple computations of \mathbb{A}^1 -homotopy types. Then we prove the affine classification and discuss the K -theoretical theory. At last we show some examples and non-examples of the affine classification.

For notations, in a simplicial model category \mathcal{M} , W, C, F will respectively be the collection of **weak equivalences**, **cofibrations**, and **fibrations**. The **function complex** or **mapping space** will be denoted as $map_{\mathcal{M}}(-, -)$. The derived function complex is denoted as $Map_{\mathcal{M}}(-, -)$. Generally, given a pair of Quillen adjunction (F, G) , we write their derived functor as $(\mathbf{L}F, \mathbf{R}G)$. The homotopy classes of morphisms will be denoted as $[X, Y]$ for $X, Y \in \mathcal{M}$.

In this thesis, multiple model structures will show up. Therefore we use subscripts or superscripts to distinguish them. For the **injective model structure** on $\mathbf{sPre}(\mathbf{Sm}_k)$, the notations will come with subscript $(-)_s$ or just without subscript. For the Nis -local (resp. \mathbb{A}^1 -local) model structure, we will use $(-)_{Nis}$ (resp. $(-)_{\mathbb{A}^1}$). More generally, the character τ will usually mean a Grothendieck topology, and the τ -local model structure will be denoted as $(-)_{\tau}$. We write $\mathbf{sPre}(\mathbf{Sm}_k)$ or just \mathbf{sPre} for the category of simplicial presheaves with injective model structure. However, conventionally we write \mathbf{Spc}_{Nis} , \mathbf{Spc} respectively for the category of simplicial presheaves with Nis -local model structure and \mathbb{A}^1 -local model structure.

Our definitions for model category are mainly from Hovey [2] and Lurie [3]. The basics for simplicial sets are from Goerss and Jardine [4]. The knowledge of \mathbb{A}^1 -homotopy theory are mainly from Morel and Voevodsky's original paper [5] and also from Asok, Hoyer, and Wendt [1][6]. Additionally, an introductory

paper by Antieau and Elmanto [7] and a new survey by Asok and Paul [8] are extremely helpful.

Model Category

In this section, we introduce the the foundation we need to build up the \mathbb{A}^1 -homotopy theory: model category structure. In the first section, we define the basic notations of a model category and some naive manipulation of cofibrations and fibrations. Then in section 2, we define the most basic model category: the category of simplicial sets. This allows us to assign simplicial structure to a model category, as what we do in section 3. Besides these, the concept of homotopy (co)limits are introduced in section 4. At last, in section 5, pointed model category is defined.

2.1 Model Category

This section will define the Model Category. And some basic tricks to operates morphisms in a model category are introduced in Example 2.1.5, Proposition 2.1.16. More importantly, we spend a rich amount of pages to justify our notations, e.g. Proposition 2.1.8, Remark 2.1.14, as in practice many notations are simplified in some conditions. Because Working with model categories, sometimes even multiple at the same time, must to respect certain restrictions. For example, a left Quillen functor does not always preserves weak equivalences, though its derived functor does, but with certain conditions: if it is a weak equivalence between cofibrant objects. One should always be careful about the language and validity of the statements in a model structural setting, even for simple statements such as ‘ A is weak equivalent to B ’.

In a category, given two morphisms $i : A \rightarrow X, p : E \rightarrow B$, we say i satisfies **left lifting property (LLP)** with respect to p , or equivalently p satisfies **right lifting property (RLP)** with respect to i if any commutative diagram as

follows admits a lifting arrow H of h :

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow i & \nearrow H & \downarrow p \\ X & \xrightarrow{h} & B \end{array}$$

Definition 2.1.1 (Model Category). Let \mathcal{M} be a category equipped with three classes of morphisms W, C, F , namely **weak equivalences**, **cofibrations** and **fibrations**, respectively denoted $\xrightarrow{\sim}, \xrightarrow{\rightarrow}, \xrightarrow{\twoheadrightarrow}$. Then \mathcal{M} is a **Model Category** if it satisfies the following axioms:

M1 \mathcal{M} has all small limits and colimits.

M2 Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$. If any two of $f, g, g \circ f$ are weak equivalences, then so is the third.

M3 Each of W, C and F is closed under retract, i.e. given a commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{p} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \xrightarrow{j} & Y & \xrightarrow{q} & B \end{array}$$

s.t. $p \circ i = id, q \circ j = id$, if g is a weak equivalence, cofibration or fibration, then so is f .

M4 Any fibration satisfies the RLP with respect to all **trivial cofibrations**, i.e. cofibrations which are also weak equivalences. Any **trivial fibration**, i.e. a fibration which is a weak equivalence, satisfies the RLP with respect to all cofibrations.

M5 Any morphism $X \rightarrow Y$ admits two functorial factorization: there exist two objects X', Y' fit into the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ \downarrow & \searrow & \downarrow \\ Y' & \xrightarrow{\twoheadrightarrow} & Y \end{array}$$

where the top and bottom horizontal arrows are weak equivalences.

A classical example would be the category of topological spaces **Top**. The weak equivalences are those continuous map which induces isomorphisms at all homotopy groups. The fibrations are Serre fibrations. The cofibrations are those 'generated' by the following principles:

Proposition 2.1.2. *We have the following statements:*

1. *Fibrations are precisely those have RLP with respect to trivial cofibrations.*
2. *Trivial fibrations are precisely those have RLP with respect to cofibrations.*
3. *Cofibrations are precisely those have LLP with respect to trivial fibrations.*
4. *Trivial cofibrations are precisely those have LLP with respect to fibrations.*

Remark 2.1.3. In some model category, e.g. the category of simplicial sets, there are two sub-collections of cofibrations and trivial cofibrations, namely **generating cofibrations** and **generating trivial cofibrations**. Such sub-collections 'generate' the (trivial) cofibrations. By this we mean: a morphism is a fibration (*resp.* trivial fibration) if it satisfies RLP with respect to all these generating trivial cofibrations (*resp.* generating cofibrations).

Definition 2.1.4. *Let \mathcal{M} be a model category. We denote \emptyset for an initial object and $*$ for a final object. We say an object X is **fibrant** if $X \rightarrow *$ is a fibration and is **cofibrant** if $\emptyset \rightarrow X$ is a cofibration.*

Example 2.1.5. Given two cofibrant objects, then their disjoint union is cofibrant as we have a commutative diagram

$$\begin{array}{ccccc}
 \emptyset & \longrightarrow & X & \longrightarrow & E \\
 \downarrow & & \downarrow & \nearrow & \downarrow \simeq \\
 Y & \longrightarrow & X \sqcup Y & \longrightarrow & B
 \end{array}$$

Since $\emptyset \rightarrow Y$ is a cofibration, hence we obtain a dotted arrow $Y \rightarrow E$ which makes the outer square commutes. Then by pushout property, we obtain the wanted lifting $X \sqcup Y \rightarrow E$. Therefore $X \rightarrow X \sqcup Y$ is a cofibration and similar things happen to $Y \rightarrow X \sqcup Y$.

Dually, given two fibrant objects X, Y , then their product is fibrant as well.

Furthermore, by a similar proof, we can show that the pushout of a cofibration along a cofibration is a cofibration. And dually the pullback of a fibration along a fibration is a fibration.

However, pushout of a weak equivalence along a cofibration may not be a weak equivalence. Lurie shows in [3, Lemma A.2.4.3] shows that the a weak equivalence can only be preserved in this way when it is a weak equivalence between cofibrant objects. And dually, pullback, along a fibration, of a weak equivalence between fibrant objects is a weak equivalence.

Remark 2.1.6. For arbitrary X in a model category \mathcal{M} , we say an object, usually denoted QX , is a **cofibrant replacement** of X if it is equipped with a trivial fibration $QX \xrightarrow{\simeq} X$. The existence of a such replacement is ensured by axiom **M5**: given the unique morphism $f : \emptyset \rightarrow X$, we have a factorization

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & QX \\ & \searrow f & \downarrow \simeq \\ & & X \end{array} \cdot$$

Dually, we say RY equipped with a trivial cofibration $Y \xrightarrow{\simeq} RY$ is a **fibrant replacement**.

Given an object X in model category \mathbf{M} . We define a **cylinder object** C_X and a **path object** P_X of X to be the objects fit into the following commutative diagrams:

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{i_0 \sqcup i_1} & C_X \\ & \searrow & \downarrow \simeq \\ & & X \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\simeq} & P_X \\ & \searrow & \downarrow p_0 \times p_1 \\ & & X \times X \end{array} \cdot$$

Definition 2.1.7. Given morphisms $f, g : X \rightarrow Y$ in a model category \mathcal{M} . A **left homotopy** between f and g is a morphism $H : C_X \rightarrow Y$ s.t. $f = H \circ i_0, g = H \circ i_1$. A **right homotopy** is a morphism $H : X \rightarrow P_Y$ s.t. $f = p_0 \circ H, g = p_1 \circ H$. In diagram, we mean:

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{i_0 \sqcup i_1} & C_X \\ & \searrow f \sqcup g & \downarrow H \\ & & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{H} & P_Y \\ & \searrow f \times g & \downarrow p_0 \times p_1 \\ & & Y \times Y \end{array}$$

Given two morphisms $f, g : X \rightarrow Y$, we say they are **left homotopic** (resp. **right homotopic**) if there exists a left (resp. right) homotopy between them, denoted $f \stackrel{l}{\simeq} g$ (resp. $f \stackrel{r}{\simeq} g$). we say they are **homotopic**, written $f \simeq g$ if they are both left and right homotopic. A **homotopy equivalence** is a morphism $f : X \rightarrow Y$ s.t. there exists $g : Y \rightarrow X$ satisfies $fg \simeq id_Y, gf \simeq id_X$.

Proposition 2.1.8. Let \mathcal{M} be a model category and $f, g : X \rightarrow Y$ be two morphisms. Then the following statements hold:

- (1) If $f \stackrel{l}{\simeq} g$ and X is fibrant, then for arbitrary morphism $h : Z \rightarrow X$, $fh \stackrel{l}{\simeq} gh$.

- (2) If X is cofibrant, then left homotopy is an equivalence relation on $\text{Hom}_{\mathcal{M}}(X, Y)$.
- (3) If X is cofibrant, then given a trivial fibration or a weak equivalence between fibrant objects $h : Y \rightarrow Z$, h induces an isomorphism:

$$\text{Hom}_{\mathcal{M}}(X, Y) / \sim \xrightarrow{\cong} \text{Hom}_{\mathcal{M}}(X, Z) / \sim$$

- (4) If X is cofibrant, then a left homotopy always induces a right homotopy.

One can dually form similar statements for fibrantness and right homotopy.

Sketch of proof. The details can be found in [2, Proposition 1.2.5]. The proof for right homotopy is purely symmetrical and dual.

- (1) Given a left homotopy $H : C_X \rightarrow Y$ between f and g , consider the diagram of the form:

$$\begin{array}{ccccc} Z \sqcup Z & \xrightarrow{h \sqcup h} & X \sqcup X & \xrightarrow{\quad} & C_X \\ \downarrow & \searrow & \searrow & & \downarrow \simeq \\ C_Z & \xrightarrow{\simeq} & Z & \xrightarrow{h} & X \end{array}$$

The cofibrantness of X allow us to assume $C_X \rightarrow X$ is a trivial fibration (otherwise we take its fibrant replacement) and so we obtain a lifting $C_h : C_Z \rightarrow C_X$. This morphism fits into a commutative diagram:

$$\begin{array}{ccccc} Z \sqcup Z & \xrightarrow{h \sqcup h} & X \sqcup X & \xrightarrow{f \sqcup g} & Y \\ \downarrow & & \downarrow & \nearrow H & \\ C_Z & \xrightarrow{C_h} & C_X & & \end{array},$$

and so $H \circ C_h$ is what we want.

- (2) The reflexive and symmetric conditions are purely tautological. Hence we are left to prove the transitivity in case of X is fibrant.

The intuition to this construction is similar to the case of classical homotopy theory for topological spaces. Let $H_{fg} : C_{fg} \rightarrow Y$ be a left homotopy between $f, g : X \rightarrow Y$ and $H_{gh} : C_{gh} \rightarrow Y$ be one between $g, h : X \rightarrow Y$. Consider the natural inclusion $i_1 : X \rightarrow C_{fg}, j_0 : X \rightarrow C_{gh}$

s.t. $H_{fg}(i_1) = H_{gh}(j_0) = g$ and the pushout induced:

$$\begin{array}{ccc}
 X & \xrightarrow{i_1} & C_{fg} \\
 \downarrow j_0 & \lrcorner & \downarrow \\
 C_{gh} & \longrightarrow & M \dashrightarrow Y \\
 & \searrow H_{gh} & \uparrow \\
 & & H_{fg}
 \end{array}$$

As shown above, we have a pushout morphism $H : M \rightarrow Y$. One can show that M is a cylinder object and H is a left homotopy from f to h in case of X cofibrant.

- (3) Suppose $h : Y \rightarrow Z$ is a trivial fibration. Then by (1)(2), the map of sets $[h_*] : \text{Hom}(X, Y) / \sim \rightarrow \text{Hom}(X, Z) / \sim$ is well-defined. Given $g : X \rightarrow Z$, then by the lifting property of $\emptyset \rightarrow X$ and h , we obtain the wanted lifting $\bar{g} : X \rightarrow Y$. Hence $[h_*]$ is surjective. On the other hand, given $f, g : X \rightarrow Y$ s.t. $hf \simeq hg$ with the corresponding left homotopy $H : C_X \rightarrow Z$. Then by the lifting property of $X \sqcup X \rightarrow C_X$ and h , we again obtain a lifting $\bar{H} : C_X \rightarrow Y$ which is the wanted left homotopy. Therefore $[h_*]$ is injective.

Then for $h : Y \rightarrow Z$ being a weak equivalence of fibrant objects, we apply the Ken Brown's Lemma in [2, Lemma 1.1.12]: if a functor between model categories sends all trivial cofibrations (*resp.* fibrations) to weak equivalences, then it sends all weak equivalences between cofibrant (*resp.* fibrant) objects to weak equivalences. Then we are done.

- (4) Let $H : C_X \rightarrow Y$ be a left homotopy between $f, g : X \rightarrow Y$ s.t. X is cofibrant. Notice the induced inclusion $i_0 : X \rightarrow C_X$ is a trivial cofibration when X is cofibrant, hence the lifting as follows:

$$\begin{array}{ccc}
 X & \longrightarrow & P_X \\
 \downarrow i_0 & \nearrow H_r & \downarrow \\
 C_X & \longrightarrow & X \times X
 \end{array}$$

The bottom arrow is induced by the trivial left homotopy at f and H . The top arrow is the trivial right homotopy at f . Then $H_r(i_1)$ is the wanted right homotopy.

□

Corollary 2.1.9. *In a model category \mathcal{M} , let $f, g : X \rightarrow Y$ be two morphisms from cofibrant object to fibrant object. Then the left (resp. right) homotopy type of f, g is independent to the choice of the cylinder (resp. path) object. And also a left homotopy between f and g coincides with a right homotopy and they are equivalence relations on $\text{Hom}_{\mathcal{M}}(X, Y)$.*

Remark 2.1.10. In many cases, the objects in the model category we work with would be either cofibrant or fibrant for free. In the cofibrant case, by prop 2.1.8, any left homotopy implies a homotopy. Therefore in this thesis, one can see for simplicial sets \mathbf{sSets} and simplicial presheaves $\mathbf{sPre}(\mathbf{Sm}_k)$, we actually define the homotopy in terms of the left homotopy.

Theorem 2.1.11. *Let $f : X \rightarrow Y$ be a morphism in a model category \mathcal{M} . Suppose X, Y are both cofibrant and fibrant. Then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. [2, Proposition 1.2.8]. □

We construct the **homotopy category** $\mathcal{H}(\mathcal{M})$ by localization at weak equivalences. By this we mean, following notations from [2, Definition 1.2.1], we first construct a category consists of all objects from \mathcal{M} and its morphisms are finite strings of composable arrows (f_1, \dots, f_m) where f_i is either an arrow in \mathcal{M} or the reversal w_i^{-1} of an weak equivalence w_i . Then $\mathcal{H}(\mathcal{M})$ is the quotient category with respect to the relations of composition and $(w_i, w_i^{-1}) = id_{source(w_i)}$, $(w_i^{-1}, w_i) = id_{target(w_i)}$. The homotopy category $\mathcal{H}(\mathcal{M})$ comes with a **localization functor** $\gamma : \mathcal{M} \rightarrow \mathcal{H}(\mathcal{M})$ which is defined as identity on objects and sends morphisms to their homotopy classes. Such a functor is characterized by the following universal property:

Lemma 2.1.12. *For a localization functor $\gamma : \mathcal{M} \rightarrow \mathcal{H}(\mathcal{M})$ as above, we have the following universal property: if $F : \mathcal{M} \rightarrow \mathcal{N}$ is a functor that sends weak equivalences to isomorphisms, then it admits a unique factorization:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \gamma \downarrow & \nearrow & \\ \mathcal{H}(\mathcal{M}) & & \mathcal{H}(\mathcal{N}) \end{array}$$

Proof. [2, Lemma 1.2.2]. □

Remark 2.1.13. The homotopy category can be constructed in many ways up to categorical equivalences. For example, let $\mathcal{M}_c, \mathcal{M}_f, \mathcal{M}_{cf}$ be the subcategory of

\mathcal{M} that consist of respectively cofibrant, fibrant, cofibrant fibrant objects, then they all lead to equivalent homotopy categories. Also consider \mathcal{M}_{cf}/\sim which is obtained by taking homotopic quotient of the morphisms $Hom(X, Y)/\sim$ by Proposition 2.1.9. Hovey shows in [2, Corollary 1.2.9] that this category is isomorphic to $\mathcal{H}(\mathcal{M}_{cf})$.

Remark 2.1.14. The categorical equivalence $\mathcal{H}(\mathcal{M}) \rightarrow \mathcal{H}(\mathcal{M}_{cf})$ is given by $\mathcal{H}(Q) \circ \mathcal{H}(R)$:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{R} & \mathcal{M}_f & \xrightarrow{Q} & \mathcal{M}_{cf} \\ \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\ \mathcal{H}(\mathcal{M}) & \xrightarrow[\mathcal{H}(R)]{\sim} & \mathcal{H}(\mathcal{M}_f) & \xrightarrow[\mathcal{H}(Q)]{\sim} & \mathcal{H}(\mathcal{M}_{cf}) \xrightarrow{\cong} \mathcal{M}_{cf}/\sim \end{array} .$$

The diagram shows we can compute the homotopy classes as

$$[X, Y] := Hom_{\mathcal{H}(\mathcal{M})}(X, Y) \cong Hom_{(\mathcal{M})}(QRX, QRY)/\sim .$$

Combined with proposition 1.1.3 (and its dual version), we can do a further simplification

$$[X, Y] \cong Hom_{\mathcal{M}}(QRX, QRY)/\sim \cong Hom_{\mathcal{M}}(QX, RY)/\overset{l}{\sim} \cong Hom_{\mathcal{M}}(QX, RY)/\overset{r}{\sim}$$

where $\overset{l}{\sim}, \overset{r}{\sim}$ stand for the equivalence relation given by left and right homotopy.

Definition 2.1.15. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a pair of adjoint functors of model categories. We say this pair is a **Quillen functor** if the following equivalent conditions hold:

1. F preserves cofibrations and trivial cofibrations.
2. G preserves fibrations and trivial fibrations.

And we say F is a **left Quillen functor** and G is a **right Quillen functor**.

Given a pair of Quillen functor (F, G) as above, we canonically have their **total left/right derived functor**:

$$\mathbf{L}F : \mathcal{H}(\mathcal{M}) \rightleftarrows \mathcal{H}(\mathcal{N}) : \mathbf{R}G$$

which is again an adjoint functor. One thing to notice is that $\mathbf{L}F(M)$ is weak equivalent to $F(QM)$ and $\mathbf{R}G(N)$ is weak equivalent to $G(RN)$. Hence in practice, we often take $\mathbf{L}F = F \circ Q$ and $\mathbf{R}G = G \circ R$. This also leads to another

abuse of notations: we might view these derived functors as functors between the original model categories and then, for instance, compute the function complex. For a much strict description, one can see in [2, Chapter 1.3].

We say a pair of Quillen functor is an **Quillen equivalence** if their total derived functor is an equivalence of the homotopy categories.

Proposition 2.1.16. *Let (F, G) be a pair of Quillen functors. Then F preserves cofibrant objects and weak equivalences between them. Dually, G preserves fibrant objects and weak equivalences between them.*

Proof. We only prove the left Quillen part. The proof to the right Quillen functor can be dually done.

Since $F \dashv G$, hence F preserves initial objects. Therefore since F preserves cofibration, one can easily tell F preserves cofibrantness. Let $f : X \rightarrow Y$ be a weak equivalence between cofibrant objects. We first prove the canonical morphism $X \rightarrow X \sqcup Y$ is a cofibration by checking the lifting property. Consider a commutative diagram:

$$\begin{array}{ccccc} \emptyset & \hookrightarrow & X & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \simeq \\ Y & \longrightarrow & X \sqcup Y & \longrightarrow & B \end{array}$$

Since $\emptyset \hookrightarrow Y$ is a cofibration, hence we obtain a morphism $Y \rightarrow E$. Then by pushout, we obtain the wanted lifted morphism $X \sqcup Y \rightarrow E$.

We consider the morphism $X \sqcup Y \rightarrow Y$ induced by f and id_Y . And we factorize it as:

$$\begin{array}{ccccc} & & & & Y \\ & & & & \uparrow \bar{f} \\ & & & & \downarrow \\ X & \xrightarrow{i_0} & X \sqcup Y & \xrightarrow{p} & Z \end{array}$$

f (curved arrow from X to Y)
 $f \sqcup id_Y$ (arrow from $X \sqcup Y$ to Y)

s.t. \bar{f} is a trivial fibration. Notice Z is cofibrant, hence since f and \bar{f} are weak equivalences, therefore $p \circ i_0$ is a weak equivalence between cofibrant objects. Then so is $F(p \circ i_0)$. Then since $F(\bar{f})$ is a weak equivalence as well, hence $F(f)$ is a weak equivalence. □

Remark 2.1.17. We potentially prove the Ken Brown's Lemma for the second part of the proposition.

Definition 2.1.18. *Let \mathcal{M} be a model category. We say \mathcal{M} is **left proper** if the*

pushout along cofibrations preserves weak equivalences, i.e. given a pushout square:

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow f & & \downarrow f' \\ B & \longrightarrow & Y \end{array}.$$

If f is a weak equivalence and i is a cofibration, then f' is a weak equivalence.

Dually \mathcal{M} is **right proper** if the pullback along fibrations preserves weak equivalences. And \mathcal{M} is **proper** if it is both left and right proper.

Remark 2.1.19. If we are working with a model category whose objects are all cofibrant, then we are given the left properness for free: by Example 2.1.5, since any weak equivalence in a such model category is one between cofibrant objects, hence it can be preserved by pushout along cofibrations. Dually, a model category whose objects are all fibrant is automatically right proper.

It is quite often that people work with a such model category because a lot of model structures arise from a diagrammatic way, which allow us to assign fibrantness or cofibrantness pointwise and then trivialize the them. Normally these construction are given names such as injective or projective model structure, just like what we will do with the homotopy limits and colimits in Section 5.

2.2 The Category of Simplicial Set

A fundamental class of objects for homotopic universe are the **simplicial sets**.

We start by setting up to an indexing category: let Δ be the category of finite ordinal numbers with order-preserving maps as morphisms. Particularly, there are two important types of order-preserving maps:

$$\begin{aligned} d^i &: \mathbf{n} - \mathbf{1} \rightarrow \mathbf{n} \quad 0 \leq i \leq n \\ s^i &: \mathbf{n} + \mathbf{1} \rightarrow \mathbf{n} \quad 0 \leq i \leq n \end{aligned}$$

namely **cofaces** and **codegeneracies**. Cofaces are unique injections given by skipping the target value i and codegeneracies are unique surjections given by repeating target value j .

Particularly, cofaces and codegeneracies can generates all other order-preserving maps by finitely many composition with certain commutativity:

Proposition 2.2.1 (Cosimplicial Identities). *Fixing $n \geq 0$, let $0 \leq i, j \leq n$. We have:*

$$\begin{cases} s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^i = s^j d^i = id & \text{if } i = j, i = j + 1 \\ s^j d^i = d^{i-1} s^j & \text{if } i > j + 1 \\ d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases}$$

Remark 2.2.2. We also have simplicial identities, which is just the dual of cosimplicial identities.

Occasionally, we will also use a special type of monotone map $\alpha^i : \mathbf{n} \rightarrow \mathbf{1}$, s.t. $\alpha^i(j) = 0$ for $j \leq i$ and $\alpha^i(j) = 1$ for $j > i$. And some identities can be checked:

Proposition 2.2.3. *We have equalities:*

$$\begin{cases} \alpha^j d^i = \alpha^{j-1} & \text{if } i \leq j \\ \alpha^j d^i = \alpha^j & \text{if } i > j \\ \alpha^j s^i = \alpha^{j+1} & \text{if } i \leq j \\ \alpha^j s^i = \alpha^j & \text{if } i > j \end{cases}$$

Definition 2.2.4 (Simplicial Objects). *Let \mathcal{M} be a category with small limits and colimits. A **simplicial object** in \mathcal{M} is a functor $X : \Delta^{op} \rightarrow \mathcal{M}$. The category of simplicial objects in \mathcal{M} with obvious morphisms is denoted \mathbf{sM} . We will simply denote $X(\mathbf{n})$ as X_n .*

*Dually, a **cosimplicial object** in \mathcal{M} is a functor $X : \Delta \rightarrow \mathcal{M}$.*

Example 2.2.5. The most fundamental example is the category of **simplicial sets**, denoted \mathbf{sSets} . By definition, $\mathbf{sSets} = Fun(\Delta^{op}, Sets)$ with natural transformations being morphisms.

A classical example of simplicial sets would be the singular set $\mathcal{S}(X)$ of a given topological space X . By definition, $\mathcal{S}(X)_n := Hom_{\mathbf{Top}}(\Delta_{top}^n, X)$ where $\Delta_{top}^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}$. The faces and degeneracies are given by pre-composition of maps induced by cofaces and codegeneracies at the vertices of Δ_{top}^n .

Similarly, we can define the **standard simplices** in \mathbf{sSets} as $\Delta^n := Hom_{\Delta}(-, \mathbf{n})$. The faces and degeneracies are given by pre-composition of cofaces and codegeneracies.

Theorem 2.2.6 (Yoneda Lemma). *Let X be a simplicial set. Then we have a natural bijection:*

$$Hom_{\mathbf{sSets}}(\Delta^n, X) \cong X_n$$

We say a map $\sigma : \Delta^n \rightarrow X$ is an n -simplex in X , just like the case for topological spaces and their singular simplices. Similarly we can think of the standard simplex Δ^n as the standard topological simplex Δ_{top}^n . And hence we define its **boundary** $\partial\Delta^n$ as the coequalizer:

$$\bigsqcup_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^n \Delta^{n-1} \longrightarrow \partial\Delta^n$$

which is given by the cosimplicial identities $d^j d^i = d^i d^{j-1}$ for $i < j$. Precisely, we label each simplices as $\Delta_{i,j}^{n-2}, \Delta_i^{n-1}$. Then as simplices in Δ^n , we have a commutative square:

$$\begin{array}{ccc} \Delta_{i,j}^{n-2} & \xrightarrow{d^{j-1}} & \Delta_i^{n-1} \\ \downarrow d^i & & \downarrow d^i \\ \Delta_j^{n-1} & \xrightarrow{d^j} & \Delta^n \end{array}$$

We also define the k -horn Λ_k^n by a almost same coequalizer except that we take out the k -face:

$$\bigsqcup_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i \neq k} \Delta^{n-1} \longrightarrow \Lambda_k^n$$

With these notations, we define the **simplicial sphere** S^n as the pushout:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \longrightarrow & S^n \end{array}$$

Observe that a simplicial set X can be obtained by gluing all its simplices in a compatible way: define the **simplex category** Δ/X consisting objects of all simplices $\sigma : \Delta^n \rightarrow X$ and morphisms are morphisms of simplicial sets over X . Then we have an isomorphism:

$$X \cong \operatorname{colim}_{\sigma \text{ in } \Delta/X} \Delta^n$$

This is directly from the fact that all presheaves on Δ is a colimit of representables. Suppose we have a cosimplicial object $F : \Delta \rightarrow \mathcal{M}$, then we naturally obtained a functor \bar{F} from its presheaves(which is **sSets**) to \mathcal{M} :

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & \mathcal{M} \\ \downarrow \text{Yoneda} & \dashrightarrow & \\ \mathbf{Pre}(\Delta) = \mathbf{sSets} & & \end{array}$$

The functor \bar{F} preserves a colimit if the target diagram does admits a colimit. This mechanic gives us the realization functor:

$$\bar{F}(X) := \operatorname{colim}_{\sigma \text{ in } \Delta/X} F(\Delta^n)$$

By setting $\mathcal{M} = \mathbf{Top}$ and $F(\mathbf{n}) = \Delta_{top}^n$, we obtain **geometric realization**, usually denoted $|\cdot|$. And since \mathbf{Top} has small colimits, then $|\cdot|$ admits a right adjoint:

$$\mathcal{S}(X) : \mathbf{n} \mapsto \operatorname{Hom}_{\mathbf{Top}}(\Delta_{top}^n, X)$$

Which is exactly the singular chain complex of a topological space.

We define in \mathbf{sSets} three classes of morphisms.

$$\left\{ \begin{array}{l} W = \text{maps whose geometric realizations are weak equivalences in } \mathbf{Top} \\ F = \text{maps with RLP with respect to } \Lambda_k^n \rightarrow \Delta^n. \\ C = \text{all inclusions of simplicial sets} \end{array} \right.$$

These form the model structure on \mathbf{sSets} , namely **Kan model structure**. Particularly, we name such fibrations as **Kan fibrations**. When we talk about a model structure of \mathbf{sSets} in this thesis, we always means the Kan model structure.

In this model structure, we interpret the homotopy at left. By this we mean: $f, g : K \rightarrow L$ is homotopic if there is a left homotopy $H : K \times \Delta^1 \rightarrow L$ s.t. $H(d^0) = f, H(d^1) = g$, denoted $f \simeq g$. A morphism $f : K \rightarrow L$ is a homotopy equivalence if there exists $g : L \rightarrow K$ s.t. $f \circ g \simeq id$ and $g \circ f \simeq id$. Since all simplicial sets are Kan cofibrant, hence the left homotopy is enough to describe homotopic relations.

Proposition 2.2.7. *The Kan model structure is proper. And it has generating cofibrations and generating trivial cofibrations:*

$$\left\{ \begin{array}{l} I = \text{the canonical inclusions } \partial\Delta^n \rightarrow \Delta^n \\ J = \text{the canonical inclusions } \Lambda_k^n \rightarrow \Delta^n \end{array} \right.$$

Proof. [3, Chapter A.2.7]. □

Theorem 2.2.8. *\mathbf{sSets} is Cartesian closed. The exponential is the simplicial sets defined as:*

$$(Y^X)_n = \operatorname{Hom}_{\mathbf{sSets}}(X \times \Delta^n, Y)$$

Proof. [4, Proposition I.5.1]. □

Remark 2.2.9. The exponential in \mathbf{sSets} is also called the **function complex**. And we will also write $\operatorname{map}(X, Y)$ as a notation.

Proposition 2.2.10. *Let $i : A \rightarrow X$ be a cofibration in \mathbf{sSets} and $p : E \rightarrow B$ be a fibration. Then the induced map:*

$$E^X \xrightarrow{(i^*, p_*)} E^A \times_{B^A} B^X$$

is a fibration.

Proof. [4, Proposition I.5.2] □

Definition 2.2.11. *Let K be a fibrant simplicial set, then we define the 0-homotopy of K as $\pi_0 K := K_0 / K_1$. The quotient here is taken by faces of 1-simplices, i.e. $x_0 \sim x_1$ in k_0 if and only if $\exists v \in K_1$ s.t. $d_0(v) = x_0, d_1(v) = x_1$.*

More generally, given a simplicial set K based by $v : \Delta^0 \rightarrow K$, we define the n -homotopy group $\pi_n(K, v)$ of K to be the set of homotopy classes of morphisms $\alpha : \Delta^n \rightarrow K$ s.t. α fits into the commutative diagram:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \alpha \\ \Delta^0 & \xrightarrow{v} & K \end{array} .$$

For non-fibrant simplicial set L , we define its n -homotopy to be the n -homotopy of its fibrant replacement.

We introduce some properties of the homotopy groups of simplicial sets. One can easily find proofs in Goerss and Jardine's book [4].

Proposition 2.2.12. *Given K a simplicial set, then its n -homotopy is indeed a group for $n \geq 1$. They are Abelian for $n \geq 2$.*

Proposition 2.2.13. *Given a simplicial set K , its n -homotopy coincides with the n -homotopy of its geometric realization $|K|$ for $n \geq 0$.*

2.3 Simplicial Model Category

In this section, we define simplicial structure on a model category as an addition. Notice we are potential requiring the simplicial structure is compatible with the model structure which is common but not necessary. Also one can see that even without **SM7**, we still can define the function complex and tensor product for a model category. In this thesis, when we mention these concept, we always assume they are compatible with each other.

Definition 2.3.1. Let \mathcal{C} be a category equipped with three functors:

1. **function complex** $\text{map}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{sSets}$;
2. **tensor product** $- \otimes - : \mathcal{C} \times \mathbf{sSets} \rightarrow \mathcal{C}$;
3. **internal function complex** $\underline{\text{Map}}(-, -) : \mathbf{sSets}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$.

s.t. the following conditions hold for any $X, Y \in \mathcal{C}$:

- (4) we have natural isomorphism $X \otimes (K \times L) \cong (X \otimes K) \otimes L$ for any $X \in \mathcal{C}, K, L \in \mathbf{sSets}$;
- (5) $\text{map}_{\mathcal{C}}(X, Y)_0 = \text{Hom}_{\mathcal{C}}(X, Y)$;
- (6) $(X \otimes -) : \mathbf{sSets} \rightleftarrows \mathcal{C} : \text{map}_{\mathcal{C}}(X, -)$ is a pair of adjoint functors;
- (7) $\text{map}_{\mathcal{C}}(-, Y) : \mathcal{C} \rightleftarrows \mathbf{sSets}^{\text{op}} : \underline{\text{Map}}(-, Y)$ is a pair of adjoint functors;

Furthermore, if \mathcal{C} satisfies axiom **SM7**: for any cofibration $i : A \rightarrow X$ and fibration $p : E \rightarrow B$, we have:

$$\text{map}_{\mathcal{C}}(X, E) \rightarrow \text{map}_{\mathcal{C}}(A, E) \times_{\text{map}_{\mathcal{C}}(A, B)} \text{map}_{\mathcal{C}}(X, B)$$

is a fibration of simplicial sets and trivial if either i or p is. Then we say \mathcal{C} is a **simplicial category**.

Remark 2.3.2. We also interpret the adjunctions as:

$$\text{Hom}_{\mathcal{C}}(X \otimes K, Y) \cong \text{Hom}_{\mathbf{sSets}}(K, \text{map}_{\mathcal{C}}(X, Y))$$

$$\text{Hom}_{\mathcal{C}}(X, \underline{\text{Map}}(K, Y)) \cong \text{Hom}_{\mathbf{sSets}}(K, \text{map}_{\mathcal{C}}(X, Y))$$

Remark 2.3.3. We also have a dual version of **SM7**, usually named **SM7b**: for a cofibration $i : A \rightarrow X$ in \mathcal{C} and a fibration $p : K \rightarrow L$ in \mathbf{sSets} , the canonical map

$$X \otimes K \cup_{A \otimes K} A \otimes L \rightarrow X \otimes L$$

is a cofibration and is trivial if either i or p is.

Proposition 2.3.4. Let \mathcal{C} be as above, then we have:

1. We also have adjunction $(- \otimes K) \dashv \underline{\text{Map}}(K, -)$, i.e.

$$\text{Hom}_{\mathcal{C}}(X \otimes K, Y) \cong \text{Hom}_{\mathcal{C}}(X, \underline{\text{Map}}(K, Y));$$

2. For Y object in \mathcal{C} , K, L simplicial sets, we have:

$$\underline{Map}(K \times L, Y) \cong \underline{Map}(K, \underline{Map}(L, Y));$$

3. For $n \geq 0$, $\text{map}_{\mathcal{C}}(X, Y)_n \cong \text{Hom}_{\mathcal{C}}(X \otimes \Delta^n, Y)$.

Proof. Directly from the adjunctions in definition. \square

Proposition 2.3.5. *Let \mathcal{C} be as above, then we have the following Quillen functors:*

1. $(- \otimes K) : \mathcal{C} \rightleftarrows \mathcal{C} : \underline{Map}(K, -)$ for any simplicial sets K .
2. $(X \otimes -) : \mathbf{sSets} \rightleftarrows \mathcal{C} : \text{map}_{\mathcal{C}}(X, -)$ for cofibrant object X .
3. $\text{map}_{\mathcal{C}}(-, Y) : \mathcal{C} \rightleftarrows \mathbf{sSets}^{op} : \underline{Map}(-, Y)$ for fibrant object Y .

Proof. Notice any object in \mathbf{sSets} is cofibrant, and $X \otimes K$ is the initial object if K is by adjunction. Therefore set apply **SM7b** to the case of $p : \emptyset \rightarrow K$ will give that $- \otimes K$ is a left Quillen functor.

Similarly for the second and the third adjunction: apply **SM7** to the case of $i : \emptyset \rightarrow X$ (resp. $p : Y \rightarrow \star$), we have $\text{map}_{\mathcal{C}}(X, -)$ (resp. $\text{map}_{\mathcal{C}}(-, Y)$) is a right (resp. left) Quillen functor. \square

Remark 2.3.6. We shall point out that how these Quillen functors induce isomorphisms of homotopy classes:

$$[X \otimes K, Y] \cong [X, \underline{Map}(K, Y)] \cong [K, \text{map}(X, Y)]$$

Proposition 2.3.7. *Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . Then it is a weak equivalence if and only if the induced map $Q(f)^* : \text{map}(QY, E) \rightarrow \text{map}(QX, E)$ is a weak equivalence for all fibrant object E .*

Dually, f is a weak equivalence if and only if $Q(f)_ : \text{map}(F, RX) \rightarrow \text{map}(F, RY)$ is a weak equivalence for any cofibrant object F .*

Proof. Assume f is a weak equivalence. Then the cofibrant replacement $Q(f)$ is again a weak equivalence. Then since the functor $\text{map}_{\mathcal{C}}(-, E)$ is a left Quillen functor, we have $\text{map}_{\mathcal{C}}(QY, E) \simeq \text{map}_{\mathcal{C}}(QX, E)$.

Conversely, assume $Q(f)^*$ is a weak equivalence for arbitrary E fibrant. Notice that $[QX, E] = \pi_0 \text{map}_{\mathcal{C}}(QX, E)$. Therefore by sending QX, QY, E into the homotopy category $\mathcal{H}(\mathcal{C})$, the assumption implies that

$$\gamma(Q(f))^* : \text{Hom}_{\mathcal{H}(\mathcal{C})}(\gamma(QY), \gamma(E)) \rightarrow \text{Hom}_{\mathcal{H}(\mathcal{C})}(\gamma(QX), \gamma(E))$$

are natural isomorphisms for any $\gamma(E)$. Since any object in $\mathcal{H}(\mathcal{C})$ has a fibrant γ -preimage, hence by Yoneda Lemma, $\gamma(Q(f))$ is an isomorphism in $\mathcal{H}(\mathcal{C})$. Equivalently, $Q(f)$ is a weak equivalence and so is f .

For the second part of the proposition, the proof is symmetrical and dual. \square

Remark 2.3.8. In this thesis, we might be more oftenly mentioning the **derived function complex** $\mathbf{Map}(-, -) := \text{map}(Q(-), R(-))$. This proposition has justified the notation as we rephrase it: a morphism $f : X \rightarrow Y$ is a weak equivalence if and only if $f^* : \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ is a weak equivalence for any (fibrant) object Z .

Proposition 2.3.9. *Let (F, G) be a Quillen functor s.t. $F(X \otimes \Delta^n) \cong FX \otimes \Delta^n, \forall n \in \mathbb{N}$ and X is cofibrant, then we have a weak equivalence of simplicial sets for any pair of objects X, Y by viewing the derived functors as functors on the model categories:*

$$\text{Map}(\mathbf{LFX}, Y) \simeq \text{Map}(X, \mathbf{RGY}).$$

Proof. Notice we have

$$\begin{aligned} \text{Map}(\mathbf{LFX}, Y) &:= \text{map}(Q\mathbf{LFX}, RY) \\ &\simeq \text{map}(QFQX, RY) \\ &\simeq \text{map}(FQX, RY) \end{aligned} \tag{2.1}$$

and similarly $\text{Map}(X, \mathbf{RGY}) \simeq \text{map}(QX, GRY)$. Therefore we are equivalent to show that $\text{map}(FX, Y) \simeq \text{map}(X, GY)$ for cofibrant object X and fibrant object Y .

In our case, the functor F commutes with the left Quillen functor $- \otimes \Delta^n$, hence

$$\begin{aligned} \text{map}(FX, Y)_n &\cong \text{Hom}(FX \otimes \Delta^n, Y) \\ &\cong \text{Hom}(F(X \otimes \Delta^n), Y) \\ &\cong \text{Hom}(X \otimes \Delta^n, GY) \\ &\cong \text{map}(X, GY)_n \end{aligned} .$$

The compatibility is straight forward, hence we have $\text{map}(FX, Y) \cong \text{map}(X, GY)$ for cofibrant X and fibrant Y . Then we are done. \square

Definition 2.3.10 (Simplicial Homotopy). *Let $f, g : X \rightarrow Y$ be two morphisms. A **simplicial homotopy** from f to g is a morphism $H : X \otimes \Delta^1 \rightarrow Y$ s.t. $H \circ (id \otimes d^0) = f, H \circ (id \otimes d^1) = g$. Denoted $f \simeq g$.*

Definition 2.3.11. *A morphism $f : X \rightarrow Y$ is an **homotopic equivalence** if there exists $g : Y \rightarrow X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.*

2.4 Homotopy Limit and Colimit

In this section, we define the homotopy (co)limits, particularly assumed small. As always, we should be careful about the objects and arrows in a diagram

because the categorical (co)limits might be incompatible with weak equivalences. A classical example would be in the category of topological spaces: the pushout diagram $\{ * \leftarrow S^0 \rightarrow D^1 \}$ is objectwise weak equivalent to the diagram $\{ * \leftarrow S^0 \rightarrow * \}$. However the pushout are not weak equivalent. A simple conclusion is: a categorical (co)limit may not be a homotopy (co)limit. Therefore we need conditions to decide which categorical diagram should computed in the homotopic sense.

Lemma 2.4.1. *Let I be a small category. Then there exists two model structure on the category \mathbf{sSets}^I of I -diagram in \mathbf{sSets} :*

1. **Projective Model Structure** given by weak equivalences and fibrations defined pointwisely.
2. **Injective Model Structure** given by weak equivalences and cofibrations defined pointwisely.

We denote them as \mathbf{sSets}_{proj}^I and \mathbf{sSets}_{inj}^I .

Proof. [3, Proposition A.2.8.2]. □

Recall for the category of small diagrams, the diagonal functor Δ which sends the objects to their constant diagram admits left and right adjoint which are colimit and limit functor. In case of model categories, we have interpret as:

$$\mathbf{sSets}_{proj}^I \begin{array}{c} \xrightarrow{\text{colim}_I} \\ \xleftarrow{\Delta} \end{array} \mathbf{sSets} \begin{array}{c} \xrightarrow{\Delta} \\ \xleftarrow{\text{lim}_I} \end{array} \mathbf{sSets}_{inj}^I$$

s.t. $\text{colim}_I \dashv \Delta \dashv \text{lim}_I$.

Proposition 2.4.2. *These two adjunctions are Quillen functors.*

Proof. Notice Δ preserves all three types of morphisms. So it is a both left and right Quillen functor. □

Definition 2.4.3. *For \mathcal{M} , I be as above The left total derived functor \mathbf{Lcolim}_I is called the **homotopy colimit** hocolim_I . Dually, the **homotopy limit** holim_I is the derived functor of the limit functor.*

*Given a I -diagram F in \mathcal{M} , we say X is a **homotopy colimit of F** if $X \simeq \text{hocolim}(F)$. Dually we say X is a **homotopy limit of F** if $X \simeq \text{holim}(F)$.*

Remark 2.4.4. One should notice that the notations work for any combinatorial model categories, which includes all model categories we concern in this thesis.

Proposition 2.4.5. *Let $F \dashv G$ be a Quillen functor between two model categories. Then $\mathbf{L}F$ commutes with homotopy colimit. And $\mathbf{R}G$ commutes with homotopy limit.*

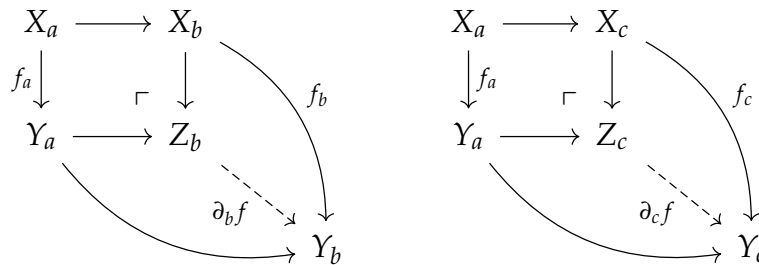
Proof. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen functor and I be a small category. Let $F^I \dashv G^I$ be the induced adjunction on the categories of diagrams. We claim G^I is a right Quillen functor by giving $\mathcal{M}^I, \mathcal{N}^I$ the projective model structure. This is straightforward since the projective fibrations are given pointwisely.

Then since we have $G^I \circ \Delta = \Delta \circ G$, we have $\mathbf{R}G^I \circ \mathbf{R}\Delta = \mathbf{R}\Delta \circ \mathbf{R}G$. Then by adjunctions, we have $\text{hocolim}_I \circ \mathbf{L}F^I = \mathbf{L}F \circ \text{hocolim}_I$. \square

Though the theory looks nice and simple, the cofibrant or fibrant replacement of the diagrams is generally difficult to describe. Luckily, for pullback and pushout, we have a relatively easy description and method to compute them.

We first fix I to be the pushout diagram, i.e. $\{b \leftarrow a \rightarrow c\}$, and then prove some statement about it. Then dually one can easily do the same things for pullback diagrams.

Proposition 2.4.6. *Let \mathcal{M} be a model category with homotopy limits and colimits, and I be the pushout diagram. Then given a morphism of diagrams $f : X \rightarrow Y$, f is a projective cofibration, i.e. a cofibration in projective model structure, if and only if $f_a, \partial_b f, \partial_c f$ are cofibrations in \mathcal{M} .*



Proof. A complete proof can be found in Dwyer and Spalinski's chapter [9, Chapter 2, Proposition 10.6] which shows that the collection of such morphisms combined with pointwise weak equivalences and pointwise fibrations indeed form a model category. \square

Corollary 2.4.7. *Given a pushout diagram $F = \{X_b \leftarrow X_a \rightarrow X_c\}$, then there is a pushout diagram $Q_{\text{proj}}F = \{QX_b \leftarrow QX_a \rightarrow QX_c\}$ equipped with a projective trivial fibration to F s.t. $Q_{\text{proj}}F$ is a projective cofibrant replacement.*

Proof. The construction is shown as follows:

$$\begin{array}{ccccc}
 & X_b' & \longrightarrow & X_b & \\
 & \uparrow & & \nearrow & \uparrow \\
 \emptyset & \twoheadrightarrow & QX_a & \xrightarrow{\simeq} & X_a \\
 & \downarrow & & \searrow & \downarrow \\
 & X_c' & \longrightarrow & X_c &
 \end{array}$$

where X_b', X_c' are obtained by functorial factorization and hence obviously cofibrant. □

Proposition 2.4.8. *Let \mathcal{M} be a model category. Given a pushout square*

$$\begin{array}{ccc}
 A & \xrightarrow{i} & X \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & Y
 \end{array}$$

this is a homotopy pushout if A, B are cofibrant and i is a cofibration.

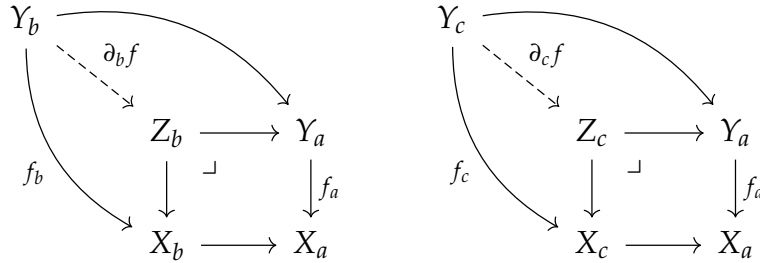
Proof. Notice that since A, B, X are all cofibrant, hence $\{QB \leftarrow A \twoheadrightarrow X\}$ is a cofibrant replacement of diagrams. Hence consider:

$$\begin{array}{ccc}
 A & \twoheadrightarrow & X \\
 \downarrow & & \downarrow \\
 QB & \xrightarrow{i} & Z \\
 \simeq \downarrow & & \downarrow p \\
 B & \longrightarrow & Y
 \end{array}$$

where Z is the homotopy colimit. By Example 2.1.5, we have $i : QB \rightarrow Z$ is a cofibration since it is a pushout of a cofibration along a cofibration. Since B is cofibrant, then $p : Z \rightarrow Y$ is a pushout of a weak equivalence between cofibrant objects along a cofibration, hence as explained in Example 2.1.5, p is a weak equivalence. □

We have similar conclusions for homotopy pullback. The proofs for the following dual proposition can be dually done.

Proposition 2.4.9. *Let I be the pullback diagram $\{b \rightarrow a \leftarrow c\}$ and \mathcal{M} be model category with homotopy limits and colimits. Given a morphism $f : Y \rightarrow X$, f is a injective fibration if and only if $f_a, \partial_b f, \partial_c f$ are fibrations.*



Corollary 2.4.10. *Given a pullback diagram $F = \{X_b \rightarrow X_a \leftarrow X_c\}$, then there is a pushout diagram $R_{inj}F = \{RX_b \twoheadrightarrow RX_a \longleftarrow RX_c\}$ equipped with an injective trivial cofibration from F s.t. $R_{inj}F$ is an injective fibrant replacement.*

Proposition 2.4.11. *Let \mathcal{M} be a model category. Given a pullback square*

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 \downarrow & \lrcorner & \downarrow \\
 B & \xrightarrow{p} & Y
 \end{array}$$

this is a homotopy pullback if X, Y are fibrant and p is a fibration.

2.5 Pointed Construction

In this section, we define the pointed model category. We shall only introduce the most basic fact in the pointed set up, since we are going to describe a more detailed set up but only for the category of simplicial presheaves.

Definition 2.5.1. *A bicomplete category \mathcal{C} is **pointed** if the unique morphism $\emptyset \rightarrow *$ is an isomorphism.*

Given a bicomplete category \mathcal{C} , we can form a new pointed category \mathcal{C}_\bullet by setting objects to be morphisms of the form $* \rightarrow \mathcal{C}$ and morphisms to be morphisms in \mathcal{C} respecting the obvious commutativity. A pointed morphism f is a weak equivalence(*resp.* cofibration/fibration) if and only if f is a weak equivalence(*resp.* cofibration/fibration) in \mathcal{C} via the forgetful functor by [2, Proposition 1.1.8]. We then naturally obtain a pair of adjoint functors:

$$(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_\bullet : F$$

where $C_+ := (* \rightarrow * \sqcup C)$ and F is a forgetful functor.

In a pointed category \mathcal{C}_\bullet , if \mathcal{C} is also a simplicial category, then we can define the **smash product** $X \wedge Y$ as the pushout square:

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \otimes Y \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & X \wedge Y \end{array}$$

where $X \vee Y$ is the pushout of $\{X \leftarrow * \rightarrow Y\}$. This will give the simplicial structure on \mathcal{C}_\bullet . Hence we can conclude:

Proposition 2.5.2. *Given a simplicial model category \mathcal{M} , then the pointed construction \mathcal{M}_\bullet is a simplicial model category. We use the footscript $(-)_\bullet$ for notations, e.g. the homotopy category \mathcal{H}_\bullet and the homotopy class of morphisms $[X, Y]_\bullet$.*

Proposition 2.5.3. *The canonical adjunction*

$$(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_\bullet : F$$

is a Quillen functor.

Proof. This is as direct as the construction of the pointed category: since $(f)_+$ is a cofibration (*resp.* trivial cofibration) if and only if f is one, therefore $(-)_+$ is a left Quillen functor. \square

Furthermore, from the construction of the pointed category, we also obtain the

Proposition 2.5.4. *Given a Quillen functor (F, G) between model categories \mathcal{M} and \mathcal{N} . Then we have an induced Quillen functor*

$$F_\bullet : \mathcal{M}_\bullet \rightleftarrows \mathcal{N}_\bullet : G_\bullet$$

Proof. The pointed functor is given by $F_\bullet(X) := (FX)_+$ for $X \in \mathcal{M}$ and $F_\bullet(f) = (f_\bullet : X_+ \rightarrow Y_+)$ for $f : X \rightarrow Y$ in \mathcal{M} . Hence F_\bullet is a left Quillen functor. \square

Corollary 2.5.5. *Given a Quillen equivalence (F, G) between model category \mathcal{M} and \mathcal{N} . If the final object in \mathcal{M} is cofibrant, then the induced Quillen functor (F_\bullet, G_\bullet) is a Quillen equivalence.*

Dually if the initial object in \mathcal{N} is fibrant, then the induced Quillen functor (F_\bullet, G_\bullet) is a Quillen equivalence.

\mathbb{A}^1 -Homotopy Theory

In this chapter, we set up the main stage: the \mathbb{A}^1 -homotopy theory. We start with the injective model category we directly construct on the category of simplicial presheaves $\mathbf{sPre}(\mathbf{Sm}_k)$ (Section 1) and then use Bousfield localization to modify the model structure as we required (Section 3). Then in Section 4, we introduce the Singular Construction which is convenient to operate. And one can see in the section, this construction actually works as a fibrant replacement. At last in Section 5, we introduce the pointed simplicial presheaves which allow us to define, as the intuition in topological theory, the homotopy of a simplicial presheaves.

3.1 Category of Presheaves

Let $\mathbf{sPre}(\mathbf{Sm}_k)$ (or simply \mathbf{sPre}) be the category of simplicial presheaves on the category \mathbf{Sm}_k of smooth schemes over a perfect field k . Notice a simplicial presheaf is exactly a bifunctor $\Delta^{op} \times \mathbf{Sm}_k^{op} \rightarrow \mathbf{Sets}$, hence we may also view it as a \mathbf{sSets} -valued presheaf. This allows us to assign the **injective model structure** on \mathbf{sPre} :

W = pointwise weak equivalences (as simplicial sets);

C = pointwise cofibrations (as simplicial sets);

And the class of fibrations F consists of those morphisms with RLP with respect to trivial cofibrations. Particularly, we say a simplicial presheaf is **constant** if it has constant values as a presheaf of \mathbf{Sm}_k and is **discrete** if it has constant values as a presheaf of Δ . For example, as we will frequently mention, a representable presheaf on schemes $X : \mathbf{Sm}_k^{op} \rightarrow \mathbf{Sets}$ is discrete and a simplicial set which is a presheaf on simplicial index Δ is constant.

Furthermore, \mathbf{sPre} carries a simplicial structure. To assign the simplicial structure to \mathbf{sPre} , given a simplicial presheaf X and a simplicial set K , we define their tensor product $X \otimes K := X \times K$ by viewing K as a constant presheaf, and the rest of functor constructed from adjunctions. But we can actually get further.

By embedding \mathbf{sSet} into \mathbf{sPre} as constant simplicial sheaves, and then define the tensor product to be simply product. Define $\underline{Map}(X, Y) : U \mapsto \text{map}(X|_U, Y|_U)$ where $X|_U$ means the restriction to object U in \mathbf{Sm}_k . We still call it internal function complex since two notations coincide when X is a simplicial set. Also we call it the exponential since this construction is the right adjoint:

$$(- \times X) \dashv \underline{Map}(X, -)$$

Proposition 3.1.1. *The above adjunction is also a Quillen functor.*

Proof. We show this by proving $X \times -$ is a left Quillen functor. Notice a cofibration $f : Y \rightarrow Z$ is a pointwise inclusion of simplicial sets, which is equivalent to f being injections of sets for every pair $(U, \mathbf{n}) \in \mathbf{Sm}_k \times \Delta$ by viewing Y, Z as presheaf of sets on $\mathbf{Sm}_k \times \Delta$. Since the product of simplicial presheaves are computed pointwise and furthermore computed for every pair (U, \mathbf{n}) , therefore $(X \times f)(U)_n : (X \times Y)(U)_n \rightarrow (X \times Z)(U)_n$ is precisely the map of sets $id \times f(U)_n : X(U)_n \times Y(U)_n \rightarrow X(U)_n \times Z(U)_n$ which is again injective. Therefore $X \times -$ preserves cofibrations of simplicial presheaves.

On the other hand, if a cofibration $f : Y \rightarrow Z$ is trivial, we are left to show $X \times f = id \times f$ is a weak equivalence. By realizing $(id \times f)(U)$, we obtain $|id| \times |f| \cong |id \times f|$ since product commutes with taking homotopy groups. Hence indeed $|id \times f|$ induces isomorphisms of all homotopy groups and therefore $X \times f$ is a weak equivalence. \square

Lemma 3.1.2 (Simplicial Yoneda Lemma). *Let X be an object in \mathbf{Sm}_k and F a simplicial presheaf. We slightly abuse the use of notation by writing X for also the (simplicial) presheaf X represents. Then we have:*

$$\text{map}(X, F) \cong F(X)$$

For the derived function complex, we have:

$$\underline{Map}(X, F) \simeq F(X)$$

Proof. For $n \geq 0$, let $f \in \text{Hom}_{\mathbf{sPre}}(X \times \Delta^n, F)$. Consider $g : U \rightarrow X$ at level $h : \mathbf{m} \rightarrow \mathbf{n}$,

$$\begin{array}{ccc} X(X) \times \Delta_n^n & \xrightarrow{f(X)_n} & F(X)_n \\ \downarrow g^* \times h^* & & \downarrow (g^*, h^*) \\ X(U) \times \Delta_m^n & \xrightarrow{f(U)_m} & F(U)_m \end{array}$$

Notice the square above is just Yoneda Lemma by viewing a simplicial presheaf as a bifunctor. Therefore we have $\text{map}(X, F)_n \cong F(X)_n$ for $n \geq 0$.

These levelwise isomorphisms respect simplicial structure as, for $g : \mathbf{m} \rightarrow \mathbf{n}$ and $f : X \times \Delta^n \rightarrow F$, the diagram below commutes since $(id \times g^* f)(id_X \times id_{\mathbf{m}}) = f(X)_m(id_X \times g) = (id^*, g^*)(f(X)_n(id_X \times id_{\mathbf{n}}))$:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{sPre}}(X \times \Delta^n, F) & \longrightarrow & F(X)_n \\ \downarrow id \times g^* & & \downarrow g^* \\ \text{Hom}_{\mathbf{sPre}}(X \times \Delta^m, F) & \longrightarrow & F(X)_m \end{array}$$

The derived result is directly from the pointwise weak equivalences and X is cofibrant. \square

Lemma 3.1.3. *Let X, Y be simplicial presheaves and $U \in \mathbf{Sm}_k$. We first have natural isomorphism of simplicial sets:*

$$\text{map}(U \times X, Y) \cong \text{map}(U, \underline{\text{Map}}(X, Y)).$$

Then

$$\underline{\text{Map}}(U, Y) \cong Y(- \times U)$$

and

$$\underline{\text{Map}}(X, Y)(U) \cong \text{map}(X, Y(- \times U))$$

Proof. By Simplicial Yoneda, we have $\underline{\text{Map}}(X, Y)(U) \cong \text{map}(U, \underline{\text{Map}}(X, Y))$. Then levelwisely,

$$\begin{aligned} \text{map}(U, \underline{\text{Map}}(X, Y))_n &\cong \text{Hom}_{\mathbf{sPre}}(U \times \Delta^n, \underline{\text{Map}}(X, Y)) \\ &\cong \text{Hom}_{\mathbf{sPre}}(U \times \Delta^n \times X, Y) \\ &\cong \text{map}(U \times X, Y)_n \end{aligned}$$

Since the adjunction is natural, hence given a map of simplicial sets $f : \Delta^n \rightarrow \Delta^m$, we have a commutative diagram of sets:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{sPre}}(U \times \Delta^m, \underline{\text{Map}}(X, Y)) & \xrightarrow{\cong} & \text{Hom}_{\mathbf{sPre}}(U \times X \times \Delta^m, Y) \\ \downarrow f^* & & \downarrow f^* \\ \text{Hom}_{\mathbf{sPre}}(U \times \Delta^n, \underline{\text{Map}}(X, Y)) & \xrightarrow{\cong} & \text{Hom}_{\mathbf{sPre}}(U \times X \times \Delta^n, Y) \end{array},$$

where the horizontal arrows are adjunctions. Hence the adjunctions induce an isomorphism of simplicial sets and so we have the following isomorphisms:

$$\underline{\text{Map}}(X, Y)(U) \cong \text{map}(U, \underline{\text{Map}}(X, Y)) \cong \text{map}(U \times X, Y).$$

Hence $\underline{\text{Map}}(U, Y)(V) \cong \text{map}(U \times V, Y) \cong Y(U \times V)$ by Simplicial Yoneda for any $Y \in \mathbf{S}\mathbf{m}_k$. The compatibility is easy to checked.

For $\underline{\text{Map}}(X, Y)$, we have

$$\begin{aligned} \underline{\text{Map}}(X, Y)(U) &\cong \text{map}(U \times X, Y) \\ &\cong \text{map}(X, \underline{\text{Map}}(U, Y)) \\ &\cong \text{map}(X, Y(- \times U)). \end{aligned}$$

□

Lemma 3.1.4. *The category $\mathbf{S}\mathbf{m}_k$ with the injective model structure is left proper. And it has generating cofibrations and generating trivial cofibrations:*

$$\begin{cases} I = \{X \otimes \partial\Delta^n \rightarrow X \otimes \Delta^n \mid X \text{ is representable, } n \geq 0\} \\ J = \{X \otimes \Lambda_k^n \rightarrow X \otimes \Delta^n \mid X \text{ is representable, } 0 \leq k \leq n\} \end{cases}$$

Proof. The left proper is given by the injective model structure for free: every object in the category is cofibrant and then use Remark 2.1.19. The rest of the proposition can be found in [10]. □

3.2 Bousfield Localization

Bousfield localization is a convenient tool to modify a model structure without annoying by-product. The point is that the (left) Bousfield localization preserves the cofibrant objects and enlarge the class of weak equivalences by reducing the amount of fibrant objects.

Definition 3.2.1. Let \mathcal{S} be a class of morphisms in a model category \mathcal{M} s.t. the function complex map $(-, -)$ is defined. We say a fibrant object E is **\mathcal{S} -local** if

$$f^* : \text{Map}(Y, E) \rightarrow \text{Map}(X, E)$$

is a weak equivalence of simplicial sets for arbitrary $f : X \rightarrow Y$ in \mathcal{S} .

A morphism $f : X \rightarrow Y$ is an **\mathcal{S} -local weak equivalence** if the induced map

$$f^* : \text{Map}(Y, E) \rightarrow \text{Map}(X, E)$$

is a weak equivalence of simplicial sets for arbitrary \mathcal{S} -local object E .

Definition 3.2.2 (Left Bousfield Localization). Let \mathcal{S} be a class of morphisms in a model category \mathcal{M} s.t. the function complex map $(-, -)$ is defined. The model structure, if it indeed is a model structure, given by:

1. *w.e.* = \mathcal{S} -local weak equivalences;
2. *cof.* = cofibrations of \mathcal{M} .

is called the **left Bousfield localization on \mathcal{M} with respect to \mathcal{S}** . We denote $L_{\mathcal{S}}\mathcal{M}$ for \mathcal{M} equipped with this \mathcal{S} -local model structure.

Theorem 3.2.3. Given a model category \mathcal{M} as above and \mathcal{S} is a class of morphisms, if the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ exists, then the identity functor:

$$\text{id} : \mathcal{M} \rightarrow L_{\mathcal{S}}\mathcal{M}$$

is a left localization of \mathcal{M} with respect to \mathcal{S} , i.e. the identity functor is a left Quillen functor satisfying:

1. its derived functor $\mathbf{Lid} : \mathcal{H}(\mathcal{M}) \rightarrow \mathcal{H}(L_{\mathcal{S}}\mathcal{M})$ takes homotopy class of any morphism in \mathcal{S} to an isomorphism in $\mathcal{H}(L_{\mathcal{S}}\mathcal{M})$;
2. the identity functor is universal, meaning given a left Quillen functor $F : \mathcal{M} \rightarrow \mathcal{N}$ s.t. it takes homotopy class of any morphism in \mathcal{S} to an isomorphism in $\mathcal{H}(\mathcal{N})$, we have a unique factorization:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow \text{id} & \nearrow \tilde{F} & \\ L_{\mathcal{S}}\mathcal{M} & & \end{array}$$

s.t. \tilde{F} is left Quillen.

proof. [11, Theorem 3.3.19]. □

Theorem 3.2.4. *Let \mathcal{S} be a set of morphisms in $\mathbf{sPre}(\mathbf{Sm}_k)$, then*

- (a) *the left Bousfield localization $L_{\mathcal{S}}\mathbf{sPre}$ exists.*
- (b) *the fibrant objects in $L_{\mathcal{S}}\mathbf{sPre}$ are exactly those \mathcal{S} -local objects.*
- (c) *$L_{\mathcal{S}}\mathbf{sPre}$ is again a left proper simplicial model category.*

Proof. Since \mathbf{Sm}_k is essentially small, by viewing $\mathbf{sPre}(\mathbf{Sm}_k)$ as the diagram category $\text{Fun}(\mathbf{Sm}_k, \mathbf{sSets})$, we can call [3, Proposition A.2.8.2] and obtain the left proper and combinatorial structure on it. Then the rest of the proposition directly from [3, Proposition A.3.7.3]. □

Remark 3.2.5. Similar proposition works for other simplicial model categories in this thesis, with a basically identical proof.

Remark 3.2.6. One thing to notice is that the cofibrant and fibrant replacement may change after localization. Therefore though the function complex $\text{map}(-, -)$ is computed in the same way, the derived function complex may differ since we need to take cofibrant or fibrant replacement.

3.3 \mathbb{A}^1 -local Model Category

In this section, we construct the \mathbb{A}^1 -homotopy theory by Bousfield localization. There are two properties we want to achieve: the contractility of the affine line \mathbb{A}^1 and a descent condition for Nisnevich topology.

For the first one, we localize by Bousfield at the morphisms of the form $X \times \mathbb{A}^1 \rightarrow X$. Then in the new model structure, these morphisms will become weak equivalences and eventually become isomorphisms in the homotopy category as we want. One thing to notice is that a collection of all such morphisms might not be a set. The solution to this is to take only those morphisms $X \times \mathbb{A}^1 \rightarrow X$ which X represents a isomorphic class of smooth schemes. This subcollection is a set since the category is essentially small. For the second one, we potentially request a condition of sheaf, i.e. a (hyper)descent condition with respect to a certain Grothendieck topology. We will mainly work in Nisnevich topology, though we sometimes also operate sheaves with respect to Zariski or Étale topology.

We first do the descent part and introduce three Grothendieck topology we will mentioned in this thesis.

Definition 3.3.1. Let X be a scheme. $\{f_i : U_i \rightarrow X\}_{i \in I}$ is a finite family of Étale morphisms. This is an **Étale covering** if X is the union of open subsets $f_i(U_i)$. This is an **Nisnevich covering** if $\forall x \in X$, there exists U_i and a point $y \in U_i$ s.t. $f_i(y) = x$ and the residue fields coincide $\kappa(x) \cong \kappa(y)$. This is an **Zariski covering** if X is the union of open subschemes $f_i(U_i)$.

Remark 3.3.2. An equivalent condition for *Nis*-covering is introduced in [5, Proposition §3 1.1]: let R^h be the henselisation of a local ring R , given $\{U_i \rightarrow X\}$, then for any point $x \in X$, the morphism $\sqcup_i (U_i \times_X \text{Spec } \mathcal{O}_{X,x}^h) \rightarrow \text{Spec } \mathcal{O}_{X,x}^h$ admits a section.

These three types of covering induces *Zariski*, *Nisnevich* and *Étale* topology on the category \mathbf{Sm}_k .

Proposition 3.3.3. The presheaf of sets represented by a schemes $X \in \mathbf{Sm}_k$ is a sheaf in *Étale* topology, and hence also a *Nisnevich* sheaf.

Proof. We cite a proposition from Milne [12]: if a Zariski sheaf respects the sheaf condition of Étale coverings in form of $U \rightarrow V$ where U, V are affine, then it is an Étale sheaf. Then we are left to show that a representable presheaf X satisfies the conditions above.

For a scheme Y , let $\{V_i \rightarrow Y\}_{i \in I}$ be a finite *Zar*-covering family. The representable presheaf X respects a such *Zar*-covering since Y can be obtained by gluing V_i along their intersection:

$$\begin{array}{ccccc}
 \sqcup V_i \cap V_j & \longrightarrow & \sqcup V_i & & \\
 \downarrow & & \downarrow & \searrow & \\
 \sqcup V_j & \longrightarrow & Y & \dashrightarrow & X
 \end{array}$$

Given $U \rightarrow V$ Étale morphism of affine schemes, which is faithfully flat, it is a strict epimorphism due to [12, Theorem 2.17]: any faithfully flat morphism of finite type is a strict epimorphism. Equivalently speaking, it induces exact sequence of sets of points for scheme Z :

$$\text{Hom}_{\mathbf{Sm}_k}(V, Z) \longrightarrow \text{Hom}_{\mathbf{Sm}_k}(U, Z) \rightrightarrows \text{Hom}_{\mathbf{Sm}_k}(U \times_V U, Z) ,$$

which is exactly the sheaf condition we want. Hence the representable presheaf X is a sheaf in *ét*-topology and so in *Nis*-topology. \square

Recall how we construct the Čech-complex of a covering of a scheme X . Say τ is the Grothendieck topology, given a τ -covering sieve induced by a finite covering family $\mathcal{U} = \{f_i : U_i \rightarrow X\}_{i \in I}$ of schemes X , we let $U := \bigsqcup_i U_i$ with a canonical morphism to X . Setting $\check{C}(\mathcal{U})_n := U^{\times_{\check{X}}^{n+1}}$, we obtain the Čech-complex of this covering which is also a simplicial schemes. By viewing X as a discrete simplicial presheaf, the canonical morphism $U \rightarrow X$ induces a morphism of simplicial presheaves $\check{C}(\mathcal{U}) \rightarrow X$.

Now we consider the collection of representatives of τ -covering sieves in $\mathbf{Pre}(\mathbf{Sm}_k)$, which is a set since \mathbf{Sm}_k is essentially small, for each sieve we construct the corresponding morphism of simplicial presheaves as above and we denote the set of them as \mathcal{S}_τ . We localize $\mathbf{sPre}(\mathbf{Sm}_k)$ by Bousfield at \mathcal{S}_τ and denote it \mathbf{Spc}_σ .

For simplicity, when $\tau = Nis$, we will simply write \mathbf{Spc} .

In case of over a finite dimensional Noetherian base scheme, they are Quillen equivalent. [DHI04]

Next we discuss the \mathbb{A}^1 -local part. We take the collection of representatives of morphisms in form of $\{X \times \mathbb{A}^1 \rightarrow X \mid X \in \mathbf{Sm}_k\}$ and again this is a set since \mathbf{Sm}_k is essentially small, denoted $\mathcal{S}_{\mathbb{A}^1}$. We localize \mathbf{Spc} at $\mathcal{S}_{\mathbb{A}^1}$ and denote the resultant $\mathbf{Spc}_{\mathbb{A}^1}$. And as a conclusion, we obtain two pairs of Quillen functors:

$$\mathbf{sPre}(\mathbf{Sm}_k) \begin{array}{c} \xrightarrow{id} \\ \xleftarrow{id} \end{array} \mathbf{Spc} \begin{array}{c} \xrightarrow{id} \\ \xleftarrow{id} \end{array} \mathbf{Spc}_{\mathbb{A}^1}$$

where the top arrows are left Quillen functors. We denote the left derived functors as \mathbf{L}_{Nis} and $\mathbf{L}_{\mathbb{A}^1}$. Notice any object is cofibrant in our discussion, hence this notation basically works for indicating which model structure we are talking about.

Notation 3.3.4. *Since we are working with three different model structure, hence it would be necessary to avoid confusion of notations. Generally, we will use footscript to distinguish the notations, e.g. $[X, Y]_{\mathbb{A}^1}$ will be the homotopy class of morphism in \mathbb{A}^1 -local sense while $[X, Y]_{Nis}$ will be in Nis -local sense.*

*In words, we will use **Nis-local** and **\mathbb{A}^1 -local** to indicate the model structure the discussion is about, e.g. a Nis -local fibrant replacement is a replacement happens in \mathbf{Spc} . Particularly, by Theorem 3.2.4, the Nis -local (resp. \mathbb{A}^1 -local) objects are precisely the fibrant objects in that model structure, hence we will basically use the term Nis -local (resp. \mathbb{A}^1 -local) standing for fibrantness.*

Notice since we are using the injective model structure, hence all objects are cofibrant for free, and so we don't need to distinguish for this. For the same reason, we rarely mention the cofibrant replacement in our discussions.

At last, we mention that the localization at Zariski or Étale covering will also shows up in this thesis. Therefore in case, we will also use notations such as $(-)_{Zar}$, $(-)_{\acute{e}t}$ or

just $(-)_\tau$ when we just refer to a Grothendieck topology τ . However notice the \mathbb{A}^1 -localization is just for Nisnevich covering.

And also since we first localize at the Nisnevich coverings, hence when we talk about an \mathbb{A}^1 -local simplicial presheaf, we always means this has to be at least a *Nis*-local simplicial presheaf. One can see in the next section, we use ‘ \mathbb{A}^1 -invariant’ for a simplicial sheaf that is somehow ‘ \mathbb{A}^1 -local’ but not necessarily *Nis*-local.

Remark 3.3.5. We make a final remark for these details. The Bousfield localization does not change the simplicial structure of the categories, hence function complex, tensor product and internal function complex is computed the same way.

However, the derived function complex might be different, since the fibrant replacement could differ in different model structures we discussed in this thesis.

And one should always be careful about the replacements. For example, the \mathbb{A}^1 -local fibrant replacement $R_{\mathbb{A}^1}\mathcal{F}$ of a simplicial presheaf \mathcal{F} is \mathbb{A}^1 -locally weak equivalent to \mathcal{F} . Hence the weak equivalence of simplicial sets $\mathcal{F}(U) \simeq R_{\mathbb{A}^1}\mathcal{F}(U)$ might need extra conditions to hold, for instance \mathcal{F} is \mathbb{A}^1 -local.

We now introduce the Nisnevich excision in this homotopy theory, in terms of squares.

Definition 3.3.6. Consider a pullback diagram in \mathbf{Sm}_k :

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where p is Étale and i is an open immersion. This is an **elementary distinguished square (EDS)** if $(X - U)_{red} \cong (p^{-1}(X - U))_{red}$ as reduced schemes.

Proposition 3.3.7. All EDS in \mathbf{Sm}_k generates the Nisnevich topology.

Proof. We prove that for a finite *Nis*-covering family $\mathcal{U} = \{p_i : X_i \rightarrow X\}$ of $X \in \mathbf{Sm}_k$, there exists a **splitting sequence** for \mathcal{U} , i.e. A sequence of closed subsets of X :

$$\emptyset = Z_{n+1} \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 = X,$$

s.t. $\bigsqcup p_i^{-1}(Z_n - Z_{n+1}) \rightarrow Z_n - Z_{n+1}$ admits a section for all n .

Let $p = \bigsqcup p_i$. We choose $x \in X$, by definition of Nisnevich topology, we let $R_1 = \text{Spec} \mathcal{O}_{X,x}^h$ and $Z_1 = X - R_1$. For Z_1 , we repeat the step and eventually

we obtain a descending sequence of closed subsets of X . Since X is Noetherian, hence the descending sequence stabilize at finite length, say length $n + 1$. Z_{n+1} must be empty otherwise we can keep reduced the size and then contradicts to the stability. And so the splitting sequence obtained.

Now we start to operate at Z_n . Let s be a section of $p^{-1}Z_n \rightarrow Z_n$. Since p is Étale, hence $p^{-1}Z_n = s(Z_n) \sqcup Y$ where Y is a closed subscheme of $\bigsqcup_i X_i$. Then let $V = \bigsqcup_i X_i - Y$, and $U = X - Z_n$, we obtain an EDS over X . Again we repeat the step, then we decompose a *Nis*-covering into a finite composition of EDS. Hence the *Nis*-topology can be generated by EDS's. \square

Corollary 3.3.8. *A presheaf F on \mathbf{Sm}_k is a sheaf in Nisnevich topology if and only if for any EDS Q , $F(Q)$ is a pullback square of sets.*

Proposition 3.3.9. *In \mathbf{Spc} , a simplicial presheaf F is Nis-local if and only if it satisfies the Nisnevich excision, i.e. for every EDS:*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

we have $F(X) \simeq \text{ho}(F(U) \times_{F(U \times_X V)} F(V))$ and $F(\emptyset) = *$.

Proof. We postpone the proof to Theorem 3.5.10 in section 5:cd-structure of this chapter. \square

Example 3.3.10. One can see a Nisnevich sheaf is indeed a *Nis*-locally fibrant discrete simplicial presheaf since we have an isomorphism $F(X) \cong F(U) \times_{F(U \times_X V)} F(V)$ as sets by Corollary 3.3.8. Furthermore this is an isomorphism of discrete simplicial sets, hence is a homotopy pullback. Particularly, any representable simplicial presheaf $X \in \mathbf{Sm}_k$ is *Nis*-local.

Proposition 3.3.11. *Any EDS*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

is a pushout square of schemes.

Furthermore, in the full subcategory of Nis-local simplicial presheaves \mathbf{Spc}_f , any EDS is also a pushout square.

Proof. It is equivalent to prove the following diagram of sets is a pullback:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Sm}_k}(X, Z) & \longrightarrow & \mathrm{Hom}_{\mathbf{Sm}_k}(V, Z) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{Sm}_k}(U, Z) & \longrightarrow & \mathrm{Hom}_{\mathbf{Sm}_k}(U \times_X V, Z) \end{array}$$

for arbitrary $Z \in \mathbf{Sm}_k$. This is straight forward since the representable presheaf Z is a *Nis*-sheaf and an EDS is a *Nis*-covering of schemes. Therefore the square is indeed a pushout of schemes.

For the second part, since we are working in the *Nis*-local simplicial sheaves, the *Nis*-local descent condition guarantee X is the colimit of the pushout diagram. \square

Corollary 3.3.12. *By viewing a scheme as a discrete simplicial presheaf, any EDS $\{U \xrightarrow{i} X \xleftarrow{p} V\}$ is a homotopy pushout in \mathbf{Spc} . Particularly, we have an *Nis*-local weak equivalence:*

$$X/U \simeq_{\mathrm{Nis}} V/(U \times_X V).$$

Same things happen in $\mathbf{Spc}^{\mathbb{A}^1}$ as we apply $\mathbf{L}_{\mathbb{A}^1}$.

Proof. By previous proposition, an EDS $\{U \xrightarrow{i} X \xleftarrow{p} V\}$ is a pushout square in \mathbf{Spc}_f , therefore by Proposition 2.4.8, this is a homotopy pushout in \mathbf{Spc}_f . By Remark 2.1.13, since this subcategory induces the same homotopy category as \mathbf{Spc} do, therefore the EDS is a homotopy pushout in \mathbf{Spc} .

We then consider a composition of pushout diagrams in \mathbf{Spc}_f :

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/U \end{array} ,$$

therefore as *Nis*-local simplicial presheaves, $X/U \cong V/(U \times_X V)$ and so they have the same *Nis*-local homotopy type.

For $\mathbf{Spc}^{\mathbb{A}^1}$, the argument is basically the same. \square

3.4 Singular Construction

We define the *algebraic standard simplices* as

$$\Delta_{alg}^n = \text{Spec}(k[X_0, \dots, X_n] / X_0 + \dots + X_n = 1).$$

The cofaces and codegeneracies are:

$$d^i : \Delta_{alg}^n \rightarrow \Delta_{alg}^{n+1}, (x_0, \dots, x_n) \mapsto (x_0, \dots, x_i, x_i, \dots, x_n)$$

$$s^j : \Delta_{alg}^n \rightarrow \Delta_{alg}^{n-1}, (x_0, \dots, x_n) \mapsto (x_0, \dots, x_j + x_{j+1}, \dots, x_n).$$

And so we obtain a cosimplicial scheme and furthermore a cosimplicial simplicial presheaf, denote Δ_{alg}^\bullet . Notice that Δ_{alg}^1 , which is isomorphic to the affine line, comes with a multiplication $\mu : \Delta_{alg}^1 \times \Delta_{alg}^1 \rightarrow \Delta_{alg}^1$ via $(x_0, x_1) \times (y_0, y_1) \mapsto (x_1 \cdot y_1)$. Then for two vertices of $d^0, d^1 : \Delta^0 \rightarrow \Delta^1$, we have the following laws (similar for $i_0 \times id$ and $i_1 \times id$):

$$\begin{array}{ccc} \Delta_{alg}^1 \times \Delta_{alg}^0 \cong \Delta_{alg}^1 & \xrightarrow{p} & \Delta_{alg}^0 \\ \downarrow id \times d^0 & & \downarrow d^0 \\ \Delta_{alg}^1 \times \Delta_{alg}^1 & \xrightarrow{\mu} & \Delta_{alg}^1 \end{array} \quad \begin{array}{ccc} \Delta_{alg}^1 \times \Delta_{alg}^0 \cong \Delta_{alg}^1 & \xrightarrow{p} & \Delta_{alg}^0 \\ \downarrow id \times d^1 & & \downarrow d^1 \\ \Delta_{alg}^1 \times \Delta_{alg}^1 & \xrightarrow{\mu} & \Delta_{alg}^1 \end{array}$$

Notice Δ_{alg}^n is isomorphic to \mathbb{A}^n by taking coordinates x_1, \dots, x_n . Then by viewing Δ_{alg}^0 and Δ_{alg}^1 as \mathbb{A}^0 and \mathbb{A}^1 , d^0, d^1 are exactly the 0 and 1-sections. Hence we will normally ignore the difference between d^0, d^1 and the section $i_0, i_1 : \text{Spec}(k) \rightarrow \mathbb{A}^1$.

Furthermore, we will normally use d^0, d^1 for the simplicial simplices, e.g. $d^0, d^1 : \Delta^0 \rightarrow \Delta^1$. Notice $\Delta^0 \cong \Delta_{alg}^0 = \text{Spec}(k[X_0] / X_0 = 1)$, hence we have two types of morphism that contains similar simplicial structures as:

$$\Delta^1 \xleftarrow{d^0, d^1} \Delta^0 \cong \Delta_{alg}^0 \xrightarrow{i_0, i_1} \Delta_{alg}^1$$

Definition 3.4.1 (Elementary Homotopy). For $f, g : X \rightarrow Y$ morphisms of simplicial presheaves. We say $H : X \times \Delta_{alg}^1 \rightarrow Y$ is an **elementary \mathbb{A}^1 -homotopy** if $H(i_0) = f, H(i_1) = g$, denoted $f \simeq_{ele} g$. We say f, g are **\mathbb{A}^1 -homotopic** if they are connected by a sequence of elementary \mathbb{A}^1 -homotopies, denoted $f \simeq_{ele, \mathbb{A}^1} g$. $f : X \rightarrow Y$ is a **strict \mathbb{A}^1 -homotopy equivalence** if there exists $h : Y \rightarrow X$ s.t. fh and hf are \mathbb{A}^1 -homotopic trivial.

Particularly, we write I^n (and I for just I^1) for Δ_{alg}^n for convenience.

Remark 3.4.2. Notice though an elementary I -homotopy may not be a simplicial homotopy, a strict I -homotopy equivalence is indeed a simplicial homotopy equivalence. This is a direct observation from \mathbb{A}^1 is \mathbb{A}^1 -locally contractible.

Definition 3.4.3 (Singular Construction). Let X be a simplicial presheaf. We define $\mathbf{Sing}^{\mathbb{A}^1} X$ or shortly $\mathbf{Sing} X$ as the presheaf given by the diagonal of the bisimplicial presheaf $X(- \times I^\bullet)$, i.e. $\mathbf{Sing} X(U)_n = X(U \times I^n)_n$.

Remark 3.4.4. This is indeed a \mathbf{sSets} -valued presheaf as the faces and degeneracies are:

$$\begin{array}{ccc} X(U \times I^n)_n & \xrightarrow{(d^i)^*} & X(U \times I^{n-1})_n \\ \downarrow d_i & & \downarrow d_i \\ X(U \times I^n)_{n-1} & \xrightarrow{(d^i)^*} & X(U \times I^{n-1})_{n-1} \end{array} \quad \begin{array}{ccc} X(U \times I^n)_n & \xrightarrow{(s^j)^*} & X(U \times I^{n+1})_n \\ \downarrow s_j & & \downarrow s_j \\ X(U \times I^n)_{n+1} & \xrightarrow{(s^j)^*} & X(U \times I^{n+1})_{n+1} \end{array}$$

which are commutative diagrams.

Remark 3.4.5. By Lemma 3.1.3, if we view a level of a simplicial presheaf as a discrete simplicial presheaf, then we have $(\mathbf{Sing} X)_n := X(- \times I^n)_n = X_n(- \times I^n) \cong \underline{\mathbf{Map}}(I^n, X_n)$.

Proposition 3.4.6. The assignment $(X \mapsto \mathbf{Sing} X)$ is a functor $\mathbf{Sing} : \mathbf{sPre} \rightarrow \mathbf{sPre}$.

Proof. For a morphism f , $\mathbf{Sing}(f)$ is simply the collection of $f(U \times I^n)_n$. \square

Proposition 3.4.7. Let $f, g : X \rightarrow Y$ be morphisms of simplicial presheaves, and $H : X \times I \rightarrow Y$ be an elementary \mathbb{A}^1 -homotopy from f to g . Then there exists a simplicial homotopy from $\mathbf{Sing}(f)$ to $\mathbf{Sing}(g)$.

Proof. Notice singular construction commutes with products, hence we obtain a morphism $\mathbf{Sing}(H) : \mathbf{Sing} X \times \mathbf{Sing} I \rightarrow \mathbf{Sing} Y$.

Also notice we have a commutative diagram:

$$\begin{array}{ccc} * \times \Delta^0 & \xrightarrow{\text{Sing}(i_0), \text{Sing}(i_1)} & \\ \downarrow d^0, d^1 & \searrow & \\ * \times \Delta^1 & \xrightarrow{G} & \mathbf{Sing} I \end{array} \quad (3.4.7)$$

where G corresponds to the identity element in $\mathbf{Sing} I(*)_1 = I(* \times I) = \mathbf{Hom}_{\mathbf{Sm}/\mathbf{k}}(I, I)$ by simplicial Yoneda 3.1.2. Hence G is a simplicial homotopy from $\mathbf{Sing}(i_0)$ to $\mathbf{Sing}(i_1)$.

Then by composing as in the commutative diagram:

$$\begin{array}{ccc}
 \text{Sing}X \times \Delta^0 & & \\
 \downarrow d^0, d^1 & \searrow \text{Sing}(f), \text{Sing}(g) & \\
 \text{Sing}X \times \Delta^1 & \longrightarrow & \text{Sing}Y \\
 \downarrow G & \nearrow \text{Sing}(H) & \\
 \text{Sing}X \times \text{Sing}I & &
 \end{array}$$

We obtain $\text{Sing}(H) \circ G$ as the wanted simplicial homotopy. \square

Proposition 3.4.8. *The canonical map $\text{Sing}X \xrightarrow{i_{0,*}} \text{Sing}(X \times I)$ is a homotopy equivalence.*

Proof. Since $p \circ i_0 = \text{id}$, hence we naturally have $\text{Sing}(p) \circ \text{Sing}(i_0) = \text{id}$. Then it suffices to show $i_0 \circ p \simeq \text{id}$ in the elementary sense which implies $\text{Sing}(i_0) \circ \text{Sing}(p) \simeq \text{id}$ by Proposition 3.4.7.

This is straightforward since we have a commutative diagram:

$$\begin{array}{ccc}
 X \times I & \xrightarrow{p} & X \\
 \downarrow \text{id} \times i_0 & & \downarrow i_0 \\
 X \times I \times I & \xrightarrow{\text{id} \times \mu} & X \times I
 \end{array}$$

where μ is the multiplication on $I \cong \mathbb{A}^1$. As $\text{id} \times \mu(\text{id} \times i_0) = i_0 \circ p$ and $\text{id} \times \mu(\text{id} \times i_1) = \text{id} \times (\mu \circ i_1) = \text{id} \times \text{id}$, hence $\text{id} \times \mu$ is the an elementary \mathbb{A}^1 -homotopy. And then the proposition proven. \square

Proposition 3.4.9. *The canonical morphism $p^* : X \rightarrow \underline{\text{Map}}(I, X)$ is a strict \mathbb{A}^1 -homotopy equivalence.*

Proof. By Lemma 3.1.3, the simplicial presheaf $\underline{\text{Map}}(I, X) \cong X(- \times I)$ and we will use this expression in this proof. Let $x : * \rightarrow I$ be a point of the representable simplicial presheaf I , notice we have a commutative diagram of simplicial presheaves:

$$\begin{array}{ccccc}
 X(- \times I) & \xrightarrow{\mu^*} & & \xrightarrow{\quad} & X(- \times I^2) \\
 \text{id} \times x \downarrow & & & \swarrow \text{id} \times x & \downarrow x^* \\
 X(- \times I) \times I & \xrightarrow{\mu^*} & X(- \times I \times I) \times I & \xrightarrow{\text{ev}} & X(- \times I)
 \end{array}$$

where ev is the evaluation morphism given by the monoidal structure of $\mathbf{sPre}(\mathbf{Sm}_k)$. When x is the 0-section i_0 , then the composition $x^* \circ \mu^* = (\mu \circ i_0)^* = (i_0 \circ p)^* = p^* \circ i_0^*$, while $x^* \circ \mu^* = (\mu \circ i_1)^* = id^*$ for $x = i_1$. Therefore $ev \circ \mu^*$ is an elementary I -homotopy from $p^* \circ i_0^*$ to id .

On the other hand, $i_0^* \circ \mu^* = id$, therefore i_0^*, p^* are strict I -homotopy equivalence. \square

Definition 3.4.10. Given a simplicial presheaf X , we say it is I -invariant or \mathbb{A}^1 -invariant if the morphism induced by projection:

$$X(-) \rightarrow X(- \times I)$$

is a weak equivalences of simplicial presheaves.

Particularly, we say a presheaf of sets X is I -invariant if the induced morphism above is an isomorphism.

Proposition 3.4.11. Given a simplicial presheaf $X \in \mathbf{sPre}(\mathbf{Sm}_k)$, $SingX$ is \mathbb{A}^1 -invariant.

Proof. Since $SingX(U \times I) \cong Sing(X(- \times I))(U) \cong Sing(\underline{Map}(I, X))(U)$, hence by Proposition 3.4.7, we have a simplicial equivalence induced from the \mathbb{A}^1 -homotopy equivalence given by Proposition 3.4.9:

$$Sing(p^*) : SingX \rightarrow Sing(\underline{Map}(I, X)).$$

Hence for arbitrary $U \in \mathbf{Sm}_k$,

$$p^* = Sing(p^*)(U) : SingX(U) \rightarrow SingX(U \times I)$$

is a weak equivalence of simplicial sets. \square

Proposition 3.4.12. The canonical map $X \rightarrow SingX$ induced by projection is an \mathbb{A}^1 -weak equivalence.

Proof. We show this levelwise, i.e. $X_n \rightarrow (SingX)_n$ is an \mathbb{A} -local weak equivalence by viewing them as discrete simplicial presheaves, then the proof is complete in [5, Proposition §2 2.14]: if a morphism of simplicial presheaves induces \mathbb{A}^1 -local weak equivalences levelwise, then it is an \mathbb{A}^1 -local weak equivalence.

At level $n = 0$, the levelwise claim automatically holds. For level $n \geq 1$, consider X_n as a discrete simplicial presheaf, then notice $SingX_n := X(- \times I^n)_n \cong \underline{Map}(I^n, X_n)$ by Remark 3.4.5, then it is equivalent to show $p^* : X_n \rightarrow \underline{Map}(I^n, X_n)$. Also since $\underline{Map}(I^n, X_n) \cong X_n(- \times I^n) = X_n(- \times I^{n-1} \times I) \cong \underline{Map}(I, \underline{Map}(I^{n-1}, X_n))$, then by induction, we are left to show $p^* : X_n \rightarrow \underline{Map}(I, X_n)$ is an \mathbb{A}^1 -local weak equivalence. By Remark ??, we are done. \square

Proposition 3.4.13. *Let X be a simplicial presheaf on \mathbf{Sm}_k . Suppose $\pi_0 X$ is I -invariant, then*

$$\pi_0 X \cong \pi_0 \text{Sing}^I X.$$

Proof. Consider the simplicial structure at $U \in \mathbf{Sm}_k$ and level 0,1, and we put an extra commutative square given by the I -invariantness.

$$\begin{array}{ccccc} X(U)_1 & \xrightarrow{p^*} & X(U \times I)_1 & \xrightarrow{i^*} & X(U)_1 \\ \downarrow d & & \downarrow d & & \downarrow d \\ X(U)_0 & \xrightarrow{p^*} & X(U \times I)_0 & \xrightarrow{i^*} & X(U)_0 \end{array} ,$$

where d is the face map and i^* is induced by inclusions. We need to show that given $a, b \in X(U)_0$, $a \stackrel{F}{\sim} b$ for some $F \in X(U)_1$ if and only if $a \stackrel{G}{\sim} b$ for some $G \in X(U \times I)_1$. Since p^* induces isomorphism of 0-homotopy, hence if $a \stackrel{F}{\sim} b$ for some $F \in X(U)_1$, then there exist $G \in X(U \times I)_1$ giving $p^*a \sim p^*b$. Since $i^* \circ p^* = id$, hence G also induces equivalence $a \sim b$ in $X(U)_0$.

Conversely, suppose $a \stackrel{G}{\sim} b$ for some $G \in X(U \times I)_1$, say $i_0^* d_0 G = a$ and $i_1^* d_1 G = b$. Then $p^*a = p^*i_0^* d_0 G = p^* d_0 i_0^* G = d_0 p^* i_0^* G$ and similarly $p^*b = d_1 p^* i_1^* G$. Therefore $p^*a \sim p^*b$ in $X(U \times I)_0$. Then again since p^* induces isomorphism at 0-homotopy, we also obtain $a \sim b$ in $X(U)_0$. \square

3.5 cd-Structure

In this section, we introduce the concept of **cd-structure** which basically rephrase the concept of Grothendieck topology. A convenience of this notation is that we can set up the concept of excision with respect to a Grothendieck topology.

Definition 3.5.1. *A cd-structure P on a small category which has initial objects is a collection of commutative squares s.t. the collection is closed under isomorphisms of squares.*

A such cd-structure P generates a Grothendieck topology τ_P . By this we mean τ_P is the coarsest Grothendieck topology s.t. the empty sieve covers the initial object \emptyset and for any squares in P :

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} ,$$

the sieve generated by U, V is a τ_P -covering sieve of X .

Example 3.5.2. As we have proven in chapter II, the Nisnevich topology on \mathbf{Sm}_k is generated by P_{Nis} which is the collection of all elementary distinguished squares(EDS), namely **Nisnevich cd-structure**.

Example 3.5.3. The Zariski topology on \mathbf{Sm}_k is generated by P_{Zar} which consists of all pullback squares:

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where p, i form a Zariski open covering of X .

Example 3.5.4. As we try to fit into the affine classification of vector bundles in chapter 3, we introduce the Nisnevich and Zariski topology restricted on category of smooth schemes \mathbf{Sm}_k^{aff} . On \mathbf{Sm}_k^{aff} , the Zariski topology is given by the **affine Zariski cd-structure** P_{Zar}^{aff} which consists of squares:

$$\begin{array}{ccc} Spec(A_{fg}) & \longrightarrow & Spec(A_g) \\ \downarrow & \lrcorner & \downarrow p \\ Spec(A_f) & \xrightarrow{i} & Spec(A) \end{array}$$

where f, g are units of the ring A . And the Nisnevich topology on \mathbf{Sm}_k^{aff} is generated by the **affine Nisnevich cd-structure** P_{Nis}^{aff} which consists of pullback squares:

$$\begin{array}{ccc} Spec(B_f) & \longrightarrow & Spec(B) \\ \downarrow & \lrcorner & \downarrow p \\ Spec(A_f) & \xrightarrow{i} & Spec(A) \end{array}$$

where p is Étale and f is a unit s.t. $A/f \cong B/f$. Section 2 in [1] provides a discussion in details that these two cd-structures indeed generate the corresponding restricting topologies.

Notice when we are talking about coverings of representable simplicial sheaves, we might construct them in two different ways. One is what we did in chapter 2 which is the Čech-complex and we denote the collection of morphisms as \mathcal{S}_τ : given a covering $\mathcal{U} = \{U_i \rightarrow X\}$ and $U := \sqcup U_i$, we construct a simplicial presheaf $\check{\mathcal{C}}(\mathcal{U})$ as $\check{\mathcal{C}}(\mathcal{U})_n := U^{\times_X^{n+1}}$ equipped with simplicial structure induced by indexes. Such a simplicial presheaf naturally maps into the representable

simplicial presheaf X . Then \mathcal{S}_τ is the collection of representatives of isomorphism classes of these morphisms.

The other one is the discrete covering sieve. A covering sieve U of X can be viewed as subpresheaf of the representable presheaf X equipped with a canonical monomorphism $U \rightarrow X$. In this way, we can also view them as discrete simplicial sheaves and then obtain a collection of representatives of these morphisms in \mathbf{sPre} , say \mathcal{S}^τ . The following lemma shows that two collection leads to the same Bousfield localized simplicial model category of simplicial presheaves.

Lemma 3.5.5. *Let \mathcal{M} be a cocomplete model category and $\mathbf{sPre}(\mathcal{M})$ be the category of simplicial presheaves equipped injective model structure. Let τ be a Grothendieck topology on \mathcal{M} and two collections of morphisms $\mathcal{S}^\tau, \mathcal{S}_\tau$ be as above.*

Then a simplicial presheaf \mathcal{F} is \mathcal{S}^τ -locally fibrant if and only if \mathcal{F} is \mathcal{S}_τ -locally fibrant.

Proof. Consider a τ -covering $\mathcal{U} = \{U_i \rightarrow X\}$, let R be the corresponding subpresheaf of the representable presheaf X . A tricky observation is that the simplicial structure on $\check{C}(\mathcal{U})$ is trivial in the sense that $d_i = d_j, \forall i, j$ at any level n . Therefore $\check{C}(\mathcal{U})$ is pointwise fibrant. Then the natural morphism $\check{C}(\mathcal{U}) \rightarrow \pi_0 \check{C}(\mathcal{U}) = \check{C}(\mathcal{U})_0$ is a pointwise weak equivalence.

However the presheaf $\check{C}(\mathcal{U})_0$ is the representable presheaf $\sqcup U_i$ which is exactly R . Therefore we have a canonical pointwise weak equivalence $\check{C}(\mathcal{U}) \rightarrow R$ and so the conclusion is straight forward. \square

Remark 3.5.6. The \mathcal{S}_τ -local condition can be described as follows:

$$\mathcal{F}(X) \rightarrow \operatorname{holim}_{n \in \mathbb{N}} \prod_{i_0, \dots, i_n \in I} \mathcal{F}(U_{i_0} \times_X U_{i_1} \times_X \cdots \times_X U_{i_n})$$

is a weak equivalence of simplicial sets for any τ -covering $\mathcal{U} = \{U_i \rightarrow X\}_{i \in I}$. We call this the **Čech-condition**.

Definition 3.5.7. *Let \mathcal{M} be a model category and P is a cd-structure on \mathcal{M} . A simplicial presheaf \mathcal{F} satisfies **P-excision** if $\mathcal{F}(\emptyset) \simeq *$ and $\mathcal{F}(Q)$ is a homotopy pullback for $\forall Q \in P$.*

Definition 3.5.8. *An initial object \emptyset in a category \mathcal{C} is **strict** if any morphism $X \rightarrow \emptyset$ is an isomorphism.*

Example 3.5.9. The category of schemes \mathbf{Sch} and the category of smooth k -schemes \mathbf{Sm}_k are examples having a strict initial object which is the spectrum of the zero ring.

We introduce the equivalence between local conditions and excisions under certain conditions. The original version is proved by Voevodsky in [13, Corollary 5.10]. Here we cite a variant in [1, Theorem 3.2.5].

Theorem 3.5.10 (Voevodsky). *Let \mathcal{M} be a model category with a strict initial object \emptyset and P be a cd-structure s.t. pullbacks of squares exist and lie in P . Suppose for any square Q in P :*

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p, \\ U & \xrightarrow{i} & X \end{array}$$

we have:

1. Q is a pullback;
2. i is a monomorphism;
3. the following square is a pullback in \mathcal{M} :

$$\begin{array}{ccc} W & \longrightarrow & V \\ \Delta \downarrow & & \downarrow \Delta \\ W \times_U W & \longrightarrow & V \times_X V \end{array} .$$

Then the P -excision condition is equivalent to τ_P -local condition.

As one might expect, cd-structures in Example 3.5.2, 3.5.3 and 3.5.4 are examples of certain nice topology. For the rest of the section, we show an example of relations between different cd-structure. These relations play an important role in the affine classification of vector bundles in chapter 3 and allow us to avoid stronger excisive condition but focus on a weaker one. In our case, the strong excision would be Nisnevich excision and the weaker one to be the Zariski one.

There is a Quillen functor induced by inclusion functor of categories $i : \mathbf{Sm}_k^{\text{aff}} \rightarrow \mathbf{Sm}_k$:

$$i^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}}) : i_*$$

The functor i^* is just the restriction to affine schemes. Hence we have the derived adjunction:

$$i^* : \mathcal{H}_k \rightleftarrows \mathcal{H}_k^{\text{aff}} : \mathbf{R}i_*$$

where $\mathbf{R}i_* \mathcal{F}(X) := \text{holim}_{U \in \mathbf{Sm}_k^{\text{aff}}/X} \mathcal{F}(U)$ is given by homotopy right Kan extension along i . The left derived functor is just i^* since all objects are cofibrant.

Proposition 3.5.11. *Let τ be the Zariski or Nisnevich topology. We assign the injective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ and assign the simplicial structure by embedding standard simplices as discrete simplicial sheaves. Then the derived adjunction i^* and $\mathbf{R}i_*$ both preserve τ -localness.*

*Particularly, we define $\mathcal{F}^{aff} := \mathbf{R}i_*i^*\mathcal{F}$, namely the **affine replacement** of \mathcal{F} .*

Proof. A simplicial τ -sheaf \mathcal{F} satisfies the corresponding Čech-conditions, hence satisfies the corresponding affine Čech-conditions. Therefore equivalently \mathcal{F} is a simplicial τ -sheaf on affine schemes.

For $\mathbf{R}i_*$, notice $i^*\mathcal{F} \otimes \Delta^n = i^*(\mathcal{F} \otimes \Delta^n)$ for arbitrary simplicial presheaf \mathcal{F} , therefore by Proposition 2.3.9, we have weak equivalences in the injective model structure:

$$\mathrm{Map}(X, \mathbf{R}i_*\mathcal{F}) \simeq \mathrm{Map}(i^*X, \mathcal{F}),$$

for representable X and arbitrary simplicial presheaf \mathcal{F} . For simplicity, we choose to localize at the discrete morphisms $\mathcal{S}^\tau = \{R \rightarrow X\}$ induced by covering sieves, then $\mathbf{R}i_*\mathcal{F}$ is τ -local if and only if $\mathrm{Map}(X, \mathbf{R}i_*\mathcal{F}) \simeq \mathrm{Map}(R, \mathbf{R}i_*\mathcal{F})$ for all $(R \rightarrow X) \in \mathcal{S}^\tau$, which is equivalent to $\mathrm{Map}(i^*X, \mathcal{F}) \simeq \mathrm{Map}(i^*R, \mathcal{F})$ for all $(R \rightarrow X) \in \mathcal{S}^\tau$.

Since i^* preserves small colimits, hence it preserves the sheaves, therefore it sends these covering sieves to τ -local isomorphisms, which indeed are τ -local weak equivalences. Hence $\mathbf{R}i_*$ preserves τ -localness. \square

Proposition 3.5.12. *In the same conditions as above, by restricting to full subcategories spanned by τ -local simplicial presheaves of homotopy categories, we have $\mathbf{R}i_*$ is a fully faithful inclusion of homotopy categories. Furthermore, $(i^*, \mathbf{R}i_*)$ is a pair of an equivalence between these homotopy subcategories.*

Proof. The first part is easy, since in \mathbf{Sm}_k/X has a final object X and hence for a τ -local simplicial presheaf \mathcal{F} , $F(X) \simeq \mathrm{holim}_{\mathbf{Sm}_k/X}(F(U))$. Therefore $i^*\mathbf{R}i_*\mathcal{F} \simeq \mathcal{F}$ and they become isomorphic in the homotopy category.

The second part is hence suffice to prove i^* reflect isomorphism in the homotopy categories. Given two τ -local simplicial presheaves \mathcal{F}, \mathcal{G} on \mathbf{Sm}_k s.t. they are pointwise weak equivalent on affines via a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$. For any separated scheme Y , it admits a finite τ -covering family $\{V_i \rightarrow Y\}$ s.t. its Čech-complex only consists of affine schemes. Then since \mathcal{F}, \mathcal{G} are τ -local, hence the family induces weak equivalence $f_Y : \mathcal{F}(Y) \rightarrow \mathcal{G}(Y)$.

For arbitrary scheme $X \in \mathbf{Sm}_k$, it admits a finite covering family $\{U_i \rightarrow X\}_{i \in I}$ s.t. the Čech-complex only consists of separated schemes. Hence similarly, we obtain equivalences everywhere and then f becomes isomorphism in the homotopy category. \square

Corollary 3.5.13. *We have a commutative diagram of categories consists of only left Quillen functors:*

$$\begin{array}{ccccc} \mathbf{sPre}(\mathbf{Sm}_k) & \xrightarrow{id} & \mathbf{Spc}_{Zar} & \xrightarrow{id} & \mathbf{Spc}_{Nis} \\ \downarrow i^* & & \downarrow i_{Zar}^* & & \downarrow i_{Nis}^* \\ \mathbf{sPre}(\mathbf{Sm}_k^{aff}) & \xrightarrow{id} & \mathbf{Spc}_{Zar}^{aff} & \xrightarrow{id} & \mathbf{Spc}_{Nis}^{aff} \end{array} .$$

Particularly, the functor i_τ^* is the same as i^* .

Proof. The composition $id \circ i^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{Spc}_0^{aff}$ sends \mathcal{S}^τ to τ -local weak equivalences and hence we can induces i_τ^* as a left Quillen functor by the universal property of left Bousfield localization. \square

Since we are going to work with Zariski and Nisnevich topology at the same times, we distinguish the notations by putting a subscript $(-)_{Zar}$ or $(-)_{Nis}$. And also since we are going to work in both \mathbf{Sm}_k and \mathbf{Sm}_k^{aff} , we distinguish the notations by putting a superscript $(-)^{aff}$ when we means something just on affine schemes.

Proposition 3.5.14. *Let \mathcal{F} be a simplicial presheaf on \mathbf{Sm}_k . If \mathcal{F} satisfies affine Zariski excision, i.e. $i^*\mathcal{F}$ is Zar-local, then a fibrant replacement $R_{Zar}\mathcal{F}$ in \mathbf{Spc}_{Zar} is weak equivalent to \mathcal{F} in $\mathbf{sPre}(\mathbf{Sm}_k^{aff})$, i.e. we have weak equivalences of simplicial sets for all affines $X \in \mathbf{Sm}_k^{aff}$*

$$\mathcal{F}(X) \simeq R_{Zar}\mathcal{F}(X).$$

Proof. Let \mathcal{F} be a simplicial presheaf on \mathbf{Sm}_k . Notice we have a commutative diagram of simplicial presheaves:

$$\begin{array}{ccc} i^*\mathcal{F} & \xrightarrow{i^*f} & i^*(R_{Zar}\mathcal{F}) \\ \downarrow & & \downarrow \\ i^*\mathbf{R}i_*i^*\mathcal{F} & \xrightarrow{i^*\mathbf{R}i_*i^*f} & i^*\mathbf{R}i_*i^*(R_{Zar}\mathcal{F}) \end{array}$$

where $f : \mathcal{F} \rightarrow R_{Zar}\mathcal{F}$ is the canonical trivial fibration. We will show all four arrows in the diagram are (injective) weak equivalences in $\mathbf{sPre}(\mathbf{Sm}_k^{aff})$.

Since $i^*\mathcal{F}$ is Zar-local by our assumption, hence one can tell the left vertical arrow is a (affinely) pointwise weak equivalence from the construction of functor $\mathbf{R}i_*$. Same thing happens to the right vertical arrow since $R_{Zar}\mathcal{F}$ is a Zar-local simplicial presheaf.

For the bottom arrow, notice i^*f is a Zar-local weak equivalence between Zar-local objects, i.e. a weak equivalence between fibrant objects in \mathbf{Spc}_{Zar}^{aff} .

Then $\mathbf{R}i_*(i^*f)$ is a weak equivalence between fibrant objects in $\mathbf{sPre}(\mathbf{Sm}_k)$ and hence is a pointwise weak equivalence. Then we apply i^* again, then $i^*\mathbf{R}i_*i^*f$ is a weak equivalence in $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$. Then i^*f is a weak equivalence in $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$. \square

Proposition 3.5.15. *Let \mathcal{F} be a simplicial presheaf on \mathbf{Sm}_k . If \mathcal{F} is affine Nis-local, then a Zariski fibrant replacement $R_{Zar}\mathcal{F}$ of \mathcal{F} is Nis-local.*

Proof. To avoid confusions, we must mention this ahead: all weak equivalences in this proof are injective weak equivalences, i.e. weak equivalences in $\mathbf{sPre}(\mathbf{Sm}_k)$ or $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$.

Since $i^*\mathcal{F}$ is Nis-local, then by Proposition 3.5.14, $i^*(R_{Zar}\mathcal{F})$ is Nis-local. By Proposition 3.5.12, $(i^*, \mathbf{R}i_*)$ preserves and reflect weak equivalences between Nis-local objects, hence $i^*\mathbf{R}i_*i^*(R_{Zar}\mathcal{F}) \simeq i^*(R_{Zar}\mathcal{F})$ implies $\mathbf{R}i_*i^*(R_{Zar}\mathcal{F}) \simeq R_{Zar}\mathcal{F}$. By Proposition 3.5.11, $\mathbf{R}i_*(i^*(R_{Zar}\mathcal{F}))$ is Nis-local and hence satisfies Nis-excision. Then since $R_{Zar}\mathcal{F}$ is pointwise weak equivalent to $\mathbf{R}i_*i^*R_{Zar}\mathcal{F}$, therefore it satisfies Nis-excision as well and so it is Nis-local. \square

Recall for a given simplicial presheaf \mathcal{F} on \mathbf{Sm}_k , one can define the singular construction $Sing^I\mathcal{F} := \text{diag}(\mathcal{F}(U \times I^\bullet), \bullet)$, where $I = \Delta_{\text{alg}}^1 \cong \mathbb{A}^1$. Now we show that not only a Nis-local fibrant replacement, but also an \mathbb{A}^1 -local fibrant replacement of \mathcal{F} can be obtained by using the Zar-replacement.

Proposition 3.5.16. *Let τ be either Nis or Zar on $\mathcal{M} = \mathbf{Sm}_k$ or $\mathbf{Sm}_k^{\text{aff}}$. Suppose \mathcal{F} is a simplicial presheaf on \mathcal{M} satisfying τ -excision s.t. $\pi_0\mathcal{F}$ is I -invariant, then $Sing^I\mathcal{F}$ satisfies τ -excision as well.*

Proof. First we need to show $Sing^I\mathcal{F}(\emptyset) \simeq *$. Notice the empty scheme \emptyset represents the initial (simplicial) presheaf since it is a strict initial object in the category of schemes \mathbf{Sch} , then $Sing^I\mathcal{F}(\emptyset)_n := \mathcal{F}(\emptyset \times I^n)_n \cong \text{Hom}_{\mathbf{sPre}}((\emptyset \times I^n) \times \Delta^n, \mathcal{F}) \cong \text{Hom}_{\mathbf{sPre}}(\emptyset, \text{Map}(I^n \times \Delta^n, \mathcal{F})) = \{*\}$ due to the monoidal adjunction. Therefore $Sing^I\mathcal{F} \cong *$.

Next we show $Sing^I\mathcal{F}$ sends any τ -excision square to a homotopy pullback. Given a square Q in τ -cd-structure, which is a pullback in \mathcal{M} by our assumption:

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & X \end{array},$$

notice $Q \times I^n$ is again a square in τ -cd-structure for $\forall n \geq 0$. Since \mathcal{F} satisfies

τ -excision, hence we have homotopy pullback for $\forall n \geq 0$:

$$\begin{array}{ccc} \mathcal{F}(X \times I^n) & \longrightarrow & \mathcal{F}(V \times I^n) \\ \downarrow & & \downarrow \\ \mathcal{F}(U \times I^n) & \longrightarrow & \mathcal{F}(W \times I^n) \end{array} .$$

Notice this gives a degreewise (at i) homotopy pullback of bisimplicial sets $\mathcal{F}(Q \times I^i)_j$ and $\pi_0^i \mathcal{F}(U \times I^i)_\bullet := \mathcal{F}(U \times I)_\bullet / \mathcal{F}(U)_\bullet$ is a discrete simplicial set since $\pi \mathcal{F}$ is I -invariant. Therefore by [14, Theorem B.4], this induces a homotopy pullback of the diagonal of the simplicial sets, which is the wanted homotopy pullback $\text{Sing}^I \mathcal{F}(Q)$. \square

Proposition 3.5.17. *Let \mathcal{F} be a Zar-local simplicial presheaf on \mathbf{Sm}_k . Suppose $i^* \mathcal{F}$ is I -invariant, then \mathcal{F} is I -invariant.*

Proof. It is equivalent to show $p^* : \mathcal{F} \rightarrow \mathcal{F}(- \times I)$ is a weak equivalence in $\mathbf{sPre}(\mathbf{Sm}_k)$. Notice $\mathcal{F}(- \times I)$ is also Zar-local: for any square Q in Zar-cd-structure, $Q \times I$ is a square in Zar-cd-structure as well, hence for $\mathcal{F}(- \times I)$ satisfies Zar-excision. Then by Proposition 3.5.12, $\mathcal{F} \simeq \mathbf{R}i_* i^* \mathcal{F} \simeq \mathbf{R}i_* i^* \mathcal{F}(- \times I) \simeq \mathcal{F}(- \times I)$ and we are done. \square

Proposition 3.5.18. *Let \mathcal{F} be a simplicial presheaf on \mathbf{Sm}_k . Suppose $i^* \mathcal{F}$ is Nis-local and $\pi_0(i^* \mathcal{F})$ is I -invariant, then $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is Nis-local and I -invariant. As a consequence, $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is an \mathbb{A}^1 -local fibrant replacement of \mathcal{F} .*

Proof. Directly from Proposition 3.5.16, we have $\text{Sing}^I \mathcal{F}$ is affine Nis-local. Then by Proposition 3.5.15, $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is Nis-local.

By Proposition 3.4.11, $\text{Sing}^I \mathcal{F}$ is I -invariant. Hence by Proposition 3.5.14, $i^* R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is pointwise weak equivalent to $i^* \text{Sing}^I \mathcal{F}$ and so $i^* R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is I -invariant which implies $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is I -invariant by Proposition 3.5.17.

At last, this is a replacement for arbitrary simplicial presheaf since $\text{Sing}^I \mathcal{F} \rightarrow R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is a Zar-local weak equivalence which is canonically an \mathbb{A}^1 -local weak equivalence. \square

3.6 Pointed Theory

In this section, we work with the pointed version of the theory. Given the injective simplicial model category $\mathbf{sPre}(\mathbf{Sm}_k)$, we have the pointed simplicial model categories equipped with Quillen functors:

$$(X, x) \wedge - : \mathbf{sSets}_\bullet \rightleftarrows \mathbf{sPre}(\mathbf{Sm}_k)_\bullet : \text{map}_\bullet(X, -),$$

where $\text{map}_\bullet(X, Y)$ is computed by $(\text{Hom}(X \wedge (\Delta^n)_+, Y))_n$.

On the other hand, we have a symmetric monoidal structure on $\mathbf{sPre}(\mathbf{Sm}_k)_\bullet$ induced from the original monoidal structure, by this we mean a pair of adjoint functors:

$$(X, x) \wedge - : \mathbf{sPre}(\mathbf{Sm}_k)_\bullet \rightleftarrows \mathbf{sPre}(\mathbf{Sm}_k)_\bullet : \underline{\text{Map}}_\bullet(X, -).$$

Proposition 3.6.1. *The above pair of adjoint functors is a pair of Quillen functor.*

Proof. This is directly by Proposition 2.5.4 since we have the Quillen functor from Proposition 3.1.1:

$$X \times - : \mathbf{sPre}(\mathbf{Sm}_k) \rightleftarrows \mathbf{sPre}(\mathbf{Sm}_k) : \underline{\text{Map}}(X, -).$$

□

Corollary 3.6.2. *We have an Quillen functor in \mathbb{A}^1 -local sense induced by the previous Quillen functor:*

$$(X, x) \wedge - : \mathbf{Spc}_\bullet^{\mathbb{A}^1} \rightleftarrows \mathbf{Spc}_\bullet^{\mathbb{A}^1} : \underline{\text{Map}}_\bullet(X, -).$$

And particularly, we have canonical isomorphism of sets in \mathbb{A}^1 -local sense:

$$[\mathbf{L}\Sigma(X, x), (Y, y)]_{\mathbb{A}^1, \bullet} \cong [(X, x), \mathbf{R}\Omega(Y, y)]_{\mathbb{A}^1, \bullet},$$

where we define $\Sigma(-) := S^1 \wedge -$ to be the **suspension** and $\Omega(-) := \underline{\text{Map}}_\bullet(S^1, -)$ to be the **loop space**.

Remark 3.6.3. The suspension $\Sigma(X, x)$ is the homotopy pushout $\{ * \leftarrow (X, x) \rightarrow * \}$. And the loop space $\Omega(X, x)$ is the homotopy pullback $\{ * \rightarrow (X, x) \leftarrow * \}$. Since we are working in (originally) the injective model structure on simplicial presheaves, we will talk more about $\mathbf{R}\Omega(X, x)$ since the loop space may not be fibrant but its derived version will always be.

For the rest of the section, we introduce the notation of homotopy sheaves which is an analogue of homotopy groups for topological spaces or simplicial sets.

Definition 3.6.4. *Given a simplicial presheaf $X \in \mathbf{sPre}(\mathbf{Sm}_k)_\bullet$ based at $x : * \rightarrow X$, we define its **simplicial n-homotopy presheaf** $\pi_n X$ to be the presheaf $(U \mapsto \pi_n(X(U), x))$ which is equivalent to $(U \mapsto [S^n \wedge U_+, X]_\bullet)$.*

*We define the **Nisnevich homotopy sheaf** $\pi_n^{\text{Nis}} X$ to be the sheaf associated to the presheaf $(U \mapsto [S^n \wedge U_+, X]_{\text{Nis}, \bullet})$. Similarly, we define the **\mathbb{A}^1 -homotopy sheaf** $\pi_n^{\mathbb{A}^1} X$ to be the presheaf $(U \mapsto [S^n \wedge U_+, X]_{\mathbb{A}^1, \bullet})$.*

Definition 3.6.5. Given a simplicial presheaf $X \in \mathbf{sPre}(\mathbf{Sm}_k)_\bullet$, we say X is \mathbb{A}^1 -connected if $\pi_0^{\mathbb{A}^1} X \cong *$. We say X is **naively \mathbb{A}^1 -connected** if $\text{Sing}^{\mathbb{A}^1} X$ is \mathbb{A}^1 -connected.

Furthermore, we say a simplicial presheaf X is **\mathbb{A}^1 - n -connected** (resp. **\mathbb{A}^1 - n -connected**) if $\pi_i X \cong *$ (resp. $\pi_i^{\mathbb{A}^1} X \cong *$) for $i \leq n$.

Proposition 3.6.6. Given a simplicial presheaf $X \in \mathbf{sPre}(\mathbf{Sm}_k)_\bullet$, if $\pi_0^{\text{Nis}} X \cong *$, then X is \mathbb{A}^1 -connected.

Proof. See in [15, Theorem 5.38]. In fact Morel proves a much stronger statement: if X is simplicial (n)-connected, then it is \mathbb{A}^1 - n -connected. \square

Proposition 3.6.7. Given a simplicial presheaf $X \in \mathbf{sPre}(\mathbf{Sm}_k)$, if X is naively \mathbb{A}^1 -connected, then X is \mathbb{A}^1 -connected.

Definition 3.6.8. Given a morphisms $f : X \rightarrow Y$ in a model category \mathcal{M} , we define its **homotopy fiber \mathbf{F}** to be the homotopy pullback of $\{ * \rightarrow Y \leftarrow X \}$. And we say a sequence $F \rightarrow X \xrightarrow{f} Y$ is a **homotopy fiber sequence** if F is weak equivalent to the homotopy fiber of f .

Notation 3.6.9. Given a homotopy fiber sequence $F \rightarrow X \rightarrow Y$ of simplicial sets in $\mathbf{sPre}(\mathbf{Sm}_k)$, we will say it is a \mathbb{A}^1 -local homotopy fiber sequence, or shortly \mathbb{A}^1 -homotopy fiber sequence, if it is also a homotopy fiber sequence in $\mathbf{Spc}_{\mathbb{A}^1}$.

Proposition 3.6.10. Given a \mathbb{A}^1 -homotopy fiber $F \rightarrow X \rightarrow Y$ in $\mathbf{Spc}^{\mathbb{A}^1}$, we have a natural long exact sequence of Nis-sheaves:

$$\cdots \rightarrow \pi_{n+1}^{\mathbb{A}^1} Y \rightarrow \pi_n^{\mathbb{A}^1} F \rightarrow \pi_n^{\mathbb{A}^1} X \rightarrow \pi_n^{\mathbb{A}^1} Y \rightarrow \cdots .$$

Proof. Dually by [2, Proposition 6.5.3], given a fiber sequence $F \rightarrow X \rightarrow Y$ in $\mathbf{sPre}(\mathbf{Sm}_k)$, we have a canonical long exact sequence of pointed sets for arbitrary object W :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [\Sigma W, F]_s & \longrightarrow & [\Sigma W, X]_s & \longrightarrow & [\Sigma W, Y]_s \\ & & & & & & \downarrow \\ & & & & & & [W, F]_s \longrightarrow [W, X]_s \longrightarrow [W, Y]_s \end{array}$$

This pointwise gives the long exact sequence of simplicial homotopy:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n F & \longrightarrow & \pi_n X & \longrightarrow & \pi_n Y \\ & & & & & & \downarrow \\ & & & & & & \pi_{n-1} F \longrightarrow \cdots \longrightarrow \pi_0 X \longrightarrow \pi_0 Y \end{array}$$

Now let $F \rightarrow X \rightarrow Y$ be an \mathbb{A}^1 -homotopy fiber and (Lid, Rid) be the canonical derived Quillen pair between $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{Spc}_{\mathbb{A}^1}$. Since Rid commutes with homotopy pullback, hence $R_{\mathbb{A}^1}F \rightarrow R_{\mathbb{A}^1}X \rightarrow R_{\mathbb{A}^1}Y$ is a homotopy fiber in $\mathbf{sPre}(\mathbf{Sm}_k)$. Then as shown above, we have a canonical long exact sequence of simplicial homotopy presheaves. Since an \mathbb{A}^1 -local fibrant replacement is also a simplicial fibrant replacement and the Nisnevich sheafification is exact, hence the exactness of the long sequence is preserved and so the wanted long exact sequence is obtained. \square

Proposition 3.6.11. *Given a homotopy fiber $F \rightarrow X \rightarrow Y$ in $\mathbf{sPre}(\mathbf{Sm}_k)$ or $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$, if $\pi_0 Y$ is \mathbb{A}^1 -invariant, then $Sing^{\mathbb{A}^1} X \rightarrow Sing^{\mathbb{A}^1} X \rightarrow Sing^{\mathbb{A}^1} Y$ is a homotopy fiber in \mathbf{sPre} .*

Proof. We show that $Sing^{\mathbb{A}^1} F(U) \rightarrow Sing^{\mathbb{A}^1} X(U) \rightarrow Sing^{\mathbb{A}^1} Y(U)$ is a homotopy fiber of simplicial sets for any $U \in \mathbf{Sm}_k$. Notice a such sequence is the diagonal of the diagram of bisimplicial sets:

$$\begin{array}{ccc} F(U \times \Delta_{\text{alg}}^\bullet) & \longrightarrow & X(U \times \Delta_{\text{alg}}^\bullet) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y(U \times \Delta_{\text{alg}}^\bullet) \end{array},$$

which is a homotopy pullback by our assumption. Then by [14, Theorem B.4], since $\pi_Y(- \times \Delta_{\text{alg}}^\bullet)$ is constant, hence the diagonal of this diagram is a homotopy pullback and so we are done. \square

If we apply this proposition to the fiber $\mathbf{R}\Omega F \rightarrow * \rightarrow F$, then we obtain the commutativity of looping functor and the singular construction.

Corollary 3.6.12. *Given a simplicial presheaf $F \in \mathbf{sPre}(\mathbf{Sm}_k)$ s.t. $\pi_0 F$ is \mathbb{A}^1 -invariant, then we have a canonical weak equivalence:*

$$Sing^{\mathbb{A}^1} \mathbf{R}\Omega F \longrightarrow \mathbf{R}\Omega Sing^{\mathbb{A}^1} F.$$

Proposition 3.6.13. *Given a homotopy fiber $F \rightarrow X \rightarrow Y$ in $\mathbf{sPre}(\mathbf{Sm}_k)$, if Y is affine Nis-local and $\pi_0 Y$ is affine \mathbb{A}^1 -invariant, then $F \rightarrow X \rightarrow Y$ is also an \mathbb{A}^1 -homotopy fiber.*

Proof. Let $i^* : \mathbf{sPre}(\mathbf{Sm}_k) \rightarrow \mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$ be the restriction to affine schemes and $\mathbf{R}i_*$ be its derived Quillen adjoint as in the previous section, then we obtain a homotopy fiber $i^* F \rightarrow i^* X \rightarrow i^* Y$ s.t. $\pi_0(i^* Y)$ is \mathbb{A}^1 -invariant. Then by Proposition 3.6.11, we have a homotopy fiber in $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$:

$$Sing^{\mathbb{A}^1} i^* F \longrightarrow Sing^{\mathbb{A}^1} i^* X \longrightarrow Sing^{\mathbb{A}^1} i^* Y.$$

Since $\mathbf{R}i_*$ and R_{Zar} commutes with homotopy pullback, hence we have a homotopy fiber:

$$\mathbf{R}i_*R_{Zar}Sing^{\mathbb{A}^1}i^*F \longrightarrow \mathbf{R}i_*R_{Zar}Sing^{\mathbb{A}^1}i^*X \longrightarrow \mathbf{R}i_*R_{Zar}Sing^{\mathbb{A}^1}i^*Y.$$

Notice $\mathbf{R}i_*R_{Zar}Sing^{\mathbb{A}^1}i^*F = \mathbf{R}i_*R_{Zar}i^*Sing^{\mathbb{A}^1}F \simeq R_{Zar}Sing^{\mathbb{A}^1}F$ by Proposition 3.5.12 and same for X, Y , therefore we further obtain a homotopy fiber which is objectwise \mathbb{A}^1 -local weak equivalent to $F \rightarrow X \rightarrow Y$:

$$R_{Zar}Sing^{\mathbb{A}^1}F \longrightarrow R_{Zar}Sing^{\mathbb{A}^1}X \longrightarrow R_{Zar}Sing^{\mathbb{A}^1}Y.$$

By Proposition 3.5.18, $R_{Zar}Sing^{\mathbb{A}^1}Y$ is *Nis*-local and \mathbb{A}^1 -invariant, hence by [5, Theorem §2 2.7], a homotopy fiber with *Nis*-local and \mathbb{A}^1 -invariant base is an \mathbb{A}^1 -homotopy fiber. \square

Application and K -Theory

In this chapter, we will apply our theory defined in Chapter 2 to some scenarios. The first section will be rather easy as we just compute some straight-forward facts. In Section 2, we introduce a motivic classification of vector bundles over affines. Then in Section 3, we prove the related facts to algebraic K -theory and then we achieve some examples in Section 4.

4.1 Some Direct Computations

In this section, we introduce some basic examples and computation. We first discuss a standard type of \mathbb{A}^1 -local simplicial presheaf: the \mathbb{A}^1 -rigid schemes.

Definition 4.1.1. Let $X \in \mathbf{Sm}_k$, we say X is an \mathbb{A}^1 -rigid scheme if

$$\mathrm{Hom}_{\mathbf{Sm}_k}(U \times_k \mathbb{A}^1, X) \rightarrow \mathrm{Hom}_{\mathbf{Sm}_k}(U, X)$$

induced by the 0-section is a bijection of sets.

Remark 4.1.2. Notice by a purely set-theoretical argument, we have an equivalent condition for \mathbb{A}^1 -rigidness: A scheme $X \in \mathbf{Sm}_k$ is \mathbb{A}^1 -rigid if and only if in the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Sm}_k}(U, X) & \xrightarrow{p^*} & \mathrm{Hom}_{\mathbf{Sm}_k}(U \times_k \mathbb{A}^1, X) \\ & \searrow \mathrm{id} & \downarrow i_0^* \\ & & \mathrm{Hom}_{\mathbf{Sm}_k}(U, X) \end{array} ,$$

we have p^* is surjective or i_0^* is injective.

Example 4.1.3. We have some classical examples:

\mathbf{G}_m : notice for $U = \text{Spec}(R)$ affine, the morphism induced by pullback $p^* : \text{Hom}_{\mathbf{Sm}_k}(U \times_k \mathbb{A}^1, \mathbf{G}_m) \rightarrow \text{Hom}_{\mathbf{Sm}_k}(U, \mathbf{G}_m)$ is the morphism of rings $R[x]^* \rightarrow R^*$ by vanishing x which is indeed an isomorphism of rings. Since the representable presheaf \mathbf{G}_m is a sheaf in Zariski topology, hence these isomorphism can be extended to arbitrary smooth schemes U . Therefore \mathbf{G}_m is \mathbb{A}^1 -rigid.

Abelian Variety: to see this, we show that the following commutative diagram of \bar{k} -points implies $f = g$:

$$\begin{array}{ccc} U(\bar{k}) & \xrightarrow{i_0^*} & U(\bar{k}) \times \mathbb{A}^1(\bar{k}) \\ \downarrow i_0^* & & \downarrow f \\ U(\bar{k}) \times \mathbb{A}^1(\bar{k}) & \xrightarrow{g} & X(\bar{k}) \end{array} ,$$

where \bar{k} be the algebraic closure of k . The key observation is that the \bar{k} -point $f(x, y)$ where $(x, y) \in U(\bar{k}) \times \mathbb{A}^1(\bar{k})$ factor as:

$$\text{Spec}(\bar{k}) \xrightarrow{y} \mathbb{A}_{\bar{k}}^1 \rightarrow X_{\bar{k}}.$$

Consider $\mathbf{G}_m \rightarrow \mathbb{A}^1 \rightarrow X$, if $\mathbb{A}^1 \rightarrow X$ is non-constant, then two group structure of \mathbf{G}_m and \mathbb{A}^1 conflict to each other, hence $\mathbb{A}^1 \rightarrow X$ is constant. Therefore $f(x, y) = f(x, 0) = g(x, 0) = g(x, y)$ and so $f = g$. Then $\text{Hom}_{\mathbf{Sm}_k}(U \times_k \mathbb{A}^1, X) \rightarrow \text{Hom}_{\mathbf{Sm}_k}(U, X)$ is an injection and by Remark 4.1.2, we are done.

Projective Curve of Positive Genus: the argument is the same as above. A non-constant morphism $\mathbb{A}^1 \rightarrow X$ extends to $\mathbb{P}^1 \rightarrow X$, then by Hurwitz Formula, the genus $g(X) \leq g(\mathbb{P}^1) = 0$ which contradicts to our assumption. Hence any morphism $\mathbb{A}^1 \rightarrow X$ is constant, and then the rest of the argument follows.

Proposition 4.1.4. An \mathbb{A}^1 -rigid scheme X represents a \mathbb{A}^1 -local (discrete) simplicial presheaf.

Proof. By Proposition 3.3.3 and Example 3.3.10, the representable presheaf X is a (discrete) Nis -local simplicial presheaf. Then by the definition of (left) Bousfield localization, X is \mathbb{A}^1 -local. \square

Proposition 4.1.5. *Given an pointed \mathbb{A}^1 -rigid scheme X , we have $\pi_0^{\mathbb{A}^1} X \cong X$ and $\pi_n^{\mathbb{A}^1} X \cong *$ for $n \geq 1$.*

Proof. Since X is \mathbb{A}^1 -local, i.e. \mathbb{A}^1 -local fibrant, and $U \in \mathbf{Sm}_k$ is \mathbb{A}^1 -local cofibrant, hence

$$[U, X]_{\mathbb{A}^1} \cong [U, X].$$

And $[U, X] = \pi_0 \text{Map}(U, X) \cong \text{Hom}_{\mathbf{Sm}_k}(U, X)$ since U, X are discrete as simplicial presheaves. Therefore $\pi_0^{\mathbb{A}^1} X(U) \stackrel{\text{def}}{=} [U, X]_{\mathbb{A}^1} \cong \text{Hom}_{\mathbf{Sm}_k}(U, X)$ and obviously $\pi_0^{\mathbb{A}^1} X \cong X$ as (pre)sheaves.

Similar arguments hold for higher homotopies. Induced from the pointed Quillen functor $(U \wedge -, \text{map}_\bullet(U, -))$ for $U \in \mathbf{Sm}_k$, we have an isomorphism of homotopy classes $[U \wedge S^n, X]_\bullet \cong [S^n, \text{Map}_\bullet(U, X)]_\bullet$. Again since U is cofibrant, then $U \wedge S^n$ is cofibrant and hence \mathbb{A}^1 -local cofibrant. Thus $[U \wedge S^n, X]_\bullet = [U \wedge S^n, X]_{\mathbb{A}^1, \bullet}$ and $\text{Map}_{\mathbf{Sm}_k, \bullet}(U, X)$ is the discrete simplicial set $\text{Hom}_\bullet(U, X)$. Since $\pi_0 S^n = *$, hence any morphism of simplicial sets $S^n \rightarrow \text{Map}_\bullet(U, X)$ is homotopic constant, hence $[S^n, \text{Map}_\bullet(U, X)]$ is a single point set. Then we are done. \square

Next example would be some basic classification of \mathbb{A}^1 -homotopy types related **motivic sphere**. We assume, as simplicial presheaves, \mathbb{A}^1 and \mathbb{G}_m is pointed by 1-section; \mathbb{P}^1 is pointed by ∞ . We define the **motivic sphere** $S^{a+b,a} := \mathbb{G}_m^{\wedge a} \wedge (S^1)^{\wedge b}$, $a, b \geq 0$.

Proposition 4.1.6. *In \mathbb{A}^1 -local sense, $\Sigma \mathbb{G}_m \simeq_{\mathbb{A}^1} \mathbb{P}^1 \simeq_{\mathbb{A}^1} \mathbb{A}^1 / \mathbb{G}_m$.*

Proof. As shown by the following three homotopy pushout diagrams:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma \mathbb{G}_m \end{array} \quad \begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\text{inclusion}} & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \quad \begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\text{inclusion}} & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{A}^1 / \mathbb{G}_m \end{array} .$$

The left diagram is canonically a homotopy pushout. Since the direct inclusion $\mathbb{G}_m \rightarrow \mathbb{A}^1$ is a monomorphism as presheaves and hence as simplicial presheaves, therefore the middle and the right diagrams are homotopy pushout by Proposition 2.4.8. Then the objectwise \mathbb{A}^1 -local weak equivalences proves the statement. \square

Remark 4.1.7. The motivic spheres can be viewed as a combination of two types of spheres: the simplicial sphere S^1 , which is the simplicial presheaf given by the simplicial set S^1 , and the Tate sphere \mathbb{G}_m , which is punctured affine line

$\mathbb{A}^1 - \{0\}$. And hence there are two corresponding suspension which we differ by subscript: $\Sigma_s := S^1 \wedge -$ and $\Sigma_t := \mathbb{G}_m \wedge -$. The Tate suspension does not show up in the thesis, and so we just write Σ for Σ_s for convenience unless necessary situations.

Proposition 4.1.8. *In \mathbb{A}^1 -local sense, $S^{2n-1,n} \simeq_{\mathbb{A}^1} \mathbb{A}^n - \{0\}$ for any integer $n \geq 1$.*

Proof. When $n = 1$, the statement is the same as Proposition 4.1.6. Then we prove the statement by induction.

Notice for any integer $n \geq 2$, we have an EDS:

$$\begin{array}{ccc} (\mathbb{A}^{n-1} - \{0\}) \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^{n-1} \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ (\mathbb{A}^{n-1} - \{0\}) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^n - \{0\} \end{array},$$

where the bottom and the right arrows are just open immersion. Therefore by Corollary 3.3.12, this is a homotopy pushout in $\mathbf{Spc}^{\mathbb{A}^1}$. On the other hand, consider the diagram:

$$\begin{array}{ccccc} * & \longleftarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{A}^{n-1} - \{0\} & \longleftarrow & (\mathbb{A}^{n-1} - \{0\}) \vee \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^{n-1} - \{0\} & \longleftarrow & (\mathbb{A}^{n-1} - \{0\}) \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \end{array}$$

If we compute the pushout of this diagram by first compute those vertical pushout, we obtain $\Sigma((\mathbb{A}^{n-1} - \{0\}) \wedge \mathbb{G}_m)$, which is \mathbb{A}^1 -locally weak equivalent to $\Sigma(S^{2n-3,n-1} \wedge \mathbb{G}_m)$ by inductive assumption. Notice this is $S^{2n-1,n}$. If we compute the pushout of this diagram by first compute those horizontal pushout, we obtain $\mathbb{A}^n - \{0\}$. Hence $S^{2n-1,n} \simeq_{\mathbb{A}^1} \mathbb{A}^n - \{0\}$. \square

Proposition 4.1.9. *In \mathbb{A}^1 -local sense, $S^{2n,n} \simeq_{\mathbb{A}^1} \mathbb{A}^n / (\mathbb{A}^n - \{0\})$ for $n \geq 1$.*

Proof. Consider the pushout of simplicial presheaves:

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathbb{A}^n \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathbb{A}^n / (\mathbb{A}^n - \{0\}) \end{array}$$

Since the top arrow is an open immersion as schemes, hence a monomorphism of simplicial presheaves, therefore by Proposition 2.4.8, this is a homotopy pushout. And then so is the following objectwise \mathbb{A}^1 -locally weak equivalent diagram:

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{A}^n / (\mathbb{A}^n - \{0\}) \end{array} .$$

Then $\mathbb{A}^n / (\mathbb{A}^n - \{0\}) \simeq_{\mathbb{A}^1} \Sigma(\mathbb{A}^n - \{0\}) \simeq_{\mathbb{A}^1} \Sigma S^{2n-1, n} = S^{2n, n}$. \square

4.2 An Affine Representability

In this section we introduce a classification of vector bundles over affine schemes. The proof using \mathbb{A}^1 -homotopy theory was first done by Prof. F Morel in [15]. Later in [1], Asok, Hoyois, and Wendt generalize the statement from the base $\text{Spec}(k)$ where k is a perfect field to the base S where S can be a quasi-compact quasi-separated scheme. This section is mainly based on the latter proof. Therefore though we work with a perfect field k , one can check that most of the arguments in this section (even in other sections) could be generalized to a quasi-compact and quasi-separated scheme.

To begin, we first recall some foundations we need in classifying theory of vector bundles. We write Gr_r for the infinite Grassmannian obtained by gluing $Gr_{r,n}, n \geq 0$ along the obvious inclusions. And we write the \mathcal{V}_r for the presheaf of isomorphism classes of rank r vector bundles over smooth schemes. We also write $B\mathbf{Vect}_r(U)$ for the nerve of the groupoid of rank r vector bundles over $U \in \mathbf{Sm}_k$.

Lemma 4.2.1. *The morphism of simplicial presheaves induced by the tautological bundle over the infinite Grassmannian*

$$\gamma_r : Gr_r \longrightarrow B\mathbf{Vect}_r$$

is an \mathbb{A}^1 -local weak equivalence.

Proof. See in [5, Proposition §4 2.6] \square

Remark 4.2.2. The assignment ($U \mapsto$ isoclasses of rank r vector bundles over U) is a pseudo-functor which only respect composition of pullback up to isomorphisms. Hence to avoid some complicated argument, we impose a easy trick introduced in [16, Section C.4]: instead of assign the isoclasses of vector bundles to $U \in \mathbf{Sm}_k$, we assign the category of so called **big vector bundles** to U .

A big vector bundle over U is a functor $P : \mathbf{Sm}_k/U \rightarrow \mathbf{Grpd}$ where \mathbf{Grpd} is the category of all vector bundles. A such functor P sends X/U to a vector bundle over X s.t. for $f : Y \rightarrow X$ over U , $f^*P(X)$ is isomorphic to $P(Y)$. In short, a big vector bundle P over U is a family of vector bundles which is compatible with respect to pullback.

A nice property to such construction is that the pull back of vector bundles can be interpreted as restrictions of domain of big vector bundles as functors. And hence \mathbf{Vect}_r is strictly functorial. Furthermore in [17] or [18], it is proven such construction is objectwise equivalent to the naively constructed pseudo-functor. This is a technical measure to avoid certain arguments, hence elsewhere in this thesis, it is safe to think of the isoclasses of vector bundles.

At last, we take the simplicial presheaf $B\mathbf{Vect}_r$ which is defined by

$$U \mapsto \text{nerve of the groupoid } \mathbf{Vect}_r(U).$$

Remark 4.2.3. A direct result is that $\pi_0 B\mathbf{Vect}_r \cong \mathcal{V}_r$.

A nice property of $B\mathbf{Vect}_r$ is that it is a simplicial presheaf that satisfies necessary conditions to progress the discussion. A detail discussion is provided in the next section.

Lemma 4.2.4. *Let \mathcal{C} be a Grothendieck site and \mathcal{E} be a category fibered in groupoid over \mathcal{C} . We write $\mathcal{E}(X)$ as the groupoid of morphisms $\text{Hom}(\mathcal{C}/X, \mathcal{E})$ in the category $\mathbf{Grpd}/\mathcal{C}$ of categories fibered in groupoid over \mathcal{C} . Then \mathcal{E} is a stack on \mathcal{C} if and only if for every cover $\{U_i \rightarrow X\}_{i \in I}$, we have an equivalence of categories:*

$$\mathcal{E}(X) \xrightarrow{\cong} \text{holim}(\Pi\mathcal{E}(U_i) \rightrightarrows \Pi\mathcal{E}(U_i \times_X U_j) \rightrightarrows \dots)$$

Proof. See in [18, Theorem 1.1] □

Corollary 4.2.5. *The classifying simplicial presheaf $B\mathbf{Vect}_r$ is Étale-local.*

Proof. Since the stack \mathbf{Vect}_r is a stack for the Étale-topology, then as a result of Corollary 4.3.6 in the next section, $B\mathbf{Vect}_r$ is ét-local as a simplicial presheaf. □

Proposition 4.2.6. *Let \mathcal{F} be a Zar-local simplicial presheaf on \mathbf{Sm}_k . Suppose $\mathcal{F}|_{\mathbf{Sm}_k^{\text{aff}}}$ is \mathbb{A}^1 -invariant, then \mathcal{F} is \mathbb{A}^1 -invariant.*

Proof. To show \mathcal{F} is \mathbb{A}^1 -invariant, is to show morphism induced by projection $p^* : \mathcal{F} \rightarrow \mathcal{F}(- \times I)$ is an pointwise weak equivalence of simplicial presheaves where $I = \Delta_{\text{alg}}^1 \cong \mathbb{A}^1$. Notice $\mathcal{F}(- \times I)$ is again a Zar-local simplicial presheaf: given a Zar-excision square Q , then $Q \times I$ is still an excision square. Then since

\mathcal{F} satisfies the excision $Q \times I$, which is equivalent to $\mathcal{F}(- \times I)$ satisfies the excision Q , hence as a consequence $\mathcal{F}(- \times I)$ satisfies *Zar*-excision and so it is *Zar*-local.

We recall the notation in the previous chapter: $(i^*, \mathbf{R}i_*)$ is a derived Quillen functor between $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\mathbf{sPre}(\mathbf{Sm}_k^{\text{aff}})$. Then by assumption, $p^* : i^* \mathcal{F} \rightarrow i^* \mathcal{F}(- \times I)$ is a pointwise weak equivalence. As shown above, p^* is weak equivalence between fibrant objects since i^* preserves *Zar*-localness. Hence $\mathbf{R}i_* p^* : \mathbf{R}i_* i^* \mathcal{F} \rightarrow \mathbf{R}i_* i^* \mathcal{F}(- \times I)$ is a weak equivalence. Then by Proposition 3.5.12, we have:

$$\mathcal{F} \simeq \mathbf{R}i_* i^* \mathcal{F} \simeq \mathbf{R}i_* i^* \mathcal{F}(- \times I) \simeq \mathcal{F}(- \times I)$$

and hence \mathcal{F} is *I*-invariant. □

Proposition 4.2.7. *Let \mathcal{F} be a simplicial presheaf on \mathbf{Sm}_k . Suppose $\mathcal{F}|_{\mathbf{Sm}_k^{\text{aff}}}$ is Nis-local and $\pi_0(\mathcal{F}|_{\mathbf{Sm}_k^{\text{aff}}})$ is \mathbb{A}^1 -invariant, then we have an isomorphism of sets:*

$$\pi_0 \mathcal{F}(U) \cong [U, \mathcal{F}]_{\mathbb{A}^1}.$$

Proof. We use $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ to prove this. By Proposition 3.5.18 in Chapter II, $R_{\text{Zar}} \text{Sing}^I \mathcal{F}$ is an \mathbb{A}^1 -local fibrant replacement of \mathcal{F} s.t. $R_{\text{Zar}} \text{Sing}^I \mathcal{F}(U) \simeq \text{Sing}^I \mathcal{F}(U)$ when U is affine. Therefore we have isomorphisms of sets for U affine:

$$\begin{aligned} [U, \mathcal{F}]_{\mathbb{A}^1} &\cong \pi_0 \text{Map}_{\mathbb{A}^1}(U, \mathcal{F}) \cong \pi_0 \text{map}(U, R_{\mathbb{A}^1} \mathcal{F}) \\ &\cong \pi_0 \text{map}(U, R_{\text{Zar}} \text{Sing}^I \mathcal{F}) \cong \pi_0 R_{\text{Zar}} \text{Sing}^I \mathcal{F}(U). \\ &\cong \pi_0 \text{Sing}^I \mathcal{F}(U) \end{aligned}$$

On the other hand, by Proposition 3.4.13, we have $\pi_0 \mathcal{F} \cong \pi_0 \text{Sing}^I \mathcal{F}$ as sheaves of sets when $\pi_0 \mathcal{F}$ is *I*-invariant. Then we obtain:

$$\pi_0 \mathcal{F}(U) \cong [U, \mathcal{F}]_{\mathbb{A}^1}$$

for $U \in \mathbf{Sm}_k$ affine. □

Theorem 4.2.8 (Bass-Quillen Conjecture). *For any affine scheme $X \in \mathbf{Sm}_k^{\text{aff}}$, the Bass-Quillen Conjecture holds: the morphism of sets*

$$\mathcal{V}_r(X) \rightarrow \mathcal{V}_r(\mathbb{A}^n \times_k X)$$

is bijective for $\forall n \geq 0$.

Proof. The case for fields, which we concern in this thesis, is proven by Quillen in [19]. □

Theorem 4.2.9. *Let $r \geq 0$ be an integer. For any smooth affine scheme $X \in \mathbf{Sm}_k$, we have a natural bijection:*

$$\mathcal{V}_r(X) \cong [X, Gr_r]_{\mathbb{A}^1}.$$

Remark 4.2.10. In this thesis, we potentially assume the Bass-Quillen Conjecture holds for $\mathcal{O}(X)$, which is true since we are working over a field. One thing to notice that it is possible to generalize the statement to certain base scheme S whenever the Bass-Quillen conjecture works.

Proof. Since $Gr_r \simeq_{\mathbb{A}^1} B\mathbf{Vect}_r$, hence it is equivalent to show $\mathcal{V}_r(X) \cong [X, B\mathbf{Vect}_r]_{\mathbb{A}^1}$ for affine X . We show that $B\mathbf{Vect}_r$ satisfies the conditions in Proposition 4.2.7 and then we obtain $[X, B\mathbf{Vect}_r]_{\mathbb{A}^1} \cong \pi_0 B\mathbf{Vect}_r(X) \cong \mathcal{V}_r(X)$ by Remark 4.2.3.

By Corollary 4.2.5, $B\mathbf{Vect}_r$ is *ét*-local and hence *Nis*-local. Also over affine schemes, $\pi_0 B\mathbf{Vect}_r \cong \mathcal{V}_r$ is *I*-invariant due to the Bass-Quillen Conjecture. Therefore $B\mathbf{Vect}_r$ satisfies the conditions required and hence we are done. \square

Even though we obtain a nice and neat classification as one might expect, to actually compute the \mathbb{A}^1 -homotopy classes is not an easy job. To do this, as we will introduce in next section, we need the representability of algebraic K -theory in \mathbb{A}^1 -homotopy theory which allows us to compute in terms of \mathbb{A}^1 -homotopy sheaves of classifying spaces as in algebraic topology.

4.3 Classifying Spaces and K -Theory

In this section, we introduce the connection between \mathbb{A}^1 -homotopy theory and algebraic K -theory. The main goal is to compute \mathbb{A}^1 -homotopy sheaves of classifying spaces BGL_r in terms of K -theories.

4.3.1 Classifying Space

We first introduce the classifying spaces of groups in form of simplicial presheaves. We will mainly follow the notations from §4 in [5]. For a sheaf of sets X on \mathbf{Sm}_k , we define $E(X)$ (or simply EX) to be the simplicial presheaf given by:

$$E(X)(U)_n := X(U)^{n+1}, \quad U \in \mathbf{Sm}_k,$$

with simplicial structure induced by (partial) projections and diagonal inclusions, i.e. the faces are $d_i(x_0, \dots, x_n) = (x_0, \dots, \hat{x}_i, \dots, x_n)$ by skipping the i^{th} entry and the degeneracies are $s_j(x_0, \dots, x_n) = (x_0, \dots, x_j, x_j, \dots, x_n)$. When X is a sheaf of groups G , then we furtherly define its **classifying space** $B(G)$ (or

simply BG) to be the simplicial presheaf ($U \mapsto \text{nerve of } G(U)$) by viewing the group $G(U)$ as a 1-object groupoid. Precisely, we set

$$BG(U)_n := G(U)^n, U \in \mathbf{Sm}_k.$$

Particularly $BG(U)_0 := \{*\}$. The degeneracies $s_i : (g_0, \dots, g_n) \mapsto (g_0, \dots, g_{i-1}, 1, g_i, \dots)$ and faces:

$$d_i(g_0, \dots, g_n) = \begin{cases} (g_1, \dots, g_n), & i = 0 \\ (g_0, \dots, g_{n-1}), & i = n \\ (g_0, \dots, g_{i-1}g_i, \dots, g_n), & \text{otherwise} \end{cases}$$

One thing to notice is that BG is a quotient of EG as:

$$G \xrightarrow{\text{diag}} EG \xrightarrow{\pi_G} \twoheadrightarrow BG,$$

where $\text{diag}(g) = (g, \dots, g)$ and $\pi_G(g_0, \dots, g_n) = (g_0g_1^{-1}, \dots, g_{n-1}g_n^{-1})$.

Proposition 4.3.1. *The simplicial presheaf EG for a sheaf of groups is homotopy contractible.*

Proof. We check this for $\forall U \in \mathbf{Sm}_k$: let X be the simplicial group $EG(U)$ and we define an extra level $X_{-1} := \pi_0 EG(U)$ with a unique face map given by the canonical morphism $d_0 : X_0 \rightarrow \pi_0 X$. Notice for each level $X_n, n \geq 0$, there is a degeneracy given by diagonal map $s_0(x_0, \dots, x_n) = (x_0, x_0, \dots, x_n)$. We further define $s_0 : X_{-1} \rightarrow X_0$ by sending the unique element to the identity of the group. Following the notations of Goerss and Jardine, these morphisms form an extra degeneracy and then by [4, Lemma III.5.1], we obtain a weak equivalence:

$$X \rightarrow K(\pi_0 X, 0)$$

where $K(\pi_0 X, 0)$ is the Eilenberg-MacLane space. In our case, this is just $K(*, 0) = *$ and hence we are done. \square

We now describe a construction, [18] of the (simplicial) presheaves associated to a category fibered over certain small category \mathcal{C} . An advantage of this construction is that it is strictly functorial and hence we can avoid some complicated pseudo-functor arguments.

Proposition 4.3.2. Let $\mathcal{E} \xrightarrow{F} \mathcal{C}$ be a category fibered over \mathcal{C} and $X \in \mathcal{C}$. Then for $\forall X' \in \mathcal{E}$ with $F(X') = X$, there is a section G :

$$\begin{array}{ccc} & & \mathcal{E} \\ & \nearrow G & \downarrow F \\ \mathcal{C}/X & \longrightarrow & \mathcal{C} \end{array}$$

s.t. G send the final object id_X to X' .

And for any two such sections G, G' s.t. $G(id_X) \xrightarrow{f} G'(id_X)$ is a morphism in the fiber category \mathcal{E}_X , then we admit a unique natural isomorphism $G \xrightarrow{\phi} G'$ over $id_{\mathcal{C}}$ s.t. $G(id_X) = f$.

Hence it follows that for each $X \in \mathcal{C}$, the natural map induced by the evaluation map at id_X as above:

$$\mathbf{Grpd}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{E}) \rightarrow \mathcal{E}_X,$$

where $\mathbf{Grpd}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{E})$ is the collection of such sections, is a surjective equivalence of groupoids. There is a left inverse which is unique up to unique natural isomorphism.

Proof. See in [18, Proposition 3.4]. □

By assigning $U \in \mathcal{C}$ the sections $\mathbf{Grpd}_{\mathcal{C}}(\mathcal{C}/U, \mathcal{E})$, we also obtain a groupoid-valued presheaf which is equivalent to the direct construction: $U \mapsto \mathcal{E}_U$. The advantage is visible: the restriction at values can be defined to be the restriction of the domain of a functor. By this we mean for a section $G : \mathcal{C}/U \rightarrow \mathcal{E}$ and a morphism $i : V \rightarrow U$ in \mathcal{C} , the restriction is just $G|_V : \mathcal{C}/V \rightarrow \mathcal{E}$.

This construction also allows us to interpret a stack in terms of homotopic conditions. Particularly, we will also refer the notation \mathcal{E} of a fibered category to its associated presheaf of groupoids. And we define a simplicial presheaf $B\mathcal{E}$ which assign $U \in \mathcal{C}$ to the nerve of $\mathcal{E}(U)$.

Definition 4.3.3. Let \mathcal{C} be a complete Grothendieck site with topology τ and \mathcal{E} be a presheaf of groupoids. We say \mathcal{E} is a **stack** in τ -topology if for every cover $\{U_i \rightarrow X\}_{i \in I}$, we have an weak equivalence of groupoids:

$$\mathcal{E}(X) \xrightarrow{\simeq} \text{holim}(\Pi \mathcal{E}(U_i) \rightrightarrows \Pi \mathcal{E}(U_i \times_X U_j) \rightrightarrows \cdots) .$$

Remark 4.3.4. This definition for stacks is proven, in [18], to be equivalent to Deligne and Mumford's definition in [20].

To justify the notation of 'holim' in this definition, we need to define a model structure on \mathbf{Grpd} the category of groupoids. Notice we have an adjunction:

$$\Pi_1 : \mathbf{sSets} \rightleftarrows \mathbf{Grpd} : N$$

where Π_1 is the fundamental groupoid construction of a simplicial set and N is the nerve of a groupoid. By [18, Theorem 2.1], there is a left proper simplicial model structure on **Grpd**:

$$\begin{cases} W = \text{equivalences of categories;} \\ C = \text{functors which are injective on objects;} \\ F = \text{functors which have RLP with respect to the map } \Pi_1(\Delta^0) \rightarrow \Pi_1(\Delta^n). \end{cases}$$

This model structure comes with generating cofibrations $\Pi_1(\Delta^0) \rightarrow \Pi_1(\Delta^1)$ and generating trivial cofibrations $\Pi_1(\partial\Delta^n \rightarrow \Delta^n)$.

Lemma 4.3.5. *Let $f : G \rightarrow H$ be a morphism of groupoids, then f is a weak equivalence if and only if Nf is.*

Similarly f is a (trivial) fibration if and only if Nf is a (trivial) fibration.

Proof. [18, Lemma 2.4]. □

Corollary 4.3.6. *Let \mathcal{E} be a presheaf of groupoid on Grothendieck site \mathcal{C} with Grothendieck topology τ . Then \mathcal{E} is a τ -stack if and only if $B\mathcal{E}$ satisfies the τ -local condition.*

Proof. By Lemma 4.3.5, we have equivalent conditions: an equivalence of groupoids for arbitrary τ -covering $\{U_i \rightarrow X\}$ in \mathcal{C} ,

$$\mathcal{E}(X) \xrightarrow{\simeq} \text{holim}(\Pi\mathcal{E}(U_i) \rightrightarrows \Pi\mathcal{E}(U_i \cap U_j) \rightrightarrows \cdots) ,$$

is equivalent to a weak equivalence of their nerve as simplicial sets:

$$N\mathcal{E}(X) \xrightarrow{\simeq} \text{holim}(\Pi N\mathcal{E}(U_i) \rightrightarrows \Pi N\mathcal{E}(U_i \cap U_j) \rightrightarrows \cdots) .$$

Since the latter one is exactly the Čech-condition of τ -localness, hence we are done. □

Example 4.3.7 (G -torsors). Let \mathcal{C} be a small site with Grothendieck topology τ and G a τ -sheaf of groups on \mathcal{C} , i.e. a discrete simplicial presheaf of groups. Given an object $X \in \mathcal{C}$ and a simplicial presheaf E , we say $E \xrightarrow{\pi} X$ is a **G-torsor** if we have a right action $a : E \times G \rightarrow E$ s.t.

1. π is a G -equivariant map;
2. $E \times G \rightarrow E \times_X E$ via π, a is an isomorphism;
3. π is τ -locally split.

We let $\mathbf{Tor}_\tau(G)$ be the category of all G -torsors over \mathcal{C} , which is fibered in groupoids. Then instead of define a groupoid-valued presheaf $Tor_\tau G(X) := (\mathbf{Tor}_\tau(G))_X$, we define by Proposition 4.3.2 an equivalent groupoid-valued presheaf

$$Tor_\tau G(X) := \mathbf{Grpd}_{\mathcal{C}}(\mathcal{C}/X, \mathbf{Tor}_\tau(G)).$$

We define $B\mathbf{Tor}_\tau G$ to be the simplicial presheaf given by nerve of the groupoid-valued presheaf $Tor_\tau G$. By this we mean $(B\mathbf{Tor}_\tau G)_n := \text{Mor}(\mathbf{Tor}_\tau G)^{\times_{\text{Ob}(\mathbf{Tor}_\tau G)}^n}$. The face $d_i : (B\mathbf{Tor}_\tau G)_{n+1} \rightarrow (B\mathbf{Tor}_\tau G)_n$ is the composition of morphisms at entry i and $i + 1$. The degeneracies are given by inserting identity morphism at the corresponding entry. We will also consider the set of iso-classes of G -torsors over X and we denote it as $\mathbf{Tor}_\tau(X, G)$.

We define the first τ -cohomology $H_\tau^1(X, G) := \pi_0 B\mathbf{Tor}_\tau G(X)$.

Let $\mathbf{sPre}(\mathcal{C})$ be the category of simplicial presheaves on a small Grothendieck site \mathcal{C} with topology τ . Then we can define a simplicial model structure as we did in previous chapter. Particularly, we again use the injective model structure. By Bousfield localizing at τ -coverings, we obtain a simplicial model category denoted $\mathbf{sPre}_\tau(\mathcal{C})$. We denote $B_\tau G$ to be a τ -local fibrant replacement of BG for a τ -sheaf of groups G .

We have a canonical morphism of simplicial presheaves:

$$BG \rightarrow B\mathbf{Tor}_\tau(G)$$

by sending the unique vertex of $BG(U)$ to the trivial torsor over U . Then by taking fibrant replacement, we obtain a morphism:

$$B_\tau G \rightarrow B\mathbf{Tor}_\tau(G).$$

Lemma 4.3.8. *The above morphism of simplicial presheaves is a (injective) weak equivalence. And canonically $\mathbf{R}\Omega B_\tau G \simeq G$.*

Proof. See in [6, Lemma 2.3.2]. □

4.3.2 Algebraic Linear Groups

Proposition 4.3.9. *In \mathbb{A}^1 -homotopy theory, we have \mathbb{A}^1 -local weak equivalences:*

$$BGL_n \simeq_{\mathbb{A}^1} B_{\text{ét}}GL_n \simeq_{\mathbb{A}^1} Gr_n$$

Proof. See in [5, Proposition §4 3.7] □

Proposition 4.3.10. *Given a Nis-sheaf of groups G , then BG is \mathbb{A}^1 -connected.*

Proof. By Proposition 3.6.6, it suffices to show $\pi_0^{Nis} BG \cong *$. Since BG is *Nis*-local, hence this is the same as $\pi_0 BG$. We check the triviality at the stalks: consider the presheaf $U \mapsto \pi_0(B\mathbf{Tor}_{Nis}(G)(U))$ which is the isoclasses of *Nis*-locally split G -torsors over U . Let $U = \text{Spec}(R)$ where R is Henselian local, which only has trivial *Nis*-locally split G -torsors, then $\pi_0(B\mathbf{Tor}_{Nis}(G)(U)) = \{*\}$. Since $\pi_0 BG$ is the *Nis*-sheafification of the considered presheaf, hence it is trivial since all its stalks are trivial. Then $\pi_0^{\mathbb{A}^1} BG \cong \pi_0 BG \cong *$. \square

Corollary 4.3.11. *Given a Nis-sheaf of groups G , if $\pi_0 BG(-)$ is \mathbb{A}^1 -invariant, then the canonical fiber $G \rightarrow EG \rightarrow BG$ is an \mathbb{A}^1 -homotopy fiber.*

Proof. This is a direct corollary of Proposition 3.6.13. \square

Now we introduce some important groups and related computations as examples.

Example 4.3.12 (The Multiplicative Group \mathbf{G}_m). As shown in Proposition 4.1.5, \mathbf{G}_m is an \mathbb{A}^1 -local Eilenberg-MacLane space $K(\mathbf{G}_m, 0)$, then consider the natural \mathbb{A}^1 -homotopy fiber $\mathbf{G}_m \rightarrow E\mathbf{G}_m \rightarrow B\mathbf{G}_m$, we obtain the long exact sequence of *Nis*-sheaves:

$$* \cong \pi_n^{\mathbb{A}^1} E\mathbf{G}_m \rightarrow \pi_n^{\mathbb{A}^1} B\mathbf{G}_m \rightarrow \pi_{n-1}^{\mathbb{A}^1} \mathbf{G}_m \cong *$$

for $n > 1$ and

$$* \cong \pi_1^{\mathbb{A}^1} E\mathbf{G}_m \rightarrow \pi_1^{\mathbb{A}^1} B\mathbf{G}_m \rightarrow \pi_0^{\mathbb{A}^1} \mathbf{G}_m \rightarrow \pi_0^{\mathbb{A}^1} E\mathbf{G}_m \cong *$$

for $n = 1$. At last, since $B\mathbf{G}_m$ is a *Nis*-sheaf, therefore it is \mathbb{A}^1 -connected by Proposition 4.3.10, therefore we conclude $B\mathbf{G}_m$ is an \mathbb{A}^1 -local Eilenberg-MacLane space $K(\mathbf{G}_m, 1)$ as one might expect.

Example 4.3.13 (The Special Linear Group SL_n). Let SL_n be the algebraic special linear group of rank n . We sketch a quick proof to SL_n is naively \mathbb{A}^1 -connected and then by Proposition 3.6.7, we obtain that SL_n is \mathbb{A}^1 -connected. Furthermore since SL_n is a sheaf in Nisnevich topology, thus by Proposition 4.3.10, BSL_n is \mathbb{A}^1 -1-connected.

We show the triviality at stalks that $\pi_0(\text{Sing}SL_n)$ is trivial, i.e.

$$[\text{Spec}(R), \text{Sing}SL_n] \cong \{*\}$$

for R Henselian local ring. Let $f : \text{Spec}(R) \rightarrow \text{Sing}SL_n$ be a morphism of simplicial presheaves, then by Yoneda, this corresponds to an element of $SL_n(R)$ (say M). Let $\mathfrak{m} \subset R$ be the maximal ideal. Notice we can factorize $M = E \cdot (I + P)$ where E is a product of elementary matrices and P only has entries in \mathfrak{m} . Then

the matrix $I + H$, where $H = (h_{ij}) \in M_n(\mathfrak{m}[t])$ s.t. $h_{ij} = t \cdot p_{ij}$ is the \mathbb{A}^1 -homotopy equivalence between I and $I + P$. Therefore M is elementary \mathbb{A}^1 -homotopic to E and hence to I .

Then similar as for \mathbb{G}_m , by the long exact sequence of the \mathbb{A}^1 -homotopy fiber $SL_n \rightarrow ESL_n \rightarrow BSL_n$, we can see $\pi_1^{\mathbb{A}^1} BSL_n \cong *$.

On the other hand, we have stability of its homotopies: the natural embedding induces morphisms of homotopies at $0 < i \leq n$

$$\pi_i^{\mathbb{A}^1} BSL_n \rightarrow \pi_i^{\mathbb{A}^1} BSL_{n+1}$$

which is isomorphisms for $i < n$ and epimorphisms for $i = n$. This is induced from the \mathbb{A}^1 -homotopy fiber $SL_{n+1}/SL_n \rightarrow BSL_n \rightarrow BSL_{n+1}$ where $SL_{n+1}/SL_n \simeq_{\mathbb{A}^1} \mathbb{A}^{n+1} - \{0\}$ which is \mathbb{A}^1 - $(n-1)$ -connected by [15]. Then if we fix $i > 0$ and let $n > i$ goes to infinity, we obtain:

$$\pi_i^{\mathbb{A}^1} BSL_n \cong \pi_i^{\mathbb{A}^1} BSL_{n+1} \cong \dots \cong \pi_i^{\mathbb{A}^1} BSL$$

where BSL is the colimit $\bigcup_{n \rightarrow \infty} BSL_n$.

Proposition 4.3.14. *Suppose the Bass-Quillen Conjecture holds for those schemes we concern, then we have isomorphisms of homotopy sheaves:*

$$\pi_i^{\mathbb{A}^1} BSL_n \cong \pi_i^{\mathbb{A}^1} BGL_n$$

for $i > 1$. And we have

$$\pi_1^{\mathbb{A}^1} BGL_n \cong \mathbb{G}_m$$

Proof. Consider the canonical \mathbb{A}^1 -homotopy fiber $SL_n \rightarrow GL_n \rightarrow \mathbb{G}_m$. Then by direct computation, we have another \mathbb{A}^1 -homotopy fiber $BSL_n \rightarrow BGL_n \rightarrow B\mathbb{G}_m$. Then by the long exact sequence, we have exactness for $i > 1$:

$$* \cong \pi_{i+1}^{\mathbb{A}^1} B\mathbb{G}_m \longrightarrow \pi_i^{\mathbb{A}^1} BSL_n \longrightarrow \pi_i^{\mathbb{A}^1} BGL_n \longrightarrow \pi_i^{\mathbb{A}^1} B\mathbb{G}_m \cong *$$

and hence we are done for $i > 1$. For $\pi_1^{\mathbb{A}^1} BGL_n$, the conclusion comes from the exactness at $i = 1$:

$$* \cong \pi_1^{\mathbb{A}^1} BSL_n \longrightarrow \pi_1^{\mathbb{A}^1} BGL_n \longrightarrow \pi_1^{\mathbb{A}^1} B\mathbb{G}_m \longrightarrow \pi_0^{\mathbb{A}^1} BSL_n \cong *,$$

hence $\pi_1^{\mathbb{A}^1} BGL_n \cong \pi_1^{\mathbb{A}^1} B\mathbb{G}_m \cong \mathbb{G}_m$. □

Corollary 4.3.15. *The above isomorphisms combined with the stability of BSL_n gives isomorphisms of homotopies between infinite classifying spaces:*

$$\pi_i^{\mathbb{A}^1} BSL \cong \pi_i^{\mathbb{A}^1} BGL$$

for $i > 1$.

4.3.3 K-Theory

Proposition 4.3.16. *We let \mathcal{K} be the Thomason-Trobaugh K-theory space in the sense [21], i.e. the K-theory of perfect complex over schemes, then by viewing \mathcal{K} as a simplicial presheaf on \mathbf{Sm}_k , it satisfies the \mathbb{A}^1 -invariance and Nis-local descent condition.*

Particularly, we have natural isomorphisms for all $X \in \mathbf{Sm}_k$ and non-negative integer i :

$$K_i(X) \cong [\Sigma_+^i X, \mathcal{K}]_{\mathbb{A}^1}$$

where K_i is the i^{th} algebraic K-group.

Proof. By [21, Proposition 6.8(i)], the map induced by pullback:

$$K(X) \rightarrow K(X \times_k \mathbb{A}^1)$$

is a weak equivalence of simplicial sets. Therefore this space is \mathbb{A}^1 -invariant.

On the other hand, [21, Theorem 10.8] shows the algebraic K-theory is satisfies the Nisnevich descent, i.e. \mathcal{K} is a Nis-local fibrant simplicial presheaf. \square

Definition 4.3.17 (*H-space*). *Let X be a simplicial set. We say X is a **H-space** if it is equipped with a multiplication $m : X \times X \rightarrow X$ and a homotopy identity $e : * \rightarrow X$, i.e. $m(e, -)$ and $m(-, e)$ are homotopic to the identity map. Furthermore, given an H-space X , we say it is **homotopy commutative** if the $m \circ t$ is homotopic to m where $t : X \times X \rightarrow X \times X$ is the swap isomorphism. And we say it is **homotopy associative** if $m \circ (id \times m)$ is homotopic to $m \circ (m \times id)$. We say an H-space X is **group-like** if the Abelian monoid $\pi_0(X)$ is a group.*

*Given a morphism of simplicial sets $f : X \rightarrow Y$, we say it is a **H-space** if it preserves the homotopy identity and the multiplication.*

Definition 4.3.18 (*Group Completion*). *Given a homotopy commutative and associative H-space X , we say an H-map $X \rightarrow Y$ is a **group completion** if*

1. $\pi_0(X) \rightarrow \pi_0(Y)$ is a group completion of the Abelian monoid $\pi_0(X)$;
2. for any commutative ring R , the induced homomorphism of homology $H_*(X; R) \rightarrow H_*(Y; R)$ is a localization of the graded commutative homology ring at the multiplicative subset $\pi_0(X) \subset H_0(X; R)$.

We usually denote the completion as $X^{\mathcal{G}P}$.

Here we cite [22, Theorem 1.6], which shows that ΩBX is a model of group completion of an H-space X with certain conditions:

Theorem 4.3.19. *Given a group-like homotopy commutative and associative H-space, then $X \rightarrow \Omega BX$ is a group completion.*

Theorem 4.3.20 (Quillen's +-construction). *Let R be a unital commutative ring. Then the canonical morphism $i : BGL(R) \rightarrow BGL(R)^+$ satisfies the following universal property: given a morphism $f : BGL(R) \rightarrow H$ where H is an H -space, then there exists a H -map $\tilde{f} : BGL(R)^+ \rightarrow H$ s.t. $\tilde{f} \circ i \simeq f$. And if $f \simeq g : BGL(R) \rightarrow H$, then $\tilde{f} = \tilde{g}$.*

Proof. See in [23, Theorem IV.1.8]. □

Proposition 4.3.21. *Let \mathcal{K} be the K-theory space. Then the canonical morphism of simplicial presheaves:*

$$\mathbb{Z} \times BGL \rightarrow \mathcal{K}$$

is an \mathbb{A}^1 -local weak equivalence where \mathbb{Z} here is viewed as a constant presheaf.

$$\text{Hence as a result, } \pi_i^{\mathbb{A}^1} BGL \cong \pi_i^{\mathbb{A}^1} (\mathbb{Z} \times BGL) \cong \pi_i^{\mathbb{A}^1} \mathcal{K}.$$

Proof. We show this at stalks: we show $Sing^I(\mathbb{Z} \times BGL(-))$ is weak equivalent to $\mathcal{K}(-)$ at all Henselian local rings. Let $U = Spec(R)$ where R is a regular Noetherian local ring. Then consider the following commutative diagram:

$$\begin{array}{ccc} K_0(R) \times Sing^I BGL(R) & \longrightarrow & \mathcal{K}(R) \\ \downarrow id \times \varphi & & \uparrow \simeq \\ K_0(R) \times Sing^I BGL(R)^+ & \xrightarrow{id \times \psi} & K_0(R) \times BGL(R)^+ \end{array}$$

where φ is the universal map of Quillen's +-construction and ψ is the map induced by projections levelwise. The right vertical is a weak equivalence for free. We show ψ, φ are weak equivalences as well and then it follows the top arrow is a weak equivalence.

Notice we can write $Sing^I BGL(R)$ as a simplicial groups $BGL(R[\Delta^\bullet])$ where $R[\Delta^n]$ is the ring corresponds to the n^{th} standard algebraic simplex. Then by [23, Exercise IV.11.9], $Sing^I BGL(R)$ is a group-like H -space. Therefore φ is a weak equivalence between group-like H -spaces since both of them are nilpotent. On the other hand, \mathcal{K} is \mathbb{A}^1 -invariant, therefore the simplicial structure on $Sing^I BGL(R)^+$, which is $BGL(R[\Delta^\bullet])$, is trivial. Thus the map induced by projection $Sing^I BGL(R)^+ \rightarrow BGL(R)^+$ is a weak equivalence.

Since R is local, hence $K_0(R) \cong \mathbb{Z}$. Then $Sing^I(\mathbb{Z} \times BGL(R)) \cong K_0(R) \times Sing^I BGL(R) \simeq \mathcal{K}(R)$ as we have shown and so $Sing^I(\mathbb{Z} \times BGL) \simeq \mathcal{K}$. Then since $\mathbb{Z} \times BGL \simeq_{\mathbb{A}^1} Sing^I(\mathbb{Z} \times BGL)$, we are done. □

Proposition 4.3.22. *Let $1 < i < n$, then we have isomorphisms:*

$$\pi_i^{\mathbb{A}^1}(BGL_n) \cong K_i.$$

Proof. Combining Proposition 4.3.21, 4.3.14, 4.3.15 and the stability of homotopy in Example 4.3.13, we have

$$\begin{aligned}\pi_i^{\mathbb{A}^1} BGL_n &\cong \pi_i^{\mathbb{A}^1} BSL_n \\ &\cong \pi_i^{\mathbb{A}^1} BSL \\ &\cong \pi_i^{\mathbb{A}^1} BGL \cong \pi_i^{\mathbb{A}^1} \mathcal{K} \cong K_i\end{aligned}$$

□

We conclude this section by the computations of \mathbb{A}^1 -homotopy of BGL_n in terms of algebraic K-theory.

Theorem 4.3.23. *For $n = 1$, $BGL_1 = B\mathbb{G}_m$ is an \mathbb{A}^1 -Eilenberg-MacLane space $K(\mathbb{G}_m, 1)$.*

For $n \geq 2$, we have isomorphisms of sheaves:

$$\begin{aligned}\pi_0^{\mathbb{A}^1} BGL_n &\cong * \\ \pi_1^{\mathbb{A}^1} BGL_n &\cong \mathbb{G}_m \\ \pi_i^{\mathbb{A}^1} BGL_n &\cong K_i\end{aligned}$$

where $2 \leq i < n$.

4.4 Examples

We start with an anti-examples to the classification Theorem 4.2.9 which the failure is due to failure of affineness.

Example 4.4.1 (Projective Line \mathbb{P}^1). An anti-example to this theorem for non-affine case actually comes easier: the most basic non-affine scheme \mathbb{P}^1 over field k breaks the equality.

It is well-known that that any vector bundle over \mathbb{P}^1 can be decomposed as a direct sum of line bundles over \mathbb{P}^1 by [23, Proposition I.5.6], i.e. given a vector bundle E (think as a quasi-coherent sheaf), we have the following isomorphism:

$$E \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$$

where $n_1 \leq \cdots \leq n_r$ and r is the rank of E .

On the other hand, since Gr_r is also represented by BGL_r in \mathbb{A}^1 -homotopy theory as shown, we have isomorphisms:

$$\begin{aligned} [\mathbb{P}^1, Gr_r]_{\mathbb{A}^1} &\cong [\mathbb{P}^1, BGL_r]_{\mathbb{A}^1} \\ &\cong [S^1 \wedge \mathbf{G}_m, BGL_r]_{\mathbb{A}^1} \\ &\cong \pi_1^{\mathbb{A}^1} BGL_r(\mathbf{G}_m) \\ &\cong Hom_{\mathbf{Sm}_k}(\mathbf{G}_m, \mathbf{G}_m) \\ &\cong Hom_{\mathbf{Rng}}(k[t, t^{-1}], k[t, t^{-1}]) \cong k^* \oplus \mathbb{Z} \end{aligned}$$

Hence, in our case, to compute the \mathbb{A}^1 -homotopy classes is to compute the endomorphisms of the ring $k[t, t^{-1}]$, which is obviously not the same as the other side.

Remark 4.4.2. Other than the direct computation above, there is a more well-known argument to non- \mathbb{A}^1 -invariance of vector bundles over the projective line. Consider $\mathbb{P}^1 \times \mathbb{A}^1$ and coordinate $(1 : t : x)$ elsewhere the infinity of \mathbb{P}^1 . And let E be a vector bundles given by coherent sheaf generated by section t, x and the transition matrix is

$$\begin{pmatrix} 1/t & x \\ 0 & t \end{pmatrix}.$$

Then by restricting to $\mathbb{P}^1 \times \{0\}$, we obtain $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ while we obtain trivial rank 2 bundle on $\mathbb{P}^1 \times \{1\}$. Hence E must not come from the pullback of a vector bundle on \mathbb{P}^1 which means $\mathcal{V}_2(\mathbb{P}^1)$ is not the same as $\mathcal{V}_2(\mathbb{P}^1 \times \mathbb{A}^1)$.

Directly from Theorem 4.2.9, one can tell that \mathbb{A}^1 -contractibility behaves better than topologically contractibility. By topological contractible variety, we mean a smooth affine scheme X over \mathbb{C} s.t. its underlying analytic space is topologically contractible. As shown in the Asok and Fasel's result below, even if a variety X over \mathbb{C} is topologically contractible, we only have sufficient triviality of $Pic(X)$. The triviality of rank 2 vector bundles needs further conditions.

Corollary 4.4.3. *With the assumption in Theorem 4.2.9, if X is \mathbb{A}^1 -local contractible, then all its vector bundles are trivial.*

Lemma 4.4.4 (Asok, Fasel [24]). *Let X be a smooth affine complex threefold over a algebraically closed field where 2 is invertible, then the Chern classes induce an isomorphism:*

$$\mathcal{V}_2(X) \xrightarrow{\cong} Pic(X) \times CH^2(X).$$

We introduce a famous question in algebraic geometry: the Zariski Cancellation problem. We recommend Kraft's paper [25] which includes a nice introduction to this problem and other involved and also a newer survey paper [26] by Gupta. Up to our concern, we present one of the simplest form of the question.

Question 4.4.5. Let X be a smooth scheme, s.t. $X \times \mathbb{A}^{N-d} \cong \mathbb{A}^N$ for some positive integer $d \leq N$, then whether X is isomorphic to an affine space \mathbb{A}^d .

The cases of lower dimension $d = 1, 2$ are solved already by [27][28][29] with positive answer. But anti-examples are found for $d = 3$: Gupta constructs an anti-example for $d = 3$ over a field of characteristic $p > 0$ in [30]. And for now, looking for anti-examples for $d = 3$ and for fields of characteristic 0 (particular for \mathbb{C}) remains open.

Lemma 4.4.6 (Dimca [31]). *The underlying analytic space of a smooth topologically contractible affine variety of dimension $d \geq 3$ is diffeomorphic to affine space \mathbb{C}^d .*

This lemma brings topologically contractible affine variety to one's sight since any concerned complex variety to the Cancellation Problem should be at least diffeomorphic to a complex affine space. Among those topologically contractible complex varieties, the Koras-Russell threefolds are one type of those expected to be an anti-example to the Cancellation Problem. This expectation shows up for reasons.

Let us introduce the specific types of Koras-Russell threefolds we concern in the following arguments.

Lemma 4.4.7 (Kaliman, Makar-Limanov [32]). *Let X be an affine variety over \mathbb{C} given by:*

$$X := \text{Spec}\left(\frac{\mathbb{C}[x, y, z, t]}{ax + x^m y + z^{a_2} + t^{a_3}}\right) \quad (1)$$

where $a \in \mathbb{C}^\times$, $m, a_2, a_3 \geq 2$ and $(a_2, a_3) = 1$, or given by:

$$X := \text{Spec}\left(\frac{\mathbb{C}[x, y, z, t]}{ax + (x^b + z^{a_2})^l y + t^{a_3}}\right) \quad (2)$$

where $a \in \mathbb{C}^\times$, $b, a_2, a_3, l \geq 2$ and $(a_2, ba_3) = 1$. Then their Makar-Limanov invariants are not the same as the coordinate ring $\mathbb{C}[X]$.

Makar-Limanov shows with his eponymous invariant defined as

$$ML(X) := \bigcap_{\mathbb{C}^+ \rightarrow \text{Aut}(X)} \mathbb{C}[X]^{\mathbb{C}^+}$$

that any non-trivial Koras-Russell threefolds X has different Makar-Limanov invariant from the affine space \mathbb{A}^3 . Hence any non-trivial Koras-Russell threefolds are indeed non isomorphic to \mathbb{A}^3 . However, Dubouloz shows in [33] that the same invariant does not distinguish $X \times \mathbb{A}^1$ from \mathbb{A}^d .

Start from a lecture in Ottawa given by Asok, which he introduced a procedure to disprove \mathbb{A}^1 -contractibility, the study of Koras-Russell threefolds came into the \mathbb{A}^1 -taste. As shown in the observation below, \mathbb{A}^1 -contractibility can be viewed as a characterization to deny the potential of a possible anti-example.

Observation 4.4.8. If X is an anti-example to the Cancellation Problem, then X is \mathbb{A}^1 -contractible.

Later in 2014, Hoyois, Krishna, and Østvær prove in [34] the Koras-Russell threefolds deny Asok's procedure, which means these varieties keep their potential. Actually, they prove Koras-Russell threefolds of type (1), (2) are stably \mathbb{A}^1 -contractible in the same paper, i.e. $\Sigma_{\mathbb{P}^1}^n X \simeq_{\mathbb{A}^1} *$ for $n \geq 0$.

Lemma 4.4.9 (Hoyois, Krishna, Østvær). *Given a Koras-Russell threefolds X of type (1) or (2), then there exists $n \geq 0$ s.t. $\Sigma_{\mathbb{P}^1}^n X \simeq_{\mathbb{A}^1} *$.*

Sketch of proof. Let X be a Koras-Russell threefolds of type (1) or (2), then it can be computed that the pullback map of higher Chow groups $CH^\bullet(Y) \rightarrow CH^\bullet(X \times Y)$ is an isomorphism for any smooth affine complex variety Y . Then by Voevodsky's computation [35], this is the same as the isomorphism of integral motivic cohomology groups

$$H^{\bullet,\bullet}(Y, \mathbb{Z}) \rightarrow H^{\bullet,\bullet}(X \times Y, \mathbb{Z})$$

for any smooth affine complex variety Y .

On the other hand, it can be proven that for complex smooth variety X pointed at a closed point x , its ∞ - \mathbb{P}^1 -spectrum $\Sigma_{\mathbb{P}^1}^\infty(X, x)$ is \mathbf{MZ} -local where \mathbf{MZ} represents the integral motivic cohomology, i.e. for any spectrum E s.t. $E \wedge \mathbf{MZ} \simeq *$, we have $\text{Hom}_{\mathbf{SH}(\mathbb{C})}(E, X)$ is trivial.

Then the isomorphisms of the integral motivic cohomology suggest isomorphisms

$$\text{Hom}_{\mathbf{SH}}(\Sigma_{\mathbb{P}^1}^\infty Y_+(i)[n], \mathbf{MZ}) \xrightarrow{\cong} \text{Hom}_{\mathbf{SH}}(\Sigma_{\mathbb{P}^1}^\infty Y_+(i)[n], \underline{\text{Map}}(\Sigma_{\mathbb{P}^1}^\infty X_+, \mathbf{MZ})),$$

which leads to an isomorphism $\mathbf{MZ} \rightarrow \underline{\text{Map}}(\Sigma_{\mathbb{P}^1}^\infty X_+, \mathbf{MZ})$. Therefore $\Sigma_{\mathbb{P}^1}^\infty(X, x) \wedge \mathbf{MZ} \simeq *$. Combined with the \mathbf{MZ} -localness, we imply $\Sigma_{\mathbb{P}^1}^\infty(X, x) \simeq *$. Then by the Quillen adjunction between unstable homotopy category and the stable homotopy category, we have $\Sigma_{\mathbb{P}^1}^n(X, x) \simeq *$ in the unstable homotopy category. \square

As one might notice, this proof requires quite much of stable \mathbb{A}^1 -homotopy theory, which is not systematically discussed in this thesis. Hence we hope to find a more 'unstable' proof to the \mathbb{A}^1 -contractibility. Luckily in [36], Dubouloz and Fasel give a such proof for the Koras-Russell threefolds of type (1).

Lemma 4.4.10 (Dubouloz, Fasel). *Given a complex Koras-Russell threefolds X given by (1), then X is \mathbb{A}^1 -contractible.*

Sketch of proof. Let $X := \text{Spec}(A)$ where $A = \mathbb{C}[x, y, z, t]/(ax + x^m y + z^s + t^r)$ s.t. $(r, s) = 1, m, r, s \geq 2$. We start by setting $a = 1$. Notice there is hypersurface in X given by $P = \{y = 0\} \subset X$ and an line $L = \{x = t = z = 0\} \subset X$. Then $P \cong \text{Spec}(\mathbb{C}[t, z]) \cong \mathbb{A}^2$ and $L \cong \text{Spec}(\mathbb{C}[y]) \cong \mathbb{A}^1$. Also since P and L intersect transversally at $(0, 0, 0, 0)$, hence the following pullback is an EDS:

$$\begin{array}{ccc} P - L & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ X - L & \longrightarrow & X \end{array} .$$

Then we obtain an induced diagram of cofibers:

$$\begin{array}{ccccc} \mathbb{A}^2 - \{0\} & \longrightarrow & \mathbb{A}^2 & \longrightarrow & (\mathbb{P}^1)^{\wedge 2} \\ \downarrow & & \downarrow & & \downarrow \\ X - L & \longrightarrow & X & \longrightarrow & X/(X - L) \end{array} .$$

There exists a quasi-affine scheme W which is a Zariski-locally trivial vector bundle over both $X - L$ and $X(1) - L$ where $X(1) := \text{Spec}(\mathbb{C}[x, y, z, t]/(x + x^m y + z + t^r)) \cong \text{Spec}(\mathbb{C}[x, y, t])$, hence since $X(1) - L \cong (\mathbb{A}^2 - \{0\}) \times \mathbb{A}^1 \simeq_{\mathbb{A}^1} \mathbb{A}^2 - \{0\}$, we then have the left vertical arrow is an \mathbb{A}^1 -weak equivalence.

On the other hand, by Motivic Purity Theorem, $X/(X - L) \simeq_{\mathbb{A}^1} N_L/(N_L - L)$ where N_L is the normal bundle. Since $N_L \cong \text{Spec}(\text{Sym}(I/I^2))$ where $I = (x, t, z)$, therefore $N_L/(N_L - L) \cong \mathbb{A}^3/(\mathbb{A}^1 \times (\mathbb{A}^2 - \{0\})) \simeq_{\mathbb{A}^1} \mathbb{A}^2/(\mathbb{A}^2 - \{0\})$. By tracking the sections generating the normal bundle, the right vertical arrow is indeed the \mathbb{A}^1 -weak equivalence we obtained.

Then combined with \mathbb{A}^2 is \mathbb{A}^1 -contractible, we obtain the middle arrow is an \mathbb{A}^1 -weak equivalence. For the case of $a \neq 1$, it follows from a similar argument as in [37] that $X \times \mathbb{A}^1$ are all isomorphic for all choice of a . Hence all Koras-Russell threefolds of type (1) are of the same \mathbb{A}^1 -homotopy type and hence \mathbb{A}^1 -contractible. \square

For now, it is still uncertain whether a Koras-Russell threefold is an anti-example to the question 4.4.5, but the new \mathbb{A}^1 -homotopical results provide a new direction and indeed motivate further researches.

Purity

When someone ask for the reason to use Nisnevich topology, a standard answer shall be representability of K -theory and the Purity Theorem: Nisnevich is the finest to ensure the previous condition and the coarsest to ensure the latter one. Since the representability of K -theory is shown in chapter 3, hence we finish the argument about purity in this chapter.

A.1 Purity Theorem

Proposition A.1.1. *Given $i : Z \rightarrow X$ closed immersion, let $Bl(X, Z)$ be the blow-up of X at Z . Then we have a pullback of schemes:*

$$\begin{array}{ccc}
 \mathbb{P}(N_Z) & \longrightarrow & Bl(X, Z) \\
 \downarrow & \lrcorner & \downarrow \\
 Z & \longrightarrow & X
 \end{array}$$

Proof. Consider locally on an affine open $V = Spec(R)$ of X , and the ideal sheaf of Z restrict to an R -ideal I . Then $Z \cap V \cong Spec(R/I)$. And so we have a pullback of schemes:

$$\begin{array}{ccc}
 \mathbb{P}((\bigoplus I^n) \otimes_R R/I) & \longrightarrow & \mathbb{P}(\bigoplus I^n) \\
 \downarrow & & \downarrow \\
 Spec(R/I) & \longrightarrow & Spec(R)
 \end{array}$$

Recall the we define the normal bundle to be the bundle, locally over $Spec(R/I)$, given by the $(I/I^2)^\vee$. This is finitely generated since X is Noetherian. Hence

by choosing an R/I -basis of I/I^2 , say e_1, \dots, e_n , we can write down an isomorphism by sending e_i to its dual. Therefore $\text{Sym}(I/I^2) \cong \text{Sym}((I/I^2)^\vee)$ as graded algebras. Then notice $\text{Sym}^n(I/I^2) \cong I^n/I^{n+1} \cong I^n \otimes_R R/I$, therefore $\mathbb{P}(N_Z) \cong \mathbb{P}(\text{Sym}(I/I^2)) \cong \mathbb{P}(\oplus(I^n \otimes_R R/I)) \cong \mathbb{P}((\oplus I^n) \otimes_R R/I)$ and then we are done. \square

Proposition A.1.2. *Consider the trivial \mathbb{A}^n -bundle $\mathbb{A}_X^n \rightarrow X$ and $i : X \rightarrow \mathbb{A}_X^n$ is the 0-section. Then the normal bundle of X is isomorphic to the trivial \mathbb{A}^1 -bundle of rank n .*

Proof. Say $\mathbb{A}^n = \text{Spec}(k[x_1, \dots, x_n])$. Notice the ideal sheaf of the 0-section is (x_1, \dots, x_n) , hence the normal bundle is $((x_1, \dots, x_n)/(x_1, \dots, x_n)^2)^\vee = \oplus \mathcal{O}_X \cdot x_i^\vee$ which is trivial of rank n . \square

Definition A.1.3. *Given a (geometric) vector bundle $p : E \rightarrow X$. We define the **Thom space** $Th(p) := E/(E - X)$ (or denoted $Th(E)$) by embedding X into E via the 0-section.*

Particularly, we define the Thom space of a closed immersion $i : Z \rightarrow X$ to be $Th(v_Z)$, where $v_Z : N_Z \rightarrow Z$ is the (geometric) normal bundle of Z .

Remark A.1.4. The quotient happens by viewing these schemes as motivic spaces, i.e. the simplicial presheaves they represent in $\mathbf{Spc}^{\mathbb{A}^1}$

Proposition A.1.5. *Given a vector bundle $p : E \rightarrow X$, we embed $\mathbb{P}(E)$ into $\mathbb{P}(E \oplus \mathcal{O})$ at the infinity. Then we have \mathbb{A}^1 -weak equivalences:*

$$Th(p) \simeq \frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O})} \simeq \frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E)}.$$

Proof. We first show the left hand side weak equivalence. The key observation is the pullback as schemes:

$$\begin{array}{ccc} E - X & \longrightarrow & \mathbb{P}(E \oplus \mathcal{O}) - X \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathbb{P}(E \oplus \mathcal{O}) \end{array}$$

Since both the left and right arrows are open immersion, hence this square is an elementary distinguished square and so this is a homotopy pushout in $\mathbf{Spc}^{\mathbb{A}^1}$. Thus this gives the \mathbb{A}^1 -weak equivalence:

$$\frac{\mathbb{P}(E \oplus \mathcal{O})}{\mathbb{P}(E \oplus \mathcal{O}) - X} \simeq Th(p).$$

Since $\mathbb{P}(\mathcal{O}) \cong X$, thus we prove the first part of the proposition.

For the second part, we show that $\mathbb{P}(E \oplus \mathcal{O}) - X \rightarrow \mathbb{P}(E)$ is a vector bundle, i.e. a Zariski-local \mathbb{A}^1 -bundle. Then they are \mathbb{A}^1 -weak equivalent and hence the second part of the proposition proven. Consider an open $U \subset X$ trivialize E , and so for $\mathbb{P}(E)$ and $\mathbb{P}(E \oplus \mathcal{O})$. Then we may assume $E \cong \mathcal{O}^n$. Notice U embeds into $\mathbb{P}(\mathcal{O}^{n+1})$ at the homogenous coordinate $(0 : \cdots : 0 : 1)$, hence we can define $p : \mathbb{P}(E \oplus \mathcal{O}) - X \rightarrow \mathbb{P}(E)$ by projection at the first n entries. Then this form an \mathbb{A}^1 -bundle. \square

Theorem A.1.6 (Purity Theorem). *Let $i : Z \rightarrow X$ be a closed immersion and N_Z be the normal bundle over Z . Then we have an \mathbb{A}^1 -weak equivalence:*

$$Th(N_Z) \simeq \frac{X}{X - Z}.$$

Before proving the theorem, we introduce the idea of the proof: we show that there are two \mathbb{A}^1 -weak equivalences:

$$Th(N_Z) \xrightarrow{\tilde{i}_0} \frac{Bl(X \times \mathbb{A}^1, Z \times \{0\})}{Bl(X \times \mathbb{A}^1, Z \times \{0\}) - Bl(Z \times \mathbb{A}^1, Z \times \{0\})} \xleftarrow{\tilde{i}_1} \frac{X}{X - Z}.$$

We write $Bl(X \times \mathbb{A}^1)$ and $Bl(Z \times \mathbb{A}^1)$ for short as there is no confusion about where we blow up. We start with the case Proposition A.1.7, then generalize the statement to Proposition A.1.9 by using Proposition A.1.8. And at last we proof the general statement by using Proposition A.1.10.

Here use the presentation $Th(N_Z) \simeq \frac{\mathbb{P}(N_Z \oplus \mathcal{O})}{\mathbb{P}(N_Z \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O})}$ and we embed Z into $X \times \mathbb{A}^1$ via the 0-section. Say \mathcal{I} is the ideal sheaf of Z in X . Therefore the normal bundle of $Z \times \{0\}$ is $N_Z \oplus \mathcal{O}$. Then as we have a commutative square:

$$\begin{array}{ccc} \mathbb{P}(N_Z \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O}) & \longrightarrow & Bl(X \times \mathbb{A}^1) - Bl(Z \times \mathbb{A}^1) \\ \downarrow & & \downarrow \\ \mathbb{P}(N_Z \oplus \mathcal{O}) & \longrightarrow & Bl(X \times \mathbb{A}^1) \end{array} .$$

Hence the map \tilde{i}_0 is well-defined.

For \tilde{i}_1 , we consider embed X and Z via the 1-section $X \times \{1\} \rightarrow X \times \mathbb{A}^1$ where the blow-up does nothing. Hence this map is obviously well-defined.

Proposition A.1.7. *Let X be the trivial \mathbb{A}^n -bundle \mathbb{A}_Z^n and $i : Z \rightarrow \mathbb{A}_Z^n$ is the 0-section, then the purity theorem holds.*

Proof. By Proposition A.1.2, we know the normal bundle N_Z is trivial of rank n . We need to show the following maps are \mathbb{A}^1 -weak equivalences:

$$\frac{\mathbb{P}(\mathbb{A}_Z^n \oplus \mathbb{A}_Z^1)}{\mathbb{P}(\mathbb{A}_Z^n \oplus \mathbb{A}_Z^1) - \mathbb{P}(\mathbb{A}_Z^1)} \xrightarrow{\tilde{i}_0} \frac{Bl(\mathbb{A}_Z^n \times \mathbb{A}^1)}{Bl(\mathbb{A}_Z^n \times \mathbb{A}^1) - Bl(Z \times \mathbb{A}^1)} \xleftarrow{\tilde{i}_1} \frac{\mathbb{A}_Z^n}{\mathbb{A}_Z^n - Z}.$$

\tilde{i}_0 is quite straight forward since the blow-up $Bl(\mathbb{A}^{n+1})$ at 0 is the canonical (geometric) line bundle over \mathbb{P}^n and hence an Zariski \mathbb{A}^1 bundle. Thus the \tilde{i}_0 is a weak equivalence terms by terms.

Let p be the projection of from the blow-up to projective space. Notice i_0 is a section to p . Hence if we show $\tilde{i}_1 \circ \tilde{p}$ is an \mathbb{A}^1 -weak equivalence, then so is \tilde{i}_1 . This is true because $\mathbb{A}^n \xrightarrow{i_1} Bl(\mathbb{A}^n \times \mathbb{A}^1) \xrightarrow{p} \mathbb{P}^n$ is just the standard open immersion. Therefore the following square is an EDS:

$$\begin{array}{ccc} \mathbb{A}_Z^n - Z & \xrightarrow{i_1 \circ p} & \mathbb{P}_Z^n - Z \\ \downarrow & & \downarrow \\ \mathbb{A}_Z^n & \xrightarrow{i_1 \circ p} & \mathbb{P}_Z^n \end{array} .$$

And so $i_1 \circ p$ is an \mathbb{A}^1 -weak equivalence. \square

Proposition A.1.8. *Let $f : Y \rightarrow X$ be an Étale morphism and $i : Z \rightarrow X$ be a closed immersion s.t.*

$$\begin{array}{ccc} Z_Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

is a pullback of schemes satisfies $Z_Y \cong Z$. Then the purity theorem holds for (X, Z) if and only if the purity holds for (Y, Z) .

Proof. Notice we have an EDS:

$$\begin{array}{ccc} Y - Z_Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X - Z & \longrightarrow & X \end{array} .$$

Therefore we have an isomorphism of cofibers $\frac{Y}{Y-Z_Y} \xrightarrow{\cong} \frac{X}{X-Z}$. Since f is flat, thus the normal bundle of Z is isomorphic to the normal bundle N_{Z_Y} of Z_Y . Hence

we also have an EDS:

$$\begin{array}{ccc} N_{Z_Y} - Z_Y & \longrightarrow & N_{Z_Y} \\ \downarrow & & \downarrow \\ N_Z - Z & \longrightarrow & N_Z \end{array} .$$

And so we obtain $Th(N_Z) \cong Th(N_{Z_Y})$. Then the proposition proven. \square

Proposition A.1.9. *Let $i : Z \rightarrow X$ be a closed immersion s.t. the square below is an EDS. Then the purity theorem holds.*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow f_Z & & \downarrow f \\ \mathbb{A}^m & \xrightarrow{j} & \mathbb{A}^n \end{array}$$

where j is the inclusion at the first m coordinates.

Proof. We consider $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^{n-m})$ with $Z \times \mathbb{A}^{n-m} \xrightarrow{f_Z \times id} \mathbb{A}^m \times \mathbb{A}^{n-m} \cong \mathbb{A}^n$. Let D be its fiber over $\mathbb{A}^m \times (0, \dots, 0)$, then

$$\begin{aligned} D &= \mathbb{A}^m \times_{\mathbb{A}^n} X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^{n-m}) \\ &\cong Z \times_{\mathbb{A}^n} (Z \times \mathbb{A}^{n-m}) \\ &\cong Z \times_{\mathbb{A}^n} Z \cong Z \times_{\mathbb{A}^m} Z \end{aligned} .$$

Since $f_Z : Z \rightarrow \mathbb{A}^m$ is Étale, then $D \cong Z \sqcup Y$ where Y is closed subscheme of $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^{n-m})$. Let $E = X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^{n-m}) - Y$ and consider the projections $p_1 : E \rightarrow X$ and $p_2 : E \rightarrow Z \times \mathbb{A}^{n-m}$. Then the pullback of p_1 along $i : Z \rightarrow X$ is the automorphism of Z since we take Y out. And same for p_2 . Therefore by Proposition A.1.8, the purity theorem holds for (X, Z) if and only if the purity holds for $(Z \times \mathbb{A}^{n-m}, Z)$ which is true by Proposition A.1.7. \square

Proposition A.1.10. *Let $U \rightarrow X$ be an Étale cover and $i : Z \rightarrow X$ be a closed immersion. If the purity theorem holds for arbitrary n -intersection $(U^{\times_n}_X, U^{\times_n}_X \times_X Z)$, $\forall n > 0$, then the purity holds for (X, Z) .*

Proof. Let $\check{C}(U)$ be the Čech complex of U over X . Since the simplicial presheaf X is a simplicial sheaf in Étale topology by Proposition 3.3.3, hence the natural morphism $\check{C}(U) \rightarrow X$ is an Nis-local weak equivalence by Lemma 3.5.5 and so an \mathbb{A}^1 -weak equivalence. Same thing happens to $\check{C}(U - U \times_X Z) \rightarrow X - Z$. \square

Finally, we use the above propositions to prove the purity theorem:

proof of purity theorem. We include [38, Théorème II.4.10]: given a closed subscheme $Z \subset X$ of codimension c along Z , there is a Zariski cover $\{f_i : U_i \rightarrow X\}_{i \in I}$ and a set of Étale morphisms $\{\varphi_i : U_i \rightarrow \mathbb{A}^{n_i}\}$ s.t. for any $i \in I$,

$$\begin{array}{ccc} U_i \times_X Z & \longrightarrow & U_i \\ \downarrow & & \downarrow \varphi_i \\ \mathbb{A}^{n_i-c} & \longrightarrow & \mathbb{A}^{n_i} \end{array}$$

is a pullback of schemes. Notice we also have a pullback of schemes for arbitrary m -intersection:

$$\begin{array}{ccccc} U_{i_1, \dots, i_m} \times_X Z & \longrightarrow & U_{i_1, \dots, i_{m-1}} \times_X Z \times \mathbb{A}^{n_{i_m}} & \longrightarrow & \mathbb{A}^{n_{i_1} + \dots + n_{i_m} - c} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U_{i_1, \dots, i_m} & \longrightarrow & U_{i_1, \dots, i_{m-1}} \times \mathbb{A}^{n_{i_m}} & \longrightarrow & \mathbb{A}^{n_{i_1} + \dots + n_{i_m}} \end{array}$$

Therefore by Proposition A.1.9, the purity theorem holds for $(U_{i_1, \dots, i_m}, U_{i_1, \dots, i_m} \times_X Z)$. Then we set $U = \bigsqcup_{i \in I} U_i$, and so this cover satisfies the conditions in Proposition A.1.10. Hence the purity theorem holds for (X, Z) . \square

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