Persistence of Stability for Piecewise Deterministic Markov Processes: Uniform Equicontinuity on Balls and Invariant Measures around Periodic Orbits

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Abstract

We generalize Alkurdi's work [1] on persistence of stability in a Piecewise Deterministic Markov Process with deterministic intervention times to allow for stochastic intervention times. The proof technique uses a famous result by Szarek linking non-expansiveness of the Markov operator P to exponential ergodicity. We establish the link between the UEB property and persistence of stability, making use of results by Szarek [2]. We first re-describe the Markov operator associated to the PDMP in a mathematically tractable form to make rigorous estimates leading up to the UEB property. As far as we are aware, this is the first work where the UEB property has been established for a class of PDMPs with an exponentially stable limit cycle. The proof of the main theorem establishing persistence of stability yields an explicit construction of an invariant region that is crater-like and contains the support of μ. We prove how the UEB property of this new operator is equivalent to the UEB property of the original problem. In the last chapter, we apply the results to a perturbed Rosenzweig-MacArthur predator-prey model to illustrate how to verify in practice the different model assumptions. Of special importance how to establish the desired exponential convergence to the stable limit cycle. For reference in future applications, we review several techniques, most notably numerical methods from *contraction analysis*. We propose several modeling options for the perturbations.

Acknowledgements

I am fortunate to have received the support and encouragement of many people during the completion of this thesis and my graduate education. I would like to thank my advisor, Dr. Sander Hille for his relentless enthusiasm and support. Sander, I appreciate how you encouraged my initially bold proposal for this project and how you have been excited about making advances along the way. A part of this thesis was written under the restrictive limitations of being in quarantine and our weekly Skype meetings played a tremendous role in my progress through the last phase of writing this work. I would also like to give a big thank you to my dear friend Emma Lucas for her extremely thorough proof-reading and her many helpful remarks about the text.

I would like to thank my friends and all the members of my family for everything that you mean to me. This space cannot do justice.

Last, I would like to thank my colleagues and students at Haarlemmermeer Lyceum. It was an honor and a privilege to return to my former school for two days a week beside my degree to teach mathematics to the school's most gifted young mathematicians. I hope I have been able to inspire my students just as much as some of my former teachers have inspired me. Now that I am leaving for Oxford I say goodbye to the school for a second time with great gratitude.

Contents

Preface

"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

- G.H. Hardy, *A Mathematician's Apology*

When I was looking for a dissertation project at the end of last year, dr. Floske Spieksma suggested that doing a project on Piecewise Deterministic Markov Processes with dr. Sander Hille would be just my cup of tea. I had never heard of PDMPs, but the name sounded exotic, so I browsed the library to find out what they were. I was intrigued by the strange interplay of dynamics and stochastics and decided to contact dr. Hille.

From the outset, dr. Hille was incredibly enthusiastic. The original idea was to extend some results about ergodic decompositions associated to a perturbed predator-prey model that had been analyzed in Taleb Alkurdi's PhD thesis [1] several years prior under dr. Hille's supervision. At the same time, I took dr. Hille's mathematical biology class in the Dutch Mastermath programme, where we discussed, from a purely analytical perspective, the same predator-prey model as Alkurdi did, save we looked at the conditions necessary for the model to exhibit an asymptotically stable limit cycle, through Hopf bifurcation.

Alkurdi had considered random perturbations at deterministic times of the predator-prey model when it exhibits stable *equilibria*. I became interested in learning to which extent Alkurdi's results would carry over to the situation with a limit cycle and started generalizing the proofs. G.H. Hardy's quote summarizes my research experience in those first few weeks. I tried to make rigorous estimates to control the deterministic dynamics, but my analytical tools were too limited. After working for a month or two, I had all definitions and machinery in place to formulate several results from what is now Chapter 1 of this thesis. The results seemed to carry over, with painstaking effort. Dr. Hille suggested that perhaps there was an easier proof route to show persistence of stability, avoiding the mathematical acrobatics needed to generalize Alkurdi's work. He suggested I investigated to which extent a property of a Markov operator on measures called "Uniform Equicontinuity on Balls" could be used in this setting. The idea was that this UEB property was weaker than the properties needed for the approach taken by Alkurdi and may be better suited for the complicated dynamics induced by the limit cycle.

The UEB property, after a few modifications, worked *like a charm*. While rigorously making the estimates was difficult and forced me to learn a great deal of deep measure theory I had not encountered so far, the results followed naturally, with much less restrictive assumptions than I had needed previously. In retrospect, I am glad that I needed to learn this theory. This was probably also the busiest time of the academic year, where I juggled my teaching obligations at Haarlemmermeer Lyceum and my graduate studies. Many of the results in Chapter 2 were first formulated during lunch breaks on the whiteboard of room 218 at the Lyceum, to the dismay of my students who came in just after the break. It definitely gave them an impression of the hard work that is doing mathematics at a graduate level.

The last part of the project contained most hurdles. I discovered a grave mistake in my estimates for the deterministic dynamics that jeopardized the proofs of Chapter 1. I worked overtime to find more, better, inequalities that could help fix this mistake. G.H. Hardy was right again. Dr. Hille generously devoted much time to understand the problem and discovered that the deterministic dynamics were by themselves incompatible with the arguments employed by Alkurdi. I had to settle for allowing that perturbation times be stochastic, in which case the arguments could be extended.

Simultaneously, I had been working on applying the theory I had developed to a specific implementation of the perturbed predator-prey model. The biggest difficulty was to show that the convergence rate I needed in the proofs related to the UEB property actually held. G.H. Hardy's quote shows its relevance a third time here: I spent a long time in the library, reading little-known papers about *contraction analysis* that proved to be very useful. I discovered how this field I had never heard about addressed exactly the inequalities I wanted to use. While I had to learn a great deal about measure theory to implement the UEB property, I now had to learn a lot about the deterministic dynamics. I am sure it made me a better mathematician.

Sometimes, the quests we - analysts - undertake to find theorems we want to use and cannot prove have serendipitous by-effects. Dr. Hille was happy about finding new tools to address problems related to PDMPs and I received the opportunity to learn about many new fields of mathematics that in some mysterious way find their traces to PDMPs. I consider myself lucky to have experienced this one other time. While doing readings for a paper in mathematical biology with Dr. Hille this semester, I came across the research of Professor Ruth Baker's group in Oxford, which captivated me. After reaching out and deciding I wanted to work in her lab, I applied for a DPhil position in her group and was admitted. The beauty of mathematics is that it implants ideas so strongly in one's mind that it seems as though it leads its own life through its practitioners.

Simón Martina-Pérez
June 2020

Introduction and Motivating Example

Dynamical systems have become a vital pillar in the mathematical modeling of phenomena in the natural sciences. Examples of models range from biology, where important insights have come from dynamical systems models of population genetics [3], epidemiology [4] and neuroscience [5], to chemistry, where fundamental advances have been made in reaction-diffusion systems [6] and pattern formation [7] to name but a few examples. The flexility of using dynamical systems, particularly ODE and PDE models, make the field of dynamical systems an important part of modern applied mathematics.

However, purely deterministic models of real-life phenomena are reductive, as they fail to capture the randomness that is inherent to many processes in real life. Such cases are abundant, one salient example is the modeling of gene expression in cells [8]. More reasons exist to incorporate randomness in simulation. Incorporating randomness into the dynamics can facilitate to take into account measurement errors or parameter mis-estimation (see for instance [9] for a treatment of uncertainty of parameter values when provided with empirical data). It it thus widely accepted that realistic models of the world ought to include some random aspect. This gives rise to a delicate interplay between the deterministic dynamics, a field of study of their own, and the model for stochastic perturbations. Some of the rigorous first advances in incorporating randomness in differential equations include work on *Stochastic Differential Equations* pioneered by Itô. We call these models noisy models and they form an important part of the models that we will compare our results to. These models are still wide-spread in many areas of applied mathematics, for instance in financial mathematics [10].

Piecewise Deterministic Markov Processes

A shortcoming of noisy differential equations is that they fail to capture instantaneous, random and state-dependent interventions in the process, which we will refer to as *jumps*. For this reason, Davis [11] pioneered the study of Piecewise Deterministic Markov Processes. In this approach, a process is defined that follows a deterministic *flow* during a random amount of time, before *jumping* randomly to a new state according to some pre-defined probability kernel. When we refer to a PDMP, this is the setting that we will have in mind. It is worth noting that a more general definition of PDMPs exists, but we will not treat those models in this work. PDMPs have received much attention in modeling across disciplines, important examples being neuron models describing the propagation of an action potential along a nerve fiber [12] or portfolio management in financial mathematics [13].

Since the work of Davis, much has been written about natural generalizations of the types of processes that exhibit the random behavior that Davis described. In the literature such processes are called *Stochastic Hybrid Systems*, or *random dynamical systems*. A notable extension to the work of Davis, who only considered deterministic evolution in between random jumps, is the work of Julien Bect, who in his doctoral thesis [14] introduced an abstract construction of piecewise-diffusive processes. Many more extensions and variations exist, but we shall not discuss them in this work. The interested reader is referred to Davis' 1993 work [15].

Stability of Randomly Perturbed Dynamical Systems

Given a non-linear dynamical system, it is possible to characterize its *qualitative* behavior when time evolves. There exists a wide repertoire of tools in dynamical systems theory to show what kind of properties the solutions of a given initial value problem have. With these tools, one can establish for instance if all solutions will converge to a single equilibrium, to a periodic solution, or to a *chaotic attractor*. From a modeling perspective, it is important to understand these dynamics, as the goal of a model is to capture the behavior of some phenomenon. An example of this is the Fitzhugh-Nagumo model, which models the time-periodic electrical impulses between neurons [5].

When the complicated behavior of a non-linear system is randomly perturbed, a natural question that arises is to which extent the randomness introduces changes the qualitative behavior of deterministic dynamics. Imagine that perturbations change the dynamics so radically that a periodic solution no longer exists with high probability. A priori, such changes in behavior are possible given the delicate interplay between the deterministic and stochastic dynamics. In such a case, introducing randomness in the model may destroy the properties of the model one was interested in. This motivates investigating to which extent stable equilibria of the dynamics remain in some sense stable when randomness is introduced. We will be concerned with the following question: granted that the deterministic system converges to some *attractor* in the state space, does it follow that the perturbed system is with high probability close to that attractor too when time progresses? More strongly, if the process has a stationary distribution, which conditions are necessary to ensure that the support of this distribution is somehow close to this attractor? This is what we informally define as *persistence of stability*.

Such questions are addressed in the setting of PDMPs by Taleb Alkurdi in his doctoral thesis [1], where the persistence amounts to finding an ergodic measure supported in a small region around the asymptotically stable equilibrium of a differential equation in a Banach space. Others, most notably Bressloff et al. [16] look at the probability that a differential equation driven by noise remains close to its limit cycle. They call these processes *noisy limit cycle oscillators*. In a similar vein, Zou and Fan [17] look at the idea of a *stochastic Hopf bifurcation*, which they informally define as

Definition 0.0.1. A Stochastic Hopf Bifurcation occurs when. the shape of stationary distribution for the stochastic system changes from peak-like into crater-like.

Zou and Fan [17] show under which conditions a specific perturbed ODE model undergoes this type of behavior. The following plot provides a visualization of the *crater-like* support of the stationary distribution of the process.

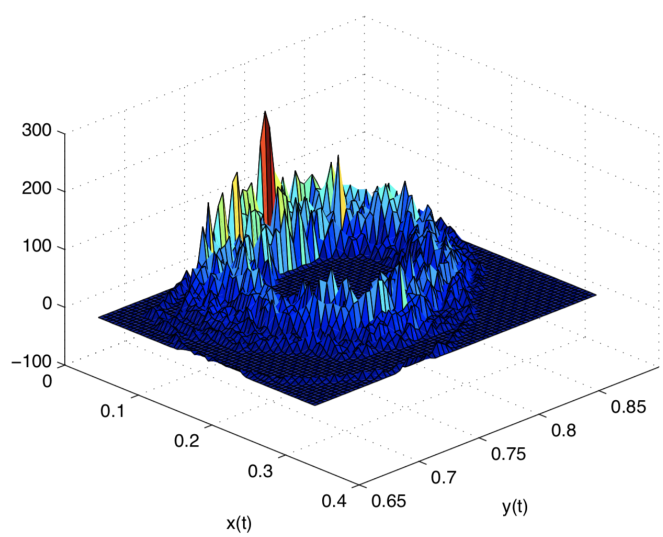

Figure 1: From [17], the stationary distribution of the process has a crater-like support.

From Noise to Jumps: a Motivating Example

The results of Zou et al. [17] and Bressloff et al. [16] concern noisy processes, which allows to express deviations from the limit cycle in terms of stochastic differential equations. The analysis of these equations shows under which conditions the desired behavior of the system takes place. Noticeably, the equations themselves give conditions for the size of perturbations. In the setting of PDMPs, such techniques no longer suffice: given that the process jumps instantaneously at random time intervals, the dynamics of the process cannot be expressed in a stochastic differential equation anymore. Rather, the dynamics of PDMPs are much better understood through their associated Markov operator. This is the approach taken by Alkurdi in [1].

However, the main topic that will be explored in this work is the behavior of the Markov operator given that the deterministic dynamics do not converge to a single point. The techniques used in the literature to analyze persistence of stability in PDMPs, most notably in [1], only apply in the case of convergence to a single point and fail when the system has an asymptotically stable limit cycle. That means that a central part of this work will be to thoroughly understand the asymptotic behavior of solutions of the deterministic system and use these to find novel ways to describe the behavior of the PDMP. While the abstract conditions are easily formulated, it will remain a challenge in applications to verify this, as will become apparent in chapter 3 of this work.

Nevertheless, heuristic simulations suggest that persistence of stability still exists in a PDMP when the underlying deterministic system has an asymptotically stable limit cycle. We heuristically simulate the following predator prey model, known as the Rosenzweig-MacArthur model, which will be thoroughly analysed in Chapter 3 of this thesis, using the code in Appendix A.

$$\dot{v} = rv\left(1 - \frac{v}{K}\right) - \frac{av}{b+v} \cdot p,$$
$$\dot{p} = -dp + h \cdot \frac{av}{b+v} \cdot p.$$

The simulation is run by fixing $a = 1, d = \frac{1}{2}, h = 1, r = K = 5$ and varying b such that the non-trivial equilibrium becomes unstable, giving rise to a limit cycle in the deterministic dynamics. One can compute that the deterministic model undergoes a Hopf bifurcation at $b = \frac{5}{3}$. By heuristically choosing small bounds for the size of the random perturbations, the following density plots in Figures 2, 3 and 4 arise. The plots seem to suggest that, provided the perturbations are chosen *small*, the shape of the invariant distribution follows the qualitative behavior of the deterministic system. More specifically, in the last plot, which passes the bifurcation point for the parameter b, the crater collapses and we obtain a peaked distribution around the now stable equilibrium of the system. This motivates the main question of this thesis: under which conditions does *persistence of stability* arise in a PDMP where the deterministic dynamics have an asymptotically stable periodic limit cycle?

Roadmap

Chapter 1 generalizes results from [1] by strengthening the arguments made in that work leading to the main theorem. In particular, we describe how perturbations can be made at stochastic instead of deterministic times and make rigorous estimates to characterize the action of the Markov operator on the support of the invariant measure. Basic properties about the associated Markov operator for the process are shown. In particular, the model for the jump and time kernels of the process is linked to the continuity of the operator. An in-depth discussion of the different modeling possibilities will follow, and it is shown that conditions can be substantially weaker than previously used in the literature. In Chapter 2 we turn our attention to the case of the deterministic dynamics have an asymptotically stable limit cycle. We describe the precise properties of such a limit cycle in Polish state space. Further, we investigate the link between the so-called UEB property and persistence of stability, making use of results by Szarek from [2]. We first recast the original Markov operator into a more mathematically tractable form and use this to make rigorous estimates leading up to the UEB property. As far as we are aware, this is the first work where the UEB property has been established for a class of PDMPs. It is also the first work to establish persistence of stability in a non-diffusive PDMP where there is a limit cycle. In the proof of the main theorem establishing persistence of stability, we give an explicit construction of an invariant region that is crater-like and where the stationary distribution is shown to have support. In chapter 3, we analyze the Rosenzweig-MacArthur predator-prey model to illustrate how to verify in practice the different model assumptions. Of special difficulty will be establishing the desired exponential convergence to the stable limit cycle. For reference in future applications, we review several techniques to show, notably from Floquet theory and *contraction analysis*. We propose several modeling options for the perturbations.

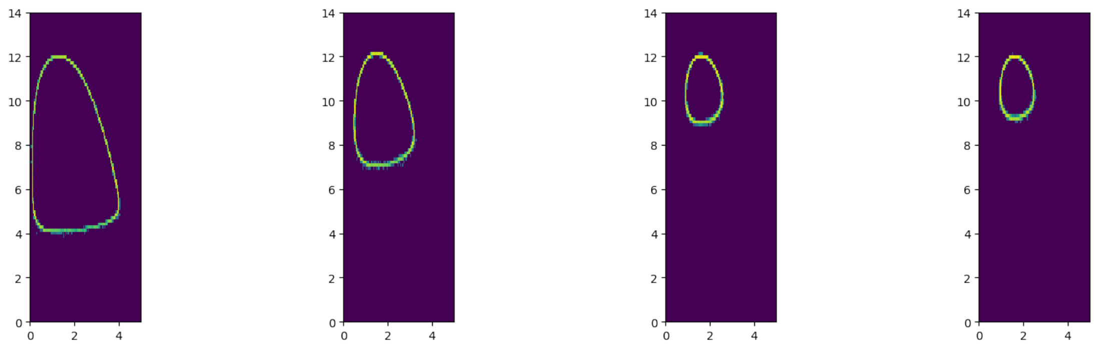

Figure 2: From left to right: $b = 1$, $b = 1.3$, $b = 1.5$, $b = 1.6$.

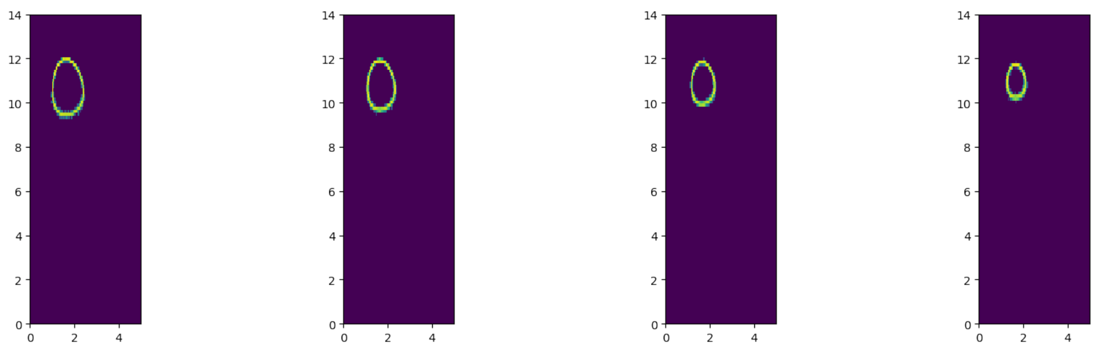

Figure 3: From left to right: $b = 1.61$, $b = 1.62$, $b = 1.63$, $b = 1.64$.

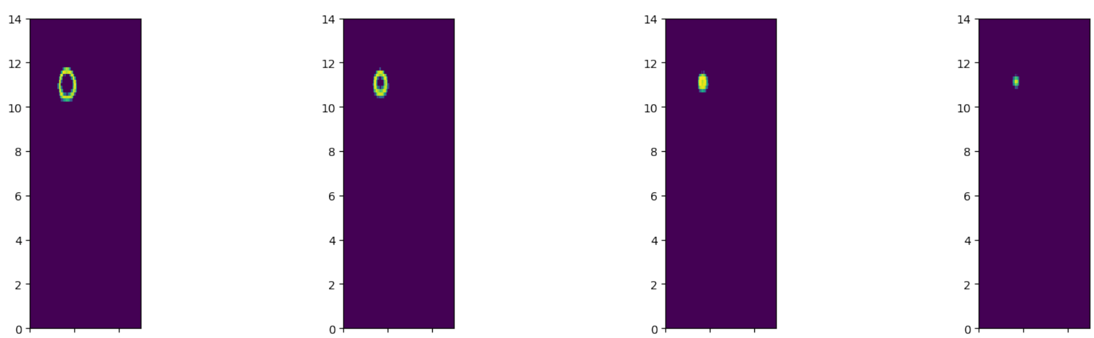

Figure 4: From left to right: $b = 1.65$, $b = 1.655$, $b = 1.665$, $b = 1.67$.

Chapter 1

Persistence of Stability in a Perturbed Dynamical System

The goal of this chapter is to generalize existing results for the existence of ergodic measures around equilibria of dynamical systems in metric spaces. The existence of invariant measures for Markov operators associated to perturbed dynamical systems is rich and many results exist. For an exposition, see for example [1]. We follow the set-up common in the literature and let S be a Polish space, d a metric that metrizes the topology on S such that (S, d) is complete and separable, and $(\Phi_t)_{t\geq 0} : S \to S$ be a semigroup of continuous maps parametrized by $t \geq 0$ that models the deterministic *flow* of the dynamical system. We assume the motion maps $t \mapsto \Phi_t(x)$ to be continuous for every $x \in S$. For each $x \in S$ we define p_x as the probability distribution of the time until the next jump when the process starts at x. Given a random time t, the process will evolve for t units of time until it reaches $\Phi_t(x)$, from where it will instantaneously jump to a new state at time t according to law $q_{\Phi_t(x)}$: for every $y \in S$, $A \subset S$, the jump probability from y into A is given by $q_y(A)$. This gives rise to the following operator P for an initial measure μ on S

$$(P\mu)(\bullet) = \int_X \int_{\mathbb{R}_+} \int_X 1_\bullet(y) q_{\Phi_t(x)}(dy) p_x(dt) \mu(dx). \tag{1.1}$$

The interpretation of equation (1.1) is that if μ is the distribution for the state of the system initially, then $P\mu$ is the probability distribution for the state of the system immediately after the first jump. Alkurdi [1] showed that for fixed jump times the Markov operator shows *persistence of stability* in a class of linear differential equations: where there is a stable point for the differential equation, for the Markov operator there exists an invariant subset B^* around this stable point such that an invariant and ergodic measure around this stable point is supported on B^*. This measure is asymptotically stable.

At this point, we do not bother about conditions about the measure-valued maps $x \mapsto p_x$ and $x \mapsto q_x$ such that the integrals in 1.1 are defined. Later on, these maps will be assumed to be Lipschitz continuous for the total variation or the Dudley norm, which both suffice.

The question we will be interested in is the behavior of the Markov operator in (1.1) around attractors of the dynamical system. Specifically, we wish to find a region around the stable equilibrium where an invariant and ergodic measure is supported and investigate the properties of this region. Generalizing the approach of Alkurdi, we will allow stochastic interventions at random times, only assuming that there exists $\Delta t > 0$ such that $p_x([0, \Delta t]) = 0$ for all x. That is, jumps cannot happen arbitrarily fast, uniformly in space.

1.1 Model Set-Up and Assumptions

A priori, specific properties of the operator in (1.1) will be difficult to prove beyond very general results. The reason for this is that each of the different model components influence the behavior of the operator. One can imagine that, for instance, the convergence behavior of the dynamical system to its asymptotic solution and the probability distributions for the jump times and jump locations will have a dramatic influence on the persistence of stability. In one way or another, it will be necessary to require some regularity on these quantities to ensure that an invariant region for the operator exists and displays nice enough properties. This will be made precise in this section.

1.1.1 Assumptions on the Dynamical System and Perturbations

In the remainder of this section we will describe in more detail the assumptions for the dynamical system and for the perturbations that will be used in this thesis. Some core notation will be developed.

Assume that there exists a set $B \subset S$ such that B is closed and invariant for the flow Φ, that is, for all $t \geq 0$, $\Phi_t(B) \subset B$. Moreover, assume that there exists a unique $x^* \in B$ such that $\Phi_t(x^*) = x^*$ for all $t \geq 0$, with other words, x^* is an equilibrium of the deterministic flow Φ and it is unique. We would also like to assume that there are no periodic solutions inside B, that is, there exist no $y \in B$, $\tau > 0$ such that $\Phi_{n\tau}(y) = y$ for all $n \in \mathbb{N}$. Assume lastly that for every $y \in B$, it holds that $d(\Phi_t(y), x^*) \to 0$ as $t \to \infty$, with other words, x^* is an *asymptotically stable* equilibrium. Note that these properties are ubiquitous in metric analysis. In metric analysis, we refer to the set

$$B_{x^*} = \{y \in S : d(\Phi_t(y), x^*) \to 0, \text{ as } t \to \infty\}$$

as the *basin of attraction* of x^*. In our setting, $B = B_{x^*}$. Furthermore, we would like to make the following assumptions about the PDMP.

Assumption 1-$\mathcal{S}$. We assume that there exists $L_\Phi > 0$ such that for all $t \geq 0$ and all $x, y \in S$:

$$d\left(\Phi_t(x), \Phi_t(y)\right) \leq L_\Phi d(x, y).$$

If $L_\Phi < 1$, then Assumption 1-$\mathcal{S}$ describes how the deterministic flow Φ is a contraction. In specific settings, such as when the flow Φ is generated by an ODE in $\mathbb{R}^n$, it can be shown that the existence of an asymptotically stable equilibrium causes the flow to be a contraction. This builds on non-trivial results that will be exposed in Chapter 3. We recall that the Hausdorff-semidistance δ_H between two sets $E, F \subset S$ is given by

$$\delta_H(E, F) \stackrel{\text{def}}{=} \sup_{x \in E} \inf_{y \in F} d(x, y), \tag{1.2}$$

and the Hausdorff distance is given then by

$$d_H(E, F) \stackrel{\text{def}}{=} \max\left(\delta_H(E, F), \delta_H(F, E)\right). \tag{1.3}$$

We would like to place the following assumptions on the perturbations.

Assumption 1-$\mathcal{P}$. The map $x \mapsto q_x$ is Lipschitz continuous with respect to the total variation norm, with Lipschitz constant L'_q. Moreover, there exists $L_q \geq 0$ such that $d_H\left(\text{supp } q_x, \text{supp } q_y\right) \leq L_q d(x, y)$ for all $x, y \in S$.

Assumption 2-$\mathcal{P}$. There exists a continuous, non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $f(t) \leq C_f t$ for all $t \geq 0$, for some $C_f > 0$, with the property that

$$d\left(\Phi_t(x), x\right) \leq f(t)$$

for all $x \in S$.

Assumption 3-$\mathcal{P}$. The map $x \mapsto p_x$ is Lipschitz continuous with respect to the total variation norm, with Lipschitz constant L'_p. Moreover,there exists $L_p \geq 0$ such that $d_H\left(\text{supp } p_x, \text{supp } p_y\right) \leq L_p d(x, y)$.

Assumption 4-$\mathcal{P}$. L'_p, L'_q, L_Φ satisfy

$$L'_p + L'_q L_\Phi < 1.$$

Moreover, L_p, L_q, C_f and L_Φ satisfy

$$L_q\left(L_\Phi + C_f L_p\right) < 1.$$

Assumption 5-$\mathcal{P}$. There exists $R^* > 0$ such that the closed ball $B^* = B[x^*, R^*]$ of radius R^* around x^* is contained in B and

$$\text{diam supp} q_x < \left(1 - \left(L_q L_\Phi + \max\{L_q C_f, 1\} L_p\right)\right) R^*$$

for all $x \in B^*$.

Remark 1: In some cases, like [1], $p_x = \delta_{\Delta t}$ for all $x \in S$, meaning that there is a deterministic jump time. This causes $L_p = 0$. In our setting, p_x is allowed to vary slightly depending on the position, controlled by the constant L_p. From a modeling perspective, this can describe uncertainty of measurement or other, place-dependent, effects for the perturbations to take place.

Remark 2: Assumption 4-$\mathcal{P}$ can fail when looking at flows that have a periodic solution, or an asymptotically stable limit cycle. This is for the following reason. Let $\gamma : \mathbb{R}_+ \to S$ be a continuous, periodic function (with period τ) such that it is a solution to the dynamical system. γ gives rise to a closed curve $\Gamma \subset S$. Let $x, y \in \Gamma$. If there exists L_Φ such that $d(\Phi_t(x), \Phi_t(y)) \leq L_\Phi d(x,y)$ for all $t \geq 0$, necessarily $L_\Phi \geq 1$. Indeed, $d(x,y) = d(\Phi_{n\tau}(x), \Phi_{n\tau}(y))$ for every integer n. If S is a vector space or Banach space and supp $q_x = x + C$ with C some fixed set, then

$$d_H(\text{supp } q_x, \text{supp } q_y) = d_H(x + C, y + C) = d(x,y),$$

which means that apparently one cannot always allow $L_q < 1$, and Assumption 4-$\mathcal{P}$ does not hold. In general, thus, if $L_\Phi > 1$, then one needs to choose q_x such that $L_q < 1$. As noted, getting L_q to be arbitrarily small may not be an available option, depending on the specific structure of the problem at hand. The implications of this are severe: in the case of a limit cycle, Assumption 4-$\mathcal{P}$ cannot be true and as such the techniques developed as an extension to those in [1] cannot be extended to the case of there being an asymptotically stable limit cycle. With other words, non-expansiveness is not the suitable property to investigate in such cases.

Remark 3: We remark that Assumption 2-$\mathcal{P}$ always is true for the solutions of $\dot{x} = f(x)$, when f is a Lipschitz continuous function on a bounded, positively invariant domain $S \subset \mathbb{R}^n$.

1.1.2 Properties of the operator P

Many results that one could wish to use for proving existence and uniqueness of ergodic measures for a given operator on measures require certain conditions on the operator. The study of the operator P gives rise, in abstract, to many interesting properties. Our goal is to first explore how several abstract properties of the perturbations give rise to such properties. In doing this, we show how relaxing or tightening assumptions on the model changes the properties of the operator.

Before introducing our results, let us introduce the following notation. Let $\mathcal{M}(S)$ be the space of Borel measures on S, $\mathcal{M}^+(S)$ the set of positive Borel measures on S and $\mathcal{P}(S)$ the set of Borel probability measures on S. We will denote $\mathcal{M}(S)$, equipped with the total variation norm, as $\mathcal{M}(S)_{\text{TV}}$ and similarly for the other spaces of measures. We let $C_b(S)$ be the collection of bounded and continuous functions $S \to \mathbb{R}$, $\text{BM}(S)$ the collection of bounded and Borel measurable functions on S and $\text{BL}(S)$ the set of bounded and Lipschitz continuous functions on S. Let finally the deterministic flow be denoted by $\Phi : \mathbb{R}_+ \times S \to \mathbb{R}$ given by $(t, x) \mapsto \Phi_t(x)$.

Recall (for example from [1]) that a Markov operator P is said to be *regular* if there exists an operator U which maps $\text{BM}(S)$ into itself such that $\langle Uf, \mu \rangle = \langle f, P\mu \rangle$ for all $f \in \text{BM}(S)$ and $\mu \in \mathcal{M}^+(S)$. Here, $\langle f, \nu \rangle = \int_S f(x)\nu(dx)$. In that case, U is unique and is called the dual of P. When P is regular and its dual maps $C_b(S)$ into itself, it is called *Markov-Feller*, when it maps $\text{BM}(S)$ into $C_b(S)$, it is called *strong-Feller* and when the map $x \mapsto P\delta_x$ from S to $\mathcal{M}(S)_{\text{TV}}$ is continuous it is called *ultra-Feller.*

Define the family of operators $\{\mathcal{Q}_t\}_{t \in \mathbb{R}_+}$ and the operator $\hat{P}$ as follows. For $t \in \mathbb{R}_+$, let $\mathcal{Q}_t : S \to \mathcal{P}(S)$ such that $x \mapsto q_{\Phi_t(x)}$ and $\hat{P} : S \to \mathcal{P}(\mathbb{R}_+)$ be given by $x \mapsto p_x$.

Proposition 1.1.1. *If the family $\{\mathcal{Q}_t\}_{t \in \mathbb{R}_+}$ is equicontinuous with respect to $\|\cdot\|_{TV}$ on $\mathcal{P}(S)$ and $\hat{P}$ is continuous with respect to $\|\cdot\|_{TV}$ on $\mathcal{P}(\mathbb{R}_+)$, then P is strong-Feller.*

Proof: Let $f \in \text{BM}(S)$. Assume that $f \neq 0$ for else there is nothing to prove. Recall that the dual operator U of P is defined by the relation $\langle f, P\mu \rangle = \langle Uf, \mu \rangle$ for all $f \in \text{BM}(S), \mu \in \mathcal{P}(S)$. Recall that

$$\langle f, P\mu \rangle = \int_S \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt) \mu(dx),$$

whence for $x \in S$, $Uf(x) = \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt)$. For the boundedness of Uf, note that - trivially,

$$|Uf(x)| = \left| \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt) \right| \leq \|f\|_\infty.$$

Hence, $\|Uf\|_\infty \leq \|f\|_\infty$. Now, fix $x \in S$ and let $\epsilon > 0$ be given. Let δ_1 be such that $\|p_x - p_z\|_{\mathrm{TV}} < \frac{\epsilon}{2\|f\|_\infty}$, whenever $d(x,z) < \delta_1$. Let δ_2 be such that $\|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} < \frac{\epsilon}{2\|f\|_\infty}$, for z such that $d(x,z) < \delta_2$. Let $t \geq 0$. Let finally $\delta = \min\{\delta_1, \delta_2\}$. Now,

$$\begin{aligned}
|Uf(x) - Uf(z)| &= \left| \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt) - \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(z)}(dy) p_z(dt) \right| \\
&\leq \left| \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt) - \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(z)}(dy) p_x(dt) \right| \\
&\qquad + \left| \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(z)}(dy) p_x(dt) - \int_{\mathbb{R}_+} \int_S f(y) q_{\Phi_t(z)}(dy) p_z(dt) \right|.
\end{aligned}$$

Let us treat each of the terms separately. Commence with the first term and note that it satisfies

$$\begin{aligned}
\left| \int_{\mathbb{R}_+} \int_S f(y) (q_{\Phi_t(x)} - q_{\Phi_t(z)})(dy) p_x(dt) \right| &\leq \int_{\mathbb{R}_+} \|f\|_\infty \cdot \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} \; p_x(dt) \\
&\leq \|f\|_\infty \sup_{t \geq 0} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} && \text{(since } p_x \in \mathcal{P}(\mathbb{R}_+)\text{)} \\
&< \frac{\epsilon}{2}. && \text{(by equicontinuity of } \mathcal{Q}_t\text{)}
\end{aligned}$$

The second term satisfies

$$\begin{aligned}
\left| \int_{\mathbb{R}_+} \int_X f(y) q_{\Phi_t(z)}(dy) (p_x - p_z)(dt) \right| &\leq \int_{\mathbb{R}_+} \|f\|_\infty |p_x - p_z|(dt) && \text{(since } q_{\Phi_t(z)} \in \mathcal{P}(S)\text{)} \\
&= \|f\|_\infty \cdot \|p_x - p_z\|_{\mathrm{TV}} \\
&< \frac{\epsilon}{2}. && \text{(by continuity of } \mathcal{P}\text{)}
\end{aligned}$$

So $|Uf(x) - Uf(z)| < \epsilon$ whenever $d(x,z) < \delta$, $Uf \in \mathcal{C}_b(S)$ and P is strong-Feller.

□

In general, the above conditions may be difficult to check or restrictive. When one is not able to control the behavior around the attractor well, the condition that

$$\sup_{t \geq 0} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} < \epsilon$$

whenever x, z are close enough may not be feasible in practice. As an example, consider a deterministic dynamical system given by an ODE in $\mathbb{R}^n$. Even though it is known that the solution to the ODE is continuous in the initial conditions, estimates such as Gronwall's inequality do not give adequate bounds on the distance between the flows when t increases. This is a problem, since then it becomes virtually impossible to make estimates about the distance between the jump distributions. For this sort of condition to hold, we need to make more assumptions on the dynamical system to relax the conditions on the perturbations. To address this, Szarek and others in the literature (for instance see [18]) have defined a property called *contractivity on average*. Let $x, z \in S$. The model is contractive on average in $\|\cdot\|_{\mathrm{TV}}$-norm if

$$\int_{\mathbb{R}+} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} p_x(dt) \leq \theta d(x,z)$$

for some $0 < \theta < 1$, uniform for all $x \in S$. One might wonder under which conditions such contractivity on average holds. It turns out that this property is natural and is used often in the literature, but in practice is difficult to demonstrate even for basic examples. This is a problem we will also encounter in Chapter 3 of this thesis.

Nevertheless, good properties of the Markov operator P can be proven with Assumptions 1-$\mathcal{P}$ through 5-$\mathcal{P}$. This is shown by the following result.

Proposition 1.1.2. *Under Assumptions 1-$\mathcal{P}$ and 3-$\mathcal{P}$, P is strong-Feller and its dual map U maps* $\mathrm{BM}(S)$ *to* $\mathrm{BL}(S)$.

Proof: From the estimate in Proposition 1.1.1, we bound

$$\begin{aligned}|Uf(x) - Uf(z)| \leq &\left|\int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(x)}(dy) p_x(dt) - \int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(z)}(dy) p_x(dt)\right| \\ &+ \left|\int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(z)}(dy) p_x(dt) - \int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(z)}(dy) p_z(dt)\right|.\end{aligned}$$

The first term can be bounded as

$$\begin{aligned}\left|\int_{\mathbb{R}_+}\int_S f(y)(q_{\Phi_t(x)} - q_{\Phi_t(z)})(dy) p_x(dt)\right| &\leq \|f\|_\infty \int_{\mathbb{R}_+} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}} p_x(dt) \\ &\leq \|f\|_\infty \int_{\mathbb{R}_+} L'_q d(\Phi_t(x), \Phi_t(z)) p_x(dt) \\ &\leq \|f\|_\infty \cdot L'_q L_\Phi d(x,z).\end{aligned}$$

We bound the second term as

$$\begin{aligned}\left|\int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(z)}(dy) p_x(dt) - \int_{\mathbb{R}_+}\int_S f(y) q_{\Phi_t(z)}(dy) p_z(dt)\right| &\leq \|f\|_\infty \int_{\mathbb{R}_+} |p_x - p_z|(dt) \\ &\leq \|f\|_\infty \|p_x - p_z\|_{\mathrm{TV}} \\ &\leq L'_p \cdot \|f\|_\infty d(x,z).\end{aligned}$$

We conclude for $K_* = \|f\|_\infty \cdot (L_p + L_q L_\Phi)$, $|Uf(x) - Uf(z)| \leq K_* d(x,z)$ and this proves the claim.

$\square$

Remark 1: The procedure of Proposition 1.1.2 only considers $\|q_x - q_{x'}\|_{\mathrm{TV}}$, where the distances no longer depend on the time. This fully decouples the jumps from the jump times and facilitates in checking model assumptions.

Remark 2: Instead of using the Lipschitz property of the maps $x \mapsto p_x$ and $x \mapsto q_x$, the estimates in the proof of Proposition 1.1.2 immediately suggest that P is Markov-Feller whenever the maps $x \mapsto p_x$ and $x \mapsto q_x$ are continuous with respect to the total variation norm. That is, there is no need for the Lipschitz property if the Markov-Feller property needs to be shown.

1.2 Persistence of Stability around the Equilibrium

Following the Markov operator approach proposed by Alkurdi, we will prove in this section the existence of an invariant ergodic measure with support in B^* that is asymptotically stable. This is what is called in the literature 'persistence of stability'.

1.2.1 Existence of a P-invariant neighborhood

An essential ingredient for the proof of the existence of the ergodic measure is the existence of a P-invariant neighborhood. If such a neighborhood would not exist, then iterating the operator P would not yield a suitable limit for the measure and so this approach would fail. For our model description, this is a direct consequence of ([1], Proposition 4.3.3) with the condition $\theta(1+L) < 1$ in the condition of the proposition replaced by the conditions on $L'_p, L'_q, L_\Phi, L_q, L_p$ in Assumption 4-$\mathcal{P}$. We can thus immediately conclude that B^* is P-invariant.

1.2.2 Non-expansiveness of P

This section will be the first step in finding an invariant measure for P on the invariant region B^* that was defined above. An important ingredient in establishing the existence of an invariant measure is the non-expansiveness of the Markov operator P. We state and prove an analogous result to ([1], Lemma 4.4.1).

Recall that P is strong-Feller, meaning that its dual operator U maps bounded and continuous functions f into the space of bounded Lipschitz continuous functions. Let L' be the Lipschitz constant of the map $\mathcal{P}$ from assumption 4-$\mathcal{P}$.

Lemma 1.2.1. *If $f \in \mathrm{BL}(B^*)$, then $Uf \in \mathrm{BL}(B^*)$ with $|Uf|_{Lip} \leq \|f\|_\infty$. Thus, $\|Uf\|_{max} \leq \|f\|_{max}$.*

Proof: In fact, any bounded and measurable function f is mapped onto the bounded Lipschitz continuous functions by U. By the proof of Proposition 1.1.2, for $x, z \in B^*$, one obtains the estimate

$$|Uf(x) - Uf(z)| \leq \|f\|_\infty K_* d(x, z)$$

for some suitable constant K_* such that

$$K_* \leq L_p + L_q L_\Phi < 1,$$

where the last, strict, inequality holds by Assumption 4-$\mathcal{P}$. Consequently, $\|Uf\|_{\max} \leq \max(1, K_*\|f\|_\infty)\|f\|_{\max}$. This finishes the proof of the claim.

□

Proposition 1.2.1. *For $\mu, \nu \in \mathcal{M}^+(B^*)$, $\|P\mu - P\nu\|_{FM} \leq \|\mu - \nu\|_{FM}$. Hence, P is non-expansive and in particular, satisfies the e-property.*

Proof: We may estimate using Lemma 1.2.1

$$|\langle f, P\mu - P\nu\rangle| = |\langle Uf, (\mu - \nu)\rangle| \leq \|Uf\|_{\max}\|\mu - \nu\|_{\mathrm{FM}} \leq \|f\|_{\max}\|\mu - \nu\|_{\mathrm{FM}},$$

since the Fortet-Mourier norm takes the supremum over the set $\{f \in \mathrm{BL}(B^*) : \|f\|_{\max} \leq 1\}$, we find that $\|P\mu - P\nu\|_{\mathrm{FM}} \leq \|\mu - \nu\|_{\mathrm{FM}}$, hence P is non-expansive, so P^n is non-expansive for $n \in \mathbb{N}$. We conclude that $\{U^n f : n \in \mathbb{N}\}$ is equicontinuous for every $f \in \mathrm{BL}(B^*)$ and therefore P satisfies the e-property.

□

1.2.3 Dynamics of the supports

From the properties of the dynamical system, it is possible to characterize the dynamics of the support of iterates of the Markov operator. This is interesting in its own right, since it characterizes the support of an invariant measure, but it is also necessary to conclude the property that P is globally concentrating, as will be shown in the following section. Following Alkurdi, we define and prove properties of the *support map* for the stochastic process. Define the support map as

$$\psi_p(E) := \overline{\bigcup_{x \in E} \mathrm{supp}(P\delta_x)}, \quad E \subset S.$$

From ([1], Proposition 2.3.7(ii)) we know that if P is Feller, then $\mathrm{supp}(P\mu) = \psi_p(\mathrm{supp}(\mu))$ for $\mu \in \mathcal{M}^+(S)$. That immediately implies that $\mathrm{supp}(P^n\delta_x) = \psi_p^n(\{x\})$ for all $x \in S$ and if $x \in \mathrm{supp}(q_x)$ for all x, then, letting $\tau_1, \ldots, \tau_n$ be run times of the system in between interventions also $\Phi_{\sum_{i=1}^n \tau_i}(x) \in \mathrm{supp}(P^n\delta_x)$ for $x \in S$.

The idea of the key result of this section is to show that Assumption 4-$\mathcal{P}$ implies that the support map ψ_p has a unique fix point in the class of closed subsets of B^*. From this, we will conclude that the support of the invariant measure will be the unique fixed point of the mapping ψ_p. We will introduce two lemmata characterizing supports, before proceeding with the proof that ψ_p has a unique fix point.

Lemma 1.2.2.

$$\text{supp}\,(P\delta_x) \subset \overline{\bigcup_{t \in \text{supp } p_x} \text{supp}(q_{\Phi_t(x)})}$$

Proof: Let $x_0 \in \text{supp}(P\delta_x)$. Then for every $\delta > 0$, $P\delta_x(B(x_0,\delta)) > 0$. Also,

$$P\delta_x(E) = \int_{\text{supp } p_x} q_{\Phi_t(x)}(E) p_x(dt),$$

so for all $\delta > 0$, $p_x(\{t \in \text{supp } p_x : q_{\Phi_t(x)}(B(x_0,\delta)) > 0\}) > 0$. Take $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$. There then exists $t_n \in \text{supp } p_x$ such that $q_{\Phi_{t_n}(x)}\left(B\left(x_0, \frac{1}{n}\right)\right) > 0$. So, $B\left(x_0, \frac{1}{n}\right) \cap \text{supp } q_{\Phi_{t_n}(x)} \neq \emptyset$, since $q_{\Phi_{t_n}(x)}$ is concentrated on its support. Pick $x_n \in B\left(x_0, \frac{1}{n}\right) \cap \text{supp } q_{\Phi_{t_n}(x)}$. Then, $x_n \to x_0$, while $x_n \in \bigcup_{t \in \text{supp } p_x} \text{supp } q_{\Phi_t(x)}$. This concludes the proof.

$\square$

Before we can provide a converse so that equality holds, we need one technical result.

Lemma 1.2.3. *Let S be a Polish space with admissible metric d. Let $\alpha \geq 0$. If $\mu \in \mathcal{M}^+(S)$, and $x_0 \in S$ such that $\mu(B(x_0, r)) > \alpha$ for some $r > 0$, then there exists $0 < r' \leq r$ such that $\mu(B(x_0, r')) > \alpha$, and $\mu(\partial B(x_0, r')) = 0$.*

Proof: For any increasing sequence $(r_n)_{n\in\mathbb{N}} \subset (0, r]$ such that $r_n \uparrow r$, $\mu(B(x_0, r_n)) \to \mu(B(x_0, r)) > \alpha$. Hence, there exists some $n_0 \in \mathbb{N}$ such that $\mu(B(x_0, r_{n_0})) > \alpha$. Put $r_0 \stackrel{\text{def}}{=} r_{n_0}$. Then, $r_0 > 0$ and $\mu(B(x_0, r')) > \alpha$ for all $r' \in [r_0, r]$.

The map $\psi : [r_0, r] \times S \to \mathbb{R}$ given by $(r', x) \mapsto \frac{d(x,x_0)}{r'}$ is separately continuous in r' and x, so it is jointly Borel measurable (see [19], Theorem 7.14.5, p. 129). It is also the case that

$$\mu(B(x_0, r')) = \int_S 1_{B(x_0,r')}(y)\mu(dy) = \int_S 1_{\left\{x: \frac{d(x,x_0)}{r'} < 1\right\}}(y)\mu(dy) = \int_S 1_{[0,1)}(\psi(r', y))\mu(dy).$$

Since ψ is jointly Borel measurable, $(r', y) \mapsto 1_{[0,1)}(\psi(r', y))$ is jointly Borel measurable. By the Fubini-Tonelli Theorem (see [19], Lemma 7.6.4, p. 93), the map $\underline{\phi}$ defined by $r' \mapsto \mu(B(x_0, r'))$ is Borel measurable on $[r_0, r]$. In a similar manner, one shows that $\overline{\phi} : r' \mapsto \mu\left(B[x_0, r']\right)$ is Borel measurable on $[r_0, r]$, where $B[x_0, r']$ is the closed ball centered at x_0 of radius r'. Put $\phi = \overline{\phi} - \underline{\phi}$.

According to Lusin's Theorem, there exists a compact subset $K \subset [r_0, r]$ of strictly positive Lebesgue measure, such that $\phi|_K$ is continuous. Since the Lebesgue measure is non-atomic, K must have at least denumerably many distinct points. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in K that consists of distinct points. Since K is a compact metric space, there is a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ that converges to $r^* \in K$ as $k \to \infty$.

From $(r_{n_k})_{k\in\mathbb{N}}$ we can construct a further subsequence, denoted the same for convenience, that is either strictly increasing, or strictly decreasing towards r^*. Put $S(x_0, r') = B[x_0, r'] \setminus B(x_0, r') = \{x \in S : d(x, x_0) = r'\}$.

First case, $r_{n_k} \uparrow r^*$. Define $A_1 \stackrel{\text{def}}{=} B(x_0, r_{n_1})$ and $A_k \stackrel{\text{def}}{=} B(x_0, r_{n_k}) \setminus B[x_0, r_{n_{k-1}}]$. Then,

$$B(x_0, r^*) = \bigsqcup_{k=1}^{\infty} A_k \sqcup S(x_0, r_{n_k}),$$

(where $\sqcup$ denotes disjoint union). So,

$$\mu(B(x_0, r^*)) = \sum_{k=1}^{\infty} \mu(A_k) + \mu(S(x_0, r_{n_k})) < \infty$$

Hence, $\lim_{k\to\infty} \mu(S(x_0, r_{n_k})) = 0$. Because $r_{n_k} \in L$ and $\phi|_K$ is continuous,

$$\mu(S(x_0, r^*)) = \lim_{k\to\infty} \mu(S(x_0, r_{n_k}) = 0.$$

Second case, $r_{n_k} \downarrow r^*$. Now define $A_1 = B(x_0, r_{n_1}) \setminus B[x_0, r_{n_2}]$ and $A_k = B(x_0, r_{n_k}) \setminus B[x_0, r_{n_{k+1}}]$. Then,

$$B(x_0, r^*) = \bigsqcup_{k=1}^{\infty} [A_k \sqcup S(x_0, r_{n_{k+1}})].$$

Hence, as above, $\lim_{k\to\infty} \mu(S(x_0, r_{n_k}) = 0$, yielding the conclusion that $\mu(S(x_0, r^*)) = 0$. Now observe that $\partial B(x_0, r^*) \subset S(x_0, r^*)$, so $\mu\left(\partial B(x_0, r^*)\right) = 0$.

□

Lemma 1.2.4.

$$\bigcup_{t\in \text{supp } p_x} \text{supp } q_{\Phi_t(x)} \subset \text{supp } P\delta_x$$

Proof: Let $t_0 \in \text{supp } p_x$, $x_0 \in \text{supp } q_{\Phi_{t_0}(x)}$. Let $\delta > 0$. Since $t_0 \in \text{supp } p_x$, $p_x\left(B\left(t_0, r\right)\right) > 0$ for all $r > 0$. If $q_{\Phi_{t_0}(x)}\left(\partial B(x_0, \delta)\right) \neq 0$, choose $0 < \delta' < \delta$ such that $q_{\Phi_{t_0}(x)}\left(\partial B(x_0, \delta')\right) = 0$. This is possible because of Lemma 1.2.3. Then, for any sequence $t_n \to t_0$,

$$\lim_{t_n\to t_0} q_{\Phi_{t_n}(x)}\left(B(x_0, \delta')\right) = q_{\Phi_{t_0}(x)}\left(B(x_0, \delta')\right) > 0.$$

There exists $r > 0$ such that $q_{\Phi_t(x)}\left(B(x_0, \delta')\right) \geq m > 0$ for every $t \in B(t_0, r)$. Then,

$$P\delta_x\left(B(x_0, \delta')\right) = \int_{\mathbb{R}_+} q_{\Phi_t(x)}\left(B(x_0, \delta')\right) p_x(dt) \geq \int_{B(t_0,r)} q_{\Phi_t(x)}\left(B(x_0, \delta')\right) p_x(dt) \geq m p_x\left(B(x_0, \delta')\right) > 0.$$

We conclude that $P\delta_x\left(B(x_0, \delta)\right) > 0$ for all $\delta > 0$. With other words, $x_0 \in \text{supp } P\delta_x$.

□

In fact, this result gives a converse, due to the following.

Corollary 1.2.1.

$$\text{supp } P\delta_x = \overline{\bigcup_{t\in \text{supp} p_x} \text{supp } q_{\Phi_t(x)}}$$

Proof: we have proven thus far that $\text{supp}\left(P\delta_x\right) \subset \overline{\bigcup_{t\in\text{supp } p_x} \text{supp}(q_{\Phi_t(x)})}$ and that $\bigcup_{t\in\text{supp } p_x} \text{supp } q_{\Phi_t(x)} \subset \text{supp } P\delta_x$. Since $\text{supp } P\delta_x$ is closed, this implies that also $\overline{\bigcup_{t\in\text{supp} p_x} \text{supp } q_{\Phi_t(x)}} \subset \text{supp } P\delta_x$, which is what remained for the equality to hold.

□

With this characterization out of the way, we can continue to prove the main result of this section. Recall first that by definition (see for instance [1], p. 63), the hyperspace $\mathcal{H}(B^*)$ is the collection of all closed and bounded subsets of B^* endowed with the Hausdorff metric d_H. Because of the continuity of the distance function d, we find that for any $E, F \subset B^*$,

$$d_H(E, F) = d_H\left(\overline{E}, \overline{F}\right).$$

From this fact it follows that, writing $\psi_p'(E) \stackrel{\text{def}}{=} \cup_{x\in E}\text{supp } P\delta_x$,

$$d_H(\psi_p(E), \psi_p(F)) = d_H(\psi_p'(E), \psi_p'(F)). \tag{1.4}$$

We also have the following result.

Proposition 1.2.2. *Let $\{E_\alpha\}_{\alpha\in A}, \{F_\beta\}_{\beta\in B}$ be two indexed collections of sets. Then,*

$$\delta_H\left(\cup_{\alpha\in A} E_\alpha, \cup_{\beta\in B} F_\beta\right) \leq \min\left\{\sup_{\alpha\in A}\inf_{\beta\in B} \delta_H(E_\alpha, F_\beta), \inf_{\beta\in B}\sup_{\alpha\in B} \delta_H(E_\alpha, F_\beta)\right\}$$

Proof: By definition,

$$\delta_H\left(\cup_{\alpha\in A}E_\alpha, F\right) = \sup_{\alpha\in A}\sup_{x\in E_\alpha} d(x,F) = \sup_{\alpha\in A}\delta_H(E_\alpha, F).$$

Also, for any E,

$$\begin{aligned}
\delta_H\left(E, \cup_{\beta\in B}F_\beta\right) &= \sup_{x\in E}\left(\inf_{\beta\in B}\inf_{y\in F_\beta} d(x,y)\right) \\
&\leq \sup_{x\in E} d(x,F_\beta) \quad \text{for all } \beta\in B \\
&= \delta_H(E,F_\beta) \quad \text{for all } \beta\in B.
\end{aligned}$$

And so we conclude that $\delta_H\left(E,\cup_{\beta\in B}F_\beta\right) \leq \inf_{\beta\in B}\delta_H(E,F_\beta)$, which finishes the proof.

□

With this result in mind, we can prove the contractivity of the support map on $\mathcal{H}(B^*)$.

Proposition 1.2.3. *ψ_p is a contraction on $(\mathcal{H}(B^*), d_H)$.*

Proof: By Assumption 1-$\mathcal{P}$, we have that

$$\begin{aligned}
\delta_H\left(\text{supp } P\delta_x, \text{supp } P\delta_y\right) &= \delta_H\left(\cup_{t\in\text{supp } p_x}\text{supp } q_{\Phi_t(x)}, \cup_{t'\in\text{supp } p_y}\text{supp } q_{\Phi_t(y)}\right) \\
&\leq \sup_{t\in\text{supp } p_x}\inf_{t'\in\text{supp } p_y}\delta_H\left(\text{supp } q_{\Phi_t(x)}, \text{supp } q_{\Phi_t(x)}\right) \qquad \text{(due to Proposition 1.2.2)} \\
&\leq L_q \sup_t\inf_{t'} d\left(\Phi_t(x), \Phi_{t'}(y)\right)
\end{aligned}$$

Then, we have for $t \geq t'$

$$\begin{aligned}
d\left(\Phi_t(x),\Phi_{t'}(y)\right) &\leq d\left(\Phi_t(x),\Phi_{t'}(x)\right) + d\left(\Phi_{t'}(x),\Phi_{t'}(y)\right) \\
&\leq d\left(\Phi_{t-t'}\left(\Phi_{t'}(x)\right),\Phi_{t'}(x)\right) + L_\Phi d(x,y) \qquad \text{(due to Assumption 1-}\mathcal{S}\text{)} \\
&\leq f(t-t') + L_\Phi d(x,y). \qquad \text{(due to Assumption 2-}\mathcal{P}\text{)}
\end{aligned}$$

If $t \leq t'$, then we obtain $d\left(\Phi_t(x),\Phi_{t'}(y)\right) \leq f(t'-t) + L_\Phi d(x,y)$, so in general we have the estimate

$$d\left(\Phi_t(x),\Phi_{t'}(y)\right) \leq f\left(|t-t'|\right) + L_\Phi d(x,y). \tag{1.5}$$

Hence we obtain that

$$\begin{aligned}
\delta_H\left(\text{supp } P\delta_x, \text{supp } P\delta_y\right) &\leq L_q \sup_{t\in\text{supp}p_x}\inf_{t'\in\text{supp}p_y}\left(f\left(|t'-t|\right) + L_\Phi d(x,y)\right) \\
&= L_q\left(f\left(\sup_t\inf_{t'}|t-t'|\right) + L_\Phi d(x,y)\right) \\
&\leq L_q\left(C_f\delta_H\left(\text{supp } p_x, \text{supp } p_y\right) + L_\Phi d(x,y)\right),
\end{aligned}$$

where the second line follows from f being continuous and non-decreasing. Invoking now Assumption 3-$\mathcal{P}$ yields that

$$\delta_H\left(\text{supp } P\delta_x, \text{supp}P\delta_y\right) \leq L_q\left(C_f\cdot L_p + L_\Phi\right)d(x,y).$$

From Assumption 4-$\mathcal{P}$, it follows that there exists $\theta < 1$ such that

$$\delta_H\left(\text{supp } P\delta_x, \text{supp}P\delta_y\right) \leq \theta d(x,y). \tag{1.6}$$

Hence, for $E, F \in \mathcal{H}\left(B^*\right)$, using Proposition 1.2.2 we obtain

$$\begin{aligned}
\delta_H\left(\psi_p(E),\psi_p(F)\right) &= \delta_H(\psi_p'(E),\psi_p(F)) \qquad \text{(by 1.4)} \\
&= \delta_H\left(\cup_{x\in E}\text{supp } P\delta_x, \cup_{y\in F}\text{supp } P\delta_y\right) \qquad \text{(by definition)} \\
&\leq \sup_{x\in E}\inf_{y\in F} d_H\left(\text{supp } P\delta_x, \text{supp } P\delta_y\right) \qquad \text{(by Proposition 1.2.2)} \\
&\leq \theta d(x,y) \qquad \text{(by (1.6))}
\end{aligned}$$

With other words, we have found that $d_H\left(\psi_p(E),\psi_p(F)\right) \leq \theta d_H(E,F)$, and so ψ_p is a contraction on $\mathcal{H}(B^*)$.

□

Since B^* is closed and ψ_p is a contraction, the Banach Fixed Point Theorem implies that ψ_p has a unique fixed point $D^* \in B^*$ with the property that $\psi_p^n(E) \to D^*$ as $n \to \infty$ whenever $E \in \mathcal{H}(B^*)$. Hence, we arrive at the following proposition.

Proposition 1.2.4. *If P has an invariant measure $\mu^* \in \mathcal{M}^+(B^*)$, then,* $\mathrm{supp}(\mu^*) = D^*$.

Proof: Since $\mathrm{supp}(P\mu^*) = \psi_p(\mathrm{supp}(\mu^*))$ given that P is Markov-Feller, it follows that $\mathrm{supp}(\mu^*)$ must be a fixed point of the map ψ_p. Since the Banach Fixed Point Theorem gives one unique fixed point of the map ψ_p, we conclude that $\mathrm{supp}(\mu^*) = D^*$.

□

1.2.4 Existence, uniqueness and stability of an invariant measure

We are finally ready to establish the main result of this section. We will use the following concept, where the definition has been taken from Alkurdi.

Definition 1.2.1. A Markov operator P on $\mathcal{M}(S)$ is called *globally* concentrating if for every $\epsilon > 0$ and every bounded Borel set $A \subset S$ there exists a bounded Borel set $B \subset S$ and an integer n_0 such that $P^n\mu(B) \geq 1-\epsilon$ for $n \geq n_0$ and $\mu \in \mathcal{P}(A)$. It is called *locally* concentrating if for every $\epsilon > 0$ there exists $\alpha > 0$ such that for every bounded Borel set $A \subset S$ there exists a Borel set $C \subset S$ with $\mathrm{diam}(C) < \epsilon$ and an integer n^* such that $P^n\mu(C) \geq \alpha$ for all $n \geq n_*$ and $\mu \in \mathcal{P}(A)$.

We will use a well-known result by Szarek ([2], Theorem 5.4) that allows us to conclude the existence of the invariant measure.

Theorem 1.2.1. *Let (S, d) be a Polish space. A non-expansive, locally and globally concentrating Markov operator is asymptotically stable.*

Proposition 1.2.1 gives that P is non-expansive, S is bounded so P is immediately globally concentrating. Therefore, all that is left to prove is that P is locally concentrating. The proof of the following theorem from Alkurdi immediately applies to our Markov operator P by Proposition 1.2.3, hence we simply state it below without proof.

Theorem 1.2.2. *Let $z \in D^*$. For all $r > 0$ there exists $N_r \in \mathbb{N}$ and $\alpha > 0$ such that $P^n\delta_x(B(z,r)) \geq \alpha$ for all $x \in B^*$ and $n \geq N_r$.*

□

We conclude that P is locally concentrating. The following corollary also comes from Alkurdi.

Corollary 1.2.2. *Let $z \in D^*$, for all $r > 0$,*

$$\inf_{x \in B^*} \liminf_{n \to \infty} P^n\delta_x(B(z,r)) > 0,$$

with which we arrive at the main result of this chapter.

Theorem 1.2.3. *Under Assumption 1-$\mathcal{S}$ and 1-$\mathcal{P}$ to 5-$\mathcal{P}$, there exists a unique ergodic measure μ^* of P restricted to B^*, which is asymptotically stable on B^*.*

Proof: P is locally concentrating, it is globally concentrating as B^* is bounded. By Proposition 1.2.1, it is non-expansive. Theorem 1.2.1 now gives the result.

□

1.3 Finding all ergodic measures on S: the KBBY-decomposition

In the proofs of this chapter, we related the uniqueness of an ergodic measure to the supports of $P^n\delta_x$ for $x \in B$ (or, more generally, for $x \in S$). It is no coincidence that this helped to give a good estimate for the uniqueness of the measure. A technique that relies on characterizing all the invariant measures on a space S under the action of a Markov operator P is the ergodic decomposition of Krylov-Bogoliubov-Beboutov-Yosida (henceforth KBBY).

1.3.1 Background: the decomposition

The KBBY decomposition is a powerful technique that *a priori* characterizes all ergodic invariant measures for a given Markov operator. The underlying theory is beautiful and the interested reader is encouraged to acquaint themselves with the theorems underlying this decomposition of the state space, most notably Zaharopol [20] and in the general case of a Polish state space Worm ([21], Chapter 5).

In short, the KBBY decomposition provides a description of all invariant ergodic measures by considering the Cesaro averages of the Markov operator. Following [1], consider the Cesaro averages $P^{(n)}$ of P and the associated measures ϵ_x for $x \in S$, defined as

$$P^{(n)} = \frac{1}{n}\sum_{k=0}^{n-1} P^k, \quad \epsilon_x = \lim_{n\to\infty}\sum_{k=0}^{n-1} P^k\delta_x,$$

whenever the latter limit exists. Since P is Markov-Feller, if the limit exists, the measure ϵ_x necessarily is invariant. The KBBY decomposition of the space provides for a set Γ_{cpie} such that ϵ_x is an ergodic measure when $x \in \Gamma_{cpie}$. The power of the KBBY decomposition is that it as a consequence of the KBBY theorem, any invariant ergodic measure is equal to some ϵ_x for $x \in \Gamma_{cpie}$ (for the details see the discussion in [20], Chapter 2 or [21], Chapter 5. This shows that the KBBY decomposition in fact does give an *a priori* description of all ergodic measures associated to the Markov operator.

1.3.2 The KBBY decomposition in the non-expansive case

Our previous results tie into the framework of the KBBY decomposition as follows. As proven for the non-expansive case, $\mathrm{supp}(P\delta_x) \subset B^*$ for all $x \in B_\Gamma$. At the same time, it was proven that all measures μ that had support on B_Γ satisfied $P^n\mu \to \mu^*$ for μ^* the unique invariant ergodic measure with support contained in B^*. This means that $\epsilon_x = \mu^*$ for all $x \in B_\Gamma$, which means that the only ergodic measure with support in the basin of attraction is equal to μ^*.

If B_Γ is the entire state space, this gives a complete characterization of the ergodic measures. In many applications, such as the setting of Chapter 2 of this thesis, B_Γ may be the interior of the state space. In such a case, one needs only consider the dynamics on the boundary. In fact, such a procedure is done in the doctoral thesis of Taleb Alkurdi [1]. In such cases, as we shall also see in the next Chapter, there exist ergodic invariant measures on the boundary as well. From the literature it is well-known (see, for instance, the treatment given in the standard reference [22]) that any invariant measure is a convex combination of these ergodic measures, extending the characterization of the invariant measures for the given Markov operator to the full state space.

1.4 Alternatives for the continuity of the kernels

In the sections above, the Lipschitz continuity of the maps $x \mapsto p_x$ and $x \mapsto q_x$ was taken to be with respect to the total variation distance on the space $\mathcal{M}^+(S)$. This is by no means the only distance that can be used. In fact, as we will see in Chapter 3, the total variation distance is quite restrictive and does not always give easy conditions to be satisfied for the continuity. Another distance on the space $\mathcal{M}^+(S)$ is given by the dual bounded Lipschitz norm. For its definition and basic properties, we refer the reader to [23].

Some of the results in this section carry over to the setting in which we take the dual bounded Lipschitz norm instead of the total variation norm on the space $\mathcal{M}^+(S)$. For instance, for the result of Proposition 1.1.2, note that the estimate

$$\int_{\mathbb{R}_+}\left|\int_X f(y)(q_{\Phi_t(x)} - q_{\Phi_t(z)})(dy)\right| p_x(dt) \le \|f\|_{\mathrm{BL}}\int_{\mathbb{R}_+}\|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|^*_{\mathrm{BL}}p_x(dt)$$

holds. In particular, since

$$\|f\|_{\mathrm{BL}}\int_{\mathbb{R}_+}\|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|^*_{\mathrm{BL}}p_x(dt) \le \|f\|_{\mathrm{BL}}\sup_{t\ge 0}\|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|^*_{\mathrm{BL}}$$

holds, we may consider the condition that for each x and each $\epsilon > 0$ there exists $\delta > 0$ such that for all z with $d(x,z) < \delta$, $\sup_{t \geq 0} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{BL}}^* < \epsilon$. This is indeed a weakening of the earlier inequality that replaced the dual bounded Lipschitz norm with the total variation norm, since by ([21], Lemma 7.2.1), if for each $f \in \mathrm{BL}(S)$, $Uf \in \mathcal{C}_b(X)$, P is Markov-Feller.

This implies that under some conditions that depend only on the deterministic dynamical system, the two norms may be interchanged for the Proposition to still be true. The major drawback of this approach is that this requires a priori knowledge of the dynamical system, which is the reason the total variation distance has been used in the previous text. In particular, the remark above shows how the logic leading up to the existence and uniqueness of the invariant ergodic measure does not hold in general when we replace the total variation distance with the dual bounded Lipschitz norm.

Chapter 2

Uniform Equicontinuity on Balls and an Invariant Ergodic Measure

A central component of the main theorem in the previous chapter was the non-expansiveness of the operator P. Non-expansiveness is a well-known property that has received plenty of attention in the literature (see for example [2] and the discussion around this property subsequently in [18]). However, the last section of Chapter 1 showed that the restrictive conditions needed to make the operator non-expansive are incompatible with the complicated dynamics associated to a PDMP where the underlying deterministic system has an asymptotically stable limit cycle. This motivates learning if conditions weaker than non-expansiveness still lead to the existence of an invariant and asymptotically stable ergodic measure μ, so that these can be verified for a model that has an asymptotically stable limit cycle.

Another mathematical motivation for developing a novel approach is that the previous section used crude bounds on total variation of measures to estimate the repeated integrals that come with the iterates of the process, whereas a more careful study of the measures against which the repeated integrals are taken may simplify computations and yield new results. The main challenge here is that in the repeated integrals seen in the expressions for $\langle f, P\mu\rangle$ and Uf, there is a delicate dependence on pairs (x,t) in the jump kernels. This was precisely the reason that restrictive conditions, such as the used equicontinuity of the families of measure-valued maps, were needed to make the necessary estimates. It is of interest to simplify the structure of the problem such that easier estimates can be made under weaker and cleverer conditions that better exploit the structure of the problem.

The Uniform Equicontinuity on Balls (UEB) property is an attractive tool in this regard. As we will see, the weaker UEB property will allow us to establish the existence of a unique invariant measure, without the difficulties associated to establishing the conditions for non-expansiveness. We will show that in fact the conditions for the UEB property to hold are very natural conditions on the random dynamical system. These conditions will hopefully be more straightforward to verify in practice, which is to the benefit of applications in modelling.

The set-up of this chapter is as follows. First we will describe the limit cycle and its basin of attraction, then we will construct a new type of process, related to the process in the previous chapter, such that the UEB property can be established. Under milder conditions than those of the previous chapter, the existence and uniqueness of an invariant ergodic measure for the corresponding new Markov operator will be proven. We extend the results back to the original setting, by finding a suitable transformation between the process of the previous chapter and that of the present chapter, to conclude that the operator from the previous chapter also has the UEB property and hence much weaker conditions are sufficient to yield the existence and uniqueness of an invariant ergodic measure for the original process.

A final, more *high-level*, note before we begin the chapter is in order. In the previous chapter, the Lipschitz continuity of the maps $x \mapsto p_x$ and $x \mapsto q_x$ was taken with respect to the total variation norm on $\mathcal{M}^+(S)$. The end of the last chapter included then a discussion about the dual bounded Lipschitz norm as an alternative assumption. In what comes, the dual bounded Lipschitz norm will be part of the theory - for instance, of the definition of the UEB property. This means that invariably, there will be an interplay of the two norms in the text. Special care has been taken to separate where each of the norms is a *model assumption* made or when it is part of the *necessary theory*.

2.1 The stable limit cycle and its basin of attraction

In this section we rigorously define what we mean by the fact that the underlying deterministic dynamics arising from the flow Φ have an asymptotically stable limit cycle.

Assumption 1-$\mathcal{N}$. There exists a continuous, periodic function $\gamma : \mathbb{R}_+ \to S$ that gives rise to a closed curve $\Gamma \subset S$ such that for some *basin of attraction* $B_\Gamma \supset \Gamma$, all $x \in B_\Gamma$ satisfy that $d(\Phi_t(x), \Gamma) \to 0$ as $t \to \infty$, where $d(\Phi_t(x), \Gamma) \stackrel{\text{def}}{=} \inf_{y \in \Gamma} d(\Phi_t(x), y)$.

Assumption 2-$\mathcal{N}$. There exist constants $c_1, c_2 > 0$ such that for all $x \in B_\Gamma$, $d(\Phi_t(x), \Gamma) \leq c_1 e^{-c_2 t}$, so Γ is *exponentially attracting.*

Assumptions 1-$\mathcal{N}$ and 2-$\mathcal{N}$ are commonplace in metric analysis, where one often investigates periodic solutions to dynamical systems. The exponential attraction is a naturally occurring phenomenon in many different settings, for example in the study of Floquet theory, which will receive more attention in Chapter 3 of this thesis. In the previous chapter, a natural condition for the map iterates that ensured convergence of measures under the map was the existence of $\theta \in [0, 1)$ such that $d(\Phi_t(x), \Phi_t(y)) \leq \theta d(x, y)$ for all $t \geq 0$. One might be tempted to have this same condition, requiring it holds whenever $t \geq \Delta t$.

In practice, this condition cannot hold with periodic solutions, however. For, let $x, y \in \Gamma, x \neq y$. Let τ be the period of the periodic solution. There exists $n \in \mathbb{N}$ such that $n\tau \geq \Delta t$. Then,

$$\theta d(x, y) \geq d(\Phi_{n\tau}(x), \Phi_{n\tau}(y)) = d(x, y),$$

which is a contradiction if $\theta < 1$. So, when considering (exponentially) attracting periodic orbits as we do in this thesis, one cannot assume (eventual) contractivity of the deterministic flow on an invariant domain that contains the periodic orbit.

In fact, such a condition is unnecessary for our results concerning the behavior of the operator. The approach in the remainder of this chapter will be to control the average properties of the jumps and jump times instead via the model description of the perturbations.

Further, let $B(x, r)$, denote the open ball in S for the distance d, centered at x and with radius r. We let $B[x, r]$ denote the closed ball in S centered at x with radius r. Given a closed curve Γ in its basin of attraction B_Γ, we can investigate sets of points that are at most a distance $\zeta \geq 0$ away from Γ. Let us call this set $\mathcal{D}_\zeta$ and define it as

$$\mathcal{D}_\zeta = \bigcup_{x \in \Gamma} B[x, \zeta] = \{y \in S : d(x, \Gamma) \leq \zeta\}$$

and, as the plots in the introduction inspired, when the perturbations are *small* (which is still to be defined!), $\mathcal{D}_\zeta$ has the appearance of a thin ring, or donut-shaped region around Γ. Having in mind that the limit cycle arises from Hopf bifurcation around an unstable equilibrium x^*, we define

$$d_1(\zeta) \stackrel{\text{def}}{=} \sup\{r \geq 0 : B(x^*, r) \cap D_\zeta = \emptyset\}$$
$$\zeta^* \stackrel{\text{def}}{=} \inf\{\zeta > 0 : d_1(\zeta) > 0\}.$$

We are ready to formulate the next assumption

Assumption 3-$\mathcal{N}$. The dynamical system satisfies $\zeta^* > 0$.

Informally, this assumption requires that the basin of attraction around the region of attraction is large enough that a ring exists around the limit cycle that lies within the basin of attraction and that the ring is thin enough that there exists a hole in the middle of the ring. Pick now $0 < \rho < \zeta^*$.

In a sense, none of the conditions on the dynamical system signify important constraints on the model. In working with PDMPs, one often has a given dynamical system, for instance defined by a system of ODEs, that later can be checked to have the properties in assumptions 1-$\mathcal{N}$-3-$\mathcal{N}$. More contentious is how one wishes to model the impulsive perturbations. We will propose the following assumptions for the perturbations, commenting on their modeling validity and effect on the study of the problem under consideration.

For the remainder of this chapter, we will assume, as in Chapter 1, that the maps $x \mapsto p_x$ and $x \mapsto q_x$ are Lipschitz continuous with respect to the total variation measure, with Lipschitz constants L'_p and L'_q respectively.

2.2 Uniform Equicontinuity on Balls

In the following, we will use two concepts from [24].

Definition 2.2.1. A Markov operator on measures Q is said to be Uniformly Equicontinuous on Balls (abbreviation: Q has the UEB-property), if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in X$,

$$\sup_{n\in\mathbb{N}} \|Q^n\mu - Q^n\nu\|^*_{\mathrm{BL}} < \epsilon$$

whenever $\mu, \nu \in \mathcal{P}(X)^{B(x,\delta)}$, where $\mathcal{P}(X)^{B(x,\delta)}$ is the collection of measures with support contained in $B(x, \delta)$.

Remark: Recall that Q is *equicontinuous* when for every μ_0 and for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|Q^n\mu - Q^n\mu_0\|^*_{\mathrm{BL}} < \epsilon$ for all μ such that $\|\mu - \mu_0\|^*_{\mathrm{BL}} < \delta$. With other words, the UEB-property implies the equicontinuity of Q. This builds on non-trivial results. Note that the converse statement is not necessarily true. In fact, as we will show below, equicontinuity as defined here for Q is equivalent to equicontinuity of the family $\{U^n f\}_{n\in\mathbb{N}}$ for each $f \in \mathrm{BL}(X)$. With that it indeed follows that

$$|U^n f(x) - U^n f(x_0)| = |\langle Q^n(\delta_x - \delta_{x_0}), f\rangle| \leq \|f\|_{\mathrm{BL}} \|Q^n\delta_x - Q^n\delta_{x_0}\|^*_{\mathrm{BL}} < \epsilon.$$

Proposition 2.2.1. *Let P be a Markov-Feller operator on the Polish space S. Let U be its dual. Let d be an admissible metric for S. The following are equivalent:*

1. *P has the UEB-property relative to d;*
2. *$\{U^n f : n \in \mathbb{N}, f \in BL(S, d), \|f\|_{BL} \leq 1\}$ is uniformly equicontinuous in $\mathcal{C}_b(S)$;*
3. *$\{U^n f : n \in \mathbb{N}\}$ is uniformly equicontinuous in $\mathcal{C}_b(S)$, for each $f \in BL(S, d)$.*

Proof: For 1. $\Longrightarrow$ 2., let $\epsilon > 0$. Since P has the UEB-property relative to d, there exists $\delta > 0$ such that for any $x_0 \in S$, and any $\mu, \nu \in \mathcal{P}(S)^{B(x_0,\delta)}$, one has

$$(\star) = \sup_{n\in\mathbb{N}} \|P^n\mu - P^n\nu\|^*_{\mathrm{BL},d} < \epsilon.$$

One has for $\mu = \delta_x$, $\nu = \delta_y$, with $x, y \in B(x_0, \delta)$; any $x_0 \in S$,

$$(\star) = \sup_{n\in\mathbb{N}} \left\{ \sup_{f\in\mathrm{BL}(S,d), \|f\|_{\mathrm{BL}}\leq 1} |U^n f(x) - U^n f(y)| \right\} < \epsilon,$$

so for any x, y such that $d(x, y) < \delta$ we have that $|U^n f(x) - U^n f(y)| < \epsilon$, which is precisely statement 2..

Note that 2. $\Longrightarrow$ 3. is immediate, since the operators U are linear.

For 3. $\implies$ 1. we must use several non-trivial results, notably the notion that "weak-implies-strong-convergence" ([21], Theorem 2.3.24, p. 32). Suppose that P does not have the UEB property relative to d. Then there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists $x_\delta \in S, \mu_\delta, \nu_\delta \in \mathcal{P}(S)^{B(x_\delta,\delta)}$ such that

$$\sup_n \|P^n\mu_\delta - P^n\nu_\delta\|^*_{\mathrm{BL},d} \geq \epsilon_0. \tag{2.1}$$

Take $\delta = \frac{1}{k}$, $x_k \in S$ and $\mu_k, \nu_k \in \mathcal{P}(S)^{B\left(x_k,\frac{1}{k}\right)}$ such that (2.1) holds. Then there exists $n = n_k$ with the property that

$$\|P^{n_k}\mu_k - P^{n_k}\nu_k\| \geq \tfrac{1}{2}\epsilon_0, \quad k \in \mathbb{N} \tag{2.2}$$

And then,

$$\begin{aligned}
\|\mu_k - \nu_k\|^*_{\mathrm{BL},d} &= \left\|\int_S\int_S \delta_x - \delta_y d\mu_k(x)d\nu_k(y)\right\|^*_{\mathrm{BL},d} \\
&\leq \int_S\int_S \|\delta_x - \delta_y\|^*_{\mathrm{BL},d}\mu_k(dx)\nu_k(dy) \\
&\leq \int_S\int_S d(x,y)\mu_k(dx)\nu_k(dy) \\
&\leq \int_S\int_S d(x,x_k) + d(x_k,y)\mu_k(dx)\nu_k(dy) \\
&\leq \frac{2}{k}. \qquad\qquad \left(\mathrm{supp}(\nu_k) \subset B\left(x_k,\tfrac{1}{k}\right), \mu_k \in \mathcal{P}(S)\right)
\end{aligned}$$

Let $f \in \mathrm{BL}(S,d)$. The family $\mathcal{F} = \{U^{n_k}f : k \in \mathbb{N}\}$ is uniformly equicontinuous in $\mathcal{C}_b(S)$ and uniformly bounded. Also, $\mu_k - \nu_k \to 0$ in $\mathcal{M}(S)_{\mathrm{BL}}$, hence, according to ([23], Theorem 8), $\mu_k - \nu_k \to 0$ $\mathcal{C}_b(S)$-weakly. A uniformly equicontinuous family like $\mathcal{F}$ is equicontinuous. ([23], Theorem 7) yields that $\mu_k - \nu_k$ converges uniformly to 0 on $\mathcal{F}$, that is,

$$\begin{aligned}
|\langle P^{n_k}\mu_k - P^{n_k}\nu_k, f\rangle| &= |\langle \mu_k - \nu_k, U^{n_k}f\rangle| \\
&\leq \sup_{g\in\mathcal{F}} |\langle \mu_k - \nu_k, g\rangle| \to 0, \text{ as } k \to \infty.
\end{aligned}$$

Clearly, $\sup_k \|P^{n_k}\mu_k - P^{n_k}\nu_k\|_{\mathrm{TV}} \leq 2$. Therefore, by "weak-implies-strong-convergence", it is implied that $\|P^{n_k}\mu_k - P^{n_k}\nu_k\|^*_{\mathrm{BL},d} \to 0$ as $k \to \infty$, which is in contradiction with (2.2). This finishes the last implication.

□

The interested reader can find more information in [24], especially in relation to non-expansiveness.

2.3 Semi-Concentrating Operators

Another important concept is that of a semi-concentrating Markov operator. In [24], we find the following result, ([24], Corollary 2.4.1 or Lemma 2.4.2 (p.12)).

Corollary 2.3.1. *Let P be a Markov operator and U be its dual. Assume that there exists a Lyapunov function V, bounded on bounded sets such that*

$$UV(x) \leq aV(x) + b, \quad \text{for all } x \in X$$

with $0 \leq a < 1$ and $b \geq 0$. Then for every $\epsilon > 0$ there exists B, a bounded Borel set, such that

$$\liminf_{n\to\infty} P^n\mu(B) \geq 1 - \epsilon$$

for every $\mu \in \mathcal{P}(X)$.

□

Remark: The term Lyapunov function in the statement of the corollary is meant only to refer to the concept of a test function whose existence guarantees a lower bound condition on the family of measures. It should not be confused with the notion of Lyapunov function as normally used in dynamical systems theory, where the concept plays an important role in the convergence of solutions of ODEs to given trajectories. See for instance [36] for this interpretation.

From the corollary, the following semi-concentrating condition was derived in [18]. Let now $\mathcal{C}_\epsilon = \{B \subset X : B \text{ can be covered with finitely many balls of radius } \epsilon\}$.

Definition 2.3.1. A Markov operator P is said to be semi-concentrating if for every $\epsilon > 0$ there exists a set $C \in \mathcal{C}_\epsilon$ and $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha$$

for all $\mu \in \mathcal{P}(X)$.

These two concepts imply the following central result to this section, which was first proven in [24] and later appeared in [18].

Theorem 2.3.1. *If a Markov operator P is semi–concentrating and uniformly equicontinuous on balls, then P admits an invariant measure.*

□

It is worth mentioning that a new approach for a proof for this theorem is proposed in the forthcoming doctoral thesis of Maya Ziemlańska [25]. The main idea is that if P is semi-concentrating and has the UEB-property, then P is tight, hence it admits an invariant measure because it is continuous (it is a Feller operator).

2.4 A new model description

Many of the ideas in the following are inspired by previous work done by Hille, Horbacz and Szarek in the article [18]. Specifically, the recursion of the probability density for the jump times they use to arrive at bounds for the difference $|U^n f(x) - U^n f(y)|$ inspired to develop a similar approach in a more general setting. In their work, the additive perturbation applied to the process is governed by one probability law, denoted ν^ϵ, with support in $B(0,\epsilon)$. In our setting, we defined a jump distribution q_x, possibly different for each point in the state space, according to which the process jumps at random times. Note that from the definition of the dual U to P, it follows that

$$U^n f(x) = \int_{\mathbb{R}^+}\int_X \cdots \int_{\mathbb{R}^+}\int_X f(z_n) q_{\Phi_{t_n}(z_{n-1})}(dz_n) p_{z_{n-1}}(dt_n) \cdots q_{\Phi_{t_1}(x)} p_x(dz_1)(dt_1),$$

and immediately the difficulty becomes apparent in estimating the difference between $U^n f(x)$ and $U^n f(y)$. Although the dependence on the initial condition appears to only concern the factor in the outer integral, the law of the process definitely still depends on the initial condition. More precisely,

$$\nu_n^x(dt_1,\dots,dt_n,dz_1,\dots z_n) = \bigotimes_{i=0}^{n-1} q_{\Phi_{t_{n-i}}(z_{n-(i+1)})}(dz_{n-i}) p_{z_{n-(i+1)}}(dt_{n-i}), \quad z_0 := x$$

depends very delicately on x. If one were to naively estimate the differences, the notation would be misleading enough to lead one to believe only the outer integral needed estimation, which is of course not the case. This because $U^n f(x) - U^n(y)$ roughly represents estimating the expected difference in $f(X^{(n)})$, where $X^{(n)}$ represents the n-th iterate of the process under the laws ν_n^x and ν_n^y.

A second difficulty in applying the methods of Hille, Horbacz and Szarek to this process is the alternating occurrence of q and p, depending on the arguments of the previous pair. This makes it difficult to group all terms with q and p together, which is essentially the approach in the proof of ([18], Lemma 4.7). To add insult to injury, the dependence on the pairs is in the parameter of the jump distribution q itself, which is bound to give big problems unless heavy conditions are placed on the jump distribution.

We suggest three modifications to the model set-up of Alkurdi [1], or [18], to work around these difficulties. The overall aim is to place as few further assumptions on the process as necessary while attempting to convert the process into a form that is more mathematically tractable.

1. Recall that the definition of the jump kernel q_x is the probability distribution on the set of all possible points that the process can jump to from x. Previous work, for instance [1], have modeled the jumps in a general Banach space as an additive perturbation, such that one can find explicit representations of the process iteratively in the perturbations. While this approach was not used in the preceding chapters, its tractability makes it more attractive to consider in the remainder. Let now $\tilde{q}_x$ be the probability distribution over the additive jumps, that is to say

$$\tilde{q}_x(A) = \mathbb{P}(\text{the process is in } x + A \text{ after the jump}) = q_x(x + A),$$

which means that $\operatorname{supp}(\tilde{q}_x) = (\operatorname{supp}(q_x)) - x$. In the remainder, we will omit the tilde and write q_x for the law governing the additive perturbation from x, meaning $q_x := \tilde{q}_x$ hence forth.

2. We will change the order of jumping and deciding a run-time for the system. Switching the application of q and p means that the process started at x is instantaneously perturbed at the start, the additive perturbation ξ governed by the law q_x and evolves deterministically according to the deterministic dynamical system for a period governed by the law $p_{x+\xi}$.

   Writing $X^{(n)}(x)$ for the n-th iterate of this process, this defines the recursion

$$X^{(n)}(x) = \Phi_{t_n}\left(X^{(n-1)}(x) + z_n\right),$$

   where $z_n \sim q_{X^{n-1}(x)}$ and $t_n \sim p_{X^{(n-1)}(x)+z_n}$.

3. In some analogy to the work of [18], we do the following. Note that the paper [18] defines the time kernel to be absolutely continuous with respect to the Lebesgue measure with some place-dependent densities. As the jump times are unbounded from above in our model, we will assume that all jump kernels are absolutely continuous with respect to a common probability density on $\mathbb{R}^+$, which is a generalization of the case considered. Secondly, rather than assuming that all perturbations are governed by a single law with support on some bounded set, we will much more generously assume that there is a probability distribution on X to which all the jump kernels are absolutely continuous. With other words, there exists $\nu \in \mathcal{P}(\mathbb{R}^+)$ and $\lambda \in \mathcal{P}(X)$ such that $p_\xi \ll \nu$ and $q_\xi \ll \lambda$ for all $\xi \in X$.

As is to be expected, the suggestions complicate the approach. For instance, one needs to prove again that the resulting operator is Markov-Feller (or stronger) and explore under which conditions such results hold. We will not do such a thing extensively. Write Q for the resulting operator of this new process on the space of measures $\mathcal{M}(X)$ and note that by definition now

$$\langle f, Q\mu\rangle = \int_X \int_X \int_{\mathbb{R}^+} f\left(\Phi_t(x+y)\right) p_{x+y}(dt) q_x(dy)\mu(dx)$$

for f a bounded and measurable function. For V the dual operator of Q, this gives that

$$Vf(x) = \int_X \int_{\mathbb{R}^+} f\left(\Phi_t(x+y)\right) p_{x+y}(dt) q_x(dy), \tag{2.3}$$

suggesting that simple continuity assumptions on q and p can be made to make the same estimates used in the preceding paragraph to conclude that for each $\epsilon > 0$ for each x there exists $\delta > 0$ such $|Vf(x) - Vf(z)| < \epsilon$ whenever $d(x, z) < \delta$. Hence, we will not go into detail as to under which conditions exactly Q is strong-Feller (or variations thereof).

We finish this section with two assumptions on the system, which will be assumed to hold from now on. First, define

$$\Delta \stackrel{\text{def}}{=} \inf_{x \in \mathcal{D}_\rho} \{d(x, \partial B_\Gamma)\}.$$

Assumption 0-$\mathcal{U}$. diam $q_x < \Delta$ for all $x \in S$.

Assumption 0-$\mathcal{U}$ is crucial, since it guarantees the invariance of $\mathcal{D}_\rho$ under the action of the PDMP. Namely, let $x \in \mathcal{D}_\rho$. Then any process started at x cannot leave B_Γ at the instantaneous jump. After Δt units of time, the process is back within distance ρ from Γ. Hence, $\mathcal{D}_\rho$ is invariant under the PDMP.

Furthermore, we would like to assume the following.

Assumption 1-$\mathcal{U}$. There exists $0 < \gamma < 1$ and $L_\Phi < \infty$ such that for all $x, y \in S$:

$$\int d\left(X^{(1)}(x), X^{(1)}(y)\right) p_{x+z_1}(dt_1) q_x(dz_1) \leq \gamma d(x,y),$$
$$\int d\left(\Phi_t(x+z), \Phi_t(y+z)\right) (\nu \otimes \lambda)(dt, dz) \leq L_\Phi d(x,y)$$

Recall, that previously the condition of contractivity on average was defined in terms of the jump kernels,

$$\int_{\mathbb{R}^+} \|q_{\Phi_t(x)} - q_{\Phi_t(z)}\|_{\mathrm{TV}}\ p_x(dt) \leq L d(x,z).$$

This formulation is more straightforward and natural to check, since we relate not the distances of the jump kernels to the contractivity of the system, but we directly relate it to the distances between iterates of the process. This should be easier to use in practice.

Remark: The second condition in Assumption 1-$\mathcal{U}$ may seem artificial at first, but in fact it follows by the following observation. Let $x, y \in S$. In general, the quantity $d\left(\Phi_t(x), \Phi_t(y)\right)$ is difficult to control. However, when the dynamical system is generated by an ODE $\dot{x} = f(x)$ for f a Lipschitz continuous function with Lipschitz constant K on a bounded domain of $\mathbb{R}^n$, the *Gronwall inequality* holds:

$$d\left(\Phi_t(x), \Phi_t(y)\right) \leq e^{Kt} d(x,y).$$

For that reason, we would like to assume the following property of the dynamical system.

Assumption 4-$\mathcal{N}$. There is a constant $K > 0$ such that the Gronwall-like bound

$$d\left(\Phi_t(x), \Phi_t(y)\right) \leq e^{Kt} d(x,y)$$

holds for all $x, y \in S$.

From the previous assumption, we make the following observation

$$\int d\left(\Phi_t(x+z), \Phi_t(y+z)\right) (\nu \otimes \lambda)(dt, dz) \leq \int e^{Kt} d(x,y) \nu(dt) \lambda(dz) = d(x,y) \int_{\mathbb{R}_+} e^{Kt} \nu(dt),$$

meaning that if $\int_{\mathbb{R}_+} e^{Kt} \nu(dt) < \infty$, the condition is met. This relates the problem of contractivity directly with the reference measures ν, λ. This is preferable from a modeling point of view. Finally, recall that a similar condition was discussed in Section 1.2, to establish the non-expansiveness of the operator. In that setting, the size of the quantity $\int_{\mathbb{R}_+} e^{Kt} \nu(dt)$ was important, since a large size of this constant impeded proving the non-expansiveness of the operator.

2.5 A measure-theoretic construction

To commence on the iterated process and exploit the probability measures to which the kernels are absolutely continuous, a question we must deal with first is the delicate dependence of the jump kernel on the jump sites in the repeated integrals. The problem here is that in the repeated integrals, the inner integral is parametrized by the variable against which there is integration in the outer integral, meaning that reversing the order of integration or expressing either of the integrals in terms of their reference measures is a subtle procedure. Therefore, it is important to establish what the (measurable) dependence is between the Radon-Nikodým derivative in the inner integral with respect to the integration variable of the outer integral. The first result we can use is exercise 6.10.72 from ([19], Part II).

Proposition 2.5.1. *Let $(X, \mathcal{A}, \mu)$ be a probability space, $\mathcal{A}$ a countably generated σ-algebra, $(T, \mathcal{B})$ a measurable space, and let μ_t, where $t \in T$, be a family of bounded measures on $\mathcal{A}$ absolutely continuous with respect to a positive measure μ such that for every $A \in \mathcal{A}$, the function $t \mapsto \mu_t(A)$ is measurable with respect to $\mathcal{B}$. There is an $\mathcal{A} \otimes \mathcal{B}$-measurable function f on $X \times T$ such that for every $t \in T$, the function $x \mapsto f(x,t)$ is the Radon-Nikodým derivative of μ_t with respect to μ.*

Proof sketch (hint from [19]): If $X = [0,1]$ and $\mathcal{A} = \mathcal{B}([0,1])$, then for every t, the Radon-Nikodým density of μ_t with respect to μ is given by the equality

$$f(x,t) = \lim_{n \to \infty} \mu_t([x - \epsilon_n, x + \epsilon_n])/\mu([x - \epsilon_n, x + \epsilon_n]),$$

with $\epsilon_n = n^{-1}$ and $f(x,t) = 0$ if $\mu([x - \epsilon_n, x + \epsilon_n]) = 0$ for some n. One can assume that the measure μ has no atoms, since for its purely atomic part the claim is obvious. It is readily seen that the functions $\mu_t([x - \epsilon_n, x + \epsilon_n])/\mu([x - \epsilon_n, x + \epsilon_n])$ are measurable with respect to $\mathcal{B}([0,1]) \otimes \mathcal{B}$, since the numerator and denominator are continuous in x due to the absence of atoms and are $\mathcal{B}$-measurable in t by assumption. The above limit exists for almost every x it t is fixed, for all other x we set $f(x,t) = 0$. In the general case, by ([19], Theorem 6.5.5) there exists an $\mathcal{A}$-measurable function $\xi : X \to [0,1]$ such that $\mathcal{A} = \left\{\xi^{-1}(B), B \in \mathcal{B}([0,1])\right\}$. Set $\nu = \mu \circ \xi^{-1}$. Then $\nu_t \ll \nu$ and by the above there exists a $\mathcal{B}([0,1]) \otimes \mathcal{B}$-measurable version $(x,t) \mapsto \rho(x,t)$ of the Radon-Nikodým densities of the measures ν_t with respect to ν. Set $f(x,t) = \rho(\xi(x), t)$. The function f is measurable with respect to $\mathcal{A} \otimes \mathcal{B}$. Let t be fixed. Given a set $A \in \mathcal{A}$, we can find a set $B \in \mathcal{B}([0,1])$ with $A = \xi^{-1}(B)$. Since $I_B(\xi(x)) = I_A(x)$, we obtain

$$\mu_t(A) = \nu_t(B) = \int_B \rho(y,t)\nu(dy) = \int_X I_B(\xi(x)) f(x,t)\mu(dx) = \int_A f(x,t)\mu(dx).$$

$\square$

Let us verify, before applying the theorem to our measures, that we are in the right setting. Remember that ν, λ are Borel measures, and that X is assumed to be Polish. Since X is second-countable, it follows that $\mathcal{B}(X)$ is countably generated. For every $x \in X$, the maps $x \mapsto p_x$ and $x \mapsto q_x$ are continuous and hence the map $x \mapsto p_x(A)$ is measurable for any $A \in \mathcal{B}(\mathbb{R}_+)$ and the map $x \mapsto q_x(A)$ is measurable for any $A \in \mathcal{B}(X)$. We will now be able to use this result as follows. By assumption, there exists ν such that $p_\xi \ll \nu$ for each $\xi \in X$. By a first application of Proposition 2.5.1, there exists a function $\psi_1 : \mathbb{R}^+ \times X \to \mathbb{R}$ such that

$$\psi_1(\cdot, \xi) = \frac{dp_\xi}{d\nu},$$

and ψ_1 is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(X)$-measurable. Hence, there exists a function $\tilde{\psi}_1 : \mathbb{R}^+ \times X \times X \to \mathbb{R}$ such that

$$\tilde{\psi}_1(\cdot, \xi_1, \xi_2) = \frac{dp_{\xi_1 + \xi_2}}{d\nu},$$

and $\tilde{\psi}_1$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$-measurable. By a second application of Proposition 2.5.1, we conclude that there exists a $\phi : X \times X \to \mathbb{R}$ such that

$$\phi(\cdot, \xi) = \frac{dq_\xi}{d\lambda},$$

and ϕ is $\mathcal{B}(X) \otimes \mathcal{B}(X)$-measurable. Define

$$\Psi(\tau, \xi_1, \xi_2) = \tilde{\psi}_1(\tau, \xi_1, \xi_2)\phi(\xi_2, \xi_1),$$

and note that Ψ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$-measurable. Finally, we have that

$$\int_X \int_{\mathbb{R}^+} f\left(\Phi_t(x+y)\right) p_{x+y}(dt) q_x(dy) = \int_X \int_{\mathbb{R}^+} f\left(\Phi_t(x+y)\right) \Psi(t,y,x)\nu(dt)\lambda(dy),$$

where the integral is well-defined due to the measurability arguments above. In particular the map

$$x \mapsto \int_X \int_{\mathbb{R}^+} f\left(\Phi_t(x+y)\right) \Psi(t,y,x)\nu(dt)\lambda(dy)$$

is measurable.

In the following, the aim is to translate the assumed Lipschitz properties of the maps $x \mapsto p_x, x \mapsto q_x$ with respect to the total variation norm to statements about continuity of the function Ψ in x. This will prove to be extremely delicate. In particular, the assumption that the Lipschitz condition holds for p and q *with respect to the BL-norm* a priori seems not to imply the desired property about the Radon-Nikodým derivative Ψ. Variants to this question have been formulated as open problems in the community before, for example at the workshop "Modeling with Measures" at the Lorentz Center in Leiden in December 2018, without a solution. In general, in the BL-norm setting, making elementary statements about the Radon-Nikodým density is involved. For instance, there are barely any results available to show that the density is in L^p instead of L^1 under the most basic assumptions. Hence, our efforts will focus on avoiding this difficulty. Nonetheless, the following statement holds.

Proposition 2.5.2. *If the maps $x \mapsto p_x$ and $x \mapsto q_x$ are Lipschitz with respect to the BL-norm, then the map*

$$F : S \to \mathcal{M}(X \times \mathbb{R}_+)_{BL}, \quad x \mapsto \Psi(z, t, x)\nu(dt)\lambda(dz)$$

is Lipschitz in x.

Proof: Let $x, y \in S$ be given. One has that for general $g \in \mathrm{BL}(X \times \mathbb{R}_+)$ such that $g(t, z) = g_1(t)g_2(z)$ - more about this choice of functions will be relevant later - the quantity

$$\left| \int g(t,z)\Psi(t,z,x)d(\nu \otimes \lambda)(t,z) - \int g(t,z)\Psi(t,z,y)d(\nu \otimes \lambda)(t,z) \right|$$

can be bounded by using the same decomposition of the differences as used previously for estimating the difference of the measures p and q (see for instance the proof of Proposition 1.1.1). This gives the following upper bound on the difference of the integrals

$$\int_S \|p_{z+x} - p_{z+y}\|^* \cdot \|g_2\|_{\mathrm{BL}(\mathbb{R}_+)} |g_1(z)| q_x(dz) + \|q_x - q_y\|^* \cdot \|g_2\|_{\mathrm{BL}(S)} \cdot \left\| z \mapsto \int_{\mathbb{R}} g_1(t) p_{z+y}(dt) \right\|_{\mathrm{BL}}$$

as for the first term, note that

$$\int_S \|p_{z+x} - p_{z+y}\|^* \cdot \|g_2\|_{\mathrm{BL}(\mathbb{R}_+)} |g_1(t)| q_x(dt) \leq L_p \|g_2\|_{\mathrm{BL}(\mathbb{R}_+)} \|g_1\|_\infty d(x,y)$$

by the Lipschitz property of the map $x \to p_x$. Since we may take $\|g_i\|_{\mathrm{BL}} \leq 1$, $i = 1, 2$, that bounds the first term as

$$\int_S \|p_{z+x} - p_{z+y}\|^* \cdot \|g_2\|_{\mathrm{BL}(\mathbb{R}_+)} |g_1(t)| q_x(dz) \leq L_p d(x,y)$$

For the second term, it is still required to estimate the Lipschitz coefficient of the functional term. To that end, estimate

$$\begin{aligned} \left| \int_{\mathbb{R}_+} g_1(t) p_{z_1+y}(dt) - \int_{\mathbb{R}_+} g_1(t) p_{z_2+y}(dt) \right| &\leq \|p_{z_1+y} - p_{z_2+y}\|^* \cdot \|g_1\|_{\mathrm{BL}} \\ &\leq L_p d(z_1 + y, z_2 + y) \cdot \|g_1\|_{\mathrm{BL}} \\ &= L_p d(z_1, z_2) \cdot \|g_1\|_{\mathrm{BL}} \end{aligned}$$

which means that $\left| z \mapsto \int_{\mathbb{R}_+} g_1(t) p_{z+y}(dt) \right|_L \leq L_p \|g_1\|_{\mathrm{BL}}$, and $\left\| z \mapsto \int_{\mathbb{R}_+} g_1(t) p_{z+y}(dt) \right\|_\infty \leq \|g_1\|_\infty$, both independent of y, so we conclude that

$$\left\| z \mapsto \int_{\mathbb{R}_+} g_1(t) p_{z+y}(dt) \right\|_{\mathrm{BL}} \leq \max(1, L_p) \cdot \|g_1\|_{\mathrm{BL}}$$

so

$$\|q_x - q_y\|^* \cdot \|g_2\|_{\mathrm{BL}(S)} \cdot \left\| z \mapsto \int_{\mathbb{R}} g_1(t) p_{z+y}(dt) \right\|_{\mathrm{BL}} \leq \max(1, L_p) \cdot d(x,y)$$

were again we have bounded $\|g_i\|_{\mathrm{BL}} \leq 1$ for $i = 1, 2$. Putting this together,

$$\left| \int g(s,t)\Psi(t,z,x)d(\nu \otimes \mu)(s,t) - \int g(s,t)\Psi(t,z,y)d(\nu \otimes \mu) \right| \leq \max(1, L_p)(1 + L_q)d(x,y)$$

which shows the required Lipschitz property.

□

Remark: The proof of the proposition above immediately reveals that the map

$$F : S \to \mathcal{M}(S \times \mathbb{R}_+)_{\mathrm{TV}}, \quad x \mapsto \Psi(t,z,x)\nu(dt)\lambda(dz)$$

is Lipschitz when $x \mapsto p_x$ and $x \mapsto q_x$ are Lipschitz with respect to the TV norm. Under that assumption, the connection between the Lipschitzianity of the map F and that of Ψ can be related via the following lemma.

Lemma 2.5.1. *Assuming that $x \mapsto p_x$ and $x \mapsto q_x$ are Lipschitz with respect to the total variation norm, the map $x \mapsto \Psi(t,z,x)$ is Lipschitz for almost all $(t,z) \in \mathbb{R}_+ \times S$.*

Proof: By the Remark, we obtain, writing $L_\Psi = \max(1, L_p)(1 + L_q)$,

$$\begin{aligned}
L_\Psi \cdot d(x,y) &\geq \sup_{f \in \mathrm{BM}(\mathbb{R}_+ \times S) : \|f\| \leq 1} \left| \int f(t,z)\Psi(t,z,x)(\nu \otimes \lambda)(dt,dz) - \int f(t,z)\Psi(t,z,y)(\nu \otimes \lambda)(dt,dz) \right| \\
&= \sup_{f \in \mathrm{BM}(\mathbb{R}_+ \times S) : \|f\| \leq 1} \left| \int f(t,z) \left\{ \Psi(t,z,x) - \Psi(t,z,y) \right\} (\nu \otimes \lambda)(dt,dz) \right| \\
&= \sup_{f \in \mathrm{BM}(\mathbb{R}_+ \times S) : \|f\| \leq 1} \int f(t,z) \left| \Psi(t,z,x) - \Psi(t,z,y) \right| (\nu \otimes \lambda)(dt,dz) && \text{(See proof below)} \\
&\geq \int \left| \Psi(t,z,x) - \Psi(t,z,y) \right| (\nu \otimes \lambda)(dt,dz) && \text{(By inclusion)}
\end{aligned}$$

and by the non-negativity of the integrand, we conclude that for $(\nu \otimes \lambda)$-almost all $(t,z) \in \mathbb{R}_+ \times S$, $|\Psi(t,z,x) - \Psi(t,z,y)| \leq L_\Psi \cdot d(x,y)$.

Proof of second equality: for the second equality in the chain above, let $x, y \in S$ and let $E_+ \subset \mathbb{R}_+ \times S$ measurable and such that $\Psi(t,z,x) - \Psi(t,z,y) > 0$ on E_+ and let E_- be such that $\Psi(t,z,x) - \Psi(t,z,y) < 0$. This follows since $\Psi(\bullet, \bullet, x)$ and $\Psi(\bullet, \bullet, y)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(S)$-measurable as (the product of) Radon-Nikodým derivatives. Note also that $E_+ \cap E_- = \emptyset$. Then,

$$g = (1_{E_+} - 1_{E_-})(t,z)f(t,z) \in \mathrm{BM}(\mathbb{R}_+ \times S),$$

and that the norm of g is equal to that of f if $f \in \mathrm{BM}(\mathbb{R}_+ \times S)$ with norm smaller or equal to 1. Replacing f by g gives the equality.

In particular, we have that for almost all $(t,z) \in \mathbb{R}_+ \times S$,

$$|x \mapsto \Psi(t,z,x)|_{\mathrm{L}} \leq \max(1, L_p)(1 + L_q) = L_\Psi,$$

and we are done.

□

Remark: Proposition 2.5.2 requires only that the maps $x \mapsto p_x$ and $x \mapsto q_x$ be Lipschitz continuous with respect to the dual bounded Lipschitz norm, while the previous Lemma necessitates the Lipschitz continuity of the maps. An adaptation of the argument above so that the desired property also is implied when there is Lipschitz continuity with respect to the dual bounded Lipschitz norm would allow for the following results to hold when only this continuity is assumed, but conform the observation prior to Proposition 2.5.2 this does not seem like a feasible approach with the existing techniques.

An alternative approach, the attractiveness of which from a modeling standpoint depending on the particular application, can be to define a model such that granted p_x, q_x for each $x \in S$, the Radon-Nikodým derivatives can be (explicitly) computed and shown to possess the desired Lipschitz properties. In some applications, this may well be possible and may in fact be easier than verifying the desired Lipschitz properties via the total variation or bounded dual Lipschitz norms.

In any case, some form of Lipschitzianity is needed, which yields the following assumption on the model

Assumption 2-$\mathcal{U}$. One of the following holds

- The maps $x \mapsto p_x$, $x \mapsto q_x$ are Lipschitz with respect to the total variation norm, with Lipschitz constants L_p, $L_q < \infty$ respectively.
- The map $x \mapsto \Psi(t, z, x)$ is Lipschitz with Lipschitz constant $L_\Psi < \infty$ for $(\nu \otimes \lambda)$-almost all $(t, z) \in \mathbb{R}_+ \times S$.

A crucial question in this section is to which extent the assumptions placed on the model earlier can be relaxed. Here we show weaker conditions than in the previous section, which we will show imply that the operator has the UEB-property and is semi-concentrating. These conditions are in full analogy to the conditions placed on a much simpler dynamical system in [18], since the system is no longer contracting to a single point, but rather has solutions converge to a periodic solution of the deterministic dynamical system. Where not mentioned, we assume the same conditions to hold as in the original model. Specifically, we still assume that the RHS of the deterministic dynamical system is Lipschitz with Lipschitz constant K and still assume that the support of the jump times is uniformly bounded from below by a constant Δt.

2.6 The product measure and its Radon-Nikodým derivative

Now we are in a position to prove the UEB-property of the operator Q. Recall that for Q to have the UEB-property, one must show that for every $\epsilon > 0$, there exists δ such that for all $x \in X$

$$\sup_{n\in\mathbb{N}} \|Q^n\mu - Q^n\nu\|^*_{\mathrm{BL}} < \epsilon$$

whenever $\mu, \nu \in \mathcal{P}(X)^{B(x,\delta)}$. Now note, for any two finite measures μ, ν, the following relation holds

$$\begin{aligned}\langle Vf, \mu - \nu\rangle = \int Vf(x)\mu(dx) - \int Vf(y)\nu(dy) &= \int\int Vf(x)\mu(dx)\nu(dy) - \int\int Vf(y)\mu(dx)\nu(dy)\\ &= \int\int Vf(x) - Vf(y)\mu(dx)\nu(dy),\end{aligned}$$

so that

$$\sup_{n\in\mathbb{N}} \|Q^n\mu - Q^n\nu\|^*_{\mathrm{BL}} = \sup_{f:\|f\|_{\mathrm{BL}}\le 1} \left|\int\int V^n f(x) - V^n f(y)\mu(dx)\nu(dy)\right|,$$

implying that we only need to estimate iterates of V^n. By the definition of the dual operator V and the construction of the process,

$$V^n f(x) = \underbrace{\int_X\int_{\mathbb{R}^+}\cdots\int_X\int_{\mathbb{R}^+}}_{n \text{ times}} f\left(\Phi_{t_n}\left(X^{(n-1)}(x) + z_n\right)\right) p_{X^{(n-1)}(x)+z_n}(dt_n) q_{X^{(n-1)}(x)} \cdots p_{x+z_1}(dt_1) q_x(dz_1).$$

By the above this gives (by abuse of notation omitting the repeated integrals; the meaning of these should be obvious given the measure against which we are integrating),

$$V^n f(x) = \int f\left(\Phi_{t_n}\left(X^{(n-1)}(x) + z_n\right)\right) \prod_{i=0}^{n-1} \Psi\left(t_{n-i}, z_{n-i}, X^{(n-(i+1))}(x)\right) \bigotimes_{i=0}^{n-1} \nu(dt_{n-i}) \otimes \lambda(dz_{n-i}),$$

where the $\bigotimes$ symbol denotes the product of measures ν and λ with their respective coordinates. Write now

$$\mathfrak{P}_n((t_i)_{i=1}^n, (z_i)_{i=1}^n, x) = \prod_{i=0}^{n-1} \Psi\left(t_{n-i}, z_{n-i}, X^{(n-(i+1))}(x)\right),$$

and notice that $\mathfrak{P}_n$ is a measurable function with respect to the n-fold direct product and that it serves as a Radon-Nikodým derivative as the n-fold product of the measures p and q. Finally, note that $\mathfrak{P}_n$ satisfies the recursions

$$\mathfrak{P}_n((t_i)_{i=1}^n, (z_i)_{i=1}^n, x) = \mathfrak{P}_{n-1}\left((t_i)_{i=2}^n, (z_i)_{i=2}^n, X^{(1)}(x)\right)\Psi(t_1, z_1, x), \tag{2.4}$$

$$\mathfrak{P}_n((t_i)_{i=1}^n, (z_i)_{i=1}^n, x) = \mathfrak{P}_{n-1}\left((t_i)_{i=1}^{n-1}, (z_i)_{i=1}^{n-1}, x\right)\Psi\left(t_n, z_n, X^{(n)}(x)\right) \tag{2.5}$$

for all x and (fixed) $(t_i)_{i=1}^n, (z_i)_{i=1}^n$ by definition. Summarizing, for $x \in X$, it holds that

$$V^n f(x) = \int f\left(\Phi_{t_n}\left(X^{(n-1)}(x) + z_n\right)\right)\mathfrak{P}_n((t_i)_{i=1}^n, (z_i)_{i=1}^n, x)\bigotimes_{i=0}^{n-1}\nu(dt_{n-i}) \otimes \lambda(dz_{n-(i+1)}).$$

Remark: Earlier it was noted that "$V^n f(x) - V^n(y)$ roughly represents estimating the expected difference in $f(X^{(n)})$, where $X^{(n)}$ represents the n-th iterate of the process under the laws ν_n^x and ν_n^y." The notation above makes this explicit.

2.7 Bounding differences of Radon-Nikodým derivatives

This section is meant to establish some properties of $\mathfrak{P}_n((t_i)_{i=1}^n, (z_i)_{i=1}^n, x)$. These properties will help in estimating the differences for the iterates of f under V, which is a key step on the road to establishing the UEB-property. In the remainder, we will use the notation $t_{k:\ell} = (t_i)_{i=k}^{\ell}$ and $z_{k:\ell} = (z_i)_{i=k}^{\ell}$ for $1 \leq k < \ell \leq n$ for the sake of convenience.

Proposition 2.7.1. *For $x, y \in S$ and for almost all $(t, z) \in (\mathbb{R}_+, S)^{\otimes n}$,*

$$\begin{aligned}\int \{\mathfrak{P}_n(t, z, x) - \mathfrak{P}_n(t, z, \ y)\}\, d(\nu \otimes \lambda)^n = &\int \{\mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, x) - \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\}\, d(\nu \otimes \mu)^{n-1} \\ &+ \int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left\{\Psi\left(t_n, z_n, X^{(n)}(x)\right) - \Psi\left(t_n, z_n, X^{(n)}(y)\right)\right\} d(\nu \otimes \lambda)^n.\end{aligned}$$

Proof: Notice that for given $(t_i)_{i=1}^n, (z_i)_{i=1}^n$, one can re-write, using the recursion (2.5), so that

$$\begin{aligned}\mathfrak{P}_n(t, z, x) - \mathfrak{P}_n(t, z, y) = &\{\mathfrak{P}_{n-1}\ (t_{1:n-1}, z_{1:n-1}, x) - \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\}\Psi\left(t_n, z_n, X^{(n)}(x)\right) \\ &+ \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left\{\Psi\left(t_n, z_n, X^{(n)}(x)\right) - \Psi\left(t_n, z_n, X^{(n)}(y)\right)\right\}\end{aligned}$$

holds for fixed z and t. Integrating both sides with respect to $(\nu \otimes \lambda)^n$ gives the equality

$$\int \{\mathfrak{P}_n(t, z, x) - \mathfrak{P}_n(t, z, y)\} d(\nu \otimes \lambda)^n = (\star),$$

and

$$\begin{aligned}(\star) = &\int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left\{\Psi\left(t_n, z_n, X^{(n)}(x)\right) - \Psi\left(t_n, z_n, X^{(n)}(y)\right)\right\} d(\nu \otimes \lambda)^n \\ &+ \int \{\mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, x) - \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\}\Psi\left(t_n, z_n, X^{(n)}(x)\right) d(\nu \otimes \mu)^n.\end{aligned}$$

Noticing that the inner integral in the second term can be integrated out, the claim is proven.

□

The equality above gives a first step in bounding the differences of $\mathfrak{P}_n$. The following lemma will be the key ingredient in establishing the UEB-property.

Lemma 2.7.1. *Assuming condition 1-$\mathcal{U}$, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$:*

$$\int |\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)|\, d(\nu \otimes \lambda)^n < C d(x,y).$$

Proof: From Proposition 2.7.1, we deduce using the triangle inequality

$$\begin{aligned}\int |\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)|\, d(\nu\otimes\lambda)^n \le \int \left|\mathfrak{P}_{n-1}\left(t_{1:n-1}, z_{1:n-1}, x\right) - \mathfrak{P}_{n-1}\left(t_{1:n-1}, z_{1:n-1}, y\right)\right| d(\nu\otimes\lambda)^{n-1} \\ + \int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left|\Psi\left(t_n, z_n, X^{(n)}(x)\right) - \Psi\left(t_n, z_n, X^{(n)}(y)\right)\right| d(\nu\otimes\lambda)^n.\end{aligned}$$

Observe that

$$\int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left|\Psi\left(t_n, z_n, X^{(n)}(x)\right) - \Psi\left(t_n, z_n, X^{(n)}(y)\right)\right| d(\nu\otimes\lambda)^n = (\star)$$

satisfies the following inequality due to the Lipschitz property

$$(\star) \le L_\Psi \cdot \int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y) d\left(X^{(n)}(x), X^n(y)\right) d(\nu\otimes\lambda)^n = (\star\star),$$

and that

$$\begin{aligned}(\star\star) &= L_\Psi \int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y)\left\{\int d\left(X^{(1)}\left(X^{(n-1)}(x)\right), X^{(1)}\left(X^{(n-1)}(y)\right) d(\nu\otimes\lambda)\right)\right\} d(\nu\otimes\lambda)^{n-1} \\ &\le L_\Psi \cdot L_\Phi \int \mathfrak{P}_{n-1}(t_{1:n-1}, z_{1:n-1}, y) d\left(X^{(n-1)}(x), X^{(n-1)}(y)\right) d(\nu\otimes\lambda)^{n-1} \\ &\le \dots \\ &\le L_\Psi L_\Phi \gamma^n d(x,y).\end{aligned}$$

From this, it follows that

$$\begin{aligned}\int |\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)|\, d(\nu\otimes\lambda)^n &\le \int |\mathfrak{P}_1(t_1,z_1,x) - \mathfrak{P}_1(t_1,z_1,y)|\, d(\nu\otimes\lambda) + L_\Psi L_\Phi \sum_{i=1}^n \gamma^i d(x,y) \\ &= \int |\Psi(t_1,z_1,x) - \Psi(t_1,z_1,y)|\, d(\nu\otimes\lambda) + L_\Psi L_\Phi \sum_{i=1}^n \gamma^i d(x,y) \\ &\le L_\Psi d(x,y) + L_\Psi L_\Phi \sum_{i=1}^\infty \gamma^i d(x,y) \\ &= \left\{L_\Psi + \frac{L_\Psi L_\Phi}{1-\gamma}\right\} d(x,y),\end{aligned}$$

which concludes the proof.

□

2.8 Establishing the UEB-property

The estimates proved in the previous section result in the main result of this section.

Theorem 2.8.1. *Assuming conditions (1-$\mathcal{N}$)-(4-$\mathcal{N}$), (1-$\mathcal{U}$) and (2-$\mathcal{U}$), Q has the UEB property.*

Proof: For the sake of notation, use the obvious shorthand $(\nu\otimes\lambda)^n$ for the measure in the integral, and we omit the indices for the t_i and z_i such that

$$|V^n f(x) - V^n f(y)| = \left|\int f\left(X^{(n)}(x)\right)\mathfrak{P}_n\left(t,z,x\right) - f\left(X^{(n)}(y)\right)\mathfrak{P}_n\left(t,z,y\right) d(\nu\otimes\lambda)^n\right|.$$

From this, it follows that

$$|V^n f(x) - V^n f(y)| \leq \left| \int \left[f\left(X^{(n)}(x)\right) - f\left(X^{(n)}(y)\right)\right] \mathfrak{P}_n(t,z,x) d(\nu \otimes \lambda)^n \right| + \left| \int f\left(X^{(n)}(y)\right) \left[\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)\right] d(\nu \otimes \lambda)^n \right|.$$

We can now treat each term separately. For the first, if f is a Lipschitz function with Lipschitz constant at most 1,

$$\begin{aligned}
\left| \int \left[f\left(X^{(n)}(x)\right) - f\left(X^{(n)}(y)\right)\right] \mathfrak{P}_n(t,z,x) d(\nu \otimes \lambda)^n \right| &\leq \int \left| f\left(X^{(n)}(x)\right) - f\left(X^{(n)}(y)\right)\right| \mathfrak{P}_n(t,z,x) d(\nu \otimes \mu)^n \\
&\leq \int d\left(X^{(n)}(x) - X^{(n)}(y)\right) \mathfrak{P}_n(t,z,x) d(\nu \otimes \lambda)^n \\
&\leq \gamma \int d\left(X^{(n-1)}(x), X^{(n-1)}(y)\right) \mathfrak{P}_{n-1}(t,z,x) d(\nu \otimes \lambda)^{n-1} \\
&\leq \cdots \leq \gamma^n d(x,y) \leq d(x,y).
\end{aligned}$$

The second term is equally simple using the estimates above. Since f is bounded and the second term only contains one term in f, we may estimate f with its supremum norm and hence only look at the functions $\mathfrak{P}$ that are inside the integral. This yields the estimate

$$\int \left| f\left(X^{(n)}(y)\right)\right| \left|\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)\right| d(\nu \otimes \mu)^n \leq \|f\|_\infty \int \left|\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)\right| d(\nu \otimes \mu)^n,$$

but by Lemma (2.7.1), $\|f\|_\infty \int |\mathfrak{P}_n(t,z,x) - \mathfrak{P}_n(t,z,y)| \, d(\nu \otimes \mu)^n \leq Cd(x,y)$, so that if f is such that $\|f\|_{\mathrm{BL}} \leq 1$, then immediately

$$\left| \int f\left(X^{(n)}(x)\right) \mathfrak{P}_n\left(t,z,x\right) - f\left(X^{(n)}(y)\right) \mathfrak{P}_n\left(t,z,y\right) d(\nu \times \lambda)^n \right| \leq (C+1) d(x,y),$$

and we conclude that Q has the UEB-property.

□

Remark: In fact, what has been proven in the theorem above is even slightly stronger than the UEB-property of the operator Q. The proof essentially yields that the family of functions $\{V^n f\}_{n\in\mathbb{N}}$ is uniformly Lipschitz. This is clearly sufficient to imply the UEB-property, but is not necessary. One may think that the conditions under which this theorem has been proven could even be weakened to show uniform equicontinuity of the family of $\{V^n f\}_{n\in\mathbb{N}}$. More explicitly, the modeling assumptions for contractivity on average give a uniform bound for γ that is independent of the base point x. One can imagine that this need not always be true, and that in some cases, integrating over x may still yield the result, even if $\sup_x \gamma_x = 1$.

2.9 Finishing the proof

Recall that by Theorem 2.3.1, the existence of the invariant ergodic measure follows by the UEB-property in conjunction with the semi-concentrating property. By Corollary (2.3.1), it suffices to find a Lyapunov function F, bounded on bounded sets such that

$$VF(x) \leq aF(x) + b$$

for $a \in [0,1), b \geq 0$. This in our case is trivial. Take $V(x) = d(x, \Gamma)$. Then,

$$VF(x) = \int d\left(\Phi_{t_1}(x + z_1), \Gamma\right) dp_{x+z_1}(dt_1) dq_x(dz_1) \leq \operatorname{diam}(B_\Gamma)$$

so the claim follows for $a = 0$ and $b = \mathrm{diam}(B_\Gamma)$, since F is clearly (totally) bounded. We remark the ease with which the semi-concentrating property follows. Note that in principle a similar estimate can be made whenever the basin of attraction is not a bounded set, in this case it can be interesting to investigate to which degree the stability assumptions made at the beginning of this section imply the existence of $a \leq 1$ such that the equation above still holds.

Now that the semi-concentrating property of the operator has been shown, we finish this section with the final result

Theorem 2.9.1. *Under the same conditions as Theorem 2.8.1, the Markov Operator Q admits a unique invariant measure*

Proof: By Theorem 2.3.1 the statement holds whenever Q has the UEB-property and it semi-concentrating. That Q is semi-concentrating has been seen from the equation above. That the conditions imply that Q has the UEB-property follows from Theorem 2.8.1. This concludes the proof.

□

Notice immediately how the assumptions needed for this final theorem are weaker than the conditions needed for the theorem derived from non-expansiveness to hold. Not only that, but the aforementioned replacement of *contractivity on average* by a condition on the distances between iterates of the system makes estimation on this new model much simpler than using non-expansiveness.

2.10 The KBBY decomposition revisited

The argument employed above fails in the model description used to show the UEB property. The difference lies in the support of the iterates of measures under the newly defined Markov operator Q. Recall that in showing the UEB property, invariance of the region $\mathcal{D}_\rho$ played a much more marginal role. Given that in this model description the jumps are done *before* the deterministic evolution of the system, there is no guarantee anymore, in contrast to the non-expansive case, that iterates from outside of $\mathcal{D}_\rho$ will always come into $\mathcal{D}_\rho$. In fact, a straightforward computation shows the following. Let Δ be given by

$$\Delta = \inf_{x \in \mathcal{D}_\rho} \{d(x, \partial B_\Gamma)\}.$$

Then clearly, since perturbations are at most of size Δ, iterates in $\mathcal{D}_\rho$ do not leave $\mathcal{D}_\rho$, but it may be true that for any y such that $d(y, \partial B_\Gamma) < \Delta$, $\mathbb{P}(y + z_1 \notin B_\Gamma) > 0$. Hence, ϵ_x is, for $x \notin \mathcal{D}_\rho$ expected to be the convex combination of the invariant measure that lives on $\mathcal{D}_\rho$, and another (unknown, perhaps intractable or ill-behaved) measure, which completely depends on whatever happens when the process leaves B_Γ. This will be a central theme in Chapter 2. Note that this gives also richer dynamics: if again there is a good description of what happens at the boundary of B_Γ, explicit expressions can be made for ϵ_x: it can even be shown to converge for all $x \in S$, and properties of invariance and it being a convex combination of ergodic measures can be shown. Numerically, the coefficients in the convex combination can even be estimated. This will all be done in the following Chapter.

2.11 The UEB property for the original operator

An important model ingredient for the development of the UEB property was the reversal of jump and time kernels from the original definition of the Markov operator as defined in (1.1). Informally, one might expect the behavior for both operators to be very similar *on average*, since for many iterations the order of jumping and then running or the other way around *should* not make a very large difference. The goal of this section is to make this precise, by defining a suitable transformation between the processes and showing under which additional mild conditions, the operator in (1.1) also has the UEB property. The ideas in this section were developed by dr. Sander Hille in personal correspondence with the author.

2.11.1 An explicit relation between U and V

Recall that P, the Markov operator defined in (1.1), was given by

$$\langle P\mu, f\rangle = \int_X \left\{ \int_{\mathbb{R}_+} \int_X f(y) q_{\Phi_t(x)}(dy) p_x(dt) \right\} \mu(dx)$$

for $f \in \mathcal{C}_b(X)$, from which we immediately deduced that the dual operator U could be expressed as

$$Uf(x) = \langle P\delta_x, f\rangle = \int_{\mathbb{R}_+} \int_X f(y) q_{\Phi_t(x)}(dy) p_x(dy).$$

In defining the perturbations as additive perturbations on the dynamical system, we make the assumption that X is a closed subset of an envelloping Banach space, which meant that for all $x \in X$ we introduced a distribution on the jump-sizes

$$\tilde{q}_x(A) := q_x(x + A)$$

so as in the discussion at the beginning of this chapter, we obtain a new, but equivalent, definition of U

$$Uf(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} \int_X f\left(\Phi_t(x) + \eta\right) \tilde{q}_{\Phi_t(x)}(d\eta) p_x(dt). \tag{2.6}$$

For notational convenience, define

$$p_x^\Phi \stackrel{\text{def}}{=} \Phi_\bullet(x) \# p_x. \tag{2.7}$$

That is, $p_x^\Phi \in \mathcal{P}(X)$ and $p_x^\Phi(E) = p_x\left(\Phi_\bullet(x)^{-1}(E)\right)$ for $E \in \mathcal{B}(X)$, where $\Phi_\bullet(x) : \mathbb{R}_+ \to X$ is given by $t \mapsto \Phi_t(x)$. Then, (2.6) can be further rewritten as

$$Uf(x) = \int_X \int_X f(\xi + \eta) \tilde{q}_\xi(d\eta) p_x^\Phi(d\xi) \tag{2.8}$$

As earlier in this chapter, writing $q_\bullet$ instead of $\tilde{q}_\bullet$, we define the two operators

$$A_q f(x) \stackrel{\text{def}}{=} \int_X f(x + \eta) q_x(d\eta), \tag{2.9}$$

$$B_p f(x) \stackrel{\text{def}}{=} \int_X f(\xi) p_x^\Phi(d\xi) = \int_{\mathbb{R}_+} f(\Phi_t(x)) p_x(dt). \tag{2.10}$$

So that immediately V as defined in (2.3) and U satisfy the pair of relations

$$\begin{aligned} Vf(x) &= \int_X \int_X f(\xi) p_{x+\eta}^\Phi(d\xi) q_x(d\eta) = A_q B_p f(x), \\ Uf(x) &= B_p A_q f(x). \end{aligned}$$

which directly implies the following key observation

Lemma 2.11.1. *For $n \geq 2$,*

$$\begin{aligned} U^n f &= B_p V^{n-1} A_q f, \\ V^n f &= A_q U^{n-1} B_p f. \end{aligned}$$

□

Lemma 2.11.1 gives the explicit relation between the two operators defined in (1.1) and (2.6). The approach from now on will establish several continuity and regularity properties of A_q and B_p to conclude the UEB property of U.

2.11.2 Some properties of A_q

An important assumption from now onward is that the metric d on X is translation invariant. That is, we assume that X is a *Fréchet space*. For all $x, y \in X$ and z in the *envelloping Banach space* such that $x+z, y+z \in X$,

$$d(x+z, y+z) = d(x,y)$$

Denote $\hat{X}$ the Banach space that envellops X. Recall that any $f \in \mathrm{BL}(X,d)$ can be extended to $\hat{f} \in \mathrm{BL}(\hat{X})$ such that $\|f\|_\infty = \|\hat{f}\|_\infty$, $|f|_L = |\hat{f}|_L$ and $\hat{f}|_X = f$.

Now, let $f \in \mathrm{BL}(X,d)$ and $x, y \in X$. Then,

$$\begin{aligned} |A_q f(x) - A_q f(y)| &= \left| \int_X f(x+\eta) q_x(d\eta) - \int_X f(y+\zeta) q_y(d\zeta) \right| \\ &\leq \left| \int_X \hat{f}(x+\eta)(q_x - q_y)(d\eta) \right| + \left| \int_X \hat{f}(x+\eta) - f(y+\eta) q_y(d\eta) \right| = (\star), \end{aligned}$$

where the last equality follows given that $y + \eta \in X$ q_y-almost surely and $x + \eta \in X$, q_x-almost surely.

Note that for any $x \in X$, $\eta \mapsto \hat{f}(x+\eta) \in \mathrm{BL}(X)$, as for any η, η'

$$|\hat{f}(x+\eta) - \hat{f}(x+\eta')| \leq |\hat{f}|_L \|\eta - \eta'\|_{\hat{X}} = |f|_L \cdot d(\eta, \eta'),$$

due to the translation invariance of the distance. Furthermore, $\sup_{\eta \in \hat{X}} |\hat{f}(x+\eta)| \leq \sup_{y \in \hat{X}} |\hat{f}(y)| = \|\hat{f}\|_\infty = \|f\|_\infty$, so that

$$\|\hat{f}(x+\bullet)\|_{\mathrm{BL}(X)} \leq \|f\|_\infty + |f|_L = \|f\|_{\mathrm{BL}(X)}.$$

This allows for the estimate

$$\begin{aligned} (\star) &\leq \|f\|_{\mathrm{BL}(X)} \cdot \|q_x - q_y\|^*_{\mathrm{BL}} + \int_X |\hat{f}(x+\eta) - \hat{f}(y+\eta)| q_y(d\eta) \\ &\leq \|f\|_{\mathrm{BL}} \cdot \|q_x - q_y\|^*_{\mathrm{BL}} + |f|_L \cdot d(x,y), \end{aligned}$$

and this implies the following result on A_q.

Lemma 2.11.2. *Assuming that $x \mapsto q_x$ is $\|\cdot\|^*_{BL}$-Lipschitz, meaning*

$$\|q_x - q_y\|^*_{BL} \leq L'_q \cdot d(x,y)$$

then, for any $f \in BL(X,d)$, $A_q f \in BL(X,d)$ and

$$\begin{aligned} \|A_q f\|_\infty &\leq \|f\|_\infty \\ |A_q f|_L &\leq L'_q \|f\|_{BL} + |f|_L \end{aligned}$$

so that in particular, $A_q \in \mathcal{L}(BL(X,d))$ and

$$\|A_q\|_{\mathcal{L}(BL(X))} \leq (1 + L'_q)$$

Proof: The first assertion is clear from the fact that q is a probability distribution. The second assertion follows from the estimate on $(\star)$. The last property follows by definition of the norm of a linear operator.

□

2.11.3 Some properties of B_p

We proceed with establishing the necessary regularity properties of B_p for the proof. For this, if the underlying deterministic dynamical system of the PDMP is defined by an ODE,

$$\begin{cases} \dot{x}(t) &= F(x(t)), \quad t \geq 0 \\ x(0) &= x_0 \in \mathbb{R}^n \end{cases} \tag{2.11}$$

for $x_0 \in X \subset \mathbb{R}^n$ a closed invariant region for the flow F and X equipped with the restriction of the Euclidean metric on $\mathbb{R}^n$, then the solution $t \mapsto x(t; x_0)$ to (2.11), which is unique when F is globally Lipschitz and defined for all $t \geq 0$, satisfies (dropping x_0 from notation)

$$x(t) = x_0 + \int_0^t F(x(s))ds. \tag{2.12}$$

Consequently, for $t \geq s$,

$$|x(t) - x(s)| = \left| \int_s^t F(x(\sigma))d\sigma \right| \leq \|F\|_\infty |t - s|$$

provided that

$$\|F\|_\infty \stackrel{\text{def}}{=} \sup_{x \in X} |F(x)| < \infty,$$

where $|F(x)|$ here denotes the ℓ^2 norm on $\mathbb{R}^n$. Note that the Lipschitz constant for $t \mapsto x(t; x_0)$ is *not* dependent on the initial condition x_0. Hence, it is quite natural in view of the application in the next chapter to assume for the flow:

$$d\left(\Phi_t(x), \Phi_s(x)\right) \leq L_\Phi |t - s| \quad \text{for any } t, s \geq 0, x \in X \tag{2.13}$$

Another assumption we would like to impose is:

$$\int_X d\left(\Phi_t(x), \Phi_t(y)\right) p_x(dt) \leq L_\Phi^p \cdot d(x, y), \quad \text{for any } x, y \in X \tag{2.14}$$

Under these two last assumptions, we can formulate the following result.

Lemma 2.11.3. *Under Assumptions (2.13) and (2.14), if $x \mapsto p_x$ is Lipschitz continuous for the $\|\cdot\|_{BL}^*$ norm on $\mathcal{M}(X)$, with Lipschitz constant L_p', then B_p maps $BL(X, d)$ into itself and for any $f \in BL(X, d)$,*

$$\begin{aligned} &\|B_p f\|_\infty \leq \|f\|_\infty \\ &|B_p f\|_L \leq |f|_L \cdot L_\Phi^p + L_p' \cdot \left(\|f\|_\infty + L_\Phi |f|_L\right). \end{aligned}$$

In particular, $B_p \in \mathcal{L}(BL(X))$ and

$$\|B_p\|_{\mathcal{L}(BL(X))} \leq \max\left(1 + L_p, L_\Phi^p + L_p L_\Phi\right).$$

Proof: Clearly, for any $f \in \mathrm{BL}(X, d)$ and $x \in X$, $|B_p f(x)| \leq \|f\|_\infty$. If $x, y \in X$, then

$$\begin{aligned} |B_p f(x) - B_p f(y)| &= \left| \int_X f\left(\Phi_t(x)\right) p_x(dt) - \int_X f\left(\Phi_s(y)\right) p_y(ds) \right| \\ &\leq \left| \int_X \left[f\left(\Phi_t(x)\right) - f\left(\Phi_t(y)\right)\right] p_x(dt) \right| + \left| \int_X f\left(\Phi_s(y)\right) [p_x - p_y](ds) \right|. \end{aligned}$$

Now, $\sup_{t \geq 0} |f\left(\Phi_t(x)\right)| \leq \|f\|_\infty$ and

$$\begin{aligned} |f\left(\Phi_t(x)\right) - f\left(\Phi_s(x)\right)| &\leq |f|_L \cdot d\left(\Phi_t(x), \Phi_s(x)\right) \\ &\leq |f|_L \cdot L_\Phi \cdot |t - s|. && \text{(by (2.13))} \end{aligned}$$

So, $s \mapsto f\left(\Phi_s(y)\right) \in \mathrm{BL}(\mathbb{R}_+)$ and $\|s \mapsto f\left(\Phi_s(y)\right)\|_{\mathrm{BL}(\mathbb{R}_+)} \leq \|f\|_\infty + |f|_L \cdot L_\Phi$.

Consequently,

$$\begin{aligned} |B_p f(x) - B_p f(y)| &\leq |f|_L \cdot \int_X d\left(\Phi_t(x), \Phi_t(y)\right) p_x(dt) + \left(\|f\|_\infty + |f|_L \cdot L_\Phi\right) \|p_x - p_y\|_{\mathrm{BL}}^* \\ &\leq |f|_L \cdot L_\Phi^p \cdot d(x, y) + L_p' \left(\|f\|_\infty + |f|_L \cdot L_\Phi\right) \cdot d(x, y). && \text{(by (2.14))} \end{aligned}$$

From this last estimate, the operator norm of B_p follows.

□

Remark: If $x \mapsto p_x$ is Lipschitz continuous with respect to the $\|\cdot\|_{\mathrm{TV}}$-norm with Lipschitz constant L_p^{TV}, then assumption (2.13) is not needed. In that case, $|B_p f|_L \leq |f|_L \cdot L_\Phi^p + \|f\|_\infty \cdot L_p^{\mathrm{TV}}$.

2.11.4 The UEB-property of U

We are now in a position to conclude the UEB property of U. We begin with this first result

Corollary 2.11.1. *Under the assumptions of Lemma 2.11.2, and Lemma 2.11.3, the family* $\{U^n f : n \in \mathbb{N}\}$ *is bounded in* $BL(X,d)$ *for every* $f \in BL(X,d)$ *if and only if the family* $\{V^n f : n \in \mathbb{N}\}$ *is bounded in* $BL(X,d)$ *for every* $f \in BL(X,d)$.

Proof: This follows immediately from Lemma 2.11.1 together with Lemmata 2.11.2 and 2.11.3.

□

In the concluding remark of Section 2.8, we observed that we have shown that $\{V^n f : n \in \mathbb{N}\}$ is uniformly Lipschitz for every $f \in \mathrm{BL}(X,d)$. Thus, the family $\{U^n f : n \in \mathbb{N}\}$ is uniformly Lipschitz, which amounts to $\{U^n f : n \in \mathbb{N}\}$ being bounded in $\mathrm{BL}(X,d)$. This allows us to conclude

Proposition 2.11.1. *Under the assumptions of Lemmata 2.11.2 and 2.11.3, P has the UEB property if and only if Q has the UEB property.*

Proof: By Proposition 2.2.1 the UEB property of the operator P (or Q) is equivalent to the family $\{U^n f : n \in \mathbb{N}\}$ (or $\{V^n f : n \in \mathbb{N}\}$) being uniformly equicontinuous in $C_b(X)$ for every $f \in \mathrm{BL}(X,d)$. Since A_q and B_p map $\mathrm{BL}(X,d)$ continuously into itself and are bounded operators on $C_b(X)$, Lemma 2.11.1 shows that $\{U^n f : n \in \mathbb{N}\}$ is uniformly equicontinuous if and only if $\{V^n f : n \in \mathbb{N}\}$ is. This finishes the proof.

□

Corollary 2.11.2. *Under Assumptions (1-$\mathcal{N}$)-(4-$\mathcal{N}$), (1-$\mathcal{U}$) and (2-$\mathcal{U}$), P has the UEB property.*

Proof: It was remarked at the end of Section 2.8 that we have shown that $\{V^n f : n \in \mathbb{N}\}$ is uniformly Lipschitz for every $f \in \mathrm{BL}(X,d)$. In particular, it is uniformly equicontinuous in $C_b(X)$. This finishes the claim in light of Proposition 2.11.1.

□

And, as desired, we arrive to the conclusion that the UEB property of the operator P should not be *too far removed* to the UEB property of the operator Q. In particular, just addition assumptions 2.13 and 2.14 to the assumptions necessary for the statement of Theorem 2.8.1 imply that the original operator also has a unique, invariant, ergodic measure. This is a significant simplification with respect to the earlier necessary assumptions for the process when dealing with the operator in the non-expansive case.

Chapter 3

An Application to a Perturbed Predator-Prey Model with Hopf Bifurcation

The purpose of this chapter is to showcase a non-trivial application of the theory developed in the previous chapters. Recall that, under some suitable conditions on the model, Chapter 1 shows how there is persistence of stability around a stable equilibrium and Chapter 2 extends this result so that persistence of stability holds around a limit cycle. However, the situation in practice necessitates that one verifies these model assumptions *from scratch*. Even for a well-known planar system such as the predator-prey model at the heart of this chapter, it will not be easy to ascertain that the needed conditions hold. In particular, exponential convergence to a limit cycle is well-known in the literature as a notoriously difficult problem in the study of ODE models (see, for instance, [26] and references therein). In the literature on dynamical systems, a range of techniques have been developed to address this issue. This theory is interesting in its own right and will be showcased in order to give the reader an impression of the available techniques to establish exponential convergence in problems of this kind.

In exposing these results, we will simultaneously discover the intimate connection between exponential convergence and the contractive properties of the deterministic model in the basin of attraction. These dynamics shape the properties of the associated Markov operator of the PDMP, as was also seen in the analysis in Chapter 2. The link between contraction theory and stochastically perturbed dynamical systems has already been established in the literature in the case of white noise (for instance in [27] and references therein). This approach is referred to as *stochastic contraction analysis*. In this work we wish to extend the work by considering the natural application of this theory to PDMPs.

We start this chapter by understanding the conditions under which the famous Rosenzweig-MacArthur model displays its qualitative properties. We are specifically interested under which conditions there is a stable and non-trivial equilibrium and when there is an asymptotic limit cycle, and how these conditions influence a suitable model for perturbations such that the PDMP has a stable, unique and invariant ergodic measure on a domain near these attractors. While the dynamics of the Rosenzweig-MacArthur model are well-studied, this is (as far as we are aware) the first work where explicit estimates on the contractivity of the system are established and used to prove persistence of stability.

3.1 The Rosenzweig-MacArthur Model and its Dynamics

The Rosenzweig-MacArthur model for the population of a predator species p and a prey (or victim) species v, combines logistic growth of the prey population with satiation of predation by the predator population, modeled with a so-called Holling type II functional response (see, for example, [38]). The model is described in the equations below

$$\dot{v} = rv\left(1 - \frac{v}{K}\right) - \frac{av}{b+v} \cdot p, \tag{3.1}$$

$$\dot{p} = -dp + h \cdot \frac{av}{b+v} \cdot p \tag{3.2}$$

The parameters satisfy $a, d, d, h, r, K > 0$ as that is the biologically relevant setting. The biologically relevant solution space for (3.1)-(3.2) is $\mathbb{R}^2_+$. We therefore limit our analysis to this state space. The goal of this section is to understand the intricate dynamics of the Rosenzweig-MacArthur model. This is needed, since we must understand when we are in the setting that the underlying deterministic system of the PDMP we want to construct by perturbing the model has an asymptotically stable limit cycle. For the remainder of the chapter, assume that $ha \neq d$ and let $\eta \stackrel{\text{def}}{=} \frac{d}{ha-d}$. We assume that $\eta > 0$.

The first question of interest when looking at the dynamics, is understanding the null-clines of the system.

Proposition 3.1.1. *The v-null-cline is given by $\{v = 0\} \cup \left\{p = \frac{rK}{a}(v+b)(K-v)\right\}$. The p-null-cline is given by $\{p = 0\} \cup \left\{v = \frac{db}{ha-d}\right\}$.*

Proof: First, $\dot{v} = 0$ if and only if $v = 0$ or $r\left(1 - \frac{v}{K}\right) = \frac{ap}{b+v}$. This is equivalent to $r(b+v)\left(1 - \frac{v}{K}\right) = ap$, so that the v-null-cline is found at $p = \frac{rK}{a}(v+b)(K-v)$. For the p-null-cline, $\dot{p} = 0$ if and only if $p = 0$ or $d = \frac{hav}{b+v}$, which is equivalent to $v = \frac{db}{ha-d}$.

□

Remark: The v-null-cline has a maximum at $v = \frac{K-b}{2}$, yielding that the corresponding $p_{\max} = \frac{rK}{a}(K+b)$.

When $\eta < \frac{K}{b}$, then the set $\{(v,p) \in \mathbb{R}^2_+ : v \leq K\}$ is positively invariant. Thus, for $\bar{p} \gg p_{\max}$ and $0 < \eta < \frac{K}{b}$, one can construct an invariant region within $\{(v,p) \in \mathbb{R}^2_+ : v \leq K, p < \bar{p}\}$ with the *right top corner removed* that is invariant. We continue, now characterizing the steady states in $\mathbb{R}^2_+$.

Proposition 3.1.2. $(0,0)$ *and* $(K,0)$ *are trivial steady states for the flow.* $(0,0)$ *is stable in* $\{v = 0\}$ *and unstable in* $\{p = 0\}$. $(K,0)$ *is a stable steady state in* $\{p = 0\}$. *There exists a non-trivial steady state* x^* *in the interior of* $\mathbb{R}^2_+$ *if and only if* $0 < \eta b < K$. x^* *is stable if* $\eta < \frac{K}{b} < 1 + 2\eta$ *and unstable when* $\eta < \frac{K-b}{2}$.

Proof: For the assertions regarding $(0,0)$ and $(K,0)$ note that they happen at the intersection of the nullclines. For the claim regarding their stability on the axes, it suffices to check the sign of the trace and determinant of the Jacobian matrix of the system. The ODE can be written as $\dot{x} = f(x)$, with $x = \begin{pmatrix} v \\ p \end{pmatrix}$, then

$$Df(X) = \begin{pmatrix} r - \frac{2rv}{K} + \frac{ab}{(b+v)^2 p} & -\frac{av}{b+v} \\ \frac{hab}{(b+v)^2} p & -d + \frac{hav}{b+v} \end{pmatrix}.$$

At $(0,0)$, we have that

$$Df(0,0) = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix}$$

with eigenvalues $r > 0$ and $-d < 0$, so $(0,0)$ is always an unstable saddle, as claimed. For $(K,0)$,

$$Df(K,0) = \begin{pmatrix} -r & -\frac{aK}{b+K} \\ 0 & -d + h\frac{aK}{b+K} \end{pmatrix},$$

with eigenvalues $-r < 0$ and $-d + h\frac{aK}{b+K}$. So $(K,0)$ is stable when $h\frac{aK}{b+K} < d$, which is equivalent to $\frac{K}{b} < \eta$ or $\eta < 0$ and unstable when $h\frac{aK}{b+K} > d$.

In [1] it is shown, provided $0 < \eta < \frac{K}{b}$, that there exists a non-trivial steady state $x^* \in \mathbb{R}^2_{>0}$ with $x^* = (v^*, p^*)$ with $v^* = b\eta$ and $p^* = \frac{rK}{a}(v^* + b)(K - v^*)$. We obtain that $Df(x^*)$ equals

$$\begin{pmatrix} r - \frac{2r}{K}v^* - \frac{ab}{(b+v^*)^2}p^* & -\frac{av^*}{b+v^*} \\ h\frac{ab}{(b+v^*)^2}p^* & -d + h\frac{av^*}{b+v^*} \end{pmatrix} = \begin{pmatrix} r\left(1 - \frac{2v^*}{K}\right) - \frac{b}{a}\frac{d}{h}r\left(1 - \frac{v^*}{K}\right)\frac{1}{v^*} & -\frac{d}{h} \\ \frac{b}{a}\frac{1}{v^*}dr\left(1 - \frac{v^*}{K}\right) & -d + h\frac{d}{h} \end{pmatrix} \quad (\text{Since } \tfrac{av^*}{b+v^*} = \tfrac{d}{h})$$

$$= \begin{pmatrix} r - \frac{2rv^*}{K} - \frac{bd}{ah}r\left(\frac{1}{v^*} - \frac{1}{K}\right) & -\frac{d}{h} \\ \frac{bd}{a}r\left(\frac{1}{v^*} - \frac{1}{K}\right) & 0 \end{pmatrix} = \begin{pmatrix} \frac{r}{K}(K - 2b\eta) - \frac{dr}{ahK\eta}(K - b\eta) & -\frac{d}{h} \\ \frac{dr}{aK\eta}(K - b\eta) & 0 \end{pmatrix},$$

where the last equality follows from

$$\frac{1}{v^*} - \frac{1}{K} = \frac{1}{b\eta} - \frac{1}{K} = \frac{K - b\eta}{Kb\eta}, \quad r - \frac{2rv^*}{K} = \frac{r}{K}(K - 2b\eta).$$

Now, note that

$$\begin{aligned}\mathrm{Tr}(Df(x^*)) &= (K - b\eta)\left\{\frac{r}{K} - \frac{dr}{ahK\eta}\right\} - \frac{r}{K}b\eta = \frac{r}{K}\left\{(K - b\eta)\left(1 - \frac{d}{ah\eta} - b\eta\right)\right\} \\ &= \frac{r}{K}b\eta\left\{\left(\frac{K}{b\eta} - 1\right)\left(1 - \frac{d}{ah\eta}\right) - 1\right\} = \frac{r}{K}b\eta\left\{\left(\frac{K}{b\eta} - 1\right)\frac{d}{ha} - 1\right\}\end{aligned}$$

and

$$\det(Df(x^*)) = \frac{d}{h}\frac{dr}{aK\eta}(K - b\eta) = \frac{d^2r}{a\eta}\left(1 - \frac{b\eta}{K}\right).$$

So, if $0 < \eta < \frac{K}{b}$, then $\det(Df(x^*)) > 0$, so either both eigenvalues are real and have the same sign, or they are both complex and have the same real part. For the trace, we remark that it is negative is and only if $\left(\frac{K}{b\eta} - 1\right)\left(1 - \frac{d}{ah\eta}\right) < 1$. Since $\left(\frac{K}{b\eta} - 1\right) > 0$, we get that this amounts to $\left(1 - \frac{d}{ah\eta}\right) < \left(\frac{K}{b\eta} - 1\right)^{-1} = \frac{b\eta}{K - b\eta}$. Now,

$$1 - \frac{d}{ah\eta} = 1 - \frac{d}{ah\frac{d}{ha-d}} = \frac{d}{ha}$$

so we conclude that the trace is negative if and only if $\left(\frac{K}{b\eta} - 1\right) < \frac{ha}{d}$, so

$$\frac{K}{b\eta} < \frac{ha}{d} + 1 = \frac{ha - d}{d} + 2 = \frac{1}{\eta} + 2$$

we conclude that x^* is stable when $\frac{K}{b} < 1 + 2\eta$ and unstable when $\eta < \frac{K-b}{2b}$.

□

In principle, the result from Proposition 3.1.2 stated is enough to conclude the existence of a unique, stable limit cycle on $\{(v,p) \leq K\}$: since there exists a positively invariant set in this region, with no stable equilibria, the theorem of Poincaré-Bendixson yields the existence of an asymptotically stable limit cycle arising here. More about the dynamics of this system can be said, however, as shown in the following result, the last of this section.

Proposition 3.1.3. *At $\eta = \frac{K-b}{2b}$ there is a Hopf bifurcation.*

Proof: To show that there is a Hopf bifurcation, we must show that both eigenvalues of $Df(x^*)$ are non-real when $\eta \to \frac{K-b}{2b}$. We have

$$Df(x^*) = \begin{pmatrix} \frac{r}{b\eta}K\left\{\left(\frac{K}{b\eta} - 1\right)\frac{d}{ha} - 1\right\} & -\frac{d}{h} \\ r\frac{db}{aK}\left(\frac{K}{b\eta} - 1\right) & 0 \end{pmatrix},$$

from which the characteristic equation follows

$$\begin{aligned}0 = \det(Df(x^*) - \lambda I) &= -\lambda\left(r\frac{b\eta}{K}\left\{\left(\frac{K}{b\eta} - 1\right)\frac{d}{ha} - 1\right\} - \lambda\right) + \frac{d}{h}r\frac{db}{aK}\left(\frac{K}{b\eta} - 1\right) \\ &= \lambda^2 - r\frac{b\eta}{K}\left\{\left(\frac{K}{b\eta} - 1\right)\frac{d}{ha} - 1\right\}\lambda + \frac{d^2rb}{aK}\left(\frac{K}{b\eta} - 1\right).\end{aligned}$$

The discriminant of the characteristic equation is given by

$$D = \left[r\frac{b\eta}{K} \left\{ \left(\frac{K}{b\eta} - 1 \right) \frac{d}{ha} - 1 \right\} \right].$$

Let $y = \frac{K}{b\eta} - 1$ and recall that $0 < \eta < \frac{K}{b}$, meaning that $y \in (0, \infty)$. At the same time, $\eta = \frac{d}{ha-d}$, so $\frac{1}{\eta} = \frac{ha}{d} - 1$ and so $\frac{d}{ha} = \frac{\eta}{\eta+1}$. Finally, $\frac{d}{ha}(y+1) = \frac{\eta}{\eta+1}\frac{K}{b\eta} = \frac{K}{b}\frac{1}{\eta+1}$. Consequently, we conclude that $D < 0$ if and only if

$$\begin{aligned}
r\left(\frac{ha}{d}y - 1\right)^2 &< 4\frac{d^2}{a}(y+1)y \\
r\frac{d^2}{h^2a^2}\left(y - \frac{ha}{d}\right)^2 &< 4\frac{d^2}{a}y(y+1) \\
\frac{r}{h^2a}\left(y - 1 - \frac{1}{\eta}\right)^2 &< 4y(y+1) \\
\frac{r}{dh}\frac{\eta}{\eta+1}\left(\left(\frac{K}{b} - 1\right)\frac{1}{\eta} - 2\right)^2 &< 4\frac{K}{b\eta}\left(\frac{K}{b}\frac{1}{\eta} - 1\right) \\
\frac{r}{dh}\left(\left(\frac{K}{b} - 1\right)\frac{1}{\eta} - 2\right)^2 &< 4\frac{K}{b}(\eta+1)\left(\frac{K}{b}\frac{1}{\eta} - 1\right) \\
\frac{r}{dh}\left(\frac{K}{b} - 1 - 2\eta\right)^2 &< 4\frac{K}{b}\eta(\eta+1)\left(\frac{K}{b} - \eta\right),
\end{aligned}$$

so the condition for Hopf bifurcation is given by

$$\frac{r}{dh}\left(\frac{K-b}{2b} - \eta\right)^2 < \frac{K}{b}\eta(\eta+1)\left(\frac{K}{b} - \eta\right). \tag{3.3}$$

When $\eta \to \frac{K-2}{2b}$, then

$$\frac{K}{b}\eta(\eta+1)\left(\frac{K}{b} - \eta\right) \to \frac{K}{b}\frac{K-b}{2b}\frac{K+b}{2b}\frac{K+b}{2b} > 0,$$

so Equation (3.3) is satisfied for η close to $\frac{K-b}{2b}$. We conclude that sufficiently close to $\eta = \frac{K-b}{2b}$, where the stability of x^* changes, both eigenvalues of $Df(x^*)$ are non-real, provided $\eta \to \frac{K-b}{2b}$ while dh remains bounded. At $\eta = \frac{K-b}{2b}$ the stability of x^* changes and so we have a Hopf bifurcation here.

□

3.2 First Attempt at Exponential Convergence to a Limit Cycle: Floquet Theory

As shown in the previous section, there is a Hopf bifurcation at $\eta = \frac{K-b}{2b}$. At the Hopf bifurcation, a limit cycle appears, but it is not clear what the stability of this limit cycle is. In particular, the most important assumption that needs to be verified in applying the results for the existence of a unique invariant measure around this cycle is the exponential convergence to the limit cycle, as for the rest we may prescribe any time and jump kernel of our choosing that satisfies the conditions for the existence of the unique ergodic invariant measure around the limit cycle. Hence, we will need tools from ODE theory to establish that indeed exponential convergence takes place. In doing so, we will encounter other details along the way that will be interesting in their own right and will receive much attention in this Chapter. We start this section with some material taken from [28]. The presentation from now on follows the lecture notes and results, unless otherwise stated, are derived from the lecture notes.

3.2.1 The Monodromy Matrix

In a general setting, we are interested in periodic orbits in a non-linear ODE. The goal of the material presented in the lecture notes is to arrive at a simple test in terms of the linearized system to arrive at conclusions about the stability of the limit cycle. The most important idea is be to exploit the movement of the whole system along with the limit cycle and make estimates about where iterates of a suitable map are located.

For $n \geq 1$, $x \in \mathbb{R}^n$, we consider the equation

$$\dot{x} = f(x)$$

where we assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. From the theory of ODEs it is well-known that then the associated flow $\phi^t(x_0)$ is also at least C^1 smooth. A central object in Floquet theory is the so called *monodromy* matrix. The following theorem, which we adopt from the lecture notes and a proof of which can be found there, defines the monodromy matrix and establishes a key property.

Theorem 3.2.1. *The matrix*

$$Y(t) = \left.\frac{\partial \phi^t(x_0)}{\partial x}\right|_{x=x_0}$$

satisfies the linear differential equation

$$\dot{Y} = f_x\left(\phi^t(x_0)\right)Y, \quad Y(0) = I_n$$

□

The importance of Y comes from the fact that solutions to the ODE depend on Y according to the proposition below. The statement and prove both come from [28].

Proposition 3.2.1. *Suppose that $\phi^t(x_0)$ is defined for $t \in [0,T]$, where $T > 0$. Let $y_0 = f(x_0)$ and $y_1 = f(\phi^{t_1}(x_0))$, where $t_1 \in [0,T]$. Then $y_1 = Y(t_1)y_0$.*

Proof: Since $x(t) = \phi_t(x_0)$ is a solution to the differential equation, it follows that

$$\frac{d}{dt}\phi^t(x_0) = f\left(\phi^t(x_0)\right).$$

So, we may differentiate this equation with respect to t to obtain that

$$\frac{d}{dt}\left(\frac{d}{dt}\phi^t(x_0)\right) = f_x\left(\phi^t(x_0)\right)\frac{d}{dt}\phi^t(x_0)$$

so that $y(t) = f\left(\phi^t(x_0)\right)$ is a solution to the linearized problem

$$\dot{y} = f_x\left(\phi^t(x_0)\right)y$$

with the initial condition $y(0) = f(x_0) = y_0$. Since any such solution has the form $y(t) = Y(t)y_0$, we get indeed that $y_1 = y(t_1) = Y(t_1)y_0$.

□

If we let Γ be a periodic orbit of the dynamical system, then there exists $T > 0$ such that for every $x_0 \in \Gamma$, we have $\phi^T(x_0) = x_0$ and $\phi^t(x_0) \neq x_0$ for $t \in (0,T)$. In this case, $\Gamma = \{x \in \mathbb{R}^n : x = \phi^t(x_0), 0 \leq t \leq T\}$ is a smooth and closed curve. Now, observe that $f\left(\phi^T(x_0)\right) = Y(T)f(x_0)$. The periodicity now yields $f(x_0) = Y(T)f(x_0)$, meaning that for any $x_0 \in \Gamma$, $f(x_0)$ is an eigenvector of $Y(T)$ corresponding to eigenvalue 1.

Floquet theory essentially draws a correspondence between the eigenvalues of the monodromy matrix and the asymptotic stability of the limit cycle. The theorems establishing this equivalence will come later, but for now it is worthwhile to explore efficient methods to actually compute the eigenvalues of the monodromy matrix given the system. One such way is provided by the Liouville equation (cf. [28], p. 86). For a general n-dimensional system this gives

$$\det(Y(T)) = \prod_{i=1}^{n} \lambda_i = \exp\left(\int_0^T \operatorname{div} f\left(\phi^t(x_0)\right) dt\right)$$

which can be made more specific when looking at a two-dimensional problem. Note, that since $f(x_0)$ is an eigenvalue of $Y(T)$ with eigenvalue 1, the formula above reduces immediately to

$$\lambda_2 = \exp\left(\int_0^T \operatorname{div} f\left(\phi^t(x_0)\right) dt\right)$$

for λ_2 the other eigenvalue of the monodromy matrix. In practice, this formula can be used to establish the desired conditions for the theorems to come later.

3.2.2 The Poincaré Map

The preliminaries from the previous section can be applied to formulate results about the asymptotic stability of the limit cycle Γ- and stronger, exponential convergence to it. Again, we follow [28] and define for $x_0 \in \Gamma$,

$$\Sigma_0 = \{\xi \in \mathbb{R}^n : \langle f(x_0), \xi \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, and introduce a *cross-section*

$$\Pi_{x_0} = \{x \in \mathbb{R}^n : x = x_0 + \xi, \xi \in \Sigma_0\}.$$

It should be obvious that the orbit starting at x_0 hits Π_{x_0} again after T units of time. We must prove that orbits starting at Π_{x_0} near x_0 hit Π_{x_0} after approximately time T. Actually, this holds for all orbits starting near x_0. This is established in the following lemma, whose statement and proof come from [28].

Lemma 3.2.1. *Fix $x_0 \in \Gamma$. There exists a C^1 map $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $\xi \mapsto \tau(\xi)$, defined in a neighborhood U_{x_0} of $\xi = 0$ such that*

- $\tau(0) = T$
- $\Phi^{\tau(\xi)}(x_0 + \xi) \in \Pi_{x_0}$

Moreover, for $\xi \in U_{x_0}$, if $t \in \mathbb{R}$ is such that $\Phi^t(x_0 + \xi) \in \Pi_{x_0}$ and $|t - T|$ is small enough, then $t = \tau(\xi)$.

Proof: Consider $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by $F(t, \xi) = \langle f(x_0), \phi^t(x_0 + \xi) - x_0 \rangle$ and consider the equation $F(t, \xi) = 0$. Clearly, $F \in C^1$ and also

$$F(T, 0) = \langle f(x_0), x_0 - x_0 \rangle = 0$$

and at the same time

$$F_t(T, 0) = \langle f(x_0), f(x_0) \rangle = \|f(x_0)\|^2 \neq 0$$

and the statement now follows from the Implicit Function Theorem.

□

Definition 3.2.1. The map $\mathcal{P} : \Sigma_0 \to \Sigma_0$, defined for ξ in the open neighborhood U_{x_0} of x_0 (as in Lemma 3.2.1) by the formula

$$\mathcal{P}(\xi) = \phi^{\tau(\xi)}(x_0 + \xi) - x_0$$

is called a Poincaré map of the periodic orbit Γ.

Many things can be stated about the Poincaré map, as it plays a large role in the analysis of dynamical systems. For interested readers, we refer to [29] for more information. The most important result that will be of use to us in the present chapter will connect the Poincaré map to the asymptotic stability of Γ together with an estimate on the speed of convergence of solutions to this limit cycle. To make this precise, we use the following definition, again from [28].

Definition 3.2.2. A cycle Γ through x_0 is called called exponentially orbitally stable with asymptotic phase if there exist $c > 0$, $K > 1$ such that for all x with $d(x, \Gamma)$ sufficiently small there exists $t_0 = t_0(x) \in [0, T)$ such that $\|\phi^t(x) - \phi^{t-t_0}(x_0)\| \leq Ke^{-ct}$, $t \geq 0$.

3.2.3 Asymptotic Phase: a Numerical Computation

In practice, Definition 3.2.2 provides even a way to compute the asymptotic phase: if one knows the period T^* of the asymptotic orbit, or can approximate this numerically, for each initial condition x_0 it is possible to compute the nearest point on the orbit after nT^* for n sufficiently large. Then one assigns the phase of that point on the period orbit to $\tau(x_0)$. An implementation of this numerical procedure can be found in the appendix. These computations are done for the Rosenzweig-MacArthur model with parameter values chosen such that there is a limit cycle as well as for the simple dynamical system given by

$$\dot{x} = -(1 - x^2 - y^2)y, \tag{3.4}$$

$$\dot{y} = (1 - x^2 - y^2)x. \tag{3.5}$$

Figures 3.1 and 3.2 show the results of these simulations, graphically. Here, the color in the plot reflects the asymptotic phase. All points in phase space that have the same asymptotic phase form what is known in the literature as an *isochron*. One can see that in the simple dynamical system, the isochrons divide the phase space *regularly*, while in the Rosenzweig-MacArthur model, the structure of the isochrons is much more difficult. We bring to the reader's attention that in general it is not possible to get a closed form expression for the asymptotic phase nor the isochrons (see also [16]), which makes simulations like those of Figure 3.1 an indispensable tool for the analysis of phase problems. It is seen that for a more complicated dynamical

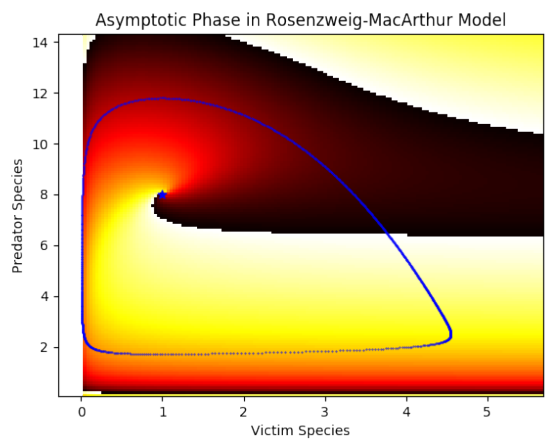

Figure 3.1: Numerical estimation of the asymptotic phase in state space for the Rosenzweig-MacArthur model for $a = 1, b = 1, d = 0.5, h = 1, r = 1.1, K = 5$. Color indicates asymptotic phase. Points with the same color have the same asymptotic phase and form an *isochron*.

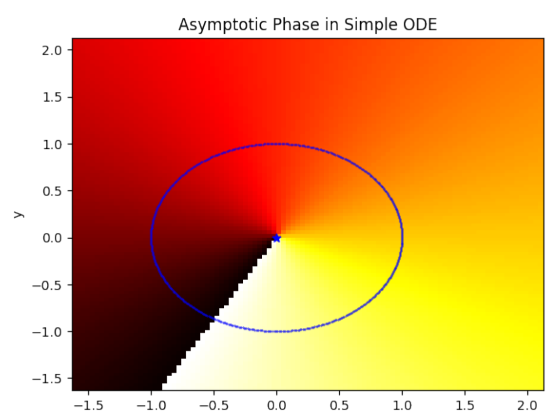

Figure 3.2: Numerical estimation of the asymptotic phase in state space. Color indicates asymptotic phase. Points with the same color have the same asymptotic phase and form an *isochron*.

system, the map assigning the asymptotic phase has a complicated structure. In particular, in the panel for the Rosenzweig-MacArthur model it can be seen that, in contrast to the simple dynamical system, solutions on the limit cycle do not move with constant velocity through the orbit: in the predator-prey model the system spends a very long time near the p-axis, while it spends a much shorter amount of time *shooting* away along the v-axis. We are aware that this kind of argument can be made even more precise with the theory of *time-scale separation*, for which there is extensive literature in applied analysis.

3.2.4 Asymptotic Stability of the Limit Cycle

Returning to the original problem, we want to establish that the limit cycle is exponentially orbitally stable by using the following important theorem, whose proof we reproduce from [28].

Theorem 3.2.2. *If all $(n-1)$ eigenvalues of the linearization $D\mathcal{P}(0)$ of the Poincaré map $\mathcal{P}$ at $\xi = 0$ satisfy $|\lambda| < 1$, then Γ is exponentially orbitally stable with asymptotic phase.*

Proof: Consider first $x = x_0 + \xi_0$, with $\xi_0 \in U_{x_0} \subset \Sigma_0$, and define for a given ξ_0:

$$\begin{aligned} \xi_k &= \mathcal{P}(\xi_{k-1}), \\ \tau_k &= \tau(\xi_{k-1}) + \tau_{k-1}, \tau_0 = 0, \end{aligned}$$

for $k = 1, 2, \ldots$. By Theorem 2.13 of the lecture notes, we know that since the linearization is hyperbolic, there exists $\delta > 0$ such that for all ξ_0 with $\|\xi_0\| \leq \delta$ the estimate $\|\xi_k\| \leq Me^{-\alpha k}\|\xi_0\|$ holds for some $M \geq 1, \alpha > 0$. Hence, $\tau(\xi_k) \to T$ and $\frac{\tau_k}{kT} \to 1$ when $k \to \infty$. But since $\tau \in \mathcal{C}^1$, we can derive a better estimate, that is

$$|\tau(\xi_{k-1}) - T| \leq C\|\xi_{k-1}\| \leq MCe^{-\alpha(k-1)}\|\xi_0\|.$$

And this implies that

$$|(\tau_k - kT) - (\tau_{k-1} - (k-1)T)| = |\tau(\xi_{k-1}) - T| \leq MCe^{-\alpha(k-1)}\|\xi_0\|$$

Thus, $\theta_k = \tau_k - kT$ is a Cauchy sequence, so it has a limit that we denote by t_0. We have

$$|\tau_{k+m} - (k+m)T - (t_k - kT)|MC\| \leq \xi_0\| \sum_{j=0}^{m-1} e^{-\alpha(k+j)} \leq MC\|\xi_0\|\frac{e^{-\alpha k}}{1 - e^{-\alpha}}$$

so taking $m \to \infty$, we find

$$|\tau_k - kT - t_0| \leq MC\|\xi_0\|\frac{e^{-\alpha k}}{1 - e^{-\alpha}}$$

For $0 \leq t \leq$ T, since $\phi^t(x)$ is a $\mathcal{C}^1$-function of (t, x) we find

$$\|\phi^{t+\tau_k}(x_0 + \xi_0) - \phi^t(x_0)\| = \|\phi^t(x_0 + \xi_k) - \phi^t(x_0)\| \leq C_1\|\xi_k\| \leq MC_1e^{-\alpha k}\|x_0\|$$

and likewise

$$\|\phi^{t+\tau_k}(x_0 + \xi_0) - \phi^{t+kT+t_0}(x_0 + \xi_0) \leq MC_2e^{-\alpha k}\|\xi_0\|$$

which together implies that

$$\|\phi^{t+t_0}(x_0 + \xi_0) - \phi^t(x_0)\| \leq M(C1 + C2)e^{-\alpha k}\|\xi_0\|$$

for $kT \leq t \leq (k+1)T$. Now take any $x \in \mathbb{R}^n$ near Γ. If this point does not belong to Π_{x_0}, consider the first intersection of the forward half-orbit starting at x with Π_{x_0} and represent it as $x_0 + \xi_0$. Apply then the proof given above.

□

Note that Theorem 3.2.2 only gives a statement for x with sufficiently small $\text{dist}(x, \Gamma)$, so some work is still required to show that the orbital stability in fact holds for a larger subset of the state space. For this, we can take a closer look at the argument in the proof of the theorem above.

In the proof, we only use that $\text{dist}(x, \Gamma)$ is small when we introduce the constants M, α for $\|\xi_0\| \leq \delta$.

In general, deciding the permitted value of δ depends delicately on the problem at hand. As far as we are aware, no results in the literature establish the global exponential convergence to the limit cycle for this system. The difficulty seems to be that results so far are concerned with asymptotic stability, that is, establishing that for $x \in \operatorname{int}\left(\mathbb{R}_+^2\right)$ it holds that $d(\phi_t(x), \Gamma) \to 0$, for instance in [30]. However, to get bounds on the rate of convergence is difficult, as even on the interior of the restricted domain $\{v, p < K\}$ not much can a priori be said about uniformity of the function $x \mapsto \frac{d(\phi_t(x),\Gamma)}{d(x,\Gamma)}$. If one could estimate that this function were strictly less than 1 and continuous on the interior, then contraction as used in the proof of the above theorem would come for free.

Remark: The above in fact holds in more generality. Floquet theory seems to be a good tool for establishing an *region* around the limit cycle where exponential convergence with asymptotic phase holds, but for general results the estimates made in the proofs are too crude. Mainly this is because the arguments essentially use linearization of the system and then some appeal to the Hartman-Grobman theorems. However, this is insufficient to conclude the behavior of the system when we are relatively far away from the limit cycle. Whenever the δ from the theorem above is large enough (depending on estimates we could make on a given deterministic dynamical system), Floquet theory may be the solution, but in full generality this approach seems not to work for all systems.

As a final note on the results in Floquet theory, we devise a computational way to establish the result for exponential attractiveness of the limit cycle. Recall, though, that the information that was available to us in terms of eigenvalues, was not explicitly about the Poincaré map itself, but about the monodromy matrix. A next aim should therefore be to relate the eigenvalues of the linearization of the Poincaré map with those of the monodromy matrix. We will simply state the theorem from [28] that expresses this relation, without proof or further theory.

Theorem 3.2.3. *$\lambda \neq 1$ is an eigenvalue of $D\mathcal{P}(0)$ if and only if λ is an eigenvalue of $Y(T)$. $\lambda = 1$ is an eigenvalue of $D\mathcal{P}(0)$ if and only if the eigenvalue 1 of $Y(T)$ has multiplicity bigger than one.*

This means that we may use the Liouville formula

$$\det(Y(T)) = \prod_{i=1}^{n} \lambda_i = \exp\left(\int_0^T \operatorname{div} f\left(\phi^t(x_0)\right) dt\right)$$

to compute the eigenvalues of the Poincaré map, and arrive at conclusions about the asymptotic orbital stability of the limit cycle Γ of the system under investigation. This will in particular be useful to derive an explicit expression to show contractivity on average for this system.

3.3 Contraction Analysis for Dynamical Systems

Floquet theory yields an adequate result for the asymptotic properties of the system, but its downside is that the available results are not informative as to the precise extent in phase space where exponential convergence takes place. Recall that the results of the previous section merely stated that for initial conditions *sufficiently close* to the limit cycle, the associated Poincaré map is a contraction. In applications, however, one is usually interested in knowing explicitly where exponential convergence takes place, so that a suitable model for the perturbations can be formulated and persistence of stability can be established in a precisely defined region of the phase space.

Another problem with Floquet theory is that it establishes a rate of convergence when the deterministic flow has a periodic solution. Recall, however, that in the Rosenzweig-MacArthur model there are also parameter combinations guaranteeing that there is a stable equilibrium to which all trajectories converge. Our results on persistence of stability necessitate that a rate of convergence *between* solutions can be established, so additional estimates about the distance between solutions of the dynamical system must be made in the case of an equilibrium. We will learn that contraction theory provides the answers about convergence between solutions both in the case of a stable equilibrium as in the case of an asymptotic limit cycle.

The literature on dynamical systems is rich in results relating the existence of so-called *Lyapunov functions* to the convergence properties of the system. The interested reader can find an exposition of such results in ([31], p. 2164). The reason that Lyapunov functions will not be further considered in this work is that the precise parametrization of Lyapunov functions is not straightforward, as it often delicately depends on the location of the equilibria or periodic orbits of the system. This issue has received plenty of attention in the literature, for instance in [26] and [31].

The difficulties in establishing exponential convergence for a dynamical system inspired the rise of *contraction analysis.* As noted by Aylward et al. in [31], "contraction analysis is an approach where stability is defined incrementally between two arbitrary trajectories, and it attempts to answer the question of whether the limiting behavior of a given dynamical system is independent of its initial conditions". The field of contraction analysis has a modest amount of results, but seems to be little-known to authors outside the field. Key papers are [32] by Slotine and Lohmiller, and [33] by Demidovich, who independently related the distance between any pair of solutions to convergence to an equilibrium state. As Pavlov et al. note in [34], the results of Demidovich were "included in one of the classical textbooks on stability theory [35], they were not translated into English and are not widely known outside Russia". A useful property of using so-called contraction metrics as in the field of contraction analysis instead of Lyapunov functions is that "a contraction metric is robust to small perturbations of the system or the metric. This means that a sufficiently good approximation to a certain contraction metric, e.g. using numerical methods, is itself a contraction metric." (Giesl in [36]).

In the classical statement, formulated by Demidovich and Slotine and Lohmiller, the following definition is needed.

Definition 3.3.1. For $n \in \mathbb{N}$ an $n \times n$ matrix X is called uniformly negative definite if there exists $\beta > 0$ such that $\frac{1}{2}\left(X + X^T\right) < -\beta I_n$, where I_n is the $n \times n$ identity matrix.

Remark: The order $<$ above is with respect to the usual order on symmetric matrices. For symmetric matrices A, B we say that $A \leq B$ if and only if for all x: $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, this notion can even be applied in a Hilbert space. With other words, $A \leq B$ if and only if the matrix $B - A$ is positive definite.

Definition 3.3.2. Given

$$\dot{x} = f(x, t) \tag{3.6}$$

for a non-linear function f with Jacobian $Df(x, t)$, a region $\mathcal{U}$ of the state space is called a *contraction region* if there exists a positive definite matrix $P > 0$ such that the matrix $P \cdot Df(x, t)$ is uniformly negative definite for all $t \geq 0$ and $x \in \mathcal{U}$.

The theorem of Demidovitch reads as follows ([34], Theorem 1).

Theorem 3.3.1. (Krakovskii's Stability Theorem) *Given the equations*

$$\dot{x} = f(x, t)$$

any trajectory, which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory. Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region.

□

Remark 1: The theorem independently proven by Slotine and Lohmiller [32] gives the statement for $P = I_n$.

Remark 2: Most of the literature on contraction analysis (see, for example, [31]) refers to the theorem above as Krakovskii's Stability Theorem.

The idea behind this result is as follows, which we take from Section 2 of [31]. Let $\delta x(t)$ be an infinitesimal displacement at a fixed time. For notational convenience we will omit the time dependence, but of course x depends on t. One obtains the following differential relation

$$\delta\dot{x}(t) = \frac{\partial f}{\partial x}(x(t))\delta x(t). \tag{3.7}$$

The infinitesimal squared distance between any two trajectories is given by $\delta x^T \delta x$ by the definition of the metric. Hence, the rate of change of the infinitesimal squared distance between two trajectories follows from 3.7 as

$$\frac{d}{dt}\left(\delta x^T \delta x\right) = 2\delta x^T \delta \dot{x} = 2\delta x^T \frac{\partial f}{\partial x}\delta x. \tag{3.8}$$

If $\lambda_1(x)$ is the largest eigenvalue of the symmetric part of the Jacobian of f at x, then $\frac{d}{dt}\left(\partial x^T \partial x\right) \leq 2\lambda_1(x)\delta x^T \delta x$. Integrating gives then that

$$\|\delta x(t)\| \leq \|\delta x(0)\| e^{\int_0^t \lambda_1(x(s))ds}. \tag{3.9}$$

The condition of uniform strict negativity now ensures that the integral in the exponent is negative and hence any infinitesimal length converges exponentially to zero when $t \to \infty$.

For the study of periodic orbits, the result above is clearly not directly applicable. For Γ the periodic orbit and $x, y \in \Gamma, x \neq y$ and τ the period of the periodic solution, $d(\Phi_{n\tau}(x), \Phi_{n\tau}(y)) = d(x, y)$ for any $n \in \mathbb{N}$, showing that the distance between trajectories cannot exponentially decrease in time. Recall that this argument was also used in Chapter 1 to justify why the techniques of [1] to show the non-expansiveness of the Markov operator could not be used in the setting of periodic orbits.

Intuitively, instead of requiring the Jacobian be a contraction in all directions, one expects that the system is contracting in directions *transversal* to the flow, while it need not contract in the direction of the flow. On the periodic orbit itself, the Jacobian cannot be contracting everywhere: if it were, then the distance between trajectories started on different points of the limit cycle would exponentially decrease, which contradicts the fact that their distance must be periodic in time. These properties were well-known in the literature, with results as early as 1960 by Borg in [37]. A rigorous description of contraction analysis in the context of periodic orbits first appears later in the literature, for a good overview of the results we refer the reader for example to [26]. Henceforth, until the end of this section, we expose results from [36].

Consider again a general *autonomous* ODE of the form (3.6), where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We denote the solution of the initial value problem (3.6) with $x(0) = \xi$ by $S_t\xi = x(t)$ and assume that it exists for all $t \geq 0$. A contraction metric for a periodic orbit can be expressed as a matrix-values function $M \in C^1(\mathbb{R}^n, \mathrm{Sym}(n))$, where $\mathrm{Sym}(n)$ denotes the symmetric $n \times n$ matrices, such that $M(x)$ is positive definite and thus $\langle v, w\rangle_x = v^T M(x)w$ defines a point-dependent scalar product for two vectors $v, w \in \mathbb{R}^n$. Consider a region of the state space where $f \neq 0$, with other words, there is no steady state in this region.

Let then,

$$V(x) = Df(x) - \frac{f(x)f(x)^T(Df(x) + Df(x)^T)}{\|f(x)\|^2} \tag{3.10}$$

and

$$L_M(x) = \max_{v\in\mathbb{R}^n, v^T M(x)v=1, v^T f(x)=0} L_M(x; v), \tag{3.11}$$

$$L_M(x; v) = \frac{1}{2} v^T \left(M'(x) + V(x)^T M(x) + M(x)V(x)\right) v \tag{3.12}$$

where $(M'(x))_{i,j=1,\ldots,n} = \left(\nabla M_{ij}(x)\right)^T f(x)$ is the matrix of the orbital derivatives of M_{ij} along solutions of (3.6) and $\|\cdot\|$ denotes the Euclidian norm. The key idea of the contraction analysis is that the function $L_M(x; v)$ for v satisfying that $v^T f(x)$ is negative, if the distance between solutions through x and $x + rv$ for $r > 0$ small, with respect to the metric $M(x)$ decreases. A heuristic explanation goes as follows. To measure the distance, synchronize the times such that the difference vector between the solutions is perpendicular to the flow. In particular, define $\theta(t)$ such that $\theta(0) = 0$ and

$$\left(S_{\theta(t)}(x + \delta v) - S_t x\right)^T f(S_t x) = 0, \quad \text{for all } t \geq 0.$$

The implicit function theorem shows that

$$\dot{\theta}(0) = \frac{\|f(x)\|^2 - rv^T Df(x)f(x)}{f(x + rv)^T f(x)} \approx 1 - r\frac{v^T \left(Df(x)^T + Df(x)\right) f(x)}{\|f(x)\|^2 - \delta v^T Df(x)^T f(x)} \tag{3.13}$$

for small $r > 0$. Consider now the squared distance between the trajectories with respect to the Riemannian metric

$$d(t) = \left(S_{\theta(t)}(x + rv) - S_t x\right)^T M(S_t x) \left(S_{\theta(t)}(x + rv) - S_t x\right)$$

and take the derivative. With first order Taylor expansion of $f(x + rv)$ one arrives at,

$$\begin{aligned}
\frac{d}{dt}d(t)\bigg|_{t=0} &= \left(\dot{\theta}(0)f(x + rv) - f(x)\right)^T M(x)rv + r^2 v^T M'(x)v + \delta v^T M(x)\left(\dot{\theta}(0)f(x + \delta v) - f(x)\right) \\
&\approx r\left(\dot{\theta}(0) - 1\right)\left[f(x)^T M(x)v + v^T M(x)f(x)\right] \\
&\qquad + \delta^2\dot{\theta}(0)\left[(Df(x)v)^T M(x)v + v^T M(x)Df(x)v\right] + \delta^2 v^T M'(x)v \\
&\approx \delta^2\left[-\frac{v^T\left(Df(x)^T + Df(x)\right)f(x)}{\|f(x)\|^2}\left[f(x)^T M(x)v + v^T M(x)f(x)\right]\right. \qquad \text{(by (3.13))} \\
&\left. +(Df(x)v)^T M(x)v + v^T M(x)Df(x)v + v^T M'(x)v\right] \\
&= \delta^2 v^T\left[V(x)^T M(x) + M(x)V(x) + M'(x)\right]v \\
&= 2\delta^2 L_M(x; v)
\end{aligned}$$

so if $L_M(x; v)$ is bounded by a negative constant $-\nu$, then $d(t)$ is exponentially decreasing. That has the following implication for the existence, uniqueness and stability of a periodic orbit and its basin of attraction, see ([26], Theorem 2.1).

Theorem 3.3.2. *Let $K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set under the flow defined by (3.6) that does not contain an equilibrium. Let $M \in C^1(\mathbb{R}^n, Sym(n))$ be such that $M(x)$ is positive definite for all $x \in \mathbb{R}^n$. Moreover, assume that $L_M(x) < -\nu < 0$ holds for all $x \in K$. Then there exists a unique periodic orbit $\Gamma \subset K$, Γ is exponentially stable and the largest real part of all nontrivial Floquet exponents is at most $-\nu$. Moreover, K is a subset of the basin of attraction of Γ.*

$\square$

This result is used in [26] to formulate matrix-valued PDEs for the contraction metric. In the subsequent paper [36] an approximate numerical solution is constructed. After analyzing the Rosenzweig-MacArthur model in the following section, it will be our aim to construct a simple contraction metric associated to the model to show the exponential convergence to the limit cycle.

3.4 Persistence of Stability in the Rosenzweig-MacArthur model with a Stable Equilibrium

The conditions of Theorem 1.2.3 prescribe when there exists an invariant, asymptotically stable measure μ, supported on a ball B^* around the stable, nontrivial equilibrium x^*. We construct a model for the perturbations and argue, through a combination of rigorous estimates and numerical approximation that the model assumptions hold.

3.4.1 Model Assumptions and Estimation Procedure

As in [1], we are interested in a population model, meaning that a biologically meaningful perturbation is one that removes individuals from the population with a certain probability distribution. Perhaps the simplest model one can define is the following. Let the jump kernels q_x have support restricted to a certain set $C_x = x + [-c_{\max}, 0] \times [-c_{\max}, 0]$ for $c_{\max} > 0$ some pre-set constant. Define q_x as the uniform distribution on C_x, for either the non-expansive case. Let p_x have a shifted exponential density with parameter λ_x such that the density is given by

$$f_x(t) = 1_{[\Delta t, \infty)}(t)\lambda_x e^{-\lambda_x(t - \Delta t)},$$

and choose $\lambda_x = \lambda$ for all x. Thus, trivially, $\|p_x - p_y\|_{\mathrm{TV}} = 0$ and $d_H(\operatorname{supp} p_x, \operatorname{supp} p_y) = 0$ for all $x, y \in \mathbb{R}^2_+$. That is, $L_p = L'_p = 0$. We will start checking the model assumptions for Theorem 1.2.3 to hold. Given parameters a, b, d, h, r, K such that $\eta < \frac{K}{b} < 1 + 2\eta$, one can estimate an invariant ball $B^* \subset \mathbb{R}^2_+$ of radius

R^* numerically such that the flow is a contraction and obtain a numerical estimate $\hat{\nu}$ of the contraction rate. In that case, since jumps can only happen after time Δt, a numerical estimate of L_Φ is given by $e^{-\hat{\nu}\Delta t}$. This satisfies Assumption 1-$\mathcal{S}$.

For Assumption 1-$\mathcal{P}$, let $\mathfrak{L}$ be the Lebesgue measure on $\mathbb{R}^2$. A simple proof-by sketch shows that

$$d_H(C_x, C_y) \leq d(x, y) \quad \text{and} \quad \mathfrak{L}(C_x \setminus C_y) \leq 2c_{\max} d(x, y).$$

Hence, $L_q = 1$ and

$$\begin{aligned}
\|q_x - q_y\|_{\text{TV}} &= \sup_{f \in \text{BM}(X), \|f\| \leq 1} \left| \int f(z) q_x(dz) - \int f(z) q_y(dz) \right| \\
&= \sup_{f \in \text{BM}(X), \|f\| \leq 1} \frac{1}{c_{\max}^2} \left| \int_{C_x} f(z) \mathfrak{L}(dz) - \int_{C_y} f(z) \mathfrak{L}(dz) \right| \\
&= \frac{\mathfrak{L}(C_x \setminus C_y) + \mathfrak{L}(C_y \setminus C_x)}{c_{\max}^2} && \text{(Choose } f(z) = 1_{C_x}(z) - 1_{C_y}(z)) \\
&\leq \frac{4}{c_{\max}} d(x, y).
\end{aligned}$$

So we conclude that $x \mapsto q_x$ is Lipschitz with respect to the total variation norm and that $L'_q \leq \frac{4}{c_{\max}}$. As for the time kernels, note that they are identical and as such their total variation distance is 0, so we satisfy condition 3-$\mathcal{P}$. Condition 2-$\mathcal{P}$ is automatically satisfied since we observe the solution of an ODE given by a continuous function. For assumption 4-$\mathcal{P}$ to hold, we must then have that $L_\Phi < 1$, which is satisfied because $L_\Phi = e^{-\hat{\nu}\Delta t}$ and $\hat{\nu} > 0$. To satisfy Assumption 5-$\mathcal{P}$ we finally deduce that we must require that $\sqrt{2} c_{\max} < \left(1 - e^{-\hat{\nu}\Delta t}\right) R^*$, considering that

$$\text{diam supp } p_x = \text{diam } C_x = \text{diam } [-c_{\max}, 0]^2 = \sqrt{2} c_{\max}.$$

These conditions can now be investigated numerically for set parameter combinations.

3.4.2 Numerical Estimation

In this section we numerically estimate the region in state space for which the Rosenzweig-MacArthur model has a contractive flow. To this end, we would like to invoke Theorem 3.3.1 to conclude the necessary contractivity. For that to hold, we must have that $\frac{1}{2}\left(Df(x) + Df(x)^T\right)$ is positive definite for all $x \in B^*$. Since it is symmetric, this is equivalent to requiring that for each $x \in B^*$,

$$\inf_{x \in B^*} \det \left(Df(x) + Df(x)^T\right) > 0$$
$$\sup_{x \in B^*} \text{Tr} Df(x) < 0.$$

Since x^* is stable, $\text{Tr}Df(x^*) < 0$. Clearly the map $x \mapsto \text{Tr}Df(x)$ is continuous, meaning that there is an open neigborhood B^* around x^* such that $\text{Tr}Df(x) < 0$ for $x \in B^*$. If in addition one can numerically find a region within B^* such that $\det\left(Df(x) + Df(x)^T\right) > 0$, then one has found a contraction region by Theorem 3.3.1.

We will demonstrate this estimation with parameter values $a = \frac{3}{4}$, $b = 4$, $d = \frac{1}{3}$, $h = 1$, $r = 3$, $K = 4$. Figure 3.3 shows a contour plot of $\text{Tr}Df(x)$ and $\det\left(Df(x) + Df(x)^T\right)$. There is a region around the stable equilibrium for which indeed $\text{Tr}Df(x) < 0$ and $\det\left(Df(x) + Df(x)^T\right) > 0$, meaning that this is a contraction region.

Remark: One would wish to analytically compute the contraction region, with the intention that estimates such as $\hat{\nu}$ and R^* can be more rigorously made. Unfortunately, though, the expression in $\det\left(Df(x) + Df(x)^T\right)$ is not directly amenable to this sort of analysis, meaning one must resort to numerical estimation.

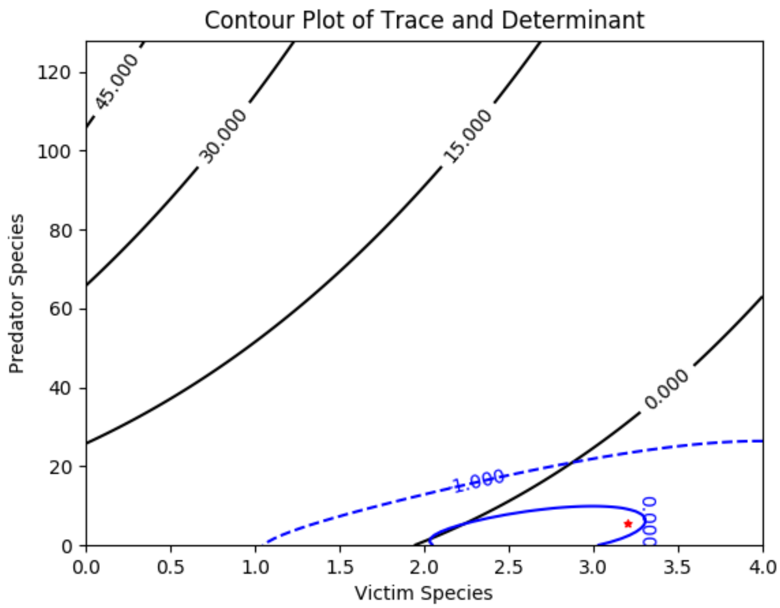

Figure 3.3: Contour plots for trace of $Df(x)$ (black) and determinant of the symmetrization of $Df(x)$ (blue). Equilibrium is plotted in red. There is a region around the equilibrium where the determinant is positive and trace negative, meaning it is a contraction region. Parameters in the simulations the Rosenzweig-MacArthur model have been fixed at $a = \frac{3}{4}$, $b = 4$, $d = \frac{1}{3}$, $h = 1$, $r = 3$, $K = 4$.

3.4.3 Persistence of Stability

We have found that there is a contraction region around the stable equilibrium x^*. Within this region, one can numerically obtain an estimate $\hat{\nu}$ for the rate of contraction. In general, the size of this contraction rate will depend on the specific subset of the contraction region one chooses. Let now perturbations be bounded by the condition $c_{\max} < \frac{\sqrt{2}}{2}\left(1 - e^{-\hat{\nu}\Delta t}\right) R^*$. We recall that R^* is the radius of the ball B^* within the contraction region. By the arguments above, we can invoke Theorem 1.2.3 and there exists a unique, invariant ergodic measure on B^*. That is, we have shown persistence of stability around the stable non-trivial equilibrium of the Rosenzweig-MacArthur model for sufficiently small stochastic perturbations, quantified by the constraint on $c_{\max}$.

3.5 Persistence of Stability in the Rosenzweig-MacArthur Model with a Limit Cycle

With the theory developed at the start of this chapter, we are now in a position to establish persistence of stability in a specific example of the Rosenzweig-MacArthur model. We will explicitly get a parameter estimate for the exponential convergence to the limit cycle first, and then we will show how numerical analysis can in fact help to establish a domain of convergence for the deterministic system. After showing several of these estimates, we turn to how to use the model parameters to make a suitable model for the perturbations, by focusing on both the time and jump kernel constructions.

3.5.1 Exponential Stability - The Liouville Formula

We remark briefly that the exponential stability of the limit cycle in the Rosenzweig-MacArthur model can be proven by using the Liouville formula, as formulated in the Lemma below.

Lemma 3.5.1. *If $\frac{2a}{b} p_{\max} < r$, where $p_{\max}$ is the maximal p on the periodic orbit, then the limit cycle is exponentially stable.*

Proof: Recall that by Theorem 3.2.3, the limit cycle is exponentially attracting if the non-trivial eigenvalue λ_2 of the monodromy matrix satisfies $|\lambda_2| < 1$. By equation (3.2.3), we obtain that

$$\lambda_2 = \exp\left(\int_0^T \mathrm{div} f\left(\phi^t(x_0)\right) dt\right).$$

Note that $\mathrm{div} f\left(\phi^t(x_0)\right) = \mathrm{Trace}\left(Df(\phi^t(x_0))\right)$. Let $(v(t), p(t))$ be a trajectory on the periodic orbit with period T and consider the integral $\int_0^T \mathrm{Trace}\left(Df(v(t), p(t))\right) dt$. First of all, one has

$$-d + \frac{hav(t)}{b + v(t)} = \frac{\dot{p}(t)}{p(t)},$$

so

$$\int_0^T -d + \frac{hav(t)}{b + v(t)} dt = \int_0^T \frac{\dot{p}(t)}{p(t)} dt = \ln(p(t))|_{t=0}^T = 0$$

Moreover,

$$r\left(1 - \frac{2v}{K}\right) = 2r\left(1 - \frac{v}{K}\right) - r = 2\frac{\dot{v}}{v} + \frac{2ap}{b+v} - r,$$

so

$$\begin{aligned}\int_0^T r\left(1 - \frac{v(t)}{K}\right) dt &= \int_0^T 2\frac{\dot{v}(t)}{v(t)} + \frac{2ap(t)}{b + v(t)} - r dt \\ &= 0 - rT + \int_0^T \frac{2ap(t)}{b + v(t)} dt \\ &\leq -rT + T\frac{2a}{b} p_{\max}.\end{aligned}$$

We conclude that if $\frac{2ap_{\max}}{b} < r$, then $\int_0^T \mathrm{Trace}(Df(v(t), p(t)))dt < 0$ and so $\lambda_2 < 1$. This implies that the limit cycle is exponentially stable under this parameter combination.

$\square$

3.5.2 Contraction Analysis of the Rosenzweig-MacArthur Model

The goal of this section is to apply Theorem 3.3.2 to the Rosenzweig-MacArthur model to obtain an explicit expression for a domain where the needed exponential convergence to the limit cycle takes place. To that end, recall from the computation leading to the characteristic equation (3.3) that

$$Df(v, p) = \begin{pmatrix} r - \frac{2rv}{K} + \frac{ab}{(b+v)^2 p} & -\frac{av}{b+v} \\ \frac{hab}{(b+v)^2} p & -d + \frac{hav}{b+v} \end{pmatrix},$$

such that $V(v, p)$, as given by (3.10), is defined by

$$V(v,p) = \begin{pmatrix} r - \frac{2rv}{K} + \frac{ab}{(b+v)^2 p} & -\frac{av}{b+v} \\ \frac{hab}{(b+v)^2} p & -d + \frac{hav}{b+v} \end{pmatrix}$$
$$- \frac{\begin{pmatrix} rv\left(1 - \frac{v}{K}\right) - \frac{avp}{b+v} \\ -dp + \frac{havp}{b+v} \end{pmatrix} \cdot \begin{pmatrix} rv\left(1 - \frac{v}{K}\right) - \frac{avp}{b+v} \\ -dp + \frac{havp}{b+v} \end{pmatrix}^T \cdot \begin{pmatrix} 2\left(r - \frac{2rv}{K} + \frac{ab}{(b+v)^2 p}\right) & \frac{hab}{(b+v)^2} p - \frac{av}{b+v} \\ \frac{hab}{(b+v)^2} p - \frac{av}{b+v} & 2\left(-d + \frac{hav}{b+v}\right) \end{pmatrix}}{\left(rv\left(1 - \frac{v}{K}\right) - \frac{avp}{b+v}\right)^2 + \left(-dp + \frac{havp}{b+v}\right)^2}.$$

It should be obvious that this expression will become mathematically intractable when computed. In fact, when evaluating the above expression symbolically in Mathematica, we encountered a recursion limit error. That means that an expression of the bound $-\nu$ in the parameter values of the model will in practice not be feasible. As shown in [36], a good approach is to approximate the solution numerically, which would give rise to a numerical approximation for a matrix-valued PDE. In [26] this PDE and its properties are described; it also gives references for numerical implementations of the algorithm used to approximate solutions. In contrast to the approach taken in [36], we will not approximate such a PDE arising from the evaluation of (3.10). Rather we will attempt to numerically verify if for given parameter values, a simple (constant) positive definite matrix M satisfies the desired condition.

Indeed, let $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ for $\alpha, \beta, \gamma > 0$ with $\alpha\gamma - \beta^2 > 0$. For the numerical simulation, we initialize the parameter values at $a = 1, b = 1, d = 0.5, h = 1, r = 1.1, K = 5$. Note that

$$1 = \frac{db}{ha - d} < 2 = \frac{K - b}{2}$$

so this parameter combination will give rise to a limit cycle by the dynamics developed in this chapter. We use the Python script in the appendix to generate the following estimates for the size of $L_M(v,p)$ in the phase space. Searching for α, β, γ such that L_M is uniformly negative can be implemented easily in any scripting language. The plots below give a good visualization of the results for different values.

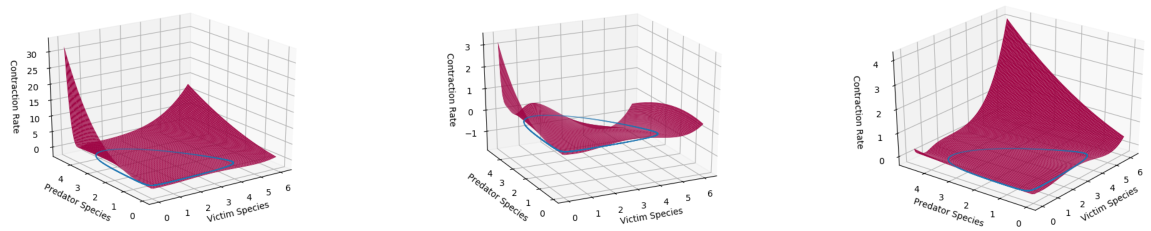

Figure 3.4: Numerical estimates of $L_M(v,p)$. Left: $\alpha = \gamma = 1, \beta = 0$. Middle: $\alpha = 0.1, \beta = 0.4, \gamma = 5$. Right: $\alpha = 0.1, \beta = 0.049, \gamma = 0.5$. Periodic orbit plotted in blue at level $z = 0$ for reference. Parameters in all three simulations of the Rosenzweig-MacArthur model have been fixed at $a = 1, b = 1, d = 0.5, h = 1, r = 1.1, K = 5$.

The periodic limit cycle plotted at height 0 shows that in the left and middle plots, L_M is not bounded away from 0 in a neighborhood around the limit cycle. For the parameter combination on the right we find that around the limit cycle, the estimated rate of contraction is bounded by -0.048, meaning that in this region there is exponential convergence to the limit cycle. At the same time, it is noteworthy that $L_M(v,p)$ is negative for a large neighborhood of the limit cycle even though the condition $\frac{2ap_{\max}}{b} < r$ is not fulfilled. Indeed, given that $p_{\max} = \frac{rK}{a}(K + b) = 82.5$ and so the condition does not hold, we find that a numerical estimate of the contraction metric yields a better result than our analysis through Floquet theory.

3.5.3 Lipschitz condition and estimates in parameter values

A necessary condition for the UEB property of the Markov operator associated to the PDMP to hold is the Lipschitz continuity of the right-hand side of the deterministic ODE model. In this section, we aim to show that the ODE is indeed Lipschitz continuous on a bounded domain. For this, let for some $M \geq K$, $\Omega_M = \{(v,p) \in \mathbb{R}^2 : |v|, |p| \leq M\}$. Recall that the region Ω_M is invariant for the deterministic flow. The following lemma holds.

Lemma 3.5.2. *On Ω_M, the right-hand side of the ODE given by equations (3.1) and (3.2) is Lipschitz continuous with Lipschitz constant L satisfying*

$$L \leq \sqrt{2} \max\left\{ r + \frac{2rM}{K} + \frac{a}{b}(1+h)M, \frac{a}{b}(1+h)M + d + \frac{a}{b^2}(1+h)M^2 \right\}$$

Proof: Let $(v,p), (\tilde{v}, \tilde{p}) \in \Omega_M$ be given. Let $\|\cdot\|_i$ denote the ℓ^i norm on $\mathbb{R}^n$ for $i = 1, 2$. Recall that for any $x \in \mathbb{R}^n$, $\|x\|_2 \leq \sqrt{2}\|x\|_1$. We will estimate the difference in ℓ^1 norm for simplicity of computation. We have the estimates

$$
\begin{aligned}
&\left|rv\left(1-\frac{v}{K}\right)-\frac{avp}{b+v}-r\tilde{v}\left(1-\frac{\tilde{v}}{K}\right)+\frac{a\tilde{v}\tilde{p}}{b+\tilde{v}}\right|+\left|-dp+h\frac{avp}{b+v}+d\tilde{p}-h\frac{a\tilde{v}\tilde{p}}{b+\tilde{v}}\right| \\
&=\left|r(v-\tilde{v})+\frac{r}{K}\left(v^2-\tilde{v}^2\right)-\frac{a}{b}\left(\frac{vp}{1+\frac{1}{b}v}-\frac{\tilde{v}\tilde{p}}{1+\frac{a}{b}\tilde{v}}\right)\right|+\left|\frac{ha}{b}\left(\frac{vp}{1+\frac{1}{b}v}-\frac{\tilde{v}\tilde{p}}{1+\frac{1}{b}\tilde{v}}\right)-d(p-\tilde{p})\right| \\
&\leq r|v-\tilde{v}|+\frac{r}{K}|v^2-\tilde{v}^2|+\frac{a}{b}(1+h)\left|\frac{vp}{1+\frac{1}{b}v}-\frac{\tilde{v}\tilde{p}}{1+\frac{1}{b}\tilde{v}}\right|+d|p-\tilde{p}| \\
&\leq r|v-\tilde{v}|+\frac{2rM}{|}v-\tilde{v}^2|\frac{a}{b}(1+h)|vp-\tilde{v}\tilde{p}|+\frac{a}{b^2}(1+h)M^2|p-\tilde{p}|+d|p-\tilde{p}| \\
&\leq\left(r+\frac{2rM}{K}+\frac{a}{b}(1+h)M\right)|v-\tilde{v}|+\left(\frac{a}{b}(1+h)M+d+\frac{a}{b^2}(1+h)M^2\right)|p-\tilde{p}|
\end{aligned}
$$

which yields the claim.

□

3.5.4 The PDMP: A Simple Model with a Serious Complication

The exponential attraction of the limit cycle is one of the necessary conditions for the existence of an invariant ergodic measure. We must ensure that the perturbations are somehow small enough so that we fit into the framework of the assertions proven in Chapter 2. In a similar vein as in the case of the stable equilibrium, the perturbations should have support restricted to a certain set $C_x = [-c_{\max}, 0] \times [-c_{\max}, 0]$ for a constant $c_{\max} > 0$.

Again, we define q_x as the uniform distribution on C_x and p_x as a shifted exponential density with parameter λ_x such that the density is given by

$$f_x(t) = 1_{[\Delta t, \infty)}(t)\lambda_x e^{-\lambda_x(t-\Delta t)},$$

and choose $\lambda_x = \lambda$ for all x. We will start checking the model assumptions for Theorem 2.8.1 to hold. Given parameters a, b, d, h, r, K such that $\eta b < (K-b)/2$, one can estimate the rate of convergence numerically as done in the previous section. Call this estimate $\hat{\nu}$. Then, given a domain $\Omega_M = \{(v,p) : |v|, |p| < M\}$ that is invariant for the flow, we assume the following conditions.

- $c_{\max} < \inf_{x \in \mathcal{D}_\rho} d\left(x, \{v=0\} \cup \{p=0\}\right) = \Delta$
- $\lambda > \sqrt{2} \max\left\{r + \frac{2rM}{K} + \frac{a}{b}(1+h)M, \frac{a}{b}(1+h)M + d + \frac{a}{b^2}(1+h)M^2\right\}$
- Δt such that $\max\{K, p_{\max}\} \cdot e^{-\hat{\nu}\Delta t} \leq \Delta$.

Start by noticing how the condition $\eta b < (K-b)/2$ guarantees the existence of a limit cycle, which is exponentially attracting by the upper bound found for the contraction rate. Granted the condition

$$\Delta = \inf_{x \in \mathcal{D}_\rho} d(x, \{v=0\} \cup \{p=0\}) > 0, \quad c_{\max} < \Delta$$

and Δt as in the conditions guarantees the invariance of $\mathcal{D}_\rho$. This makes the process satisfy Assumption 2-$\mathcal{P}$. Furthermore, let L be the Lipschitz coefficient of the ODE as in Lemma 3.5.2 and observe that

$$
\begin{aligned}
\sup_x \int_{\mathbb{R}_+} e^{Lt} p_x(dt) = \int_{\mathbb{R}_+} \lambda 1_{[\Delta t, \infty)}(t) e^{-\lambda(t-\Delta t)} e^{Lt} dt = \lambda e^{\lambda \Delta t} \int_{\Delta t}^{\infty} e^{-(\lambda - L)t} dt \\
= \frac{\lambda e^{\lambda \Delta t} e^{-(\lambda-L)\Delta t}}{\lambda - L} \\
= \frac{\lambda}{\lambda - L} e^{L \Delta t} < \infty
\end{aligned}
$$

By the remark after assumption 1-$\mathcal{U}$ the second condition in 1-$\mathcal{U}$ is met.

As for assumption 2-$\mathcal{U}$, note that the Lipschitz continuity of the maps $x \mapsto q_x$ and $x \mapsto p_x$ follows by the same argument as in the case of the stable equilibrium. We recall that $L_q \leq \frac{4}{c_{\max}}$ and $L_p = 0$.

To invoke Theorem 2.8.1 it is left to prove that the operator is *contractive on average*, in the sense of Assumption 1-$\mathcal{U}$. Unfortunately, this is where the available estimates start to break down. Consider the following estimate based on Gronwall's inequality

$$\int d\left(\Phi_t(x+z_1), \Phi_t(y+z_1)\right) p_{x+z_1} d(t_1) q_x(dz_1) \leq \int e^{Lt} d\left(x+z_1, y+z_1\right) p_{x+z_1} d(t_1) q_x(dz_1)$$
$$= d(x,y) \int_{\Delta t}^{\infty} e^{Lt} \lambda e^{-\lambda(t-\Delta t)} = d(x,y) \frac{e^{Lt}\lambda}{\lambda - L}.$$

The problem with this approximation is that it is wholly uninformative since $\frac{e^{Lt}\lambda}{\lambda - L} < 1$ and as such it does not lead to contractivity. The question is then to which extent better bounds can be found. In general, estimating the differences of solutions of non-linear ODEs is an intractable problem: the literature does not provide many results that go beyond the Gronwall estimate.

The difficulty in finding a better approximation analytically motivates to look at this problem numerically. One can numerically integrate solutions to find better bounds on the problem at hand. In this specific case, we revisit the parameter values used in Figure 3.5, namely $a = 1, b = 1, d = 0.5, h = 1, r = 1.1, K = 5$. For these parameter values we compute the ratio $\frac{d(\Phi_t(x), \Phi_t(y))}{d(x,y)}$ over time. In the following two displays we take initial conditions $x = (2,2)$ and $(2.5, 2.5)$ and $x = (2,2)$ and $y = (4,4)$ respectively.

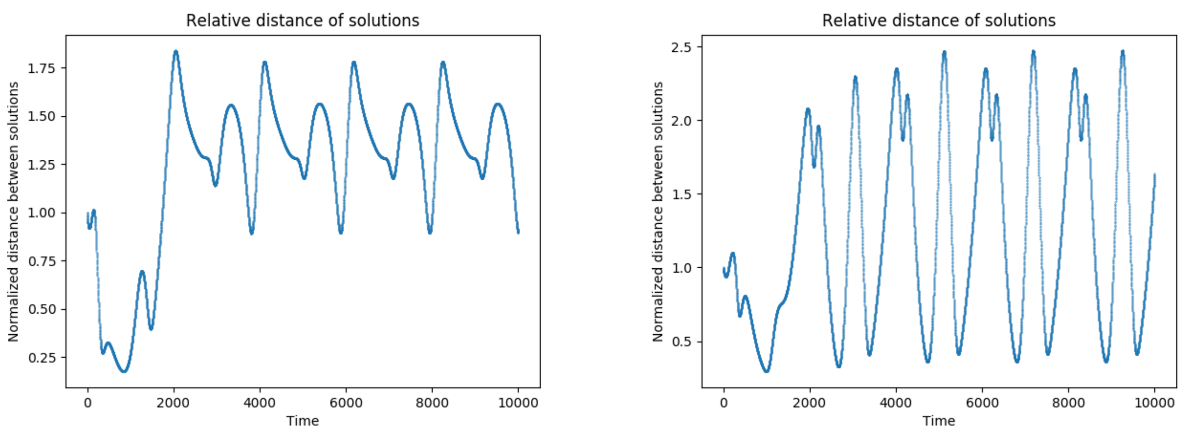

Figure 3.5: Numerical estimates of $\frac{d(\Phi_t(x), \Phi_t(y))}{d(x,y)}$ plotted against time. Left: $x = (2,2), y = (4,4)$. Right: $x = (2,2), y = (2.5, 2.5)$.

As expected, the ratio of the distances is periodic in time, as solutions converge to the limit cycle. The plots also demonstrate how the distances between solutions can oscillate quite severely. The effect on showing contractivity on average is the following: note that for the initial conditions in the left display, the quantity $\frac{d(\Phi_t(x), \Phi_t(x))}{d(x,y)} \geq 1$ eventually, meaning that the average distance between solutions does not become smaller. In the right hand side display there are times for which the ratio is smaller than 1, but the quantity $\frac{d(\Phi_t(x), \Phi_t(y))}{d(x,y)}$ averaged over a period, is larger than 1. We conclude that for this predator-prey system, we cannot guarantee that the system is contractive on average, since the deterministic part fails to bring solutions together close enough. It does not matter which kernel is implemented for the jumps in this case.

The preceding arguments show that the condition of *contractivity on average* is an amenable and innocent modeling assumption from a theoretical point of view, but in practice may not reflect the extremely delicate dynamics of the PDMP. In the particular setting of this perturbed predator-prey model, for instance, the complicated non-linear dynamics fail to give contractivity on average.

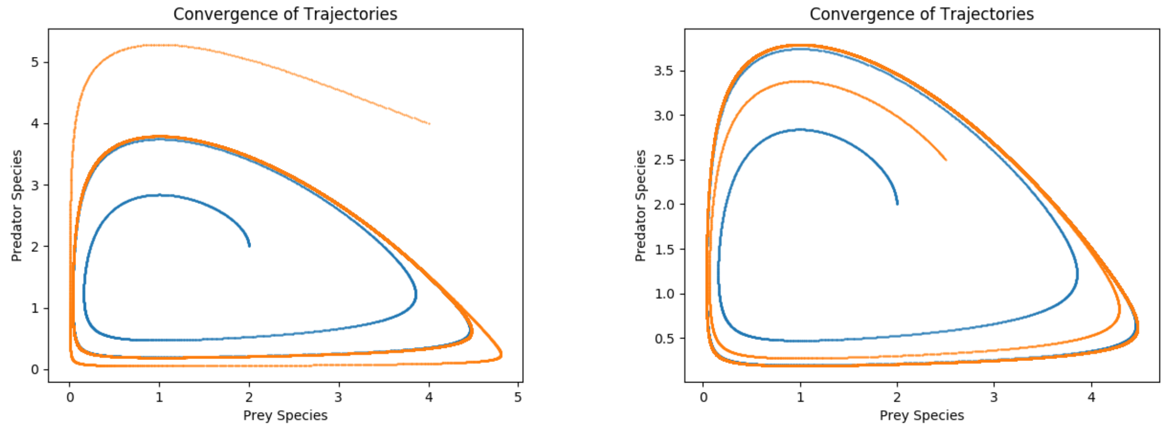

Figure 3.6: Numerical estimates of solution paths $\Phi_t(x), \Phi_t(y)$. Left: $x = (2,2), y = (4,4)$. Right: $x = (2,2), y = (2.5, 2.5)$.

3.5.5 An explicit KBBY ergodic decomposition for the perturbed predator-prey model

We revisit the decomposition of Krylov-Bogolioubov-Beboutoff-Yosida to classify the possible ergodic invariant measures for the Markov operator P corresponding to the perturbed predator-prey model. What the previous section failed to establish was the existence of a unique invariant ergodic measure restricted to $\mathcal{D}_\rho$. That does not mean that we cannot draw conclusions about possible invariant measures in the interior.

Nevertheless, we recall from Chapter 2 that the dynamics of the Cesaro averages ϵ_x for $x \in \text{int}[0,K]^2$ can possibly be quite complicated. Indeed, when we define

$$\Xi = \{x \in \text{int } [0,K]^2 : \mathbb{P}(x + z_1 \in \{v \leq 0\} \cup \{p \leq 0\}) > 0\}$$

for z_1 as usual the instantaneous perturbation be the collection of x in the interior of the state space that can jump onto the axes. Then for each $x \in \Xi$ we have that

$$\begin{aligned} Q\delta_x = \mathbb{P}(x + z_1 \in \{v = 0\}) \cdot \delta_{\pi_p(x+z_1)} + \mathbb{P}(x + z_1 \in \{p = 0\}) \cdot \delta_{\pi_v(x+z_1)} \\ + [1 - \mathbb{P}(x + z_1 \in \{v = 0\}) - \mathbb{P}(x + z_1 \in \{p = 0\})] \cdot \mu_x \end{aligned}$$

where π_p and π_v are the projections on the p and v axes, respectively and μ_x is some probability measure supported on $\{x + \text{supp}(q_x)\} \cap \text{int}[0,K]^2$. Since we know that for every $x \notin \mathcal{D}_\rho$ this argument holds, while we know that $\mathcal{D}_\rho$ is invariant for Q, it follows that we can iterate the argument above and arrive at

$$\epsilon_x = \begin{cases} \delta_0 & x \in \{v = 0\} \\ \mu_v & x \in \{p = 0\} \\ \mu^* & x \in \mathcal{D}_\rho \\ \mu = p_{1,x}\delta_0 + p_{2,x}\mu_v + (1 - p_{1,x} - p_{2,x})\mu^* & x \in \text{int}[0,K]^2 \setminus \mathcal{D}_\rho \end{cases}$$

where μ^* is an unknown (possible not unique) measure on $\mathcal{D}_\rho$ if the Cesaro average ϵ_x converges, and ϵ_x may not converge otherwise. With other words, if the Cesaro averages converge, for $x \in \text{int}[0,K]^2 \setminus \mathcal{D}_\rho$, there will be for $p_{1,x}, p_{2,x} \in (0,1)$ depending on x, describing that ϵ_x is the convex combination of ergodic measures, so ϵ_x is itself not ergodic as it is the non-trivial convex combination of ergodic measures. In an analogous setting, Alkurdi shows in ([1], Corollary 3.5.3) that the coefficients are continuous in x.

This gives us a modest characterization of the ergodic measures for this Markov operator. These considerations suggest that for each $x \in \text{int}[0,K]^2$, there exists $\pi_x = p_{1,x} + p_{2,x}$ that is the *extinction probability*, the probability that, on average, the iterates will be on one of the axes. If then the Cesaro averages converge, they are the coefficients in the non-trivial convex combination of ergodic measures. We can estimate these probabilities numerically to obtain a plot like Figure 3.7. We initialize the parameters as $a = 1, b = 1, d = 0.5, h = 1, r = 5, K = 5$ and $c_{\max} = 0.05$. We numerically simulate 100 trials per gridpoint and estimate the frequency that the iterates hit the axes. This results in the following plot.

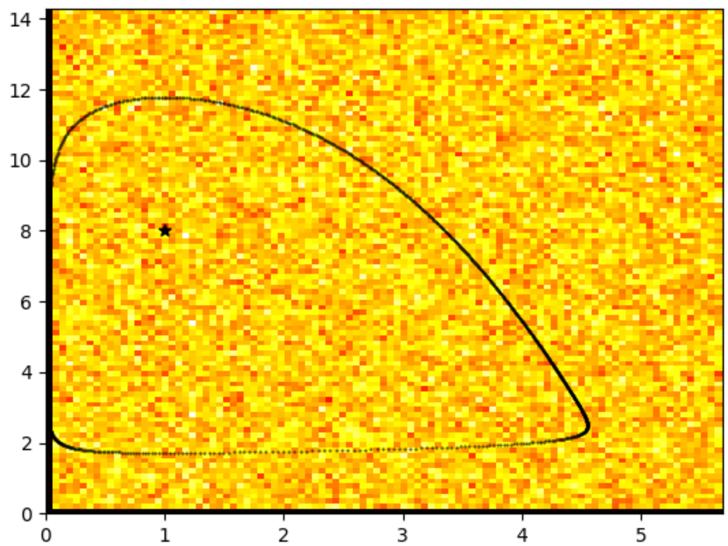

Figure 3.7: Numerical estimate for the extinction probabilities for $a = 1, b = 1, d = 0.5, h = 1, r = 5, K = 5$ and $c_{\max} = 0.05$ and 100 iterations per gridpoint.

3.6 Discussion and Outlook

In this Chapter, persistence of stability is shown to hold in the case of a stable equilibrium, while in the case of a limit cycle, the conditions of Theorem 2.8.1 could not be shown. Specifically, the numerical analysis provided evidence against the requirement that the PDMP be *contractive on average* with respect to the Euclidian distance on $\mathbb{R}^2$. As we have encountered several times in this work, the problem seems to lie at the estimation of the distance with respect to this ℓ^2 norm when we let solutions run in time. Estimates do exist, but the most generally applicable ones, such as Gronwall's estimate, have been shown in this work to be too coarse to be useful in estimating. This is especially true since in the applications considered, the domains of the PDMPs have been bounded, meaning that an exponential bound is very quickly uninformative.

The answer to this problem may entail recasting the condition that the PDMP be contractive on average with respect to the given distance d on the Polish state space into a condition that makes the PDMP contractive on average with respect to a different (sufficiently well-behaved) metric. In general, though, it is not easy to construct a metric with the desired property that it be contractive on average. This problem actually was at the heart of Chapter 1, which at the outset of the project considered persistence of stability along periodic solutions. The failure there to construct a metric that was contractive on the hyperspace $\mathcal{H}(B^*)$ implied that Theorem 1.2.3 could only be shown for stable equilibria, since in general the condition that $L_\Phi < 1$ as in Assumption 1-$\mathcal{S}$ could not be shown.

The field of contraction analysis may provide answers to this problem by providing a family of different metrics on the state space. It may be true that by considering a metric that is in a sense already contractive, it could be possible to (numerically) establish contractivity on average. This would surely amplify the range of models that can be shown to exhibit persistence of stability along a periodic solution of the deterministic system. The key characteristic of the norms considered in contraction analysis, we recall, are given in some way by inner products on state space. One may even consider more general Riemannian distances. It would be interesting to see if more sophisticated approaches could simplify our understanding of the delicate interplay between dynamics and stochastics.

Appendix

Appendix A: Numerical Simulation for a Perturbed Predator-Prey Model

Numerical simulation

The numerical simulation procedure below uses a fourth-order Runge-Kutta method for the numerical integration of the ODE. Jump times are exponentially chosen and the jump distribution is a Beta-distribution.

```
import numpy as np
import math
from scipy import stats
import matplotlib.pyplot as plt
from tqdm import tnrange
import time


cmax = 0.05
step_size= 0.01

def dv(v,p,a,b,d,h,r,K):
    return r*v*(1-v/K) - (a*v)/(b+v)*p

def dp(v,p,a,b,d,h,r,K):
    return -d*p +h*(a*v)/(b+v)*p

def integrate(t, de, v_init, p_init,a,b,d,h,r,K):
    n= math.floor(float(t)/float(de))
    values = np.zeros((2,n+2))
    values[0,0] = v_init
    values[1,0] = p_init
    for i in range(1,n+1):
        vi= values[0,i-1]
        pi= values[1,i-1]
        v1= de*dv(vi,pi,a,b,d,h,r,K)
        p1= de*dp(vi,pi,a,b,d,h,r,K)
        v2= de*dv(vi + v1/2, pi + p1/2,a,b,d,h,r,K)
        p2= de*dp(vi + v1/2, pi + p1/2,a,b,d,h,r,K)
        v3= de*dv(vi + v2/2, pi + p2/2,a,b,d,h,r,K)
        p3= de*dp(vi + v2/2, pi + p2/2,a,b,d,h,r,K)
        v4= de*dv(vi + v3, pi + p3,a,b,d,h,r,K)
        p4= de*dp(vi + v3, pi + p3,a,b,d,h,r,K)
        values[0,i] = max(vi + (v1 + 2*v2 + 2*v3 +v4)/6,0)
        values[1,i] = max(pi + (p1 + 2*p2 + 2*p3 +p4)/6,0)

    D = t - (n*de)
```

```
    vn= values[0,n]
    pn= values[1,n]
    v1= D*dv(vn,pn,a,b,d,h,r,K)
    p1= D*dp(vn,pn,a,b,d,h,r,K)
    v2= D*dv(vn + v1/2, pn + p1/2,a,b,d,h,r,K)
    p2= D*dp(vn + v1/2, pn + p1/2,a,b,d,h,r,K)
    v3= D*dv(vn + v2/2, pn + p2/2,a,b,d,h,r,K)
    p3= D*dp(vn + v2/2, pn + p2/2,a,b,d,h,r,K)
    v4= D*dv(vn + v3, pn + p3,a,b,d,h,r,K)
    p4= D*dp(vn + v3, pn + p3,a,b,d,h,r,K)
    values[0,n+1] = max(vn + (v1 + 2*v2 + 2*v3 +v4)/6,0)
    values[1,n+1] = max(pn + (p1 + 2*p2 + 2*p3 +p4)/6,0)
    return values

def generate_perturbation(v,p):
    try:
        return stats.beta.rvs(1, v/(v+p), loc = -1*cmax, scale = cmax)
    except:
        return 0

def apply_kernel(v_init,p_init,a,b,d,h,r,K):
    random_time = np.random.exponential(10)
    v_init_c = v_init
    p_init_c = p_init
    v_init = max(v_init_c + generate_perturbation(v_init_c, p_init_c),0)
    p_init = max(p_init_c + generate_perturbation(v_init_c, p_init_c),0)
    solution = integrate(random_time, step_size, v_init, p_init,a,b,d,h,r,K)
    n = solution.shape[1]-1
    sol = np.transpose(solution[:,n])
    return sol


def num_exp(v_init, p_init, N,a,b,d,h,r,K):
    values= np.zeros((2,1))
    values[0,0]= v = v_init
    values[1,0]= p = p_init

    for n in range(N):
        sol = apply_kernel(v,p,a,b,d,h,r,K)
        append= np.zeros((2,1))
        append[:,0] = sol
        values = np.append(values,append,axis=1)
        v = sol[0]
        p = sol[1]
    return values
```

Creating the samples

```
import numpy as np
import math
from scipy import stats
import matplotlib.pyplot as plt
from astropy.visualization import simple_norm
import feeder as f
```

```
parameters = [1,1.3,1.5,1.6,1.61,1.62,1.63,1.64,1.65,1.655,1.66,1.665,1.67,1.7]
a= 1
d= 1/2
h= 1
r= 5
K= 5

for i in range(0,len(parameters)):
    print(i)
    b = parameters[i]
    eta = d/(h*a - d)
    stead_v = b*eta
    stead_p = (r*b)*(1+eta)*(1-b*eta/K)/a
    feed = f.num_exp(stead_v,stead_p,15000,a,b,d,h,r,K)
    xlocs = feed[0,5000:14999]
    ylocs = feed[1,5000:14999]
    xz = [0,5]
    yz = [0,14]
    xlocs = np.append(xlocs,xz)
    ylocs = np.append(ylocs,yz)
    heatmap, xedges, yedges = np.histogram2d(xlocs,ylocs, bins = 100)
    extent= [xlocs.min(), xlocs.max(),ylocs.min(),ylocs.max()]
    np.savetxt("sample%i"%i, feed)
    plt.clf()
    norm = simple_norm(heatmap, 'log')
    plt.imshow(heatmap.T, origin='lower', norm=norm, extent=extent)
    plt.savefig("sample%i"%i, bbox_inches="tight",transparent=True)
```

Computing the extinction probabilities

With the same numerical integration scheme, it is computed for a fixed number of iterates per gridpoint if the endpoint is contained in $\mathcal{D}_d$.

```
import feeder as f
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import phase3
a= 1
b= 1
d= 1/2
h= 1
r= 5
K= 5
gridsize = 10
eta = d/(h*a - d)
stead_v = b*eta
stead_p = (r*b)*(1+eta)*(1-b*eta/K)/a
tot_time = 4000
step_size = f.step_size
iterations = 50

def is_in_D(orbit,width,gridpoint):
    dist = min([np.linalg.norm(gridpoint-orbit[:,i]) for i in range(0,orbit.shape[1]-1)])
    return dist < width

def iterate(gridpoint):
```

```
    v = gridpoint[0]
    p = gridpoint[1]
    for i in range(1,iterations):
        v = max(v + f.generate_perturbation(v, p),0)
        p = max(p + f.generate_perturbation(v, p),0)
        random_time = np.random.exponential(10)
        it = f.integrate(random_time,step_size,v,p,a,b,d,h,r,K)
        v=it[0,it.shape[1]-1]
        p=it[1,it.shape[1]-1]
    return [v,p]

def compute_persistence_prob(gridpoint, orbit, width, reps):
    persistence_count = 0.0
    for i in range(1,reps):
        persistence_count += int(is_in_D(orbit,width,iterate(gridpoint)))
    return float(persistence_count/reps)

def generate_probs(grid, orbit, width,reps):
    p = ["" for g in grid]
    for g in grid:
        p[grid.index(g)] = compute_persistence_prob(g,orbit,width,reps)
    return p

#Creating the grid
values = f.integrate(tot_time, step_size, 4,4,a,b,d,h,r,K)
phases = phase3.assignperiod(values[:,(values.shape[1]-3000):(values.shape[1]-1)])
xmin = min(values[0,:])
xmax = max(values[0,:])
ymin = min(values[1,:])
ymax = max(values[1,:])
xw = xmax-xmin
yw = ymax -ymin
xs = np.linspace(max(xmin-0.25*xw,0), xmax+0.25*xw, gridsize)
ys = np.linspace(max(ymin-0.25*yw,0), ymax+0.25*yw, gridsize)
grid = [(x,y) for x in xs for y in ys]

reps = 100
pre_set_width = 0.5
width = min(min(phases[0][0,:]),min(phases[0][1,:]),pre_set_width)
probs = generate_probs(grid,phases[0],width,reps)

P = np.empty([gridsize,gridsize])
for x in xs:
        for y in ys:
                gr = x,y
                P[np.where(xs==x)[0][0],np.where(ys==y)[0][0]]= probs[grid.index(gr)]

plt.pcolormesh(xs,ys,P.transpose(),cmap="hot")
plt.scatter(phases[0][0,:],phases[0][1,:],color="k",s=0.15)
plt.scatter(stead_v,stead_p,color="k",marker="*",s=40)
plt.show()
```

Appendix B: Code for asymptotic phase computations

The heuristic employed in this numerical approximation is to first estimate the period of the limit cycle by letting the data run for a long time, find the first index for which the data repeats itself, and integrate the deterministic ODE model for each initial condition on a grid. The gridpoint is then assigned as aymptotic phase the phase of the closest point on the periodic orbit.

```
import feeder as f
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import matplotlib
a= 1
b= 1
d= 1/2
h= 1
r= 5
K= 5
gridsize = 150
eta = d/(h*a - d)
stead_v = b*eta
stead_p = (r*b)*(1+eta)*(1-b*eta/K)/a

tot_time = 8000
step_size = f.step_size
n = int(tot_time/step_size)

def findperiod(data):
    T= 100000
    for i in range(1,data.shape[1]-2):
        for j in range(max(data.shape[1]-1-int(T/step_size),i+1),min(data.shape[1]-1,
                i+int(T/step_size))):
            if(np.linalg.norm(data[:,i]-data[:,j]) < 0.005):
                if((j-i)*step_size <T):
                    T = (j-i)*step_size
    return int(T/step_size)

def assignperiod(data):
    period = findperiod(data)
    l_ind = data.shape[1]-1
    orbit = data[:,(l_ind - period):l_ind]
    phases = [t*step_size for t in range(0,period)]
    return [orbit,phases]

def assignphase(grid,orbit):
    limitphases = ["" for g in grid]
    for g in grid:
        T=orbit[0].shape[1]-1
        li = f.integrate(4*T,step_size,g[0],g[1],a,b,d,h,r,K)
        errs = [np.linalg.norm(li[:,li.shape[1]-1]-orbit[0][:,i])
        for i in range(0,orbit[0].shape[1]-1)]
        limitphases[grid.index(g)] = int(np.where(errs==np.min(errs))[0])*step_size
    return [grid,limitphases]

values = f.integrate(tot_time, step_size, 4,4,a,b,d,h,r,K)
phases = assignperiod(values[:,(values.shape[1]-3000):(values.shape[1]-1)])
xmin = min(values[0,:])
```

```
xmax = max(values[0,:])
ymin = min(values[1,:])
ymax = max(values[1,:])
xw = xmax-xmin
yw = ymax -ymin
xs = np.linspace(max(xmin-0.25*xw,(xmax-xmin)/gridsize), xmax+0.25*xw, gridsize)
ys = np.linspace(max(ymin-0.25*yw,(ymax-ymin)/gridsize), ymax+0.25*yw, gridsize)
grid = [(x,y) for x in xs for y in ys]
limitphases = assignphase(grid,phases)
limits=np.empty([gridsize,gridsize])
for x in xs:
        for y in ys:
                gr = x,y
                limits[np.where(xs==x)[0][0],np.where(ys==y)[0][0]]=
                limitphases[1][grid.index(gr)]

np.savetxt("limitphases", limitphases[1])
plt.pcolormesh(xs,ys,limits.transpose(),cmap="hot")
plt.scatter(phases[0][0,:],phases[0][1,:],color="b",s=0.15)
plt.scatter(stead_v,stead_p,color="b",marker="*",s=40)
plt.savefig("phase", bbox_inches="tight",transparent=True)
plt.show()
```

Appendix C: Numerical search for a contraction metric

We include the code used for the computation of L_M for the constant contraction metrics. This code was used to generate the plots.

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
import importlib
from mpl_toolkits.mplot3d import Axes3D
import feeder as f
step_size = 0.01
tot_time = 8000
a= 1
b= 1
d= 1/2
h= 1
r= 1.1
K= 5
s=20
gridsize = 150

def LM(v,p):
     f = np.array([[r*v*(1-v/K) - (a*v*p)/(b+v)],
          [-d*p + (h*a*v*p)/(b+v)]])
     x = np.array([[-d*p + (h*a*v*p)/(b+v)],
          [-(r*v*(1-v/K) - (a*v*p)/(b+v))]])
     Df = np.array([[r - (2*r*v)/K + (a*b*p)/((b+v)**2), -(a*v)/(b+v)],
           [(h*a*b*p)/((b+v)**2), -d + (h*a*v)/(b+v)]])
     V = Df - (1/np.linalg.norm(f)**2)*(f*(f.transpose()))*(Df + Df.transpose())
     M = np.array([[alpha, beta],
          [beta,gamma]])
```

```
    return (0.5*np.inner(x.transpose(),(np.dot((M*V)+(M*V).transpose()
        ,x)).transpose()))[0][0]

def ploth(x,y):
    return 0

vLM= np.vectorize(LM)
vploth = np.vectorize(ploth)

orbit = f.integrate(tot_time, step_size, 4,4, a,b,d,h,r,K)[:,5000:8000]
xs2 = orbit[0,:]
ys2 = orbit[1,:]
xw = max(xs2)- min(xs2)
yw = max(ys2)- min(ys2)
x_extent = np.linspace(0, K+xw/4, 100)
y_extent= np.linspace(0, max(ys2)+ yw/4,100)
x_extent, y_extent = np.meshgrid(x_extent,y_extent)
Z2 = vploth(xs2, ys2)

alpha=1
gamma = 1
beta = 0
Z= vLM(x_extent, y_extent)

fig = plt.figure()
ax = plt.axes(projection="3d")
ax.plot_surface(x_extent,y_extent,Z,rstride =1, cstride =1, cmap="Spectral")
ax.scatter(xs2,ys2,Z2, s=0.1)
ax.set_xlabel("Victim Species")
ax.set_ylabel("Predator Species")
ax.set_zlabel("Contraction Rate")
plt.show()
```

Bibliography

[1] T. Alkurdi. Piecewise deterministic Markov processes : an analytic approach. *Doctoral Thesis, Universiteit Leiden*, 2013.

[2] A. Lasota and J. Myjak. Fractals, Semifractals and Markov operators. *International Journal of Bifurcation and Chaos*, 9(2):307–325, 1999.

[3] A. Etheridge. *Some mathematical models from population genetics. École d'Été de Probabilités de Saint-Flour XXXIX-2009*, volume 2012. 01 2011.

[4] F. Brauer. Mathematical epidemiology: Past, present, and future. *Infectious Disease Modelling*, 2, 02 2017.

[5] J. Cronin. *Mathematical Aspects of Hodgkin-Huxley Neural Theory*. Cambridge Studies in Mathematical Biology. Cambridge University Press, 1987.

[6] C.H. Lee and H.G. Othmer. A multi-time-scale analysis of chemical reaction networks: I. Deterministic systems. *J. Math. Biology*, 60:387–450, 2010.

[7] M. Mimura and Y. Nishiura. Spatial patterns for an interaction-diffusion equation in morphogenesis. *J. Math. Biology*, 7:243–263, 1979.

[8] V. Shahrezaei and P. Swain. Analytical distributions for stochastic gene expression. *Proceedings of the National Academy of Sciences*, 105(45):17256–17261, 2008.

[9] Matthew J Simpson, Ruth E Baker, Sean T Vittadello, and Oliver J Maclaren. Parameter identifiability analysis for spatiotemporal models of cell invasion. *bioRxiv*, 2019.

[10] Y. Zeng. Stochastic calculus for finance i : The binomial asset pricing model solution of exercise problems. 2014.

[11] M. H. A. Davis. Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 46(3):353–388, 1984.

[12] A. Genadot and M. Thieullen. Multiscale Piecewise Deterministic Markov Process in infinite dimension: central limit theorem and Langevin approximation. *ESAIM: Probability and Statistics*, 18:541–569, 2014.

[13] N. Bäuerle and U. Rieder. Optimal control of piecewise deterministic markov processes with finite time horizon. 01 2010.

[14] J. Bect. Processus de Markov diffusifs par morceaux: outils analytiques et numériques. *Doctoral thesis, Université Paris-Sud - Paris XI*, 2007.

[15] M.H.A. Davis. *Markov Models & Optimization*. Taylor & Francis, 2017.

[16] Paul C. Bressloff and James MacLaurin. A variational method for analyzing limit cycle oscillations in stochastic hybrid systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 28(6):063105, 2018.

[17] Dejun Fan Xiaoling Zou and Ke Wang. Stationary distribution and stochastic Hopf bifurcation for a predator-prey system with noises, 2013.

[18] T. Szarek S. Hille, K. Horbacz. Existence of a unique invariant measure for a class of equicontinuous Markov operators with application to a stochastic model for an autoregulated gene. *Annales mathématiques Blaise Pascal.*, (2):171–217, 2016.

[19] V.I. Bogachev. *Measure Theory*. Measure Theory. Springer Berlin Heidelberg, 2007.

[20] R. Zaharopol. *Invariant Probabilities of Markov-Feller Operators and Their Supports*. Frontiers in Mathematics. Birkhäuser Basel, 2005.

[21] D.T.H. Worm. Semigroups on spaces of measures. *Doctoral Thesis, Universiteit Leiden*, 2010.

[22] J. Diestel and J.J. Uhl. *Vector Measures*. Mathematical surveys and monographs. American Mathematical Society, 1977.

[23] R. Dudley. Convergence of baire measures. *Studia Mathematica*, 27(3):251–268, 1966.

[24] T. Szarek. Invariant measures for nonexpansive Markov operators on Polish spaces. *Dissertationes Mathematicae - DISS MATH*, 415:1–62, 01 2003.

[25] M.A. Ziemlańska. Approach to Markov Operators on Spaces of Measures by means of Equicontinuity. *PhD thesis at Leiden University, in preparation*, 2020.

[26] P. Giesl. On a matrix-valued PDE characterizing a contraction metric for a periodic orbit, 2018.

[27] P. Giesl E. Scalas H. Björnsson, S. Hafstein and S. Gudmundsson. Computation of the stochastic basin of attraction by rigorous construction of a lyapunov function. *Discrete & Continuous Dynamical Systems - B*, 24:4247, 2019.

[28] O. Diekmann Yu.A. Kuznetsov and W.-J. Beyn. Dynamical Systems Lecture Notes. *Universiteit Utrecht*, 2011.

[29] Thomas S. Parker and Leon O. Chua. *Poincaré Maps*, pages 31–56. Springer New York, New York, NY, 1989.

[30] K.-S. Cheng. Uniqueness of a limit cycle for a predator-prey system. *SIAM J.Math.Anal.*, 12(4):541–548, 1981.

[31] P. A. Parrilo E.M. Aylward and J. E. Slotine. Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming. *Automatica*, 44(8):2163 – 2170, 2008.

[32] W. Lohmiller and J.E. Slotine. On Contraction Analysis for Non-linear Systems. *Automatica*, 34(6):683–696, 1998.

[33] B.P. Demidovich. Dissipativity of a system of nonlinear differential equations in the large. *Uspekhi Mat. Nauk*, 16:216, 1961.

[34] N. van de Wouw A. Pavlov, A. Pogromsky and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters*, 52(3):257 – 261, 2004.

[35] B.P. Demidovich. *Lectures on Stability Theory*. Nauka, 1967.

[36] P. Giesl. Computation of a contraction metric for a periodic orbit using meshfree collocation. *SIAM Journal on Applied Dynamical Systems*, 18(3):1536–1564, 2019.

[37] G. Borg. *A condition for the existence of orbitally stable solutions of dynamical systems*. Kungl. Tekniska Högskolans Handlingar. Elander, 1960.

[38] G. Söderbacka N.L.P. Lundström. Estimates of Size of Cycle in a Predator-Prey System. *Differ Equ Dyn Syst*, 2018.