Analysis of Welter's game

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Contents

1 Introduction

Welter's game is a game that is played on an infinite strip of squares, numbered $0, 1, 2, 3, \ldots$. A number of coins are placed on some of the squares, with no two coins on the same square. Two players play alternately. In each turn, they take a coin from a square and move it to a square with a lower number, that is not yet occupied by a coin. If there are no moves left at the start of a player's turn, that player loses.

This is an example of a so-called impartial combinatorial game. For each initial position of such a game, there exists a strategy that is winning for one of the players, regardless of what moves the other player chooses. This is related to the Sprague-Grundy function, a function with all possible positions in the game as domain and $\mathbb{N} = \{0, 1, 2, \ldots\}$ as range. The Sprague-Grundy function has value 0 in a position if and only if there is a winning strategy starting from that position for the player whose turn it is not. Such positions are called P-positions.

The Sprague-Grundy function is a recursive function. The value at a certain position depends on the values at all positions that can be reached from that position in one move. This recursive expression for the Sprague-Grundy function does not lead to a computationally efficient algorithm for calculating the value at a certain position. So if we want to use the Sprague-Grundy function to efficiently calculate the winning strategy of a game, we need to find a closed-form expression for the Sprague-Grundy function, or a computationally more efficient recursive expression. We will further discuss P-positions and the Sprague-Grundy function in Chapter 2.

Next, we discuss the Sprague-Grundy function for Welter's game. In the literature, we encountered four different methods for finding a computationally efficient expression for this function (see [10], [2] and [1]). Each of these consists of defining a candidate Sprague-Grundy function based on a list of one or more properties. To prove that a candidate function is indeed the Sprague-Grundy function for Welter's game, we first need to prove that it satisfies the properties of at least two of the other candidate functions, thereby proving that the three candidate functions are equal.

Because the properties that define the four candidate functions are quite different, it is not immediately obvious that the functions are all equal. In order to better understand the relations between these defining properties, we examined the pairwise equivalence of the four sets of defining properties. We tried to prove directly that each of the candidate functions satisfies the properties of each of the three other functions. We discuss the four candidate functions and their equivalence in Chapter 3. Some of the proofs in this chapter come from [10] and [2], others are new.

Once it has been established that the four candidate functions are equal, we prove in Chapter 4 that this function is indeed the Sprague-Grundy function for Welter's game. In order to do so, we first discuss a few interesting properties that this function satisfies. We follow Conway's proof, which can be found in [2]. The same result was first proved in a different way by Welter, in [10].

In Chapter 5, we discuss an interesting method for determining P-positions for Welter's game. This method uses tables filled with such positions, and can also be applied to multiple similar games. It also leads to a new proof that the Sprague-Grundy function for Welter's game must satisfy one of the properties discussed in Chapter 4, directly from the definition of Welter's game.

We discuss some further properties of Welter's game in Chapter 6. First, we discuss a method that can be used to determine a move in the optimal strategy, by completing a triangle of numbers. Next, we discuss an interesting property that the Sprague-Grundy function for Welter's game has when the amount of coins is a multiple of 4. This can help to find a move in the optimal strategy, even in cases where the amount of coins is not a multiple of 4. We also check whether a similar property holds when the amount of coins is not a multiple of 4. Then, we discuss the optimal strategy when Welter's game is played with at most 4 coins. Finally, we discuss the optimal solution for the situation where there are exactly 5 coins, and all are on positions in $\{0, \ldots, 15\}$.

In Chapter 7, we further discuss a non-intuitive property of Welter's original candidate function. We prove directly from the definition of Welter's game that Welter's candidate function is indeed the Sprague-Grundy function for Welter's game played with 2 coins. Then, we discuss some interesting patterns that appear in tables with Sprague-Grundy values for Welter's game with 3 coins, that are related to this candidate function.

Finally, in Chapter 8 we discuss the optimal strategy for the misère version for Welter's game. In this variant, if there are no moves left at the start of a player's turn, that player wins.

In later chapters, we will often use the definitions and the main results of Chapters 2, 3, and 4, without explicitly referring back to those chapters.

2 Impartial combinatorial games

In this chapter, we discuss some important properties of impartial combinatorial games. First we give the definition in Section 2.1, then we discuss a method for finding winning strategies in Section 2.2. In Section 2.3 we look at the game Nim, which is an example of an impartial combinatorial game. Here we also define the so-called nim-sum, which plays an important role in finding winning strategies for many games, including Welter's game. Finally, in Section 2.4 we discuss the Sprague-Grundy function, which can be used to analyse impartial combinatorial games. A more detailed description of impartial combinatorial games and their properties, with many examples, can be found in Chapter 1 of [3].

2.1 Definition

Combinatorial games are games with two players where no chance is involved. In each turn a player moves the game state from one position to another, and the players alternate turns. At all times both players know in which position the game is and which moves from one position to another are possible. Below we give a more precise definition.

Definition 2.1 (Combinatorial game)**.** A combinatorial game is a game that satisfies the following conditions:

- There are two players.
- There is a set of feasible positions of the game.
- For each combination of player and position, there is either a set of feasible moves to other positions that the player may choose from if it is his turn, or no moves are possible. If there is a feasible move from position A to position B, we call B a follower of A.
- The game ends when a position is reached from which no moves are possible for the player whose turn it is. Such a position is called a terminal position.
- The players alternate moving. At each time, the player who will move next is called the next player, and the other player is called the previous player.

Definition 2.2 (Impartial combinatorial game)**.** A combinatorial game is called impartial if, given that the game is in a certain position, the set of available moves is the same regardless of whose turn it is to play.

Example 2.3. Chess is a combinatorial game, but it is not impartial, because one player may only move the white pieces while the other may only move the black pieces.

Definition 2.4 (Normal play and misère play)**.** Under the normal play rule, if a player moves to a terminal position, that player wins. Under the misère play rule, a player who moves to a terminal position loses. We assume the normal play rule is used, unless we state otherwise.

Note that under these conditions a combinatorial game does not necessarily end in a finite number of turns. If a game does not end in a finite number of turns, there is a draw. However, for this thesis we only look at impartial combinatorial games where there is a finite maximum number of turns before the game ends.

2.2 Winning strategies

In all combinatorial games in which a draw cannot occur, there is a winning strategy for one of the players. If the player follows this strategy, he will win, regardless of what the other player does. We will prove this for impartial combinatorial games that end within a finite maximum number of turns, using the normal play rule. It can be proven in a similar way for other combinatorial games.

First, we separate the positions into those in which there is a winning strategy for the player whose turn it is, and those in which there is a winning strategy for the other player.

Definition 2.5 (N-positions and P-positions)**.** A position in an impartial combinatorial game is called an N-position if there is a strategy with which the next player, that is the player whose turn it is, can win no matter what moves the other player will make. It is called a P-position if there is such a winning strategy for the other player, the previous player, instead.

Theorem 2.6. *In an impartial combinatorial game for which there is a maximum number of turns before the game ends, there is always a winning strategy for one of the players.*

Proof. We use the following algorithm to label each position as either a P-position or an N-position.

Step 1: Give every terminal position the label P.

Step 2: Give the label N to every position which has a feasible move to a position with label P.

Step 3: Give the label P to every position for which all feasible moves lead to positions with label N.

Step 4: If every position has a label, stop. Otherwise return to step 2.

Note that, if we follow the algorithm above, each position gets at most one label. To show that this algorithm is correct, we first need to show that every position gets a label. For each position A we can define $d(A)$ as the maximum amount of moves to get from that position to a terminal position. Once all the followers of a position have been labelled, either a follower has received label P or all the followers have label N. So each position will receive its label no later than the round after all of its followers are labelled. Since we start by labelling all the terminal positions, this means that position A will receive its label in round $d(A)$ at the latest.

Now we need to show that each P-position gets the label P, and each N-position gets the label N. We prove this by induction on $d(A)$. If $d(A) = 0$, then A is a terminal position. If a terminal position is reached, the player who made the last move has won. So every terminal position is a P-position and should get label P. This means that the algorithm correctly labels these positions.

Now assume that $d(A) > 0$ and that B is labelled correctly for each position B with $d(B) < d(A)$. If A received label N, then it must have a follower B which has label P. Because B is a follower of A, we must have $d(B) < d(A)$. So by the induction hypothesis, B is a P-position. Now we can find a winning strategy for the next player starting from A. That player can move to position B. After this move, he will be the previous player. He can then follow the winning strategy for the previous player starting from position B. So A is an N-position.

Assume that $d(A) > 0$ and that B is labelled correctly for all positions B with $d(B) < d(A)$. If A received label P, then its followers must all have label N. A follower B of A must satisfy $d(B) < d(A)$. So by the induction hypothesis, all followers are N-positions. Now we can find a winning strategy for the previous player starting from A. For any follower B of A, if the next player moves to position B, the previous player can follow the winning strategy for the next player from there. So position A is a P-position.

□

Corollary 2.7. *Every position in an impartial combinatorial game for which there is a maximum number of turns before the game ends is either a P-position or an N-position. Further, the partition into P-positions and N-positions is the unique one with the following three properties:*

- *All terminal positions are P-positions.*
- *From every N-position, at least one of the feasible moves is to a P-position.*
- *From every P-position, all feasible moves are to N-positions.*

Corollary 2.8. *The winning strategy is to always move to a P-position, so that the other player will be forced to move to an N-position. Repeat until a terminal P-position is reached.*

If we want to know the winning strategy for a particular game, we need to find a computationally efficient method for determining which positions are P-positions and which are N-positions. For some games, this can be done using the Sprague-Grundy function, which we discuss in Section 2.4.

2.3 Nim

Nim is an impartial combinatorial game played with piles of coins. Each turn, a player must remove coins from exactly one of the piles. At least one coin must be removed, and at most the whole pile. If a player cannot make a move because all the coins have been removed, that player loses.

Definition 2.9 (Nim)**.** Nim is an impartial combinatorial game. The positions are $(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n \in \mathbb{N}$. For such a position, we say that there are n piles of coins, of sizes $x_1, \ldots, x_n$.

The followers of a position $(x_1, \ldots, x_n)$ are all positions

$$(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$$

with $i \in \{1, \ldots, n\}$ and $x_i' < x_i$.

To analyse this game, we first define the binary expansion and the nim-sum.

Definition 2.10 (Binary expansion)**.** Let $x \in \mathbb{N}$. Then there exist $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in \{0, 1\}$ such that $x = x_0 + 2x_1 + 2^2x_2 + \cdots + 2^nx_n$. We call $x_nx_{n-1}\ldots x_1x_0$ the binary expansion of x. For any $m > n$, $x_mx_{m-1}\ldots x_1x_0$ with $x_{n+1} = \cdots = x_m = 0$ is also the binary expansion of x. For any $k \in \mathbb{N}$, the last k digits of the binary expansion of x are $x_{k-1}x_{k-2}\ldots x_1x_0$.

Definition 2.11 (Nim-sum)**.** For any two non-negative integers x and y, their nim-sum $x \oplus y$ is the sum without carry in base 2. Suppose that the binary expansion of x is $x_nx_{n-1}\ldots x_1x_0$ and the binary expansion of y is $y_ny_{n-1}\ldots y_1y_0$. Then $x \oplus y$ has binary expansion $z_nz_{n-1}\ldots z_1z_0$, where $z_i = x_i + y_i \mod 2$ for all $i \in \{1, \ldots, n\}$. We call $x_1 \oplus \cdots \oplus x_n$ the nim-sum of the position $(x_1, \ldots, x_n)$.

We now give a few useful properties of nim-addition.

Lemma 2.12. *The nim-sum is commutative and associative, so* $x \oplus y = y \oplus x$ *and* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *for all* $x, y, z \in \mathbb{N}$.

Lemma 2.13. $x \oplus y = x \oplus z$ *if and only if* $y = z$.

Proof. By nim-adding x to both sides, we find that $y = z$ implies $x \oplus y = x \oplus z$. Further, note that $x \oplus x = 0$ and $0 \oplus x = x$ for all $x \in \mathbb{N}$. This means that $x \oplus y = x \oplus z$ implies

$$y = 0 \oplus y = x \oplus x \oplus y = x \oplus x \oplus z = 0 \oplus z = z.$$

□

Example 2.14. We calculate $3 \oplus 5 \oplus 7$. The binary expansion of 3 is 11 or 011, that of 5 is 101 and that of 7 is 111. When adding in base 2 without carry we get

$$\begin{array}{ll} 011 & \\ 101 & \\ 111 & + \\ \hline 001 & \end{array}$$

So the binary expansion of $3 \oplus 5 \oplus 7$ is 001, which means that $3 \oplus 5 \oplus 7 = 1$.

Theorem 2.15. *A position* $(x_1, \ldots, x_n)$ *in Nim is a P-position if and only if* $x_1 \oplus \cdots \oplus x_n = 0$.

Proof. Note that Nim is an impartial combinatorial game, and that starting from position $(x_1, \ldots, x_n)$, there are at most $x_1 + \cdots + x_n$ moves before the game ends. This means we can use Corollary 2.7. We check the three conditions.

• The only terminal positions are the ones with $x_1 = \cdots = x_n = 0$, for any n. These positions all satisfy $x_1 \oplus \cdots \oplus x_n = 0$.

• From each position $(x_1, \dots, x_n)$ with $x_1 \oplus \cdots \oplus x_n \neq 0$, we can move to a position with nim-sum 0 as follows: write the nim-sum using column addition as in Example 2.14, and look at the leftmost column with an odd number of 1s. Pick an $i \in \{1, \dots, n\}$ such that x_i that has a 1 in that column, and replace x_i by a number x_i' such that there are an even number of 0s in each column. Then in the first position on which the binary expansions of x_i and x_i' differ, x_i has a 1 and x_i' has a 0. So we have $x_i' < x_i$. This means that $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$ is a follower of $(x_1, \dots, x_n)$. By our choice of x_i', it has nim-sum 0.

• From a position $(x_1, \dots, x_n)$ with $x_1 \oplus \cdots \oplus x_n = 0$, every feasible move is to a position $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$ for some $i \in \{1, \dots, n\}$ and with $x_i' < x_i$. Suppose that such a position satisfies $x_1 \oplus \cdots \oplus x_{i-1} \oplus x_i' \oplus x_{i+1} \cdots \oplus x_n = 0$. Then we have $x_i = x_i'$ by Lemma 2.13. This gives a contradiction, so all followers of $(x_1, \dots, x_n)$ have a nim-sum unequal to 0.

□

Example 2.16. Suppose Nim is played, and the current position is $(1, 3, 5)$. We calculate the nim-sum:

$$\begin{array}{rl} 001 & \\ 011 & \\ 101 & + \\ \hline 111 & \end{array}$$

The nim-sum has binary expansion 111, so this is not a P-position. So the player who starts can win by moving to a P-position. Using the column addition we see that the player needs to remove coins from the third pile. To get an even number of 1s in each column, this 101 pile needs to be changed to a 010 pile. This means that the player should remove coins from the third pile until there are 2 coins left in this pile. This corresponds to a move to position $(1, 3, 2)$.

2.4 Sprague-Grundy function

One way of finding the P-positions and N-positions is by using the Sprague-Grundy function, which we will define now.

Definition 2.17 (Minimal excludant)**.** For any set $A \subseteq \mathbb{N}$, the minimal excludant of A is

$$\text{mex}(A) := \min\{n \in \mathbb{N} : n \notin A\}.$$

Definition 2.18 (Sprague-Grundy function)**.** For any position x, let $F(x)$ be the set of followers of x. Then the Sprague-Grundy function g is the unique function such that for all x,

$$g(x) = \min\{n \in \mathbb{N} : n \neq g(y) \text{ for all } y \in F(x)\} = \text{mex}\{g(y) : y \in F(x)\}.$$

We call $g(x)$ the Sprague-Grundy value of x.

For any impartial combinatorial game with a maximum number of turns before the game ends, the Sprague-Grundy function is well-defined. In such a game, we have $g(x) = \text{mex}(\emptyset) = 0$ for any terminal position x. If all followers of position x are terminal positions, then $g(x) = 1$. In general, if $d(x)$ is the maximum number of moves to get from position x to a terminal position, then $g(x) \leq d(x)$.

Example 2.19. Let g be the Sprague-Grundy function for the game Nim. We calculate $g((1,2))$. Note that for any $x, y \in \mathbb{N}$, the position (x, y) in Nim equals the position (y, x). So

$$\begin{aligned} g((1,2)) &= \text{mex}\{g((0,2)), g((1,1)), g((1,0))\} \\ &= \text{mex}\{g((0,2)), g((1,1)), g((0,1))\}. \end{aligned}$$

Further, we have

$$\begin{aligned} &g((0,0)) = \text{mex}(\emptyset) = 0, \\ &g((0,1)) = \text{mex}\{g((0,0))\} = \text{mex}\{0\} = 1, \\ &g((1,1)) = \text{mex}\{g((0,1)), g((1,0))\} = \text{mex}\{g((0,1))\} = \text{mex}\{1\} = 0, \\ &g((0,2)) = \text{mex}\{g((0,1)), g((0,0))\} = \text{mex}\{1,0\} = 2. \end{aligned}$$

So

$$g((1,2)) = \text{mex}\{2,0,1\} = 3.$$

If the Sprague-Grundy value can be calculated efficiently for a particular game, then it is easy to find a winning strategy.

Theorem 2.20. *For any impartial combinatorial game for which there is a maximum number of turns before the game ends, the P-positions are precisely those positions with Sprague-Grundy value 0.*

Proof. We check the three conditions from Corollary 2.7.

- If x is a terminal position, then $F(x)$ is the empty set. In that case

$$\text{mex}\{n \in \mathbb{N} : n \neq g(y) \text{ for all } y \in F(x)\} = \min\{n \in \mathbb{N}\} = 0.$$

So all terminal positions have Sprague-Grundy value 0.

- Suppose a position x has Sprague-Grundy value unequal to 0. Then we have $\text{mex}\{g(y) : y \in F(x)\} \neq 0$, so $0 \in \{g(y) : y \in F(x)\}$. This means that x has a follower with Sprague-Grundy value 0.

- If a position x has Sprague-Grundy value 0, then $\text{mex}\{g(y) : y \in F(x)\} = 0$, so $0 \notin \{g(y) : y \in F(x)\}$. This means that all followers of x have Sprague-Grundy value unequal to 0.

□

The recursive expression $g(x) = \text{mex}\{g(y) : y \in F(x)\}$ for the Sprague-Grundy function does not lead to a computationally efficient algorithm for finding the value at a certain position. However, for some games there exists a closed-form expression for the Sprague-Grundy function, or a computationally more efficient recursive expression. This is true for the game Nim, as well as for Welter's game. We will discuss the Sprague-Grundy function of Welter's game in Chapter 3. We now give the Sprague-Grundy function for Nim.

Theorem 2.21. *The Sprague-Grundy value of any position* $(x_1, \ldots, x_n)$ *in Nim is equal to its nim-sum,* $x_1 \oplus \cdots \oplus x_n$.

Proof. For any position A, let $d(A)$ be the maximum amount of moves needed to get from position A to a terminal position. We use induction on $d(A)$. The proof is similar to that of Theorem 2.15, except that we now work with nim-sum k' for any $k' \in \mathbb{N}$, instead of only with nim-sum 0.

If $d(A) = 0$, then A is a terminal position. By Theorem 2.20, this means that A has Sprague-Grundy value 0. By Theorem 2.15, A also has nim-sum 0. So the Sprague-Grundy value of A is equal to its nim-sum.

Now let $A = (x_1, \ldots, x_n)$ with $d(A) > 0$, and assume that for each position B with $d(B) < d(A)$, the Sprague-Grundy value of B is equal to its nim-sum. Let $k = x_1 \oplus \cdots \oplus x_n$. We need to show that there is a follower of $(x_1, \ldots, x_n)$ with Sprague-Grundy value k' for each $k' < k$, and that there is no follower of $(x_1, \ldots, x_n)$ with Sprague-Grundy value k.

Assume that $k > 0$. Let $k' \in \mathbb{N}$ with $k' < k$. We write the nim-sum using column addition as in Example 2.14, and look at the first position in which the binary expansion of the nim-sum k differs from that of k'. Since $k > k'$, k has a 1 in this position while k' has a 0 in this position. This means that the number of 1s in the corresponding column must be odd. We can pick an $i \in \{1, \ldots, n\}$ such that x_i has a 1 in this column, and replace x_i by a number x_i' so that the nim-sum becomes equal to k'. Then in the first position on which the binary expansions of x_i and x_i' differ, x_i has a 1 and x_i' has a 0. So we have $x_i' < x_i$. This means that $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ is a follower of $(x_1, \ldots, x_n)$. By our choice of x_i', this follower has nim-sum equal to k'. By the induction hypothesis, its Sprague-Grundy value also equals k'.

Suppose that a follower of $(x_1, \ldots, x_n)$ has Sprague-Grundy value k. Then there exist $i \in \{1, \ldots, n\}$ and $x_i' < x_i$ such that the follower is the position $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$. By the induction hypothesis, this means that $x_1 \oplus \cdots \oplus x_{i-1} \oplus x_i' \oplus x_{i+1} \oplus \cdots \oplus x_n = k$. Then we have $x_i = x_i'$ by Lemma 2.13. This gives a contradiction, so all followers of $(x_1, \ldots, x_n)$ have Sprague-Grundy value unequal to k.

□

Example 2.22. We show that we can move from the position $(1, 3, 5)$ to a position with Sprague-Grundy value 2. We write the nim-sum using column addition:

$$\begin{array}{rl} 001 & \\ 011 & \\ 101 & + \\ \hline 111 & \end{array}$$

The binary expansion of 2 is 010. The first difference with the nim-sum is in the left-most column. From the column addition we see that we need to remove coins from the third pile. To get nim-sum 010, this 101 pile needs to be changed to a 000 pile, so all coins from this pile need to be removed. This corresponds to a move to position $(1, 3, 0)$.

3 Sprague-Grundy function for Welter's game

In this chapter, we discuss several candidate functions for the Sprague-Grundy function corresponding to Welter's game. First we describe Welter's game in Section 3.1. Next, we look at four equivalent candidates for the Sprague-Grundy function in Section 3.2, though we do not yet prove that they are equivalent and that they are related to Welter's game. In Section 3.3, we prove some lemmas which will be useful for the proofs in the following sections. Then, in Section 3.4, we prove that the functions discussed are symmetric in their arguments. We will need this property for some of the proofs in Section 3.5, where we prove the equivalence of the four candidates, and conclude that they in fact equal a single function. In Chapter 4 we will prove that this function is indeed the Sprague-Grundy function for Welter's game. In Chapter 6 we will discuss some other properties of Welter's game. These lead to two methods for determining P-positions, and to the optimal strategy for the game played with at most 4 coins, or with 5 coins on the squares $\{0, \ldots, 15\}$.

3.1 Welter's game

Welter's game is an impartial combinatorial game played on an infinite strip of squares, numbered $0, 1, 2, 3, \ldots$. A number of coins are placed on some of the squares, and no two coins may be on the same square. In each turn, a player picks up one coin and moves it to a lower-numbered square that is not yet occupied by a coin. The player who moves last, wins.

Definition 3.1 (Welter's game)**.** Welter's game is an impartial combinatorial game. The positions are $(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n \in \mathbb{N}$ are distinct. For such a position, we say that there are n coins on squares $x_1, \ldots, x_n$.

The followers of a position $(x_1, \ldots, x_n)$ are all positions

$$(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$$

with $i \in \{1, \ldots, n\}$, $x_i' < x_i$ and $x_i' \in \mathbb{N} \setminus \{x_1, \ldots, x_n\}$.

Note that, for any starting position $(x_1, \ldots, x_n)$, there are at most $x_1 + \cdots + x_n$ moves before the terminal position $(0, 1, \ldots, n-1)$ is reached. So by Theorem 2.20, the winning strategies can be found by calculating the Sprague-Grundy value.

Welter's game is similar to Nim, because Nim can also be played with coins on an infinite strip of squares instead of with piles of coins. In this variant, a coin on square n is equivalent to a pile of n coins, and removing coins from a pile of n coins corresponds to picking up a coin on square n and moving it to a lower-numbered square. So the only difference between Nim and Welter's game is that in Nim multiple coins may be on the same square.

Welter discussed this game in [10] and found the optimal strategy essentially by finding a computationally efficient recursive expression for the Sprague-Grundy function, although he did not call it that. At the time, some variants of the game had already been solved. In [8], Sprague defined and solved the variant with exactly 5 coins, and played on the squares $\{0, \ldots, 15\}$. We will discuss this variant in Section 6.4. In [9], Welter solved the variant with exactly 5 coins and an infinite strip of squares.

3.2 Four candidate functions

In this section, we discuss candidate functions for the Sprague-Grundy function for Welter's game. We write $[\, x_1 \mid \cdots \mid x_n \,]$ for the value of the Sprague-Grundy function of Welter's game on position $(x_1, \ldots, x_n)$. By definition, we have

$$[\, x_1 \mid \cdots \mid x_n \,] = \operatorname{mex}\{[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] : \\ i \in \{1, \ldots, n\}, x_i' \in \mathbb{N} \setminus \{x_1, \ldots, x_n\}, x_i' < x_i\}.$$

However, this does not lead to a computationally efficient method to calculate $[\, x_1 \mid \cdots \mid x_n \,]$.

We will discuss several methods for finding a computationally efficient expression for the Sprague-Grundy function. Each of these consists of defining a candidate Sprague-Grundy function based on a list of one or more properties. We discuss four candidates for the Sprague-Grundy function for Welter's game. We call these functions Welter's candidate, the Mating candidate, the Animating candidate and the Triangle candidate. In Section 3.5, we will prove that these functions are equal. In Chapter 4, we will prove that they equal the Sprague-Grundy function for Welter's game. For convenience of notation, we will use the notation $[\, x_1 \mid \cdots \mid x_n \,]$ for each of the functions.

Welter's candidate We first discuss Welter's candidate, which Welter defined in [10].

Definition 3.2 (Welter's candidate)**.** $(x_1, \ldots, x_n) \mapsto [\, x_1 \mid \cdots \mid x_n \,]$ is the unique real-valued function defined on any position $(x_1, \ldots, x_n)$ of Welter's game that satisfies the following three conditions:

1. $[\, 0 \,] = 0$.
2. $[\, 0 \mid x_1 \mid \cdots \mid x_n \,] = [\, x_1 - 1 \mid \cdots \mid x_n - 1 \,]$ when $x_1, \ldots, x_n \neq 0$.
3. $[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] = [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_n$ for all $x \in \mathbb{N}$.

Here, $(x)_n$ is the result of nim-adding n copies of x, i.e. $(x)_n = x$ when n is odd and $(x)_n = 0$ when n is even.

Lemma 3.3. *There exists a unique function satisfying the three properties of Welter's candidate.*

Proof. We first show that there exists a function satisfying the three properties. We define $f : \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathbb{N}^n \to \mathbb{N}$ as follows. Let $f((x_1)) = x_1$ for all $x_1 \in \mathbb{N}$, and let $f((x_1, \ldots, x_n)) = f((x_2 \oplus x_1 - 1, \ldots, x_n \oplus x_1 - 1)) \oplus (x_1)_n$ for all $n \in \mathbb{N} \setminus \{0, 1\}$. Then $f((0)) = 0$, so the first property is satisfied. Further, for all distinct $x_1, \ldots, x_n \in \mathbb{N}$, we have

$$\begin{aligned} f((0, x_1, \ldots, x_n)) &= f((x_1 \oplus 0 - 1, \ldots, x_n \oplus 0 - 1)) \oplus (0)_n \\ &= f((x_1 - 1, \ldots, x_n - 1)), \end{aligned}$$

so f satisfies the second property. Finally, we have

$$\begin{aligned}
&f((x_1 \oplus x, \dots, x_n \oplus x)) \\
&= f((x_2 \oplus x \oplus x_1 \oplus x - 1, \dots, x_n \oplus x \oplus x_1 \oplus x - 1)) \\
&= f((x_2 \oplus x_1 - 1, \dots, x_n \oplus x_1 - 1)) \\
&= f((x_1, \dots, x_n)),
\end{aligned}$$

so f satisfies the third property.

Next we show unicity. Let $f, g : \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathbb{N}^n \to \mathbb{N}$ be two functions satisfying the three properties of Welter's candidate. For all $n \in \mathbb{N} \setminus \{0\}$, let $f_n : \mathbb{N}^n \to \mathbb{N}$ be the function f restricted to the domain of positions with n coins. Similarly, let $g_n : \mathbb{N}^n \to \mathbb{N}$ be the restriction of the function g to the domain of positions with n coins. We prove by induction that $f_n = g_n$ for all $n \in \mathbb{N}$.

Suppose that $f_1((x_1)) \neq g_1((x_1))$ for some $x_1 \in \mathbb{N}$. Then, by the third condition, we have $f_1((0)) \oplus x_1 \neq g_1((0)) \oplus x_1$. By the first condition, it follows that $0 \oplus x_1 \neq 0 \oplus x_1$, but this gives a contradiction. So $f_1 = g_1$.

Now let $n \in \mathbb{N}$ and assume that $f_m = g_m$ for all $m < n$. Suppose that $f_n((x_1, \dots, x_n)) \neq g_n((x_1, \dots, x_n))$ for some distinct $x_1, \dots, x_n \in \mathbb{N}$. Then, by the third condition, we have

$$f_n((0, x_2 \oplus x_1, \dots, x_n \oplus x_1)) \neq g_n((0, x_2 \oplus x_1, \dots, x_n \oplus x_1)).$$

By the second condition, it follows that

$$f_{n-1}((x_2 \oplus x_1 - 1, \dots, x_n \oplus x_1 - 1)) \neq g_{n-1}((x_2 \oplus x_1 - 1, \dots, x_n \oplus x_1 - 1)),$$

but this contradicts the induction hypothesis. So we may conclude that $f_n = g_n$.

□

Example 3.4. Consider the position $(1, 5, 6)$. We have

$$\begin{aligned}
[\,1 \mid 5 \mid 6\,] &= [\,0 \mid 4 \mid 7\,] \oplus (1)_3 = [\,0 \mid 4 \mid 7\,] \oplus 1 = [\,3 \mid 6\,] \oplus 1 \\
&= [\,0 \mid 5\,] \oplus 1 \oplus (3)_2 = [\,0 \mid 5\,] \oplus 1 \oplus 0 = [\,4\,] \oplus 1 \\
&= [\,0\,] \oplus 1 \oplus (4)_1 = [\,0\,] \oplus 1 \oplus 4 = 0 \oplus 1 \oplus 4 = 5.
\end{aligned}$$

The first two properties of Welter's candidate are easy to understand. Since the position (0), where one coin is placed on the 0 square, is a terminal position, it must have Sprague-Grundy value 0. The following lemma shows that the Sprague-Grundy function for Welter's game must satisfy the second property of Welter's candidate. We discuss the third property further in Chapter 7.

Lemma 3.5. *The second property of Welter's candidate follows directly from the rules of Welter's game.*

Proof. Let $(x_1, \dots, x_n) \mapsto [\ x_1 \mid \cdots \mid x_n\ ]$ for all distinct $x_1, \dots, x_n$ be the Sprague-Grundy function for Welter's game.

Let $n \in \mathbb{N}$. Because $(0, 1, \dots, k)$ is a terminal position for all $k \in \mathbb{N}$, we have

$$[\,0 \mid 1 \mid \cdots \mid n\,] = 0 = [\,0 \mid 1 \mid \cdots \mid n-1\,].$$

Now let $x_1, \dots, x_n \in \mathbb{N} \setminus \{0\}$ be distinct and assume that for all $i \in \{1, \dots, n\}$ and $x_i' < x_i$,

$$\begin{aligned}&[\ 0 \mid x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] \\ &= [\ x_1 - 1 \mid \cdots \mid x_{i-1} - 1 \mid x_i' - 1 \mid x_{i+1} - 1 \mid \cdots \mid x_n - 1\ ].\end{aligned}$$

Then

$$\begin{aligned}&[\ 0 \mid x_1 \mid \cdots \mid x_n\ ] \\ &= \operatorname{mex}\{[\ 0 \mid x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] : i \in \{1, \dots, n\}, x_i' < x_i\} \\ &= \operatorname{mex}\{[\ x_1 - 1 \mid \cdots \mid x_{i-1} - 1 \mid x_i' - 1 \mid x_{i+1} - 1 \mid \cdots \mid x_n - 1\ ] : \\ &\qquad i \in \{1, \dots, n\}, x_i' < x_i\} \\ &= [\ x_1 - 1 \mid \cdots \mid x_{i-1} - 1 \mid x_i - 1 \mid x_{i+1} - 1 \mid \cdots \mid x_n - 1\ ].\end{aligned}$$

□

In [10], Welter proved that Welter's candidate gives the Sprague-Grundy function for Welter's game. We will discuss this result in Chapter 4.

Mating candidate A second candidate for the Sprague-Grundy function for Welter's game is the Mating candidate, which is described in Chapter 13 of [2].

Definition 3.6 (Mating candidate)**.** For all distinct $x_1, x_2 \in \mathbb{N}$, we define $[\ x_1\ ] := x_1$ and $[\ x_1 \mid x_2\ ] := x_1 \oplus x_2 - 1$.

Let $n \geq 3$ and let $x_1, \dots, x_n \in \mathbb{N}$ be distinct. Let $i_1, i_2 \in \{1, \dots, n\}$ be such that out of the n numbers, x_{i_1} and x_{i_2} are congruent to each other modulo the highest power of 2. Then let $i_3, i_3 \in \{1, \dots, n\}$ be such that out of the numbers $\{x_1, \dots, x_n\} \setminus \{x_{i_1}, x_{i_2}\}$, x_{i_3} and x_{i_4} are congruent modulo the highest power of 2, et cetera. Then if n is even, we define

$$[\ x_1 \mid \cdots \mid x_n\ ] := [\ x_{i_1} \mid x_{i_2}\ ] \oplus [\ x_{i_3} \mid x_{i_4}\ ] \oplus \cdots \oplus [\ x_{i_{n-1}} \mid x_{i_n}\ ],$$

and if n is odd,

$$[\ x_1 \mid \cdots \mid x_n\ ] := [\ x_{i_1} \mid x_{i_2}\ ] \oplus [\ x_{i_3} \mid x_{i_4}\ ] \oplus \cdots \oplus [\ x_{i_{n-2}} \mid x_{i_{n-1}}\ ] \oplus [\ x_{i_n}\ ].$$

We call this the Mating candidate because the numbers $x_1, \dots, x_n$ are separated into pairs of mates.

Example 3.7. Consider the position $(1, 5, 6)$. The numbers 1 and 5 are mates because they are congruent modulo 4, while 1 and 6 and 5 and 6 are not. So we have

$$[\ 1 \mid 5 \mid 6\ ] = [\ 1 \mid 5\ ] \oplus [\ 6\ ] = (1 \oplus 5 - 1) \oplus 6 = 3 \oplus 6 = 5.$$

Example 3.8. Consider the position $(1, 3, 7, 8, 11, 19)$. The numbers 3 and 19 are congruent modulo 16, while no pair is congruent modulo 32. When looking at the remaining numbers $\{1, 7, 8, 11\}$, we see that 7 and 11 are congruent modulo 4 while no pair is congruent modulo 8. So we have

$$\begin{aligned}[\ 1 \mid 3 \mid 7 \mid 8 \mid 11 \mid 19\ ] &= [\ 3 \mid 19\ ] \oplus [\ 7 \mid 11\ ] \oplus [\ 1 \mid 8\ ] \\ &= (3 \oplus 19 - 1) \oplus (7 \oplus 11 - 1) \oplus (1 \oplus 8 - 1) \\ &= (16 - 1) \oplus (12 - 1) \oplus (9 - 1) \\ &= 15 \oplus 11 \oplus 8 = 12.\end{aligned}$$

Animating candidate In Chapter 13 of [2], Conway discussed a class of functions he called animating functions, and proved that a particular animating function, the Animating candidate, is the Sprague-Grundy function for Welter's game. He also proved that this function is the same as Welter's candidate.

Definition 3.9 (Mating function)**.** For all distinct $x, y \in \mathbb{N}$, let

$$(x \mid y) = 2^{n+1} - 1$$

if x and y are congruent modulo 2^n but not modulo 2^{n+1}. The function defined by $(x, y) \mapsto (x \mid y)$ for all distinct $x, y \in \mathbb{N}$ is the mating function.

Definition 3.10 (Animating candidate)**.** For all distinct $x_1, \ldots, x_n \in \mathbb{N}$, we define

$$\begin{aligned}&[\, x_1 \mid \cdots \mid x_n \,]\\ &:= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_1 \mid x_n) \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_{n-1} \mid x_n).\end{aligned}$$

We will often write $(x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n)$ instead of $(x_1 \mid x_2) \oplus \cdots \oplus (x_1 \mid x_n) \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_{n-1} \mid x_n)$.

Example 3.11. Consider the position $(1, 5, 6)$. We have

$$\begin{aligned}[\, 1 \mid 5 \mid 6 \,] &= 1 \oplus 5 \oplus 6 \oplus (1 \mid 5) \oplus (1 \mid 6) \oplus (5 \mid 6)\\ &= 2 \oplus (2^{2+1} - 1) \oplus (2^{0+1} - 1) \oplus (2^{0+1} - 1)\\ &= 2 \oplus 7 \oplus 1 \oplus 1 = 5.\end{aligned}$$

Conway named $(x, y) \mapsto (x \mid y)$ the mating function because the Animating candidate satisfies the properties of the Mating candidate, meaning that the functions are the same. He called functions of the type

$$f(x) = (\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m)$$

for some $m \in \mathbb{N}$ and $c'_1, c'_2, \ldots c'_m \in \mathbb{Z}$, where m is odd, animating functions. This is because they are defined in terms of *a*ddition and *nim*-addition, and they preserve the *mating* function, i.e. satisfy $(f(x)|f(y)) = (x|y)$ for all $x, y \in \mathbb{Z}$. The latter fact follows from Lemma 3.16 below.

Note that $x_1 \mapsto [\, x_1 \mid \cdots \mid x_n \,]$ is a function of the type

$$f(x) = x \oplus c \oplus (x \mid c_1) \oplus \cdots \oplus (x \mid c_n)$$

for some $c, c_1, \ldots, c_n \in \mathbb{Z}$. Functions of this type can also be written as

$$f(x) = (\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m)$$

for some $m \in \mathbb{N}$ and $c'_1, c'_2, \ldots c'_m \in \mathbb{Z}$, where m is odd. So the Animating candidate is an animating function. In order to prove this, the nim-sum and mating function first need to be defined for negative numbers, and $(x|x)$ needs to be defined for any $x \in \mathbb{Z}$. We will further discuss this in Section 4.1.

Triangle candidate We give one more candidate function. It is based on an interesting property that Welter proved holds for the Sprague-Grundy function for Welter's game.

Lemma 3.12. *For all $n \geq 3$ and distinct $x_1, \dots, x_n \in \mathbb{N}$, the Sprague-Grundy function for Welter's game satisfies*

$$[\, x_1 \mid \cdots \mid x_n \,] = [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,],$$

or, more generally,

$$\begin{aligned} &[\, x_1 \mid \cdots \mid x_n \,] \\ &= [\, [\, x_1 \mid \cdots \mid x_{i-1} \mid x_{i+1} \mid \cdots \mid x_n \,] \mid [\, x_1 \mid \cdots \mid x_{j-1} \mid x_{j+1} \mid \cdots \mid x_n \,] \,] \\ &\quad \oplus [\, x_1 \mid \cdots \mid x_{i-1} \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_{j+1} \mid \cdots \mid x_n \,]. \end{aligned}$$

The proof can be found in Lemma 2 of [10]. We will also give the proof in Section 3.5.

Definition 3.13 (Triangle candidate)**.** For all distinct $x_1, x_2 \in \mathbb{N}$, we have $[\, x_1 \,] := x_1$ and $[\, x_1 \mid x_2 \,] := x_1 \oplus x_2 - 1$. If $n \geq 3$ and $x_1, \dots, x_n \in \mathbb{N}$ are distinct, we have

$$[\, x_1 \mid \cdots \mid x_n \,] := [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,].$$

Note that it is not immediately obvious that this function is well-defined. For the function to be well-defined, we need to have

$$[\, x_1 \mid \cdots \mid x_{n-1} \,] \neq [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]$$

for all distinct $x_1, \dots, x_n \in \mathbb{N}$. By the Unique Prime Lemma, which we will prove in Section 4.1, this is true.

Example 3.14. Consider the position $(1, 5, 6)$. We have

$$\begin{aligned} [\, 1 \mid 5 \mid 6 \,] &= [\, [\, 1 \mid 5 \,] \mid [\, 1 \mid 6 \,] \,] \oplus [\, 1 \,] = [\, (1 \oplus 5 - 1) \mid (1 \oplus 6 - 1) \,] \oplus 1 \\ &= [\, 3 \mid 6 \,] \oplus 1 = (3 \oplus 6 - 1) \oplus 1 = 4 \oplus 1 = 5. \end{aligned}$$

We call this the Triangle candidate because with it, the value $[\, x_1 \mid \cdots \mid x_n \,]$ for any position $(x_1, \dots, x_n)$ can be found by completing a triangle of numbers, as is explained in Chapter 15 of [1].

Suppose we want to find $[\, x_1 \mid \cdots \mid x_n \,]$, for some distinct $x_1, \dots, x_n \in \mathbb{N}$. We start by writing down $n + 1$ copies of the number 0 in a row, leaving a space between each 0. We fill the row below with $x_1, \dots, x_n$, where each number is diagonally below two of the 0s. While filling the rest of the triangle, we ensure that whenever there is a diamond pattern

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

the numbers satisfy $(a \oplus d) = (b \oplus c) + 1$, so $c = ((a \oplus d) - 1) \oplus b$. Whenever three of the locations of a diamond are filled, a unique fourth number can be added so that the equation holds. If we do this until we can finish no more diamonds, the value $[\, x_1 \mid \cdots \mid x_n \,]$ is at the bottom of the triangle. We will further discuss this method in Section 6.1.

Example 3.15. Suppose we want to find [1 | 3 | 7 | 8 | 11 | 19]. Using the method described above, we get

$$
\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & 1 & & 3 & & 7 & & 8 & & 11 & & 19 & \\
 & & 1 & & 3 & & 14 & & 2 & & 23 & & \\
 & & & 2 & & 11 & & 3 & & 31 & & & \\
 & & & & 11 & & 9 & & 25 & & & & \\
 & & & & & 10 & & 12 & & & & & \\
 & & & & & & 12 & & & & & &
\end{array}
$$

So [1 | 3 | 7 | 8 | 11 | 19] = 12.

In Section 3.5, we will discuss the fact that the four candidates above are the same function. In Chapter 4, we prove that this function is the Sprague-Grundy function for Welter's game. This was first proved by Welter in [10].

In many cases, it is possible to directly prove that one of the four candidates satisfies the properties of another. If candidate A satisfies the properties of candidate B, we say that A implies B. In order to get a better grip on the four candidates, we discuss many of these direct implications in Section 3.5. Some of these have been proved before. For example, in [10], Welter proved using his function that Lemma 3.12 holds. Welter's candidate also satisfies $[\ x_1\ ] = x_1$ and $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$. So Welter's candidate satisfies the properties of the Triangle candidate, meaning the two functions are equal. Welter also proved that Welter's candidate implies the Mating candidate. In [2], Conway used the Animating candidate, and discussed the fact that it implies the Mating candidate, the Triangle candidate and Welter's candidate.

3.3 Some useful lemmas

In this section, we discuss some lemmas that will be useful in the following sections. First, we look at a few properties of the mating function. These were used by Conway in [2] for his proof that the Animating candidate is the Sprague-Grundy function for Welter's game, although he did not mention the sixth property explicitly.

Lemma 3.16. *The mating function has the following properties:*

1. *For all $a, b, x \in \mathbb{N}$ with $a \neq b$, we have $(a \oplus x \mid b \oplus x) = (a \mid b)$.*

2. *For all $a, b, x \in \mathbb{N}$ with $a \neq b$, we have $(a + x \mid b + x) = (a \mid b)$.*

3. *For all $n \in \mathbb{N} \setminus \{0\}$, we have $n \oplus (n \mid 0) = n - 1$.*

4. *For all $a, b \in \mathbb{N}$ with $a \neq b$, we have $a \oplus b \oplus (a \mid b) = a \oplus b - 1$.*

5. *For all distinct $a, b, x \in \mathbb{N}$ such that out of $\{a, b, x\}$, the numbers a and b are congruent modulo the highest power of 2, we have $(a \mid x) = (b \mid x) < (a \mid b)$.*

6. *For all distinct $a, b, c \in \mathbb{N}$, we have $(a \oplus (a \mid c) \mid b \oplus (b \mid c)) = (a \mid b)$.*

Proof. 1. Suppose that a and b are congruent modulo 2^n, but not modulo 2^{n+1}. This means that the binary expansions of a and b have their last n digits in common, but not the last $n+1$ digits. Nim-adding x to a and b changes their binary expansions in the same positions. So the binary expansions of $a \oplus x$ and $b \oplus x$ also have their last n digits in common, but not the last $n+1$ digits. It follows that $(a \oplus x \mid b \oplus x) = (a \mid b)$.

2. Suppose that a and b are congruent modulo 2^n, but not modulo 2^{n+1}. Assume without loss of generality that $a < b$. Then $b - a$ is a multiple of 2^n, but not a multiple of 2^{n+1}. Since $(b + x) - (a + x) = b - a$, the numbers $a + x$ and $b + x$ are also congruent modulo 2^n but not modulo 2^{n+1}.

3. Let ℓ be the amount of 0s at the end of the binary expansion of n, which may be zero. So the last $\ell + 1$ digits of the binary expansion of n are $10\ldots0$. The last $\ell+1$ digits of the binary expansion of $n-1$ are $01\ldots1$, and the binary expansions of $n - 1$ and n are equal on the positions before that.

n has its last ℓ digits in common with 0, but not its last $\ell+1$ digits. This means that $(n|0) = 2^{\ell+1} - 1 = 1 + 2 + \cdots + 2^{\ell}$. This has binary expansion $11\ldots1$, with $\ell + 1$ digits. So the last $\ell + 1$ digits of the binary expansion of $n \oplus (n \mid 0)$ are $01\ldots1$, and the binary expansions of n and $n \oplus (n \mid 0)$ are equal on the positions before that.

This means that $n - 1$ and $n \oplus (n \mid 0)$ have the same binary expansion, so they are equal.

4. Using Lemmas 3.16.1 and 3.16.3 we get

$$a \oplus b \oplus (a \mid b) = a \oplus b \oplus (a \oplus b \mid b \oplus b) = a \oplus b \oplus (a \oplus b|0) = a \oplus b - 1.$$

5. Suppose that a and b are congruent modulo 2^n but not modulo 2^{n+1}. By our assumption, a and x are not congruent modulo 2^{n+1}. Assume that a and x are congruent modulo 2^m for some $m \leq n$. Then b is congruent to a modulo 2^m, which means that b and x are also congruent modulo 2^m. Similarly, if b and x are congruent modulo 2^m, then so are a and x. We conclude that $(a \mid x) = (b \mid x)$.

Now suppose that $(a \mid x) = (b \mid x) = (a \mid b)$. Then the binary expansions of a, b and x are all equal on the last n digits, but no two are the same on the $(n+1)$th last digit. This is not possible, so we conclude that $(a \mid x) = (b \mid x) < (a \mid b)$.

6. Assume that $(a \mid c) = 2^{n_1+1} - 1$ and $(b \mid c) = 2^{n_2+1} - 1$. If $n_1 = n_2$, then $(a \mid c) = (b \mid c)$, and the result follows from Lemma 3.16.1. Otherwise, assume without loss of generality that $n_1 < n_2$. Then because b is congruent to c modulo 2^{n_2}, and $n_2 \geq n_1 + 1$, b is also congruent to c modulo 2^{n_1+1} and modulo 2^{n_1}. We know that a is congruent to c modulo 2^{n_1}, but not modulo 2^{n_1+1}. It follows that a must also be congruent to b modulo 2^{n_1}, but not modulo 2^{n_1+1}.

The binary expansion of $a \oplus (a \mid c)$ and the binary expansion of a differ on each of the last n_1+1 digits. The binary expansion of $b \oplus (b \mid c)$ and the one of b differ on each of the last $n_2 + 1 > n_1 + 1$ digits. This means that on the last $n_1 + 1$ positions, a and b have the same digit if and only if $a \oplus (a \mid c)$ and $b \oplus (b \mid c)$ have the same digit. We conclude that $a \oplus (a \mid c)$ and $b \oplus (b \mid c)$ are congruent modulo 2^{n_1}, but not modulo 2^{n_1+1}, so $(a \oplus (a \mid c) \mid b \oplus (b \mid c)) = (a \mid b)$.

□

The following lemma will be used multiple times in the following sections. It was used by Welter in [10] for his proof that Welter's candidate is the Sprague-Grundy function for Welter's game.

Lemma 3.17. *If* $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ *and out of* $\{x_1, x_2, x_3\}$*, the numbers* x_1 *and* x_2 *are congruent modulo the highest power of 2, then*

$$[\ [\ x_1 \mid x_3\ ] \mid [\ x_2 \mid x_3\ ]\ ] = [\ x_1 \mid x_2\ ].$$

Proof. Let n be such that $(x_1 \mid x_2) = 2^{n+1} - 1$. By Lemma 3.13.5, there is some $m < n$ such that $(x_1 \mid x_3) = (x_2 \mid x_3) = 2^{m+1} - 1$.

The last $m+1$ digits of the binary expansion of $x_1 \oplus x_3$ are $10\dots0$. So the last $m+1$ digits of the binary expansion of $[\ x_1 \mid x_3\ ]$ are $01\dots1$, and the binary expansions of $x_1 \oplus x_3$ and $[\ x_1 \mid x_3\ ]$ are equal on the positions before that. Similarly, the last $m+1$ digits of the binary expansion of $[\ x_2 \mid x_3\ ]$ are $01\dots1$, and the binary expansions of $x_2 \oplus x_3$ and $[\ x_2 \mid x_3\ ]$ are equal on the positions before that.

So the last $m+1$ digits of the binary expansion of $[\ x_1 \mid x_3\ ] \oplus [\ x_2 \mid x_3\ ]$ are $00\dots0$, and the binary expansions of $[\ x_1 \mid x_3\ ] \oplus [\ x_2 \mid x_3\ ]$ and of $x_1 \oplus x_3 \oplus x_2 \oplus x_3 = x_1 \oplus x_2$ are equal on the positions before that. The last $n+1$ digits of the binary expansion of $x_1 \oplus x_2$ are $10\dots0$, so the last $m+1$ digits are $00\dots0$. It follows that $[\ x_1 \mid x_3\ ] \oplus [\ x_2 \mid x_3\ ] = x_1 \oplus x_2$. This implies that $[\ [\ x_1 \mid x_3\ ] \mid [\ x_2 \mid x_3\ ]\ ] = [\ x_1 \mid x_2\ ]$.

□

3.4 Symmetry

In this section, we discuss the fact that the Sprague-Grundy function for Welter's game is symmetric in its arguments $x_1, \dots, x_n$, for any n. That is, we have $[\ x_1 \mid \cdots \mid x_n\ ] = [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n)}\ ]$ for any distinct $x_1, \dots, x_n \in \mathbb{N}$ and any permutation π on $\{1, \dots, n\}$. The Sprague-Grundy function must have this property because the order of the coins does not matter in Welter's game. Therefore, we need to show that the candidate functions discussed in Section 3.2 also satisfy this property. Some of the proofs in Section 3.5, where we prove that the four functions are the same, rely on the Symmetry property.

Definition 3.18 (Symmetry)**.** Let $(x_1, \dots, x_n) \mapsto [\ x_1 \mid \cdots \mid x_n\ ]$ be a function defined on all positions of Welter's game. The function is symmetric if and only if

$$[\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n)}\ ] = [\ x_1 \mid \cdots \mid x_n\ ]$$

for all distinct $x_1, \dots, x_n$ and any permutation π on $\{1, \dots, n\}$.

The Animating candidate is clearly symmetric. So if we start from this candidate, we can directly prove that Symmetry holds without first showing that the candidate is equivalent to one of the others. The Mating candidate is also symmetric by definition, because the result only depends on which numbers are congruent modulo the highest power of 2, and on the values $[\ x_1 \mid x_2\ ]$ for all distinct $x_1, x_2 \in \mathbb{N}$. These satisfy

$$[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1 = x_2 \oplus x_1 - 1 = [\ x_2 \mid x_1\ ].$$

In Lemma 3 of [10], Welter proved that any function satisfying the properties of both Welter's candidate and the Triangle candidate is symmetric. In Section 3.5, we will show that these two candidate functions are equivalent. Thus, using either Welter's candidate or the Triangle candidate, Symmetry also follows.

Figure 1 provides an overview of the results in this section. The proof that uses two candidates is denoted using a + next to the corresponding implication arrows in the figure.

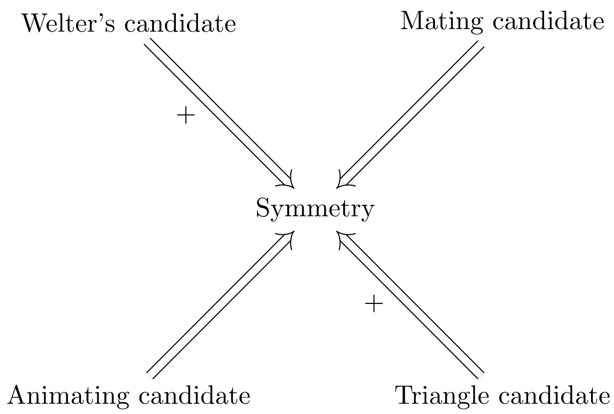

Figure 1: Overview of direct implications. A + signifies that two candidates are used for the proof.

In order to prove that Welter's candidate and the Triangle candidate together imply Symmetry, we first need the following lemma. This is Lemma 2.2 in [9].

Lemma 3.19. *Let* $x, y \in \mathbb{N} \setminus \{0\}$ *be such that the last 1 in the binary expansion of* y *is later than the last 1 in the binary expansion of* x*. That is, such that there exist* $m, n \in \mathbb{N}$ *with* $m < n$ *such that the last* n *digits of the binary expansion of* x *are* $10\ldots0$ *and the last* m *digits of the binary expansion of* y *are* $10\ldots0$*. Then* $(x-1) \oplus y + 1 = (x-1) \oplus (y-1)$*.*

Proof. Let m be such that the last m digits of the binary expansion of y are $10\ldots0$. Then the last m digits of the binary expansion of $y-1$ are $01\ldots1$, and the binary expansions of y and $y-1$ are equal on all other positions. So the binary expansion of $y \oplus (y-1)$ equals $11\ldots1$, with m 1s.

Let n be such that the last n digits of the binary expansion of x are $10\ldots0$. Then the last n digits of the binary expansion of $x-1$ are $01\ldots1$. We have $n > m$, so it follows that the last m digits of the binary expansion of $(x-1) \oplus y$ are $01\ldots1$. Then the last m digits of the binary expansion of $(x-1) \oplus y + 1$ are $10\ldots0$, and the binary expansions of $(x-1) \oplus y$ and $(x-1) \oplus y + 1$ are equal on all other positions. It follows that the binary expansion of $(x-1) \oplus y \oplus ((x-1) \oplus y + 1)$ equals $11\ldots1$, with m 1s.

So we have

$$y \oplus (y-1) = (x-1) \oplus y \oplus ((x-1) \oplus y + 1).$$

By nim-adding $(x-1) \oplus y$ to both sides, we get $(x-1) \oplus y + 1 = (x-1) \oplus (y-1)$.

□

Next, we prove that Welter's candidate and the Triangle candidate together imply Symmetry. We use a more detailed version of Welter's proof.

Lemma 3.20. *If the function* $(x_1, \ldots, x_n) \mapsto [\ x_1 \mid \cdots \mid x_n\ ]$ *for all distinct* $x_1, \ldots, x_n \in \mathbb{N}$ *satisfies the properties of both Welter's candidate and the Triangle candidate, it is symmetric.*

Proof. We will prove this by induction on n.

If $x_1, x_2 \in \mathbb{N}$ are distinct, we get

$$[\ x_2 \mid x_1\ ] = x_2 \oplus x_1 - 1 = x_1 \oplus x_2 - 1 = [\ x_1 \mid x_2\ ].$$

Now let $x_1, x_2, x_3 \in \mathbb{N}$ be distinct. To prove that

$$[\ x_{\pi(1)} \mid x_{\pi(2)} \mid x_{\pi(3)}\ ] = [\ x_1 \mid x_2 \mid x_3\ ]$$

for any permutation π on $\{1, 2, 3\}$, we need to show that any two numbers can be interchanged. We will show that $[\ x_1 \mid x_3 \mid x_2\ ] = [\ x_1 \mid x_2 \mid x_3\ ]$ and $[\ x_3 \mid x_2 \mid x_1\ ] = [\ x_1 \mid x_2 \mid x_3\ ]$ for any distinct $x_1, x_2, x_3 \in \mathbb{N}$. Then it follows that

$$[\ x_2 \mid x_1 \mid x_3\ ] = [\ x_3 \mid x_1 \mid x_2\ ] = [\ x_3 \mid x_2 \mid x_1\ ] = [\ x_1 \mid x_2 \mid x_3\ ].$$

Because Symmetry holds for $n = 2$, we have

$$\begin{aligned}[\ x_1 \mid x_3 \mid x_2\ ] &= [\ [\ x_1 \mid x_3\ ] \mid [\ x_1 \mid x_2\ ]\ ] \oplus [\ x_1\ ] \\ &= [\ [\ x_1 \mid x_2\ ] \mid [\ x_1 \mid x_3\ ]\ ] \oplus [\ x_1\ ] = [\ x_1 \mid x_2 \mid x_3\ ].\end{aligned}$$

To show that $[\ x_3 \mid x_2 \mid x_1\ ] = [\ x_1 \mid x_2 \mid x_3\ ]$, we use Lemma 3.19. We can apply this to our problem as follows. Suppose that x_i, x_j are congruent modulo a strictly higher power of 2 than x_i', x_j' for some $x_i, x_j, x_i', x_j' \in \mathbb{N}$ with $x_i \neq x_j$ and $x_i' \neq x_j'$. Then,

$$(x_i \oplus x_j - 1) \oplus x_i' \oplus x_j' + 1 = (x_i \oplus x_j - 1) \oplus (x_i' \oplus x_j' - 1). \tag{1}$$

Assume that out of $\{x_1, x_2, x_3\}$, the numbers x_1 and x_2 are congruent modulo the highest power of 2. Then by Lemma 3.16.5, x_1 and x_2 are congruent modulo a strictly higher power of 2 than x_1 and x_3.

Using Lemma 3.17 and the fact that Symmetry holds for $n = 2$, we get

$$\begin{aligned}&[\ x_3 \mid x_2 \mid x_1\ ] = [\ [\ x_3 \mid x_2\ ] \mid [\ x_3 \mid x_1\ ]\ ] \oplus [\ x_3\ ] \\ &= [\ [\ x_1 \mid x_3\ ] \mid [\ x_2 \mid x_3\ ]\ ] \oplus [\ x_3\ ] = [\ x_1 \mid x_2\ ] \oplus x_3.\end{aligned}$$

We rewrite this using equation (1).

$$\begin{aligned}[\ x_1 \mid x_2\ ] \oplus x_3 &= ((x_1 \oplus x_2 - 1) \oplus x_1 \oplus x_3) \oplus x_1 \\ &= ((x_1 \oplus x_2 - 1) \oplus (x_1 \oplus x_3 - 1) - 1) \oplus x_1 \\ &= [\ [\ x_1 \mid x_2\ ] \mid [\ x_1 \mid x_3\ ]\ ] \oplus [\ x_1\ ] \\ &= [\ x_1 \mid x_2 \mid x_3\ ].\end{aligned}$$

So in this case $[\ x_3 \mid x_2 \mid x_1\ ] = [\ x_1 \mid x_2 \mid x_3\ ]$. Similarly, if x_2 and x_3 are congruent modulo the highest power of 2, we can start from $[\ x_1 \mid x_2 \mid x_3\ ]$ and show that it is equal to $[\ x_3 \mid x_2 \mid x_1\ ]$.

Now assume that x_1 and x_3 are congruent modulo the highest power of 2 out of $\{x_1, x_2, x_3\}$. Using Lemma 3.16.5, we find that x_1 and x_3 are congruent modulo a strictly higher power of 2 than x_1 and x_2. Then we get, using (1),

$$\begin{aligned}[\ x_1 \mid x_2 \mid x_3\ ] &= [\ [\ x_1 \mid x_2\ ] \mid [\ x_1 \mid x_3\ ]\ ] \oplus [\ x_1\ ] \\ &= ((x_1 \oplus x_2 - 1) \oplus (x_1 \oplus x_3 - 1) - 1) \oplus x_1 \\ &= ((x_1 \oplus x_2) \oplus (x_1 \oplus x_3 - 1)) \oplus x_1 \\ &= x_2 \oplus (x_1 \oplus x_3 - 1).\end{aligned}$$

Similarly, we have

$$[\ x_3 \mid x_2 \mid x_1\ ] = x_2 \oplus (x_3 \oplus x_1 - 1).$$

So $[\ x_1 \mid x_2 \mid x_3\ ]$ and $[\ x_3 \mid x_2 \mid x_1\ ]$ are equal.

Now let $n > 3$ and assume that the symmetry property holds for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct and let π be a permutation on $\{1, \ldots, n-2\}$. Then, using the induction hypothesis, we find that

$$\begin{aligned}&[\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_{n-1} \mid x_n\ ] \\ &= [\ [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_{n-1}\ ] \mid [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_n\ ]\ ] \\ &\quad \oplus [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)}\ ] \\ &= [\ [\ x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1}\ ] \mid [\ x_1 \mid \cdots \mid x_{n-2} \mid x_n\ ]\ ] \oplus [\ x_1 \mid \cdots \mid x_{n-2}\ ] \\ &= [\ x_1 \mid \cdots \mid x_n\ ],\end{aligned}$$

and

$$\begin{aligned}&[\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_n \mid x_{n-1}\ ] \\ &= [\ [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_n\ ] \mid [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)} \mid x_{n-1}\ ]\ ] \\ &\quad \oplus [\ x_{\pi(1)} \mid \cdots \mid x_{\pi(n-2)}\ ] \\ &= [\ [\ x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1}\ ] \mid [\ x_1 \mid \cdots \mid x_{n-2} \mid x_n\ ]\ ] \oplus [\ x_1 \mid \cdots \mid x_{n-2}\ ] \\ &= [\ x_1 \mid \cdots \mid x_n\ ].\end{aligned}$$

We assumed that $n \geq 4$. So by the above, it follows that we can interchange the first two positions. Using this fact, we get

$$\begin{aligned}&[\ x_1 \mid x_2 \mid \cdots \mid x_n\ ] \\ &= [\ x_2 \mid x_1 \mid x_3 \mid \cdots \mid x_n\ ] \\ &= [\ 0 \mid x_1 \oplus x_2 \mid x_3 \oplus x_2 \mid \cdots \mid x_n \oplus x_2\ ] \oplus (x_2)_n \\ &= [\ x_1 \oplus x_2 - 1 \mid x_3 \oplus x_2 - 1 \mid \cdots \mid x_n \oplus x_2 - 1\ ] \oplus (x_2)_n\end{aligned}$$

By the induction hypothesis, this equals

$$\begin{aligned}&[\ x_n \oplus x_2 - 1 \mid x_3 \oplus x_2 - 1 \mid \cdots \mid x_{n-1} \oplus x_2 - 1 \mid x_1 \oplus x_2 - 1\ ] \oplus (x_2)_n \\ &= [\ 0 \mid x_n \oplus x_2 \mid x_3 \oplus x_2 \mid \cdots \mid x_{n-1} \oplus x_2 \mid x_1 \oplus x_2\ ] \oplus (x_2)_n \\ &= [\ x_2 \mid x_n \mid x_3 \mid \cdots \mid x_{n-1} \mid x_1\ ] \\ &= [\ x_n \mid x_2 \mid x_3 \mid \cdots \mid x_{n-1} \mid x_1\ ],\end{aligned}$$

Now we only need to show that we can interchange any $i \in \{x_1, \ldots, x_{n-2}\}$ with any $j \in \{x_{n-1}, x_n\}$. We can do this by using a permutation on $\{x_1, \ldots, x_{n-2}\}$ which places i in the first position, permuting $\{x_{n-1}, x_n\}$ so that j is in the last position, interchanging the first and last positions and then applying the inverses of the permutations.

So any pair of numbers from $\{x_1, \ldots, x_n\}$ can be interchanged, which implies that the Sprague-Grundy function for Welter's game is symmetric.

□

3.5 Equivalence of the candidates

In this section, we will prove that the four candidates for the Sprague-Grundy function given in Section 3.2 are equivalent.

Theorem 3.21. *The following are equivalent:*

- *Welter's candidate: Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. We have*

1. $[\, 0 \,] := 0$.

2. $[\, 0 \mid x_1 \mid \cdots \mid x_n \,] := [\, x_1 - 1 \mid \cdots \mid x_n - 1 \,]$ *when* $x_1, \ldots, x_n \neq 0$.

3. $[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] := [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_n$ *for all* $x \in \mathbb{N}$.

- *Mating candidate: For all distinct $x_1, x_2 \in \mathbb{N}$, we have $[\; x_1 \;] := x_1$ and $[\; x_1 \mid x_2 \;] := x_1 \oplus x_2 - 1$. Let $n \geq 3$ and let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Let $i_1, i_2 \in \{1, \ldots, n\}$ be such that out of the n numbers, x_{i_1} and x_{i_2} are congruent modulo the highest power of 2. Then let $i_3, i_3 \in \{1, \ldots, n\}$ be such that out of $\{x_1, \ldots, x_n\} \setminus \{x_{i_1}, x_{i_2}\}$, x_{i_3} and x_{i_4} are congruent modulo the highest power of 2, et cetera. Then if n is even,*

$$[\, x_1 \mid \cdots \mid x_n \,] := [\, x_{i_1} \mid x_{i_2} \,] \oplus [\, x_{i_3} \mid x_{i_4} \,] \oplus \cdots \oplus [\, x_{i_{n-1}} \mid x_{i_n} \,],$$

and if n is odd,

$$[\, x_1 \mid \cdots \mid x_n \,] := [\, x_{i_1} \mid x_{i_2} \,] \oplus [\, x_{i_3} \mid x_{i_4} \,] \oplus \cdots \oplus [\, x_{i_{n-2}} \mid x_{i_{n-1}} \,] \oplus [\, x_{i_n} \,].$$

- *Animating candidate: For all distinct $x_1, \ldots, x_n \in \mathbb{N}$, we have*

$$[\, x_1 \mid \cdots \mid x_n \,] := x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n).$$

- *Triangle candidate: For all distinct $x_1, x_2 \in \mathbb{N}$, we have $[\; x_1 \;] := x_1$ and $[\, x_1 \mid x_2 \,] := x_1 \oplus x_2 - 1$. If $n \geq 3$ and $x_1, \ldots, x_n \in \mathbb{N}$ are distinct, we have*

$$[\, x_1 \mid \cdots \mid x_n \,] := [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,].$$

In order to prove this, it is enough to show, for example, that the Animating candidate satisfies the properties of each of the three other candidates. If we can show that the Animating candidate satisfies the properties of Welter's candidate, then we can conclude that this function is equal to Welter's candidate. However, in this thesis we prove in multiple ways that the equivalence holds, in order to better understand the relation between these candidate functions. In many cases, it is possible to directly prove that one candidate satisfies the properties of another, without first showing that the candidate is equivalent to one of the remaining two. If candidate A satisfies the properties of candidate B, we say that A implies B. Below, we discuss many of these direct implications. For some of the proofs, we need to assume that Symmetry holds.

Figure 2 provides an overview of the results in this section. If there is an S next to an implication arrow, it signifies that Symmetry is used for the proof.

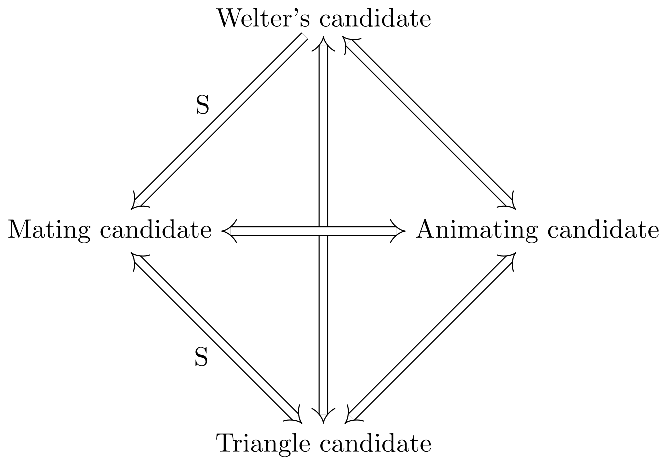

Figure 2: Overview of direct implications. An S signifies that Symmetry is used for the proof.

We first prove that the four candidates are equivalent when $n \leq 2$.

Lemma 3.22. *Each of the four candidate functions satisfies the properties of the other three candidates when $n \leq 2$.*

Proof. First we start from Welter's candidate. Assume that $[\ 0\ ] = 0$ and that for all distinct $x_1, x_2 \in \mathbb{N}$ and all $x \in \mathbb{N}$, we have $[\ 0 \mid x_1\ ] = [\ x_1 - 1\ ]$, $[\ x_1 \oplus x\ ] = [\ x_1\ ] \oplus x$ and $[\ x_1 \oplus x \mid x_2 \oplus x\ ] = [\ x_1 \mid x_2\ ]$. Then

$$[\ x_1\ ] = [\ 0 \oplus x_1\ ] = [\ 0\ ] \oplus x_1 = 0 \oplus x_1 = x_1$$

for all $x_1 \in \mathbb{N}$, and

$$[\ x_1 \mid x_2\ ] = [\ 0 \mid x_2 \oplus x_1\ ] = [\ x_2 \oplus x_1 - 1\ ] = x_2 \oplus x_1 - 1 = x_1 \oplus x_2 - 1$$

for all distinct $x_1, x_2 \in \mathbb{N}$. This also implies that

$$[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 \oplus (x_1 \mid x_2)$$

by Lemma 3.16.4.

Now, we will show that the other two candidates follow from either the Mating candidate or the Triangle candidate, which are equivalent for $n \leq 2$. Assume that for all distinct $x_1, x_2 \in \mathbb{N}$, we have $[\ x_1\ ] = x_1$ and $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$.

Then $[\ 0\ ] = 0$ and $[\ 0 \mid x_1\ ] = x_1 - 1 = [\ x_1 - 1\ ]$ for all $x_1 \in \mathbb{N}$. Also, for any $x \in \mathbb{N}$ and distinct $x_1, x_2 \in \mathbb{N}$ we have

$$[\ x_1 \oplus x\ ] = x_1 \oplus x = [\ x_1\ ] \oplus x,$$

and

$$[\ x_1 \oplus x \mid x_2 \oplus x\ ] = (x_1 \oplus x) \oplus (x_2 \oplus x) - 1 = x_1 \oplus x_2 - 1 = [\ x_1 \mid x_2\ ].$$

Further, by Lemma 3.16.4 we have $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 \oplus (x_1 \mid x_2)$.

Finally, we show at the Animating candidate implies the other three when $n \leq 2$. Assume that $[\ x_1\ ] = x_1$ and $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 \oplus (x_1 \mid x_2)$ for all distinct $x_1, x_2 \in \mathbb{N}$. Then $[\ 0\ ] = 0$, and by Lemma 3.16.3 we have

$$[\ 0 \mid x_1\ ] = x_1 \oplus (0 \mid x_1) = x_1 - 1$$

for any $x_1 \in \mathbb{N}$. Further, for any $x \in \mathbb{N}$ and distinct $x_1, x_2 \in \mathbb{N}$ we get

$$[\ x_1 \oplus x\ ] = x_1 \oplus x = [\ x_1\ ] \oplus x,$$

and, using Lemma 3.16.1,

$$[\ x_1 \oplus x \mid x_2 \oplus x\ ] = (x_1 \oplus x) \oplus (x_2 \oplus x) \oplus (x_1 \oplus x \mid x_2 \oplus x) = x_1 \oplus x_2 \oplus (x_1 \mid x_2).$$

Finally, $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ by Lemma 3.16.4.

□

We start by proving that Welter's candidate implies the Mating candidate, assuming that Welter's candidate satisfies the Symmetry property. This is Lemma 5 of [10].

Lemma 3.23. *Welter's candidate implies the Mating candidate, assuming that Symmetry holds for Welter's candidate.*

Proof. We prove this using induction on n. By Lemma 3.22, the statement holds for $n \leq 2$. Let $n \geq 3$ and assume that for all distinct $x_1, \ldots, x_{n-1}$ such that x_1 and x_2 are congruent modulo the highest power of 2, we have $[\ x_1 \mid \cdots \mid x_{n-1}\ ] = [\ x_1 \mid x_2\ ] \oplus [\ x_3 \mid \cdots \mid x_{n-1}\ ]$.

Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Assume without loss of generality that x_1 and x_2 are congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n\}$. Because of Symmetry, we have

$$\begin{aligned}[\ x_1 \mid \cdots \mid x_n\ ] &= [\ x_1 \oplus x_n \mid \cdots \mid x_{n-1} \oplus x_n \mid 0\ ] \oplus (x_n)_n \\ &= [\ x_1 \oplus x_n - 1 \mid \cdots \mid x_{n-1} \oplus x_n - 1\ ] \oplus (x_n)_n.\end{aligned}$$

By Lemmas 3.16.1 and 3.16.2, $x_1 \oplus x_n - 1$ and $x_2 \oplus x_n - 1$ are congruent modulo the highest power of 2 out of $\{x_1 \oplus x_n - 1, \ldots, x_{n-1} \oplus x_n - 1\}$. So by the induction hypothesis, the above equals

$$[\ x_1 \oplus x_n - 1 \mid x_2 \oplus x_n - 1\ ] \oplus [\ x_3 \oplus x_n - 1 \mid \cdots \mid x_{n-1} \oplus x_n - 1\ ] \oplus (x_n)_n.$$

By Lemma 3.22, we have $[\, x_1 \mid x_2 \,] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$. So we can use Lemma 3.17. Then the above is equal to

$$\begin{aligned}
&[\, [\, x_1 \mid x_n \,] \mid [\, x_2 \mid x_n \,] \,] \oplus [\, x_3 \oplus x_n - 1 \mid \cdots \mid x_{n-1} \oplus x_n - 1 \,] \oplus (x_n)_{n-2} \\
&= [\, x_1 \mid x_2 \,] \oplus [\, x_3 \oplus x_n - 1 \mid \cdots \mid x_{n-1} \oplus x_n - 1 \,] \oplus (x_n)_{n-2} \\
&= [\, x_1 \mid x_2 \,] \oplus [\, x_3 \oplus x_n \mid \cdots \mid x_{n-1} \oplus x_n \mid 0 \,] \oplus (x_n)_{n-2} \\
&= [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,].
\end{aligned}$$

□

Next, we show that Welter's candidate and the Animating candidate imply each other.

Lemma 3.24. *Welter's candidate implies the Animating candidate.*

Proof. We prove this using induction. For $n \leq 2$, the statement holds by Lemma 3.22. Now let $n \in \mathbb{N}$ and assume that

$$[\, x_1 \mid \cdots \mid x_{n-1} \,] = x_1 \oplus \cdots \oplus x_{n-1} \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})$$

for all distinct $x_1, \ldots, x_{n-1} \in \mathbb{N}$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. We have

$$\begin{aligned}
[\, x_1 \mid \cdots \mid x_n \,] &= [\, 0 \mid x_1 \oplus x_2 \cdots \mid x_1 \oplus x_n \,] \oplus (x_1)_n \\
&= [\, x_1 \oplus x_2 - 1 \mid \cdots \mid x_1 \oplus x_n - 1 \,] \oplus (x_1)_n.
\end{aligned}$$

By the induction hypothesis, this is equal to

$$\begin{aligned}
&(x_1 \oplus x_2 - 1) \oplus \cdots \oplus (x_1 \oplus x_n - 1) \\
&\quad \oplus (x_1 \oplus x_2 - 1 \mid x_1 \oplus x_3 - 1) \oplus \cdots \oplus (x_1 \oplus x_{n-1} - 1 \mid x_1 \oplus x_n - 1) \oplus (x_1)_n.
\end{aligned}$$

By Lemmas 3.16.2 and 3.16.1 this can be simplified to

$$(x_1 \oplus x_2 - 1) \oplus \cdots \oplus (x_1 \oplus x_n - 1) \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_{n-1} \mid x_n) \oplus (x_1)_n.$$

Using Lemma 3.16.4, we get

$$\begin{aligned}
&(x_1 \oplus x_2 \oplus (x_1 \mid x_2)) \oplus \cdots \oplus (x_1 \oplus x_n \oplus (x_1 \mid x_n)) \\
&\quad \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_{n-1} \mid x_n) \oplus (x_1)_n \\
&= (x_1)_{2n-1} \oplus x_2 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n) \\
&= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n).
\end{aligned}$$

□

Lemma 3.25. *The Animating candidate implies Welter's candidate.*

Proof. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. By Lemmas 3.16.3 and 3.16.2 we have

$$\begin{aligned}
&[\, 0 \mid x_1 \mid \cdots \mid x_n \,] \\
&= x_1 \oplus (0 \mid x_1) \oplus \cdots \oplus x_n \oplus (0 \mid x_n) \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n) \\
&= (x_1 - 1) \oplus \cdots \oplus (x_n - 1) \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n) \\
&= (x_1 - 1) \oplus \cdots \oplus (x_n - 1) \\
&\quad \oplus (x_1 - 1 \mid x_2 - 1) \oplus \cdots \oplus (x_{n-1} - 1 \mid x_n - 1) \\
&= [\, x_1 - 1 \mid \cdots \mid x_n - 1 \,].
\end{aligned}$$

By Lemma 3.16.1 we have

$$\begin{aligned}
&[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] \\
&= x_1 \oplus x \oplus \cdots \oplus x_n \oplus x \oplus (x_1 \oplus x \mid x_2 \oplus x) \oplus \cdots \oplus (x_{n-1} \oplus x \mid x_n \oplus x) \\
&= x_1 \oplus x \oplus \cdots \oplus x_n \oplus x \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n) \\
&= [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_n.
\end{aligned}$$

□

Next, we show that Welter's candidate implies the Triangle candidate. This is Lemma 2 of [10].

Lemma 3.26. *Welter's candidate implies the Triangle candidate.*

Proof. We prove this using induction on n. By Lemma 3.22, the statement holds for $n \leq 2$, so we have $[\, x_1 \,] = x_1$ and $[\, x_1 \mid x_2 \,] = x_1 \oplus x_2 - 1$ for all $x_1, x_2 \in \mathbb{N}$. Now let $x_1, x_2, x_3 \in \mathbb{N}$ be distinct. Then

$$\begin{aligned}
&[\, [\, x_1 \mid x_2 \,] \mid [\, x_1 \mid x_3 \,] \,] \oplus [\, x_1 \,] = [\, x_1 \oplus x_2 - 1 \mid x_1 \oplus x_3 - 1 \,] \oplus x_1 \\
&= [\, 0 \mid x_1 \oplus x_2 \mid x_1 \oplus x_3 \,] \oplus x_1 = [\, x_1 \mid x_2 \mid x_3 \,].
\end{aligned}$$

Now let $n > 3$ and assume that the statement holds for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Then we find that

$$\begin{aligned}
&[\, x_1 \mid \cdots \mid x_n \,] = [\, 0 \mid x_1 \oplus x_2 \mid x_1 \oplus x_3 \mid \cdots \mid x_1 \oplus x_n \,] \oplus (x_1)_n \\
&= [\, x_1 \oplus x_2 - 1 \mid x_1 \oplus x_3 - 1 \mid \cdots \mid x_1 \oplus x_n - 1 \,] \oplus (x_1)_n.
\end{aligned}$$

By the induction hypothesis, this equals

$$\begin{aligned}
&[\, [\, x_1 \oplus x_2 - 1 \mid \cdots \mid x_1 \oplus x_{n-1} - 1 \,] \\
&\quad \mid [\, x_1 \oplus x_2 - 1 \mid \cdots \mid x_1 \oplus x_{n-2} - 1 \mid x_1 \oplus x_n - 1 \,] \,] \\
&\quad \oplus [\, x_1 \oplus x_2 - 1 \mid \cdots \mid x_1 \oplus x_{n-2} - 1 \,] \oplus (x_1)_n \\
&= [\, [\, 0 \mid x_1 \oplus x_2 \mid \cdots \mid x_1 \oplus x_{n-1} \,] \mid [\, 0 \mid x_1 \oplus x_2 \mid \cdots \mid x_1 \oplus x_{n-2} \mid x_1 \oplus x_n \,] \,] \\
&\quad \oplus [\, 0 \mid x_1 \oplus x_2 \mid \cdots \mid x_1 \oplus x_{n-2} \,] \oplus (x_1)_{n-2} \\
&= [\, [\, x_1 \mid x_2 \mid \cdots \mid x_{n-1} \,] \oplus (x_1)_{n-1} \mid [\, x_1 \mid x_2 \mid \cdots \mid x_{n-2} \mid x_n \,] \oplus (x_1)_{n-1} \,] \\
&\quad \oplus [\, x_1 \mid x_2 \mid \cdots \mid x_{n-2} \,] \\
&= [\, [\, x_1 \mid x_2 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid x_2 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid x_2 \mid \cdots \mid x_{n-2} \,].
\end{aligned}$$

□

Now, we show that the Triangle candidate also implies Welter's candidate.

Lemma 3.27. *The Triangle candidate implies Welter's candidate.*

Proof. First, we will prove by induction that for all $x \in \mathbb{N}$ and all distinct $x_1, \ldots, x_n \in \mathbb{N}$, we have $[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] = [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_n$. By Lemma 3.22, the statement holds when $n \leq 2$. Let $n \geq 3$ and assume that the statement holds for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Then, by the induction hypothesis,

$$\begin{aligned}
&[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] \\
&= [\, [\, x_1 \oplus x \mid \cdots \mid x_{n-1} \oplus x \,] \mid [\, x_1 \oplus x \mid \cdots \mid x_{n-2} \oplus x \mid x_n \oplus x \,] \,] \\
&\quad \oplus [\, x_1 \oplus x \mid \cdots \mid x_{n-2} \oplus x \,] \\
&= [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus (x)_{n-1} \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \oplus (x)_{n-1} \,] \\
&\quad \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \oplus (x)_{n-2}.
\end{aligned}$$

If n is odd, this is

$$[\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \oplus (x)_{n-2}.$$

If n is even, it is

$$[\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus x \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \oplus x \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \oplus (x)_{n-2}.$$

By the induction hypothesis, the statement holds for $n = 2$, so this is equal to

$$[\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \oplus (x)_{n-2}.$$

So in both cases we find that

$$\begin{aligned}
&[\, x_1 \oplus x \mid \cdots \mid x_n \oplus x \,] \\
&= [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \oplus (x)_{n-2} \\
&= [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_{n-2} = [\, x_1 \mid \cdots \mid x_n \,] \oplus (x)_n.
\end{aligned}$$

Next, we prove by induction that $[\, 0 \mid x_1 \mid \cdots \mid x_n \,] = [\, x_1 - 1 \mid \cdots \mid x_n - 1 \,]$ whenever $x_1, \ldots, x_n \in \mathbb{N} \setminus \{0\}$ are distinct. Let $x_1 \in \mathbb{N}$. Then, by Lemma 3.22, $[\, 0 \mid x_1 \,] = [\, x_1 - 1 \,]$. Now let $n \geq 2$ and assume that the statement holds for all $m < n$. Then we have

$$\begin{aligned}
&[\, 0 \mid x_1 \mid \cdots \mid x_n \,] \\
&= [\, [\, 0 \mid x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, 0 \mid x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, 0 \mid x_1 \mid \cdots \mid x_{n-2} \,] \\
&= [\, [\, x_1 - 1 \mid \cdots \mid x_{n-1} - 1 \,] \mid [\, x_1 - 1 \mid \cdots \mid x_{n-2} - 1 \mid x_n - 1 \,] \,] \\
&\quad \oplus [\, x_1 - 1 \mid \cdots \mid x_{n-2} - 1 \,] \\
&= [\, x_1 - 1 \mid \cdots \mid x_n - 1 \,].
\end{aligned}$$

□

Next, we show that the Mating candidate and the Animating candidate directly imply each other. Conway proved this in Chapter 13 of [2].

Lemma 3.28. *The Mating candidate and the Animating candidate are equivalent.*

Proof. We prove this by induction on n. For $n \leq 2$, the result follows from Lemma 3.22. Let $n \geq 3$ and assume that the Mating candidate and the Animating candidate are equivalent for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct and let $i, j \in \{1, \ldots, n\}$ be such that x_i and x_j are congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n\}$. Without loss of generality, assume that $i = 1$ and $j = 2$. Then, by the induction hypothesis, we have

$$\begin{aligned}&[\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,] \\ &= x_1 \oplus x_2 \oplus (x_1 \mid x_2) \oplus x_3 \oplus \cdots \oplus x_n \oplus (x_3 \mid x_4) \oplus \cdots \oplus (x_{n-1} \mid x_n).\end{aligned}$$

By Lemma 3.16.5, we have $(x_1 \mid x_i) = (x_2 \mid x_i)$ for all $i \in \{3, \ldots, n\}$. So the above equals

$$\begin{aligned}&x_1 \oplus x_2 \oplus (x_1 \mid x_2) \oplus x_3 \oplus \cdots \oplus x_n \oplus (x_3 \mid x_4) \oplus \cdots \oplus (x_{n-1} \mid x_n) \\ &\quad \oplus (x_1 \mid x_3) \oplus (x_2 \mid x_3) \oplus (x_1 \mid x_4) \oplus (x_2 \mid x_4) \oplus \cdots \oplus (x_1 \mid x_n) \oplus (x_2 \mid x_n) \\ &= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n).\end{aligned}$$

So $[\, x_1 \mid \cdots \mid x_n \,] = [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,]$ if and only if

$$[\, x_1 \mid \cdots \mid x_n \,] = x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n).$$

□

Now, we show that the Mating candidate and the Triangle candidate directly imply each other, assuming that the Triangle candidate satisfies the Symmetry property.

Lemma 3.29. *The Mating candidate and the Triangle candidate are equivalent, assuming that Symmetry holds for the Triangle candidate.*

Proof. Recall that the Mating candidate satisfies the Symmetry property.

We prove this by induction on n. For $n \leq 2$, the Mating candidate and the Triangle candidate are equal. Let $x_1, x_2, x_3 \in \mathbb{N}$ be distinct and assume that out of $\{x_1, x_2, x_3\}$, the numbers x_1 and x_2 are congruent modulo the highest power of 2. Then, by Lemma 3.17, we have

$$[\, [\, x_1 \mid x_3 \,] \mid [\, x_2 \mid x_3 \,] \,] \oplus [\, x_3 \,] = [\, x_1 \mid x_2 \,] \oplus [\, x_3 \,].$$

Now let $n \geq 3$ and assume that the Mating candidate and the Triangle candidate are equivalent for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Because of Symmetry, we may assume that x_1 and x_2 are congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n\}$. Then, by the induction hypothesis,

$$\begin{aligned}&[\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \\ &= [\, [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \\ &\quad \oplus [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_{n-2} \,].\end{aligned}$$

By Lemma 3.22, we have $[\, x_1 \oplus x \mid x_2 \oplus x \,] = [\, x_1 \mid x_2 \,]$ for any distinct $x_1, x_2 \in \mathbb{N}$. So the above can be simplified to

$$\begin{aligned}&[\, [\, x_3 \mid \cdots \mid x_{n-1} \,] \mid [\, x_3 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_{n-2} \,] \\ &= [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,].\end{aligned}$$

So

$$[\, x_1 \mid \cdots \mid x_n \,] = [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,]$$

if and only if $[\, x_1 \mid \cdots \mid x_n \,] = [\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,]$.

□

Finally, we show that the Animating candidate and the Triangle candidate directly imply each other. Conway proved this in Chapter 13 of [2].

Lemma 3.30. *The Animating candidate and the Triangle candidate are equivalent.*

Proof. We prove this by induction on n. By Lemma 3.22, the two candidates are equivalent for $n \leq 2$. Let $n \geq 3$, and assume that the candidates are equivalent for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Because $[\, x \mid y \,] = x \oplus y \oplus (x \mid y)$ for all distinct $x, y \in \mathbb{N}$, we have

$$\begin{aligned}&[\,[\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]\,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,]\\ &= [\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]\\ &\quad \oplus ([\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]) \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,].\end{aligned}$$

By the induction hypothesis, this equals

$$\begin{aligned}&x_1 \oplus \cdots \oplus x_{n-1} \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})\\ &\quad \oplus x_1 \oplus \cdots \oplus x_{n-2} \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-3} \mid x_{n-2})\\ &\quad \oplus (x_1 \mid x_n) \oplus \cdots \oplus (x_{n-2} \mid x_n)\\ &\quad \oplus x_1 \oplus \cdots \oplus x_{n-2} \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-3} \mid x_{n-2})\\ &\quad \oplus ([\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,])\\ &= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})\\ &\quad \oplus (x_1 \mid x_n) \oplus \cdots \oplus (x_{n-2} \mid x_n) \oplus ([\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]).\end{aligned}$$

Using the induction hypothesis again, we get

$$\begin{aligned}&([\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,])\\ &= (x_1 \oplus \cdots \oplus x_{n-1} \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})\\ &\quad \mid x_1 \oplus \cdots \oplus x_{n-2} \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-3} \mid x_{n-2})\\ &\quad \oplus (x_1 \mid x_n) \oplus \cdots \oplus (x_{n-2} \mid x_n)).\end{aligned}$$

Using Lemma 3.16.1, this simplifies to

$$(x_{n-1} \oplus (x_1 \mid x_{n-1}) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})) \mid x_n \oplus (x_1 \mid x_n) \oplus \cdots \oplus (x_{n-2} \mid x_n)).$$

This is equal to $(x_{n-1} \mid x_n)$ by Lemma 3.16.6. Combining this with the above, we get

$$\begin{aligned}&[\,[\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]\,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,]\\ &= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-2} \mid x_{n-1})\\ &\quad \oplus (x_1 \mid x_n) \oplus \cdots \oplus (x_{n-2} \mid x_n) \oplus (x_{n-1} \mid x_n)\\ &= x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n).\end{aligned}$$

So

$$[\, x_1 \mid \cdots \mid x_n \,] = [\,[\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,]\,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,]$$

if and only if $[\, x_1 \mid \cdots \mid x_n \,] = x_1 \oplus \cdots \oplus x_n \oplus (x_1 \mid x_2) \oplus \cdots \oplus (x_{n-1} \mid x_n)$.

□

The above lemmas together prove Theorem 3.21. So the four candidates are the same function. Further, using the results of Section 3.4 we can conclude that this function satisfies the Symmetry property. In Chapter 4, we will show that this function is the Sprague-Grundy function for Welter's game.

4 Welter's Theorem

Recall that the four candidate functions for the Sprague-Grundy function for Welter's game are equal. In this section, we will prove Welter's Theorem, which states that the function described in Chapter 3 is the Sprague-Grundy function for Welter's game.

In order to do so, we first discuss some properties that Welter's game satisfies, and that are used in both Welter's and Conway's proofs of Welter's Theorem. We discuss the Unique Prime Lemma in Section 4.1 and the Even Alteration Theorem in Section 4.2. Then, we prove Welter's Theorem in Section 4.3.

Welter's and Conway's proofs start from different candidate functions, and it is not immediately obvious that these functions are related. In an attempt to better understand the relation between the four candidates and the interesting properties that the function has, we will provide proofs of the Unique Prime Lemma and the Even Alteration Theorem in multiple ways, starting from different candidate functions.

4.1 Unique Prime Lemma

Welter's game has the following property. If we start from any position and want to move to a position with a given Sprague-Grundy value, it is possible to do so by moving any one of the coins, possibly to a higher-numbered position. We will need this for our proof of Welter's Theorem. So we need to show that the function discussed in Chapter 3 satisfies this property.

Lemma 4.1 (Unique Prime Lemma)**.** *If* $[\ x_1 \mid \cdots \mid x_n\ ] = k$ *and* $k' \neq k$*, there exist unique* $x'_1, \ldots, x'_n \in \mathbb{N}$*, with* $x'_i \neq x_j$ *and* $x'_i \neq x'_j$ *whenever* $i \neq j$*, such that*

$$[\ x'_1 \mid x_2 \mid \cdots \mid x_n\ ] = [\ x_1 \mid x'_2 \mid x_3 \mid \cdots \mid x_n\ ] = \cdots = [\ x_1 \mid \cdots \mid x_{n-1} \mid x'_n\ ] = k'.$$

We call this the Unique Prime Lemma because of the unicity of the corresponding primed numbers $x'_1, \ldots, x'_n$. Note that the lemma implies that

$$[\ x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1}\ ] \neq [\ x_1 \mid \cdots \mid x_{n-2} \mid x_n\ ]$$

for all distinct $x_1, \ldots, x_n \in \mathbb{N}$, so that the Triangle candidate is well-defined.

We will give several proofs of the lemma, starting from multiple candidate functions. Figure 3 provides an overview of the results in this section. Some of the proofs use the Symmetry property. When that is the case, there is an S next to the corresponding implication arrow in the figure.

First, we give a direct proof that Welter's candidate implies the Unique Prime Lemma.

Lemma 4.2. *Welter's candidate implies the Unique Prime Lemma, assuming that Symmetry holds for Welter's candidate.*

Proof. We prove this by induction on n. Suppose we want to find x'_1 such that $[\ x'_1\ ] = k'$. Since

$$[\ x\ ] = [\ 0 \oplus x\ ] = [\ 0\ ] \oplus x = x$$

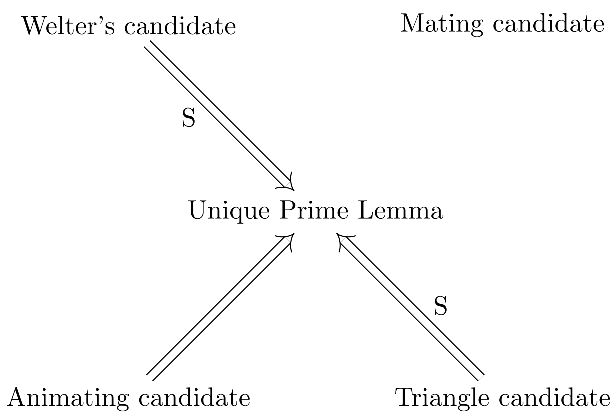

Figure 3: Overview of direct implications. An S signifies that Symmetry is used for the proof.

for all $x \in \mathbb{N}$, we find that the only option is $x_1' = k'$.

Now let $n \geq 2$, and suppose that the Unique Prime Lemma holds for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct and let $k' \in \mathbb{N}$ be such that $k' \neq [\ x_1 \mid \cdots \mid x_n\ ]$. We need to show that for all $i \in \{1, \ldots, n\}$, there exists a unique x_i' such that $x_i' \neq x_j$ and $x_i' \neq x_j'$ whenever $i \neq j$, and

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] = k'.$$

Because of Symmetry, it is enough to show that this is true when $i \neq 1$.

Note that, for all $i \in \{2, \ldots, n\}$

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] = k'$$

is true if and only if

$$[\ 0 \mid x_2 \oplus x_1 \mid \cdots \mid x_{i-1} \oplus x_1 \mid x_i' \oplus x_1 \mid x_{i+1} \oplus x_1 \mid \cdots \mid x_n \oplus x_1\ ] = k' \oplus (x_1)_n,$$

which is true if and only if

$$\begin{aligned} &[\ x_2 \oplus x_1 - 1 \mid \cdots \mid x_{i-1} \oplus x_1 - 1 \mid x_i' \oplus x_1 - 1 \mid x_{i+1} \oplus x_1 - 1 \\ &\quad \mid \cdots \mid x_n \oplus x_1 - 1\ ] \\ &= k' \oplus (x_1)_n. \end{aligned}$$

By the induction hypothesis, there exist unique $x_2'', \ldots, x_n'' \in \mathbb{N}$ such that for all $i \in \{2, \ldots, n\}$, we have

$$[\ x_2 \oplus x_1 - 1 \mid \cdots \mid x_{i-1} \oplus x_1 - 1 \mid x_i'' \mid x_{i+1} \oplus x_1 - 1 \mid \cdots \mid x_n \oplus x_1 - 1\ ] = k' \oplus (x_1)_n,$$

where $x_i'' \neq x_1 \oplus x_j - 1$ and $x_i'' \neq x_j''$ for all $j \in \{2, \ldots, n\} \setminus \{i\}$. For all $i \in \{2, \ldots, n\}$, let $x_i' = (x_i'' + 1) \oplus x_1$. Then $x_i' \in \mathbb{N}$, and x_i' must be the unique number such that

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] = k'$$

Further, we have $x'_i \neq x_j$ and $x'_i \neq x'_j$ for all $j \in \{2, \ldots, n\} \setminus \{i\}$. Note that $x''_i + 1 > 0$, so

$$x'_i = (x''_i + 1) \oplus x_1 \neq x_1.$$

By Symmetry, we can conclude that there also exists a unique $x'_1 \in \mathbb{N}$ such that

$$[\ x'_1 \mid x_2 \mid \cdots \mid x_n\ ] = k',$$

and $x'_1 \neq x_j$ and $x'_1 \neq x'_j$ for all $j \in \{1, \ldots, n\} \setminus \{1, \ell\}$ for any $\ell \in \{2, \ldots, n\}$. So $x'_i \neq x'_1$.

□

Next, we give a direct proof that the Animating candidate implies the Unique Prime Lemma. We use a more detailed version of the proof Conway used in Chapter 13 of [2].

We first need to define the nim-sum for negative numbers. We define the binary expansion of -1 as a string of 1s, infinite to the left. Then we can add, subtract and nim-add just like we would with non-negative integers. So for all $n \in \mathbb{N} \setminus \{0\}$, we can find the binary expansion of $-n$ using the equation $-n = (-1) - (n-1)$. For example, since $-2 = (-1) - 1$, its binary expansion is $\ldots 1110$, which is equal to that of $(-1) \oplus 1$. The binary expansion of $-3 = (-2) - 1$ is $\ldots 1101$, which is equal to that of $(-1) \oplus 2$.

Remember that for any $a, b \in \mathbb{N}$, a and b are congruent modulo 2^n if and only if the last n digits of their binary expansions are equal. Because of how we defined the binary expansions of negative integers, the same holds for any $a, b \in \mathbb{Z}$.

We define $(x \mid x) = -1$ for any $x \in \mathbb{Z}$. Note that with our new definitions, Lemmas 3.16.1, 3.16.2 and 3.16.3 hold more generally. For all $a, b, x \in \mathbb{Z}$, we have $(a \oplus x \mid b \oplus x) = (a \mid b)$ and $(a + x \mid b + x) = (a \mid b)$, and for all $n \in \mathbb{Z}$, we have $n \oplus (n \mid 0) = n - 1$.

Now, we need the following lemma.

Lemma 4.3. *Any function of the form*

$$f(x) = x \oplus c \oplus (x \mid c_1) \oplus \cdots \oplus (x \mid c_n),$$

with $n \in \mathbb{N}$ and $c, c_1, \ldots, c_n \in \mathbb{Z}$, can also be written as

$$f(x) = (\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m)$$

for some $c'_1, c'_2, \ldots c'_m \in \mathbb{Z}$, with $m = 4n + 1$.

Proof. We prove this by induction on n. For any $c \in \mathbb{Z}$, the function $x \mapsto x \oplus c$ for all $x \in \mathbb{Z}$ is already in the required form, so the statement holds when $n = 0$. Now let $n \in \mathbb{N} \setminus \{0\}$ and assume that the statement holds for all $m \leq n$. Let $c, c_1, \ldots, c_n, c_{n+1} \in \mathbb{Z}$, and let f be the function defined by

$$f(x) = x \oplus c \oplus (x \mid c_1) \oplus \cdots \oplus (x \mid c_n)$$

for all $x \in \mathbb{Z}$. By the induction hypothesis, f can be written in the required form. Let $c'_1, \ldots, c'_m$ be such that

$$f(x) = (\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m).$$

We need to show that the function g defined by

$$g(x) = x \oplus c \oplus (x \mid c_1) \oplus \cdots \oplus (x \mid c_n) \oplus (x \mid c_{n+1})$$

for all $x \in \mathbb{Z}$ can also be written in the required form.

Let $c'_{m+1} = 0$. Then we have

$$f(x) = ((\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m) + c'_{m+1}).$$

Let $h(x) = f(x) \oplus f(c_{n+1})$, for all $x \in \mathbb{Z}$. By Lemmas 3.16.1 and 3.16.2, we have $(h(a) \mid h(b)) = (f(a) \mid f(b)) = (a \mid b)$ for all $a, b \in \mathbb{Z}$. So for any $x \in \mathbb{Z}$,

$$(x \mid c_{n+1}) = (h(x) \mid h(c_{n+1})) = (h(x) \mid 0).$$

Using Lemma 3.16.3, it follows that

$$h(x) - 1 = h(x) \oplus (h(x) \mid 0) = h(x) \oplus (x \mid c_{n+1}) = f(x) \oplus f(c_{n+1}) \oplus (x \mid c_{n+1}),$$

so

$$\begin{aligned} g(x) &= f(x) \oplus (x \mid c_{n+1}) = (h(x) - 1) \oplus f(c_{n+1}) \\ &= ((f(x) \oplus f(c_{n+1})) - 1) \oplus f(c_{n+1}). \end{aligned}$$

This is a function of the required form, with $c'_{m+2} = f(c_{n+1})$, $c'_{m+3} = -1$ and $c'_{m+4} = f(c_{n+1})$.

□

Lemma 4.4. *The Animating candidate implies the Unique Prime Lemma.*

Proof. Let $i \in \{1, \ldots, n\}$ and let $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{N}$ be distinct. Then $x_i \mapsto [\ x_1 \mid \cdots \mid x_n\ ]$ is a function of the form

$$x \mapsto x \oplus c \oplus (c \mid c_1) \oplus \cdots \oplus (x \mid c_n)$$

for some $c, c_1, \ldots, c_n \in \mathbb{Z}$. By Lemma 4.3, it can be written in the form

$$x \mapsto (\ldots((((x \oplus c'_1) + c'_2) \oplus c'_3) + c'_4) \oplus \cdots \oplus c'_m)$$

for some $c'_1, \ldots, c'_m \in \mathbb{Z}$. Functions of this form clearly have an inverse, so this implies that for every $k' \in \mathbb{N}$ there exists a unique $x'_i \in \mathbb{Z}$ such that

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n\ ] = k'.$$

Now we only need to show that x'_i is non-negative, and unequal to x_j and x'_j for all $j \in \{1, \ldots, n\} \setminus \{i\}$.

Assume that $x'_i = x_j$ for some $j \neq i$. Then $x'_i \geq 0$, so if we use the definition of the Animating candidate to write out $[\ x_1 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n\ ]$ as a nim-sum, the only negative term is $(x'_i \mid x_j) = -1$. Then the result of the nim-addition is negative, but by assumption it should be equal to $k' \in \mathbb{N}$. This gives a contradiction, so $x'_i \neq x_j$ whenever $j \neq i$.

Now assume that x'_i is negative. Because $x'_i \neq x_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$, the numbers $(x'_i \mid x_j), \ldots, (x'_i \mid x_n)$ are all non-negative. So if we write out

$[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,]$ as a nim-sum, the only negative term is x_i'. Then the result of the nim-addition is negative. This again gives a contradiction.

Finally, assume that $x_i' = x_j'$ for some $j \in \{1, \dots, n\} \setminus \{i\}$. Then

$$\begin{aligned}[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] &= [\, x_1 \mid \cdots \mid x_{j-1} \mid x_j' \mid x_{j+1} \mid \cdots \mid x_n \,] \\ &= [\, x_1 \mid \cdots \mid x_{j-1} \mid x_i' \mid x_{j+1} \mid \cdots \mid x_n \,]\end{aligned}$$

Because the Animating candidate is symmetric, it follows that

$$\begin{aligned}&[\, x_1 \mid \cdots \mid x_{i-1} \mid x_j \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_i' \mid x_{j+1} \mid \cdots \mid x_n \,] \\ &= [\, x_1 \mid \cdots \mid x_{i-1} \mid x_i \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_i' \mid x_{j+1} \mid \cdots \mid x_n \,].\end{aligned}$$

We can conclude that

$$\begin{aligned}&x_j \oplus (x_j \mid x_1) \oplus \cdots \oplus (x_j \mid x_{i-1}) \oplus (x_j \mid x_{i+1}) \oplus \dots \\ &\quad \oplus (x_j \mid x_{j-1}) \oplus (x_j \mid x_i') \oplus (x_j \mid x_{j+1}) \oplus \cdots \oplus (x_j \mid x_n) \\ &= x_i \oplus (x_i \mid x_1) \oplus \cdots \oplus (x_i \mid x_{i-1}) \oplus (x_i \mid x_{i+1}) \oplus \dots \\ &\quad \oplus (x_i \mid x_{j-1}) \oplus (x_i \mid x_i') \oplus (x_i \mid x_{j+1}) \oplus \cdots \oplus (x_i \mid x_n).\end{aligned}$$

So the function

$$\begin{aligned}x \mapsto & x \oplus (x \mid x_1) \oplus \cdots \oplus (x \mid x_{i-1}) \oplus (x \mid x_{i+1}) \oplus \dots \\ & \oplus (x \mid x_{j-1}) \oplus (x \mid x_i') \oplus (x \mid x_{j+1}) \oplus \cdots \oplus (x \mid x_n)\end{aligned}$$

maps both x_i and x_j to the same number. But $x_i \neq x_j$, and we proved above that functions of this type are bijective, so this gives a contradiction.

□

Finally, we give a direct proof that the Triangle candidate implies the Unique Prime Lemma.

Lemma 4.5. *The Triangle candidate implies the Unique Prime Lemma, assuming that Symmetry holds for the Triangle candidate.*

Proof. We prove this by induction on n. For any $k' \in \mathbb{N}$, we have $[\, x_1' \,] = k'$ if and only if $x_1' = k'$. Now let $x_1, x_2 \in \mathbb{N}$ be distinct, let $k' \in \mathbb{N}$, and assume that we want to find x_1' such that $[\, x_1' \mid x_2 \,] = k'$. Since $[\, x_1' \mid x_2 \,] = x_1' \oplus x_2 - 1$, the only option is $x_1' = (k' + 1) \oplus x_2$. Similarly, we have $x_2' = (k' + 1) \oplus x_1$.

Let $i \in \{1, 2\}$. Then $(k' + 1) \oplus x_i \in \mathbb{N}$. Also, we have

$$x_1' = (k' + 1) \oplus x_1 \neq (k' + 1) \oplus x_2 = x_2'.$$

Now assume that $x_1' = x_2$. Then

$$[\, x_1' \mid x_2 \,] = x_1' \oplus x_2 - 1 = 0 - 1 = -1,$$

which gives a contradiction. Similarly, we have $x_2' \neq x_1$.

Let $n \geq 3$, and assume that the Unique Prime Lemma holds for all $m < n$. Let $x_1, \dots, x_n \in \mathbb{N}$ be distinct, let $k' \in \mathbb{N}$ and let $i \in \{1, \dots, n\}$. We will find x_i' such that $[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] = k'$. Because of the assumption that

Symmetry holds, we may assume without loss of generality that $i = n$. Then we need to find x'_n such that

$$\begin{aligned} k' &= [\, x_1 \mid \cdots \mid x_{n-1} \mid x'_n \,] \\ &= [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,]. \end{aligned}$$

This is equivalent to

$$[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,] = (k' \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] + 1) \oplus [\, x_1 \mid \cdots \mid x_{n-1} \,].$$

By the induction hypothesis, there is a unique $x'_n \in \mathbb{N}$ with this property, and we have $x'_n \neq x_1, \ldots, x_{n-2}$ and $x'_n \neq x'_1, \ldots, x'_{n-2}$. Suppose that $x'_n = x_{n-1}$. Then

$$[\, x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1} \,] = (k' \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] + 1) \oplus [\, x_1 \mid \cdots \mid x_{n-1} \,],$$

so $k' \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] + 1 = 0$, which gives a contradiction. Finally, assume that $x'_n = x'_{n-1}$. Then

$$[\, x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1} \mid x'_{n-1} \,] = k' = [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \mid x_n \,].$$

Using the Triangle candidate, we find that

$$\begin{aligned} &[\, [\, x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \,] \\ &= [\, [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,]. \end{aligned}$$

Then

$$[\, x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1} \,] = [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,],$$

which violates the assumption that the Unique Prime Lemma holds for $n - 1$.

□

Welter's game is not the only game that satisfies a version of the Unique Prime Lemma. For example, Nim has the same property. The Unique Prime Lemma also holds for the game Antonim, which is similar to Nim and Welter's game. Another similar game, Antimatter, satisfies a slightly different property. We will discuss this further in Chapter 5. There, we also prove directly from the definition of Welter's game that it must satisfy the Unique Prime Lemma.

4.2 Even Alteration Theorem

An interesting property of Welter's game is that whenever an even number of elements in a position is exchanged by their primed counterparts as defined in the Unique Prime Lemma, the Sprague-Grundy value remains the same. We will need this property for our proof of Welter's Theorem. So we will show that the function discussed in Chapter 3 satisfies this property.

Theorem 4.6 (Even Alteration Theorem)**.** *Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct and let $k \in \mathbb{N}$ be such that $[\, x_1 \mid \cdots \mid x_n \,] = k$. Let $k' \in \mathbb{N}$ be unequal to k, and let $x'_1, \ldots, x'_n$ be as defined in Lemma 4.1. Then the equation $[\, x_1 \mid \cdots \mid x_n \,] = k$ remains true whenever an even number of $x_1, \ldots, x_n, k$ are replaced by the corresponding numbers $x'_1, \ldots, x'_n, k'$. We express this with the following notation:*

$$\left[\begin{array}{c|c|c} x_1 & \cdots & x_n \\ x'_1 & \cdots & x'_n \end{array}\right] = \begin{array}{c} k \\ k' \end{array}$$

We give two proofs of this theorem, starting from different candidate functions. Figure 4 provides an overview of the results in this section. When a proof uses the Symmetry property, there is an S next to the corresponding implication arrow in the figure.

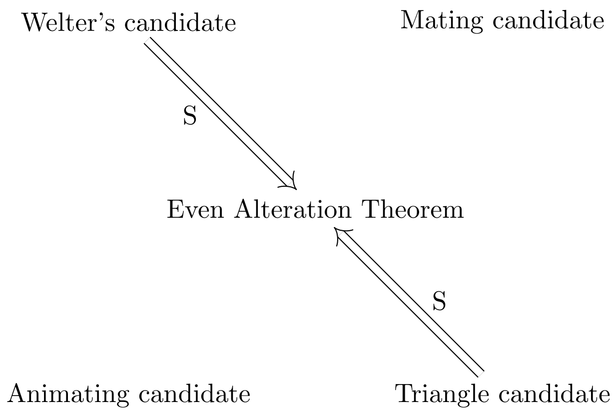

Figure 4: Overview of direct implications. An S signifies that Symmetry is used for the proof.

Note that it is enough to show that

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_j' \mid x_{j+1} \mid \cdots \mid x_n\ ] = [\ x_1 \mid \cdots \mid x_n\ ]$$

for all distinct $x_1, \ldots, x_n \in \mathbb{N}$ and all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then we also have

$$\begin{aligned}&[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_j' \mid x_{j+1} \mid \cdots \mid x_{k-1} \mid x_k' \mid x_{k+1} \mid \cdots \mid x_n\ ] \\ &= [\ x_1 \mid \cdots \mid x_{k-1} \mid x_k' \mid x_{k+1} \mid \cdots \mid x_n\ ] = k'\end{aligned}$$

for all distinct $x_1, \ldots, x_n \in \mathbb{N}$ and all distinct $i, j, k \in \{1, \ldots, n\}$. The statement then follows by induction on the amount of primed elements.

First, we give a direct proof that Welter's candidate implies the Even Alteration Theorem.

Lemma 4.7. *Welter's candidate implies the Even Alteration Theorem, assuming that Symmetry holds for Welter's candidate.*

Proof. Let $x_1, x_2 \in \mathbb{N}$ be distinct and assume that $[\ x_1 \mid x_2\ ] = k$, $[\ x_1' \mid x_2\ ] = k'$ and $[\ x_1 \mid x_2'\ ] = k'$. Let $x = x_1 \oplus x_1'$. Then $x_1' = x_1 \oplus x$, and

$$k' = [\ x_1' \mid x_2\ ] = [\ x_1 \oplus x \mid x_2\ ] = [\ x_1 \mid x_2 \oplus x\ ].$$

By the Unique Prime Lemma, we can conclude that $x_2' = x_2 \oplus x$. So

$$[\ x_1' \mid x_2'\ ] = [\ x_1 \oplus x \mid x_2 \oplus x\ ] = [\ x_1 \mid x_2\ ] = k.$$

Now let $n \geq 3$ and assume that the Even Alteration Theorem holds for all $m < n$. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct and let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. By Symmetry, we may assume that $i, j \neq 1$. Assume without loss of generality that $i = n-1$

and $j = n$. Suppose that $[\, x_1 \mid \cdots \mid x_n \,] = k$, $[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \mid x_n \,] = k'$ and $[\, x_1 \mid \cdots \mid x_{n-1} \mid x'_n \,] = k'$. We have

$$\begin{aligned}[\, x_1 \mid \cdots \mid x_n \,] &= [\, 0 \mid x_2 \oplus x_1 \mid \cdots \mid x_n \oplus x_1 \,] \oplus (x_1)_n \\ &= [\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_n \oplus x_1 - 1 \,] \oplus (x_1)_n,\end{aligned}$$

so

$$[\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_n \oplus x_1 - 1 \,] = k \oplus (x_1)_n.$$

Similarly,

$$[\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_{n-2} \oplus x_1 - 1 \mid x'_{n-1} \oplus x_1 - 1 \mid x_n \oplus x_1 - 1 \,] = k' \oplus (x_1)_n$$

and

$$[\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_{n-1} \oplus x_1 - 1 \mid x'_n \oplus x_1 - 1 \,] = k' \oplus (x_1)_n.$$

By the induction hypothesis, we can conclude that

$$[\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_{n-2} \oplus x_1 - 1 \mid x'_{n-1} \oplus x_1 - 1 \mid x'_n \oplus x_1 - 1 \,] = k \oplus (x_1)_n.$$

So

$$\begin{aligned}&[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \mid x'_n \,] \\ &= [\, 0 \mid x_2 \oplus x_1 \mid \cdots \mid x_{n-2} \oplus x_1 \mid x'_{n-1} \oplus x_1 \mid x'_n \oplus x_1 \,] \oplus (x_1)_n \\ &= [\, x_2 \oplus x_1 - 1 \mid \cdots \mid x_{n-2} \oplus x_1 - 1 \mid x'_{n-1} \oplus x_1 - 1 \mid x'_n \oplus x_1 - 1 \,] \oplus (x_1)_n \\ &= k.\end{aligned}$$

□

Next, we give a direct proof that the Triangle candidate implies the Even Alteration Theorem. This is Lemma 8 of [10].

Lemma 4.8. *The Triangle candidate implies the Even Alteration Theorem, assuming that Symmetry holds for the Triangle candidate.*

Proof. Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct, and let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. By Symmetry, we may assume without loss of generality that $i = n - 1$ and $j = n$. Assume that $[\, x_1 \mid \cdots \mid x_n \,] = k$, $[\, x_1 \mid \cdots \mid x_{n-1} \mid x'_{n-1} \mid x_n \,] = k'$ and $[\, x_1 \mid \cdots \mid x_{n-1} \mid x'_n \,] = k'$. Then

$$\begin{aligned}&[\, [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \\ &= [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \mid x_n \,] = [\, x_1 \mid \cdots \mid x_{n-1} \mid x'_n \,] \\ &= [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,].\end{aligned}$$

This implies that

$$\begin{aligned}&[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \\ &= [\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,],\end{aligned}$$

so by nim-adding $[\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,]$ to both sides we find that

$$\begin{aligned}&[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,] \\ &= [\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,].\end{aligned}$$

We conclude that

$$
\begin{aligned}
&[\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \mid x'_n \,] \\
&= [\, [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x'_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \\
&= [\, [\, x_1 \mid \cdots \mid x_{n-2} \mid x_{n-1} \,] \mid [\, x_1 \mid \cdots \mid x_{n-2} \mid x_n \,] \,] \oplus [\, x_1 \mid \cdots \mid x_{n-2} \,] \\
&= [\, x_1 \mid \cdots \mid x_n \,] = k.
\end{aligned}
$$

□

Welter's game is not the only game for which the Even Alteration Theorem holds. It also holds for the game Nim, as we show below.

Theorem 4.9. *The Even Alteration Theorem holds for Nim.*

Proof. Let $x_1, \ldots, x_n \in \mathbb{N}$, and let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Because the Sprague-Grundy function for Nim is symmetric, we may assume without loss of generality that $i = 1$ and $j = 2$. Let $k, k' \in \mathbb{N}$ be distinct, and assume that $x_1 \oplus \cdots \oplus x_n = k$, $x'_1 \oplus x_2 \oplus \cdots \oplus x_n = k'$, and $x_1 \oplus x'_2 \oplus x_3 \oplus \cdots \oplus x_n = k'$. By combining the second and third statements, we get

$$x'_1 \oplus x_1 \oplus x_2 \oplus x'_2 = 0.$$

By nim-adding this to the first statement, we find that $x'_1 \oplus x'_2 \oplus x_3 \oplus \cdots \oplus x_n = k$.

□

There exist other games that are also similar to Welter's game, but for which the Even Alteration Theorem does not hold. We will discuss some of these games in Chapter 5.

4.3 Proof of Welter's Theorem

In this section, we prove Welter's Theorem, which says that the function discussed in Chapter 3 is the Sprague-Grundy function for Welter's game. We use a more detailed version of Conway's proof, which can be found in Chapter 13 of [2]. We first need the following lemma.

Lemma 4.10. *Let $n \geq 3$, and let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct, such that out of $\{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}$, x_1 and x_2 are congruent modulo the highest power of 2. Let $k' \in \mathbb{N}$, and let x'_1 and x'_2 be as defined in Lemma 4.1. Then $x'_1 \oplus x'_2 = x_1 \oplus x_2$.*

Proof. We have $[\, x'_1 \mid x'_2 \mid x_3 \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,]$, by the Even Alteration Theorem. By using the properties of the Triangle candidate and the Symmetry property, we find that

$$
\begin{aligned}
&[\, [\, x'_1 \mid x_3 \mid \cdots \mid x_n \,] \mid [\, x'_2 \mid x_3 \mid \cdots \mid x_n \,] \,] \oplus [\, x_3 \mid \cdots \mid x_n \,] \\
&= [\, x'_1 \mid x'_2 \mid x_3 \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] \\
&= [\, [\, x_1 \mid x_3 \mid \cdots \mid x_n \,] \mid [\, x_2 \mid x_3 \mid \cdots \mid x_n \,] \,] \oplus [\, x_3 \mid \cdots \mid x_n \,].
\end{aligned}
$$

This implies that

$$\begin{aligned}&[\, x_1' \mid x_3 \mid \cdots \mid x_n \,] \oplus [\, x_2' \mid x_3 \mid \cdots \mid x_n \,] \\ &= [\, x_1 \mid x_3 \mid \cdots \mid x_n \,] \oplus [\, x_2 \mid x_3 \mid \cdots \mid x_n \,].\end{aligned}$$

Using the properties of the Animating candidate, we conclude that

$$\begin{aligned}&x_1' \oplus (x_1' \mid x_3) \oplus \cdots \oplus (x_1' \mid x_n) \oplus x_2' \oplus (x_2' \mid x_3) \oplus \cdots \oplus (x_2' \mid x_n) \\ &= x_1 \oplus (x_1 \mid x_3) \oplus \cdots \oplus (x_1 \mid x_n) \oplus x_2 \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_2 \mid x_n).\end{aligned}$$

Using Lemmas 3.16.6 and 3.16.1, we get

$$\begin{aligned}&(x_1 \mid x_2) \\ &= (x_1 \oplus (x_1 \mid x_3) \oplus \cdots \oplus (x_1 \mid x_n) \mid x_2 \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_2 \mid x_n)) \\ &= (0 \mid x_1 \oplus (x_1 \mid x_3) \oplus \cdots \oplus (x_1 \mid x_n) \oplus x_2 \oplus (x_2 \mid x_3) \oplus \cdots \oplus (x_2 \mid x_n)) \\ &= (0 \mid x_1' \oplus (x_1' \mid x_3) \oplus \cdots \oplus (x_1' \mid x_n) \oplus x_2' \oplus (x_2' \mid x_3) \oplus \cdots \oplus (x_2' \mid x_n)) \\ &= (x_1' \oplus (x_1' \mid x_3) \oplus \cdots \oplus (x_1' \mid x_n) \mid x_2' \oplus (x_2' \mid x_3) \oplus \cdots \oplus (x_2' \mid x_n)) \\ &= (x_1' \mid x_2').\end{aligned}$$

It now follows that x_1' and x_2' are also congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$. Then, using the properties of the Mating candidate and the Even Alteration Theorem, we get

$$\begin{aligned}&[\, x_1 \mid x_2 \,] \oplus [\, x_3 \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] \\ &= [\, x_1' \mid x_2' \mid x_3 \mid \cdots \mid x_n \,] = [\, x_1' \mid x_2' \,] \oplus [\, x_3 \mid \cdots \mid x_n \,],\end{aligned}$$

so $[\, x_1 \mid x_2 \,] = [\, x_1' \mid x_2' \,]$, which implies that $x_1 \oplus x_2 = x_1' \oplus x_2'$.

□

Next, we will show that if

$$\left[\begin{array}{c|c|c} x_1 & \ldots & x_n \\ x_1' & \ldots & x_n' \end{array}\right] = \begin{array}{l} k \\ k', \end{array}$$

then an even amount of the inequalities $x_1' < x_1, \ldots, x_n' < x_n, k' < k$ are true. This implies that if $[\, x_1 \mid \cdots \mid x_n \,] = k$ and $k' < k$, there is at least one feasible move from $(x_1, \ldots, x_n)$ to a position $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ with $[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] = k'$. In order to prove this, we need two more lemmas.

Lemma 4.11. *If*

$$\left[\begin{array}{c|c|c} x_1 & \ldots & x_n \\ x_1' & \ldots & x_n' \end{array}\right] = \begin{array}{l} k \\ k', \end{array}$$

then for any x, an even number of the inequalities

$$\begin{aligned}&x_1 \oplus x_1' \oplus x < x \\ &\ldots \\ &x_n \oplus x_n' \oplus x < x \\ &k \oplus k' \oplus (x)_n < (x)_n\end{aligned}$$

are true.

Proof. We prove this by induction on n. Let x_1, x_1', k, k' be such that

$$\left[\begin{array}{c} x_1 \\ x_1' \end{array}\right] = \begin{array}{c} k \\ k' \end{array}$$

and let $x \in \mathbb{N}$. Then the inequalities are $x_1 \oplus x_1' \oplus x < x$ and $k \oplus k' \oplus x < x$. Here, $k = x_1$ and $k' = x_1'$, so the inequalities are equivalent.

Next, let $x_1, x_2, x_1', x_2', k, k'$ be such that

$$\left[\begin{array}{c|c} x_1 & x_2 \\ x_1' & x_2' \end{array}\right] = \begin{array}{c} k \\ k' \end{array}$$

and let $x \in \mathbb{N}$. Now, the inequalities are

$$\begin{aligned} & x_1 \oplus x_1' \oplus x < x \\ & x_2 \oplus x_2' \oplus x < x \\ & k \oplus k' \oplus 0 < 0. \end{aligned}$$

The last inequality is certainly false, because $k, k' \in \mathbb{N}$. So we need to show that an even number of the first two inequalities hold. By the Even Alteration Theorem, we have $[\, x_1' \mid x_2' \,] = [\, x_1 \mid x_2 \,]$. This implies that $x_1' \oplus x_2' = x_1 \oplus x_2$, so $x_1 \oplus x_1' = x_2 \oplus x_2'$. So the two inequalities are equivalent.

Now let $n \geq 3$ and assume that the statement holds for all $m < n$. Let $x_1, \ldots, x_n, x_1', \ldots, x_n', k, k'$ be such that

$$\left[\begin{array}{c|c|c} x_1 & \ldots & x_n \\ x_1' & \ldots & x_n' \end{array}\right] = \begin{array}{c} k \\ k', \end{array}$$

and let $x \in \mathbb{N}$. Note that if we interchange an even amount of the elements with their primed counterparts in the statement

$$\left[\begin{array}{c|c|c} x_1 & \ldots & x_n \\ x_1' & \ldots & x_n' \end{array}\right] = \begin{array}{c} k \\ k', \end{array}$$

the statement remains true and the corresponding set of inequalities remains the same. For example, if x_1' and x_2 are congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$, we can exchange x_1' with x_1 and k' with k, and our problem remains the same. Because of this fact and because of Symmetry, we may assume without loss of generality that x_1 and x_2 are congruent modulo the highest power of 2 out of $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\}$. Then, by the properties of the Mating candidate, we have

$$\left[\begin{array}{c|c|c} x_3 & \ldots & x_n \\ x_3' & \ldots & x_n' \end{array}\right] = \begin{array}{l} k \oplus [\, x_1 \mid x_2 \,] \\ k' \oplus [\, x_1 \mid x_2 \,]. \end{array}$$

By the induction hypothesis, an even amount of the inequalities

$$\begin{aligned} & x_3 \oplus x_3' \oplus x < x \\ & \ldots \\ & x_n \oplus x_n' \oplus x < x \\ & k \oplus [\, x_1 \mid x_2 \,] \oplus k' \oplus [\, x_1 \mid x_2 \,] \oplus (x)_{n-2} < (x)_{n-2} \end{aligned}$$

hold. This set of inequalities is equivalent to

$$\begin{aligned}
&x_3 \oplus x_3' \oplus x < x \\
&\dots \\
&x_n \oplus x_n' \oplus x < x \\
&k \oplus k' \oplus (x)_n < (x)_n,
\end{aligned}$$

so now it is enough to show that an even amount of the inequalities

$$\begin{aligned}
&x_1 \oplus x_1' \oplus x < x \\
&x_2 \oplus x_2' \oplus x < x
\end{aligned}$$

hold. By Lemma 4.10, we have $x_1 \oplus x_2 = x_1' \oplus x_2'$. By nim-adding $x_1' \oplus x_2$ to both sides, we find that $x_1 \oplus x_1' = x_2 \oplus x_2'$. So the two inequalities are equivalent.

□

Lemma 4.12. *Let $y_1, \dots, y_n, y \in \mathbb{N}$. Then an even amount of the inequalities*

$$\begin{aligned}
y_1 \oplus y &< y_1 \\
&\dots \\
y_n \oplus y &< y_n \\
y_1 \oplus \cdots \oplus y_n \oplus y &< y_1 \oplus \cdots \oplus y_n
\end{aligned}$$

are true for any $y \in \mathbb{N}$.

Proof. Suppose that $y_1 \oplus \cdots \oplus y_n \oplus y < y_1 \oplus \cdots \oplus y_n$. Then the first 1 from the left in the binary expansion of y is in a location on which the binary expansion of $y_1 \oplus \cdots \oplus y_n$ also has a 1. So an odd amount of the y_i have a 1 on this position, and these are exactly the ones for which the inequality $y_i \oplus y < y_i$ is true. So in total, an even amount of the inequalities are true.

Now suppose that $y_1 \oplus \cdots \oplus y_n \oplus y \geq y_1 \oplus \cdots \oplus y_n$. Then the first 1 from the left in the binary expansion of y is on a location at which $y_1 \oplus \cdots \oplus y_n$ has a 0. Then there are an even amount of y_i with a 1 on this position, and these are exactly the ones for which the inequality $y_i \oplus y < y_i$ is true. Again, it follows that an even amount of the inequalities are true.

□

Theorem 4.13. *If*

$$\left[\begin{array}{c|c|c} x_1 & \dots & x_n \\ x_1' & \dots & x_n' \end{array}\right] = \begin{array}{l} k \\ k', \end{array}$$

then an even number of the inequalities $x_1' < x_1, \dots, x_n' < x_n, k' < k$ are true.

Proof. We prove this by induction. Let x_1, x_1', k, k' be such that

$$\left[\begin{array}{c} x_1 \\ x_1' \end{array}\right] = \begin{array}{l} k \\ k'. \end{array}$$

Then $k = x_1$ and $k' = x_1'$, so the inequalities $x_1' < x_1$ and $k' < k$ are equivalent.

Now let $n \geq 2$, and assume that the statement holds for all $m < n$. Let $x \in \mathbb{N}$, and let $x_1, \dots, x_{n-1}, x'_1, \dots, x'_{n-1}, k, k'$ be such that

$$\left[\begin{array}{c|c|c|c} 0 & x_1 & \dots & x_{n-1} \\ x & x'_1 & \dots & x'_{n-1} \end{array}\right] = \begin{array}{l} k \\ k'. \end{array} \tag{2}$$

Let $y_i \in \{x_i, x'_i\}$ for all $i \in \{1, \dots, n-1\}$. Then

$$[\, 0 \mid y_1 \mid \cdots \mid y_{n-1} \,] = [\, y_1 - 1 \mid \cdots \mid y_{n-1} - 1 \,],$$

and this equals k if an even amount of the elements are primed, and k' if an odd amount of the elements are primed. So we have

$$\left[\begin{array}{c|c|c} x_1 - 1 & \dots & x_{n-1} - 1 \\ x'_1 - 1 & \dots & x'_{n-1} - 1 \end{array}\right] = \begin{array}{l} k \\ k'. \end{array} . \tag{3}$$

We compare equations (2) and (3). For the corresponding inequalities, the only difference is that in the first case, there is an extra inequality $x < 0$. But this inequality is certainly false, because $x \in \mathbb{N}$. So in both cases, the number of true inequalities is the same.

Now let $x_1, \dots, x_n, x'_1, \dots, x'_n, k, k'$ be such that

$$\left[\begin{array}{c|c|c} x_1 & \dots & x_n \\ x'_1 & \dots & x'_n \end{array}\right] = \begin{array}{l} k \\ k'. \end{array} \tag{4}$$

Let $x \in \mathbb{N}$. Then

$$\left[\begin{array}{c|c|c} x_1 \oplus x & \dots & x_n \oplus x \\ x'_1 \oplus x & \dots & x'_n \oplus x \end{array}\right] = \begin{array}{l} k \oplus (x)_n \\ k' \oplus (x)_n \end{array} \tag{5}$$

is also true. We will show that the parity of the number of true inequalities is the same in both cases. In order to do so, we first note that for all $\ell \in \{x_1, \dots, x_n, k\}$ and all $x \in \mathbb{N}$, an even amount of the inequalities

$$\begin{aligned} \ell' &< \ell \\ \ell' \oplus x &< \ell \oplus x \\ \ell \oplus \ell' \oplus x &< x \end{aligned}$$

are true. This follows from Lemma 4.12, with the choice $n = 2$, $y_1 = \ell$, $y_2 = \ell \oplus x$ and $y = \ell \oplus \ell'$.

Now we can conclude that an even amount of

$$\begin{aligned} x'_1 &< x_1 \\ &\dots \\ x'_n &< x'_n \\ k' &< k \end{aligned}$$

are true if and only if an even amount of

$$\begin{aligned} x_1' \oplus x &< x_1 \oplus x \\ x_1 \oplus x_1' \oplus x &< x \\ &\dots \\ x_n' \oplus x &< x_n \oplus x \\ x_n \oplus x_n' \oplus x &< x \\ k' \oplus (x)_n &< k \oplus (x)_n \\ k \oplus k' \oplus (x)_n &< (x)_n \end{aligned}$$

are true. By Lemma 4.11, we find that this is true if and only if an even amount of

$$\begin{aligned} x_1' \oplus x &< x_1 \oplus x \\ &\dots \\ x_n' \oplus x &< x_n \oplus x \\ k' \oplus (x)_n &< k \oplus (x)_n \end{aligned}$$

are true. We conclude that for equations (4) and (5), the parity of the amount of true inequalities is equal.

Now let $x_1, \dots, x_n, x_1', \dots, x_n', k, k'$ be such that

$$\left[\begin{array}{c|c|c} x_1 & \dots & x_n \\ x_1' & \dots & x_n' \end{array} \right] = \begin{array}{l} k \\ k'. \end{array}$$

By what we proved above, the parity of the amount of corresponding true inequalities is the same as that for

$$\left[\begin{array}{l|c|c|c} 0 & x_2 \oplus x_1 & \dots & x_n \oplus x_1 \\ x_1' \oplus x_1 & x_2' \oplus x_1 & \dots & x_n' \oplus x_1 \end{array} \right] = \begin{array}{l} k \oplus (x_1)_n \\ k' \oplus (x_1)_n, \end{array}$$

which is the same as that for

$$\left[\begin{array}{c|c|c} x_2 \oplus x_1 - 1 & \dots & x_n \oplus x_1 - 1 \\ x_2' \oplus x_1 - 1 & \dots & x_n' \oplus x_1 - 1 \end{array} \right] = \begin{array}{l} k \oplus (x_1)_n \\ k' \oplus (x_1)_n. \end{array}$$

By the induction hypothesis, this is even.

□

Now we can prove Welter's Theorem.

Theorem 4.14 (Welter's Theorem)**.** *The function described in Chapter 3 is the Sprague-Grundy function for Welter's game.*

Proof. Let $x_1, \dots, x_n \in \mathbb{N}$. Note that $[\ x_1 \mid \cdots \mid x_n\ ] \in \mathbb{N}$. Let $k' \in \mathbb{N}$ with $k' < [\ x_1 \mid \cdots \mid x_n\ ]$. Then, by the Unique Prime Lemma, there exist $x_i' \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ for all $i \in \{1, \dots, n\}$ such that

$$[\ x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n\ ] = k'.$$

By Theorem 4.13, there exists an $i \in \{1, \ldots, n\}$ such that $x_i' < x_i$. Then $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ is a follower of $(x_1, \ldots, x_n)$.

Now let $x_i' \in \mathbb{N} \setminus \{x_1, \ldots, x_n\}$ with $x_i' < x_i$ for some $i \in \{1, \ldots, n\}$. Then, by the Unique Prime Lemma, we must have

$$[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] \neq [\, x_1 \mid \cdots \mid x_n \,].$$

Together, this proves that

$$\begin{aligned}[\, x_1 \mid \cdots \mid x_n \,] = \operatorname{mex}\{&[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] \\ &: x_i' < x_i, x_i' \notin \{x_1, \ldots, x_n\}, i \in \{1, \ldots, n\}\}.\end{aligned}$$

So the function is the Sprague-Grundy function for Welter's game.

□

Conway's version of the proof is similar to Welter's version. Welter also used Theorem 4.13, but gave a different proof of that theorem. While Conway's proof uses the properties of all four candidate functions, Welter's proof does not use the properties of the Animating candidate.

5 Table method

In this section we discuss the Table method, a method for determining a P-position using a table with previously known P-positions. This is based on a method used for analysing the game Antonim in [7], but can also be applied to Welter's game and to several other games. In Section 5.1 we describe how to apply the Table method to Welter's game. This also leads to a direct proof of the Unique Prime Lemma, starting from the definition of Welter's game. We have not encountered this in the literature. In Section 5.2 we discuss how to apply the Table method to some other games, including Nim.

5.1 Application to Welter's game

Let $x_1, \dots, x_n \in \mathbb{N}$ be distinct, and suppose that we know all the P-positions of the form $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, x)$ with $x \in \mathbb{N}$, $i \in \{1, \dots, n\}$ and $x_i' < x_i$. Then, by the following lemma, we can find a unique z such that $(x_1, \dots, x_n, z)$ is a P-position.

Lemma 5.1. *Let $x_1, \dots, x_n \in \mathbb{N}$ be distinct, and let*

$$\mathcal{A} = \{z \in \mathbb{N} : \exists i \in \{1, \dots, n\}, x_i' < x_i \text{ such that } (x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, z) \text{ is a P-position}\}.$$

Let $z = \text{mex}(\mathcal{A} \cup \{x_1, \dots, x_n\})$. Then $(x_1, \dots, x_n, z)$ is a P-position, and $(x_1, \dots, x_n, z')$ is an N-position for all $z' \neq z$.

Proof. For any distinct $x_1, \dots, x_n \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{N} \backslash \{x_1, \dots, x_n\}$ with $z_1 < z_2$, the position $(x_1, \dots, x_n, z_1)$ is a follower of $(x_1, \dots, x_n, z_2)$. This means that they cannot both be P-positions. So there is at most one $z' \in \mathbb{N}$ such that $(x_1, \dots, x_n, z')$ is a P-position. This also implies that $\mathcal{A}$ has at most $x_1 x_2 \dots x_n$ elements, so $z := \text{mex}(\mathcal{A} \cup \{x_1, \dots, x_n\})$ is well-defined.

To prove that $(x_1, \dots, x_n, z)$ is a P-position, we need to show that its followers are all N-positions. First we look at followers of the type $(x_1, \dots, x_n, z')$, with $z' < z$. Let $z' \in \mathbb{N}$ be such that $z' < z$. Then we have $z' \in \mathcal{A} \cup \{x_1 \dots, x_n\}$. If $z' \in \{x_1, \dots, x_n\}$, then $(x_1, \dots, x_n, z')$ is not a feasible position. Otherwise, we have $z' \in \mathcal{A}$. Then there exists a position of the type $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, z')$ with $x_i' < x_i$ that is a P-position. Such a position is a follower of $(x_1, \dots, x_n, z')$, so $(x_1, \dots, x_n, z')$ must be an N-position.

If a follower of $(x_1, \dots, x_n, z)$ of the type $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, z)$ for some $i \in \{1, \dots, n\}$ and $x_i' < x_i$ is a P-position, then $z \in \mathcal{A}$. This gives a contradiction. So all followers of this type are also N-positions.

□

Now we can prove directly from the definition, so without assuming knowledge of the relation between the Sprague-Grundy function and Welter's candidate, the Mating candidate, the Animating candidate and the Triangle candidate, that the Unique Prime Lemma must hold for Welter's game. As usual, we write $[\ x_1 \mid \cdots \mid x_n\ ]$ for the Sprague-Grundy value of position $(x_1, \dots, x_n)$.

Lemma 5.2. *Welter's game satisfies the conditions of the Unique Prime Lemma.*

Proof. We start by proving by induction on k that if $x_1, \dots, x_n \in \mathbb{N}$ are distinct and $k \in \mathbb{N}$, then there exists a unique $z \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ such that $[\, x_1 \mid \cdots \mid x_n \mid z \,] = k$.

For any distinct $x_1, \dots, x_n \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ such that $z_1 < z_2$, the position $(x_1, \dots, x_n, z_1)$ is a follower of $(x_1, \dots, x_n, z_2)$. This means they cannot have the same Sprague-Grundy value. So for any k, if a z such that $[\, x_1 \mid \cdots \mid x_n \mid z \,] = k$ exists, it is unique.

Let $x_1, \dots, x_n \in \mathbb{N}$ be distinct. For $k = 0$, the statement follows from Lemma 5.1. Now let $k \in \mathbb{N} \setminus \{0\}$ and assume that the statement holds for all $k' < k$. Note that it is enough to show that there exists a value z such that each follower of $(x_1, \dots, x_n, z)$ of the type $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, z)$ for some $i \in \{1, \dots, n\}$ and $x_i' < x_i$ has Sprague-Grundy value unequal to k, and such that for each $k' < k$, there exists a follower of $(x_1, \dots, x_n, z)$ with Sprague-Grundy value equal to k'. Then we have $[\, x_1 \mid \cdots \mid x_n \mid z \,] = k$ for the minimal z satisfying these requirements.

Let

$$\begin{aligned} \mathcal{A} = \{z \in \mathbb{N} : & \exists i \in \{1, \dots, n\}, x_i' < x_i \text{ such that} \\ & [\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \mid z \,] = k\}. \end{aligned}$$

For each $i \in \{1, \dots, n\}$ and $x_i' < x_i$, there is at most one z such that

$$[\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \mid z \,] = k,$$

so $\mathcal{A}$ is finite. Let z' be the maximal element of $\mathcal{A}$, if $\mathcal{A} \neq \emptyset$. Otherwise, let $z' = -1$. For all $x \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ with $x > z'$, no follower of the position $(x_1, \dots, x_n, x)$ of the type $(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n, x)$ has Sprague-Grundy value k.

For all $i < k$, let z_i be such that $[\, x_1 \mid \cdots \mid x_n \mid z_i \,] = i$. Such z_i exist by the induction hypothesis. Let $z'' = \max\{z_i : 0 \leq i < k\}$. Then for all $x \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ with $x > z''$ and for all $k' < k$, the position $(x_1, \dots, x_n, x)$ has a follower with Sprague-Grundy value k', namely the position $(x_1, \dots, x_n, z_{k'})$.

We can conclude that the z as described in the statement exists. This equals x_{n+1}' as described in the Unique Prime Lemma. Because Welter's game is symmetric, it follows that x_i' exists for all i. Further, assume that $x_i' = x_j'$ for some $i, j \in \{1, \dots, n+1\}$ with $i \neq j$. Then,

$$\begin{aligned} [\, x_1 \mid \cdots \mid x_{i-1} \mid x_i' \mid x_{i+1} \mid \cdots \mid x_n \,] &= [\, x_1 \mid \cdots \mid x_{j-1} \mid x_j' \mid x_{j+1} \mid \cdots \mid x_n \,] \\ &= [\, x_1 \mid \cdots \mid x_{j-1} \mid x_i' \mid x_{j+1} \mid \cdots \mid x_n \,], \end{aligned}$$

which, by Symmetry, implies that

$$\begin{aligned} & [\, x_1 \mid \cdots \mid x_{i-1} \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_{j+1} \mid \cdots \mid x_n \mid x_i' \mid x_j \,] \\ & = [\, x_1 \mid \cdots \mid x_{i-1} \mid x_{i+1} \mid \cdots \mid x_{j-1} \mid x_{j+1} \mid \cdots \mid x_n \mid x_i' \mid x_i \,]. \end{aligned}$$

Because $x_i \neq x_j$, one of the corresponding positions is a follower of the other, which gives a contradiction. So $x_i' \neq x_j'$ whenever $i \neq j$.

□

Using Lemma 5.1, we can fill tables with P-positions. We call this the Table method. Below we provide some examples.

Example 5.3. If we want to find the P-positions for Welter's game with 3 coins, we can fill a table where the first column gives x_1, the first row gives x_2, and the unique z such that (x_1, x_2, z) is a P-position can be read off the table. The elements on the diagonal are empty, as they do not correspond to feasible positions in Welter's game. By Lemma 5.1, each other position of the table must be filled with the minimal excludant of the row elements up to that point and the column elements up to that point, including the row and column headers x_1 and x_2. This gives Table 1.

	0	1	2	3	4	5	6	7	8	9	10
0		2	1	4	3	6	5	8	7	10	9
1	2		0	5	6	3	4	9	10	7	8
2	1	0		6	5	4	3	10	9	8	7
3	4	5	6		0	1	2	11	12	13	14
4	3	6	5	0		2	1	12	11	14	13
5	6	3	4	1	2		0	13	14	11	12
6	5	4	3	2	1	0		14	13	12	11
7	8	9	10	11	12	13	14		0	1	2
8	7	10	9	12	11	14	13	0		2	1
9	10	7	8	13	14	11	12	1	2		0
10	9	8	7	14	13	12	11	2	1	0	

Table 1: P-positions for Welter's game with 3 coins.

Example 5.4. If we want to find the P-positions for Welter's game with 4 coins, we can fill multiple tables where the element in the upper left gives the value of x_1, the first column gives x_2 and the first row gives x_3, and the unique x_4 such that (x_1, x_2, x_3, x_4) is a P-position can be read off the table. Then the elements that do not correspond to feasible positions are empty. The other positions of the tables are filled with the minimal excludant of the earlier elements in the same row or column of the same table, the row and column headers, the element in the upper left corner, and the elements in the same position in earlier tables. For $x_1 \in \{0, 1\}$ and $x_2, x_3 \in \{0, \ldots, 10\}$, this results in Tables 2 and 3.

From the above example, we see that when $x_1 \in \{0, 1\}$ and $x_2, x_3 \in \{0, \ldots, 10\}$, a position (x_1, x_2, x_3, x_4) is a P-position if and only if $x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$. In Section 6.3, we will discuss the fact that this is always true.

We can use a similar method to find the Sprague-Grundy values of each position, because the Sprague-Grundy value is also the minimal excludant of earlier values. In this case, the table elements that do not correspond with feasible positions again remain empty. All other elements of the table are filled with the minimal excludant of the row elements up to that point and the column elements up to that point, and the earlier elements in the same position in earlier tables.

Example 5.5. If we want to find the Sprague-Grundy values for Welter's game with 2 coins, we can fill a table where the first column gives x_1, the first row

0	0	1	2	3	4	5	6	7	8	9	10
0											
1			3	2	5	4	7	6	9	8	11
2		3		1	6	7	4	5	10	11	8
3		2	1		7	6	5	4	11	10	9
4		5	6	7		1	2	3	12	13	14
5		4	7	6	1		3	2	13	12	15
6		7	4	5	2	3		1	14	15	12
7		6	5	4	3	2	1		15	14	13
8		9	10	11	12	13	14	15		1	2
9		8	11	10	13	12	15	14	1		3
10		11	8	9	14	15	12	13	2	3	

Table 2: P-positions for Welter's game with 4 coins, with one coin on 0.

1	0	1	2	3	4	5	6	7	8	9	10
0			3	2	5	4	7	6	9	8	11
1											
2	3			0	7	6	5	4	11	10	9
3	2		0		6	7	4	5	10	11	8
4	5		7	6		0	3	2	13	12	15
5	4		6	7	0		2	3	12	13	14
6	7		5	4	3	2		0	15	14	13
7	6		4	5	2	3	0		14	15	12
8	9		11	10	13	12	15	14		0	3
9	8		10	11	12	13	14	15	0		2
10	11		9	8	15	14	13	12	3	2	

Table 3: P-positions for Welter's game with 4 coins, with one coin on 1.

gives x_2, and the Sprague-Grundy value of (x_1, x_2) can be read off the table. Then the diagonal is empty, and each other position of the table must be filled with the minimal excludant of the row elements up to that point and the column elements up to that point. When we do this, we get value $x_2 - 1$ at position $(0, x_2)$ and value $x_1 - 1$ at position $(x_1, 0)$ or all $x_1, x_2 \in \mathbb{N}$, as can be seen in Table 4. This means that after the first row and column, the table is a shifted version of Table 1 from Example 5.3.

	0	1	2	3	4	5	6	7	8	9	10
0		0	1	2	3	4	5	6	7	8	9
1	0		2	1	4	3	6	5	8	7	10
2	1	2		0	5	6	3	4	9	10	7
3	2	1	0		6	5	4	3	10	9	8
4	3	4	5	6		0	1	2	11	12	13
5	4	3	6	5	0		2	1	12	11	14
6	5	6	3	4	1	2		0	13	14	11
7	6	5	4	3	2	1	0		14	13	12
8	7	8	9	10	11	12	13	14		0	1
9	8	7	10	9	12	11	14	13	0		2
10	9	10	7	8	13	14	11	12	1	2	

Table 4: Sprague-Grundy values for Welter's game with 2 coins.

By combining Examples 5.3 and 5.5, we get the following result.

Lemma 5.6. *Let $x_1, x_2 \in \mathbb{N}$ be distinct. The unique z such that (x_1, x_2, z) is a P-position in Welter's game is equal to the Sprague-Grundy value of the position $(x_1 + 1, x_2 + 1)$, so to $(x_1 + 1) \oplus (x_2 + 1) - 1$.*

We will discuss a different proof of this lemma in Section 6.3.

Suppose we want to use the Table method to determine whether a given position $(x_1, \ldots, x_n, x_{n+1})$ is a P-position or an N-position. Then we need to know exactly which of the positions $(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n, z)$ for all $x'_i < x_i$ and $z \in \mathbb{N}$ are P-positions. So while the Table method provides some insight into the behaviour of the P-positions, it does not provide an efficient algorithm for determining which positions are P-positions.

5.2 Application to other games

In this section, we discuss three other games to which a version of the Table method can be applied, and which satisfy a version of the Unique Prime Lemma. We discuss the games Nim, Antonim and Antimatter.

Nim For the game Nim, which we discussed before in Section 2.3, we use a slightly different version of Lemma 5.1.

Lemma 5.7. *Let $x_1, \ldots, x_n \in \mathbb{N}$, and let*

$$\mathcal{A} = \{z \in \mathbb{N} : \exists i \in \{1, \ldots, n\}, x'_i < x_i \text{ such that } (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n, z) \text{ is a P-position}\}.$$

Let $z = \text{mex}(\mathcal{A})$*. Then* $(x_1, \dots, x_n, z)$ *is a P-position, and* $(x_1, \dots, x_n, z')$ *is an N-position for all* $z' \neq z$*.*

The proof is similar to that of Lemma 5.1. When we apply the Table method to Nim, no table elements are empty. Each position in the tables is filled with the minimal excludant of the earlier elements in the same row or column, and the earlier elements in the same position in previous tables. The row and column headers may be ignored while filling the tables. This leads to the following result, which also follows from Theorem 2.21.

Lemma 5.8. *Let* $x_1, \dots, x_n \in \mathbb{N}$*. The unique* z *such that* $(x_1, \dots, x_n, z)$ *is a P-position in Nim is equal to the Sprague-Grundy value of* $(x_1, \dots, x_n)$*.*

Using a similar proof to that of Lemma 5.2, we can prove the following result, which is the Unique Prime Lemma for the game Nim. Here, we write $[x_1, \dots, x_n]$ for the Sprague-Grundy value of the position $(x_1, \dots, x_n)$.

Lemma 5.9. *Let* $(x_1, \dots, x_n)$ *be a position in Nim. If* $[\ x_1 \mid \cdots \mid x_n\ ] = k$ *and* $k' \neq k$*, there exist unique* $x'_1, \dots, x'_n \in \mathbb{N}$*, with* $x'_i \neq x'_j$ *whenever* $i \neq j$*, such that*

$$[\ x'_1 \mid x_2 \mid \cdots \mid x_n\ ] = [\ x_1 \mid x'_2 \mid x_3 \mid \cdots \mid x_n\ ] = \cdots = [\ x_1 \mid \cdots \mid x_{n-1} \mid x'_n\ ] = k'.$$

Antonim Antonim is a variant of Nim. Antonim is played like Nim, except no two piles of coins may have the same size. It can also be seen as a variant of Welter's game in which multiple coins may be on the 0 square.

Definition 5.10 (Antonim)**.** Antonim is an impartial combinatorial game. The positions are $(x_1, \dots, x_n)$ where $x_i \neq x_j$ whenever $x_j \neq 0$. For such a position, we say that there are n coins on squares $x_1, \dots, x_n$.

The followers of a position $(x_1, \dots, x_n)$ are all positions

$$(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$$

with $i \in \{1, \dots, n\}$, $x'_i < x_i$ and $x'_i \in \mathbb{N} \setminus \{x_1, \dots, x_n\}$ or $x'_i = 0$.

We write $[\ x_1 \mid \cdots \mid x_n\ ]$ for the Sprague-Grundy value corresponding to position $(x_1, \dots, x_n)$.

For this game, we can prove the following, using a proof similar to that of Lemma 5.1. This was first proved in [7].

Lemma 5.11. *Let* $x_1, \dots, x_n \in \mathbb{N}$ *such that if* $x_i = x_j$ *and* $i \neq j$*, then* $x_i = x_j = 0$*. Let*

$$\begin{aligned}\mathcal{A} = \{z \in \mathbb{N} : & \exists i \in \{1, \dots, n\}, x'_i < x_i \text{ such that} \\ & (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n, z) \text{ is a P-position}\}.\end{aligned}$$

Let $z = \text{mex}(\mathcal{A} \cup (\{x_1, \dots, x_n\} \setminus \{0\}))$*. Then* $(x_1, \dots, x_n, z)$ *is a P-position, and* $(x_1, \dots, x_n, z')$ *is an N-position for all* $z' \neq z$*.*

This leads to the result that the Unique Prime Lemma also holds for Antonim, using a similar proof to that of Lemma 5.2.

Lemma 5.12. *Let $(x_1, \dots, x_n)$ be a position in Antonim. If $[\, x_1 \mid \cdots \mid x_n \,] = k$ and $k' \neq k$, there exist unique $x'_1, \dots, x'_n \in \mathbb{N}$, with $x'_i \neq x'_j$ whenever $i \neq j$, such that*

$$[\, x'_1 \mid x_2 \mid \cdots \mid x_n \,] = [\, x_1 \mid x'_2 \mid x_3 \mid \cdots \mid x_n \,] = \cdots = [\, x_1 \mid \cdots \mid x_{n-1} \mid x'_n \,] = k'.$$

Antimatter Another game that is similar to Welter's game is Antimatter. This game was defined by Fraenkel in [4]. It is played using two types of particles, positrons and electrons, instead of coins. Two particles of the same type may not be on the same square. If two particles of different types end up on the same square, both are annihilated.

Definition 5.13 (Antimatter)**.** Antimatter is an impartial combinatorial game. The positions are $(p_1, \dots, p_n|e_1, \dots, e_m)$ with distinct $p_1, \dots, p_n, e_1, \dots, e_m \in \mathbb{N}$. For such a position, we say that there are n positrons on squares $p_1, \dots, p_n$, and m electrons on squares $e_1, \dots, e_m$.

Positions $(p_1, \dots, p_n|e_1, \dots, e_m)$ with $p_i = e_j$ for some $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ are also feasible. In this case, we have

$$(p_1, \dots, p_n|e_1, \dots, e_m) = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n|e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_m),$$

and we say that the positron and the electron on square $p_i = e_j$ are annihilated.

The followers of a position $(p_1, \dots, p_n|e_1, \dots, e_m)$ are all positions

$$(p_1, \dots, p_{i-1}, p'_i, p_{i+1}, \dots, p_n|e_1, \dots, e_m)$$

with $i \in \{1, \dots, n\}$, $p'_i < p_i$ and $p'_i \in \mathbb{N} \setminus \{p_1, \dots, p_n, e_1, \dots, e_m\}$ and

$$(p_1, \dots, p_n|e_1, \dots, e_{i-1}, e'_i, e_{i+1}, \dots, e_m)$$

with $i \in \{1, \dots, m\}$, $e'_i < e_i$ and $e'_i \in \mathbb{N} \setminus \{p_1, \dots, p_n, e_1, \dots, e_m\}$. In addition, the positions

$$(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n|e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_m)$$

with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ are followers.

We write $[\, p_1 \mid \cdots \mid p_n \parallel e_1 \mid \cdots \mid e_m \,]$ for the Sprague-Grundy value corresponding to position $(p_1, \dots, p_n|e_1, \dots, e_m)$.

For this game, we can prove the following variant of Lemma 5.1.

Lemma 5.14. *Let $p_1, \dots, p_n \in \mathbb{N}$ and $e_1, \dots, e_m \in \mathbb{N}$ be such that the position $(p_1, \dots, p_n|e_1, \dots, e_m)$ is feasible in Antimatter. Let*

$$\begin{aligned}\mathcal{A} = \{z \in \mathbb{N} :\, & \exists i \in \{1, \dots, n\}, p'_i < p_i \text{ such that} \\ & (p_1, \dots, p_{i-1}, p'_i, p_{i+1}, \dots, p_n|e_1, \dots, e_m, z) \text{ is a P-position} \\ & \text{or } \exists j \in \{1, \dots, m\}, e'_j < e_j \text{ such that} \\ & (p_1, \dots, p_n|e_1, \dots, e_{j-1}, e'_j, e_{j+1}, \dots, e_m, z) \text{ is a P-position.} \\ & \text{or } \exists i \in \{1, \dots, n\}, j \in \{1, \dots, m\} \text{ such that} \\ & (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n \mid e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_m, z) \\ & \text{is a P-position.}\}\end{aligned}$$

Let $X \subseteq \{e_1, \ldots, e_m\}$ be the set of electrons that are not annihilated in the position $(p_1, \ldots, p_n|e_1, \ldots, e_m)$, and let $z = \text{mex}(\mathcal{A} \cup X)$. Then the position $(p_1, \ldots, p_n|e_1, \ldots, e_m, z)$ is a P-position, and $(p_1, \ldots, p_n|e_1, \ldots, e_m, z')$ is an N-position for all $z' \neq z$.

In the above lemma, we can interchange the positrons and electrons to find that there is also a unique z such that $(p_1, \ldots, p_n, z|e_1, \ldots, e_m)$ is a P-position, for any $p_1, \ldots, p_n, e_1, \ldots, e_m$ such that $(p_1, \ldots, p_n|e_1, \ldots, e_n)$ is a feasible position.

Because of annihilation, it is possible that $(p_1, \ldots, p_n|e_1, \ldots, e_m, e_{m+1})$ is a feasible position, but $(p_1, \ldots, p_n|e_1, \ldots, e_m)$ is not. This happens only when the position $(p_1, \ldots, p_n|e_1, \ldots, e_m, e_{m+1})$ has multiple positrons on position e_{m+1}, of which all but one are annihilated. In this case, $(p_1, \ldots, p_n|e_1, \ldots, e_m, z)$ is only a feasible position when $z = e_{m+1}$, as otherwise two positrons that are not annihilated remain on the same position. So for these positions, it is not true that for all $k' \neq [\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m \mid e_{m+1}\ ]$ there exists a unique e'_{m+1} such that $[\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m \mid e'_{m+1}\ ] = k'$. So the Unique Prime Lemma can only hold for positions such that, if any one particle is removed, the resulting position is still feasible. This is certainly true for positions which do not contain any annihilated particles. It is possible to prove the following, using a similar proof to that of Lemma 5.2.

Lemma 5.15. *Let $p_1, \ldots, p_n \in \mathbb{N}$ and $e_1, \ldots, e_m \in \mathbb{N}$ be such that the position $(p_1, \ldots, p_n|e_1, \ldots, e_m)$ is feasible in Antimatter, and contains no annihilated particles. If $[\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ] = k$ and $k' \neq k$, there exist unique $p'_1, \ldots, p'_n \in \mathbb{N}$ and $e'_1, \ldots, e'_m \in \mathbb{N}$ such that*

$$\begin{aligned}
&[\ p'_1 \mid p_2 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ] = [\ p_1 \mid p'_2 \mid p_3 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ] = \ldots \\
&= [\ p_1 \mid \cdots \mid p_{n-1} \mid p'_n \mid\mid e_1 \mid \cdots \mid e_m\ ] = [\ p_1 \mid \cdots \mid p_n \mid\mid e'_1 \mid e_2 \mid \cdots \mid e_m\ ] = \ldots \\
&= [\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_{m-1} \mid e'_m\ ] = k'.
\end{aligned}$$

In order to apply the Table method to Antimatter, we need to look separately at the positions that are not feasible, before filling in the other positions in the tables. This is because if $(p_1, \ldots, p_n|e_1, \ldots, e_m)$ is not a feasible position, a z satisfying the requirements of Lemma 5.14 may still exist. If one does exist, it must be unique because in this case there is at most one z such that $(p_1, \ldots, p_n|e_1, \ldots, e_m, z)$ becomes a feasible position.

Example 5.16. Take $n = 2$, $m = 1$ and $p_1 = p_2$. Then (p_1, p_2) is not a feasible position. There is only one feasible position of the form $(p_1, p_2|z)$, that is the one with $z = p_1 = p_2$. In that case one of the positrons is annihilated, so we get $(p_1, p_2|z) = (p_1)$, and this is a P-position if and only if $p_1 = 0$.

Example 5.17. If we want to find the P-positions for Antimatter with 2 positrons and one electron, we can fill a table where the first column gives p_1, the first row gives p_2, and the unique e such that $(p_1, p_2|e)$ is a P-position can be read off the table. We first look at the situations where the two positrons are in the same position. In Example 5.16 we found that $(0, 0|0)$ is a P-position, and there are no other P-positions where the two positrons are on the same square. Each other position in the table must be filled with the minimal excludant of the row elements up to that point and the column elements up to that point. This leads to Table 5.

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1		0	2	3	4	5	6	7	8	9
2	2	0		1	5	3	4	8	6	7	11
3	3	2	1		0	6	7	4	5	10	8
4	4	3	5	0		1	2	9	10	6	7
5	5	4	3	6	1		0	2	9	11	12
6	6	5	4	7	2	0		1	3	12	13
7	7	6	8	4	9	2	1		0	3	5
8	8	7	6	5	10	9	3	0		1	2
9	9	8	7	10	6	11	12	3	1		0
10	10	9	11	8	7	12	13	5	2	0	

Table 5: P-positions for Antimatter with 2 positrons and 1 electron.

Example 5.18. If we want to find the P-positions for Antimatter with 2 positrons and 2 electrons, we can fill multiple tables where the element in the upper left gives the value of e_1, the first column gives p_1, the first row gives p_2, and the unique e_2 such that $(p_1, p_2|e_1, e_2)$ is a P-position can be read off the table. We again have to look separately at the case $p_1 = p_2$, as in this case the position $(p_1, p_2|e_1)$ might not be feasible. If $e_1 = p_1 = p_2$, then $(p_1, p_2|e_1)$ is a feasible position. If $e_1 \neq p_1 = p_2$, then $(p_1, p_2|e_1)$ is not feasible. Then the position $(p_1, p_2|e_1, e_2)$ is only feasible for $e_2 = p_1 = p_2$. Then $(p_1, p_2|e_1, e_2) = (p_1|e_1)$, which is a P-position if $e_1 = p_1$. So by Theorem 5.14, it is not a P-position when $e_1 \neq p_1$. So if $e_1 \neq p_1 = p_2$, there is no e_2 such that $(p_1, p_2|e_1, e_2)$ is a feasible position.

The rest of the positions can be filled in with the minimal excludant of the earlier elements in the same row or column of the same table, the elements in the same position in earlier tables, and the element e_1 in the upper left corner, if that element is not equal to the row header or the column header. For $p_1, p_2 \in \{1, \ldots, 10\}$ and $e_1 \in \{1, 2\}$, using the method above results in Tables 6 and 7.

0	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1		3	2	5	4	7	6	9	8	11
2	2	3		1	6	7	4	5	10	11	8
3	3	2	1		7	6	5	4	11	10	9
4	4	5	6	7		1	2	3	12	13	14
5	5	4	7	6	1		3	2	13	12	15
6	6	7	4	5	2	3		1	14	15	12
7	7	6	5	4	3	2	1		15	14	13
8	8	9	10	11	12	13	14	15		1	2
9	9	8	11	10	13	12	15	14	1		3
10	10	11	8	9	14	15	12	13	2	3	

Table 6: P-positions for Antimatter with 2 positrons and 2 electrons, with one electron on 0.

1	0	1	2	3	4	5	6	7	8	9	10
0		0	3	2	5	4	7	6	9	8	11
1	0	1	2	3	4	5	6	7	8	9	10
2	3	2		0	7	6	5	4	11	10	9
3	2	3	0		6	7	4	5	10	11	8
4	5	4	7	6		0	3	2	13	12	15
5	4	5	6	7	0		2	3	12	13	14
6	7	6	5	4	3	2		0	15	14	13
7	6	7	4	5	2	3	0		14	15	12
8	9	8	11	10	13	12	15	14		0	3
9	8	9	10	11	12	13	14	15	0		2
10	11	10	9	8	15	14	13	12	3	2	

Table 7: P-positions for Antimatter with 2 positrons and 2 electrons, with one electron on 1.

In Section 4.2, we mentioned that the Even Alteration Theorem does not hold for all games that are similar to Welter's game. In particular, it does not hold for Antonim or for Antimatter. For example, some Sprague-Grundy values for Antimatter are

$$\begin{aligned}
[\,6 \mid 7 \mid 9 \mid\mid 3 \mid 4\,] &= 16, \\
[\,6 \mid 7 \mid \mathbf{8} \mid\mid 3 \mid 4\,] &= 2, \\
[\,6 \mid 7 \mid 9 \mid\mid \mathbf{0} \mid 4\,] &= 2, \\
[\,6 \mid 7 \mid \mathbf{8} \mid\mid \mathbf{0} \mid 4\,] &= 13.
\end{aligned}$$

So when $(p_1, p_2, p_3|e_1, e_2) = (6, 7, 9|3, 4)$ and $k' = 2$, we have $p_3' = 8$ and $e_1' = 0$. But in this case,

$$[\,p_1 \mid p_2 \mid p_3' \mid\mid e_1' \mid e_2\,] \neq [\,p_1 \mid p_2 \mid p_3 \mid\mid e_1 \mid e_2\,].$$

With the same values for p_1, p_2, p_3, e_1, e_2 and k', we have $p_1' = 2$. In this case,

$$[\,p_1' \mid p_2 \mid p_3' \mid\mid e_1 \mid e_2\,] \neq [\,p_1 \mid p_2 \mid p_3 \mid\mid e_1 \mid e_2\,],$$

so exchanging two particles of the same type with their primed counterparts also does not give the same Sprague-Grundy value. For Antonim, we get a counterexample when $(x_1, x_2, x_3, x_4, x_5) = (5, 6, 7, 8, 9)$, $k' = 7$, $x_2' = 1$ and $x_4' = 4$.

6 Further properties of Welter's game

We will discuss a few more properties of Welter's game. In Section 6.1, we discuss a method for finding a move to a P-position, which uses the properties of the Triangle candidate. In Section 6.2, we prove that the Sprague-Grundy value of a position is congruent to 0 modulo 16 if and only if the nim-sum of the position is congruent to 0 modulo 16, if the amount of coins is a multiple of 4. We discuss how this property can be used to find a move to a P-position. We also discuss variants modulo other powers of 2, and where the amount of coins is not necessarily a multiple of 4. Next, in Section 6.3, we discuss the solution to Welter's game played with at most 4 coins. In this case, Welter's game turns out to be very similar to Nim. Finally, in Section 6.4 we discuss the solution to Welter's game if there are 5 coins and they are all on the positions $\{0, \dots, 15\}$.

6.1 Using the Triangle candidate

In Section 3.2, we discussed that the Sprague-Grundy value of a given position can be found by completing a triangle of numbers. Here, we prove this result and discuss how to apply this method to find a move to a P-position. This is based on Chapter 15 of [1].

Lemma 6.1. *If we fill a triangle as described in Section 3.2, we get the Sprague-Grundy value* $[x_1, \dots, x_n]$ *at the bottom position.*

Proof. We prove this by induction, using the properties of the Triangle candidate and the Symmetry property. If $n = 1$, we get

$$\begin{array}{ccc} 0 & & 0 \\ & x_1 & \end{array}$$

and $[x_1] = x_1$. Now let $n > 1$ and assume that the method works for all $m < n$. Let

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

be the bottom diamond. Then $(a \oplus d) = (b \oplus c) + 1$. We need to prove that $c = [\, x_1 \mid \cdots \mid x_n \,]$. By the induction hypothesis, we have $b = [\, x_2 \mid \cdots \mid x_{n-1} \,]$, $a = [\, x_1 \mid \cdots \mid x_{n-1} \,]$ and $d = [\, x_2 \mid \cdots \mid x_n \,]$. So, using the properties of the Triangle candidate and the Symmetry property, we find that

$$\begin{aligned} c &= ((a \oplus d) - 1) \oplus b \\ &= (([\, x_1 \mid \cdots \mid x_{n-1} \,] \oplus [\, x_2 \mid \cdots \mid x_n \,]) - 1) \oplus [\, x_2 \mid \cdots \mid x_{n-1} \,] \\ &= [\, [\, x_1 \mid \cdots \mid x_{n-1} \,] \mid [\, x_2 \mid \cdots \mid x_n \,] \,] \oplus [\, x_2 \mid \cdots \mid x_{n-1} \,] = [\, x_1 \mid \cdots \mid x_n \,]. \end{aligned}$$

□

Now, assume that $[\, x_1 \mid \cdots \mid x_n \,] = k$, and let $k' \in \mathbb{N}$. Suppose we want to find $x'_1, \dots, x'_n$ such that

$$\left[\begin{array}{c|c|c} x_1 & \dots & x_n \\ x'_1 & \dots & x'_n \end{array}\right] = \begin{array}{l} k \\ k'. \end{array}$$

We can do this by filling a pattern in the same way as before, but starting with different rows, as follows.

- The first row should now have $2n + 1$ 0s, with an empty space between each pair.

- The second row should have $x_1, \ldots, x_n$ to the right and below the first n 0s, so that each number is diagonally below two 0s.

- The $(n+1)$th row should have $n+1$ numbers, alternating between the numbers k and k'. Between each number should be an empty space, and the first number should be below $x_{\frac{n+1}{2}}$ if that number exists, and in the column between the one with $x_{\frac{n}{2}}$ and the one with $x_{\frac{n+2}{2}}$ otherwise.

By completing diamonds, we can eventually fill in the second row. This row then has the numbers $x_1, \ldots, x_n, x'_1, \ldots, x'_n$, as we prove below.

Example 6.2. Suppose we want to find a move to a P-position from $(1, 5, 7)$. We use the following pattern, where the numbers in bold are the ones we start with.

$$\begin{array}{ccccccccccccc}
\mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} \\
& \mathbf{1} & & \mathbf{5} & & \mathbf{7} & & 13 & & 9 & & 3 & \\
& & 3 & & 1 & & 9 & & 3 & & 9 & & \\
& & & \mathbf{4} & & \mathbf{0} & & \mathbf{4} & & \mathbf{0} & & &
\end{array}$$

We conclude that

$$\left[\begin{array}{c|c|c} 1 & 5 & 7 \\ 13 & 9 & 3 \end{array}\right] = \begin{array}{l} 4 \\ 0. \end{array}$$

This means that the only feasible way to move to a P-position is by moving the coin on square 7 to square 3.

Lemma 6.3. *If we fill a table as described above, we get $x'_1, \ldots, x'_n$ in the positions in the second row below and between the last $n + 1$ 0s.*

Proof. Call the numbers in the described final positions $y_1, \ldots, y_n$. We will use induction on i to show that $y_i = x'_i$ for all $i \in \{1, \ldots, n\}$. By Lemma 6.1, we have

$$[\, y_1 \mid x_2 \mid \cdots \mid x_n \,] = [\, x_2 \mid \cdots \mid x_n \mid y_1 \,] = k'.$$

So $y_1 = x'_1$ by the Unique Prime Lemma. Now let $i \in \{1, \ldots, n-1\}$ and assume that $y_1 = x'_1, \ldots, y_{i-1} = x'_{i-1}$. Then we find that

$$[\, x_{i+1} \mid \cdots \mid x_n \mid x'_1 \mid \cdots \mid x'_{i-1} \mid y_i \,] = \begin{cases} k & \text{if } i \text{ is even,} \\ k' & \text{if } i \text{ is odd.} \end{cases}$$

If i is even, we can use Symmetry and the Even Alteration Theorem to find that

$$[\, x_1 \mid \cdots \mid x_{i-2} \mid x'_{i-1} \mid y_i \mid x_{i+1} \mid \cdots \mid x_n \,] = k.$$

By the Unique Prime Lemma and the Even Alteration Theorem, we conclude that $y_i = x'_i$. If i is odd, we find by using Symmetry and the Even Alteration Theorem that

$$[\, x_1 \mid \cdots \mid x_{i-1} \mid y_i \mid x_{i+1} \mid \cdots \mid x_n \,] = k'.$$

By the Unique Prime Lemma, it follows that $y_i = x'_i$.

□

6.2 Congruence modulo 16

In this section, we will discuss an interesting property of Welter's game. If there are a multiple of four coins, then the nim-sum of a position is congruent to 0 modulo 16 if and only if the Sprague-Grundy value of the position is congruent to 0 modulo 16. This property can be used to find a move to a P-position. We also discuss variants modulo other powers of 2, and the situation when the amount of coins is not a multiple of 4.

$4k$ **coins** We start by discussing the case where the amount of coins is a multiple of 4. Before we prove that the nim-sum of a position is congruent to 0 modulo 16 if and only if the Sprague-Grundy value is congruent to 0 modulo 16, we first show that the equivalence holds modulo 8. The proof we use comes from Section 2.4 of [5]. We will also show that the equivalence holds modulo 2 and modulo 4. We first need the following lemma.

Lemma 6.4. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \dots, x_{4k} \in \mathbb{N}$ be distinct. Suppose that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2^n$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2^n$, for any $n \in \mathbb{N} \setminus \{0\}$. Then each pair of mates is congruent modulo 2, and possibly modulo a higher power of 2.*

Proof. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \dots, x_{4k}\}$, et cetera. Then, by the properties of the Mating candidate, we have

$$[\, x_1 \mid \cdots \mid x_{4k} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,].$$

First, we show that there can be at most one pair of mates that is congruent modulo $2^0 = 1$, but not modulo a higher power of 2. Suppose there are two such pairs. Then exactly one of x_{4k-1} and x_{4k} is odd, and exactly one of x_{4k-3} and x_{4k-2} is odd. These odd numbers are congruent to each other modulo 2, which gives a contradiction.

Now assume that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2^n$. Suppose that there is a pair of mates that is not congruent modulo 2, so only modulo $2^0 = 1$. Then the binary expansion of $[\, x_{4k-1} \mid x_{4k} \,]$ ends in 0, and the binary expansions of $[\, x_1 \mid x_2 \,]$, $\dots$, $[\, x_{4k-3} \mid x_{4k-2} \,]$ all end in 1. So there are $2k - 1$ pairs of mates (x_i, x_j) for which the binary expansion of $[\, x_i \mid x_j \,]$ ends in 1. This means that the binary expansion of $[\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,]$ ends in 1. Then $[\, x_1 \mid \cdots \mid x_{4k} \,] \not\equiv 0 \mod 2^n$, which gives a contradiction.

Next, suppose that $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2^n$, and that there is a pair of mates that is congruent modulo $2^0 = 1$ but not modulo a higher power of 2. Then the binary expansions of $x_1 \oplus x_2, \dots, x_{4k-3} \oplus x_{4k-2}$ end in 0, while the binary expansion of $x_{4k-1} \oplus x_{4k}$ ends in 1. This means that the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ ends in 1, so that $x_1 \oplus \cdots \oplus x_{4k} \not\equiv 0 \mod 2^n$. So in this case we also get a contradiction.

□

Using the lemma above, we can show that the equivalence holds modulo 2.

Lemma 6.5. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \dots, x_{4k} \in \mathbb{N}$ be distinct. Then we have $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2$ if and only if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2$.*

Proof. Assume that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2$. In both cases, all pairs of mates are congruent modulo 2 by Lemma 6.4. So whenever (x_i, x_j) is a pair of mates, the last digit of the binary expansion $x_i \oplus x_j$ equals 0, and the last digit of the binary expansion of $[\, x_i \mid x_j \,]$ equals 1. It follows that the last digit of $x_1 \oplus \cdots \oplus x_{4k}$ equals 0. Further, using the properties of the Mating candidate and the fact that the amount of pairs is even, it follows that the last digit of $[\, x_1 \mid \cdots \mid x_{4k} \,]$ equals 0. So if $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2$, then both $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2$ and $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 2$.

□

We now show that the equivalence holds modulo 8.

Lemma 6.6. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \dots, x_{4k} \in \mathbb{N}$ be distinct. Then we have $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 8$ if and only if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 8$.*

Proof. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \dots, x_{4k}\}$, et cetera. Then, by the properties of the Mating candidate, we have

$$[\, x_1 \mid \cdots \mid x_{4k} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,].$$

Note that by Lemma 6.4, in both cases, all mates are congruent to each other modulo 2. This means that for any pair of mates (x_i, x_j), the binary expansion of $x_i \oplus x_j$ can end in 000, 010, 100 or 110. Say there are w pairs of the first type, x pairs of the second type, y of the third type and z of the last type. Since $[\, x_1 \mid x_2 \,] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, this means that there are w pairs for which the binary expansion of $[\, x_i \mid x_j \,]$ ends in 111, x types for which it ends in 001, y pairs for which it ends in 011 and z for which it ends in 101. This is summarised below.

	$x_i \oplus x_j$	$[\, x_i \mid x_j \,]$
w	000	111
x	010	001
y	100	011
z	110	101

Assume that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 8$. Then

$$[\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,] \equiv 0 \mod 8,$$

so the last 3 digits of its binary expansion are 0s. The third last digit is 0 if and only if $w + z$ is even, the second last digit is 0 if and only if $w + y$ is even, and the last digit is 0 if and only if $w + x + y + z$ is even. Since $w + z$ and $w + y$ are even, $w + z + w + y$ is also even, so $y + z$ is even. Because $w + y$ and $w + x + y + z$ are even, $x + z$ is even. Since $y + z$ and $x + z$ are even, the binary expansion of $x_1 \oplus x_2 \oplus \cdots \oplus x_{4k-1} \oplus x_{4k}$ must end in 000. It follows that $x_1 \oplus x_2 \oplus \cdots \oplus x_{4k-1} \oplus x_{4k} \equiv 0 \mod 8$.

Next, assume that $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 8$. Then, because the last 3 digits of its binary expansion are 0s, $y + z$ and $x + z$ are even. There is an even amount of pairs, so $w + x + y + z$ is also even. Then because $x + z$ is even, so is $w + y$. Because $w + y$ and $y + z$ is even, $w + y + y + z$ is also even, so $w + z$ is even. Since $w + z$, $w + y$ and $w + x + y + z$ are even, we can conclude that the binary expansion of $[\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ]$ ends in 000. It follows that $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 8$.

□

The same method as in the proof of Lemma 6.6 can be used to prove the equivalence modulo 4. In that case, the corresponding table is the following.

	$x_i \oplus x_j$	$[\ x_i \mid x_j\ ]$
x	00	11
y	10	01

This leads to the following result.

Lemma 6.7. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \ldots, x_{4k} \in \mathbb{N}$ be distinct. Then we have $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 4$ if and only if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 4$.*

For the congruence modulo 16, the same method cannot be used, as we show below.

Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \ldots, x_{4k} \in \mathbb{N}$ be distinct. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k}\}$, et cetera. Assume that $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$. Then by Lemma 6.4, the nim-sum of each pair of mates has a binary expansion that ends in 0000, 0010, 0100, 0110, 1000, 1010, 1100 or 1110. Let us assume the amount of pairs of each type is $n_1, n_2, \ldots, n_8$, respectively. The corresponding Sprague-Grundy values end in 1111, 0001, 0011, 0101, 0111, 1001, 1011 and 1101, respectively. This is summarised below.

	$x_i \oplus x_j$	$[\ x_i \mid x_j\ ]$
n_1	0000	1111
n_2	0010	0001
n_3	0100	0011
n_4	0110	0101
n_5	1000	0111
n_6	1010	1001
n_7	1100	1011
n_8	1110	1101

If $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$, so that the last 4 digits of the binary expansion of $[\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ]$ are 0s, then $n_1 + n_6 + n_7 + n_8$, $n_1 + n_4 + n_5 + n_8$, $n_1 + n_3 + n_5 + n_7$ and $n_1 + \cdots + n_8$ are even. If we take two such sums, add them together, and then remove the numbers that appear twice, we get another sum that must be even. Repeating this process leads to a list of subsets of $\{n_1, \ldots, n_8\}$ that have an even sum. These can be found in Table 8. Here, each row contains one of the subsets.

n_1					n_6	n_7	n_8
n_1			n_4	n_5			n_8
n_1		n_3		n_5		n_7	
n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
			n_4	n_5	n_6	n_7	
		n_3		n_5	n_6		n_8
	n_2	n_3	n_4	n_5			
		$\mathbf{n_3}$	$\mathbf{n_4}$			$\mathbf{n_7}$	$\mathbf{n_8}$
	n_2	n_3			n_6	n_7	
	$\mathbf{n_2}$		$\mathbf{n_4}$		$\mathbf{n_6}$		$\mathbf{n_8}$
n_1		n_3	n_4		n_6		
n_1	n_2	n_3					n_8
n_1	n_2		n_4			n_7	
n_1	n_2			n_5	n_6		
	n_2			n_5		n_7	n_8

Table 8: Subsets of $\{n_1, \ldots, n_8\}$ which have an even sum if $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 16$. Each row contains one of the subsets.

Using this method, we can show that $n_3 + n_4 + n_7 + n_8$ and $n_2 + n_4 + n_6 + n_8$ are even. However we cannot prove that $n_5 + n_6 + n_7 + n_8$ is even. This means that we cannot use this method to show that the fourth last digit in the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ is 0. So we cannot use this method to prove that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 16$ implies $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$.

Now assume that $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$, so that the last 4 digits of its binary expansion are 0s. Then $n_5+n_6+n_7+n_8$, $n_3+n_4+n_7+n_8$ and $n_2+n_4+n_6+n_8$ are even. $n_1 + \cdots + n_8$ is also even, because there is an even number of pairs. Using the method described above, we can show that the subsets of $\{n_1, \ldots, n_8\}$ in Table 9 have an even sum. Here, each row contains one of the subsets.

Using this method, we can prove that $n_1 + n_4 + n_5 + n_8$ and $n_1 + n_3 + n_5 + n_7$ are even. However, we cannot prove that $n_1 + n_6 + n_7 + n_8$ is even. So this method is not enough to show that the fourth last digit of the binary expansion of $[\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,]$ is a 0. So we cannot use this method to prove that $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$ implies $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 16$.

So, we use a different method to prove the equivalence modulo 16. This result was also mentioned in [5].

Theorem 6.8. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \ldots, x_{4k} \in \mathbb{N}$ be distinct. Then we have $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 16$ if and only if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$.*

Proof. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k}\}$, et cetera. Then, by the properties of the Mating candidate, we have

$$[\, x_1 \mid \cdots \mid x_{4k} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,].$$

We define $n_1, \ldots, n_8$ as before. Assume that $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 16$. Then we also have $[\, x_1 \mid \cdots \mid x_{4k} \,] \equiv 0 \mod 8$. By Lemma 6.6, it then follows that

				n_5	n_6	n_7	n_8
		n_3	n_4			n_7	n_8
	n_2		n_4		n_6		n_8
n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
		n_3	n_4	n_5	n_6		
	n_2		n_4	n_5		n_7	
n_1	n_2	n_3	n_4				
	n_2	n_3			n_6	n_7	
n_1	n_2			n_5	n_6		
$\mathbf{n_1}$		$\mathbf{n_3}$		$\mathbf{n_5}$		$\mathbf{n_7}$	
	n_2	n_3		n_5			n_8
n_1	n_2					n_7	n_8
n_1		n_3			n_6		n_8
$\mathbf{n_1}$			$\mathbf{n_4}$	$\mathbf{n_5}$			$\mathbf{n_8}$
n_1			n_4		n_6	n_7	

Table 9: Subsets of $\{n_1, \dots, n_8\}$ which have an even sum if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$. Each row contains one of the subsets.

$x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 8$. Similarly, if we have $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$, then also $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 8$. This implies $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 8$ by Lemma 6.6. So in both cases, the last 3 digits of the binary expansion are equal, and we only need to show that they are equal on the fourth last position.

Let (x_i, x_j) be a pair of mates. We look at the value in the fourth last position in the binary expansions of $x_i \oplus x_j$ and of $[\ x_i \mid x_j\ ]$. Both numbers have the same value in this position except when the binary expansion of $x_i \oplus x_j$ ends in 0000 and the binary expansion of $[\ x_i \mid x_j\ ]$ ends in 1111, which happens n_1 times, or when the binary expansion of $x_i \oplus x_j$ ends in 1000 and that of $[\ x_i \mid x_j\ ]$ ends in 0111, which happens n_5 times. It follows that if n_1 and n_5 have the same parity, the binary expansions of $[\ x_1 \mid \cdots \mid x_{4k}\ ]$ and $x_1 \oplus \cdots \oplus x_{4k}$ are the same in the fourth last position.

So it is enough to show that $n_1 + n_5$ is even whenever $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$. This means there are an even amount of pairs of mates for which the binary expansion of the nim-sum ends in 000. As in the proof of Lemma 6.6, we call w the number of pairs of mates for which the nim-sum ends in 000, x the amount for which it ends in 010, y the amount for which it ends in 100, and z the amount for which it ends in 110. The corresponding Sprague-Grundy values have binary expansions ending in 111, 001, 011 and 101, respectively. Then $w = n_1 + n_5$, so we need to show that w is even.

Assume that $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$ or $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$, and that w is odd. By Lemma 6.6, we have $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 8$ and $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 8$, so the last 3 digits of the binary expansions of $x_1 \oplus \cdots \oplus x_{4k}$ and of $[\ x_1 \mid \cdots \mid x_{4k}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [x_{4k-1} \mid x_{4k}\ ]$ are 0s. It follows that $y + z$, $x + z$, $w + z$, $w + y$ and $w + x + y + z$ are even. Then $w + x = w + z + x + z$ is also even. Since $w + x$, $w + y$ and $w + z$ are even and w is odd, x, y and z are also odd. In particular, this means that x, y and z are all at least 1. Let (x_1, x_2), (y_1, y_2), and (z_1, z_2) be pairs of mates with nim-sum ending in 010, 100 and 110, respectively. Note that both pairs of numbers (x_1, x_2) and (z_1, z_2) are congruent modulo 2, but not modulo 4. This means that for $i, j \in \{1, 2\}$, x_i and z_j cannot be congruent modulo 4, as otherwise x_i and z_j would form a pair of mates. Similarly, no two of the numbers can be congruent modulo 8.

Let $i \in \{0, \ldots, 7\}$ be such that $z_1 \equiv i \mod 8$. Then $z_2 \equiv (i \oplus 6) \mod 8$. It follows that $x_1 \not\equiv i, i \oplus 6 \mod 8$, because otherwise x_1 would be congruent to z_1 or z_2 modulo 8. Similarly, $x_2 \not\equiv i, i \oplus 6 \mod 8$. We also have $x_1 \not\equiv i \oplus 4, i \oplus 6 \oplus 4 \mod 8$, because otherwise x_1 would be congruent modulo 4 with z_1 or z_2. Similarly, $x_2 \not\equiv i \oplus 4, i \oplus 6 \oplus 4 \mod 8$. So we have

$$x_1, x_2 \not\equiv i, i \oplus 2, i \oplus 4, i \oplus 6 \mod 8.$$

This means that if z_1 and z_2 are even then x_1 and x_2 are odd, and if z_1 and z_2 are odd then x_1 and x_2 are even.

Now let $j \in \{0, \ldots, 7\}$ be such that $x_1 \equiv j \mod 8$. Then $x_2 \equiv j \oplus 2 \mod 8$. We must have $y_1, y_2 \not\equiv j, j \oplus 2, i, i \oplus 6 \mod 8$, as otherwise y_1 or y_2 would be congruent to one of x_1, x_2, z_1, z_2 modulo 8. Because $y_2 \equiv y_1 \oplus 4 \mod 8$, we also have $y_1 \not\equiv i \oplus 4, i \oplus 6 \oplus 4, j \oplus 4, j \oplus 2 \oplus 4 \mod 8$. This means that $y_1 \not\equiv i \oplus 4, i \oplus 2, j \oplus 4, j \oplus 6 \mod 8$. Combined with the previous result, we now have

$$y_1 \not\equiv i, i \oplus 2, i \oplus 4, i \oplus 6, j, j \oplus 2, j \oplus 4, j \oplus 6 \mod 8.$$

Because $i, j \in \{0, \ldots, 7\}$ and they do not have the same parity,

$$\{i, i \oplus 2, i \oplus 4, i \oplus 6, j, j \oplus 2, j \oplus 4, j \oplus 6\} = \{0, \ldots, 7\}.$$

So this gives a contradiction.

□

Modulo 32, the equivalence does not hold. If we try to follow the proof of Theorem 6.8, we get the following. Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \ldots, x_{4k} \in \mathbb{N}$ be distinct. Let $n_1, \ldots, n_8$ be the amount of pairs of mates for which the binary expansion of the nim-sum ends in 0000, 0010, 0100, 0110, 1000, 1010, 1100 and 1110, respectively. The corresponding Sprague-Grundy values have binary expansions that end in 1111, 0001, 0011, 0101, 0111, 1001, 1011 and 1101, respectively. To show that the equivalence holds modulo 32, we would need to show that n_1 cannot be odd. From the proof of Theorem 6.8, we know that $n_2 + n_6$, $n_3 + n_7$ and $n_4 + n_8$ cannot all be odd.

If $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$ and $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$, then we find that $n_5 + n_6 + n_7 + n_8$, $n_3 + n_4 + n_7 + n_8$, $n_2 + n_4 + n_6 + n_8$, $n_1 + n_6 + n_7 + n_8$, $n_1 + n_4 + n_5 + n_8$, $n_1 + n_3 + n_5 + n_7$ and $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8$ are all even. By repeatedly adding two of the sums together and removing the terms that appear twice, we can find a list of subsets of $\{n_1, \ldots, n_8\}$ that have an even sum. These subsets are listed in Table 10, where each row contains one of the subsets.

Here, even if n_1 is odd, it is not necessarily true that all of $n_2 + n_6$, $n_3 + n_7$ and $n_4 + n_8$ are odd. For example, it is possible for n_1, n_2, n_5 and n_6 to be odd while n_3, n_4, n_7 and n_8 are even.

				n_5	n_6	n_7	n_8
		n_3	n_4			n_7	n_8
	n_2		n_4		n_6		n_8
n_1					n_6	n_7	n_8
n_1			n_4	n_5			n_8
n_1		n_3		n_5		n_7	
n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
		n_3	n_4	n_5	n_6		
	n_2		n_4	n_5		n_7	
n_1				n_5			
n_1			n_4		n_6	n_7	
n_1		n_3			n_6		n_8
n_1	n_2	n_3	n_4				
	n_2	n_3			n_6	n_7	
n_1		n_3	n_4		n_6		
n_1	n_2			n_5	n_6		
n_1	n_2		n_4			n_7	
			n_4	n_5	n_6	n_7	
		n_3		n_5	n_6		n_8
	n_2	n_3	n_4	n_5			
	n_2	n_3		n_5			n_8
n_1		n_3	n_4	n_5		n_7	n_8
n_1	n_2					n_7	n_8
n_1	n_2		n_4	n_5	n_6		n_8
			n_4				n_8
		n_3				n_7	
	n_2	n_3	n_4		n_6	n_7	n_8
n_1	n_2	n_3					n_8
	n_2			n_5		n_7	n_8
n_1	n_2	n_3		n_5	n_6	n_7	
	n_2				n_6		

Table 10: Subsets of $\{n_1, \ldots, n_8\}$ which have an even sum if $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 16$ and $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 16$. Each row contains one of the subsets.

The examples below show that the equivalence does not hold modulo 2^n for any $n \geq 5$. The examples satisfy $n_1 = n_2 = n_5 = n_6 = 1$ and $n_3 = n_4 = n_7 = n_8 = 0$ when $n = 5$ and $k = 2$. The first example shows that $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 2^n$ when $n \geq 5$, while the second example shows that $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2^n$ when $n \geq 5$.

Example 6.9. Let $n \geq 5$. We consider the position

$$(1, 9, 2, 8, 5, 7, 2^n, 2 \cdot 2^n, \ldots, (4k-6) \cdot 2^n),$$

with $k \in \mathbb{N} \setminus \{0, 1\}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k}\}$, et cetera.

The numbers $2^n, 2 \cdot 2^n, \ldots, (4k-6) \cdot 2^n$ are split into $2k-3$ pairs of mates, all of which with a congruence modulo 2^n, and possibly modulo a higher power of 2. That means that the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k-7} \oplus x_{4k-6}$ are $0 \ldots 0$. Further, the numbers 1 and 9, 2 and 8, and 5 and 7 form pairs of mates. So we have $x_{4k-5} \oplus x_{4k-4} = 1 \oplus 9 = 8$, $x_{4k-3} \oplus x_{4k-2} = 2 \oplus 8 = 10$ and $x_{4k-1} \oplus x_{4k} = 5 \oplus 7 = 2$. It follows that the last n digits of the binary expansions of $x_{4k-5} \oplus x_{4k-4}$, $x_{4k-3} \oplus x_{4k-2}$ and $x_{4k-1} \oplus x_{4k}$ are $0 \ldots 01000$, $0 \ldots 01010$, and $0 \ldots 010$, respectively. Then the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ are $0 \ldots 0$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k-7} \mid x_{4k-6}\ ]$ are $1 \ldots 1$ Further, the last n digits of the binary expansions of $[\ x_{4k-5} \mid x_{4k-4}\ ]$, $[\ x_{4k-3} \mid x_{4k-2}\ ]$ and $[\ x_{4k-1} \mid x_{4k}\ ]$ are $0 \ldots 0111$, $0 \ldots 01001$, and $0 \ldots 01$, respectively. So the last n digits of the binary expansion of

$$\begin{aligned}&[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \\ &= [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-7} \mid x_{4k-6}\ ] \\ &\quad \oplus [\ x_{4k-5} \mid x_{4k-4}\ ] \oplus [\ x_{4k-3} \mid x_{4k-2}\ ] \oplus [\ x_{4k-1} \mid x_{4k}\ ]\end{aligned}$$

are $1 \ldots 10000$.

So $x_1 \oplus \cdots \oplus x_{4k} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k}\ ] \equiv 0 \mod 2^n$ when $n \geq 5$.

Example 6.10. Let $n \geq 5$. We consider the position

$$(1, 1 + 2^{n-2}, 2^{n-2}, 2 + 2^{n-1}, 5, 7, 2^n, 2 \cdot 2^n, \ldots, (4k-6) \cdot 2^n),$$

with $k \in \mathbb{N} \setminus \{0, 1\}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k}\}$, et cetera.

The numbers $2^n, 2 \cdot 2^n, \ldots, (4k-6) \cdot 2^n$ are split into $2k-3$ pairs of mates, all of which with a congruence modulo 2^n, and possibly modulo a higher power of 2. That means that the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k-7} \oplus x_{4k-6}$ are $0 \ldots 0$. The numbers 1 and $1 + 2^{n-2}$, 2^{n-2} and $2 + 2^{n-1}$, and 5 and 7 form pairs of mates. This means that $x_{4k-5} \oplus x_{4k-4} = 1 \oplus (1 + 2^{n-2}) = 2^{n-2}$, $x_{4k-3} \oplus x_{4k-2} = 2^{n-2} \oplus (2 + 2^{n-1}) = 2^{n-1} + 2^{n-2} + 2$ and

$x_{4k-1} \oplus x_{4k} = 5 \oplus 7 = 2$. It follows that the last n digits of the binary expansions of $x_{4k-5} \oplus x_{4k-4}$, $x_{4k-3} \oplus x_{4k-2}$ and $x_{4k-1} \oplus x_{4k}$ are $010\ldots0$, $110\ldots010$ and $0\ldots010$, respectively. So the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ are $10\ldots0$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k-7} \mid x_{4k-6}\ ]$ are $1\ldots1$. Further, the last n digits of the binary expansions of $[\ x_{4k-5} \mid x_{4k-4}\ ]$, $[\ x_{4k-3} \mid x_{4k-2}\ ]$ and $[\ x_{4k-1} \mid x_{4k}\ ]$ are $001\ldots1$, $110\ldots01$ and $0\ldots01$, respectively. So the last n digits of the binary expansion of

$$\begin{aligned}&[\ x_1 \mid \cdots \mid x_{4k+1}\ ]\\ &= [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-7} \mid x_{4k-6}\ ]\\ &\quad \oplus [\ x_{4k-5} \mid x_{4k-4}\ ] \oplus [\ x_{4k-3} \mid x_{4k-2}\ ] \oplus [\ x_{4k-1} \mid x_{4k}\ ]\end{aligned}$$

are $0\ldots0$.

So $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2^n$ when $n \geq 5$.

Note that both of the examples above need k to be at least 2, so that the number of coins is at least 8. Suppose $x_1, x_2, x_3, x_4 \in \mathbb{N}$ are distinct, such that x_1 and x_2 are congruent modulo the highest power of 2. Let $n \in \mathbb{N}$. Then $x_1 \oplus x_2 \oplus x_3 \oplus x_4 \equiv 0 \mod 2^n$ if and only if $x_1 \oplus x_2 \equiv x_3 \oplus x_4 \mod 2^n$, which is true if and only if $[\ x_1 \mid x_2\ ] \equiv [\ x_3 \mid x_4\ ] \mod 2^n$, so if and only if $[\ x_1 \mid x_2 \mid x_3 \mid x_4\ ] = [\ x_1 \mid x_2\ ] \oplus [\ x_3 \mid x_4\ ] \equiv 0 \mod 2^n$. So the equivalence modulo 2^n does hold when $k = 1$, for any $n \in \mathbb{N}$.

$4k + 1$ coins We will now discuss the case where the amount of coins is not a multiple of four. First, we prove the equivalence modulo 2 and modulo 4 in the situation where the amount of coins is $4k + 1$, for some $k \in \mathbb{N} \setminus \{0\}$.

Lemma 6.11. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \ldots, x_{4k+1} \in \mathbb{N}$ be distinct. Then we have $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2$ if and only if $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2$.*

Proof. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k+1}\}$, et cetera. Then, by the properties of the Mating candidate,

$$\begin{aligned}[\ x_1 \mid \cdots \mid x_{4k+1}\ ] &= [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus [\ x_{4k+1}\ ]\\ &= [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus x_{4k+1}.\end{aligned}$$

Suppose that x_{4k-1} and x_{4k} are congruent modulo $2^0 = 1$, but not congruent modulo 2. Then x_{4k+1} is congruent modulo 2 to either x_{4k-1} or x_{4k}. Then x_{4k-1} and x_{4k} would not be mates, so this gives a contradiction. So each pair of mates is congruent modulo 2, and possibly modulo a higher power of 2. This means that for any pair of mates (x_i, x_j), the last digit of the binary expansion of $x_i \oplus x_j$ is 0, and the last digit of the binary expansion of $[\ x_i \mid x_j\ ]$ is 1. Since there are $2k$ pairs of mates, it follows that the last digit of the binary expansion of $[\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ]$ is 0. So

$$[\ x_1 \mid \cdots \mid x_{4k+1}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus x_{4k+1} \equiv 0 \mod 2$$

if and only if the binary expansion of x_{4k+1} ends in 0.

The last digit of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ is 0. This implies that

$$x_1 \oplus \cdots \oplus x_{4k} \oplus x_{4k+1} \equiv 0 \mod 2$$

if and only if the binary expansion of x_{4k+1} ends in 0.

So $[\, x_1 \mid \cdots \mid x_{4k+1} \,] \equiv 0 \mod 2$ if and only if $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2$.

□

Lemma 6.12. *Let $k \in \mathbb{N} \setminus \{0\}$ and let $x_1, \dots, x_{4k+1} \in \mathbb{N}$ be distinct. Then we have $[\, x_1 \mid \cdots \mid x_{4k+1} \,] \equiv 0 \mod 4$ if and only if $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 4$.*

Proof. Assume that x_1 and x_2 are congruent modulo the highest power of 2, that x_3 and x_4 are congruent modulo the highest power of 2 out of $\{x_3, \dots, x_{4k+1}\}$, et cetera. Then

$$[\, x_1 \mid \cdots \mid x_{4k+1} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,] \oplus x_{4k+1}.$$

As in the proof of Lemma 6.11, we can show that for any pair of mates (x_i, x_j), the last digit of the binary expansion of $x_i \oplus x_j$ is 0. Now let x be the amount of pairs of mates (x_i, x_j) such that the binary expansion of $x_i \oplus x_j$ ends in 00, and y the amount such that the binary expansion ends in 10. Then there are x pairs of mates (x_i, x_j) such that the binary expansion of $[\, x_i \mid x_j \,]$ ends in 11, and y pairs such that it ends in 01. Since there are $2k$ pairs of mates, we have $x + y = 2k$, so x and y have the same parity.

Assume that x and y are both odd. Then the last two digits of the binary expansion of $[\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,]$ are 10, and the last two digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ are also 10. This implies that

$$[\, x_1 \mid \cdots \mid x_{4k+1} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,] \oplus x_{4k+1} \equiv 0 \mod 4$$

if and only if the binary expansion of x_{4k+1} ends in 10, and

$$x_1 \oplus \cdots \oplus x_{4k} \oplus x_{4k+1} \equiv 0 \mod 4$$

if and only if the binary expansion of x_{4k+1} ends in 10. So in this case, $[\, x_1 \mid \cdots \mid x_{4k+1} \,] \equiv 0 \mod 4$ if and only if $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 4$.

Now assume that x and y are both even. Then the last two digits of the binary expansion of $[\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,]$ are 00, and the last two digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k}$ are also 00. This implies that

$$[\, x_1 \mid \cdots \mid x_{4k+1} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k-1} \mid x_{4k} \,] \oplus x_{4k+1} \equiv 0 \mod 4$$

if and only if the binary expansion of x_{4k+1} ends in 00, and

$$x_1 \oplus \cdots \oplus x_{4k} \oplus x_{4k+1} \equiv 0 \mod 4$$

if and only if the binary expansion of x_{4k+1} ends in 00. So also in this case, $[\, x_1 \mid \cdots \mid x_{4k+1} \,] \equiv 0 \mod 4$ if and only if $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 4$.

□

The examples below show that the equivalence does not hold modulo 2^n for any $n \geq 3$, when the amount of coins is $4k+1$ for some $k \in \mathbb{N}\setminus\{0\}$. The first example shows that $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2^n$ when $n \geq 3$, while the second example shows that $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2^n$ when $n \geq 3$.

Example 6.13. Let $n \geq 3$. We consider the position

$$(5, 7, 2, 2^n + 1, 2 \cdot 2^n + 1, \ldots, (4k-2) \cdot 2^n + 1),$$

with $k \in \mathbb{N} \setminus \{0\}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k+1}\}$, et cetera.

The numbers $2^n + 1, 2 \cdot 2^n + 1, \ldots, (4k-2) \cdot 2^n + 1$ are split into $2k-1$ pairs of mates, all of which with a congruence modulo 2^n, and possibly modulo a higher power of 2. That means that the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k-3} \oplus x_{4k-2}$ are $0 \ldots 0$. Further, the numbers 5 and 7 form a pair, so the last n digits of the binary expansion of $x_{4k-1} \oplus x_{4k} = 1 \oplus 3 = 2$ are $0 \ldots 010$. Finally, the last n digits of the binary expansion of $x_{4k+1} = 2$ are $0 \ldots 010$. So the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k+1}$ are $0 \ldots 0$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k-3} \mid x_{4k-2}\ ]$ are $1 \ldots 1$, and the last n digits of the binary expansion of $[\ x_{4k-1} \mid x_{4k}\ ]$ are $0 \ldots 01$. So the last n digits of the binary expansion of

$$[\ x_1 \mid \cdots \mid x_{4k+1}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-3} \mid x_{4k-2}\ ] \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus x_{4k+1}$$

are $1 \ldots 100$.

So $x_1 \oplus \cdots \oplus x_{4k+1} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2^n$ when $n \geq 3$.

Example 6.14. Let $n \geq 3$. We consider the position

$$(5, 7, 2^n - 2, 2^n + 1, 2 \cdot 2^n + 1, \ldots, (4k-2) \cdot 2^n + 1),$$

with $k \in \mathbb{N} \setminus \{0\}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k+1}\}$, et cetera.

The numbers $2^n + 1, 2 \cdot 2^n + 1, \ldots, (4k-2) \cdot 2^n + 1$ are split into $2k-1$ pairs of mates, all of which with a congruence modulo 2^n, and possibly modulo a higher power of 2. That means that the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k-3} \oplus x_{4k-2}$ are $0 \ldots 0$. Further, the numbers 5 and 7 form a pair, so the last n digits of the binary expansion of $x_{4k-1} \oplus x_{4k} = 5 \oplus 7 = 2$ are $0 \ldots 010$. Finally, the last n digits of the binary expansion of $x_{4k+1} = 2^n - 2$ are $1 \ldots 10$. This means that the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k+1}$ are $1 \ldots 100$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k-3} \mid x_{4k-2}\ ]$ are $1 \ldots 1$, and the last n digits of the binary expansion of $[\ x_{4k-1} \mid x_{4k}\ ]$ are $0\dots01$. So the last n digits of the binary expansion of

$$[\ x_1 \mid \cdots \mid x_{4k+1}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-3} \mid x_{4k-2}\ ] \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus x_{4k+1}$$

are $0\dots0$.

We conclude that $[\ x_1 \mid \cdots \mid x_{4k+1}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \dots, x_{4k+1} \equiv 0 \mod 2^n$ when $n \geq 3$.

Note that both of the examples above need k to be at least 1. For any $x_1 \in \mathbb{N}$, we have $[\ x_1\ ] = x_1$, so the equivalence modulo 2^n does hold when $k = 0$, for any $n \in \mathbb{N}$.

$4k + 2$ **coins** Next, we discuss the case where there are $4k + 2$ coins, for some $k \in \mathbb{N}$. In this case, the equivalence modulo 2^n does not hold for any $n \in \mathbb{N} \setminus \{0\}$, as the examples below show. The first example shows that $x_1 \oplus \cdots \oplus x_{4k+2} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+2}\ ] \equiv 0 \mod 2^n$, while the second example shows that $[\ x_1 \mid \cdots \mid x_{4k+2}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+2} \equiv 0 \mod 2^n$.

Example 6.15. Let $n \in \mathbb{N} \setminus \{0\}$. We consider the position

$$(2^n, 2 \cdot 2^n, \dots, (4k + 2) \cdot 2^n),$$

with $k \in \mathbb{N}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, let x_3 and x_4 be the numbers congruent modulo the highest power of 2 out of $\{x_3, \dots, x_{4k+2}\}$, et cetera.

The $4k+2$ numbers are split into $2k+1$ pairs of mates, each with a congruence modulo 2^n, and possibly modulo a higher power of 2. So the last n digits of each of the binary expansions of $x_1 \oplus x_2, \dots, x_{4k+1} \oplus x_{4k+2}$ are $0\dots0$, which means that the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k+2}$ are $0\dots0$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \dots, [\ x_{4k+1} \mid x_{4k+2}\ ]$ are $1\dots1$. So the last n digits of the binary expansion of

$$[\ x_1 \mid \cdots \mid x_{4k+2}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus [\ x_{4k+1} \mid x_{4k+2}\ ]$$

are $1\dots1$.

It follows that $x_1 \oplus \cdots \oplus x_{4k+2} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+2}\ ] \equiv 0 \mod 2^n$, for any $n \in \mathbb{N} \setminus \{0\}$.

Example 6.16. Let $n \in \mathbb{N} \setminus \{0\}$. We consider the position

$$(1, 2^n, 2 \cdot 2^n \dots, (4k + 1) \cdot 2^n),$$

with $k \in \mathbb{N}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \dots, x_{4k+2}\}$, et cetera.

In this case, $4k$ of the numbers on positions $2^n, \dots, (4k + 1) \cdot 2^n$ form $2k$ pairs of mates, all with a congruence modulo 2^n, and possibly modulo a higher power of 2. The remaining pair consists of the number 1 and one number that is congruent to 0 modulo 2^n. This means that the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k-1} \oplus x_{4k}$ are $0 \ldots 0$, while the last n digits of the binary expansion of $x_{4k+1} \oplus x_{4k+2}$ are $0 \ldots 01$. So the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k+2}$ are $0 \ldots 01$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k-1} \mid x_{4k}\ ]$ are $1 \ldots 1$, while the last n digits of the binary expansion of $[\ x_{4k+1} \mid x_{4k+2}\ ]$ are $0 \ldots 0$. This means that the last n digits of the binary expansion of

$$[\ x_1 \mid \cdots \mid x_{4k+2}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k-1} \mid x_{4k}\ ] \oplus [\ x_{4k+1} \mid x_{4k+2}\ ]$$

are $0 \ldots 0$.

So $[\ x_1 \mid \cdots \mid x_{4k+2}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+2} \equiv 0 \mod 2^n$, for any $n \in \mathbb{N} \setminus \{0\}$.

$4k+3$ coins Finally, we discuss the case where there are $4k+3$ coins, for some $k \in \mathbb{N}$. The examples below show that also in this case, the equivalence modulo 2^n does not hold for any $n \in \mathbb{N} \setminus \{0\}$. The first example shows that $x_1 \oplus \cdots \oplus x_{4k+3} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+3}\ ] \equiv 0 \mod 2^n$, and the second example shows that $[\ x_1 \mid \cdots \mid x_{4k+3}\ ] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+3} \equiv 0 \mod 2^n$.

Example 6.17. Let $n \in \mathbb{N} \setminus \{0\}$. We consider the position

$$(2^n, 2 \cdot 2^n, \ldots, (4k+3) \cdot 2^n),$$

with $k \in \mathbb{N}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the ones congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k+3}\}$, et cetera.

In this case, all mates are congruent modulo 2^n, and possibly modulo a higher power of 2. So the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k+1} \oplus x_{4k+2}$ are $0 \ldots 0$. Since all of the $4k+3$ numbers are congruent to 0 modulo 2^n, the last n digits of the binary expansion of x_{4k+3} are also $0 \ldots 0$.

Since $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\ x_1 \mid x_2\ ], \ldots, [\ x_{4k+1} \mid x_{4k+2}\ ]$ are $1 \ldots 1$. So the last n digits of the binary expansion of

$$[\ x_1 \mid \cdots \mid x_{4k+3}\ ] = [\ x_1 \mid x_2\ ] \oplus \cdots \oplus [\ x_{4k+1} \mid x_{4k+2}\ ] \oplus x_{4k+3}$$

are $1 \ldots 1$.

So $x_1 \oplus \cdots \oplus x_{4k+3} \equiv 0 \mod 2^n$ does not imply $[\ x_1 \mid \cdots \mid x_{4k+3}\ ] \equiv 0 \mod 2^n$, for any $n \in \mathbb{N} \setminus \{0\}$.

Example 6.18. Let $n \in \mathbb{N} \setminus \{0\}$. We consider the position

$$(2^n - 1, 2^n, 2 \cdot 2^n, \ldots, (4k+2) \cdot 2^n),$$

with $k \in \mathbb{N}$. Let x_1 and x_2 be the numbers congruent modulo the highest power of 2, x_3 and x_4 the numbers congruent modulo the highest power of 2 out of $\{x_3, \ldots, x_{4k+3}\}$, et cetera.

The numbers $2^n, 2 \cdot 2^n, \ldots, (4k+2) \cdot 2^n$ are split into $2k+1$ pairs of mates, all with a congruence modulo 2^n, and possibly modulo a higher power of 2. So the last n digits of each of the binary expansions of $x_1 \oplus x_2, \ldots, x_{4k+1} \oplus x_{4k+2}$ are $0 \ldots 0$. The last n digits of the binary expansion of $x_{4k+3} = 2^n - 1$ are $1 \ldots 1$. It follows that the last n digits of the binary expansion of $x_1 \oplus \cdots \oplus x_{4k+2} \oplus x_{4k+3}$ are $1 \ldots 1$.

Since $[\, x_1 \mid x_2 \,] = x_1 \oplus x_2 - 1$ for all distinct $x_1, x_2 \in \mathbb{N}$, the last n digits of each of the binary expansions of $[\, x_1 \mid x_2 \,], \ldots, [\, x_{4k+1} \mid x_{4k+2} \,]$ are $1 \ldots 1$. It follows that the last n digits of the binary expansion of

$$[\, x_1 \mid \cdots \mid x_{4k+3} \,] = [\, x_1 \mid x_2 \,] \oplus \cdots \oplus [\, x_{4k+1} \mid x_{4k+2} \,] \oplus x_{4k+3}$$

are $0 \ldots 0$.

So $[\, x_1 \mid \cdots \mid x_{4k+3} \,] \equiv 0 \mod 2^n$ does not imply $x_1 \oplus \cdots \oplus x_{4k+3} \equiv 0 \mod 2^n$, for any $n \in \mathbb{N} \setminus \{0\}$.

Finding a move to a P-position We will now describe a method which uses Theorem 6.8 in order to find a move from an N-position to a P-position. First we discuss the situation where the amount of coins is a multiple of 4, and then the case where the amount of coins is not a multiple of 4. This is based on Chapter 15 of [1].

Suppose we are given an N-position $(x_1, \ldots, x_n)$, where n is a multiple of 4, and we want to find a move to a P-position. In order to do so, we will create a list of moves consisting of taking a coin from an occupied square x_i for some $i \in \{1, \ldots, n\}$, and placing it on another square x_i'. Each of the moves in the list must lead to a situation where the nim-sum is congruent to 0 modulo 16, but not all moves need to be feasible moves in Welter's game. We first calculate the nim-sum $k = x_1 \oplus \cdots \oplus x_n$ of the position. Because $(x_1, \ldots, x_n)$ is not a P-position, k is not congruent to 0 modulo 16 by Theorem 6.8. We nim-add k to each of the elements x_i in the position. The result is a list of various ways to move a coin x_i to a different square x_i' so that the nim-sum becomes 0. By nim-adding multiples of 16 to the values x_i', we can find all ways to make the nim-sum congruent to 0 modulo 16. Since we are only interested in moves to lower-numbered squares, we only nim-add a multiple of 16 to x_i' when the resulting value is lower than the corresponding value x_i. We also remove the move from x_i to x_i' from the list if x_i' itself is larger than x_i. The result is a list of ways to make the nim-sum congruent to 0 modulo 16 by moving a coin to a lower-numbered square.

Not all of the moves in the list are feasible in Welter's game. We remove the options where a coin would be placed on a square that is already occupied. What remains is a set of all feasible moves to positions with nim-sum congruent to 0 modulo 16. By Theorem 6.8, these positions also have Sprague-Grundy value congruent to 0 modulo 16. Since $(x_1, \ldots, x_n)$ is an N-position, and so must have a follower with Sprague-Grundy value equal to 0, at least one of these positions must have Sprague-Grundy value equal to 0.

Example 6.19. Suppose we want to move to a P-position, starting from the position $(0, 1, 3, 4, 8, 12, 16, 23)$. The nim-sum of the position is 5, so we start by nim-adding 5 to each of the numbers. The results of this are below.

0	1	3	4	8	12	16	23
5	4	6	1	13	9	21	18

In the corresponding moves, only the coins on squares 4, 12 and 23 would move to a lower-numbered square. We can make a larger list of moves by nim-adding multiples of 16 to the numbers in the second row. Since we are only interested in moves to lower-numbered squares, we only nim-add 16 to the numbers in the last two columns. This, way, we find two more options: we can move the coin on square 16 to square 5, or move the coin on square 23 to square 2. So in total, there are five ways to make the nim-sum congruent to 0 modulo 16 by moving a coin to a lower-numbered square.

We cannot move the coin on square 4 to square 1, as that square is already occupied. So the feasible moves are to the following positions:

$$(0, 1, 3, 4, 8, \mathbf{9}, 16, 23)$$
$$(0, 1, 3, 4, 8, 12, \mathbf{5}, 23)$$
$$(0, 1, 3, 4, 8, 12, 16, \mathbf{18})$$
$$(0, 1, 3, 4, 8, 12, 16, \mathbf{2})$$

All of these have a nim-sum congruent to 0 modulo 16, so they must have a Sprague-Grundy value congruent to 0 modulo 16 as well. To find a feasible move to a P-position, we need to check which of these have Sprague-Grundy value 0. Using the properties of the Mating candidate, we find that

$$\begin{aligned}
[\,0 \mid 1 \mid 3 \mid 4 \mid 8 \mid 9 \mid 16 \mid 23\,] &= [\,0 \mid 16\,] \oplus [\,1 \mid 9\,] \oplus [\,4 \mid 8\,] \oplus [\,3 \mid 23\,] \\
&= 15 \oplus 7 \oplus 11 \oplus 19 = 16, \\
[\,0 \mid 1 \mid 3 \mid 4 \mid 8 \mid 12 \mid 5 \mid 23\,] &= [\,0 \mid 8\,] \oplus [\,4 \mid 12\,] \oplus [\,1 \mid 5\,] \oplus [\,3 \mid 23\,] \\
&= 7 \oplus 7 \oplus 3 \oplus 19 = 16, \\
[\,0 \mid 1 \mid 3 \mid 4 \mid 8 \mid 12 \mid 16 \mid 18\,] &= [\,0 \mid 16\,] \oplus [\,4 \mid 12\,] \oplus [\,8 \mid 18\,] \oplus [\,1 \mid 3\,] \\
&= 15 \oplus 7 \oplus 25 \oplus 1 = 16, \\
[\,0 \mid 1 \mid 3 \mid 4 \mid 8 \mid 12 \mid 16 \mid 2\,] &= [\,0 \mid 16\,] \oplus [\,4 \mid 12\,] \oplus [\,1 \mid 3\,] \oplus [\,8 \mid 2\,] \\
&= 15 \oplus 7 \oplus 1 \oplus 9 = 0.
\end{aligned}$$

So in this case, the only feasible way to move to a P-position is by moving the coin on square 23 to square 2.

We can also use this method if the amount of coins is not a multiple of 4. Suppose there are $4k + (4 - i)$ coins on the positions $x_1, \ldots, x_{4k+(4-i)}$, for some $i \in \{1, 2, 3\}$. We add squares numbered $-1, \ldots, -i$, and place a coin on each of them. We then renumber so that $-i$ becomes 0, $-i + 1$ becomes 1, et cetera. The resulting position is $(0, \ldots, i - 1, x_1 + i, \ldots, x_n + i)$. This position has $4(k + 1)$ coins on squares numbered $0, 1, 2, \ldots$, so we can apply the method described above to find a move to a P-position. Since none of the coins on squares $0, \ldots, i - 1$ can be moved to a lower-numbered unoccupied square, the resulting position equals $(0, \ldots, i-1, x_1+i, \ldots, x_{j-1}+i, x'_j+i, x_{j+1}+i, \ldots, x_n+i)$ for some $j \in \{1, \ldots, n\}$ and $x'_j \in \mathbb{N}$ with $x'_j < x_j$. Now we simply renumber again so that 0 becomes $-i$, 1 becomes $-i+1$, et cetera. Finally, we remove the coins on the negative squares from the position. The resulting position is $(x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_n)$. The following lemma shows that this resulting position is a P-position.

Lemma 6.20. *Let $x_1, \ldots, x_n \in \mathbb{N}$ be distinct. Let $i \in \mathbb{N} \setminus \{0\}$. If $(0, \ldots, i-1, x_1 + i, \ldots, x_n + i)$ is a P-position, then so is $(x_1, \ldots, x_n)$.*

Proof. By repeatedly applying the second property of Welter's candidate, we get

$$\begin{aligned} 0 &= [\ 0 \mid \cdots \mid i-1 \mid x_1 + i \mid \cdots \mid x_n + i\ ] \\ &= [\ 0 \mid \cdots \mid i-2 \mid x_1 + i - 1 \mid \cdots \mid x_n + i - 1\ ] \\ &= \cdots = [\ x_1 \mid \cdots \mid x_n\ ]. \end{aligned}$$

□

Example 6.21. Suppose we want to find a move to a P-position, starting from the position $(1, 3, 7, 8, 11, 19)$. In order to do so, we first add coins on squares -2 and -1 so that we have 8 coins. This leads to position $(-2, -1, 1, 3, 7, 8, 11, 19)$. Next, we relabel the squares so that -2 becomes 0, -1 becomes, 1, et cetera. Our position then becomes $(0, 1, 3, 5, 9, 10, 13, 21)$. Now we can apply the same method as before. The nim-sum of this position is 28, so we nim-add 28 to each of the numbers in the position. The results are below.

0	1	3	5	9	10	13	21
28	29	31	25	21	22	17	9

In the corresponding moves, only the coin on square 21 moves to a lower-numbered position. By nim-adding 16 to some of the numbers in the second row, we find three other options. We can move the coin on square 9 to square 5, the coin on square 10 to square 6 or the coin on square 13 to square 1. So there are four ways to make the nim-sum congruent to 0 modulo 16 by moving a coin to a lower-numbered square.

We cannot move the coin on square 21 to square 9, the coin on square 9 to square 5 or the coin on square 13 to square 1, because these squares are already occupied. So there is only one feasible move to a position with nim-sum congruent to 0 modulo 16, namely the one where the coin on square 10 is moved to square 6. So $(0, 1, 3, 5, 9, 6, 13, 21)$ must be a P-position. Then $(1, 3, 7, 4, 11, 19)$ is a P-position by Lemma 6.20. So to move to a P-position, we need to move the coin on square 8 to square 4.

6.3 4 or fewer coins

If there are at most 4 coins, Welter's game is easy to play. If there is one coin at position x_1, we have $[\ x_1\ ] = x_1$, so Welter's game is just Nim. With two coins, we get $[\ x_1 \mid x_2\ ] = x_1 \oplus x_2 - 1$. It follows that the P-positions are those with $x_1 \oplus x_2 = 1$, which are the positions $[\ 2k \mid 2k+1\ ]$ for any $k \in \mathbb{N}$. Below, we show that Welter's game with 4 coins and with 3 coins also have easy solutions. This is based on Chapter 15 of [1].

Lemma 6.22. *For all distinct $a, b, c \in \mathbb{N}$, the unique z such that (a, b, c, z) is a P-position in Welter's game is $a \oplus b \oplus c$.*

Proof. By the Unique Prime Lemma, there is a unique z such that (a, b, c, z) is a P-position. Assume without loss of generality that $[\ a \mid b \mid c \mid z\ ] = [\ a \mid b\ ] \oplus [\ c \mid z\ ]$. Then

$$0 = [\ a \mid b \mid c \mid z\ ] = (a \oplus b - 1) \oplus (c \oplus z - 1),$$

so $a \oplus b - 1 = c \oplus z - 1$. This simplifies to $z = a \oplus b \oplus c$.

□

By Lemma 6.22, the P-positions for Welter's game with 4 coins are the same as those for Nim, with the only exception being that positions of the form (a, a, b, b) for any $a, b \in \mathbb{N}$ are not feasible in Welter's game. So the winning strategy for Welter's game with 4 coins is the same as that for Nim with 4 coins.

Lemma 6.23. *For all distinct $a, b \in \mathbb{N}$, the unique z such that (a, b, z) is a P-position in Welter's game is $(a + 1) \oplus (b + 1) - 1$.*

Proof. By the second property of Welter's candidate, we have $[\ a \mid b \mid z\ ] = [\ 0 \mid a + 1 \mid b + 1 \mid z + 1\ ]$. By Lemma 6.22, we have

$$0 = [\ a \mid b \mid z\ ] = [\ 0 \mid a + 1 \mid b + 1 \mid z + 1\ ]$$

if and only if $z + 1 = 0 \oplus (a + 1) \oplus (b + 1)$. Then $z = (a + 1) \oplus (b + 1) - 1$.

□

Lemma 6.23 shows that we can use the strategy for Nim with 4 coins to solve Welter's game with 3 coins. If (a, b, c) is an N-position in Welter's game, then $(0, a + 1, b + 1, c + 1)$ is also an N-position. By Lemma 6.22, we can use the winning strategy for Nim with 4 coins to move from $(0, a + 1, b + 1, c + 1)$ to a P-position. In this new position, only one coin will have moved to a different position, and this is not the coin on 0. This means that the new position can be converted to a position with 3 coins, and that position is a follower of (a, b, c).

Example 6.24. Suppose we want to move to a P-position starting from position $(1, 5, 7)$. We note that $(1, 5, 7)$ is a P-position in Welter's game if and only if $(0, 2, 6, 8)$ is a P-position in Welter's game, which is true if and only if $(0, 2, 6, 8)$ is a P-position in Nim. By Theorem 2.21, $(0, 2, 6, 8)$ is an N-position in Nim. To move to a P-position, we need to move the coin on square 8 to square 4. So $(0, 2, 6, 4)$ is a P-position in Nim and in Welter's game, which means that $(1, 5, 3)$ is a P-position in Welter's game. So to move to a P-position from $(1, 5, 7)$, we need to move the coin on square 7 to square 3.

6.4 5 coins on $\{0, \ldots, 15\}$

If Welter's game is played with 5 coins, and all the coins are on squares in $\{0, \ldots, 15\}$, then Welter's game is easy to play. This special case was originally solved by Sprague in [8]. Although Sprague did not use nim-addition, the result can be expressed in terms of nim-addition. The result was also discussed in Chapter 9 of [6]. The proof we give below is very similar to Sprague's proof.

For this variant of Welter's game, we renumber the squares so that 1 becomes 15, 2 becomes 14, et cetera. So we play on the squares $\{0, 15, 14, \ldots, 1\}$. For any distinct $x, y \in \{0, 15, 4, \ldots, 1\}$, we write $x \prec y$ if $x > y$ or $x = 0$. Then, a coin on square y may be moved to square x if and only if x is unoccupied and $x \prec y$.

Definition 6.25. Let $k \in \{15, 14, \ldots, 1\}$. The numbers corresponding to k are all $x \in \{0, 15, 14, \ldots, 1\}$ such that $x \oplus k \prec x$.

We first prove a few lemmas concerning corresponding numbers.

Lemma 6.26. *Let $k \in \{15, 14, \ldots, 1\}$. The numbers corresponding to k satisfy the following properties:*

1. *The numbers corresponding to k are k and the 7 numbers in $\{15, \ldots, 1\} \setminus \{k\}$ which have a 0 in their binary expansion at the first location from the left where the binary expansion of k has a 1. The numbers not corresponding to k are 0 and the 7 numbers in $\{15, \ldots, 1\} \setminus \{k\}$ which have a 1 in their binary expansion at the first location where the binary expansion of k has a 1.*

2. *The nim-sum of all numbers not corresponding to k equals k.*

3. *If x_1, x_2, x_3 are distinct numbers not corresponding to k, then $x_1 \oplus x_2 \oplus x_3 \neq 0$.*

Proof. 1. For each position in the binary expansion, there are 8 numbers out of $\{0, 15, 14, \ldots, 1\}$ which have a 0 in this position and 8 which have a 1. If k has a 1 in the position, then there are 7 numbers remaining in $\{15, 14, \ldots, 1\} \setminus \{k\}$ with a 1 in this position, and 7 with a 0.

We have $k \oplus k = 0$, and $0 \prec k$. So k corresponds to k. Note that $0 \oplus k = k \not\prec 0$, so 0 does not correspond to k. Now let $x \in \{15, 14, \ldots, 1\} \setminus \{k\}$. Then $x \oplus k \in \{15, \ldots, 1\}$, so $x \oplus k \prec k$ if and only if $x \oplus k > x$. Note that the first position from the left at which the binary expansions of $x \oplus k$ and x differ is the first position where the binary expansion of k has a 1. It follows that $x \oplus k > x$ if and only if the binary expansion of x has a 0 at the first position from the left at which the binary expansion of k has a 1. This proves the first statement. The second statement follows immediately.

2. The nim-sum of all 8 numbers in $\{15, \ldots, 1\}$ with a 1 in a specific position of their binary expansion is 0. Using Lemma 6.26.1, it follows that the nim-sum of the 8 numbers not corresponding to k is k.

3. Let x_1, x_2, x_3 be distinct numbers not corresponding to k, and assume that $x_1 \oplus x_2 \oplus x_3 = 0$. If $x_1 = 0$, then $x_2 = x_3$, which gives a contradiction. So $x_1 \neq 0$. By Lemma 6.26.1, it follows that the binary expansion of x_1 has a 1 at the first position from the left where the binary expansion of k has a 1. Similarly, the binary expansions x_2 and x_3 have a 1 in that position. Then the binary expansion of $x_1 \oplus x_2 \oplus x_3$ also has a 1 in this position, so it does not equal 0. This gives a contradiction, so $x_1 \oplus x_2 \oplus x_3 \neq 0$.

□

Lemma 6.27. *Let (x_1, x_2, x_3) be a position of Welter's game played on the squares $\{0, 15, 14, \ldots, 1\}$, such that $x_1 \oplus x_2 \oplus x_3 = 0$. Then for each $k \in \{1, \ldots, 15\}$ there is a follower of (x_1, x_2, x_3) with nim-sum k.*

Proof. Let $k \in \{1, \ldots, 15\}$. If $x_1 = 0$, then $x_2 = x_3$, which gives a contradiction. So $x_1 \neq 0$, and, similarly, $x_2, x_3 \neq 0$.

First assume that $x_i = k$ for some $i \in \{1, 2, 3\}$. Without loss of generality, assume that $i = 1$. Then $x_2 \oplus x_3 = k$. We can move the coin on square $x_i = k$ to square 0, as square 0 is unoccupied and $0 \prec x$ for all $x \in \{15, \ldots, 1\}$. The resulting position has nim-sum k. Now assume that $x_i \neq k$ for all $i \in \{1, 2, 3\}$. By Lemma 6.26.3, there exists i such that x_i corresponds to k, so $x_i \oplus k \prec x_i$. Suppose that $x_i \oplus k = x_j$ for some $j \in \{1, 2, 3\} \setminus \{i\}$. Without loss of generality, assume that $i = 1$ and $j = 2$. Then $k = x_1 \oplus x_2 = x_3$, which gives a contradiction. So the square $x_i \oplus k$ is unoccupied. Moving the coin on square x_i to $x_i \oplus k$ nim-adds k to the nim-sum, so it leads to a position with nim-sum equal to k.

□

Now we can solve this special case of Welter's game.

Theorem 6.28. *Let $x_1, \ldots, x_5 \in \{0, \ldots, 15\}$ be distinct. Renumber so that 1 becomes 15, 2 becomes 14, et cetera. $(x_1, \ldots, x_5)$ is a P-position if and only if its nim-sum after renumbering is 0.*

Proof. We check the conditions of Corollary 2.7.

• The only terminal position in Welter's game with 5 coins is $(0, 1, 2, 3, 4)$. After renumbering, this becomes $(0, 15, 14, 13, 12)$. The nim-sum is $0 \oplus 15 \oplus 14 \oplus 13 \oplus 12 = 0$.

• Suppose a position has nim-sum unequal to 0 after the renumbering, and that the position after renumbering is $(x_1, x_2, x_3, x_4, x_5)$. Let

$$k = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5.$$

Note that $k \in \{15, 14, \ldots, 1\}$. We need to show that there is a feasible move to a position with nim-sum equal to 0. Moving a coin from square x_i to square x_i' nim-adds $x_i \oplus x_i'$ to the nim-sum. This means that the only way to move to a position with nim-sum equal to 0 is by moving a coin from from square x_i to square $x_i \oplus k$ for some i. So we need to find $i \in \{1, \ldots, 5\}$ such that square $x_i \oplus k$ is unoccupied and $x_i \oplus k \prec x_i$. The second requirement is satisfied if and only if x_i corresponds to k.

Assume that $x_1, \ldots, x_5$ all do not correspond to k. Then by Lemma 6.26.1, there exist 3 other numbers that do not correspond to k. By Lemma 6.26.2, the nim-sum of these three equals 0, but this contradicts Lemma 6.26.3.

So at least one of $x_1, \ldots, x_5$ corresponds to k. Without loss of generality, assume that x_1 corresponds to k. If $x_1 \oplus k$ is unoccupied, we are done. Otherwise, there is a $j \neq 1$ such that $x_1 \oplus k = x_j$. Without loss of generality, assume that $x_1 \oplus k = x_2$. Then $k = x_1 \oplus x_2$, which implies that $x_3 \oplus x_4 \oplus x_5 = 0$. By

Lemma 6.27, there is a follower of (x_3, x_4, x_5) with nim-sum k. Without loss of generality, assume that $x_3' \oplus x_4 \oplus x_5 = k$, for some x_3' with $x_3' \prec x_3$, such that $x_3' \neq x_4, x_5$. Then $x_3' = x_3 \oplus k$. Suppose that square $x_3 \oplus k$ is occupied. Then there is a $j \in \{1, 2\}$ such that $x_3 \oplus k = x_j$. Assume without loss of generality that $x_3 \oplus k = x_1$. Then

$$x_1 \oplus x_3 = k = x_1 \oplus x_2,$$

so $x_2 = x_3$. This contradicts the fact that $(x_1, x_2, x_3, x_4, x_5)$ is a feasible position. So $x_3' = x_3 \oplus k$ is not occupied.

So in all cases, there is an $i \in \{x_1, x_2, x_3, x_4, x_5\}$ such that x_i corresponds to k and $x_i \oplus k$ is unoccupied, so that moving the coin on square x_i to square x_i' leads to a position with nim-sum equal to 0.

• Suppose a position has nim-sum equal to 0 after the renumbering, and that the position after renumbering is $(x_1, x_2, x_3, x_4, x_5)$. We need to show that all feasible moves are to positions with nim-sum unequal to 0.

Every move changes exactly one of $x_1, \ldots, x_5$. Without loss of generality, assume that x_1 is replaced by some $x_1' \neq x_1$. Then

$$x_1' \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 \neq x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 = 0,$$

so the resulting position has nim-sum unequal to 0.

□

The proof of the above theorem states that if $(x_1, x_2, x_3, x_4, x_5)$ is an N-position, then there is an $i \in \{1, \ldots, 5\}$ such that $x_i \oplus k \prec x_i$ and $x_i \oplus k$ is unoccupied, and that moving the coin on square x_i to square $x_i \oplus k$ leads to a P-position. The examples below clarify this.

Example 6.29. Consider the position $(1, 3, 6, 8, 14)$. After renumbering the squares, this becomes $(15, 13, 10, 8, 2)$. We have

$$15 \oplus 13 \oplus 10 \oplus 8 \oplus 2 = 2.$$

Since the nim-sum equals 2, this is not a P-position by Theorem 6.28. We can get to a P-position by moving the coin in square 2 to square $2 \oplus 2 = 0$. This leads to the position $(15, 13, 10, 8, 0)$. After renumbering the squares again, we find that this equals position $(1, 3, 6, 8, 0)$ in the usual notation.

Example 6.30. Consider the position $(2, 3, 6, 8, 14)$. After renumbering the squares, we get $(14, 13, 10, 8, 2)$. We find the nim-sum of the five occupied squares using column addition:

$$\begin{array}{ll} 1110 & \\ 1101 & \\ 1010 & \\ 1000 & \\ 0010 & + \\ \hline 0011 & \end{array}$$

The nim-sum k equals 3, so by Theorem 6.28 this is not a P-position. We need to find an occupied square x_i such that $x_i \oplus k \prec x_i$. Because square 3 is not occupied, we need an occupied square whose binary expansion has a 0 at the first location where the binary expansion of 3 has a 1. The options are square 13 and square 8. We have $13 \oplus 3 = 14$. Because square 14 is occupied, we cannot move the coin on square 13 to square 14. So we move the coin on square 8 to square $8 \oplus 3 = 11$. This leads to the P-position $(14, 13, 10, 11, 2)$. After renumbering the squares again, we find that this is position $(2, 3, 6, 5, 14)$ in the usual notation.

7 Understanding Welter's third property

While the first two properties of Welter's candidate are easy to understand, the third is not so intuitive. In this chapter, we try to get a better grip on this property. In order to do so, we first prove that the property holds for Welter's game with 2 coins, directly from the rules of Welter's game, in Section 7.1. In Section 7.2, we discuss some interesting patterns that occur in tables filled with Sprague-Grundy values for Welter's game with 3 coins, and that are related to Welter's third property.

7.1 2 coins

In this section, we prove directly from the definition of Welter's game, so without assuming knowledge of the relation between the Sprague-Grundy function and Welter's candidate, the Mating candidate, the Animating candidate and the Triangle candidate, that Welter's third property holds when $n = 2$.

As usual, we write $[\ x_1 \mid \cdots \mid x_n\ ]$ for the Sprague-Grundy value of position $(x_1, \ldots, x_n)$. First, we note that when there are 2 coins, Welter's game is similar to Nim. Because of this fact, it is easy to find an explicit expression for the Sprague-Grundy function for Welter's game played with 2 coins.

Lemma 7.1. $[\ x \mid y\ ] = x \oplus y - 1$ *for all distinct* $x, y \in \mathbb{N}$.

Proof. We prove this by induction. By Welter's second and first properties, we have

$$[\ 0 \mid 1\ ] = [\ 0\ ] = 0 = 0 \oplus 1 - 1.$$

Because Welter's game is symmetric, we also have $[\ 1 \mid 0\ ] = [\ 0 \mid 1\ ] = 1 \oplus 0 - 1$.

Now let $x, y \in \mathbb{N}$ be distinct, and assume that $[\ x' \mid y\ ] = x' \oplus y - 1$ for all $x' < x$ with $x' \neq y$, and that $[\ x \mid y'\ ] = x \oplus y' - 1$ for all $y' < y$ with $y' \neq x$. Then we have

$$\begin{aligned} [\ x \mid y\ ] &= \operatorname{mex}(\{[\ x' \mid y\ ] : x' < x, x' \neq y\} \cup \{[\ x \mid y'\ ] : y' < y, y' \neq x\}) \\ &= \operatorname{mex}(\{x' \oplus y - 1 : x' < x, x' \neq y\} \cup \{x \oplus y' - 1 : y' < y, y' \neq x\}). \end{aligned}$$

By Lemma 2.21, the Sprague-Grundy value of any position in Nim is its nim-sum. So

$$x \oplus y = \operatorname{mex}(\{x' \oplus y : x' < x\} \cup \{x \oplus y' : y' < y\}).$$

Because $x \neq y$, we have $x \oplus y \neq 0$, so

$$x \oplus y - 1 = \operatorname{mex}(\{x' \oplus y : x' < x\} \cup \{x \oplus y' : y' < y\}) - 1 \neq -1.$$

Note that $x' \oplus y - 1 = -1$ if and only if $x' = y$, and $x \oplus y' - 1 = -1$ if and only if $x = y'$. It follows that

$$\begin{aligned} &\operatorname{mex}(\{x' \oplus y : x' < x\} \cup \{x \oplus y' : y' < y\}) - 1 \\ &= \operatorname{mex}(\{x' \oplus y - 1 : x' < x, x' \neq y\} \cup \{x \oplus y' - 1 : y' < y, y' \neq x\}). \end{aligned}$$

So

$$\begin{aligned}[\, x \mid y \,] &= \operatorname{mex}(\{x' \oplus y - 1 : x' < x, x' \neq y\} \cup \{x \oplus y' - 1 : y' < y, y' \neq x\}) \\ &= \operatorname{mex}(\{x' \oplus y : x' < x\} \cup \{x \oplus y' : y' < y\}) - 1 \\ &= x \oplus y - 1.\end{aligned}$$

□

By Lemma 7.1, we have

$$[\, x_1 \oplus x \mid x_2 \oplus x \,] = (x_1 \oplus x) \oplus (x_2 \oplus x) - 1 = x_1 \oplus x_2 - 1 = [\, x_1 \mid x_2 \,]$$

for all distinct $x_1, x_2 \in \mathbb{N}$, so Welter's second property holds when $n = 2$.

Since the above proof relies on the fact that Welter played with 2 coins is similar to Nim, and there is no such similarity for Welter played with 3 or more coins, we cannot apply a similar argument to the situation with more coins. So this proof does not help in understanding Welter's third property in general.

We provide a second proof of Welter's third property. For this proof, we look at the followers of each position and use induction. We first need the following lemma.

Lemma 7.2. *Let $x, y \in \mathbb{N}$ with $x < y$, and let $k \in \mathbb{N}$. Suppose that $y \oplus 2^k = y - 2^k$, and that $x \oplus 2^k = x + 2^k \geq y - 2^k$. Then $x \oplus (y - 2^k) < 2^k$.*

Proof. Let $m = x \oplus (y - 2^k)$. We need to show that $m < 2^k$. If the binary expansions of x and $y - 2^k$ only differ on the last k digits, then $m < 2^k$. So suppose the binary expansions of x and $y - 2^k$ differ at an earlier position. Let $\ell > k$ be such that the ℓth last digit is the first position from the left at which the binary expansions of x and $y - 2^k$ differ.

Since $y - 2^k = y \oplus 2^k$, the binary expansion of $y - 2^k$ only differs from that of y on the kth last digit. It follows that the first position from the left where the binary expansions of x and y differ is the ℓth last position. Since $x < y$, the binary expansion of x must have a 0 at the ℓth last position, while that of y has a 1.

The binary expansion of $x + 2^k = x \oplus 2^k$ only differs from that of x on the kth last digit. So the first position where the binary expansions of $x + 2^k$ and $y - 2^k$ differ is also on the ℓth last digit, where the binary expansion of $x + 2^k$ has a 0 and the binary expansion of $y - 2^k$ has a 1. This implies that $x + 2^k < y - 2^k$, which gives a contradiction.

□

Now, we again prove Welter's third property for $n = 2$.

Lemma 7.3. $[\, a \oplus x \mid b \oplus x \,] = [\, a \mid b \,]$ *for all $a, b, x \in \mathbb{N}$ with $a \neq b$.*

Proof. Note that it is enough to show that $[\, a \oplus x \mid b \oplus x \,] = [\, a \mid b \,]$ for all distinct $a, b \in \mathbb{N}$ whenever x is a power of 2. We will show this by induction. First note that $[\, 0 \mid 1 \,] = [\, 1 \mid 0 \,]$, so $[\, 0 \mid 1 \,] = [\, 0 \oplus 2^0 \mid 1 \oplus 2^0 \,]$ and $[\, 1 \mid 0 \,] = [\, 1 \oplus 2^0 \mid 0 \oplus 2^0 \,]$.

Now let $a, b, k \in \mathbb{N}$ with $a \neq b$. Assume that $[\, x \oplus 2^i \mid y \oplus 2^i \,] = [\, x \mid y \,]$ for all distinct $x, y \in \mathbb{N}$ and $i < k$, and that $[\, a' \oplus 2^k \mid b' \oplus 2^k \,] = [\, a' \mid b' \,]$ for all distinct $a', b' \in \mathbb{N}$ with $a' \leq a$, $b' \leq b$ and $(a', b') \neq (a, b)$. We will prove that $[\, a \oplus 2^k \mid b \oplus 2^k \,] = [\, a \mid b \,]$.

Note that if $[\, x \oplus 2^i \mid y \oplus 2^i \,] = [\, x \mid y \,]$ for all distinct $x, y \in \mathbb{N}$ and $i < k$, we also have $[\, a \oplus x \mid b \oplus x \,] = [\, a \mid b \,]$ for all $x < 2^k$.

Suppose that $(a \oplus 2^k, b \oplus 2^k) = (a - 2^k, b - 2^k)$. Let $a' = a - 2^k$ and $b' = b - 2^k$. Then, by the induction hypothesis, we have $[\, a' \oplus 2^k \mid b' \oplus 2^k \,] = [\, a' \mid b' \,]$, which implies that

$$\begin{aligned}[\, a \mid b \,] &= [\, (a \oplus 2^k) \oplus 2^k \mid (b \oplus 2^k) \oplus 2^k \,] = [\, a' \oplus 2^k \mid b' \oplus 2^k \,] \\ &= [\, a' \mid b' \,] = [\, a \oplus 2^k \mid b \oplus 2^k \,].\end{aligned}$$

So if $(a \oplus 2^k, b \oplus 2^k) = (a - 2^k, b - 2^k)$, we are done.

Recall that Welter's game is symmetric by definition. This means that there are two distinct cases we still need to consider: either $(a \oplus 2^k, b \oplus 2^k) = (a + 2^k, b + 2^k)$, or $(a \oplus 2^k, b \oplus 2^k) = (a + 2^k, b - 2^k)$. In both cases, we look at each follower of (a, b) and show that there is a follower of $(a \oplus 2^k, b \oplus 2^k)$ with the same Sprague-Grundy value. Then, we show that for each follower of $(a \oplus 2^k, b \oplus 2^k)$, there is either a follower of (a, b) which has the same Sprague-Grundy value, or a position that has (a, b) as a follower which has the same Sprague-Grundy value. In the latter case, the Sprague-Grundy value is certainly not equal to that of (a, b). We then have

$$\begin{aligned}&\{\{[\, a' \mid b \,] : a' < a, a' \neq b\} \cup \{[\, a \mid b' \,] : b' < b, b' \neq a\}\} \\ &\subseteq \{\{[\, a' \mid b \oplus 2^k \,] : a' < a \oplus 2^k, a' \neq b \oplus 2^k\} \\ &\qquad \cup \{[\, a \oplus 2^k \mid b' \,] : b' < b \oplus 2^k, b' \neq a \oplus 2^k\}\}\end{aligned}$$

and

$$\begin{aligned}[\, a \mid b \,] \notin &\{\{[\, a' \mid b \oplus 2^k \,] : a' < a \oplus 2^k, a' \neq b \oplus 2^k\} \\ &\qquad \cup \{[\, a \oplus 2^k \mid b' \,] : b' < b \oplus 2^k, b' \neq a \oplus 2^k\}\}.\end{aligned}$$

Then we can conclude that

$$\begin{aligned}[\, a \mid b \,] &= \operatorname{mex}(\{[\, a' \mid b \,] : a' < a, a' \neq b\} \cup \{[\, a \mid b' \,] : b' < b, b' \neq a\}) \\ &= \operatorname{mex}(\{[\, a' \mid b \oplus 2^k \,] : a' < a \oplus 2^k, a' \neq b \oplus 2^k\} \\ &\qquad \cup \{[\, a \oplus 2^k \mid b' \,] : b' < b \oplus 2^k, b' \neq a \oplus 2^k\}) \\ &= [\, a \oplus 2^k \mid b \oplus 2^k \,].\end{aligned}$$

• First, we assume that $a \oplus 2^k = a + 2^k$ and $b \oplus 2^k = b + 2^k$. We will first show that for each follower of (a, b), there is a follower of $(a \oplus 2^k, b \oplus 2^k)$ with the same Sprague-Grundy value.

Let $a' < a$ be such that $a' \neq b$, so that (a', b) is a follower of (a, b). Then $a' \oplus 2^k \neq b \oplus 2^k$, so $(a' \oplus 2^k, b \oplus 2^k)$ is also a feasible position. Also,

$$a' \oplus 2^k \leq a' + 2^k < a + 2^k.$$

This means that $(a' \oplus 2^k, b \oplus 2^k) = (a' \oplus 2^k, b+2^k)$ is a follower of $(a+2^k, b+2^k)$. By the induction hypothesis, $[\ a' \oplus 2^k \mid b \oplus 2^k\ ] = [\ a' \mid b\ ]$. So for each follower of (a, b) of the type (a', b), the position $(a + 2^k, b + 2^k)$ has a follower with the same Sprague-Grundy value. The same holds for followers of the type (a, b').

Next, we show that for each follower of $(a \oplus 2^k, b \oplus 2^k)$, there is either a follower of (a, b) with the same Sprague-Grundy value, or a position which has (a, b) as a follower.

Let $a' < a + 2^k$ be such that $a' \neq b + 2^k$, so that $(a', b + 2^k)$ is a follower of $(a+2^k, b+2^k)$. Note that $a' \oplus 2^k \neq b$, as otherwise we would have $a' = b \oplus 2^k = b + 2^k$.

If $a' \oplus 2^k < a$, then $(a' \oplus 2^k, b)$ is a follower of (a, b), and by the induction hypothesis it has the same Sprague-Grundy value as $(a', b + 2^k)$. We certainly have $a' \oplus 2^k < a$ when $a' \oplus 2^k = a' - 2^k$, because then

$$a' \oplus 2^k = a' - 2^k < a + 2^k - 2^k = a.$$

If $a' \oplus 2^k = a' + 2^k$ we may have $a' \oplus 2^k \geq a$. In this case, let $x = a' \oplus a$. We have $a' < a + 2^k$, $(a + 2^k) \oplus 2^k = (a + 2^k) - 2^k$ and $a' \oplus 2^k = a' + 2^k \geq (a + 2^k) - 2^k$. So by Lemma 7.2, $x = a' \oplus (a + 2^k - 2^k) < 2^k$.

By the induction hypothesis, $(a', b + 2^k)$ has the same Sprague-Grundy value as $(a' \oplus x, (b + 2^k) \oplus x) = (a, (b + 2^k) \oplus x)$. Note that

$$(b + 2^k) \oplus x \geq (b + 2^k) - x > b.$$

This means that the position $(a, (b + 2^k) \oplus x)$ has (a, b) as a follower unless $a = (b + 2^k) \oplus x$, but in that case we would have $a' = b + 2^k$, which gives a contradiction.

We can conclude that each follower of $(a + 2^k, b + 2^k)$ of the type $(a', b + 2^k)$ corresponds to either a follower of (a, b) or a position that has (a, b) as a follower. The same holds for followers of the type $(a + 2^k, b')$. So if $a \oplus 2^k = a + 2^k$ and $b \oplus 2^k = b + 2^k$, then $[\ a \mid b\ ] = [\ a \oplus 2^k \mid b \oplus 2^k\ ]$.

• Now, we assume that $a \oplus 2^k = a + 2^k$ and $b \oplus 2^k = b - 2^k$. We will show that each follower of (a, b) corresponds to a follower of $(a \oplus 2^k, b \oplus 2^k)$ with the same Sprague-Grundy value, and that each follower of $(a \oplus 2^k, b \oplus 2^k)$ corresponds to a follower of (a, b) with the same Sprague-Grundy value.

Similarly to what we proved in the previous case, a follower of (a, b) of the type (a', b) corresponds to the position $(a' \oplus 2^k, b - 2^k)$, which is a follower of $(a + 2^k, b - 2^k)$.

Now let $b' < b$ be such that $b' \neq a$, so that (a, b') is a follower of (a, b). Suppose that $b' \oplus 2^k < b - 2^k$ and $a + 2^k \neq b' \oplus 2^k$. Then, by the induction hypothesis, the position (a, b') has the same Sprague-Grundy value as $(a + 2^k, b' \oplus 2^k)$, which is a follower of $(a + 2^k, b - 2^k)$. If $a + 2^k = b' \oplus 2^k$, then $a = b'$, which gives a contradiction. The inequality $b' \oplus 2^k < b - 2^k$ certainly holds when $b' \oplus 2^k = b' - 2^k$.

If $b' \oplus 2^k = b' + 2^k$ we may have $b' \oplus 2^k \geq b - 2^k$. In this case, let $x = b' \oplus (b - 2^k)$. We have $b' < b$, $b \oplus 2^k = b - 2^k$ and $b' \oplus 2^k = b' + 2^k \geq b - 2^k$. So by Lemma 7.2, $x = b' \oplus (b - 2^k) < 2^k$.

By the induction hypothesis, (a, b') has the same Sprague-Grundy value as $(a \oplus x, b' \oplus x) = (a \oplus x, b - 2^k)$. Note that

$$a \oplus x \leq a + x < a + 2^k.$$

This means that the position $(a \oplus x, b - 2^k)$ is a follower of $(a + 2^k, b - 2^k)$ unless $a \oplus x = b - 2^k$. But if $a \oplus x = b - 2^k$ then $a = b'$, which gives a contradiction.

So we find that each follower of (a, b) corresponds to a follower of $(a+2^k, b-2^k)$. Similarly, all followers of $(a + 2^k, b - 2^k)$ correspond to followers of (a, b). It follows that $[\ a \mid b\ ] = [\ a \oplus 2^k \mid b \oplus 2^k\ ]$ if $a \oplus 2^k = a + 2^k$ and $b \oplus 2^k = b - 2^k$.

□

In short, the idea of the argument above is as follows. Let $a, b, k \in \mathbb{N}$. Then each follower (a', b) of (a, b) has the same Sprague-Grundy value as $(a' \oplus 2^k, b \oplus 2^k)$ by the induction hypothesis, and the same Sprague-Grundy value as $(a' \oplus x, b \oplus x)$ for all $x < 2^k$. Either one of these is a follower of $(a \oplus 2^k, b \oplus 2^k)$, or one of the positions of the second type has $(a \oplus 2^k, b \oplus 2^k)$ as a follower. For the proof, it is relevant that a position $(x_1 \oplus x, x_2 \oplus x)$ is not feasible in Welter's game if and only if (x_1, x_2) is not a feasible position.

A similar argument might help to understand Welter's third property for the situation with more than 2 coins. However, the argument above is not enough by itself. If we try to apply it to the situation with more coins, we run into a problem, as the example below shows.

Example 7.4. We look at the positions $(2, 4, 6, 8)$ and $(2 \oplus 1, 4 \oplus 1, 6 \oplus 1, 8 \oplus 1) = (3, 5, 7, 9)$. The followers of both positions are listed below.

Position	(2, 4, 6, 8)		(3, 5, 7, 9)	
Followers	(1, 4, 6, 8)	(2, 3, 6, 8)	(2, 5, 7, 9)	(3, 4, 7, 9)
	(0, 4, 6, 8)	(2, 1, 6, 8)	(1, 5, 7, 9)	(3, 2, 7, 9)
		(2, 0, 6, 8)	(0, 5, 7, 9)	(3, 1, 7, 9)
				(3, 0, 7, 9)
	(2, 4, 5, 8)	(2, 4, 6, 7)	(3, 5, 6, 9)	(3, 5, 7, 8)
	(2, 4, 3, 8)	(2, 4, 6, 5)	(3, 5, 4, 9)	(3, 5, 7, 6)
	(2, 4, 1, 8)	(2, 4, 6, 3)	(3, 5, 2, 9)	(3, 5, 7, 4)
	(2, 4, 0, 8)	(2, 4, 6, 1)	(3, 5, 1, 9)	(3, 5, 7, 2)
		(2, 4, 6, 0)	(3, 5, 0, 9)	(3, 5, 7, 1)
				(3, 5, 7, 0)

Assume that $[\ a' \oplus 1 \mid b' \oplus 1 \mid c' \oplus 1 \mid d' \oplus 1\ ] = [\ a' \mid b' \mid c' \mid d'\ ]$ for all $a' \leq 2$, $b' \leq 4$, $c' \leq 6$, $d' \leq 8$ with $(a', b', c', d') \neq (2, 4, 6, 8)$.

Then the position $(1, 4, 6, 8)$ has the same Sprague-Grundy value as $(0, 5, 7, 9) = (0 \oplus 1, 4 \oplus 1, 6 \oplus 1, 8 \oplus 1)$. Similarly, we can pair up each of the other followers of $(2, 4, 6, 8)$ with a follower of $(3, 5, 7, 9)$, as in the proof of Lemma 7.3. This leaves four followers of $(3, 5, 7, 9)$ without a paired element: $(2, 5, 7, 9)$, $(3, 4, 7, 9)$, $(3, 5, 6, 9)$ and $(3, 5, 7, 8)$.

Ideally, we would show that each of these has the same Sprague-Grundy value as a position which has $(2, 4, 6, 8)$ as a follower. However, we cannot use the induction hypothesis to show for example that $[\ 2 \mid 5 \mid 7 \mid 9\ ] = [\ 2 \oplus 1 \mid 5 \oplus 1 \mid 7 \oplus 1 \mid 9 \oplus 1\ ] = [\ 3 \mid 4 \mid 6 \mid 8\ ]$.

For Welter's game with 3 coins, we run into yet another problem. Here, Welter's third property states that $[\ a \oplus x \mid b \oplus x \mid c \oplus x\ ] = [\ a \mid b \mid c\ ] \oplus x$ for all distinct $a, b, c \in \mathbb{N}$, and all $x \in \mathbb{N}$. So in this case, the property cannot be used to show that two followers have the same Sprague-Grundy value.

Lemma 7.3 does not necessarily hold for other games that are similar to Welter's game. We discuss the games Antonim and Antimatter, which are described in Section 5.2. In the game Antonim, the position $(0, 0)$ is feasible, but all other positions of the type (a, a) are not feasible. This can lead to a situation where a feasible follower of one of the positions corresponds to a position that is not feasible, as in the following example.

Example 7.5. We look at the game Antonim with 2 coins. Let $a = 1$, $b = 2$ and $k = 0$. Then $(a \oplus 2^k, b \oplus 2^k) = (0, 3)$. The followers of both positions are listed below.

Position	(1, 2)		(0, 3)	
Followers	(0, 2)	(1, 0)		(0, 2) (0, 1) (0, 0)

The position $(0, 2)$ corresponds to $(0, 2)$. These positions have Sprague-Grundy value 2. The position $(1, 0)$ corresponds to $(0, 1)$. These two positions have Sprague-Grundy value 1. However, the position $(0, 3)$ has a third follower, $(0, 0)$, with Sprague-Grundy value 0. This corresponds to $(1, 1)$, but that is not a feasible position in Antonim. So $(1, 2)$ has Sprague-Grundy value 0 while $(0, 3)$ has Sprague-Grundy value 3.

The lemma does hold for the game Antimatter played with 1 positron and 1 electron, because here all positions of the type $(a|a)$ are feasible. It also holds for Antimatter played with 2 positrons or with 2 electrons, because that is equivalent to Welter's game with 2 coins.

There is no property similar to Welter's third property that holds for Antimatter or Antonim played with more than 2 particles or coins. We checked whether there exists a function f, such that

$$[\ p_1 \oplus x \mid \cdots \mid p_n \oplus x \mid\mid e_1 \mid \cdots \mid e_m\ ] = f([\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ])$$

for all positions $(p_1, \ldots, p_n | e_1, \ldots, e_m)$ of Antimatter with no annihilated particles, and all $x \in \mathbb{N}$. Here, we checked functions f of the type $f(x) = (x+a) \oplus b$ or $f(x) = (x \oplus b) + a$, with $a \in \{c_1, c_1+x, c_1-x, c_1 \oplus x\}$ for some $c_1 \in \{-5, -4, \ldots, 5\}$ and $b \in \{c_2, c_2+x, c_2-x, c_2 \oplus x\}$ for some $c_2 \in \{-5, -4, \ldots, 5\}$. We found that no such property holds for Antimatter. This also implies that there is no similar property of the type

$$[\ p_1 \oplus x \mid \cdots \mid p_n \oplus x \mid\mid e_1 \oplus x \mid \cdots \mid e_m \oplus x\ ] = f([\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ])$$

We also checked whether there exists a function f, such that

$$[\ x_1 \oplus x \mid \cdots \mid x_n \oplus x\ ] = [x_1, \ldots, x_n]$$

for all positions $(x_1, \ldots, x_n)$ of Antonim and all $x \in \mathbb{N}$. Again, we checked functions f of the type $f(x) = (x+a) \oplus b$ or $f(x) = (x \oplus b) + a$, with $a \in \{c_1, c_1 + x, c_1 - x, c_1 \oplus x\}$ for some $c_1 \in \{-5, -4, \ldots, 5\}$ and $b \in \{c_2, c_2 + x, c_2 - x, c_2 \oplus x\}$ for some $c_2 \in \{-5, -4, \ldots, 5\}$. This also did not lead to a property of this type.

Note that Antonim and Antimatter do satisfy Welter's first property, and each have a property that is similar to Welter's second property. For Antonim this is

$$[\ 0 \mid x_1 \mid \cdots \mid x_n\ ] = [\ x_1 \mid \cdots \mid x_n\ ]$$

for all distinct $x_1, \ldots, x_n \in \mathbb{N} \setminus \{0\}$, and for Antimatter this is

$$[\ 0 \mid p_1 \mid \cdots \mid p_n \mid\mid 0 \mid e_1 \mid \cdots \mid e_m\ ] = [\ p_1 \mid \cdots \mid p_n \mid\mid e_1 \mid \cdots \mid e_m\ ]$$

whenever $(0, p_1, \ldots, p_n | 0, e_1, \ldots, e_m)$ is a feasible position for Antimatter.

7.2 3 coins

In order to better understand Welter's game with 3 coins, we made some tables. In each table, the upper left element is x_1, the first column gives x_2, the first row gives x_3, and the corresponding element in the table is $[\ x_1 \mid x_2 \mid x_3\ ]$. These tables turn out to show some interesting patterns. In this section, we describe these patterns, and prove that some of them hold more generally.

0	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**
0																																				
1			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
2		0		2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31	34
3		1	2		0	5	6	3	4	9	10	7	8	13	14	11	12	17	18	15	16	21	22	19	20	25	26	23	24	29	30	27	28	33	34	31
4		2	1	0		6	5	4	3	10	9	8	7	14	13	12	11	18	17	16	15	22	21	20	19	26	25	24	23	30	29	28	27	34	33	32
5		3	4	5	6		0	1	2	11	12	13	14	7	8	9	10	19	20	21	22	15	16	17	18	27	28	29	30	23	24	25	26	35	36	37
6		4	3	6	5	0		2	1	12	11	14	13	8	7	10	9	20	19	22	21	16	15	18	17	28	27	30	29	24	23	26	25	36	35	38
7		5	6	3	4	1	2		0	13	14	11	12	9	10	7	8	21	22	19	20	17	18	15	16	29	30	27	28	25	26	23	24	37	38	35
8		6	5	4	3	2	1	0		14	13	12	11	10	9	8	7	22	21	20	19	18	17	16	15	30	29	28	27	26	25	24	23	38	37	36
9		7	8	9	10	11	12	13	14		0	1	2	3	4	5	6	23	24	25	26	27	28	29	30	15	16	17	18	19	20	21	22	39	40	41
10		8	7	10	9	12	11	14	13	0		2	1	4	3	6	5	24	23	26	25	28	27	30	29	16	15	18	17	20	19	22	21	40	39	42
11		9	10	7	8	13	14	11	12	1	2		0	5	6	3	4	25	26	23	24	29	30	27	28	17	18	15	16	21	22	19	20	41	42	39
12		10	9	8	7	14	13	12	11	2	1	0		6	5	4	3	26	25	24	23	30	29	28	27	18	17	16	15	22	21	20	19	42	41	40
13		11	12	13	14	7	8	9	10	3	4	5	6		0	1	2	27	28	29	30	23	24	25	26	19	20	21	22	15	16	17	18	43	44	45
14		12	11	14	13	8	7	10	9	4	3	6	5	0		2	1	28	27	30	29	24	23	26	25	20	19	22	21	16	15	18	17	44	43	46
15		13	14	11	12	9	10	7	8	5	6	3	4	1	2		0	29	30	27	28	25	26	23	24	21	22	19	20	17	18	15	16	45	46	43
16		14	13	12	11	10	9	8	7	6	5	4	3	2	1	0		30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	46	45	44
17		15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	47	48	49
18		16	15	18	17	20	19	22	21	24	23	26	25	28	27	30	29	0		2	1	4	3	6	5	8	7	10	9	12	11	14	13	48	47	50
19		17	18	15	16	21	22	19	20	25	26	23	24	29	30	27	28	1	2		0	5	6	3	4	9	10	7	8	13	14	11	12	49	50	47
20		18	17	16	15	22	21	20	19	26	25	24	23	30	29	28	27	2	1	0		6	5	4	3	10	9	8	7	14	13	12	11	50	49	48
21		19	20	21	22	15	16	17	18	27	28	29	30	23	24	25	26	3	4	5	6		0	1	2	11	12	13	14	7	8	9	10	51	52	53
22		20	19	22	21	16	15	18	17	28	27	30	29	24	23	26	25	4	3	6	5	0		2	1	12	11	14	13	8	7	10	9	52	51	54
23		21	22	19	20	17	18	15	16	29	30	27	28	25	26	23	24	5	6	3	4	1	2		0	13	14	11	12	9	10	7	8	53	54	51
24		22	21	20	19	18	17	16	15	30	29	28	27	26	25	24	23	6	5	4	3	2	1	0		14	13	12	11	10	9	8	7	54	53	52
25		23	24	25	26	27	28	29	30	15	16	17	18	19	20	21	22	7	8	9	10	11	12	13	14		0	1	2	3	4	5	6	55	56	57
26		24	23	26	25	28	27	30	29	16	15	18	17	20	19	22	21	8	7	10	9	12	11	14	13	0		2	1	4	3	6	5	56	55	58
27		25	26	23	24	29	30	27	28	17	18	15	16	21	22	19	20	9	10	7	8	13	14	11	12	1	2		0	5	6	3	4	57	58	55
28		26	25	24	23	30	29	28	27	18	17	16	15	22	21	20	19	10	9	8	7	14	13	12	11	2	1	0		6	5	4	3	58	57	56
29		27	28	29	30	23	24	25	26	19	20	21	22	15	16	17	18	11	12	13	14	7	8	9	10	3	4	5	6		0	1	2	59	60	61
30		28	27	30	29	24	23	26	25	20	19	22	21	16	15	18	17	12	11	14	13	8	7	10	9	4	3	6	5	0		2	1	60	59	62
31		29	30	27	28	25	26	23	24	21	22	19	20	17	18	15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2		0	61	62	59
32		30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0		62	61	60
33		31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62		0	1
34		32	31	34	33	36	35	38	37	40	39	42	41	44	43	46	45	48	47	50	49	52	51	54	53	56	55	58	57	60	59	62	61	0		2
35		33	34	31	32	37	38	35	36	41	42	39	40	45	46	43	44	49	50	47	48	53	54	51	52	57	58	55	56	61	62	59	60	1	2	

Table 11: Sprague-Grundy values for Welter's game with 3 coins, with one coin on square 0.

First, we discuss the situation where $x_1 = 0$. From Table 11, we can see that $[\ 0 \mid x \mid y\ ] = (x - 1) \oplus (y - 1) - 1$ when $x, y \in \{1, \ldots, 35\}$. The fact that this holds in general follows from Welter's properties. So Table 11 has a pattern that is very similar to that found in the table for the Sprague-Grundy values of Nim with 2 coins. The difference is that the pattern is shifted one place to the right and one place down, and each number is decreased by 1. The pattern is as follows.

Each 2×2 block of the form

	$2\ell - 1$	2ℓ
$2k - 1$		
$2k$		

with $k, \ell \in \mathbb{N} \setminus \{0\}$ is filled as follows:

	$2\ell - 1$	2ℓ
$2k - 1$	$2m - 1$	$2m$
$2k$	$2m$	$2m - 1$

for some $m \in \mathbb{N}$. If $m = 0$, then the fields with $2m - 1$ are empty, and the corresponding positions are not feasible in Welter's game. This happens if and only if $k = \ell$.

So for each 4×4 block

	$4\ell - 3$	$4\ell - 2$	$4\ell - 1$	4ℓ
$4k - 3$	a		b	
$4k - 2$				
$4k - 1$	c		d	
$4k$				

with $k, \ell \in \mathbb{N} \setminus \{0\}$, it is sufficient to know the elements a, b, c and d. These are, respectively, m, $m + 2$, $m + 2$ and m, for some $m \in \mathbb{N}$, where m is odd. In general, we find that for every $n \in \mathbb{N} \setminus \{0\}$, the $2^n \times 2^n$ blocks are determined by the elements in the top-left corners of the four $2^{n-1} \times 2^{n-1}$ blocks that we get if the block is split into four parts of equal size. These elements are m, $m + 2^{n-1}$, $m + 2^{n-1}$ and m, for some $m \in \mathbb{N}$, where m is odd.

Next, we discuss the situation where one coin is on square 1. This corresponds to Table 12. Here, we used colours to signify some changes when comparing this table to Table 11. Let $k, \ell \in \mathbb{N}$. Then the 2×2 block

	2ℓ	$2\ell + 1$
$2k$		
$2k + 1$		

is filled with the values $[\ 0 \mid 2k \mid 2\ell\ ]$, $[\ 0 \mid 2k \mid 2\ell + 1\ ]$, $[\ 0 \mid 2k + 1 \mid 2\ell\ ]$ and $[\ 0 \mid 2k + 1 \mid 2\ell + 1\ ]$ in Table 11 and the values $[\ 1 \mid 2k \mid 2\ell\ ]$, $[\ 1 \mid 2k \mid 2\ell + 1\ ]$, $[\ 1 \mid 2k + 1 \mid 2\ell\ ]$ and $[\ 1 \mid 2k + 1 \mid 2\ell + 1\ ]$ in Table 12. Depending on these values, we assign a colour. These are as follows.

Table 11	Table 12	Colour
$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2m+1 & 2m \\ 2k+1 & 2n & 2n+1 \end{array}$ for some $m, n \in \mathbb{N}$	$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2n & 2n+1 \\ 2k+1 & 2m+1 & 2m \end{array}$	Red
$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2m+1 & 2n \\ 2k+1 & 2m & 2n+1 \end{array}$ for some $m, n \in \mathbb{N}$	$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2n & 2m+1 \\ 2k+1 & 2n+1 & 2m \end{array}$	Blue
$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2m+1 & 2n \\ 2k+1 & 2n & 2m+1 \end{array}$ for some $m, n \in \mathbb{N}$	$\begin{array}{c\|c\|c} & 2\ell & 2\ell+1 \\ \hline 2k & 2m & 2n+1 \\ 2k+1 & 2n+1 & 2m \end{array}$	Green

Here, the fields with $2m$ and $2m+1$ may be empty, if the corresponding positions are not feasible in Welter's game.

For example, the block

	2	3
4	4	5
5	1	0

in Table 12 is coloured red because the corresponding block in Table 11 is

	2	3
4	1	0
5	4	5

These 2×2 blocks are related to Welter's third property. It states that if a block in Table 11 is

	2ℓ	$2\ell+1$
$2k$	a	b
$2k+1$	c	d

then the corresponding block in Table 12 is

	2ℓ	$2\ell+1$
$2k$	$d \oplus 1$	$c \oplus 1$
$2k+1$	$b \oplus 1$	$a \oplus 1$

Here, a, b, c, and d may be empty if the corresponding positions are not feasible in Welter's game.

There is an interesting pattern in Table 12. For each $n \in \mathbb{N} \setminus \{0\}$, the block consisting of the first $2^n + 1$ rows and $2^n + 1$ columns of 2×2 blocks can be coloured as follows. The block in the first row and the first column is not coloured. The rest of the first row is coloured red, while the rest of the first column is coloured blue. The block in the bottom row and the rightmost column is green. The

1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1																																				
2	0			3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18	23	24	21	22	27	28	25	26	31	32	29	30	35
3	1		3		5	0	7	2	9	4	11	6	13	8	15	10	17	12	19	14	21	16	23	18	25	20	27	22	29	24	31	26	33	28	35	30
4	2		4	5		7	0	1	10	3	12	13	6	15	8	9	18	11	20	21	14	23	16	17	26	19	28	29	22	31	24	25	34	27	36	37
5	3		1	0	7		5	4	11	2	9	8	15	6	13	12	19	10	17	16	23	14	21	20	27	18	25	24	31	22	29	28	35	26	33	32
6	4		2	7	0	5		3	12	1	10	15	8	13	6	11	20	9	18	23	16	21	14	19	28	17	26	31	24	29	22	27	36	25	34	39
7	5		7	2	1	4	3		13	0	15	10	9	12	11	6	21	8	23	18	17	20	19	14	29	16	31	26	25	28	27	22	37	24	39	34
8	6		8	9	10	11	12	13		15	0	1	2	3	4	5	22	7	24	25	26	27	28	29	14	31	16	17	18	19	20	21	38	23	40	41
9	7		5	4	3	2	1	0	15		13	12	11	10	9	8	23	6	21	20	19	18	17	16	31	14	29	28	27	26	25	24	39	22	37	36
10	8		6	11	12	9	10	15	0	13		3	4	1	2	7	24	5	22	27	28	25	26	31	16	29	14	19	20	17	18	23	40	21	38	43
11	9		11	6	13	8	15	10	1	12	3		5	0	7	2	25	4	27	22	29	24	31	26	17	28	19	14	21	16	23	18	41	20	43	38
12	10		12	13	6	15	8	9	2	11	4	5		7	0	1	26	3	28	29	22	31	24	25	18	27	20	21	14	23	16	17	42	19	44	45
13	11		9	8	15	6	13	12	3	10	1	0	7		5	4	27	2	25	24	31	22	29	28	19	26	17	16	23	14	21	20	43	18	41	40
14	12		10	15	8	13	6	11	4	9	2	7	0	5		3	28	1	26	31	24	29	22	27	20	25	18	23	16	21	14	19	44	17	42	47
15	13		15	10	9	12	11	6	5	8	7	2	1	4	3		29	0	31	26	25	28	27	22	21	24	23	18	17	20	19	14	45	16	47	42
16	14		16	17	18	19	20	21	22	23	24	25	26	27	28	29		31	0	1	2	3	4	5	6	7	8	9	10	11	12	13	46	15	48	49
17	15		13	12	11	10	9	8	7	6	5	4	3	2	1	0	31		29	28	27	26	25	24	23	22	21	20	19	18	17	16	47	14	45	44
18	16		14	19	20	17	18	23	24	21	22	27	28	25	26	31	0	29		3	4	1	2	7	8	5	6	11	12	9	10	15	48	13	46	51
19	17		19	14	21	16	23	18	25	20	27	22	29	24	31	26	1	28	3		5	0	7	2	9	4	11	6	13	8	15	10	49	12	51	46
20	18		20	21	14	23	16	17	26	19	28	29	22	31	24	25	2	27	4	5		7	0	1	10	3	12	13	6	15	8	9	50	11	52	53
21	19		17	16	23	14	21	20	27	18	25	24	31	22	29	28	3	26	1	0	7		5	4	11	2	9	8	15	6	13	12	51	10	49	48
22	20		18	23	16	21	14	19	28	17	26	31	24	29	22	27	4	25	2	7	0	5		3	12	1	10	15	8	13	6	11	52	9	50	55
23	21		23	18	17	20	19	14	29	16	31	26	25	28	27	22	5	24	7	2	1	4	3		13	0	15	10	9	12	11	6	53	8	55	50
24	22		24	25	26	27	28	29	14	31	16	17	18	19	20	21	6	23	8	9	10	11	12	13		15	0	1	2	3	4	5	54	7	56	57
25	23		21	20	19	18	17	16	31	14	29	28	27	26	25	24	7	22	5	4	3	2	1	0	15		13	12	11	10	9	8	55	6	53	52
26	24		22	27	28	25	26	31	16	29	14	19	20	17	18	23	8	21	6	11	12	9	10	15	0	13		3	4	1	2	7	56	5	54	59
27	25		27	22	29	24	31	26	17	28	19	14	21	16	23	18	9	20	11	6	13	8	15	10	1	12	3		5	0	7	2	57	4	59	54
28	26		28	29	22	31	24	25	18	27	20	21	14	23	16	17	10	19	12	13	6	15	8	9	2	11	4	5		7	0	1	58	3	60	61
29	27		25	24	31	22	29	28	19	26	17	16	23	14	21	20	11	18	9	8	15	6	13	12	3	10	1	0	7		5	4	59	2	57	56
30	28		26	31	24	29	22	27	20	25	18	23	16	21	14	19	12	17	10	15	8	13	6	11	4	9	2	7	0	5		3	60	1	58	63
31	29		31	26	25	28	27	22	21	24	23	18	17	20	19	14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3		61	0	63	58
32	30		32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61		63	0	1
33	31		29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	63		61	60
34	32		30	35	36	33	34	39	40	37	38	43	44	41	42	47	48	45	46	51	52	49	50	55	56	53	54	59	60	57	58	63	0	61		3
35	33		35	30	37	32	39	34	41	36	43	38	45	40	47	42	49	44	51	46	53	48	55	50	57	52	59	54	61	56	63	58	1	60	3	

Table 12: Sprague-Grundy values for Welter's game with 3 coins, with one coin on square 1.

rest of the bottom row is coloured red, while the rest of the rightmost column is coloured blue. The middle block in the remaining uncoloured area is green, the rest of the middle row is coloured red and the rest of the middle column is coloured blue. This separates the area into 4 uncoloured areas of the same size. The middle rows and columns of each of these areas are coloured in the same way: the middle block is coloured green, the rest of the middle row is coloured red, and the rest of the middle column is coloured blue. This process is repeated until the entire area is coloured. Note that this pattern follows from the pattern in Table 11 combined with Welter's third property.

The table for the situation where one coin is on square 3 can be coloured in a similar way. This time the colour signifies the change when comparing to the situation where one coin is on square 2. Doing this leads to Table 13. Interestingly, this has the same pattern as that in Table 12, but shifted one row of 2×2 blocks down and one column of 2×2 blocks to the right. The second row of 2×2 blocks is now red, while the second column is blue. The colouring to the left of the blue line is the mirror image of the colouring on the right of it, and the colouring above the red line is the mirror image of the colouring below it.

If we colour the table for the situation with one coin on square 5 in the same way, we get Table 14. We find that the pattern is again shifted one row of 2×2 blocks down and one column of 2×2 blocks to the right. Again, the colouring to the left of the blue line is the mirror image of the colouring on the right of it, and the colouring above the red line is the mirror image of the colouring below it.

In general, we can show that this pattern repeats in all tables corresponding to the situation where one coin is on a specific odd-numbered square, assuming

3	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0		1	2		0	5	6	3	4	9	10	7	8	13	14	11	12	17	18	15	16	21	22	19	20	25	26	23	24	29	30	27	28	33	34	31
1	1		3		5	0	7	2	9	4	11	6	13	8	15	10	17	12	19	14	21	16	23	18	25	20	27	22	29	24	31	26	33	28	35	30
2	2	3			6	7	0	1	10	11	4	5	14	15	8	9	18	19	12	13	22	23	16	17	26	27	20	21	30	31	24	25	34	35	28	29
3																																				
4	0	5	6			1	2	7	8	13	14	3	4	9	10	15	16	21	22	11	12	17	18	23	24	29	30	19	20	25	26	31	32	37	38	27
5	5	0	7		1		3	6	13	8	15	2	9	4	11	14	21	16	23	10	17	12	19	22	29	24	31	18	25	20	27	30	37	32	39	26
6	6	7	0		2	3		5	14	15	8	1	10	11	4	13	22	23	16	9	18	19	12	21	30	31	24	17	26	27	20	29	38	39	32	25
7	3	2	1		7	6	5		11	10	9	0	15	14	13	4	19	18	17	8	23	22	21	12	27	26	25	16	31	30	29	20	35	34	33	24
8	4	9	10		8	13	14	11		1	2	15	0	5	6	3	20	25	26	7	24	29	30	27	12	17	18	31	16	21	22	19	36	41	42	23
9	9	4	11		13	8	15	10	1		3	14	5	0	7	2	25	20	27	6	29	24	31	26	17	12	19	30	21	16	23	18	41	36	43	22
10	10	11	4		14	15	8	9	2	3		13	6	7	0	1	26	27	20	5	30	31	24	25	18	19	12	29	22	23	16	17	42	43	36	21
11	7	6	5		3	2	1	0	15	14	13		11	10	9	8	23	22	21	4	19	18	17	16	31	30	29	12	27	26	25	24	39	38	37	20
12	8	13	14		4	9	10	15	0	5	6	11		1	2	7	24	29	30	3	20	25	26	31	16	21	22	27	12	17	18	23	40	45	46	19
13	13	8	15		9	4	11	14	5	0	7	10	1		3	6	29	24	31	2	25	20	27	30	21	16	23	26	17	12	19	22	45	40	47	18
14	14	15	8		10	11	4	13	6	7	0	9	2	3		5	30	31	24	1	26	27	20	29	22	23	16	25	18	19	12	21	46	47	40	17
15	11	10	9		15	14	13	4	3	2	1	8	7	6	5		27	26	25	0	31	30	29	20	19	18	17	24	23	22	21	12	43	42	41	16
16	12	17	18		16	21	22	19	20	25	26	23	24	29	30	27		1	2	31	0	5	6	3	4	9	10	7	8	13	14	11	44	49	50	15
17	17	12	19		21	16	23	18	25	20	27	22	29	24	31	26	1		3	30	5	0	7	2	9	4	11	6	13	8	15	10	49	44	51	14
18	18	19	12		22	23	16	17	26	27	20	21	30	31	24	25	2	3		29	6	7	0	1	10	11	4	5	14	15	8	9	50	51	44	13
19	15	14	13		11	10	9	8	7	6	5	4	3	2	1	0	31	30	29		27	26	25	24	23	22	21	20	19	18	17	16	47	46	45	12
20	16	21	22		12	17	18	23	24	29	30	19	20	25	26	31	0	5	6	27		1	2	7	8	13	14	3	4	9	10	15	48	53	54	11
21	21	16	23		17	12	19	22	29	24	31	18	25	20	27	30	5	0	7	26	1		3	6	13	8	15	2	9	4	11	14	53	48	55	10
22	22	23	16		18	19	12	21	30	31	24	17	26	27	20	29	6	7	0	25	2	3		5	14	15	8	1	10	11	4	13	54	55	48	9
23	19	18	17		23	22	21	12	27	26	25	16	31	30	29	20	3	2	1	24	7	6	5		11	10	9	0	15	14	13	4	51	50	49	8
24	20	25	26		24	29	30	27	12	17	18	31	16	21	22	19	4	9	10	23	8	13	14	11		1	2	15	0	5	6	3	52	57	58	7
25	25	20	27		29	24	31	26	17	12	19	30	21	16	23	18	9	4	11	22	13	8	15	10	1		3	14	5	0	7	2	57	52	59	6
26	26	27	20		30	31	24	25	18	19	12	29	22	23	16	17	10	11	4	21	14	15	8	9	2	3		13	6	7	0	1	58	59	52	5
27	23	22	21		19	18	17	16	31	30	29	12	27	26	25	24	7	6	5	20	3	2	1	0	15	14	13		11	10	9	8	55	54	53	4
28	24	29	30		20	25	26	31	16	21	22	27	12	17	18	23	8	13	14	19	4	9	10	15	0	5	6	11		1	2	7	56	61	62	3
29	29	24	31		25	20	27	30	21	16	23	26	17	12	19	22	13	8	15	18	9	4	11	14	5	0	7	10	1		3	6	61	56	63	2
30	30	31	24		26	27	20	29	22	23	16	25	18	19	12	21	14	15	8	17	10	11	4	13	6	7	0	9	2	3		5	62	63	56	1
31	27	26	25		31	30	29	20	19	18	17	24	23	22	21	12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5		59	58	57	0
32	28	33	34		32	37	38	35	36	41	42	39	40	45	46	43	44	49	50	47	48	53	54	51	52	57	58	55	56	61	62	59		1	2	63
33	33	28	35		37	32	39	34	41	36	43	38	45	40	47	42	49	44	51	46	53	48	55	50	57	52	59	54	61	56	63	58	1		3	62
34	34	35	28		38	39	32	33	42	43	36	37	46	47	40	41	50	51	44	45	54	55	48	49	58	59	52	53	62	63	56	57	2	3		61
35	31	30	29		27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	63	62	61	

Table 13: Sprague-Grundy values for Welter's game with 3 coins, with one coin on square 3.

5	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0		3	4	5	6		0	1	2	11	12	13	14	7	8	9	10	19	20	21	22	15	16	17	18	27	28	29	30	23	24	25	26	35	36	37
1	3		1	0	7		5	4	11	2	9	8	15	6	13	12	19	10	17	16	23	14	21	20	27	18	25	24	31	22	29	28	35	26	33	32
2	4	1		7	0		6	3	12	9	2	15	8	5	14	11	20	17	10	23	16	13	22	19	28	25	18	31	24	21	30	27	36	33	26	39
3	5	0	7		1		3	6	13	8	15	2	9	4	11	14	21	16	23	10	17	12	19	22	29	24	31	18	25	20	27	30	37	32	39	26
4	6	7	0	1			4	5	14	15	8	9	2	3	12	13	22	23	16	17	10	11	20	21	30	31	24	25	18	19	28	29	38	39	32	33
5																																				
6	0	5	6	3	4			7	8	13	14	11	12	1	2	15	16	21	22	19	20	9	10	23	24	29	30	27	28	17	18	31	32	37	38	35
7	1	4	3	6	5		7		9	12	11	14	13	0	15	2	17	20	19	22	21	8	23	10	25	28	27	30	29	16	31	18	33	36	35	38
8	2	11	12	13	14		8	9		3	4	5	6	15	0	1	18	27	28	29	30	7	24	25	10	19	20	21	22	31	16	17	34	43	44	45
9	11	2	9	8	15		13	12	3		1	0	7	14	5	4	27	18	25	24	31	6	29	28	19	10	17	16	23	30	21	20	43	34	41	40
10	12	9	2	15	8		14	11	4	1		7	0	13	6	3	28	25	18	31	24	5	30	27	20	17	10	23	16	29	22	19	44	41	34	47
11	13	8	15	2	9		11	14	5	0	7		1	12	3	6	29	24	31	18	25	4	27	30	21	16	23	10	17	28	19	22	45	40	47	34
12	14	15	8	9	2		12	13	6	7	0	1		11	4	5	30	31	24	25	18	3	28	29	22	23	16	17	10	27	20	21	46	47	40	41
13	7	6	5	4	3		1	0	15	14	13	12	11		9	8	23	22	21	20	19	2	17	16	31	30	29	28	27	10	25	24	39	38	37	36
14	8	13	14	11	12		2	15	0	5	6	3	4	9		7	24	29	30	27	28	1	18	31	16	21	22	19	20	25	10	23	40	45	46	43
15	9	12	11	14	13		15	2	1	4	3	6	5	8	7		25	28	27	30	29	0	31	18	17	20	19	22	21	24	23	10	41	44	43	46
16	10	19	20	21	22		16	17	18	27	28	29	30	23	24	25		3	4	5	6	31	0	1	2	11	12	13	14	7	8	9	42	51	52	53
17	19	10	17	16	23		21	20	27	18	25	24	31	22	29	28	3		1	0	7	30	5	4	11	2	9	8	15	6	13	12	51	42	49	48
18	20	17	10	23	16		22	19	28	25	18	31	24	21	30	27	4	1		7	0	29	6	3	12	9	2	15	8	5	14	11	52	49	42	55
19	21	16	23	10	17		19	22	29	24	31	18	25	20	27	30	5	0	7		1	28	3	6	13	8	15	2	9	4	11	14	53	48	55	42
20	22	23	16	17	10		20	21	30	31	24	25	18	19	28	29	6	7	0	1		27	4	5	14	15	8	9	2	3	12	13	54	55	48	49
21	15	14	13	12	11		9	8	7	6	5	4	3	2	1	0	31	30	29	28	27		25	24	23	22	21	20	19	18	17	16	47	46	45	44
22	16	21	22	19	20		10	23	24	29	30	27	28	17	18	31	0	5	6	3	4	25		7	8	13	14	11	12	1	2	15	48	53	54	51
23	17	20	19	22	21		23	10	25	28	27	30	29	16	31	18	1	4	3	6	5	24	7		9	12	11	14	13	0	15	2	49	52	51	54
24	18	27	28	29	30		24	25	10	19	20	21	22	31	16	17	2	11	12	13	14	23	8	9		3	4	5	6	15	0	1	50	59	60	61
25	27	18	25	24	31		29	28	19	10	17	16	23	30	21	20	11	2	9	8	15	22	13	12	3		1	0	7	14	5	4	59	50	57	56
26	28	25	18	31	24		30	27	20	17	10	23	16	29	22	19	12	9	2	15	8	21	14	11	4	1		7	0	13	6	3	60	57	50	63
27	29	24	31	18	25		27	30	21	16	23	10	17	28	19	22	13	8	15	2	9	20	11	14	5	0	7		1	12	3	6	61	56	63	50
28	30	31	24	25	18		28	29	22	23	16	17	10	27	20	21	14	15	8	9	2	19	12	13	6	7	0	1		11	4	5	62	63	56	57
29	23	22	21	20	19		17	16	31	30	29	28	27	10	25	24	7	6	5	4	3	18	1	0	15	14	13	12	11		9	8	55	54	53	52
30	24	29	30	27	28		18	31	16	21	22	19	20	25	10	23	8	13	14	11	12	17	2	15	0	5	6	3	4	9		7	56	61	62	59
31	25	28	27	30	29		31	18	17	20	19	22	21	24	23	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7		57	60	59	62
32	26	35	36	37	38		32	33	34	43	44	45	46	39	40	41	42	51	52	53	54	47	48	49	50	59	60	61	62	55	56	57		3	4	5
33	35	26	33	32	39		37	36	43	34	41	40	47	38	45	44	51	42	49	48	55	46	53	52	59	50	57	56	63	54	61	60	3		1	0
34	36	33	26	39	32		38	35	44	41	34	47	40	37	46	43	52	49	42	55	48	45	54	51	60	57	50	63	56	53	62	59	4	1		7
35	37	32	39	26	33		35	38	45	40	47	34	41	36	43	46	53	48	55	42	49	44	51	54	61	56	63	50	57	52	59	62	5	0	7	

Table 14: Sprague-Grundy values for Welter's game with 3 coins, with one coin on square 5.

that Welter's third property holds and that $[\,0 \mid x \mid y\,] = (x-1) \oplus (y-1) - 1$ for all $x, y \in \mathbb{N}$.

Theorem 7.6. *The colour patterns in all the tables corresponding to a situation where one coin is on a specific odd-numbered square are as described above.*

Proof. We call the table corresponding to the situation where one coin is on square x the x-table, for all $x \in \mathbb{N}$.

The pattern of the 0-table follows from the fact that

$$[\,0 \mid x \mid y\,] = (x-1) \oplus (y-1) - 1$$

for all $x, y \in \mathbb{N}$, and directly leads to the colouring of the 1-table.

Now let $n \in \mathbb{N} \setminus \{0\}$. Let $k, \ell \in \mathbb{N}$. Then if the 0-table has the 2×2 block

	2ℓ	$2\ell+1$
$2k$	a	b
$2k+1$	c	d

then the 2^n-table has the 2×2 block

	$(2\ell) \oplus (2^n)$	$(2\ell+1) \oplus (2^n)$
$(2k) \oplus (2^n)$	$a \oplus (2^n)$	$b \oplus (2^n)$
$(2k+1) \oplus (2^n)$	$c \oplus (2^n)$	$d \oplus (2^n)$

This means that the (2^n+1)-table has the 2×2 block

	$(2\ell) \oplus (2^n)$	$(2\ell+1) \oplus (2^n)$
$(2k) \oplus (2^n)$	$d \oplus (2^n) \oplus 1$	$c \oplus (2^n) \oplus 1$
$(2k+1) \oplus (2^n)$	$b \oplus (2^n) \oplus 1$	$a \oplus (2^n) \oplus 1$

The numbers in the 2×2 blocks of the 2^n-table and (2^n+1)-table relate to each other in the same way as those in the 2×2 blocks of the 0-table and the 1-table. So the 2×2 block

	$(2\ell) \oplus (2^n)$	$(2\ell+1) \oplus (2^n)$
$(2k) \oplus (2^n)$		
$(2k+1) \oplus (2^n)$		

of the (2^n+1)-table has the same colour as the 2×2 block

	2ℓ	$2\ell+1$
$2k$		
$2k+1$		

of the 1-table.

Now we know that each coloured block in the (2^n+1)-table corresponds to a block

	2ℓ	$2\ell+1$
$2k$		
$2k+1$		

in the 1-table with the same colour, for some $k, \ell \in \mathbb{N}$. The block in the (2^n+1)-table is 2^{n-1} 2×2 blocks to the right of the one in the 1-table if $2\ell \equiv i \mod 2^{n+1}$ for some $i \in \{0, \ldots, 2^n-1\}$, and 2^{n-1} 2×2 blocks to the left if $2\ell \equiv i \mod 2^{n+1}$ for some $i \in \{2^n, \ldots, 2^{n+1}-1\}$. Similarly, the block is 2^{n-1} 2×2 blocks down from the one in the 1-table if $2k \equiv i \mod 2^{n+1}$ for some $i \in \{0, \ldots, 2^n-1\}$, and 2^{n-1} blocks up from the one in the 1-table if $2k \equiv i \mod 2^{n+1}$ for some $i \in \{2^n, \ldots, 2^{n+1}-1\}$.

This means that we can divide the area into larger blocks consisting of 2^n by 2^n coloured blocks in the 1-table, each starting at a position $(2k, 2\ell)$ with $2k, 2\ell \equiv 0 \mod 2^{n+1}$. Such a block can be divided into 4 blocks of equal size, each consisting of 2^{n-1} by 2^{n-1} coloured blocks. We call these A, B, C and D, as follows.

A	B
C	D

Then the (2^n+1)-table has these same coloured blocks, but in a different order, namely

D	C
B	A

These blocks have another interesting property. Each of these $2^n \times 2^n$ blocks is symmetric in the 2^{n-2}nd row of 2×2 blocks, so that the second row is equal to the last row, the third row is equal to the second last row, et cetera. Similarly, the second column is equal to the last column, the third column is equal to the second last column, et cetera.

Looking at the 1-table, we see that the blocks A, B, C, D can differ only on the colour of the block in the upper-left. This is always red for B, blue for C and green for D. For A, it can be red, blue, green or empty if none of the positions in that 2×2 block are feasible. We now call a $2^{n-1} \times 2^{n-1}$ block of coloured blocks B, R or G, depending on whether the upper-left 2×2 block is blue, red or green, respectively. Then the 1-table has the pattern

Y_{11}	R	Y_{12}	R	Y_{13}	R	$\ldots$
B	G	B	G	B	G	$\ldots$
Y_{21}	R	Y_{22}	R	Y_{23}	R	$\ldots$
B	G	B	G	B	G	$\ldots$
Y_{31}	R	Y_{32}	R	Y_{33}	R	$\ldots$
B	G	B	G	B	G	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

for some colours $Y_{i,j}$, $i, j \in \mathbb{N} \setminus \{0\}$. So the (2^n+1)-table has the following pattern:

G	B	G	B	G	B	$\dots$
R	Y_{11}	R	Y_{12}	R	Y_{13}	$\dots$
G	B	G	B	G	B	$\dots$
R	Y_{21}	R	Y_{22}	R	Y_{23}	$\dots$
G	B	G	B	G	B	$\dots$
R	Y_{31}	R	Y_{32}	R	R_{33}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For all $i, j \in \mathbb{N} \setminus \{0\}$, the Y_{ij} block is moved 2^{n-1} blocks to the right and 2^{n-1} blocks down. To the right of the Y_{ij} block, we get an R block. Below it we get a B block, and to the bottom right a G block. This means that the pattern of the 1-table is moved 2^{n-1} blocks to the right and 2^{n-1} blocks down. Note that $Y_{1i} = R$ for all $i \geq 2$, and $Y_{i1} = B$ for all $i \geq 2$. In the 1-table, both the 2nd and 4th rows of $2^{n-1} \times 2^{n-1}$ coloured blocks consist of alternating B blocks and G blocks, starting with a B block. So in the $(2^n + 1)$-table, both the 1st and 3rd rows of $2^{n-1} \times 2^{n-1}$ coloured blocks consist of alternating G blocks and B blocks, starting with a G block. Recall that each $2^{n-1} \times 2^{n-1}$ block is symmetric in its 2^{n-2}nd line, and the blocks only differ on the colour of the upper-left 2×2 block. this means that the colouring above the red line is the mirror image of the colouring below it. Similarly, the colouring to the left of the blue line is the mirror image of the colouring to the right of it.

So for each n, the $(2^n + 1)$-table has the colouring as described. Now let $m < n$. Then the $(2^n + 1)$-table has the correct colouring. We will show that the $(2^n + 2^m + 1)$-table also has the correct colouring.

This time, we compare the $(2^n + 1)$-table to the $(2^n + 2^m + 1)$-table. As before, the area is divided into $2^{m-1} \times 2^{m-1}$ blocks. Coloured $2^{m-1} \times 2^{m-1}$ blocks that appear in the $(2^n + 1)$-table also appear in the $(2^n + 2^m + 1)$-table, but in a different position. In the $(2^n + 1)$-table, the empty 2×2 block at the start of the shifted pattern is in the $(2^{n-1} + 1)$th row and the $(2^{n-1} + 1)$th column of 2×2 blocks. Because 2^m is a divisor of 2^{n-1}, this block is at the top left of a group of four $2^{m-1} \times 2^{m-1}$ blocks. This means that the $2^{m-1} \times 2^{m-1}$ blocks in the $(2^n + 1)$-table follow the same pattern as before, where for each group of four blocks, the one on the top right is coloured R, the one on the bottom left is coloured B and the one on the bottom right is coloured G. Because the colouring above the red line is the mirror image of the colouring below it, the rows above the red line also follow this pattern. Similarly, the columns to the left of the blue line follow this pattern. So the $2^{m-1} \times 2^{m-1}$ blocks in the $(2^n + 1)$-table have the following pattern:

Y_{33}	R	Y_{32}	R	B	R	Y_{32}	R	Y_{33}	R	...
B	G	B	G	B	G	B	G	B	G	...
Y_{23}	R	Y_{22}	R	B	R	Y_{22}	R	Y_{23}	R	...
B	G	B	G	B	G	B	G	B	G	...
R	R	R	R		R	R	R	R	R	...
B	G	B	G	B	G	B	G	B	G	...
Y_{23}	R	Y_{22}	R	B	R	Y_{22}	R	Y_{23}	R	...
B	G	B	G	B	G	B	G	B	G	...
Y_{33}	R	Y_{32}	R	B	R	Y_{32}	R	Y_{33}	R	...
B	G	B	G	B	G	B	G	B	G	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

So in the $(2^n + 2^m + 1)$-table, we find the following pattern:

G	B	G	B	G	B	G	B	G	B	G	...
R	Y_{33}	R	Y_{32}	R	B	R	Y_{32}	R	Y_{33}	R	...
G	B	G	B	G	B	G	B	G	B	G	...
R	Y_{23}	R	Y_{22}	R	B	R	Y_{22}	R	Y_{23}	R	...
G	B	G	B	G	B	G	B	G	B	G	...
R	R	R	R	R		R	R	R	R	R	...
G	B	G	B	G	B	G	B	G	B	G	...
R	Y_{23}	R	Y_{22}	R	B	R	Y_{22}	R	Y_{23}	R	...
G	B	G	B	G	B	G	B	G	B	G	...
R	Y_{33}	R	Y_{32}	R	B	R	Y_{32}	R	Y_{33}	R	...
G	B	G	B	G	B	G	B	G	B	G	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Then, as before, we find that the $(2^n + 2^m + 1)$-table contains the colouring of the $(2^n + 1)$-table, shifted 2^{m-1} 2×2 blocks to the right and 2^{m-1} 2×2 blocks down. Further, because all even rows of $2^{m-1} \times 2^{m-1}$ coloured blocks in the $(2^n + 1)$-table consist of alternating B blocks and G blocks, starting with a B block, and the $2^{m-1} \times 2^{m-1}$ blocks are symmetric in their 2^{m-2}nd lines, the colouring above the red line in the $(2^n + 2^m + 1)$-table is still the mirror image of the colouring below it. Similarly, the colouring to the left of the blue line is the mirror image of the colouring to the right of it.

By repeatedly adding smaller powers of 2, we find that all x-tables for odd x have the colouring as described.

□

While the tables give some insight into the effect of Welter's third property, they do not seem to help in proving that this property holds. However, we did notice a few more things in these tables.

The tables corresponding to the situation where one coin is on an even-numbered square also have some interesting patterns, though it is not so easy to see why they have these patterns. As an example, we describe the pattern in the table corresponding to the situation where one coin is on square 2. For this table, we colour the 2×2 blocks

	$2\ell-1$	2ℓ
$2k-1$		
$2k$		

for $k, \ell \in \mathbb{N} \setminus \{0\}$. We use the following colours.

Table 12	Table 15	Colour
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2m-1 \\ 2k & 2n-1 & 2n \end{array}$ for some $m, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2n-1 & 2n \\ 2k & 2m & 2m-1 \end{array}$	Red
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2n-1 \\ 2k & 2m-1 & 2n \end{array}$ for some $m, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m-1 & 2n \\ 2k & 2m & 2n-1 \end{array}$	Red
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2n-1 \\ 2k & 2m-1 & 2n \end{array}$ for some $m, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2n-1 & 2m \\ 2k & 2n & 2m-1 \end{array}$	Blue
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2m-1 \\ 2k & 2n-1 & 2n \end{array}$ for some $m, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m-1 & 2m \\ 2k & 2n & 2n-1 \end{array}$	Blue
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2n-1 \\ 2k & 2n-1 & 2m \end{array}$ for some $m, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m-1 & 2n \\ 2k & 2n & 2m-1 \end{array}$	Green
$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2m & 2n-1 \\ 2k & 2n-1 & 2m \end{array}$ for some $m \in \mathbb{N}, n \in \mathbb{N} \setminus \{0\}$	$\begin{array}{ccc} & 2\ell-1 & 2\ell \\ 2k-1 & 2n-1 & 2m \\ 2k & 2m & 2n-1 \end{array}$	Purple

Here, the fields with $2m$ and $2m-1$ may be empty, if the corresponding positions are not feasible in Welter's game.

While the coloured 2×2 blocks for the situation where one coin is on a specific odd-numbered square are clearly related to Welter's third property, this is not so obvious when a coin is on an even-numbered square. For example, Welter's third property states that if a block in Table 12 is

	$4\ell-1$	4ℓ
$4k-1$	a	b
$4k$	c	d

for some $k, \ell \in \mathbb{N} \setminus \{0\}$, $a, b, c, d \in \mathbb{N}$, then Table 15 has

	$4(\ell-1)$	$4\ell-3$	$4\ell-2$	$4\ell-1$	4ℓ	$4\ell+1$	$4\ell+2$	$4\ell+3$
$4(k-1)$	$a \oplus 3$							$b \oplus 3$
$4k-3$								
$4k-2$								
$4k-1$								
$4k$								
$4k+1$								
$4k+2$								
$4k+3$	$c \oplus 3$							$d \oplus 3$

But that does not indicate a relation between a 2×2 block

	$2\ell-1$	2ℓ
$2k-1$		
$2k$		

in Table 12 and the corresponding block

	$2\ell-1$	2ℓ
$2k-1$		
$2k$		

in Table 15. Even so, the coloured blocks form interesting patterns.

2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0		0		2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31	34
1	0			3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18	23	24	21	22	27	28	25	26	31	32	29	30	35
2																																				
3	2	3			6	7	0	1	10	11	4	5	14	15	8	9	18	19	12	13	22	23	16	17	26	27	20	21	30	31	24	25	34	35	28	29
4	1	4		6		0	7	2	9	12	3	14	5	8	15	10	17	20	11	22	13	16	23	18	25	28	19	30	21	24	31	26	33	36	27	38
5	4	1		7	0		6	3	12	9	2	15	8	5	14	11	20	17	10	23	16	13	22	19	28	25	18	31	24	21	30	27	36	33	26	39
6	3	2		0	7	6		4	11	10	1	8	15	14	5	12	19	18	9	16	23	22	13	20	27	26	17	24	31	30	21	28	35	34	25	32
7	6	7		1	2	3	4		14	15	0	9	10	11	12	5	22	23	8	17	18	19	20	13	30	31	16	25	26	27	28	21	38	39	24	33
8	5	8		10	9	12	11	14		0	15	2	1	4	3	6	21	24	7	26	25	28	27	30	13	16	31	18	17	20	19	22	37	40	23	42
9	8	5		11	12	9	10	15	0		14	3	4	1	2	7	24	21	6	27	28	25	26	31	16	13	30	19	20	17	18	23	40	37	22	43
10	7	6		4	3	2	1	0	15	14		12	11	10	9	8	23	22	5	20	19	18	17	16	31	30	13	28	27	26	25	24	39	38	21	36
11	10	11		5	14	15	8	9	2	3	12		6	7	0	1	26	27	4	21	30	31	24	25	18	19	28	13	22	23	16	17	42	43	20	37
12	9	12		14	5	8	15	10	1	4	11	6		0	7	2	25	28	3	30	21	24	31	26	17	20	27	22	13	16	23	18	41	44	19	46
13	12	9		15	8	5	14	11	4	1	10	7	0		6	3	28	25	2	31	24	21	30	27	20	17	26	23	16	13	22	19	44	41	18	47
14	11	10		8	15	14	5	12	3	2	9	0	7	6		4	27	26	1	24	31	30	21	28	19	18	25	16	23	22	13	20	43	42	17	40
15	14	15		9	10	11	12	5	6	7	8	1	2	3	4		30	31	0	25	26	27	28	21	22	23	24	17	18	19	20	13	46	47	16	41
16	13	16		18	17	20	19	22	21	24	23	26	25	28	27	30		0	31	2	1	4	3	6	5	8	7	10	9	12	11	14	45	48	15	50
17	16	13		19	20	17	18	23	24	21	22	27	28	25	26	31	0		30	3	4	1	2	7	8	5	6	11	12	9	10	15	48	45	14	51
18	15	14		12	11	10	9	8	7	6	5	4	3	2	1	0	31	30		28	27	26	25	24	23	22	21	20	19	18	17	16	47	46	13	44
19	18	19		13	22	23	16	17	26	27	20	21	30	31	24	25	2	3	28		6	7	0	1	10	11	4	5	14	15	8	9	50	51	12	45
20	17	20		22	13	16	23	18	25	28	19	30	21	24	31	26	1	4	27	6		0	7	2	9	12	3	14	5	8	15	10	49	52	11	54
21	20	17		23	16	13	22	19	28	25	18	31	24	21	30	27	4	1	26	7	0		6	3	12	9	2	15	8	5	14	11	52	49	10	55
22	19	18		16	23	22	13	20	27	26	17	24	31	30	21	28	3	2	25	0	7	6		4	11	10	1	8	15	14	5	12	51	50	9	48
23	22	23		17	18	19	20	13	30	31	16	25	26	27	28	21	6	7	24	1	2	3	4		14	15	0	9	10	11	12	5	54	55	8	49
24	21	24		26	25	28	27	30	13	16	31	18	17	20	19	22	5	8	23	10	9	12	11	14		0	15	2	1	4	3	6	53	56	7	58
25	24	21		27	28	25	26	31	16	13	30	19	20	17	18	23	8	5	22	11	12	9	10	15	0		14	3	4	1	2	7	56	53	6	59
26	23	22		20	19	18	17	16	31	30	13	28	27	26	25	24	7	6	21	4	3	2	1	0	15	14		12	11	10	9	8	55	54	5	52
27	26	27		21	30	31	24	25	18	19	28	13	22	23	16	17	10	11	20	5	14	15	8	9	2	3	12		6	7	0	1	58	59	4	53
28	25	28		30	21	24	31	26	17	20	27	22	13	16	23	18	9	12	19	14	5	8	15	10	1	4	11	6		0	7	2	57	60	3	62
29	28	25		31	24	21	30	27	20	17	26	23	16	13	22	19	12	9	18	15	8	5	14	11	4	1	10	7	0		6	3	60	57	2	63
30	27	26		24	31	30	21	28	19	18	25	16	23	22	13	20	11	10	17	8	15	14	5	12	3	2	9	0	7	6		4	59	58	1	56
31	30	31		25	26	27	28	21	22	23	24	17	18	19	20	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4		62	63	0	57
32	29	32		34	33	36	35	38	37	40	39	42	41	44	43	46	45	48	47	50	49	52	51	54	53	56	55	58	57	60	59	62		0	63	2
33	32	29		35	36	33	34	39	40	37	38	43	44	41	42	47	48	45	46	51	52	49	50	55	56	53	54	59	60	57	58	63	0		62	3
34	31	30		28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	63	62		60
35	34	35		29	38	39	32	33	42	43	36	37	46	47	40	41	50	51	44	45	54	55	48	49	58	59	52	53	62	63	56	57	2	3	60	

Table 15: Sprague-Grundy values for Welter's game with 3 coins, with one coin on square 2.

The table corresponding to the situation with one coin on square 2 is Table 15. For each $n \in \mathbb{N} \setminus \{0\}$, the block with the first 2^n rows and 2^n columns of 2×2 blocks is coloured as follows. The first row is red, while the first column is blue. The block in the last row and the last column is green. The rest of the bottom row is blue, while the rest of the rightmost column is red. The four middle blocks in the remaining uncoloured area are coloured as follows: the top left of these four is green, the top right is purple, the bottom left is purple and the bottom right is green. Of the middle two rows, the rest of the bottom row is coloured red, while that of the top row is coloured blue. The rest of left middle column is coloured red and the rest of the right one is coloured blue. The area is now separated into 4 uncoloured areas of the same size. The middle rows and columns of each of these areas are coloured in the same way as before. This process is repeated until the entire area is coloured.

In the tables for the situation where one coin is on an even-numbered square as well as the ones for the situation where one coin is on an odd-numbered square, we noticed the following. In all the tables, the elements in the 2×2 blocks

	2ℓ	$2\ell+1$
$2k$		
$2k+1$		

for all $k, \ell \in \mathbb{N}$ and

	$2\ell-1$	2ℓ
$2k-1$		
$2k$		

for all $k, \ell \in \mathbb{N} \setminus \{0\}$ seem to have one of the five possible forms. These are

x	y
y	x

;

$x+1$	x
y	$y+1$

;

$x+1$	y
x	$y+1$

;

x	$x+1$
$y+1$	y

;

x	$y+1$
$x+1$	y

for some $x, y \in \mathbb{N}$. However, we have not been able to prove this.

By looking at the three possible 2×2 patterns for the tables corresponding to the situation where one coin is on an odd-numbered square, we see that for all $x, y, z \in \mathbb{N}$, the Sprague-Grundy value $[\, x \mid y \mid z \,]$ is even if and only if $x + y + z$ is odd. We can also prove this using Welter's properties.

Lemma 7.7. *For any distinct $x, y, z \in \mathbb{N}$, $[\ x \mid y \mid z\ ]$ is even if and only if $x + y + z$ is odd.*

Proof. Let $x, y, z \in \mathbb{N}$ be distinct. Assume that $x = 0$. Then we have

$$[\, 0 \mid y \mid z \,] = [\, y-1 \mid z-1 \,] = [\, 0 \mid (y-1) \oplus (z-1) \,] = (y-1) \oplus (z-1) - 1.$$

This is even if and only if $(y-1) \oplus (z-1)$ is odd, which is true if and only if the parity of $y-1$ is unequal to that of $z-1$. This occurs if and only if $0 + y + z$ is odd.

Now assume that $x > 0$. Then $[\ x \mid y \mid z\ ] = [\ 0 \mid y \oplus x \mid z \oplus x\ ] \oplus x$. This is even if and only if $[\, 0 \mid y \oplus x \mid z \oplus x \,]$ is even and x is even or $[\, 0 \mid y \oplus x \mid z \oplus x \,]$ is odd and x is odd. Because of what we proved already, this is equivalent to: $y \oplus x + z \oplus x$ is odd and x is even, or $y \oplus x + z \oplus x$ is even and x is odd.

Note that $y \oplus x$ has the same parity as $z \oplus x$ if and only if y has the same parity as z. It follows that $y \oplus x + z \oplus x$ has the same parity as $y + z$. So $[\ x \mid y \mid z\ ]$ is even if and only if either $y + z$ is odd and x is even, or $y + z$ is even and x is odd. So $[\ x \mid y \mid z\ ]$ is even if and only if $x + y + z$ is odd.

□

8 Misère Welter's game

In this chapter, we describe the optimal strategy for the misère version of Welter's game. This is a more detailed version of the proof in [1]. The solution for misère Welter uses a strategy on the Abacus positions, which we define first.

Definition 8.1. A position $(x_1, \ldots, x_n)$ is called an Abacus position if $x_i \in \{i-1, 2n-i\}$ for all $i \in \{1, \ldots, n\}$.

While we have previously used the Sprague-Grundy function to find the optimal strategy for Welter's game played with the normal play rule, we will now use the same function in order to solve the misère version of Welter's game. The Sprague-Grundy values for all Abacus positions follow from the following lemma and the Even Alteration Theorem.

Lemma 8.2. *For all $n \in \mathbb{N}$, we have*

$$\left[\begin{array}{l|l|l|l|l|l|l} 0 & 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 2n-1 & 2n-2 & 2n-3 & \ldots & n+2 & n+1 & n \end{array}\right] = \begin{array}{c} 0 \\ 1. \end{array}$$

Proof. Note that $[\ 0 \mid 1 \mid \cdots \mid n-1\ ]$ is a terminal position, so it has Sprague-Grundy value 0. By the Even Alteration Theorem, it is then enough to show that

$$\begin{aligned} [\ 2n-1 \mid 1 \mid 2 \mid \cdots \mid n-1\ ] &= [\ 0 \mid 2n-2 \mid 2 \mid 3 \mid \cdots \mid n-1\ ] \\ &= \cdots = [\ 0 \mid 1 \mid \cdots \mid n-2 \mid n\ ] = 1. \end{aligned}$$

We will use induction to show that

$$[\ 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i' \mid i+1 \mid \cdots \mid n-1\ ] = 1$$

for all $i \in \{0, \ldots, n-1\}$, where $i' = 2n-1-i$ for all i. We start with $i = n-1$. The position $(0, 1, \ldots, n-2, n)$ only has one follower, $(0, 1, \ldots, n-2, n-1)$, which has Sprague-Grundy value 0. So $[\ 0 \mid 1 \mid \cdots \mid n-2 \mid n\ ] = 1$.

Now let $i < n-1$ and assume that

$$[\ 0 \mid 1 \mid 2 \mid \cdots \mid j-1 \mid j' \mid j+1 \mid \cdots \mid n-1\ ] = 1$$

for all $j > i$. We will show that

$$[\ 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i' \mid i+1 \mid \cdots \mid n-1\ ] = 1.$$

In the position $(0, 1, 2, \ldots, i-1, i', i+1, \ldots, n-1)$, the coins on the squares $0, 1, 2, \ldots, i-1$ cannot be moved. The coin on square i' can be moved to square i, or to any square in $\{n, n+1, \ldots, i'-1\} = \{j' : j > i\}$. The coins on squares $i+1, \ldots, n-1$ can only be moved to square i. So the followers of the position $(0, 1, 2, \ldots, i-1, i', i+1, \ldots, n-1)$ are

$$(0, 1, 2, \ldots, i-1, i, i+1, \ldots, n-1),$$

which has Sprague-Grundy value 0,

$$(0, 1, 2, \ldots, i-1, j', i+1, \ldots, n-1)$$

for any $j > i$, and

$$(0, 1, 2, \ldots, i-1, i', i+1, \ldots, j-1, i, j+1, \ldots, n-1)$$

for any $j > i$.

Let $j > i$. By the induction hypothesis, we have

$$[\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i \mid i+1 \mid \cdots \mid j-1 \mid j' \mid j+1 \mid \cdots \mid n-1 \,] = 1.$$

Recall that, if one position is a follower of another, their Sprague-Grundy values are not the same. Using Symmetry, we can conclude that

$$\begin{aligned} &[\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid j' \mid i+1 \mid \cdots \mid n-1 \,] \\ &= [\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid j \mid i+1 \mid \cdots \mid j-1 \mid j' \mid j+1 \mid \cdots \mid n-1 \,] \\ &\neq [\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i \mid i+1 \mid \cdots \mid j-1 \mid j' \mid j+1 \mid \cdots \mid n-1 \,] = 1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &[\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i' \mid i+1 \mid \cdots \mid j-1 \mid i \mid j+1 \mid \cdots \mid n-1 \,] \\ &= [\, 0 \mid 1 \mid 2 \mid \cdots \mid j-1 \mid i' \mid j+1 \mid \cdots \mid n-1 \,] \\ &\neq [\, 0 \mid 1 \mid 2 \mid \cdots \mid j-1 \mid j' \mid j+1 \mid \cdots \mid n-1 \,] = 1. \end{aligned}$$

So $(0, 1, 2, \ldots, i-1, i', i+1, \ldots, n-1)$ has a follower with Sprague-Grundy value 0 and does not have a follower with Sprague-Grundy value 1, which means that

$$[\, 0 \mid 1 \mid 2 \mid \cdots \mid i-1 \mid i' \mid i+1 \mid \cdots \mid n-1 \,] = 1.$$

□

Next, we discuss a winning strategy on the Abacus positions.

Lemma 8.3. *If the game is in a non-terminal Abacus position with Sprague-Grundy value 0, and the opposing player makes a move, it is possible to move to another Abacus position with Sprague-Grundy value 0. Also, if the game is in an Abacus position with Sprague-Grundy value 1, and the opposing player makes a move to a non-terminal position, it is possible to move to another Abacus position with Sprague-Grundy value 1.*

Proof. Let $(x_1, \ldots, x_n)$ be an Abacus position. Let $x'_i = 2n - 1 - x_i$ for all $i \in \{1, \ldots, n\}$. Assume that the opponent moves the coin on square x_i to square x, for some $i \in \{1, \ldots, n\}$ and $x < x_i$. Without loss of generality, assume that $i = 1$. Because $\{x_1, \ldots, x_n, x'_1, \ldots, x'_n\} = \{0, 1, \ldots, 2n-1\}$, we have $x \in \{x'_1, \ldots, x'_n\}$.

Assume that $[\, x_1 \mid \cdots \mid x_n \,] = 0$, and that $(x_1, \ldots, x_n)$ is not a terminal position. By Lemma 8.2 we have

$$\left[\begin{array}{c|c|c|c|c|c|c} 0 & 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 2n-1 & 2n-2 & 2n-3 & \ldots & n+2 & n+1 & n \end{array} \right] = \begin{array}{c} 0 \\ 1, \end{array}$$

so an even number of $x_1, \ldots, x_n$ are in $\{n, n+1, \ldots, 2n-1\}$. It follows that an even number of the inequalities $x_1 > x'_1, \ldots, x_n > x'_n$ are true.

Suppose that $x = x'_1$. Then $x_1 > x'_1$, so an odd number of the inequalities $x_2 > x'_2, \ldots, x_n > x'_n$ hold. So there is an $i \in \{2, \ldots, n\}$ such that the move from square x_i to x'_i is feasible. The resulting position is an Abacus position, and by Lemma 8.2, we have

$$[\, x'_1 \mid x_2 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] = 0.$$

Now assume that $x \neq x'_1$. Then $x \in \{x'_2, \ldots, x'_n\}$. Let $i \in \{2, \ldots, n\}$ be such that $x = x'_i$. Then $x_1 > x'_i$, so $x_i > x'_1$. Then we can move the coin in square x_i to square x'_1 to get the position

$$(x'_i, x_2, \ldots, x_{i-1}, x'_1, x_{i+1}, \ldots, x_n) = (x'_1, x_2, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n),$$

where we used Symmetry. This is an Abacus position, and by Lemma 8.2 we have

$$[\, x'_1 \mid x_2 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] = 0.$$

So in both cases, we can move to an Abacus position with Sprague-Grundy value 0.

Now suppose that $[\, x_1 \mid \cdots \mid x_n \,] = 1$. Then by Lemma 8.2, an odd number of $x_1, \ldots, x_n$ are in $\{n, n+1, \ldots, 2n-1\}$, so an odd number of the inequalities $x_1 > x'_1, \ldots, x_n > x'_n$ hold.

Suppose that $x = x'_1$. Then $x_1 > x'_1$, so an even amount of the inequalities $x_2 > x'_2, \ldots, x_n > x'_n$ are true. If at least one of them is true, say $x_i > x'_i$, then we can move from x_i to x'_i. By Lemma 8.2, we have

$$[\, x'_1 \mid x_2 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] = 1,$$

so the resulting position is an Abacus position with Sprague-Grundy value 1. If none of the inequalities $x_2 > x'_2, \ldots, x_n > x'_n$ are true, then $(x'_1, x_2, \ldots, x_n) = (0, 1, \ldots, n-1)$. In this case, the opponent has made the final move.

Now suppose that $x \neq x'_1$. Let $i \in \{2, \ldots, n\}$ be such that $x = x'_i$. Then $x_1 > x'_i$, so $x_i > x'_1$. So we can move the coin on square x_i to square x'_1, which gives the position

$$(x'_i, x_2, \ldots, x_{i-1}, x'_1, x_{i+1}, \ldots, x_n) = (x'_1, x_2, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n),$$

using Symmetry. This is an Abacus position, and by Lemma 8.2 we have

$$[\, x'_1 \mid x_2 \mid \cdots \mid x_{i-1} \mid x'_i \mid x_{i+1} \mid \cdots \mid x_n \,] = [\, x_1 \mid \cdots \mid x_n \,] = 1.$$

□

Note that each terminal position is an Abacus position, so the game will reach an Abacus position with Sprague-Grundy value 0 at some point. By Lemma 8.3, once the game has reached such a position, the winning strategy for Welter's game with the normal play rule is to return to an Abacus position with value 0 after every move the opponent makes. Then the player will be the one to eventually move to a terminal position.

In addition, Lemma 8.3 gives a result for the misère version of Welter's game. If the game has reached an Abacus position with Sprague-Grundy value 1, the winning strategy is to return to an Abacus position with value 1 after every move the opponent makes. Terminal positions do not have Sprague-Grundy value 1, so this way the player will not be the one to move to a terminal position.

Lemma 8.4. *If there is a move from a non-Abacus position to an Abacus position, then there is also a move to a different Abacus position with a different Sprague-Grundy value.*

Proof. Let $(x, x_2, \ldots, x_n)$ be a non-Abacus position which has a follower that is an Abacus position. Without loss of generality, we assume that the move to the Abacus position is made by moving the coin on square x. Let $x_1 < x$ be such that $(x_1, \ldots, x_n)$ is an Abacus position.

Let $x'_i = 2n - 1 - x_i$ for all $i \in \{1, \ldots, n\}$. If $x > x'_1$, then $(x'_1, x_2, \ldots, x_n)$ is also a follower of $(x, x_2, \ldots, x_n)$, and its Sprague-Grundy value is not the same as that of $(x_1, x_2, \ldots, x_n)$ by Lemma 8.2.

Assume that $x \leq x'_1$. Note that $x \neq x'_1$, as $(x'_1, x_2, \ldots, x_n)$ is an Abacus position. So $x < x'_1$. Because $\{0, \ldots, 2n-1\} = \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}$, we have $x \in \{x'_2, \ldots, x'_n\}$. Let $i \in \{2, \ldots, n\}$ be such that $x = x'_i$. Then $x'_i = x < x'_1$, so $x_i > x_1$. So

$$\begin{aligned}&(x, x_2, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_n) = (x'_i, x_2, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_n)\\ &= (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n),\end{aligned}$$

where we used Symmetry, is a follower of $(x, x_2, \ldots, x_n)$. This is an Abacus position, and by Lemma 8.2, it has a different Sprague-Grundy value than $(x_1, \ldots, x_n)$.

□

By the lemma above, whenever it is possible to move from a non-Abacus position to an Abacus position with Sprague-Grundy value 0, it is also possible to move to an Abacus position with Sprague-Grundy value 1, and vice versa.

Remember that under the normal play rule, the game will at some point reach an Abacus position with Sprague-Grundy value 0. As we argued above, there is a winning strategy for the player who makes the first move to such a position. So under the normal play rule, the player who can move to an Abacus position first will move to an Abacus position with Sprague-Grundy value 0, and then keep moving to Abacus positions with Sprague-Grundy value 0 until the game ends.

So the winning strategy under the misère play rule is to follow the winning strategy for the normal play rule until the first time that would lead to an Abacus position. Then, the player should move to an Abacus position with Sprague-Grundy value 1 instead of to one with Sprague-Grundy value 0. After that, the player should keep moving to Abacus positions with Sprague-Grundy value 1 until the game ends.

We have now proved the following theorem.

Theorem 8.5. *In the misère version for Welter's game, the positions which have a winning strategy for the player whose turn it is are the Abacus positions with Sprague-Grundy value 1, and the non-Abacus positions that are N-positions.*

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