Quasi-normal mode damping by matter in Schwarzschild spacetime

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THESIS

submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
PHYSICS

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Leiden, The Netherlands, August 4, 2023

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August 4, 2023

Abstract

We present a formalism to calculate matter induced damping of the quasi-normal modes of perturbed Schwarzschild spacetime. Potential damping effects might be relevant for high precision models and data analysis of a black hole ringdown gravitational wave signal. In our model the surrounding matter is assumed to be relevant only at the perturbative level. We employ a kinetic approach and consider an initial distribution of massive particles on circular geodesics. Via the Boltzmann equation we derive the perturbed distribution function and energy-momentum tensor, including collisions in the collision time approximation. A new wave equation, which is especially useful for circular orbits, is derived for odd (axial) perturbations. Numerically solving for the quasi-normal mode frequencies is left for future work.

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Chapter 1

Introduction

The last decades have witnessed spectacular progress in the theory and detection methods of gravitational waves (GWs).* Worldwide effort has culminated in the first ever observed GW event by the LIGO collaboration in 2015 [1]. This momentous milestone and subsequent observations of other GW events with LIGO and VIRGO are far from the final destination of GW physics. Rather, they have become an impetus for further improvement of the GW detector sensitivity [2, 3], and modelling the generation and waveforms of GWs.

Predicting the GW waveform generated by a specific event, e.g. a binary black hole merger, is essential for the detection of GWs. In interferometers a typical GW signal is buried under noise orders of magnitudes larger than the signal itself [4]. The signal can still be extracted, however, with a matched filtering technique. This technique consists of searching for a match between the interferometer data and GW waveform templates. Hence accurate GW waveform predictions are essential for GW detection.

Accurate GW waveform predictions are also crucial to maximize the scientific knowledge that can be extracted from an observed GW signal. For instance, parameters such as the black hole (BH) masses, final BH spin and luminosity distance to a GW event can be inferred by fitting a waveform to the GW signal of inspiralling and merging binary BHs [5] (which together with compact neutron star (NS) binaries are the most common GW sources). For binary BHs with similar masses the leading-order quadropole moment generally is sufficient to accurately reconstruct these quantities, but for highly unequal component masses the parameter estimation is biased when only using the leading-order contribution [6]. Beyond leading-order waveforms and observations are not only necessary

*For a very brief overview of the history of GW physics, see the introduction of [1].

for accurate parameter estimation, but also for tests of general relativity (GR) in the strong field regime relying on compact merger events [7].

The scientific value of observing higher modes and prospects of improving and new GW detectors[†] have stimulated both the modelling of higher modes and the development of data analysis techniques that can extract them. It has been proposed that no-hair properties of vacuum BHs (i.e. all Kerr-type BHs are uniquely described by their spin and mass) can be tested with coherent stacking, or beyond GR ringdown parametrizations, of multiple GW events to extract overtones and higher modes [9, 10]. Tests of GR can also be performed by identifying overtones in individual GW events [11–13]. A recently proposed method is to use rational filters that clean out specific modes from the time domain signal [14]. Application to the first GW150914 event has revealed the first overtone [15]. Other studies that model and reanalyze several GW events have shown that subdominant higher multipoles and precession effects are observable [6, 16–19]. It is therefore justified to claim that GW physics is entering the stage at which spectroscopy of BHs becomes possible [13, 20, 21].

There might be a caveat, however. As pointed out by Barausse et al. in 2014, until then almost all GW waveform models developed for binary mergers assume an isolated environment, i.e. no matter near the sources [22]. However, it is well known that in reality BHs are surrounded by accretion disks and NSs by magnetospheres filled with plasma. To ascertain whether surrounding matter jeopardizes the accurate modelling of GW waveforms and the claimed detection of higher multipoles, overtones, etc. the effect of matter must be quantitatively estimated. The conclusion of [22] is that precision GW astrophysics typically is not impaired by environments such as accretion disks, electromagnetic fields, charges, etc.

The work of [22] and most subsequent related studies (see Sec. 1.2.1 for a discussion) do not incorporate, however, the effect that a matter distribution perturbed by the passing GWs has on the propagation of the GWs. This matter backreaction effect[‡] has been shown to cause considerable damping of cosmological GWs by neutrinos after decoupling [23]. In a study by Baym et al. this work on GW damping by matter is extended

[†]In 2019 Baibhav and Berti claim that future space-based interferometer *"LISA could detect so many modes that current numerical relativity simulations do not have enough resolution (or do not contain enough higher harmonics) to extract all available science from the data."* [8].

[‡]The term 'backreaction' in this context indicates the effect of the GW induced perturbed matter distribution on the passing GWs, i.e. the coupling of GWs to the perturbed matter energy-momentum tensor via the Einstein equations. Here we do not mean the self-interaction effects of GWs coupling to their own contribution to the energy-momentum tensor, which is a second order effect (see e.g. Ch. 2 in [4]).

by adding collisional effects, both in flat space and a cosmological background [24]. To the best of our knowledge, in the astrophysical context of a BH the matter backreaction effect has only been (implicitly) taken into account in [22, 25–27] for specific matter distributions and mainly in the fluid approximation (further discussion below).

It is the aim of this research to calculate the matter induced damping of GWs generated near a BH. By means of a relatively simple toy model calculation we intend to estimate whether GW damping by matter has a significant effect on these GWs, and therefore if it must be included in the data analysis of GW events. We will restrict ourselves to GWs from a single BH. For GWs from a binary merger this means we only consider GWs from the final merger product in the so-called *ringdown* phase.

Before proceeding with a calculation in Chapter 2 we must first introduce matter damping of GWs and gravitational perturbations in BH spacetimes a little further. This we do in the next two sections, where we also summarize previous and related work on these topics. Unless specified, all quantities are in $G = c = 1$ units.

1.1 Gravitational wave damping by matter

The subject of damping of BH ringdown GWs by surrounding matter is inspired by the works of Weinberg [23] and Baym et al. [24]. These works consider the damping of GWs by matter in a Minkowski and cosmological background. We briefly summarize their approach and results.

Weinberg considers the matter backreaction of free streaming neutrinos on cosmological GWs originating from inflation, which up to his work has been largely neglected [23]. The evolution equation for cosmological tensor modes h_{ij}, given by

$$\ddot{h}_{ij} + 3\frac{\dot{a}}{a}\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 16\pi G\pi_{ij} \, ,$$

where a the scale factor, indicates that any anisotropic part of the stress tensor, denoted by π_{ij}, affects the GW evolution. To zeroth order $\pi_{ij} = 0$ by assumption of spatial homogeneity and isotropy. A perfect fluid also has $\pi_{ij} = 0$. Freely streaming neutrinos (being the dominant contribution to the matter budget at the relevant cosmological times) can, however, develop anisotropic stresses through GW induced perturbations. Weinberg computes the anisotropic stress perturbation through an explicit computation of the perturbed phase space density of neutrinos. Hence a kinetic approach in terms of phase space densities is used, which is more general than a macroscale fluid description. To zeroth order the neutrinos are in thermal equilibrium at a cosmologically redshifting temperature. The first order perturbation to the phase space density is determined by solving the perturbed Boltzmann equation for the first order perturbation to the distribution function. The resulting anisotropic stress perturbation π_{ij} leads to an integro-differential equation for the propagating GW h_{ij}, which is solved numerically. The squared amplitude of h_{ij} is reduced by 35.6% for wavelengths entering the horizon during radiation domination. At later horizon entry times the effect is smaller, with a $\sim$ 10% reduction for the largest wavelengths.

Baym et al. have extended the calculation by Weinberg in several respects [24]. They include collisions through a collision term (in the collision time approximation) in the perturbed Boltzmann equation. Besides, the GW damping by matter is also considered in a flat background. In both the flat and expanding cosmological background the calculation is for both massive and massless particles, in contrast to Weinberg who regarded neutrinos as massless. Baym et al. use a kinetic approach because it includes both the hydrodynamic and nearly collisionless limit. To zeroth order in both backgrounds the matter is taken to be in thermal equilibrium.

The results lead to the identification of two damping mechanisms: damping through collisions and Landau damping [24]. The result of Weinberg, not considering collisions, is interpreted as Landau damping. Landau damping can occur in a cosmological background where the energy transfer between the GW and matter, propagating at slightly different velocities, does not completely cancel out because of the expansion. The expansion in the presence of matter effectively spreads the frequencies of a GW wavepacket. Collisions might inhibit the Landau damping effect and suppress the amplitude reduction computed by Weinberg.

In flat space Landau damping is impossible [24]. Collisional effects in the intergalactic and interstellar medium only significantly damp GWs for frequencies $\omega \lesssim 1/\tau_U \sim 10^{-18}\ \mathrm{s}^{-1}$, where τ_U the age of the Universe. These frequencies are extremely small, and such GWs are not generated by astrophysical processes. Using the flat space result for GW damping, the maximum collisional damping in dense astrophysical environments, e.g. near BHs, is estimated. Collisional damping is maximum when $\omega\tau \sim 1$, where τ the collision time. Although realistically $\omega\tau \ll 1$, even when assuming $\omega\tau \sim 1$ an unrealistic amount of mildly non-relativistic matter is required for significant damping. For the unrealistic scenario of $\omega\tau \sim 1$ and extremely relativistic matter the damping might be significant.

Moving beyond the estimates of [24] of damping in dense astrophysical environments, it is natural to ask whether a more realistic spacetime, such as Schwarzschild, allows for Landau damping and whether the effect is significant. In that case the unfeasible condition $\omega\tau \sim 1$ for the above estimates could be circumvented. If such Landau damping is possible, we can use an analogy based hypothesis to estimate that damping might be significant in Schwarzschild spacetime. Recall that for flat space damping is significant when $\omega \sim 1/\tau_U \sim H = \dot{a}/a$ where H the Hubble constant (Eq. 42 in [24]). In a homogeneous, isotropic spacetime filled with matter the intrinsic curvature decoded by the Riemann tensor is $\mathcal{O}(H^2)$. In Schwarzschild spacetime the Riemann tensor is $\mathcal{O}(M/r^3)$. We then hypothesize that by analogy we get $\omega \sim \sqrt{M/r^3} \sim 1/M$ near the central source, e.g. a BH. Since GWs from BHs have frequency $\omega = \mathcal{O}(M^{-1})$, as discussed in the next section, this is a hint that damping might be significant. This motivates our study of damping of GWs from BHs by matter.

1.2 Black hole spacetime perturbations: quasi-normal modes

Metric perturbations of BH spacetimes (Schwarzschild, Reissner-Nordström, Kerr, depending on whether the BH has spin and/or charge) have been studied extensively since the seminal work on gravitational perturbations of Schwarzschild spacetime by Regge and Wheeler in 1957 [28]. They, and subsequently all other work in this area, exploited the spherical symmetry of the spacetime to separate the radial from the angular coordinates and write the perturbation $h_{\mu\nu}$ as a sum of radial functions multiplied by spherical harmonics. For each angular quantum number ℓ, m the perturbation $h_{\mu\nu}^{\ell m}$ is further decomposed as the sum of an even (*polar*) and odd (*axial*) component under the spatial parity operation. By defining a clever combination of the components of $h_{\mu\nu}^{\ell m}$ into a master function $\Psi^{(\mathrm{o})}$ they rewrote the linearized Einstein equations into a single, Schrödinger-type wave equation for odd modes in vacuum, of the form

$$-\frac{\partial^2}{\partial r_*^2}\Psi^{(\mathrm{o})} + \frac{\partial}{\partial t^2}\Psi^{(\mathrm{o})} + V_\ell^{(\mathrm{o})}\Psi^{(\mathrm{o})} = 0\,, \tag{1.1}$$

where r_* the radial (tortoise) coordinate and $V_\ell^{(\mathrm{o})}$ the potential of odd perturbations, depending on the radial coordinate and multipole number ℓ. Zerilli has derived a very similar wave equation for even perturbations, i.e. for $\Psi^{(\mathrm{e})}$ and $V_\ell^{(\mathrm{e})}$ [29]. In later work BH perturbation theory has been extended in multiple directions[§]: Kerr spacetimes were also studied [31, 32], the formalism was made gauge invariant and coordinate independent [33, 34], and matter sources were included (leading to Eq. (1.1) with a source term $S^{(o)}$ resp. $S^{(e)}$ term on the right side) [34, 35].

From the study of numerical and approximate solutions to Eq. (1.1), and similar evolution equations for other BH spacetimes, it was realized that perturbations of both non-spinning and spinning BHs are stable, implying that they always decay [36–38]. For this reason they are called *quasi-normal modes* (QNMs) [39]. Quasi-normality implies that the frequencies of the solutions have both a real part and imaginary part, leading to oscillation resp. decay. In essence the quasi-normality of BH spacetimes reflects the breaking of time symmetry by the event horizon [30].

All radiative perturbation modes ($\ell \geq 2$) are radiated away as GWs [40, 41]. The leading interpretation is that the GW perturbations are on spiral orbits close to the unstable circular orbit (light ring) at $r = 3M$ (Schwarzschild), and slowly leak out to infinity as observable GW radiation [42, 43]. The GW perturbations are excited when particles cross the maximum of the potential V_ℓ, which is close to the light ring [30, 44].

To find the frequencies of QNMs, and hence of the radiated GWs, Eq. (1.1) must be solved. In Fourier-space Eq. (1.1) becomes an ordinary differential equation. Physical solutions must have only ingoing waves at the event horizon and only outgoing waves at infinity. Upon imposing these boundary conditions, the solutions allow for a discrete infinity of QNM frequencies [30]. In Schwarzschild spacetime these QNM frequencies only depend on the BH mass M, $\omega_{\rm QNM} \propto M^{-1}$ [45]. This fact is quite remarkable, implying that irrespective of the source of the perturbation the QNM spectrum is the same. In vacuum odd and even Schwarzschild QNMs have the same frequencies, i.e. are isospectral [30]. The lowest Schwarzschild QNM modes for $\ell = 2, 3$ can be found in Figure 2 in [46]. Kerr QNMs are more involved, as they dependent on the value of the BH spin parameter. Extremal Kerr QNM frequencies are computed in [47].

1.2.1 Environmental effects

Although the theory and computation of Schwarzschild QNM frequencies in vacuum is firmly established, the effect of matter on QNMs is less well studied. The 2014 study by Barausse et al. mentions that most studies until that moment have considered compact objects in isolation [22]. As

[§]In the introduction of [30] a timeline of important milestones in the study of quasi-normal modes is given.

mentioned above, they conclude that for a broad class of scenarios environmental effects are not significant. In the wake of their lengthy study others have also studied the effects of astrophysical environments of compact binaries on GW generation and propagation. Below we summarize the findings of [22] and subsequent work. Many nonvacuum QNM studies focus on the observational imprint and detectability of the environment of compact binaries, and much attention is given to the inspiral phase. With regards to the aims of this project, however, we limit ourselves to effects during the ringdown phase of a single BH.[¶] At the end we briefly point to some findings for the inspiral phase.

Barausse et al. deploy a number of toy models to study environmental effects on ringdown QNM frequencies. Three simple yet insightful mathematical models considered are [22, p. 11-21]:

1. A wave equation of the form of Eq. (1.1) with a potential consisting of two rectangular or delta function barriers. The two barriers represent the maximum of the vacuum odd/even potential (near $r = 3M$) and the presence of composite matter at a larger distance. For these potentials the wave equation can be solved analytically in a similar fashion as potential barrier problems in quantum mechanics.

2. A wave equation of the form of Eq. (1.1) for the exact metric solution to the Einstein equations for a nonrotating BH and a discontinuous thin shell of matter. The metric only depend on the radial coordinate, and outside the shell is simply given by the Schwarzschild solution with $M_{\text{BH}} \to M_{\text{BH}} + M_{\text{shell}}$.

3. A *scalar*[||] wave equation derived from an approximate solution for the metric in the presence of an extended mass distribution in the fluid approximation.

[¶]We remark in passing that the distinction can be subtle. The QNM signal of a single BH typically is numerically calculated by performing a 'scattering experiment' of an incoming (Gaussian) wavepacket by a BH [27, 48]. After all, QNMs must always be excited by some source. The difference between the GW signal of extreme mass ratio inspirals and a single BH ringdown is that the former has a continuous, 'ringing' signal instead of a rapidly decaying ringdown signal from an incoming, scattered wavepacket. In both cases, however, the frequencies are determined by the fundamental BH parameters.

[||]The analysis is restricted to scalar QNMs for the following reason: *"Computing the gravitational perturbations of this spacetime would require an explicit stress-energy tensor for this matter distribution. In addition, the metric perturbations and the fluid perturbations would be coupled, rendering the analysis unnecessarily involved."* It is the coupling between metric and matter perturbations that will be the subject of our work, see Chapter 2, although we do not use the fluid approximation. The qualitative behavior of scalar and gravitational perturbations is expected to be similar since the respective potentials are similar [26, p. 8].

In all models the matter distribution functions are spherically symmetric. For this reason explicit source terms for the wave equation in a spherically symmetric background can be circumvented. For model 2 and 3 the eigenfrequencies are computed by direct integration.

The results of the three models are in qualitative agreement and lead to the following conclusions [22]. First, matter-BH QNMs can differ significantly from vacuum and are typically located beyond the light ring. The QNM spectrum is much richer than in vacuum. For each vacuum mode there is a parametric correction to the vacuum frequency and, additionally, an infinite set of matter driven modes. The isospectrality of even and odd modes is broken.

Second, even though the QNM spectrum is different from vacuum, in the time domain signal the vacuum BH QNMs dominate, especially when the matter is located far away. This paradox can be understood as follows. QNMs are excited in dynamical situations, such as a compact binary mergers, during which the matter-BH modes are only slightly excited and at later times compared to the vacuum mode. At later times, however, the matter-BH modes can dominate over the power-law tail of the vacuum ringdown signal. The matter-BH modes can be more strongly excited by inspiralling matter, or an extreme mass ratio inspiral.

A very remarkable result supported by all three models is that at large distances the deviation from vacuum QNM frequencies increases with the radius r_0 at which the matter is localized. This conclusion is in agreement with rare earlier works on dirty BH QNMs [49, 50]. The effect is attributed to the exponential sensitivity of QNMs, given by $e^{i\omega r_*}$, to small corrections. At smaller radii, $r_0 \lesssim \mathcal{O}(10M)$, the QNM frequencies do not deviate monotonically from vacuum.

The fourth and observationally most relevant conclusion is that even for very conservative estimates, i.e. unrealistically 'dirty' environments, the imprint on the GW ringdown signal gives a correction to the vacuum QNM frequencies of less than 0.1%, requiring signal-to-noise ratios of at least $\mathcal{O}(10^3)$. The changes to the QNM frequencies for different scenarios of 'dirt' are summarized in Table 1 in [22] and Table 2 in [51]. Accretion effects on ringdown frequencies are negligible.

Degollado and Heirdero have studied GWs induced by a massive scalar field falling into a Schwarzschild BH [52]. The gravitational perturbations coupled to the Klein-Gordon equation, which is evolved in the zeroth order background metric only, result in usual quasi-normal ringing, but now followed by a late time tail of small amplitude 'wiggles'. The wiggle frequency depends on the BH and scalar field mass. Because of their small amplitude using late time wiggles to identify environmental effects in observations is technically challenging.

The backreaction effect of matter perturbations on the perturbing GW is incorporated in a study by Bishop et al. [25]. They investigate a source of GWs surrounded by a thin shell of dust. The background metric is determined inside, in and outside the shell. Inside the shell the background spacetime is Minkowski, outside the shell it is equivalent to Schwarzschild. Through an elaborate computer-based analytical calculation the coupled perturbation equations for the matter and GWs on these backgrounds are solved in the three regions. Only the dominant quadropole ($\ell = 2$) component of GW radiation is taken into account. With the help of constraint equations and matching conditions at the boundaries of the three regions the GW strain $h_+ + ih_\times$ is found to be (Eq. 26 in [25])

$$\frac{h_+ + ih_\times}{H_{M0} e^{i\omega u} \, {}_2Z_{2,2}} = 1 + \frac{2M_S}{r_0} + \frac{2iM_S}{r_0^2 \omega} + \frac{iM_S e^{-2ir_0\omega}}{2r_0^2\omega} + \mathcal{O}\left(\frac{M_S \Delta}{r_0^2}, \frac{M_S}{r_0^3 \omega^2}\right) ,$$

where H_{M0} some constant, u the time coordinate, ${}_2Z_{2,2}$ the spin-weighted $\ell = 2$ spherical harmonic, M_S the shell mass, r_0 the minimum radius at which the shell is located, and Δ the radial extent of the shell. The three largest corrections (the second, third and fourth term on the right side) have the following interpretation. The first term represents the gravitational redshift of the GW travelling through a gravitational potential. The second term changes the phase but not the magnitude of the strain. The third term modifies the magnitude and is interpreted as an incoming GW generated by the shell, which alters the geometry near the source. This might lead to a GW echo. It is argued that there is no net energy exchange between the shell and the GW. Regarding the observational imprint of the three effects, the magnitude of the three terms depends on the ratio M_S/r_0, which was assumed small to solve the linearized equations. There is no formal constraint on ω.[**] The effect of a thick shell can be obtained by integrating over many thin concentric shells.

In a follow-up to the work by Bishop et al., the question is addressed whether potential GW echos observed in LIGO data can be explained by the above effect of a matter shell [53]. The answer is negative. The effect of a matter shell on the GW signal from a post-merger product and a core-collapse supernova is also estimated. In the former scenario the modification to a GW signal is visible at high signal-to-noise ratios, but

[**] If we assume that the GW is a Schwarzschild QNM leaking out from the light ring at $r = 3M_{\mathrm{BH}}$, then $\omega = \mathcal{O}(M_{\mathrm{BH}}^{-1})$ for the fundamental $\ell = 2$ mode [46, p. 14]. Since $r_0 \gtrsim M_{\mathrm{BH}}$ this means that $\omega r_0 \gtrsim \mathcal{O}(1)$. Hence $M_S/r_0^2\omega \ll 1$ for $M_s/r_0 \ll 1$. If the shell is close to the source (BH) the linearization condition is $M_S \ll M_{\mathrm{BH}}$.

the assumption of a static spherical shell forming under these condition is unlikely. In the latter scenario the shell effect is the largest and the shell model is most reliable.

A recent study by Cardoso et al. focusses on GW generation and propagation in spacetimes of a BH surrounded by anisotropic fluids [27]. These spacetimes might be used, for instance, to model galaxies with a central supermassive BH (SMBH) and a spherically symmetric mass distribution, or astrophysical BHs with surrounding matter if the fluid approximation is justified. Cardoso et al. explicitly calculate the spacetime metric for the combination of a BH and a spherically symmetric Hernquist-type mass density distribution (the total mass can be much larger than the BH mass, so there evidently is a deviation from Schwarzschild). The assumption of a spherically symmetric distribution effectively means that the particles are taken to be on all possible circular geodesics. The resulting metric evidently also is spherically symmetric. Similar to vacuum QNM studies linearized perturbations to this metric are considered. A radial wave equation for odd perturbations is derived where the potential $V_\ell^{(o)}$ (Eq. 18 in [27]) is similar albeit not identical to the vacuum potential for Schwarzschild. The equation is solved by direction integration to determine the QNM frequencies. For a total external mass $M \ll M_{\mathrm{BH}}$ extending over the typical scale $a_0 \gg M_{\mathrm{BH}}$ they find to first order that the fundamental quadropolar mode changes as (Eq. 20 in [27])

$$\frac{\omega_{20}(M, a_0)}{\omega_{20}(0,0)} = 1 - 1.1\frac{M}{a_0} + \mathcal{O}\left(\frac{M^2}{a_0^2}\right) .$$

This leading order frequency change is interpreted as a gravitational redshift of GWs leaking out from the BH light ring and climbing the gravitational potential of the surrounding matter. It is noticed that this interpretation might not hold when M/a_0 is large, i.e. when the matter is localized close to the BH. Although the calculation is for arbitrary ratio M/a_0, for galaxies it is estimated as $M/a_0 = \mathcal{O}(10^{-4})$ [27, 54].

As is emphasized, Cardoso et al. purely study the spacetime effect and do not consider BH accretion effects. Their approach is particularly elegant because it allows to circumvent the mathematical challenge of calculating the perturbations to the matter distribution explicitly. All matter effects are simply captured by the potential $V_\ell^{(o)}$. Our research, which also considers the response of matter to a GW perturbation, will use a mathematically more complicated approach. We will not use the fluid approximation, however.

Comparable to Cardoso et al. Daghigh and Kunstatter explicitly solve for the spacetime metric, now for the scenario of a nonrotating BH and a dark matter spike [26]. The dark matter is approximated as a perfect fluid. To construct the metric the Tolman-Oppenheimer-Volkoff equations are solved for a radial power law density distribution. The corresponding Regge-Wheeler equation is derived for *scalar* perturbations, as it is expected that these will lead to similar behavior as for gravitational perturbations. To estimate the effect of dark matter spikes on the ringdown waveform, the BHs Sagittarius A* and M87 are taken as concrete examples and the analysis is restricted to $\ell = 2$ modes. The real (oscillation) resp. imaginary (damping) part of the QNM frequency decreases resp. increases with increasing total spike mass. Although the effects expectedly are more pronounced for M87, with current techniques a dark matter spike is observationally indistinguishable from vacuum Schwarzschild ringdown, taking into account the observational estimates of such a spike for M87. For heavier BHs the spike might be observable, because the deviation increases with BH mass.

The ringing of Schwarzschild BHs surrounded by dark matter is also studied by Zhang et al. [55]. The perturbations of the dark matter distribution are neglected, however. Complex QNM frequencies are calculated for different spherically symmetric matter profiles. Deviations from the vacuum Schwarzschild QNM frequencies increase with the central matter density and typical (length) scale over which the matter is distributed, and can in some scenarios give $\sim 10\%$ corrections. The isospectrality of the axial and polar modes is not broken. The analysis has been generalized to Kerr BHs [56].

Finally, a few comments on environmental effects on the GW signal from the inspiral phase. Cardoso et al. consider the inspiralling of two objects with an extreme mass ratio, e.g. a SMBH and a star, in a nonisolated environment. They show that conversion between GWs and density waves occurs [57]. The GW signal is a BH ringdown followed by a long-lived, fluid mode at later times due to the coupling between the GW and the matter halo. The GW flux receives corrections compared to vacuum, which is interpreted as a redshift effect. The flux correction is within the reach of future GW detectors. A similar analysis of extreme mass ratio inspirals surrounded by matter also shows that the flux can receive significant corrections and the GW frequency is redshifted [58]. The imprint of environmental effects on the GW signal is further studied for intermediate mass BH binaries with gas (accretion) discs by [59, 60] and with dark matter by [61], and for extreme mass ratio inspirals by [62–65], mostly focussing on gaseous accretion disks.

The significance of environmental effects on inspiral waveforms is confirmed by Zwick et al., who state that for low redshifts environmental effects during the inspiral stage of massive binaries are a bigger model uncertainty than the current 5 PN (Post-Newtonian) limit to the vacuum GW waveform templates [66]. Astrophysical environments are especially important at the mHz frequencies of future detectors, as compared to the Hz frequencies of current, ground-based detectors [66]. Cole et al. show that for future GW detectors such as LISA, environmental effects can be observationally distinguished because of the very long detectability (up to multiple years) of the inspiral signal [67].

Chapter 2

Quasi-normal modes in dirty Schwarzschild

The aim of this work and this chapter is to calculate the backreaction effect of matter ('dirt') on GWs near a BH. Specifically, we want to know how the QNM frequencies and GW ringdown signal are affected by matter.

This chapter is structured as follows. In Sec. 2.1 we lay down the formalism and model assumptions to attack this problem, and in Sec. 2.2 we review circular Schwarzschild geodesics. In Sec. 2.3 the perturbed energy-momentum tensor is computed and in Sec. 2.4 we use this to derive an evolution equation for odd QNMs. Because of the transient nature of time solving for the QNM frequencies is deferred to future work. Nevertheless, we discuss some prospects for the future in Sec. 2.5.

2.1 Formalism and model assumptions

The metric is decomposed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ , \tag{2.1}$$

where $\bar{g}_{\mu\nu}$ the background metric and $h_{\mu\nu}$ a gravitational perturbation. The evolution equations for small perturbations, $|h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$, are derived from the linearized Einstein equations

$$\bar{G}_{\mu\nu} + G_{\mu\nu}[h] = 8\pi \left(\bar{T}_{\mu\nu} + \delta T_{\mu\nu} \right) \ , \tag{2.2}$$

where $\bar{G}_{\mu\nu}$ the background Einstein tensor, $G_{\mu\nu}[h]$ the linearized Einstein tensor for first order perturbations in $h_{\mu\nu}$, $\bar{T}_{\mu\nu}$ the energy-momentum tensor for matter content following background geodesics and $\delta T_{\mu\nu}$ the first order perturbation of the energy-momentum tensor due to the gravitational perturbation. The matter backreaction on GWs is encoded by the term $\delta T_{\mu\nu}$ which is linear in $h_{\mu\nu}$.

We consider the exterior spacetime of a BH. It is assumed that the BH is not spinning and the matter content is small compared to the BH mass, so that to zeroth order $\bar{G}_{\mu\nu} = 0$ and to first order $G_{\mu\nu}[h] = 8\pi \left(\bar{T}_{\mu\nu} + \delta T_{\mu\nu}\right)$. These two assumptions imply that the background is Schwarzschild spacetime, which is the spacetime of a nonrotating BH in vacuum. In t, r, θ, ϕ coordinates the line element is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \, . \tag{2.3}$$

Employing a kinetic approach we consider a dilute gas of particles [24]. The phase space density distribution function of particles is denoted by

$$f(r^i, p_i, t) = f_0(r^i, p_i, t) + \delta f(r^i, p_i, t) \, , \tag{2.4}$$

where f_0 the zeroth order distribution and δf the first order perturbation due to the GW. In the Hamiltonian formalism the coordinates $r^i = \{r, \theta, \phi\}$ and covariant momenta $p_i = \{p_r, p_\theta, p_\phi\}$ are independent. Collisions will be taken into account, cf. Sec. 2.3.1.

We consider particles that to zeroth order are on circular orbits, which are Schwarzschild geodesics. Hence there is no radial motion, $p_r = 0$. We assume that the distribution is spherically symmetric so that it only depends on the radial coordinate r. A circular orbit in a spherically symmetric geometry can be uniquely specified by one of the following parameters [68]: radius r, energy ϵ_c, total angular momentum L or circular frequency Ω. A convenient choice for our analysis is ϵ_c, the zeroth order energy of particles orbiting at radius r. Using this there are multiple ways to write an expression for f_0 that satisfies the assumptions, for instance [69]

$$f_0(\epsilon_0, \epsilon_c) = N(\epsilon_0)\delta\left(\epsilon_0 - \epsilon_c\right) = N(\epsilon_c)\delta\left(\epsilon_0 - \epsilon_c\right) \, , \tag{2.5}$$

where ϵ_0 the zeroth order energy in the background spacetime. Here $N(\epsilon_0)$ encodes the radial profile and the number density. We consider circular orbits and expressions for ϵ_0 and ϵ_c in the next section.

2.1.1 Comparison with the literature

Before continuing we briefly relate our formalism and model to the relevant literature discussed in the previous chapter.

Although we employ a kinetic approach like Baym et al., we do not consider the particles to initially be in thermal equilibrium [24]. Instead we assume circular orbits, i.e. an anisotropic distribution function in momentum space. This has two reasons. First, it is not obvious that matter surrounding a BH merger and the final merger product has enough time to achieve thermal equilibrium during the ringdown GW emission. The second reason is of a more practical nature. Because Schwarzschild spacetime 'has a central force', constructing a thermal equilibrium function is nontrivial as it would include motion in the radial direction. As we will see in Sec. 2.3, it is the negligence of radial motion that greatly simplifies the calculations.

Compared to Bishop et al. we extend the analysis to arbitrary ℓ (not only $\ell = 2$) [25]. Bishop et al. consider a GW source and a shell placed at a large distance from the source, effectively assuming spacetime inside the shell to be flat. In contrast, we allow for the matter to be close to the GW source and hence we take the relevant geometry into account.

Compared to Barausse et al. (toy model 3, see Sec. 1.2.1), Bishop et al., Daghigh and Kunstatter, and Cardoso et al. we do not determine the (approximate) background metric for a BH and a specific matter distribution [22, 25–27]. Instead we assume the matter content to be smalll compared to the BH mass. Although for perturbation theory to be self-consistent this limits the allowed amount of matter, the considerable upshot is that the radial profile $N(r)$ is much more flexible than the very specific ones considered by these four works. Besides, in our approach the zeroth order assumption of spherical symmetry can be relaxed. In the other four works this assumption is essential to obtain a reasonably solveable background metric. Another difference is that Barausse et al. (toy model 3) and Daghigh and Kunstatter only consider scalar perturbations [22, 26].

Importantly, the other works have mainly worked in the fluid approximation, except for Bishop et al. who consider dust. In none of these four works collisions are taken into account. We, on the contrary, use a kinetic approach and incorporate collisions. Therefore our analysis will not depend on the validity of the fluid approximation of matter near a BH.

2.2 Circular Schwarzschild geodesics

We consider particles subject to no other force than gravity. Hence to zeroth order the particles follow Schwarzschild geodesics. The geodesic equations $p^\nu \nabla_\nu p^\mu = 0$ for a (test) particle can be rewritten in the form

$$\frac{dp_\mu}{d\lambda} = \frac{1}{2}\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^\mu} p^\alpha p^\beta \quad , \quad \frac{dx^\mu}{d\lambda} = p^\mu = \bar{g}^{\mu\nu} p_\nu \, , \tag{2.6}$$

where λ an affine parameter, i.e. $\lambda = \tau/m$ for massive particles. Since $\bar{g}_{\mu\nu}$ is time independent the covariant energy $\epsilon \equiv p_0$ is conserved (to zeroth order). An explicit expression for $\epsilon \equiv p_0$ can be found from the dispersion relation

$$g^{\mu\nu} p_\mu p_\nu = -m^2 \, , \tag{2.7}$$

where m the particle mass, which we can simply set to zero in the massless case. To zeroth order

$$\epsilon_0 = \sqrt{-\bar{g}_{00}\left(-\bar{g}_{00} p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} + m^2\right)} \, , \tag{2.8}$$

where $\bar{g}_{00} = -(1 - 2M/r)$.

From Eq. (2.6) it is also obvious that $L_z \equiv p_\phi$ is a conserved quantity, which is related to the angular momentum in the z-direction. A third constant of motion is given by

$$L^2 \equiv p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \, , \tag{2.9}$$

as can be straightforwardly checked with Eq. (2.6). This quantity is related to the squared total angular momentum.

It can be shown that circular orbits, $p_r = 0$, are solutions to the geodesic equations. For massive circular orbits the total angular momentum L^2 only depends on the radius r. The relation between L^2 and r can be found using the geodesic equations for p_r and demanding both $p_r = 0$ and $dp_r/d\lambda = 0$. From the latter requirement,

$$\frac{dp_r}{d\lambda} = -\frac{M}{r^2\left(1 - \frac{2M}{r}\right)}\left(\frac{L^2}{r^2} + m^2\right) + \frac{L^2}{r^3} = 0 \, , \tag{2.10}$$

follows

$$L^2(r) = \frac{m^2 M r}{1 - \frac{3M}{r}} \, , \tag{2.11}$$

and thus the energy of circular orbits, which only depends on r, is

$$\epsilon_c(r) = \sqrt{\left(1 - \frac{2GM}{r}\right)\left(\frac{L^2(r)}{r^2} + m^2\right)} \, . \tag{2.12}$$

By positivity of L^2 massive circular orbits only exist at radii $r > 3M$. In the Newtonian limit $r \gg 3M$ we retrieve the well known Newtonian angular momentum, $L_N = mv_{\text{circ}}r = m\sqrt{Mr}$. Massive circular orbits are stable when $\frac{d^2 p_r}{d\lambda^2} < 0$. Combining Eq. (2.10) and (2.11) it can be shown that the stability criterion is met for $r > 6M$.

The Schwarzschild geodesic equations for massless particles, such as photons, only allow for circular orbits at $r = 3M$. These orbits are unstable. Therefore our model assumption of an extended distribution of particles travelling on circular orbits restricts the applicability to massive particles.

2.3 Energy-momentum tensor perturbation

The energy-momentum tensor $T_{\mu\nu}$ of a gas of particles described by the distribution function $f(r^i, p_i, t)$ is [23, 70–72]

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \int_p \frac{p_\mu p_\nu}{-p^0} f(r^i, p_i, t) \,, \tag{2.13}$$

where $g = \det(g_{\mu\nu})$ and $\int_p \equiv \tilde{g} \int d^3p/(2\pi)^3$ for $\tilde{g}$ the internal degrees of freedom. The integration is over covariant momenta p_i, and the minus sign in front of p^0 accounts for the fact that $p^0 = g^{0\mu} p_\mu < 0$ as $\epsilon \equiv p_0 > 0$.

To compute the first order perturbation of the energy-momentum tensor we need $\delta\epsilon$, δp^0, $\delta\left(1/\sqrt{-g}\right)$ and δf. To calculate the perturbation to the energy $\epsilon \equiv p_0$ we write

$$\epsilon = \epsilon_0 + \delta\epsilon \tag{2.14}$$

and perturb Eq. (2.7) to first order. We find

$$\delta\epsilon = \frac{\bar{g}_{00}}{2\epsilon_0} h^{\alpha\beta} p_\alpha p_\beta \,, \tag{2.15}$$

using that to first order the inverse metric is $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$. Writing $p^0 = \bar{p}^0 + \delta p^0$ the perturbation δp^0 can be determined in a similar fashion to $\delta\epsilon$, yielding

$$\delta p^0 = -\frac{h^{00}\epsilon_0}{2} + \frac{h^{ij} p_i p_j}{2\epsilon_0} \,. \tag{2.16}$$

Finally, from

$$\sqrt{-g} = \sqrt{-\bar{g}} \left(1 + \frac{h^\mu{}_\mu}{2}\right) \tag{2.17}$$

we observe

$$\delta\left(\frac{1}{\sqrt{-g}}\right) = -\frac{h^{\mu}{}_{\mu}}{2\sqrt{-\bar{g}}} = -\frac{\bar{g}_{\mu\nu}h^{\mu\nu}}{2\sqrt{-\bar{g}}} \,. \tag{2.18}$$

The perturbation δf is calculated explicitly in Sec. 2.3.1.

The first order perturbation of the energy-momentum tensor is

$$\begin{aligned}
\delta T_{00} &= \frac{1}{\sqrt{-\bar{g}}}\int_p \frac{\epsilon_0^2}{-\bar{p}^0}\left[\left(2\frac{\delta\epsilon}{\epsilon_0} - \frac{\delta p^0}{\bar{p}^0} - \frac{h^{\mu}{}_{\mu}}{2}\right) f_0 + \delta f\right] \\
&= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p \epsilon_0 \left[\left(\frac{\bar{g}_{00}}{\epsilon_0^2}h^{\alpha\beta}p_\alpha p_\beta - \frac{\bar{g}_{00}}{2\epsilon_0^2}h^{mn}p_m p_n - \frac{1}{2}\bar{g}_{mn}h^{mn}\right) f_0 + \delta f\right] , \\
\delta T_{0i} &= \frac{1}{\sqrt{-\bar{g}}}\int_p \frac{\epsilon_0 p_i}{-\bar{p}^0}\left[\left(\frac{\delta\epsilon}{\epsilon_0} - \frac{\delta p^0}{\bar{p}^0} - \frac{h^{\mu}{}_{\mu}}{2}\right) f_0 + \delta f\right] \\
&= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p p_i \left[\left(\frac{1}{2}\bar{g}_{00}h^{00} + \frac{\bar{g}_{00}}{\epsilon_0}h^{0m}p_m - \frac{1}{2}\bar{g}_{mn}h^{mn}\right) f_0 + \delta f\right] , \\
\delta T_{ij} &= \frac{1}{\sqrt{-\bar{g}}}\int_p \frac{p_i p_j}{-\bar{p}^0}\left[\left(-\frac{\delta p^0}{\bar{p}^0} - \frac{h^{\mu}{}_{\mu}}{2}\right) f_0 + \delta f\right] \\
&= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p \frac{p_i p_j}{\epsilon_0}\left[\left(-\frac{\bar{g}_{00}}{2\epsilon_0^2}h^{mn}p_m p_n - \frac{1}{2}\bar{g}_{mn}h^{mn}\right) f_0 + \delta f\right] ,
\end{aligned}$$

using $\bar{p}^0 = \bar{g}^{00}\epsilon_0$. We can further simplify the expressions for $\delta T_{\mu\nu}$ by observing that f_0 and ϵ_0 are even under parity $\mathbf{p} \to -\mathbf{p}$. Hence, terms that have an odd number of p_i vanish when integrated over momentum. This results in

$$\begin{aligned}
\delta T_{00} &= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p \epsilon_0 \left[\left(\bar{g}_{00}h^{00} + \frac{\bar{g}_{00}}{2\epsilon_0^2}h^{mn}p_m p_n - \frac{1}{2}\bar{g}_{mn}h^{mn}\right) f_0 + \delta f\right] , \\
\delta T_{0i} &= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p p_i \left[\frac{\bar{g}_{00}}{\epsilon_0}h^{0m}p_m f_0 + \delta f\right] , \\
\delta T_{ij} &= \frac{-\bar{g}_{00}}{\sqrt{-\bar{g}}}\int_p \frac{p_i p_j}{\epsilon_0}\left[\left(-\frac{\bar{g}_{00}}{2\epsilon_0^2}h^{mn}p_m p_n - \frac{1}{2}\bar{g}_{mn}h^{mn}\right) f_0 + \delta f\right] .
\end{aligned} \tag{2.19}$$

2.3.1 Perturbed Boltzmann equation

The perturbation δf to the distribution function can be determined from the Boltzmann equation. The general relativistic Boltzmann equation in terms of of the independent variables r^i, p_i, t is [24]

$$\left(\frac{\partial}{\partial t} + \frac{\partial \epsilon}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \epsilon}{\partial x^i}\frac{\partial}{\partial p_i}\right) f = \mathcal{C} \,, \tag{2.20}$$

where $\mathcal{C}$ a collision term. As argued before, circular orbits can be uniquely specified by $\epsilon_0 = \epsilon_c$ (up to initial conditions) and therefore the zeroth order distribution function f_0 is purely a function of ϵ_0 and ϵ_c, see Eq. (2.5) for a possible parametrization. Neglecting collisions, to zeroth order Eq. (2.20) is then trivially satisfied (since all $\partial/\partial p_r$ derivatives vanish).

At first order we include collisions in the collision time approximation. Similar to Baym et al. we write the collision term as

$$\mathcal{C} = -\frac{1}{\tau}\left(\delta f - \delta\epsilon \frac{\partial f_0}{\partial \epsilon_0}\right) \,, \tag{2.21}$$

where τ the collision time. The term in the brackets encodes the deviation of δf from the first order perturbation of f_0 through its dependence on ϵ. Baym et al. consider this collision term as only $\ell \geq 2$ perturbations are relevant [24].

The first order perturbation of Eq. (2.20) is

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \underline{\frac{\partial \epsilon_0}{\partial p_r}\frac{\partial}{\partial r}} + \frac{\partial \epsilon_0}{\partial p_\theta}\frac{\partial}{\partial \theta} + \frac{\partial \epsilon_0}{\partial p_\phi}\frac{\partial}{\partial \phi} - \frac{\partial \epsilon_0}{\partial r}\underset{\sim}{\frac{\partial}{\partial p_r}} - \frac{\partial \epsilon_0}{\partial \theta}\frac{\partial}{\partial p_\theta} + \frac{1}{\tau}\right)\delta f \\ &= \left(\frac{\delta\epsilon}{\tau} + \underline{\frac{\partial \delta\epsilon}{\partial r}\frac{\partial \epsilon_0}{\partial p_r}} + \frac{\partial \delta\epsilon}{\partial \theta}\frac{\partial \epsilon_0}{\partial p_\theta} + \frac{\partial \delta\epsilon}{\partial \phi}\frac{\partial \epsilon_0}{\partial p_\phi} - \underset{\sim}{\frac{\partial \delta\epsilon}{\partial p_r}}\frac{\partial \epsilon_0}{\partial r} - \frac{\partial \delta\epsilon}{\partial p_\theta}\frac{\partial \epsilon_0}{\partial \theta}\right)\frac{\partial f_0}{\partial \epsilon_0} - \underset{\sim}{\frac{\partial \delta\epsilon}{\partial p_r}\frac{\partial \epsilon_c}{\partial r}\frac{\partial f_0}{\partial \epsilon_c}} \,,\end{aligned} \tag{2.22}$$

using $f_0 = f_0(\epsilon_0, \epsilon_c)$ and $\epsilon_c = \epsilon_c(r)$. In perturbation theory all p_i's are independent and not perturbed, and hence p_r remains zero. For this reason we will neglect all underlined terms, which contain a derivative $\partial/\partial p_r$. Loosely speaking all derivatives $\partial/\partial p_r$ can be ignored because physically ϵ_0, δf and $\delta\epsilon$ simply do not depend on p_r for circular orbits.*

*More formally this can be seen as follows. The f_0 for circular orbits effectively contains a delta function $\delta(p_r)$ which kills all terms proportional to p_r such as $\partial\epsilon_0/\partial p_r$. This is expected to hold on both sides, so the solid underlined terms vanish. The terms $\partial\delta f/\partial p_r$ and $\partial\delta\epsilon/\partial p_r$ are more subtle, however. The latter (only) vanishes if we set $p_r = 0$ in the expression for $\delta\epsilon$. In fact, for consistency with circular orbits the expression for $\delta\epsilon$, Eq. (2.15), must have $p_r = 0$ because GW perturbations leave p_i invariant (although not p^i). This can be incorporated by multiplication of $\delta\epsilon$ by $\delta(p_r)$. Then also δf will be proportional to $\delta(p_r)$ and (upon integration by parts) all the underlined terms vanish. In essence we consider the Boltzmann equation for 2D momentum space, discarding all p_r.

To further simplify the above equation we Fourier transform in time, $\partial_t \to -i\omega$. Typically the θ, ϕ dependence of QNM perturbations $h_{\mu\nu}$ is written in terms of scalar, vector and tensor spherical harmonics. All these spherical harmonics depend on ϕ via $e^{im\phi}$, and thus $\partial_\phi \to im$. Furthermore

$$\frac{\partial \epsilon_0}{\partial p_\theta} = \frac{-\bar{g}_{00} p_\theta}{r^2 \epsilon_0}\,, \quad \frac{\partial \epsilon_0}{\partial p_\phi} = \frac{-\bar{g}_{00} p_\phi}{r^2 \sin^2\theta \epsilon_0}\,, \quad \frac{\partial \epsilon_0}{\partial \theta} = \frac{\bar{g}_{00} p_\phi^2}{r^2 \epsilon_0} \frac{\cos\theta}{\sin^3\theta}\,. \tag{2.23}$$

Combining everything and dividing by $\partial\epsilon_0/\partial p_\theta$, Eq. (2.22) becomes

$$\begin{aligned}
&\left(\frac{\partial}{\partial\theta} + \frac{p_\phi^2 \cos\theta}{p_\theta \sin^3\theta}\frac{\partial}{\partial p_\theta} + \frac{imp_\phi}{p_\theta \sin^2\theta} + \frac{r^2\epsilon_0}{\bar{g}_{00} p_\theta}\left(i\omega - \frac{1}{\tau}\right)\right)\delta f \\
&= \frac{\partial f_0}{\partial \epsilon_0}\left(\frac{\partial}{\partial\theta} + \frac{p_\phi^2 \cos\theta}{p_\theta \sin^3\theta}\frac{\partial}{\partial p_\theta} + \frac{imp_\phi}{p_\theta \sin^2\theta} - \frac{r^2\epsilon_0}{\bar{g}_{00} p_\theta}\frac{1}{\tau}\right)\delta\epsilon \\
&= \left(\frac{\partial}{\partial\theta} + \frac{p_\phi^2 \cos\theta}{p_\theta \sin^3\theta}\frac{\partial}{\partial p_\theta} + \frac{imp_\phi}{p_\theta \sin^2\theta} - \frac{r^2\epsilon_0}{\bar{g}_{00} p_\theta}\frac{1}{\tau}\right)\left[\delta\epsilon \frac{\partial f_0}{\partial \epsilon_0}\right]\,,
\end{aligned} \tag{2.24}$$

where in the last line we used that $\left(\frac{\partial\epsilon_0}{\partial p_\theta}\frac{\partial}{\partial\theta} - \frac{\partial\epsilon_0}{\partial\theta}\frac{\partial}{\partial p_\theta}\right)\frac{\partial f_0}{\partial\epsilon_0} = 0$ as $f_0 = f_0(\epsilon_0, \epsilon_c)$ and $\epsilon_c = \epsilon_c(r)$. For $\omega = 0$ the solution is $\delta f = \delta\epsilon \frac{\partial f_0}{\partial \epsilon_0}$. For nonzero frequencies, solving Eq. (2.24) is a nontrivial task, which we take on in Appendix A. We find (Eq. (A.12))

$$\delta f = \delta\epsilon \frac{\partial f_0}{\partial \epsilon_0} - \frac{i\omega r^2 \epsilon_0}{\bar{g}_{00}}\frac{\partial f_0}{\partial \epsilon_0}\exp(...) \int_{\theta_0}^{\theta} d\theta' \frac{\exp(-...[\theta'])\delta\epsilon}{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}}\,, \tag{2.25}$$

where

$$\exp\,(...) \equiv \exp\left(\frac{r^2\epsilon_0\left(-i\omega + \frac{1}{\tau}\right)}{\bar{g}_{00} L}\arctan\left(\frac{p_\theta}{L\cot\theta}\right) - im\arctan\left(\frac{p_\theta \tan\theta}{p_\phi}\right)\right)\,, \tag{2.26}$$

$$\begin{aligned}
\exp\,(-...[\theta']) \equiv \exp\Bigg(&-\frac{r^2\epsilon_0\left(-i\omega + \frac{1}{\tau}\right)}{\bar{g}_{00} c_1}\arctan\left(\frac{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}}{c_1\cot\theta'}\right) \\
&+ im\arctan\left(\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}\frac{\tan\theta'}{p_\phi}\right)\Bigg)\,,
\end{aligned} \tag{2.27}$$

and *inside* the integral over θ' in the expression for $\delta\epsilon$ we substitute p_θ and ϵ_0 by

$$p_\theta(\theta') = \pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}} \quad , \, \epsilon_0 = \sqrt{-\bar{g}_{00}\left(-\bar{g}_{00}p_r^2 + \frac{c_1^2}{r^2} + m^2\right)} \, . \tag{2.28}$$

The $\pm$ corresponds to the sign of p_θ. The constant c_1^2, which corresponds to the conserved squared total angular momentum L^2, and ϵ_0 are constant w.r.t. θ'. *Outside* the integral we have resubstituted $\pm\sqrt{c_1^2 - p_\phi^2/\sin^2\theta} \to p_\theta$, $c_1 \to L$ and ϵ_0 as given by Eq. (2.8) in the expression for exp(...), see Eq. (A.13).

The GW perturbation enters at time $t = t_0$ when some particles under consideration are at the angle $\theta = \theta_0$, and δf satisfies the boundary condition $\delta f(\theta_0) = 0$. The angle θ_0 can be related to $t - t_0$ by using

$$\frac{d\theta}{dt} = \pm\frac{\bar{g}_{00}\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}}{r^2\epsilon_0} \, , \tag{2.29}$$

leading to the implicit equation for θ_0

$$\arctan\left(\frac{\pm\sqrt{L^2 - \frac{p_\phi^2}{\sin^2\theta}}}{L\cot\theta}\right) - \arctan\left(\frac{\pm\sqrt{L^2 - \frac{p_\phi^2}{\sin^2\theta_0}}}{L\cot\theta_0}\right) = \frac{\bar{g}_{00}L}{r^2\epsilon_0}(t - t_0) \, . \tag{2.30}$$

The θ-dependence of θ_0 does not spoil the solution δf, since by definition $h_{\mu\nu}(\theta_0) = 0$, i.e. the integrand is zero at $\theta = \theta_0$.

To be consistent with our assumptions about vanishing $\partial/\partial p_r$ derivatives for circular orbits, we recall that $\delta\epsilon, \delta f$ do not depend on p_r.

2.3.2 Do even and odd perturbations mix?

The spherical symmetry of Schwarzschild spacetime enables a convenient separation of variables for $h_{\mu\nu}$. For that reason the perturbations $h_{\mu\nu}$ are commonly written as the sum over angular numbers ℓ, m of the product of a function depending on radius and a scalar, vector or tensor spherical harmonic encoding the dependence on θ, ϕ. Based on the angular behavior of the spherical harmonics under parity, QNM perturbations are further decomposed into even (polar) and odd (axial) perturbations as

$$h_{\mu\nu} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(h_{\mu\nu}^{\ell m,(\mathrm{e})} + h_{\mu\nu}^{\ell m,(\mathrm{o})} \right) . \tag{2.31}$$

The decomposition into even and odd functions means that under the spatial parity transformation $\mathbf{r} \to -\mathbf{r}$, in spherical coordinates $(r, \theta, \phi) \to (r, \pi - \theta, \phi + \pi)$, the functions transform as

$$h_{\mu\nu}^{\ell m,(\mathrm{e})} \to (-1)^{\ell+\sigma} h_{\mu\nu}^{\ell m,(\mathrm{e})} , \quad h_{\mu\nu}^{\ell m,(\mathrm{o})} \to (-1)^{\ell+1+\sigma} h_{\mu\nu}^{\ell m,(\mathrm{o})} , \tag{2.32}$$

where $\sigma \in \{0, 1, 2\}$ encodes the number of indices μ, ν that are θ (e.g. $\sigma = 1$ for $\mu = \theta$, $\nu \neq \theta$). Because of the spherical symmetry of Schwarzschild spacetime, at the linear level even and odd perturbations are decoupled, and the perturbed Einstein equations, Eq. (2.2), separate into even and odd equations.

It is expected that the perturbed energy-momentum tensor $\delta T_{\mu\nu}$ preserves the parity decoupling, meaning that the evolution equations of even resp. odd modes only couple to even resp. odd $\delta T_{\mu\nu}$. As a sanity check of our solution δf we check and confirm the preservation of parity decoupling in Appendix B.

2.4 Odd parity perturbations

We have derived an explicit expression for δf and hence also $\delta T_{\mu\nu}$, Eq. (2.19). To compute the effect of $\delta T_{\mu\nu}$ on $h_{\mu\nu}$ an evolution equation is needed, which we derive in this section. Only odd parity perturbations are considered, because their evolution equations are less complicated than for even perturbations. For the parametrization of the odd perturbations we rely on Martel and Poisson, and Nagar and Rezzolla [34, 45].

The odd perturbations can be parametrized as[†] [34, 45]

$$h_{\mu\nu}^{(\mathrm{o})} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\mu\nu}^{\ell m,(\mathrm{o})} , \tag{2.33}$$

$$h_{\mu\nu}^{\ell m,(\mathrm{o})} = \begin{pmatrix} 0 & 0 & h_t^{\ell m} X_\theta^{\ell m} & h_t^{\ell m} X_\phi^{\ell m} \\ 0 & 0 & h_r^{\ell m} X_\theta^{\ell m} & h_r^{\ell m} X_\phi^{\ell m} \\ h_t^{\ell m} X_\theta^{\ell m} & h_r^{\ell m} X_\theta^{\ell m} & h_2^{\ell m} X_{\theta\theta}^{\ell m} & h_2^{\ell m} X_{\theta\phi}^{\ell m} \\ h_t^{\ell m} X_\phi^{\ell m} & h_r^{\ell m} X_\phi^{\ell m} & h_2^{\ell m} X_{\theta\phi}^{\ell m} & h_2^{\ell m} X_{\phi\phi}^{\ell m} \end{pmatrix} . \tag{2.34}$$

[†]For GWs only the radiative modes $\ell \geq 2$ are of interest. Odd perturbations cannot have $\ell = 0$ [34].

The functions $h_t^{\ell m}, h_r^{\ell m}, h_2^{\ell m}$ depend on radius r and time t (from now on we suppress the ℓm index notation of h_0, h_1, h_2). The angular dependence is encoded in the angular structure functions, which are defined as

$$X_\theta^{\ell m} = -\frac{1}{\sin\theta}\partial_\phi Y^{\ell m} \,, \tag{2.35}$$

$$X_\phi^{\ell m} = \sin\theta\partial_\theta Y^{\ell m} \,, \tag{2.36}$$

$$X_{\theta\theta}^{\ell m} = -\frac{1}{\sin\theta}\left(\partial^2_{\theta\phi} - \cot\theta\partial_\phi\right) Y^{\ell m} \,, \tag{2.37}$$

$$X_{\theta\phi}^{\ell m} = \frac{1}{2}\left(\sin\theta\partial^2_\theta - \frac{1}{\sin\theta}\partial^2_\phi - \cos\theta\partial_\theta\right) Y^{\ell m} \,, \tag{2.38}$$

$$X_{\phi\phi}^{\ell m} = \sin\theta\left(\partial^2_{\theta\phi} - \cot\theta\partial_\phi\right) Y^{\ell m} \,. \tag{2.39}$$

Here $Y^{\ell m}(\theta, \phi)$ is the well known (scalar) spherical harmonic function.

Martel and Poisson derive that under an odd parity gauge transformation, $S_\alpha = (0, 0, S_\theta, S_\phi)$, where

$$S_\theta = \sum_{\ell m} \xi^{\ell m} X_\theta^{\ell m} \,,\; S_\phi = \sum_{\ell m} \xi^{\ell m} X_\phi^{\ell m} \,, \tag{2.40}$$

the functions h_0, h_1, h_2 transform as

$$\begin{aligned} h_t &\to h_t' = h_t - \partial_t\xi \,, \\ h_r &\to h_r' = h_r - \partial_r\xi + \frac{2\xi}{r} \,, \\ h_2 &\to h_2' = h_2 - 2\xi \,, \end{aligned} \tag{2.41}$$

such that the quantities

$$\begin{aligned} \tilde{h}_t &= h_t - \frac{1}{2}\partial_t h_2 \,, \\ \tilde{h}_r &= h_r - \frac{1}{2}\partial_r h_2 + \frac{h_2}{r} \end{aligned} \tag{2.42}$$

are gauge invariant [34]. As can be observed from Eq. (2.41) the gauge ξ can be chosen such that $h_2 = 0$, yielding $\tilde{h}_t = h_t$ and $\tilde{h}_r = h_t$. The $h_2 = 0$ gauge is called the Regge-Wheeler gauge [28].

The perturbed Einstein equations for the gauge-invariant fields $\tilde{h}_t, \tilde{h}_r$ are given in Appendix C of [34]. However, the source terms are given in contravariant form.[‡] To avoid any potential confusion when raising and lowering the indices when also first order source term perturbations are incorporated, we have derived the perturbed Einstein equations for a covariant source term. For convenience we work in the Regge-Wheeler gauge, $h_2 = 0$, $\tilde{h}_t = h_t$ and $\tilde{h}_r = h_r$.

The odd perturbed Einstein equations in the Regge-Wheeler gauge are

$$-\frac{\partial^2}{\partial t \partial r} h_r + \frac{\partial^2}{\partial r^2} h_t - \frac{2}{r}\frac{\partial}{\partial t} h_r - \frac{\Lambda r - 4M}{r^3 k} h_t = S_t \, , \tag{2.43}$$

$$\frac{\partial^2}{\partial t^2} h_r - \frac{\partial^2}{\partial t \partial r} h_t + \frac{2}{r}\frac{\partial}{\partial t} h_t + \frac{(\Lambda - 2)k}{r^2} h_r = S_r \, , \tag{2.44}$$

$$-\frac{1}{k}\frac{\partial}{\partial t} h_t + k\frac{\partial}{\partial r} h_r + \frac{2M}{r^2} h_r = S \, , \tag{2.45}$$

where $\Lambda \equiv \ell(\ell+1)$ and $k \equiv (1 - 2M/r) = -\bar{g}_{00}$. The functions S_t, S_r, S are angular integrals over components of the energy-momentum tensor,

$$\begin{aligned} S_t &= -\frac{16\pi}{\ell(\ell+1)k} \int d\Omega \left(T_{t\theta} \bar{X}^{\ell m}_{\theta} + T_{t\phi} \frac{\bar{X}^{\ell m}_{\phi}}{\sin^2\theta} \right) \, , \\ S_r &= \frac{16\pi k}{\Lambda} \int d\Omega \left(T_{r\theta} \bar{X}^{\ell m}_{\theta} + T_{r\phi} \frac{\bar{X}^{\ell m}_{\phi}}{\sin^2\theta} \right) \, , \\ S &= \frac{16\pi}{\Lambda(\Lambda-2)} \int d\Omega \left(T_{\theta\theta} \bar{X}^{\ell m}_{\theta\theta} + 2T_{\theta\phi} \frac{\bar{X}^{\ell m}_{\theta\phi}}{\sin^2\theta} + T_{\phi\phi} \frac{\bar{X}^{\ell m}_{\phi\phi}}{\sin^4\theta} \right) \, . \end{aligned} \tag{2.46}$$

A bar over the angular functions denotes complex conjugation.

Eq. (2.43), (2.44), and (2.45) are identical to the evolution equations for $\tilde{h}_t$, $\tilde{h}_r$ in appendix C of Martel and Poisson, except that we give the source term in covariant form [34]. The three equations can be combined into a single 'master' wave equation. A common choice in the literature is to define the Cunningham-Price-Moncrief (CPM) function [34, 45, 73]

$$\Psi^{(o)} \equiv \frac{2r}{\Lambda - 2} \left(\partial_r \tilde{h}_t - \partial_t \tilde{h}_r - \frac{2}{r} \tilde{h}_t \right) \, , \tag{2.47}$$

such that the three equations can be combined into the master equation [34, 45]

$$\left(-\partial_t^2 + \partial_{r_*}^2 - V^{(o)}_{\ell} \right) \Psi^{(o)} = S^{(o)} \, , \tag{2.48}$$

[‡]Martel and Poisson mention that $T_{\mu\nu}$ is $T^{\mu\nu}$ lowered with the background metric [34, p. 7]. However, inconsistencies might arise when also taking into account the perturbation $\delta T_{\mu\nu}$. At this level terms like $\bar{g}^{\mu\alpha} h^{\nu\beta} \bar{T}_{\mu\nu}$ must also be taken into account to be consistent for first order source perturbations.

where

$$r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{2.49}$$

the tortoise coordinate, and the potential and source term are

$$V_\ell^{(o)} = k\left(\frac{\Lambda}{r^2} - \frac{6M}{r^3}\right), \tag{2.50}$$

$$S^{(o)} = \frac{2rk}{\Lambda - 2}\left(\frac{1}{k}\partial_t S_r + k\partial_r S_t + \frac{2M}{r^2}S_t\right). \tag{2.51}$$

When the source term $S^{(o)}$ does not depend on the perturbation $h_{\mu\nu}$ the CPM master function is a convenient choice as it leads to a single, relatively simple wave equation. However, in this work we also consider $\delta T_{\mu\nu}$ which depends on h_t, h_r, but clearly not in the combination of Eq. (2.47). Therefore $\delta T_{\mu\nu}$ cannot be written purely in terms of $\Psi^{(o)}$ defined by Eq. (2.47). A different master function than the CPM function is required in order to derive a decoupled, single wave equation. We proceed to derive a new master equation below.

The assumption of circular orbits simplifies the otherwise daunting task by the following observation: all dependence of $\delta T_{\mu\nu}$ is through contractions $h^{\mu\nu}p_\mu p_\nu$, or the trace $h^\mu{}_\mu$, see Eq. (2.19) and Eq. (2.25). The latter vanishes in the Regge-Wheeler gauge, cf. Eq. (2.34). Since circular orbits have $p_r = 0$, the contractions kill the dependence on h_r. Therefore $\delta T_{\mu\nu}$ and hence S_t, S_r, S can only depend on h_t, and a convenient master variable simply is h_t. An evolution equation for h_t in Fourier space can be arrived at as follows: (1) Solve Eq. (2.45) for $-i\omega\partial_r h_r$ and substitute in Eq. (2.43); (2) Solve Eq. (2.44) for h_r and substitute in the result from the previous point. After multiplication by k^2, the resulting wave equation for h_t is

$$k^2\frac{\partial^2}{\partial r^2}h_t - \frac{\omega^2(6M-2r)k}{-\omega^2 r^2 + (\Lambda-2)k}\frac{\partial}{\partial r}h_t + \left(\omega^2 - V_\ell^h\right)h_t = S_h^{(o)}, \tag{2.52}$$

where

$$V_\ell^h \equiv \frac{k}{r^3}\left(\Lambda r - 4M - \frac{2\omega^2(6M-2r)}{-\omega^2 + \frac{(\Lambda-2)k}{r^2}}\right), \tag{2.53}$$

$$S_h^{(o)} \equiv k\left(kS_t - i\omega S + \frac{i\omega(6M-2r)}{-\omega^2 r^2 + (\Lambda-2)k}S_r\right). \tag{2.54}$$

The function h_r is related to h_t as

$$h_r = \frac{1}{-\omega^2 + \frac{(\Lambda-2)k}{r^2}}\left(S_r - i\omega\frac{\partial}{\partial r}h_t + i\omega\frac{2}{r}h_t\right). \tag{2.55}$$

Hence, once h_t is known h_r can be computed straightforwardly.

By the assumption of circular orbits $S_r = \bar{S}_r + \delta S_r = 0$. To zeroth order $T_{\mu\nu} = \bar{T}_{\mu\nu}$ is spherically symmetric and diagonal, and all angular integrals in Eq. (2.46) for multipole numbers $\ell \geq 1$ vanish. Hence for radiative modes, $\ell \geq 2$, only $\delta T_{\mu\nu}$ is relevant.

2.5 Discussion and prospects

Because of time limitations, solving Eq. (2.52) is left for future work. We finish this chapter with a few remarks on our results and indicate some pathways for future work.

Importantly, since δf and hence $\delta T_{\mu\nu}$ contains an integral over angle θ' between (θ_0, θ) it is to be expected that the perturbed source terms mix modes with different multipole numbers. Although this severely complicates the computation of $h_t^{\ell m}$, difficulties can be evaded when only focussing on the fundamental $\ell = 2$ mode.

In all likelihood Eq. (2.52) is not analytically solveable. Since the source terms are not zero, semi-analytical techniques developed to solve the homogeneous CPM equation, Eq. (2.48) with $S^{(o)} = 0$, probably are not directly applicable to Eq. (2.52). A straightforward method is to solve Eq. (2.52) subject to physics informed boundary conditions by direct integration with a 'shooting method', as discussed in Chandrasekhar and Detweiler, and Völkel [74, 75].

Although numerical calculations are necessary to precisely estimate the GW damping effect for a specific radial profile of f_0, a rough estimate of the relevant conditions can be made. We observe that the function $h_t = h_t(t, r)$ is not affected by the θ' and p_i integrals of $\delta T_{\mu\nu}$. The angular integral in δf will not change the order of magnitude, as is also assumed for the angular integrals of the source terms of Eq. (2.46). The momentum integrals yield (energy) densities and pressures. In essence we therefore expect the components of $\delta T_{\mu\nu}$ and hence S_t, S to be $\mathcal{O}(h_t \rho_0, h_t P_0)$ for the energy density ρ_0 and pressure P_0 of the zeroth order distribution. The terms in the potential V_ℓ^h are $\mathcal{O}(M h_t / r^3, h_t / r^2)$, cf. Eq. (2.53), and the potential is maximum at $r \sim \mathcal{O}(M)$. Therefore a significant change to the vacuum solution is expected if (at least locally) ρ_0 and P_0 are not much smaller than $1/M^2$. If these conditions hold the consistency of treating the matter at the perturbative level must be verified.

Chapter 3

Conclusion

In recent times advanced models and data analysis techniques have been developed to extract overtones and higher multipole modes beyond the fundamental mode in GW signals from compact merging binaries, cf. Chapter 1. In this era of increasing precision GW science and BH spectroscopy, environmental effects on GW (ringdown) waveforms must be quantified.

This work has presented a formalism to compute the potential damping effect of matter on quasi-normal modes from a nonspinning black hole, cf. Chapter 2. Our toy model has the following ingredients:

- Schwarzschild spacetime as a background. The external matter is assumed to be relevant only at the perturbative level.
- A kinetic approach in terms of a phase space distribution function $f(r^i, p_i, t)$. The kinetic approach enables to go beyond fluid approximations often employed in the literature.
- To zeroth order a spherically symmetric distribution of massive particles on circular orbits (geodesics).
- Collisions (at first order) in the collision time approximation.

We have derived the GW induced first order perturbation to the energy-momentum tensor, $\delta T_{\mu\nu}$. The perturbed distribution function δf is calculated by solving the perturbed Boltzmann equation, including collisions. An explicit check has confirmed that the decoupling of even and odd parity modes is preserved by $\delta T_{\mu\nu}$.

A new master wave equation for odd perturbations is derived, suitable to incorporate $\delta T_{\mu\nu}$ in terms of $h_{\mu\nu}$ for circular orbits. In the future we will solve this equation numerically to quantify the damping effect for a variety of radial matter distribution profiles.

3.1 Acknowledgements

I am grateful to dr. Subodh Patil for suggesting this project. Besides being an exceptionally gifted teacher he also is a devoted supervisor with an eye for the human behind the work and word salads. His time, patience, encouragement and hope have been fundamental to the joy and success in this project.

Also thanks to PhD students Anna Negro and Alex Mikulenko for their time and effort to help me think through the problems. This project would have been less fun without my office mates, Raúl, Tinatin and Ivan. Thank you for your company. Finally, I should also thank for the elevating support of Sergei Rachmaninoff, who - as many brilliant composers - is a true Schrödingers cat: dead yet immortal.

"'Man has Nature whacked,' said someone to a friend of mine not long ago. (...) the speaker was dying of tuberculosis."
C.S. Lewis, *The Abolition of Man*

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Appendix A

Derivation solution perturbed Boltzmann equation

In this appendix the perturbed Boltzmann equation, Eq. (2.24), is solved.

Eq. (2.24) has two derivative operators, $\partial/\partial\theta$ and $\partial/\partial p_\theta$, in the combination

$$\frac{\partial}{\partial\theta} + \frac{p_\phi^2 \cos\theta}{p_\theta \sin^3\theta}\frac{\partial}{\partial p_\theta} \, .$$

In general a first order PDE can be solved with the method of characteristics, which for a parameter $u = u(\theta, p_\theta)$ satisfying

$$\frac{d\theta}{du} = 1 \quad , \quad \frac{dp_\theta}{du} = \frac{\cos\theta}{\sin^3\theta}\frac{p_\phi^2}{p_\theta^2} \tag{A.1}$$

reduces the PDE to an ODE in terms of u,

$$\frac{\partial}{\partial\theta} + \frac{p_\phi^2 \cos\theta}{p_\theta \sin^3\theta}\frac{\partial}{\partial p_\theta} \;\rightarrow\; \frac{d}{du} \, . \tag{A.2}$$

The two equations of Eq. (A.1) have the solution $u = \theta + c_2$ and $p_\theta = \pm\sqrt{c_1^2 - p_\phi^2/\sin^2(u - c_2)}$. We can choose our initial curve to be such that $c_2 = 0$, i.e. $u = \theta$, which fundamentally means that we get an ODE in terms of θ when we write every p_θ in terms of θ as

$$p_\theta = \pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}} \, , \tag{A.3}$$

which also implies that

$$\epsilon_0 = \sqrt{-\bar{g}_{00}\left(-\bar{g}_{00}p_r^2 + \frac{c_1^2}{r^2} + m^2\right)} \, . \tag{A.4}$$

From Eq. (2.9) it is apparent that $c_1^2 = L^2$, but for now we write c_1 to indicate that it is a constant along integration, and hence also ϵ_0.*

To solve Eq. (2.24) with p_θ written in terms of θ we employ a Green's function approach. Instead of δf we consider $G(\theta,\theta')$ and replace the right side of Eq. (2.24) by $\delta(\theta-\theta')$, yielding

$$\frac{dG}{d\theta} + \left(\frac{imp_\phi}{\pm\sin^2\theta\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}} + \frac{r^2\epsilon_0\left(i\omega - \frac{1}{\tau}\right)}{\pm\bar{g}_{00}\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}}\right) G = \delta(\theta-\theta') \, , \tag{A.5}$$

where ϵ_0 independent of θ, cf. Eq. (A.4). Using the anti-derivatives

$$\begin{aligned}\int d\theta \left(c_1^2 - \frac{p_\phi^2}{\sin^2\theta}\right)^{-1/2} &= \frac{1}{c_1}\arctan\left(\frac{\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}}{c_1\cot\theta}\right) + C_1 \, , \\ \int \frac{d\theta}{\sin^2\theta}\left(c_1^2 - \frac{p_\phi^2}{\sin^2\theta}\right)^{-1/2} &= \frac{1}{p_\phi}\arctan\left(\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}\,\frac{\tan\theta}{p_\phi}\right) + C_2 \, ,\end{aligned} \tag{A.6}$$

the solution to Eq. (A.5) is

$$G = C_3(\theta')\exp\left(...[\theta]\right) + \exp\left(...[\theta]\right)\int_0^\theta dx\,\exp\left(-...[x]\right)\delta(x-\theta') \, , \tag{A.7}$$

where $C_3(\theta')$ an integration constant and

$$\begin{aligned}\exp\left(...[\theta]\right) \equiv \exp&\left(\frac{r^2\epsilon_0\left(-i\omega + \frac{1}{\tau}\right)}{\bar{g}_{00}c_1}\arctan\left(\frac{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}}{c_1\cot\theta}\right)\right. \\ &\left. -im\arctan\left(\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta}}\,\frac{\tan\theta}{p_\phi}\right)\right) \, .\end{aligned} \tag{A.8}$$

*Fundamentally, $c_1^2 = L^2$ is a constant w.r.t. the integration because L^2 is conserved along geodesics, i.e. $\left[\frac{\partial}{\partial\theta} + \frac{p_\phi^2\cos\theta}{p_\theta\sin^3\theta}\frac{\partial}{\partial p_\theta}\right]L^2 = 0$. The same applies for ϵ_0.

When $G(\theta,\theta')$ satisfies a boundary condition, this condition is also satisfied by its convolution with a source term. Therefore we can impose the relevant boundary condition of $\delta f(\theta)$ on $G(\theta,\theta')$. The appropriate boundary condition is that at the time $t = t_0$ at which the GW reaches the (radial) position under consideration the perturbation to the zeroth order distribution f_0 is zero, because the distribution is only perturbed by the GW. At $t = t_0$ we indicate the position of a particle by (r_0, θ_0, ϕ_0). Thus we impose $\delta f(\theta_0) = 0$ and $G(\theta_0, \theta') = 0$. Therefore

$$G(\theta,\theta') = \exp\left(...[\theta]\right) \int_{\theta_0}^{\theta} dx \exp\left(-...[x]\right) \delta(x-\theta') \, . \tag{A.9}$$

The expression for $G(\theta,\theta')$ can be further simplified by using the delta function to perform the integral, yielding

$$G(\theta,\theta') = \begin{cases} \Theta(\theta-\theta')\Theta(\theta'-\theta_0)\exp(...[\theta])\exp(-...[\theta']) & \theta > \theta_0 \, , \\ \Theta(\theta'-\theta)\Theta(\theta_0-\theta')\exp(...[\theta])\exp(-...[\theta']) & \theta < \theta_0 \, . \end{cases} \tag{A.10}$$

where the step functions ensure that θ' is in the integration domain. The above $G(\theta,\theta')$ solves eq. (A.5) with $\delta(\theta-\theta')\Theta(\pm(\theta'-\theta_0))$ on the right side, the sign corresponding to $\theta > \theta_0$ resp. $\theta < \theta_0$. The boundary condition has therefore led us to only consider an impulse response when θ' in the integration domain (θ_0, θ) resp. (θ, θ_0).

The full solution to Eq. (2.24) is obtained by convoluting $G(\theta,\theta')$ with the right side of Eq. (2.24) (recall Eq. (A.2), (A.3), (A.4))

$$\begin{aligned} \delta f &= \int_0^{\pi} d\theta' G(\theta,\theta') \frac{\partial f_0}{\partial \epsilon_0} \left(\frac{d\delta\epsilon}{d\theta'} + \frac{imp_\phi \delta\epsilon}{\pm \sin^2\theta' \sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} - \frac{r^2 \epsilon_0 \frac{1}{\tau} \delta\epsilon}{\pm \bar{g}_{00} \sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} \right) \\ &= \frac{\partial f_0}{\partial \epsilon_0} \int_{\theta_0}^{\theta} d\theta' \frac{\exp(...[\theta])}{\exp(...[\theta'])} \left(\frac{d\delta\epsilon}{d\theta'} + \frac{imp_\phi \delta\epsilon}{\pm \sin^2\theta' \sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} - \frac{r^2 \epsilon_0 \frac{1}{\tau} \delta\epsilon}{\pm \bar{g}_{00} \sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} \right) \\ &= \left[\frac{\exp(...[\theta])}{\exp(...[\theta'])} \delta\epsilon \frac{\partial f_0}{\partial \epsilon_0} \right]_{\theta_0}^{\theta} - \frac{i\omega r^2 \epsilon_0}{\bar{g}_{00}} \frac{\partial f_0}{\partial \epsilon_0} \int_{\theta_0}^{\theta} d\theta' \frac{\exp(...[\theta])}{\exp(...[\theta'])} \frac{\delta\epsilon}{\pm \sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} \end{aligned} \tag{A.11}$$

where in the last line we integrated by parts the term $d\delta\epsilon/d\theta'$. Since at $t = t_0$ and the corresponding angular coordinate $\theta = \theta_0$ the GW enters,

$h^{\mu\nu}(\theta_0) = 0$ and $\delta\epsilon(\theta_0) = 0$. We therefore end up with

$$\delta f = \delta\epsilon \frac{\partial f_0}{\partial \epsilon_0} - \frac{i\omega r^2 \epsilon_0}{\bar{g}_{00}} \frac{\partial f_0}{\partial \epsilon_0} \exp(...[\theta]) \int_{\theta_0}^{\theta} d\theta' \frac{\exp(-...[\theta'])\delta\epsilon}{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}}\,, \tag{A.12}$$

where in the expression for $\delta\epsilon$ inside the integral the relations Eq. (A.3) and (A.4) must be taken into account. Outside the integral, however, the substitutions can be undone, and the function $\exp(...[\theta])$ can be written as

$$\exp\,(...) = \exp\left(\frac{r^2\epsilon_0\left(-i\omega + \frac{1}{\tau}\right)}{\bar{g}_{00}L} \arctan\left(\frac{p_\theta}{L\cot\theta}\right) - im\arctan\left(\frac{p_\theta\tan\theta}{p_\phi}\right)\right)\,. \tag{A.13}$$

Eq. (A.12) can be checked by restoring $c_1 \to L$ and $\pm\sqrt{L^2 - p_\phi^2/\sin^2\theta} \to p_\theta$ (only *outside* the integral over θ') and plugging it into Eq. (2.24). To do this explicitly we rewrite Eq. (2.24) to

$$\left(-i\omega + \frac{1}{\tau} + \hat{L}\right)\delta f - \left(\frac{1}{\tau} + \hat{L}\right)\left[\delta\epsilon\frac{\partial f_0}{\partial\epsilon_0}\right] = 0\,,$$

where

$$\hat{L} \equiv \frac{\partial\epsilon_0}{\partial p_\theta}\frac{\partial}{\partial\theta} + im\frac{\partial\epsilon_0}{\partial p_\phi} - \frac{\partial\epsilon_0}{\partial\theta}\frac{\partial}{\partial p_\theta}\,.$$

Observing that

$$\left(-i\omega + \frac{1}{\tau} + \hat{L}\right)\exp(...) = 0\,,$$
$$\left[\frac{\partial\epsilon_0}{\partial p_\theta}\frac{\partial}{\partial\theta} - \frac{\partial\epsilon_0}{\partial\theta}\frac{\partial}{\partial p_\theta}\right]L^2 = \left[\frac{\partial\epsilon_0}{\partial p_\theta}\frac{\partial}{\partial\theta} - \frac{\partial\epsilon_0}{\partial\theta}\frac{\partial}{\partial p_\theta}\right]\epsilon_0 = 0\,,$$

one can check that (see also Eq. (2.23))

$$\begin{aligned}
&\left(-i\omega + \frac{1}{\tau} + \hat{L}\right)\delta f - \left(\frac{1}{\tau} + \hat{L}\right)\left[\delta\epsilon\frac{\partial f_0}{\partial\epsilon_0}\right] \\
&= -i\omega\delta\epsilon\frac{\partial f_0}{\partial\epsilon_0} \\
&\quad - \frac{i\omega r^2\epsilon_0}{\bar{g}_{00}}\frac{\partial f_0}{\partial\epsilon_0}\exp(...)\frac{\partial\epsilon_0}{\partial p_\theta}\frac{\partial}{\partial\theta}\int_{\theta_0}^{\theta} d\theta' \exp(-...[\theta'])\frac{\delta\epsilon}{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}} \\
&= 0\,.
\end{aligned}$$

Appendix B

Parity check

The perturbed Einstein equations for even resp. odd perturbations $h_{\mu\nu}^{\ell m}$ single out energy-momentum terms $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$ which gain a factor $(-1)^{\ell+\sigma}$ resp. $(-1)^{\ell+1+\sigma}$ under parity (σ the number of indices that are θ). In this appendix we check whether the parts of $\delta T_{\mu\nu}$ that gain a factor $(-1)^{\ell+\sigma}$ resp. $(-1)^{\ell+1+\sigma}$ under parity are only due to $h_{\mu\nu}^{\ell m,(\mathrm{e})}$ resp. $h_{\mu\nu}^{\ell m,(\mathrm{o})}$.

Spatial parity is defined by

$$(r, \theta, \phi) \to (r, \pi - \theta, \phi + \pi) \; . \tag{B.1}$$

The behavior of the following quantities under spatial parity is:

1. $\sin(\theta) \to \sin(\theta)$, $\cos(\theta) \to -\cos(\theta)$, $\tan(\theta) \to -\tan(\theta)$.

2. $(p_r, p_\theta, p_\phi) \to (p_r, -p_\theta, p_\phi)$. This can be understood from $p_i \propto dp^i/d\tau$; only $d\theta/d\tau$ changes sign. Hence also $\pm\sqrt{L^2 - p_\phi^2/\sin^2\theta} \to \mp\sqrt{L^2 - p_\phi^2/\sin^2\theta}$. Clearly ϵ_0 and f_0 are invariant under parity.

3. Combining 1 and 2: $\exp(...) \to \exp(...)$, see Eq. (2.26).

4. Combining 1 and 2: $\delta\epsilon \propto h_{\ell m}^{\alpha\beta} p_\alpha p_\beta$ and $h^{ij} p_i p_j$ gain a factor $(-1)^\ell$ resp. $(-1)^{\ell+1}$ for even resp. odd modes.

5. Similar to $\theta \to \pi - \theta$, $\theta_0 \to \pi - \theta_0$. Therefore, the parity operation on the integral boundaries of δf can be moved inside the integral $\int_{\theta_0}^{\theta} d\theta' ... \to \int_{\pi-\theta_0}^{\pi-\theta} d\theta' ...$ by redefining $\theta'' = \pi - \theta'$. Combined with the above observations

$$\int_{\theta_0}^{\theta} d\theta' \frac{\exp(-...[\theta'])\delta\epsilon}{\pm\sqrt{c_1^2 - \frac{p_\phi^2}{\sin^2\theta'}}}$$

gains a factor $(-1)^\ell$ resp. $(-1)^{\ell+1}$ for even resp. odd modes. Similar to exp(...), exp(−...[θ'']) is invariant under parity inside the integral.

Based on the above we now consider each term of the expressions for $\delta T_{\mu\nu}$, given by Eq. (2.19), individually. It is straightforward to verify that

$$\bar{g}_{00}h^{00}\ ,\ \frac{\bar{g}_{00}}{2\epsilon_0^2}h^{ij}p_ip_j\ ,\ \frac{1}{2}\bar{g}_{ij}h^{ij}\ ,\ \frac{\bar{g}_{00}}{\epsilon_0}h^{0m}p_m\ ,\ \delta f$$

all transform as $(-1)^\ell$ resp. $(-1)^{\ell+1}$ for even resp. odd modes. To determine the total transformation property of $\delta T_{\mu\nu}$ also the transformation of $p_\mu p_\nu$ must be taken into account. For (only) one of the indices μ, ν equal to θ this yields an additional minus sign.

The resulting behavior of $\delta T_{\mu\nu}$ under parity for even resp. odd $h^{\ell m}_{\mu\nu}$ is summarized in Table B.1. Recalling that the even resp. odd perturbed Einstein equations filter out the $(-1)^{\ell+\sigma}$ resp. $(-1)^{\ell+1+\sigma}$ part of $\delta T_{\mu\nu}$, we conclude that parity decoupling is preserved: $h^{\ell m,(\mathrm{e})}_{\mu\nu}$ resp. $h^{\ell m,(\mathrm{o})}_{\mu\nu}$ only couple to $\delta T^{\ell m,(\mathrm{e})}_{\mu\nu}$ resp. $\delta T^{\ell m,(\mathrm{o})}_{\mu\nu}$. This confirms our expectations.

Perturbation type	δT_{00}	δT_{0i}	δT_{ij}
$h^{(\mathrm{e})}_{\mu\nu}$	$(-1)^\ell$	$(-1)^{\ell+\sigma}$	$(-1)^{\ell+\sigma}$
$h^{(\mathrm{o})}_{\mu\nu}$	$(-1)^{\ell+1}$	$(-1)^{\ell+1+\sigma}$	$(-1)^{\ell+1+\sigma}$

Table B.1: *Behavior of $\delta T_{\mu\nu}$, Eq. (2.19), under the spatial parity transformation. $\sigma \in \{0,1,2\}$ is the number of indices i, j that are θ.*