Measurable cardinal numbers

Ferrer, Camilo

C. Ferrer
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Thesis supervisors: prof.dr. K.P. Hart

Leiden University
Mathematical Institute

Preface

In this paper we start off with a very simple question: what are the of homomorphism of powers $\mathbb{Z}$ to $\mathbb{Z}$ itself. This, in theory, seems straightforward enough, but it turns out that the aforementioned homomorphisms, i.e. for a set I, the homomorphisms $\varphi : \mathbb{Z}^I \to \mathbb{Z}$, give some fascinating and unexpected results. This, at the beginning seems like a fundamental abstract algebra problem, but somehow once we uncover the mysteries behind such homomorphisms we end up instead in the world of set theory. It turns out that these homomorphisms depend on the concept of ultrafilters, and, in particular we end up discovering beneath it all the so-called 'measurable' cardinal numbers. Once we discover said measurable cardinal numbers, we will look at their properties. It turns out that these measurable cardinal numbers are not only interesting in their own right, but they have very fascinating implications. From proving ZFC's consistency, to implications in combinatorics, and, we even get a look at how 'big' these cardinal numbers are if they exist.

As just implied, in this paper we will not only be using Zermelo-Fraenkel (ZF) set theory, but we will also be using the Axiom of Choice (AC). In other words, throughout this whole paper, we will be using ZFC set theory. This of course implies Zorn's Lemma.

Knowledge of some abstract algebra and set theory is expected from the reader. Basic definitions such as homomorphisms, cardinals, and more are therefore not defined rigorously, but more so informally, in this paper. All relevant sources will be listed in the appendix, and there one can also find the relevant background information, if needed.

Last but definitely not least, I would like to give lots of thanks to my supervisor K.P. Hart for all the helpful feedback and good conversation throughout the whole process of writing this thesis.

Contents

1 Preliminaries

To start things off, we will be looking at and describing homomorphims from powers of $\mathbb{Z}$ to $\mathbb{Z}$. When we say 'homomorphims' we mean the all too well known group homomorphisms from abstract algebra. In other words, we will focus on homomorphisms $\varphi : \mathbb{Z}^I \to \mathbb{Z}$, where I can be any (infinite) set. In particular, for $x, y \in \mathbb{Z}^I$ we have that $\varphi(x + y) = \varphi(x) + \varphi(y)$.

If we consider the trivial case where φ is a homomorphism from $\mathbb{Z}$ to itself, it becomes quite clear what the homomorphisms are. These are all maps $\varphi : \mathbb{Z} \to \mathbb{Z}$ such that $\varphi(x) = yx$, for some $y \in \mathbb{Z}$; in fact, $y = \varphi(1)$, so $\varphi(x) = \varphi(1)x$. Clearly these are all homomorphism since, by distributivity we have $y(x+z) = yx+yz$, for $x, y, z \in \mathbb{Z}$. Thus, we know how all the homomorphims of $\mathbb{Z}$ to $\mathbb{Z}$ look like, but as we will soon see, we are not satisfied with this. The next evolution of our attempt to find new homomorphims therefore is: let I be a finite set. In other words we will consider maps φ from $\mathbb{Z}^N$ to $\mathbb{Z}$ where N is a natural number. Now, what does a homomorphism $\varphi : \mathbb{Z}^N \to \mathbb{Z}$ look like. One such homomorphism is clear: the trivial map, the map that maps all elements of $\mathbb{Z}^N$ to the identity element; nothing interesting here. But of course we are not satisfied with this, we would like to think of some more interesting, perhaps even exciting ones. But now, since in this paper we will be focusing on arbitrary (infinite) powers of $\mathbb{Z}$ we will recall what powers of $\mathbb{Z}$ exactly are.

The *direct product* of $\mathbb{Z}$, denoted as $\mathbb{Z}^I$, where I is an arbitrary set, is defined as the set of all functions $x : I \to \mathbb{Z}$ such that $x(i) \in \mathbb{Z}$ for all $i \in I$. The notion of direct products can naturally be generalized to any set T, so that T^I, where I is an arbitrary set, is analogous to the aforementioned definition. Furthermore, we say that for any $x \in \mathbb{Z}^I$ the support of x, denoted as $\operatorname{supp}(x)$, is defined to be the set of indices $i \in I$ such that $x(i)$ is non-zero. Finally, we recall that the *direct sum* of $\mathbb{Z}$, which is denoted as $\mathbb{Z}^{(I)}$, is the subset of $\mathbb{Z}^I$ such that $\operatorname{supp}(x)$ is finite. When I is finite we may simply write $\mathbb{Z}^{(I)}$ as $\mathbb{Z}^N$, where $N \in \mathbb{N}$ is the size of I. This, of course, can also be seen as the Cartesian product.

Now that the structure of $\mathbb{Z}^N$ is made clear, it is evident what a 'natural' homomorphism from $\mathbb{Z}^N$ to $\mathbb{Z}$ would look like. The restriction of our homomorphism $\varphi : \mathbb{Z}^N \to \mathbb{Z}$ to each summand induces a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ (which we just discussed), and together these homomorphisms determine φ. Thus, the homomorphisms from $\mathbb{Z}^N$ to $\mathbb{Z}$ are the functions $\varphi_y : x \mapsto \sum_{i=0}^N x_i y_i$ with $y \in \mathbb{Z}^N$.

So far we have been making use of the natural numbers without rigorously explaining what we mean by them. Let us therefore define them, but, before we do that, let us recall what an ordinal number is.

Definition 1.1. An *ordinal number* (or simply, an ordinal) is a set T which is well-ordered by inclusion $\in$, and, it has the property that every element of T is a subset of T.

It is common practice to use lower case Greek letters to denote ordinals, starting with α. Furthermore, we will denote the class of all ordinals as Ord. Of course ordinal numbers have very useful properties, such as $\alpha < \beta$ if and only if $\alpha \in \beta$ and $\alpha = \{\beta : \beta < \alpha\}$.

In particular, we can see every well-ordered set as a unique ordinal number.

Theorem 1.2. *Every well-ordered set is isomorphic to a unique ordinal number.*

Since we are using ZFC set theory, in particular, the axiom of choice, we can state the Theorem above as "every set has a bijection to some ordinal". We define the *successor* of α as $\alpha + 1 = \alpha \cup \{\alpha\}$. If $\alpha = \beta + 1$, then we call α a *successor ordinal*. An ordinal $\alpha \neq 0$ which is not a successor ordinal is called a *limit ordinal*. In particular, we can define what we so far have called "the natural numbers" with ordinals.

Definition 1.3. We denote the least nonzero limit ordinal as ω. The ordinals less than ω are called *finite ordinals*.

We should note that ω is non-other than the set of natural numbers $\mathbb{N}$, and, furthermore, finite ordinals are simply natural numbers $n \in \mathbb{N}$. Of course ordinal numbers are very interesting in their own right, but in this paper we have our interests in cardinal numbers. Two sets X and Y are said to have the same cardinality $X \approx Y$ if there exists a bijection between X and Y. As a consequence of the axiom of choice every well-ordered set can be associated to its cardinal number by means of a bijection. Now that we have defined both ordinals and cardinal numbers we can see the relationship between them.

Definition 1.4. An ordinal number α is called a cardinal number (or simply, a cardinal) if $\alpha \not\approx \beta$ for all $\beta < \alpha$.

Of course, if T is a well-ordered set, then there exists an ordinal α such that the cardinality of T is equal to the cardinality of α. We define $|T|$ to be α, where α is the least ordinal such that $T \approx \alpha$. It goes without saying that $|T|$ is indeed a cardinal number. In particular, the ordinal ω is the *least infinite cardinal*. Without going too much into detail, something that will be very useful for the upcoming chapters is the notion of *limit cardinals*, where the definition of limit cardinals is analogous to that of limit ordinals.

Now, back to our main question. What are all the homomorphism of $\mathbb{Z}^I \to \mathbb{Z}$. We already explored finite I, where we showed that the homomorphisms are all in a sense quite straightforward. Clearly the next step is to look at I when it is a non-finite cardinal number.

To start with, we need to look at the least infinite cardinal, that being ω. We will therefore for now be interested in the homomorphims $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}$. The following Theorem tells us that such φ is determined by a finite set of coordinates.

Theorem 1.5. *For every homomorphism $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}$ there is an $m \in \omega$ such that φ is determined by the coordinates below m, that is, if $x, y \in \mathbb{Z}^\omega$ are such that $x \restriction m = y \restriction m$ then $\varphi(x) = \varphi(y)$.*

Proof. For $m \in \omega$ we write $U_m = \{x \in \mathbb{Z}^\omega : x(i) = 0, \text{for all } i < m\}$, the subgroup of $\mathbb{Z}^\omega$, such that $x \in U_m$ is zero before the m-th element. Notice, that the statement in the Theorem is equivalent to the statement that there is an m such that U_m in contained in the kernel of φ. It is necessary to find an $m \in \omega$, $a \in \mathbb{Z}^\omega$ and a nonzero $r \in \mathbb{Z}$ such that φ is constant on $a + rU_m$ to prove the aforementioned statement. In fact, it is not only necessary, but also sufficient. This follows from the fact that if we take $x \in U_m$, then $a + rx \in a + rU_m$ and so $\varphi(a) = \varphi(a + rx)$. But $\varphi(a + rx) = \varphi(a) + r\varphi(x)$ and so $r\varphi(x) = 0$, which implies that $\varphi(x) = 0$.

For the sake of contradiction, suppose that no such m, a, and r exist. We construct sequences $\langle a_k : k < \omega \rangle$ in $\mathbb{Z}^\omega$ and $\langle r_k : k < \omega \rangle$ in $\mathbb{Z} \setminus \{0\}$ as follows. First enumerate $\mathbb{Z}$ as $\langle z_k : k < \omega \rangle$.

Start with $a_0 = 0$ and $r_0 = 1$. To find a_{k+1} and r_{k+1} we use that φ is not constant on $a_k + (r_0 \cdots r_k r_{k+1})U_{k+1}$ and take $a_{k+1} \in a_k + (r_0 \cdots r_k)U_{k+1}$ such that $\varphi(a_{k+1}) \neq z_k$. Next take r_{k+1} so large that $z_k \notin \varphi(a_{k+1}) + r_{k+1}\mathbb{Z}$. It is clear that

$$\varphi[a_{k+1} + (r_0 \cdots r_k r_{k+1})U_{k+2}] \subset \varphi(a_{k+1}) + r_{k+1}\mathbb{Z}$$

and hence $z_k \notin \varphi[a_{k+1} + (r_0 \cdots r_k)U_{k+2}]$.

By construction of the sequence $\langle a_k : k < \omega \rangle$ we see that $a_{k+1}(i) = a_k(i)$ whenever $i \leq k$. This implies that the pointwise limit, a, of the sequence exists. If we fix some k then now we know that $a(i) = a_k(i)$ for $i \leq k$ and that $a(i) - a_k(i)$ is divisible by $r_0 \cdots r_k$ for all $i > k$. This means that $a \in a_k + (r_0 \cdots r_k)U_{k+1}$ for all k. Thus, for all k we have $\varphi(a) \in \varphi(a_{k+1}) + r_{k+1}\mathbb{Z}$ and hence $\varphi(a) \neq z_k$. This is a contradiction because $\varphi(a) \in \mathbb{Z}$. □

With help of this Theorem we find just what we were looking for, that is, the classification of all homomorphisms from $\mathbb{Z}^\omega$ to $\mathbb{Z}$.

Corollary 1.6. *Every homomorphisms $\varphi : \mathbb{Z}^\omega \to \mathbb{Z}$ is determined by a finite sequence $\langle c_k : k < m \rangle$ of integers in the sense that $\varphi(x) = \sum_{k<m} c_k x(k)$ for all $x \in \mathbb{Z}^\omega$.*

Proof. Let m be as in Theorem 1.5 and notice that for all $x \in \mathbb{Z}^\omega$ the difference $x - (x \restriction m)$ belongs to U_m, where $(x \restriction m)(k) = 0$ if $k \geq m$ and $(x \restriction m)(k) = x(k)$ if $k < m$. Furthermore, $x \restriction m = \sum_{k<m} e_k x(k)$, where e_k is the k-th 'unit vector' in $\mathbb{Z}^\omega$, i.e. all coordinates of e_k are zero exept the k-th one which is equal to 1. It follows that $\varphi(x) = \varphi(x \restriction m) = \sum_{k<m} \varphi(e_k)x_k$, and thus φ is determined by the sequence $\langle \varphi(e_k) : k < m \rangle$. □

So much for our "bigger thinking" then. This characteristical way of defining such a homomorphism is more general than one might initially think. In fact, it follows that if κ is an infinite cardinal, F is a finite subset of κ, and $\langle c_\alpha : \alpha \in F \rangle$ a finite sequence of integers, then $\varphi_{F,c}(x) = \sum_{\alpha \in F} c_\alpha x_\alpha$ defines a homomorphism from $\mathbb{Z}^\kappa \to \mathbb{Z}$. This will be the basis for the next section.

2 Homomorphisms from $\mathbb{Z}^\kappa$ to $\mathbb{Z}$

From now on we will focus our attention on infinite cardinals κ. Clearly then, it seems that no matter how 'big' we make the exponent of the integers, no matter how much we increase our cardinal κ to bigger and bigger infinities – we are doomed to have only such characteristically defined homomorphisms such as above. Right? It turns out the answer to this question is more involved than originally thought.

This is where the so called 'ultrafilters' come and miraculously save us from such simply defined homomorphisms.

Definition 2.1. A *filter* F on a nonempty set I is a collection of subsets of I that satisfies the following properties:

(i) $\emptyset \notin F$, $I \in F$;

(ii) if $X, Y \in F$, then $X \cap Y \in F$;

(iii) if $X \in F$ and $X \subset Y \subset I$, then $Y \in F$.

An *ideal* N on a nonempty set I is a collection of subsets of I that satisfies the following properties:

(i) $\emptyset \in N$, $I \notin N$;

(ii) if $X, Y \in N$, then $X \cup Y \in N$;

(iii) if $X \in N$ and $Y \subset X \subset I$, then $Y \in N$.

We can see that filters and ideals are each other's opposite. Indeed, if F is a filter on I, then $\overline{F} = \{I \setminus X : X \in F\}$ is an ideal; we call $\overline{F}$ the *dual ideal of* F. On the other hand, when N is an ideal on I, then $\overline{N} = \{I \setminus X : X \in N\}$ is a filter on I; likewise, we call $\overline{N}$ the *dual filter of* N.

Filters give a meaning of "largeness", in the sense that members of the filters are considered large (and their compliment small). Conversely, and ideal gives a meaning for "smallness".

Example 2.2. If Y is a subset of I, then the filter

$$F_Y := \{X \in \mathcal{P}(I) : Y \subset X\},$$

is called the *principal filter* generated by Y. If F is a filter on I which is not equal to F_Y for any $Y \subset I$, then we call F a *nonprincipal filter*.

One of the (if not the) most important notions in this paper is the following. An *ultrafilter* on I is a filter F on I such that for every subset X of I, either $X \in F$ or $I \setminus X \in F$. That is, if we have an ultrafilter F on I, then every subset X of I gets a tag: X is either 'big' or 'small', depending on whether $X \in F$ or $I \setminus X \in F$, respectively. Note that principal filters that are generated by one element are in fact ultrafilters; just not very interesting ones.

Definition 2.3. Let κ be an infinite cardinal. We call a filter F on I *κ-complete* if for every subset E of F with cardinality strictly less than κ, the intersection over all E belongs to F.

In particular, we will be focusing on 'σ-completeness', that is, a filter F on I is called *σ-complete* if for every countable subset E of F the intersection over all E belongs to F. In addition to this we say that an ideal N is called a *σ-ideal* whenever every countable subset S of N the union over all S belongs to N.

An important example of a filter is the following.

Example 2.4. If $|I| \geq \kappa$, then $C_\kappa = \{X \subset O : |I \setminus X| < \kappa\}$ is called the *co-κ filter* on I. In particular, we call C_ω the *cofinite filter*

Behind all of this talk about ultrafilters, what is going to be surprising, is that they seem to have a relationship with cardinals. That may not be so surprising since informally we could say that cardinals (or better still, ordinals) are just sets, but still, it doesn't seem very natural. Before that, we need some tools to help us get where we need to be. We say that a non-empty family of subsets S of I has the *finite intersection property* (FIP) whenever the intersection of any finite subset of S is non-empty.

Lemma 2.5. *Let S be a family of subsets of I. Then S is contained in a filter on I if and only if S has FIP.*

Proof. First, it is clear that a filter, and therefore any subset of a filter, has FIP. Conversely, let S have FIP. Now if F is the set of all subsets of I which contain the intersection of a finite subset of S, then F is a filter. □

In fact, there is a similar result for ultrafilters. First, we say that a filter F is a *maximal filter* whenever there is no filter on I which properly contains F.

Lemma 2.6. *A filter F on I is an ultrafilter if and only if F is a maximal filter.*

Proof. First let F be an ultrafilter on I. Furthermore, let F' be a filter on I which contains F. Then, if $X \in F'$ then also $X \in F$. If you suppose the contrary, then $I \setminus X \in F$, and then we would have that $\emptyset = X \cap (I \setminus X) \in F'$, which is a contradiction. Thus, F is maximal filter. Now suppose that F is a maximal filter on I. For any $Y \subset I$, if $\{Y\} \cup F$ has FIP, then $\{Y\} \cup F$ is, by Lemma 2.5, contained in a filter F'. But then, by the maximality of F we have that $F' = F$, so $Y \in F$. But for any $X \subset I$ either $\{X\} \cup F$ or $\{I \setminus X\} \cup F$ has FIP. Thus, F is an ultrafilter. □

Here is the (maybe not so) surprising result on the existence of nonprincipal ultrafilters.

Theorem 2.7. *Every filter on I is contained in an ultrafilter. Therefore, if I is an infinite set, then there exists a nonprincipal ultrafilter.*

Proof. If F is a filter on I, then by Zorn's Lemma there is a maximal filter F' which contains F. By Lemma 2.6 F' is an ultrafilter. Finally, if I is an infinite set, then F contains the cofinite filter, and thus D is nonprincipal. □

So we shouldn't concern ourselves too much about the existence of nonprincipal ultrafilters (for now). And now we will finally discuss the fundamental definition for this particular paper.

Definition 2.8. A cardinal κ is called *measurable* if there is a nonprincipal σ-complete ultrafilter on κ (or simply on a set I of size κ).

Now with all the important background information out of the way, we can go back to the question at hand: what are all the homomorphisms $\varphi : \mathbb{Z}^\kappa \to \mathbb{Z}$, where κ is any infinite cardinal.

If $\mathcal{U}$ is a σ-complete ultrafilter on κ, then, for every $x \in \mathbb{Z}^\kappa$ we have that $\kappa = \bigcup_{k\in\mathbb{Z}}\{\alpha : x_\alpha = k\}$ and by σ-completeness there is exactly one such k_x such that $\{\alpha : x_\alpha = k_x\} \in \mathcal{U}$. Now, since

$$\{\alpha : x_\alpha = k_x\} \cap \{\alpha : y_\alpha = k_y\} \subset \{\alpha : x_\alpha + y_\alpha = k_x + k_y\},$$

it follows that $k_{x+y} = k_x + k_y$. Therefore, the map $\varphi_\mathcal{U} : \mathbb{Z}^\kappa \to \mathbb{Z}$ given by $x \mapsto k_x$ is a well-defined homomorphim. First, we will consider the case when κ is a non-measurable cardinal. By definition then, every σ-complette ultrafilter, $\mathcal{U}$, is principal, in other words, $\mathcal{U} = \{U : \alpha \in U\}$, for some $\alpha \in \kappa$. But then we have that the homomorphism $\varphi_\mathcal{U}$ is simply the projection $x \mapsto x_\alpha$. More Interestingly, if $\mathcal{U}$ is nonprincipal then we have a new type of homomorphism. Namely, for any finite $E \subset \kappa$ and sequence $c = \langle c_\alpha : \alpha \in E\rangle$ we have that $\varphi_{E,c} \neq \varphi_\mathcal{U}$ as seen at the end of Chapter 1. To see this take $x \in \mathbb{Z}^\kappa$ such that $x_\alpha = 0$ if $\alpha \in E$ and $x_\alpha = 1$ otherwise. Then $\varphi_{E,c}(x) = 0$ but $\varphi_\mathcal{U}(x) = 1$. The question thus becomes: are there any other types of homomorphisms?

Before we answer this question we will need some more material first. First for any subset $Y \subset \kappa$ we will write $\mathbb{Z}_Y = \{x \in \mathbb{Z}^\kappa : \text{supp}(x) \subset Y\}$. We define

$$\mathcal{I}_\varphi = \{Y \subset \kappa : \varphi[\mathbb{Z}_Y] = \{0\}\},$$

that is, the set of subsets Y of κ such that the image of $\mathbb{Z}_Y$ under φ is equal to the zero set. For reasons that will become obvious at the end of this chapter we will always assume that φ is not the zero homomorphism.

Lemma 2.9. *The family $\mathcal{I}_\varphi$ is an ideal on κ.*

Proof. Since we assumed that φ is not the zero homomorphism, we have that $\kappa \notin \mathcal{I}_\varphi$. Furthermore, if $Y \subset X \subset \kappa$ then $\mathbb{Z}_Y \subset \mathbb{Z}_X$, so that if $X \in \mathcal{I}_\varphi$, then $Y \in \mathcal{I}_\varphi$. Finally, since $\mathbb{Z}_{X\cup Y} = \mathbb{Z}_X + \mathbb{Z}_Y$, we have that $X \cup Y \in \mathcal{I}_\varphi$, whenever $X, Y \in \mathcal{I}_\varphi$. □

Since we know what the homomorphisms from $\mathbb{Z}^\omega$ to $\mathbb{Z}$ look like it would be very handy to be able to use this. Thankfully it is also the case that we will be using these homomorphisms. Therefore, we need to define homomorphisms from $\mathbb{Z}^\omega$ to $\mathbb{Z}^\kappa$ as an 'intermediate' that will make our later proofs possible.

Lemma 2.10. *Let $\langle a_k : k < \omega\rangle$ be a sequence of elements of $\mathbb{Z}^\kappa$ with disjoint supports, i.e. supp$(a_i) \cap$ supp$(a_j) = \emptyset$ for all $i, j < \omega$ unequal to each other. Then, the map $\psi : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa$ given by $x \mapsto \sum_{k<\omega} x_k a_k$ is a homomorphism.*

Proof. That the sum is well-defined follows from the fact that at every coordinate α all or all but one of the terms $x_k a_k(\alpha)$ are zero. Finally, since $x_k a_k(\alpha) + y_k a_k(\alpha) = (x_k + y_k)a_k(\alpha)$ for all $k < \omega$, we have that $\psi(x) + \psi(y) = \psi(x + y)$. □

For the following lemmas we will recall that if $a \in \mathbb{Z}^\kappa$ and $X \subset \kappa$, then $a \restriction X$ is the *restriction* of a to X.

Lemma 2.11. *The family $\mathcal{I}_\varphi$ is a σ-ideal on κ.*

Proof. Let $\{X_k : k < \omega\}$ be a countable subfamily of $\mathcal{I}_\varphi$. We have to show that $X := \bigcup_{k<\omega} X_k$ is in $\mathcal{I}_\varphi$. We can assume that the X_k are pairwise disjoint: replace each X_k by $X_k \setminus \bigcup_{l<k} X_l$, if necessary. Let $a \in \mathbb{Z}_X$ and write $a_k = a \restriction X_k$ for all k. Now let $\psi : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa$ be the homomorphism defined as in Lemma 2.10. Notice that the composition $\varphi \circ \psi$ is a homomorphism from $\mathbb{Z}^\omega$ to $\mathbb{Z}$, therefore, we may apply Corollary 1.6. We find that $\varphi(\psi(x)) = \sum_{k<m} x(k)\varphi(\psi(e_k))$ for some m. But since $\psi(e_k) = a_k$ and $\varphi(a_k) = 0$ for all k we have that $\varphi \circ \psi$ is the zero homomorphism. But $a = \psi(\mathbf{1})$, where $\mathbf{1} \in \mathbb{Z}^\omega$ is the point with all coordinates equal to 1; thus, $\varphi(a) = 0$. Since a was arbitrary we get that $\varphi[\mathbb{Z}_X] = \{0\}$. □

Lemma 2.12. *Let $\{Y_k : k < \omega\}$ be an infinite pairwise disjoint family of subsets of κ. Then there is an m such that $Y_k \in \mathcal{I}_\varphi$ for $k \geq m$.*

Proof. Assume for the sake of a contradiction that none of the Y_k belong to $\mathcal{I}_\varphi$. For all k choose $a_k \in Y_k$ such that $\varphi(a_k) \neq 0$. Now, define $\psi : \mathbb{Z}^\omega \to \mathbb{Z}^\kappa$ as in Lemma 2.10 and apply Corollary 1.6 to the composition $\varphi \circ \psi$, just as before. Again, we find that $\varphi(\psi(x)) = \sum_{k<m} x(k)\varphi(\psi(e_k))$ for some m. But since $\psi(e_k) = a_k$ we get that $\varphi(a_k) = \varphi(\psi(e_k)) = 0$ for all $k \geq m$; this is a contradiction. □

In other words, Lemma 2.12 tells us that the ideal $\mathcal{I}_\varphi$ is quite big. The remainder of this chapter concerns the filter $\mathcal{F}_\varphi := \overline{\mathcal{I}_\varphi}$ dual to $\mathcal{I}_\varphi$.

Lemma 2.13. *There are only finitely many ultrafilters that extend $\mathcal{F}_\varphi$.*

Proof. Let F denote the set of ultrafilters that extend $\mathcal{F}_\varphi$ and assume it is infinite. Let $\mathcal{E} \in F$ and, as $\mathcal{F}_\varphi$ is not equal to $\mathcal{E}$, we take $X \in \mathcal{E} \setminus \mathcal{F}_\varphi$. The set X divides F into two pieces, namely, $\{\mathcal{D} \in F : X \in \mathcal{D}\}$ and $\{\mathcal{D} \in F : \kappa \setminus X \in \mathcal{D}\}$. One of the pieces is infinite; let X_0 be the one of X (the former set) or $\kappa \setminus X$ (the latter) for which $F_0 := \{\mathcal{D} \in F : X_0 \in \mathcal{D}\}$ is infinite; let Y_0 be the (finite) complement of F_0; also take $\mathcal{D}_0 \in \mathcal{D}$ with $Y_0 \in \mathcal{D}_0$.

Repeat this step recursively: given X_k such that $F_k = \{\mathcal{D} \in F : X_k \in \mathcal{D}\}$ is infinitely; split X_k into X_{k+1} and Y_{k+1} such that $F_{k+1} = \{\mathcal{D} \in F_k : X_{k+1} \in \mathcal{D}\}$ is infinite; and there is $\mathcal{D}_{k+1} \in F_k$ with $Y_{k+1} \in \mathcal{D}_{k+1}$.

The resulting family $\{Y_k : k < \omega\}$ is pairwise disjoint, and, since for each k we have that $Y_k \in \mathcal{D}_k$ and $\mathcal{D}_k \in F$, we get that the set Y_k is not in $\mathcal{I}_\varphi$. This contradicts our previous Lemma. □

This tells us that we can write the family of ultrafilters that extend $\mathcal{F}_\varphi$ as $\{\mathcal{D}_i : i < m\}$ for some $m \in \omega$. Now if $x \in \mathbb{Z}^\kappa$ is such that $\operatorname{supp}(x) \notin \mathcal{D}_i$ for all i, then $\operatorname{supp}(x) \in \mathcal{I}_\varphi$, and so $\varphi(x) = 0$. Next, observe that if $\mathcal{D}$ and $\mathcal{E}$ are distinct ultrafilters, then one can find $X \in \mathcal{D} \setminus \mathcal{E}$ and $Y \in \mathcal{E} \setminus \mathcal{D}$; therefore, $X \setminus Y \in \mathcal{D}$, and $Y \setminus X \in \mathcal{E}$, and lastly $(X \setminus Y) \cap (Y \setminus X) = \emptyset$.

Applying this a finite number of times, we obtain pairwise disjoint sets $\{Y_i : i < m\}$ such that $Y_i \in \mathcal{D}_i$ for all i. Note that since F, the set of ultrafilters that extend $\mathcal{F}_\varphi$, is equal to $F = \{\mathcal{D}_i : i < m\}$, we have that the union $\bigcup_{i<m} Y_i$ belongs to $\mathcal{F}_\varphi$.

Furthermore, $\mathcal{D}_i$ is σ-complete. To see this, let $\langle X_k : k < \omega \rangle$ be a sequence of elements of $\mathcal{D}_i$, where we assume that $X_k \subset Y_i$ for all k. Then $Y_i \setminus X_k \in \mathcal{I}_\varphi$ for all k, thus $\bigcup_{k<\omega}(Y_i \setminus X_k) \in \mathcal{I}_\varphi$. But this means that $\bigcap_{k<\omega} X_k \in \mathcal{D}_i$.

Finally, with this, we get the characterization of the homomorphisms from $\mathbb{Z}^\kappa$ to $\mathbb{Z}$.

Theorem 2.14. *Let κ be an infinite cardinal and let $\varphi : \mathbb{Z}^\kappa \to \mathbb{Z}$ be a homomorphim. Then, there are finitely many σ-complete ultrafilters $\{\mathcal{D}_i : i < n\}$ such that $\varphi(x) = 0$, whenever $supp(x) \notin \mathcal{D}_i$, for all i.* □

In particular, we find that we can write φ out explicitly in terms of these σ-complete ultrafilters.

Corollary 2.15. *With the notation as above we can write $\varphi = \sum_{i<m} c_i \varphi_{\mathcal{D}_i}$, where $c_i = \varphi(\mathbf{1} \restriction Y_i)$.*

Proof. Let $\{Y_i : i < m\}$ be as above and define $W := \kappa \setminus \bigcup_{i<m} Y_i$. Now since $\{W\} \cup \{Y_i : i < m\}$ is a partition of $\mathbb{Z}^\kappa$, we can write every $x \in \mathbb{Z}^\kappa$ as a sum of restrictions:

$$x = x \restriction W + \sum_{i<m} x \restriction Y_i.$$

We know that $\varphi(x \restriction W) = 0$, so we need to show that $\varphi(x \restriction Y_i) = c_i \varphi_{\mathcal{D}_i}$, for all i. Now, for each i let $k_i \in \mathbb{Z}$ be such that $Z_i := \{\alpha \in Y_i : x(\alpha) = k_i\} \in \mathcal{D}_i$. Then we have that $\varphi(x \restriction Y_i) = \varphi(x \restriction Z_i)$ because $Y_i \setminus Z_i \in \mathcal{I}_\varphi$. Furthermore, $\varphi(x \restriction Z_i) = k_i \varphi(\mathbf{1} \restriction Z_i)$ and also $k_i \varphi(\mathbf{1} \restriction Z_i) = k_i \varphi(\mathbf{1} \restriction Y_i)$. The latter can be written as $\varphi(\mathbf{1} \restriction Y_i)\varphi_{\mathcal{D}_i}(x)$, that is, $c_i \varphi_{\mathcal{D}_i}(x)$. □

Here we find ourselves in either two different cases. First, the case when all our σ-complete ultrafilters on κ are principal: we see that the homomorphisms $\varphi : \mathbb{Z}^\kappa \to \mathbb{Z}$ are in fact the same as the ones in Corollary 1.6. On the other hand, if there are nonprincipal σ-complete ultrafilter on κ, then as we already saw right after Definition 2.8, we do in fact get a new type of homomorphism. Therefore, the only two types of homomorphims from powers of $\mathbb{Z}$ to $\mathbb{Z}$ are exactly the ones described above, depending on whether κ is measurable or not.

It should be pointed out that when κ is a non-measurable cardinal, then it follows from what we have seen so far, that the dual group of $\mathbb{Z}^\kappa$ is nothing else but $\mathbb{Z}^{(\kappa)}$. And since the dual group of the direct sum, $\mathbb{Z}^{(\kappa)}$, is always the direct product, i.e., our original group $\mathbb{Z}^\kappa$, we can simply state these two facts as: the direct product and the direct sum of $\mathbb{Z}$ are each other dual groups when raised to a non-measurable cardinal κ. But as we also saw, the dual group of $\mathbb{Z}^\kappa$ cannot be $\mathbb{Z}^{(\kappa)}$ when κ is measurable. In fact, it has to be much larger. Indeed, we hypothesize that the dual group of $\mathbb{Z}^\kappa$ has to have cardinality at least 2^κ when κ is measurable.

All of this is in fact a consequence of a much more general result. The name of this general result is called the Łoś-Eda Theorem: for further information on the Łoś-Eda and other related results on should see the book Almost Free Modules: Set-theoretic Methods by Eklof and Mekler, [8]; the Theorem is stated in Chapter III.3 of the book.

From this point forward, we will be focusing on such type of question such as: if such a σ-measurable cardinal exist, then what kind of consequences are there? It turns out that there are quite a few very interesting, even fascinating results.

3 On measurable cardinals

We will begin first by looking at a natural question that arises—well, at least from a set theoretic point of view—about these measurable cardinals. The question is: if κ is any measurable cardinal, then, how 'big' is κ? We will soon see that measurable cardinals are absolutely not small at all.

First, it should be noted that the name measurable in measurable cardinal comes from a very natural place. Indeed, originally, measurable cardinals were directly related to the problem of measures. Stanislaw Ulam was interested in these cardinals back the 1930's because he was researching what kind of cardinal admitted a non-trivial κ-additive two-valued measure. What this exactly means is out of the scope of this paper, but it should be noted that measurable cardinals originally had nothing to do with the existence of nonprincipal σ-complete complete ultrafilters, but were completely a measure theoretic question in the set theoretic world. It wasn't until the course of time that the definition of measurable cardinal was adapted to the definition used in this paper. Indeed, the question of a cardinal admitting a non-trivial κ-additive two-valued measure and of a cardinal that has a σ-complete nonprincipal ultrafilter are equiconsistent.

We shall now consider the least measurable cardinal. What is meant by that is simply the smallest cardinal number, κ, that has a nonprincipal σ-complete ultrafilter. Back in Chapter 2 we gave a definition for measurable cardinals, and, while this definition is correct, it is not the best, since, in fact, it doesn't show us the bigger picture. But to get an idea of what I mean, we will need the following result.

Lemma 3.1. *Let κ be the least measurable cardinal, that is, let κ be the least cardinal such that there exists a nonprincipal σ-complete ultrafilter U on κ. Then U is κ-complete.*

Proof. Let U be a nonprincipal σ-complete ultrafilter on κ. For the sake of a contradiction assume that U is not κ-complete. But then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ of κ, where $\gamma < \kappa$ and $X_\alpha \notin U$ for all $\alpha < \gamma$. Now let $f : \kappa \to \gamma$ be the map defined by $f(x) = \alpha$ if and only if $x \in X_\alpha$, for all $x \in \kappa$.

Furthermore, if we define $D \subset \gamma$ by $Y \in D$ if and only if $f^{-1}(Y) \in U$, then D is a σ-complete ultrafilter on γ. In particular, it follows that our ultrafilter D is nonprincipal. To see this, we assume the contrary, that is, assume $\{\alpha\} \in D$ for some $\alpha < \gamma$. But then $X_\alpha \in U$, which contradicts our assumptions on X_α. By D being a nonprincipal σ-complete ultrafilter on γ, we contradict the fact that κ is the least cardinal that carries such a nonprincipal σ-complete ultrafilter. Thus, U is κ-complete. □

So by letting κ be a measurable cardinal as in Definition 2.8 and U being the respective nonprincipal σ-complete ultrafilter, we automatically get that U is in fact much more than that, indeed U is κ-complete. Which makes us question why we didn't in the first place simply give the definition of measurable cardinals, by saying that their respective ultrafilter have to be κ-complete instead of σ-complete. And if fact, that's exactly what we'll do from this point onwards, so we redefine "measurable cardinals" accordingly.

Definition 3.2. We call an uncountable cardinal κ *measurable* if there exists a κ-complete nonprincipal ultrafilter on κ.

From this point forwards, measurable cardinals are to be defined as in Definition 3.2. Cardinal numbers with σ-complete ultrafilters are now called σ-*measurable.* Now, with this in mind, we realize that our measurable cardinals have to be quite big. Indeed, notice that if U is a κ-complete nonprincipal ultrafilter on κ, then every element of U has cardinality κ, because of the simple fact that every set of cardinality less than κ is the union of fewer than κ singletons.

Before we may begin talking about 'bigness' in a different sense, we will first need to introduce a few concepts. First, we should recall that in Chapter 1 we touched a bit upon the notion of limit ordinals/cardinals; this will be expanded in the current chapter. We call a cardinal κ a *strong limit* cardinal if $2^\lambda < \kappa$ for all $\lambda < \kappa$. Notice that every strong limit cardinal is automatically a limit cardinal. We say that a subset X of limit cardinal κ is bounded if $\sup X < \kappa$; on the other hand, we say that X is unbounded if $\sup X = \kappa$.

Now, if $\kappa > 0$ is a limit ordinal and $\langle \alpha_\nu : \nu < \beta \rangle$ is an increasing β-sequence, where β is a limit ordinal, then we call $\langle \alpha_\nu : \nu < \beta \rangle$ *cofinal* in α if $\lim_{\nu \to \beta} \alpha_\nu = \alpha$. Furthermore, if α is an infinite limit ordinal, then we define the *cofinality* of α as

$$\begin{aligned}\mathrm{cf}(\alpha) := &\text{ the least limit ordinal } \beta \text{ such that there is an increasing } \beta\text{-sequence}\\ &\langle \alpha_\nu : \nu < \beta \rangle, \text{ with } \lim_{\nu \to \beta} \alpha_\nu = \alpha.\end{aligned}$$

In particular, we say that an infinite ordinal is *singular* if $\mathrm{cf}(\alpha) = \alpha$. If α is not singular, then it is said to be *regular.* Now, of course we aren't working — strictly speaking — with ordinals, but we are more so interested in cardinals. Thankfully there are analogous definitions of the aforementioned ones with respect to cardinals. In particular, we say that a cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$. Likewise, we say that κ is *singular* if it is not regular; that is, κ is singular if $\mathrm{cf}(\kappa) < \kappa$. A very interesting thing to note is that (in ZFC) every singular cardinal is a limit cardinal. Furthermore, notice that if κ is measurable, then, for the same reasons as when were discussing the cardinality of every set in U, we find that κ is regular: indeed, suppose that κ is singular, then κ is the union of fewer than κ many sets each of cardinality less than κ.

But are there any regular limit cardinals? There's a very simple answer to this question, namely $\aleph_0$, since it is indeed a regular limit cardinal. Therefore, the answer to the question above is a yes. Curiously enough $\aleph_0$ is a countable regular limit cardinal, but what about uncountable regular limit cardinals? The answer to this question will, for now, be left unanswered, but we will briefly touch upon it again in the last Chapter. Nonetheless, we are still very interested in these cardinals — i.e., uncountable regular limit cardinal numbers, — and, in fact, they are going to be the focus of this chapter. Accordingly, we will be giving these special cardinals a name: we will give the name *weakly inaccessible cardinals* to the aforementioned uncountable regular limit cardinals. An interesting fact to note is that if κ is a weakly inaccessible cardinal, then $\kappa = \aleph_\kappa$; however, the converse is not generally true. Now, as hinted by the name, we can actually make this definition a bit stronger.

Definition 3.3. Let κ be an uncountable cardinal. We call κ *(strongly) inaccessible* if it is regular and

$$\forall \lambda < \kappa : 2^\lambda < \kappa.$$

Usually, instead of calling them strongly inaccessible cardinals, we just say inaccessible cardinals. Clearly every inaccessible cardinal is weakly inaccessible; the converse is not always true. By looking at the definition one can easily see that if κ is inaccessible then κ is in a sense very big. Indeed, we actually call inaccessible cardinals *large.* This is how we rigorously refer to a cardinal that gives us a sense of largeness or bigness. In this paper I will be using the words large and big interchangeably in this sense. In the previous chapter we said that we would be interested in measurable cardinals, we never mentioned inaccessible cardinals, so what do they have to do with each other?

Theorem 3.4. *A measurable cardinal is inaccessible.*

Proof. Let κ be a measurable cardinal and let U be a non-principle κ-complete ultrafilter on κ. First we will show that κ is regular. For the sake of a contradiction assume the contrary. That is, $\mathrm{cf}(\kappa) < \kappa$ and let $\{\kappa_\nu : \nu < \mathrm{cf}(\kappa)\}$ be a family of cardinals such that

$$\begin{aligned} &\kappa_\nu < \kappa, \text{ for } \nu < \mathrm{cf}(\kappa); \\ &\kappa_\mu < \kappa_\nu, \text{ for } \mu < \nu < \mathrm{cf}(\kappa); \text{ and} \\ &\sum_{\nu < \mathrm{cf}(\kappa)} \kappa_\nu = \kappa. \end{aligned}$$

Now if $\kappa \setminus \kappa_\nu \in U$, for $\nu < \mathrm{cf}(\kappa)$ then

$$\bigcap_{\nu < \mathrm{cf}(\kappa)} \kappa \setminus \kappa_\nu = 0 \in U,$$

since $\mathrm{cf}(\kappa) < \kappa$ and U is κ-complete. Thus, $\kappa_\nu \in U$ for some $\nu < \mathrm{cf}(\kappa)$. Now, because U is nonprincipal, we have that $\kappa_\nu \setminus \eta \in U$, for $\eta < \kappa_\nu$. Therefore,

$$\bigcap_{\eta < \kappa_\nu} \kappa_\nu \setminus \{\eta\} = 0 \in U,$$

again, since $\mathrm{cf}(\kappa) < \kappa$ and U is κ-complete. This clearly contradicts the definition of an ultrafilter. Thus, κ is indeed regular.

For the second part, again, we will argue by contradiction. So suppose that there exists some $\lambda < \kappa$ such that $2^\lambda > \kappa$. Now let S be a set of functions $f : \lambda \to \{0, 1\}$ such that $|S| = \kappa$; furthermore, let U be a nonprincipal κ-complete ultrafilter on S. For each $\alpha < \lambda$ let X_α be either $\{f \in S : f(\alpha) = 0\}$ or $\{f \in S : f(\alpha) = 1\}$ depending on which one is in U; let ε_α be either 0 or 1 according to the choice of X_α. Since U is κ-complete we have that the set $X := \bigcap_{\alpha<\lambda} X_\alpha \in U$. But notice that X has at most one element, that being the function f which has the value $f(\alpha) = \varepsilon_\alpha$; a contradiction. Therefore, we have as required, namely, for all $\lambda < \kappa$ we have that $2^\lambda < \kappa$. □

In particular, what we learn from Theorem 3.4 is, just as we previously suggested, that our measurable cardinals from Chapter 2 are very big indeed. It goes without saying that, a priori, just by looking at the definition of measurable cardinals, one could not tell at all that measurable cardinals have to be big. By going through the whole process of chapter 2 of creating a new type of homomorphism from $\mathbb{Z}^\kappa \to \mathbb{Z}$ we could get an idea that κ has to be somewhat big. So, in a sense, one could only get an intuition of the bigness of measurable cardinals by having played around with them a bit. The definition is completely vague about its size. Indeed, since a measurable cardinal is inaccessible we say that they are large cardinals. But exactly how big are they? A natural question would be the converse of Theorem 3.4. But it turns out that the least inaccessible cardinal is not measurable. In particular, the least measurable cardinal is greater than the least inaccessible cardinal. The proof of this fact is out of the reach of this paper.

Indeed, measurable cardinals seem to be bigger than inaccessible. But if we know that the least measurable cardinal is greater than the least inaccessible cardinal, then what about the second-smallest inaccessible cardinal? Is the second-smallest inaccessible cardinal as big as the least measurable? It turns out, that measurable cardinals are in fact way bigger than one might originally expect, the previous sentences do a disservice to how big measurable cardinals are in relation to inaccessible cardinals.

4 On Mahlo cardinals

Before we go further on with this discussion we will need to introduce another type of big cardinal, namely, Mahlo cardinals. First, we need to define the notion of limit points. Let X be a set of limit ordinals and let $\alpha > 0$ be a limit ordinal. Then, we call α a *limit point* of X if $\sup(X \cap \alpha) = \alpha$.

Definition 4.1. If κ is a regular uncountable cardinal, then we say that a set $C \subset \kappa$ is *closed unbounded* both when C is unbounded in κ and C contains all its limit points less than κ.

In addition to this, we say that a set $S \subset \kappa$ is *stationary* if the intersection of S and C is non-empty for every closed unbounded set $C \subset \kappa$.

Now, with this in mind, we can define the next big cardinals of interest.

Definition 4.2. If κ is an inaccessible cardinal, then we say that κ is a *Mahlo cardinal* if the set of all regular cardinals below κ is stationary.

By definition Mahlo cardinals are large cardinals. In particular, it follows that if κ is Mahlo, then the set of all inaccessible cardinals below κ is stationary, and thus, κ is "the κ-th inaccessible cardinal". That κ is the *κ-th inaccessible* simply means that if you enumerate the inaccessible as $\langle \mu_\alpha : \alpha \in \mathrm{Ord} \rangle$, then we get that $\kappa = \mu_\kappa$. In the literature, cardinals κ that are the κ-th inaccessible are usually called *hyper-inaccessible*. Not only are Mahlo cardinals hyper-inaccessible, but they are in fact hyper-hyper-inaccessible. What this exactly means is out of the reach of this paper, but one can already imagine that they are definitely bigger then the already very big hyper-inaccessible cardinals.

Thus, indeed, Mahlo cardinal are huge, but what do they have to do with our measurable cardinals? Before answering this question directly, we will need to take a detour into another type of large cardinal; and for this we will need some notation.

If $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ, then we say that the *diagonal intersection* of X_α, for $\alpha < \kappa$, is:

$$\triangle_{\alpha<\kappa} X_\alpha := \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}.$$

Furthermore, we call a filter F on a cardinal κ *normal* if it is closed under the diagonal intersection, that is: if for every $\alpha < \kappa$, we have that X_α is contained in F, then $\triangle_{\alpha<\kappa} X_\alpha$ is contained in F. We can define the same concept to ideals, namely, we call an ideal on κ normal whenever the dual filter is normal.

Lemma 4.3. *Let κ be a regular uncountable cardinal and let F be a normal filter on κ which contains the sets $\{\alpha : \alpha_0 < \alpha < \kappa\}$, for $\alpha_0 < \kappa$. Then, F contains all closed unbounded sets.*

Proof. Consider such a cardinal κ and filter F as above, then notice that the set C_0 of all limit ordinals is contained in F. Indeed, C_0 is the diagonal intersection of the sets $X_\alpha = \{\xi : \alpha + 1 < \xi < \kappa\}$, that is, $C_0 = \triangle_{\alpha<\kappa} X_\alpha$.

Furthermore, if C is a closed unbounded set and $C = \{a_\alpha : \alpha < \kappa\}$ is its increasing enumeration, and we let $X_a = \{\xi : a_\alpha < \xi < \kappa\}$, then it follows that $C_0 \cap \triangle_{\alpha<\kappa} X_\alpha \subset C$. □

To relate the notion of normal filters to measurable cardinals we will call normal κ-complete nonprincipal ultrafilters *normal measures* on κ. A result that follows immediately from Lemma 4.3 is the following.

Lemma 4.4. *Let D be a normal measure on κ. Then every set in D is stationary.* □

Thankfully, these normal measures aren't only hypothetical, indeed, we may talk about their existence when considering measurable cardinals.

Theorem 4.5. *If κ is a measurable cardinal, then κ carries a normal measure.*

Proof. First, let U be a nonprincipal κ-complete ultrafilter on κ. We will define a relation on κ^κ as follows: we say that $f, g \in \kappa^\kappa$ are equivalent, i.e. $f \equiv g$, if and only if $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$. Then clearly $\equiv$ is an equivalence relation on κ^κ.

Now let $f < g$ be if and only if $\{\alpha < \kappa : f(\alpha) < f\alpha)\} \in U$. Then not only is $<$ a linear ordering on κ^κ, but $<$ is also a linear ordering on $\kappa^\kappa / \equiv$. Furthermore, $<$ is a well-ordering on $\kappa / \equiv$. Indeed, if we argue by contradiction, then suppose that there is an infinite descending sequence $f_0 > f_1 > \cdots > f_n > \ldots$. Now define $X_n = \{\alpha < \kappa : f_n(\alpha) > f_{n+1}(\alpha)\}$, and $X = \bigcap_{n=0}^{\infty} X_n$; then, notice that X is non-empty. Finally, if $\alpha \in X$, then $f_0(\alpha) > f_1(\alpha) > \cdots > f_n(\alpha) > \ldots$; which is a contradiction.

Now, in this well-ordering, let f be the least function that has the property that $\{\alpha < \kappa : f(\alpha) > \gamma\} \in U$, for every $\gamma < \kappa$. Let $D = \{X \subset \kappa : f^{-1}(X) \in U\}$; then, clearly D is a κ-complete ultrafilter, and furthermore, D is nonprincipal. To see the latter simply see that for $\gamma < \kappa$, the inverse image of $\{\gamma\}$ under f is not in U, therefore $\{\gamma\}$ is not in D.

Now, we need to mention the following fact: "a measure is normal if and only if every regressive function on a set of measure one is constant on a set of measure one." This fact will not be proven, but we will use it to show that D is a normal measure. Indeed, let X be a subset of D and let h be a function (in this well-ordering) with the property that $f(\alpha) < \alpha$ for all $\alpha \in X$ and $\alpha > 0$. Then h is constant on D. To see this define the function g as $g(\alpha) = (h \circ f)(\alpha)$. Now, since $g(\alpha) < f(\alpha)$ for every $a \in f^{-1}(X)$, we have that $g < f$. By the minimality of f we have that g has to be constant on a subset Y of U. Therefore, h is constant on $f(Y)$ and $f(Y) \in D$. Thus, D is a normal measure. □

Lemma 4.6. *Let κ be a measurable cardinal. Then κ is also Mahlo.*

Proof. Let κ be a measurable cardinal. Since κ is a strong limit, the set of all strong limit cardinals below κ is closed bounded. Therefore, it is enough if we prove that the set of regular cardinals below κ is stationary. Now, thanks to Theorem 4.5 we may consider D, a normal measure on κ, so let us do that. Furthermore, by Lemma 4.4 we only need show that the set of all regular cardinals below κ is in our normal measure D.

For the sake of a contradiction assume the contrary, that is, let $\{\alpha : \mathrm{cf}(\alpha) < \alpha\} \in D$. Since D is normal there is a $\lambda < \kappa$ such that $E_\lambda = \{\alpha : \mathrm{cf}(\alpha) = \lambda\} \in D$. Now for every $\alpha \in E_\varepsilon$ let $\langle x_{\alpha,\xi} : \xi < \lambda \rangle$ be an increasing sequence with limit α. Furthermore, for every $\xi < \lambda$ there are y_ξ and $A_\xi \in D$ with the property that $x_{\alpha,\xi} = y_\xi$ for all $\alpha \in A_\xi$. Finally, define $A = \bigcap_{\xi < \lambda} X_\xi$, and notice that $A \in D$. But then it follows that the only element that is contained in A is the limit $\lim_{\xi \to \lambda} y_\xi$; a contradiction. □

Indeed, it turns out that κ is measurable, then not only is κ inaccessible, but it is also hyper-inaccessible, etc. The next phase of this chapter will show us that this is not the only increase in largeness that our measurable cardinals will suffer; they will get larger.

5 An even larger cardinal

In this chapter we continue with our search for large cardinals. In particular, we will be introducing a cardinal that is even larger than the previously discussed Mahlo cardinals. Let A be any set and n a natural number. Then $[A]^n$ will denote the family of all subsets of A that have cardinality equal to n, that is, $[A]^n = \{X \subset A : |X| = n\}$. Furthermore, if $\{X_i : i \in I\}$ is a partition of $[A]^n$, then say that a subset H of A is *homogeneous* (for the partition) if $[H]^n$ is included in X_i for an $i \in I$. In other words, H is homogeneous if all the n-element subsets of H are in the same piece of the partition.

Definition 5.1. Let κ, λ be infinite cardinal number, where $\lambda \leq \kappa$. Furthermore, let $n \in \omega$, and let m be a cardinal number smaller than κ. The symbol

$$\kappa \to (\lambda)^n_m$$

denotes what is called the *partition property*; that is, every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ.

For the sake of simplicity, when $m = 2$ we will just leave the subscript empty, that is, $\kappa \to (\lambda)^n_m$ denotes the same thing as $\kappa \to (\lambda)^n$. As is well known in the literature, Ramsey's Theorem, as seen in Theorem 9.1 of [7], states that if $n, k \in \omega$, then every partition $\{X_1, X_2, ..., X_k\}$ of $[\omega]^n$ into k pieces has an infinite homogeneous set. In arrow notation this can be simply stated as $\aleph_0 \to (\aleph_0)^n_k$, for n and κ natural numbers. Furthermore, the Erdős-Rado Theorem, as seen in Theorem 9.6 of [7], and its consequences show that the size of the homogeneous set is generally smaller than the size of the set that is being partitioned.

But now a natural question arises, that is, does the partition property $\kappa \to (\kappa)^2$ hold for any κ other than $\kappa = \omega$, as in Ramsey's Theorem. It turns out that this question cannot be answered in simple terms, and, in fact, a cardinal that satisfies such a condition has its own special name.

Definition 5.2. We call an uncountable cardinal number κ *weakly compact* if it satisfies the partition property $\kappa \to (\kappa)^2$.

Even though the definition of weakly compact cardinals given here is purely combinatorial, it turns out that originally weak compactness comes from a less combinatorial question, namely, it comes from a question in logic, that is, for which cardinal does a certain Compactness Theorem hold: namely, weakly compact cardinals. What exactly this particular Compactness Theorem is, is completely out of the reach of this paper. One can find more information on this matter in [7]. In particular for us though, it turns out that these weakly compact cardinals have some interesting combinatorial properties which we will cover briefly. As stated previously, Ramsey's Theorem states that the partition property $\aleph_0 \to (\aleph_0)^n_k$ holds for any natural numbers n and k. A natural question arises, namely, does this also hold for $\aleph_1$?

Lemma 5.3. *For all cardinals κ, the partition property $2^\kappa \to (\kappa^+)^2$ does not hold. That is,*

$$2^\kappa \not\to (\kappa^+)^2.$$

Before we prove this we will need to give another short Lemma.

Lemma 5.4. *The lexicographically ordered set $\{0,1\}^\kappa$ has neither an increasing nor decreasing κ^+-sequence.*

Proof. We will only consider the increasing case, since the decreasing case is very similar to the increasing case. Assume that the set $X = \{f_\alpha : \alpha < \kappa^+\} \subset \{0,1\}^\kappa$ has the property that $f_\alpha < f_\beta$ if $\alpha < \beta$. Let $\lambda \leq \kappa$ be the least cardinal such that the cardinality of the set $\{f_\alpha \upharpoonright \lambda : \alpha < \kappa^+\}$ is equal to κ^+. Furthermore, let Y be a subset of X such that the cardinality of Y is equal to κ^+ and $f \upharpoonright \lambda \neq g \upharpoonright \lambda$ for all $f, g \in Y$. Without loss of generality we may assume that $X = Y$.

Now for every $\alpha < \kappa^+$, let ξ_α be such that: $f_\alpha \upharpoonright \xi_\alpha = f_{\alpha+1} \upharpoonright \xi_\alpha$, $f_\alpha(\xi_\alpha) = 0$, and $f_{\alpha+1}(\xi_\alpha) = 1$. Then, it follows that $\xi_\alpha < \lambda$ for all $\alpha < \kappa^+$. Thus, there is a $\xi < \lambda$ such that $\xi = \xi_\alpha$ for κ^+ elements $f_\alpha \in X$. But if both $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha \upharpoonright \xi = f_\beta \upharpoonright$, then it follows that $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$. Therefore, $f_\alpha = f_\beta$ for all $\alpha, \beta < \kappa^+$. Thus, the cardinality of the set $\{f_\alpha \upharpoonright \xi : \alpha < \kappa^+\}$ is equal to κ^+; but this a contradiction by the minimality assumption on λ.

Proof of Lemma 5.3. First define $\{f_\alpha : \alpha < 2^\kappa\}$ to be an enumeration of $\{0,1\}^\kappa$. Furthermore, let $\prec$ be a linear ordering of 2^κ induced by the lexicographic ordering of $\{0,1\}^\kappa$, that is, for $\alpha, \beta \in \{0,1\}^\kappa$ and the respective $f_\alpha, f_\beta \in \{f_\gamma : \gamma < 2^\kappa\}$ we have that $\alpha \prec \beta$ whenever $f_\alpha < f_\beta$. Now we define a partition $F : [2^\kappa]^2 \to \{0,1\}$ as follows:

$$F(\{\alpha,\beta\}) = \begin{cases} 1, & \text{if the ordering } \prec \text{ of } \{\alpha,\beta\} \text{ agrees with the natural ordering;} \\ 0, & \text{otherwise.} \end{cases}$$

But if $H \subset 2^\kappa$ is a homogeneous set of order κ^+, then it follows that $\{f_\alpha : \alpha \in H\}$ is either an increasing or decreasing κ^+-sequence in $(\{0,1\}^\kappa, <)$; this contradicts Lemma 5.4. □

In particular, this tells us that our previous attempt at generalizing Ramsey's Theorem fails. Indeed, whenever $\kappa \leq 2^{\aleph_0}$ we have that $\kappa \not\to (\aleph_1)^2$, and thus also $\aleph_1 \not\to (\aleph_1)^2$. On the other hand, if $k > 2^{\aleph_0}$, then as we know from the Erdős-Rado Theorem, the partition property $\kappa \to (\aleph_1)^2$ does hold.

We took this detour, of course, because we wanted to show the link between the aforementioned Mahlo and Measurable cardinals. Indeed, it turns out that thanks to weakly compact cardinals we can actually show the link between all of these.

Lemma 5.5. *If κ is a weakly compact cardinal, then κ is inaccessible.*

Proof. Let κ be a weakly compact cardinal. We need to show that κ is both regular and a strong limit cardinal. To show the former, assume that κ is not regular, that is, let κ be the disjoint union $\bigcup\{A_\gamma : \gamma < \lambda\}$ where both $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for all $\gamma < \kappa$. Now, define a partition $F : [\kappa]^2 \to \{0,1\}$ as follows:

$$F(\{\alpha,\beta\}) = \begin{cases} 0, & \text{if } \alpha, \beta \in A_\gamma, \text{ for some } \gamma < \kappa; \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, this partition does not have a homogeneous set of cardinality κ; a contradiction.

Now to show that κ is a strong limit cardinal assume the contrary, that is, let $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$. By Lemma 5.3 we have that $2^\lambda \not\to (\lambda^+)^2$, therefore it follows that $\kappa \not\to (\lambda^+)^2$, and thus $\kappa \not\to (\kappa)^2$; again, a contradiction. □

The converse of this Lemma doesn't hold in general: an inaccessible cardinal is in general not weakly compact. But this isn't the only link between these large cardinals. In fact, just as measurable cardinals, weakly compact cardinals are also related to the previously discussed Mahlo cardinals.

Lemma 5.6. *If κ is a weakly compact cardinal, then κ is Mahlo.*

Unfortunately the proof of this Lemma is out of the reach of this paper. Nonetheless, this shows that weakly compact aren't only inaccessible, but they're in fact (hyper-)hyper-inaccessible. One should note however that the converse of this Lemma does not hold: not every Mahlo cardinal is weakly compact. Thus, it turns out that if one originally thought that inaccessible cardinals where big, then weakly compact cardinals blow this bigness out of the water. But now we have two cardinals—measurable and weakly compact cardinals—that are larger than Mahlo and inaccessible cardinals. Naturally, a natural question is which one is larger than the other (if that's even possible). The following result will tell us a bit about the relationship between measurable and weakly compact cardinals. To keep things in line with the focus of this paper we will only be giving a sketch of the proof which uses a notion we are already familiar with: that of nonprincipal κ-complete ultrafilters.

Lemma 5.7. *If k is a measurable cardinal, then κ is also weakly compact.*

(sketch). First partition $[\kappa]^2$ into two sets, namely, sets where every paired edge has the same colour. So let $[\kappa]^2 = A \cup B$, where, say, every paired edges in A has the colour red, and, likewise every paired edges in B has the colour blue. Now let U be a nonprincipal κ-complete ultrafilter on κ. For all $\alpha < \kappa$ define $A_\alpha = \{\beta > \alpha : \{\alpha, \beta\} \in A\}$ and $B_\alpha = \{\beta > \alpha : \{\alpha, \beta\} \in B\}$; and let

$$U_\alpha = \begin{cases} A_\alpha, & \text{if } A_\alpha \in U; \\ B_\alpha, & \text{if } B_\alpha \in U. \end{cases}$$

Now we define a sequence, $x = \{x_\alpha : \alpha < \kappa\}$, where the first element is $x_0 = 0 \in U_0$, and for $0 < \alpha < \kappa$ we let

$$x_\alpha = \min \bigcap_{\gamma < \alpha} U_{x_\gamma} \in U.$$

For each $\alpha < \kappa$ we identify every x_α with a respective ε_α. Namely,

$$\varepsilon_\alpha = \begin{cases} A, & \text{if } A_{x_\alpha} = U_{x_\alpha}; \\ B, & \text{if } B_{x_\alpha} = U_{x_\alpha}. \end{cases}$$

And thus for $\alpha < \beta < \kappa$, we have $\{x_\alpha, x_\beta\} \in A$ if $\varepsilon_\alpha = A$. Repeating this infinitely many times $|A|$, we get a set of indices $I = \{\alpha : \varepsilon_\alpha = A\}$; this gives us the required homogeneous set of cardinality κ, namely $H = \{x_\alpha : \alpha \in I\}$, where $\{x_\alpha, x_\beta\} \in I$ for $\alpha, \beta \in I$. □

That is, measurable cardinals—even though one could a-priori not tell at all that they should be large—are very big cardinals. The converse of this Theorem as one could probably predict already does not hold in general. In fact, it turns out that, again, measurable cardinals are way larger than weakly compact cardinals.

Lemma 5.8. *If κ is a measurable cardinal, then κ is the κ-th weakly compact cardinal.*

Therefore, even though we have seen that inaccessible, Mahlo, and weakly compact cardinals are huge, measurable cardinals simply are unthinkably huge. To put their largeness into words would be an impossible task.

6 On the bizarreness of measurable cardinals

Back in Chapter 3 we briefly touched upon the existence of (weakly) inaccessible cardinals. We can finally give an answer to this question.

Theorem 6.1. *It cannot be shown that the existence of inaccessible cardinals is consistent with ZFC. Furthermore, the existence of inaccessible cardinals cannot be proven in ZFC.*

Moreover, the existence of weakly inaccessible cardinals cannot be proven in ZFC, but since we are more interested in (strongly) inaccessible cardinals, we will only be focusing on them. Interestingly enough, the first statement of the Theorem above follows from the second part of Gödel's incompleteness Theorem. Even though neither the existence nor the consistency of said existence is provable in ZFC, that doesn't mean that we should stop here and simply negate their existence. On the contrary, as stated previously (albeit indirectly) it is highly believed that in ZFC it isn't possible to prove the non-existence of either of the aforementioned claims. If we were to stop here on our search for knowledge of these big cardinals, then, we would lose the ability to learn of so many interesting details and consequences of said cardinals.

A very surprising and exciting consequence of inaccessible cardinals is that their existence proves the consistency of ZFC.

Lemma 6.2. *Let κ be an inaccessible cardinal. Then V_κ is a model of ZFC.*

In other words, what this tells us is the simple fact that if we were to take ZFC as we know it and add to it the axiom "there exist an inaccessible cardinal", then with these two sets of axioms we would be able to prove the consistency of ZFC. We should not get confused by this statement however, since this doesn't mean that we have proved the consistency of 'everything'. No, since we are now working in the 'universe'/set of axioms ZFC + "there exist an inaccessible cardinal" we still need to prove the consistency of this universe itself. Unfortunately, anyone who has heard of Gödel's Second Incompleteness Theorem knows very well that a universe can never prove its own consistency. In particular, this means that even though we may be able to prove the consistency of ZFC by being in this bigger universe ZFC + "there exist an inaccessible cardinal", we cannot prove the consistency of this universe by means of its own set of axioms. Meaning that even though we may be able to prove that ZFC is consistent using ZFC + "there exist an inaccessible cardinal", there could very well be the chance that this bigger universe itself is inconsistent, basically making our previous statement of "ZFC is consistent" irrelevant, a sentence of nothingness. This isn't as far at it goes fortunately (or unfortunately as will soon be made clear). Adding more large cardinals to our set of axioms creates bigger universes which can prove the consistency of the smaller universes. For example, the universe with ZFC and two inaccessibles can prove the consistency of the universe with ZFC with one inaccessible which in turn can prove the consistency of ZFC. One can imagine a ridiculously infinitely large tower of these universes of axioms where after adding so many inaccessible cardinals we reach the realm of the Mahlo cardinals, and, after adding so many of these Mahlo cardinals we end up even further into the realm of the weakly compact, and, as you can expect, after adding so many weakly compact cardinals where we end up in the realm of the measurable cardinals, and, of course, it doesn't have to stop here, so one could simply never stop; continue ad infinitum... But, and it goes without saying, that we end up were we began since obviously we can't prove the consistency of the biggest universe (with its own set of axioms, of course). We have taken such a big step that we ended up going around the whole world just to end up at the same spot.

With this though we can give a proof of the first statement of Theorem **??**.

Proof of Theorem 6.1 (first statement). For simplicity's sake we will let T identify the statement "there exist an inaccessible cardinal". Now, assume that it can be shown that the existence of inaccessible cardinals is consistent with ZFC. Furthermore, we assume that ZFC itself is consistent.

Now, since T is consistent with ZFC, it follows that ZFC + T is consistent. By Lemma 6.2. we have that in ZFC + T there is a model of ZFC. In particular, the sentence "ZFC is consistent" is therefore provable in ZFC + T. Notice that we assumed that the sentence "T is consistent with ZFC" is provable. But then it follows that the sentence "ZFC + T is consistent" is provable in ZFC + T, which contradicts non-other than Gödel's Second Incompleteness Theorem. □

A very bizarre consequence of the existence of inaccessible cardinals is the following. If κ is an inaccessible cardinal, then in V_κ it is consistent that there is no inaccessible cardinal. The details of this weird but fascinating paradox are sadly left out since they are out of the reach of this paper.

Notice however that the second statement of Theorem 6.1 can be proven with the aforementioned information since if it can be proven that if an inaccessible cardinal, κ, exists, then we would also be immediately showing that a measurable cardinal does not exist (in a smaller universe of V_κ); but this is a glaring contradiction. Hence, the existence of inaccessible cardinals cannot be proven in ZFC.

Even though that it goes without saying that the same holds for larger cardinals, we still haven't shown anything specific on the existence of measurable cardinals. And there is good reason for this. Measurable cardinals are quite fascinating things, but unfortunately they require quite advanced knowledge of set theory that is sadly out of the reach of this paper. Nonetheless, we will show (without proof) a quite interesting consequence of (the existence of) measurable cardinals. In basic terms we say that a set is *constructable* if it can be defined entirely in terms of 'simpler' sets.

Lemma 6.3. *If there exists a measurable cardinal then $V \neq L$, that is, not all sets are constructable.*

The reason for giving this last Lemma is that I would like to impress upon the reader that these measurable cardinals are, quite informally if I'm allowed to say, simply amazing. From the simple question of asking for which sets there exists non-trivial κ-additive two-valued measures to the (albeit not fully realized) consistency of ZFC to finally the constructability of sets in ZFC. This shows how bizarre and fascinating measurable cardinals can be. It goes without saying that this is but the tip of the iceberg when it comes to measurable cardinals and their consequences, there is still so much more left to uncover.

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