The Knopp cocycle and cycle integrals of the j-function
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Chapter 1

Introduction

1.1 Singular moduli and their differences

By complex multiplication (CM) theory, the value of the modular j-function at an imaginary quadratic number τ is an algebraic integer generating the ring class field of the order generated by τ. This algebraic integer is called a *singular modulus*. Complex multiplication gives one of the only constructive methods to generate class fields. It is the focus of Hilbert's twelfth problem to generalise this to general number fields.

In the paper 'On singular moduli', Gross and Zagier provide an explicit factorisation of the norm of the difference between two singular moduli [GZ84]. For example the difference of the singular moduli of the CM-values $\frac{1+\sqrt{-163}}{2}$, $\frac{1+\sqrt{-67}}{2}$ is equal to

$$\begin{aligned} j\left(\tfrac{1+\sqrt{-163}}{2}\right) - j\left(\tfrac{1+\sqrt{-67}}{2}\right) &= -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3 + 2^{15} \cdot 3^7 \cdot 5^3 \cdot 11^3 \\ &= -2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331. \end{aligned}$$

Notice that the primes occuring in this factorisation are not split in the number fields of discriminants -163, and -67, and that they divide a positive integer of the form $\frac{163\cdot 67 - x^2}{4} > 0$. Gross and Zagier give an explicit formula for the multiplicity of any such prime q in terms of the embeddings of the associated imaginary quadratic fields in the quaternion algebra $B_{q,\infty}$ ramified at q and ∞.

In [GZ86], Gross and Zagier carefully reconsider the arguments of [GZ84] to give an explicit description of the height of a *Heegner point* in terms of the derivative of the L-series of an elliptic curve E at $s = 1$. If this derivative is non-zero, then Heegner points can be used to construct a rational point on the elliptic curve of infinite order, hence showing that if E has analytic rank 1, then it has algebraic rank $\geqslant 1$. Combined with later results of Kolyvagin [Kol89], this solves the Birch and Swinnerton-Dyer conjecture for elliptic curves of analytic rank ≤ 1.

1.2 Real quadratic singular moduli: an archimedean attempt

Singular moduli are connected to two central problems in number theory, as mentioned above. Generalising their theory to other number fields K is therefore of key importance. We will discuss two attempts to generalise singular moduli to the case where K is real quadratic, the first of which is complex analytic in nature. If $\tau \in K \setminus \mathbb{Q}$ is a real quadratic irrationality, then, since it is not an element of the upper half-plane, we cannot evaluate the j-function at τ directly. We can however consider the geodesic in the upper half-plane connecting τ and its conjugate τ'. We will integrate over this geodesic to define a way to 'evaluate' j at a real quadratic number in terms of *cycle integrals*.

(a) Imaginary quadratic points in the upper half-plane

(b) The geodesic in the upper half-plane connecting a real quadratic irrationality τ and its conjugate τ'.

Figure 1.1: Quadratic points with respect to the upper half-plane

Consider the stabiliser of the geodesic associated to τ under the action of $\mathrm{SL}_2(\mathbb{Z})$. It is generated by $-I$ and a free element γ, called an *automorph* of τ. The eigenvalues of γ are fundamental units of norm 1 in a quadratic order associated to τ (cf. [Bue89]). We make the choice of γ unique by flipping the sign and possibly inverting such that its trace is positive and the eigenvalue corresponding to τ is greater than 1.

Definition 1.1. Let τ be a real quadratic irrationality and let $f \in \mathbb{C}[j]$ be a modular function that is holomorphic on $\mathcal{H}$. Let $Q(z) = az^2 + bz + c$ be the unique primitive quadratic polynomial with discriminant $\Delta = b^2 - 4ac$ such that τ is its *principal* root, i.e.

$$\tau = \frac{-b + \sqrt{\Delta}}{2a}.$$

Furthermore let γ_τ be the automorph of τ. Then we define the *value* of f in τ to be

$$f[\tau] := \int_{z_0}^{\gamma_\tau z_0} f(z) \frac{\sqrt{\Delta} dz}{Q(z)}$$

for any z_0 on the geodesic in $\mathcal{H}$ connecting τ and τ'.

This definition is independent of the base point z_0. Moreover, cycle integrals are invariant under $\mathrm{SL}_2(\mathbb{Z})$: for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $f[\gamma\tau] = f[\tau]$. Cycle integrals of the j-function have gained attention in recent decades after a paper by Kaneko [Kan09] who made some interesting observations about their numerical values, and a paper by Duke, Imamoğlu and Tóth who studied the traces of cycle integrals of modular functions [DIT11]. They use them to generalise several results of Zagier on the traces of singular moduli [Zag02] to the real quadratic case.

Although they exhibit many interesting properties analogous to CM-singular moduli, no algebraic values have been recognised in them, neither theoretically nor numerically, in spite of serious efforts. This makes it unclear whether they have a role to play in explicit class field theory or the Birch and Swinnerton-Dyer conjecture.

1.3 Real quadratic singular moduli: a non-archimedean attempt

The distinguishing property of a real quadratic field K is that the place ∞ is split in K instead of inert. One might therefore try to replace the complex analytic (and hence ∞-adic) j-function with a p-adic analytic object, where p is inert in K. This is the approach of Darmon and Vonk in [DV21], who replace the j function with a certain p-adic limit of the *Knopp cocycles*, defined below. For any two real quadratic irrationalities τ_1, τ_2 and any prime p inert in both respective real quadratic number fields, Darmon and Vonk define a quantity

$$\Theta_p(\tau_1, \tau_2) \in \mathbb{C}_p$$

which (conjecturally) behaves in all key respects like the differences of singular moduli studied by Gross and Zagier. For example, for the real quadratic irrationalities $\tau_1 = \frac{1+\sqrt{5}}{2}$, and $\tau_2 = \frac{-6+\sqrt{44}}{2}$ with discriminants $(\Delta_1, \Delta_2) = (5, 44)$ and $p = 3$, the quantity $\Theta_3(\tau_1, \tau_2)$ is verified to satisfy the polynomial

$$48841x^8 + 115280x^6 + 164562x^4 + 115280x^2 + 48841$$

up to precision $O(3^{200})$. The roots of this polynomial generate the compositum of the corresponding Hilbert class fields H_1 and H_2. The constant coefficient of this polynomial factors as $48841 = 13^2 \cdot 17^2$. Notice that both of these primes are inert in K_1 and K_2 and that they divide a positive integer of the form

$$\frac{\Delta_1\Delta_2 - x^2}{4p}.$$

Darmon and Vonk also conjecture an explicit formula for the multiplicity of in any finite place $\mathfrak{q}$ of H_1H_2 in $\Theta_p(\tau_1, \tau_2)$. This formula is related to the embeddings of K_1 and K_2 into the indefinite quaternion algebra that is ramified at p and q.

1.4 Archimedean real quadratic singular moduli

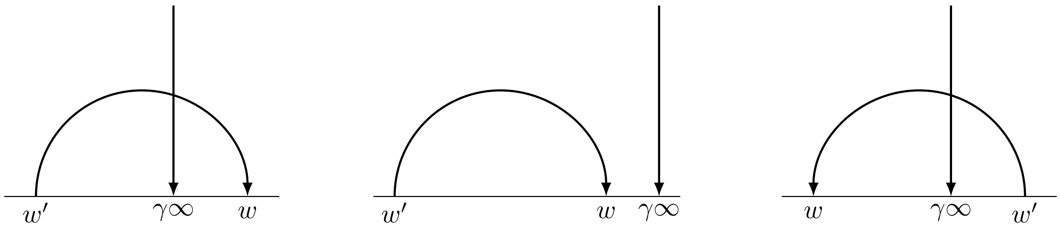
Having successfully defined a p-adic theory of differences of real quadratic singular moduli, there are now two questions to be asked. Firstly one may wonder if there is an analogous p-adic theory of differences of CM-singular moduli. This question was answered affirmatiely by Daas' PhD thesis [Daa24]. Secondly, we ask if there is an ∞-adic construction of real quadratic singular moduli. We start exploring this question in this thesis, and hope to make some useful observations.

ν	∞	p
K imaginary	Gross-Zagier [GZ84]	Daas [Daa24]
K real	?	Darmon-Vonk [DV21]

The construction of Darmon and Vonk crucially relies on p-adic limits of the *Knopp cocycle*. Given a real quadratic irrationality τ_1, the (multiplicative) Knopp cocycle Kn_{τ_1} is a cocycle for $\mathrm{SL}_2(\mathbb{Z})$ whose logarithmic derivative is given by

$$\mathrm{kn}_\tau : \gamma \mapsto \sum_{w \in \mathrm{SL}_2(\mathbb{Z})\tau_1} \frac{\delta_{\infty,\gamma\infty}(w)}{z - w}, \tag{1.2}$$

where $\delta_{\infty,\gamma\infty}(w)$ is the right-handed signed intersection number between the geodesics connecting ∞ with $\gamma\infty$ and the conjugate w' of w with w.



(a) $\delta_{\infty,\gamma\infty}(w) = 1$ (b) $\delta_{\infty,\gamma\infty}(w) = 0$ (c) $\delta_{\infty,\gamma\infty}(w) = -1$

Figure 1.2: Examples of all possible values of the intersection number $\delta_{\infty,\gamma\infty}(w) \in \{-1, 0, 1\}$.

In Chapter 4, we prove (following [DIT11; DIT10a]) that the additive Knopp cocycle also appears as the period of the logarithmic derivative of the difference of two modular j functions $j(z_1) - j(z_2)$. More explicitly: for τ a real quadratic irrationality with automorph γ_τ and corresponding primitive polynomial Q, we define an analytic continuation $F(z)$ to the upper half-plane of the function given by

$$\int_{w_0}^{\gamma_\tau w_0} \frac{j'(z)}{j(z) - j(u)} \frac{\sqrt{\Delta} du}{Q(u)} \tag{1.3}$$

in a neighbourhood of the cusp ∞. This function satisfies the transformation formula

$$\frac{1}{(cz+d)^2} F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = F(z) + (\mathrm{kn}_\tau + \mathrm{kn}_{\tau'})(\gamma^{-1})(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

We hope that this connection between the p-adic construction of Darmon and Vonk and the ∞-adic cycle integrals will eventually lead us to an ∞-adic analogue of real quadratic singular moduli. We will explore this connection as follows.

- In Chapter 2 we will explore the first approximation Θ of the p-adic limits defining Θ_p. To do this, we will first construct the multiplicative Knopp cocycle Kn_{τ_1}. This construction is different from the construction given in [DV21], as we do not rely on any cohomological arguments. This allows us to remove an ambiguity in the original construction.

  Let τ_1 and τ_2 be two real quadratic irrationalities not in the same $\mathrm{SL}_2(\mathbb{Z})$ orbits. Let $K_1 = \mathbb{Q}(\tau_1)$ and $K_2 = \mathbb{Q}(\tau_2)$ and let γ_2 be the automorph of τ_2. We define

  $$\Theta(\tau_1, \tau_2) = \mathrm{Kn}_{\tau_1}(\gamma_2)(\tau_2). \tag{1.4}$$

  This map already shares various properties with the differences of singular moduli. For example, Θ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on either argument: for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $\Theta(\tau_1, \tau_2) = \Theta(\gamma\tau_1, \tau_2) = \Theta(\tau_1, \gamma\tau_2)$. We will prove that the primes dividing the norm of $\Theta(\tau_1, \tau_2)$ also divide a positive integer of the form

  $$\frac{\Delta_1 \Delta_2 - x^2}{4},$$

  where Δ_1 and Δ_2 are the discriminants of the orders associated to τ_1 and τ_2. These primes are however not all inert in K_1 and K_2. Also in contrast to the differences of singular moduli, $\Theta(\tau_1, \tau_2)$ does not generate the compositum of the respective Hilbert class fields $H_1 H_2$; instead it always lies in $K_1 K_2$. We end the chapter by conducting many computations of various values of Θ using the programming language Sage. We include an appendix which explains the algorithms used to make these computations. Informed by these computations we conjecture that Θ is antisymmetric:

$$\Theta(\tau_1,\tau_2)\times\Theta(\tau_2,\tau_1)=1.$$

- In Chapter 3 we discuss the various properties of cycle integrals of modular functions. In [DIT11], Duke, Imamoḡlu and Tóth use these cycle integrals to extend many of the results of Zagier on the traces of CM-singular moduli [Zag02] to include also positive discriminants. We will also discuss the numerical computations of cycle integrals of the j-function made by Kaneko [Kan09] and discuss his observations. Inspired by (1.3) we also define a possible candidate construction of differences of real quadratic singular moduli: the 'double' cycle integral
$$\Psi(\tau_1,\tau_2)=\int_{z_0}^{\gamma_1 z_0}\int_{w_0}^{\gamma_2 w_0}(j(z)-j(w))\frac{\sqrt{\Delta_1}dz}{Q_1(z)}\frac{\sqrt{\Delta_2}dw}{Q_2(w)},$$
where $\gamma_1,\gamma_2,Q_1,Q_2,\Delta_1,\Delta_2$ are the automorphs, primitive polynomials and discriminants associated to the real quadratic irrationalities τ_1,τ_2. However we are not able to recognise any algebraic values in Ψ.

- In Chapter 4 we show the connection between the previous two chapters as is sketched above. The analytic continuation F of (1.3) is related via the logarithmic derivative to a real quadratic Borcherds product that has the multiplicative Knopp cocycle Kn_τ as its period. We are able to use this to give explicit formulae for symmetrised versions of the Knopp cocycle and Θ.

Chapter 2

The Knopp Cocycle

The construction of $\Theta_p(\tau_1, \tau_2) \in \mathbb{C}_p$ by Darmon and Vonk [DV21] yields p-adic real quadratic differences of singular moduli. It relies crucially on taking p-adic limits of the Knopp cocycle. As a first step of trying to define a similar ∞-adic construction of real quadratic differences of singular moduli, we will study the first approximation $\Theta(\tau_1, \tau_2)$ of $\Theta_p(\tau_1, \tau_2)$. This is a global quantity lying in the compositum of the numbers fields $\mathbb{Q}(\tau_1), \mathbb{Q}(\tau_2)$. A similar complex analytic construction of differences of real quadratic singular moduli would likely consist of an archimedean limit procedure involving $\Theta(\tau_1, \tau_2)$. We will define Θ and study its properties.

Let $\mathcal{M}_k$ be the space of meromorphic functions on the upper half-plane $\mathcal{H}$, endowed with the left weight k action of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \star_k f := \frac{1}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right).$$

Notice that the j-function is an element of $Z^0(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_0^\times)$. Looking at the next cohomological group $Z^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_0^\times)$, Darmon and Vonk find that it contains an element associated to any $\mathrm{SL}_2(\mathbb{Z})$-class of real quadratic irrationalities $[\tau]$: the *(multiplicative) Knopp cocycle* Kn_τ. This cocycle even lies in the subspace $Z^1(\mathrm{SL}_2(\mathbb{Z}), K(z)^\times)$, where $K = \mathbb{Q}(\tau)$. A way of constructing this cocycle is by lifting the *additive Knopp cocycle*

$$\begin{aligned} \mathrm{kn}_\tau : \mathrm{SL}_2(\mathbb{Z}) &\longrightarrow K(z) \subset \mathcal{M}_2 \\ \gamma &\longmapsto \sum_{w \in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\gamma\infty}(w)}{z-w} \end{aligned} \tag{2.1}$$

under the logarithmic derivative

$$\begin{aligned} \mathrm{dlog} : K(z)^\times &\to K(z) \\ f &\mapsto f'/f, \end{aligned}$$

where the $\mathrm{SL}_2(\mathbb{Z})$-action on $K(z)^\times$ is induced by $K(z)^\times \subset \mathcal{M}_0^\times$. In Theorem 2.18, we will prove that such a lift exists.

We 'evaluate' the Knopp cocycle Kn_{τ_1} at a second real quadratic irrationality τ_2 by first evaluating it at the automorph γ_2 of τ_2:

$$\Theta(\tau_1, \tau_2) := \mathrm{Kn}_{\tau_1}(\gamma_2)(\tau_2).$$

To study the properties of Θ, we will first construct the Knopp cocycle. Afterwards we make some computations of the values of Θ.

2.1 Lifting the additive Knopp cocycle

Let τ be a real quadratic irrationality. The Knopp cocycle is constructed by lifting the additive Knopp cocycle under the logarithmic derivative map. We first prove that the additive Knopp cocycle is indeed a cocycle and then construct its lifts.

Proposition 2.2. *The map* kn_τ *defined in* (2.1) *is well-defined and a 1-cocycle.*

Before proving this proposition, we will first need some more knowledge about the intersection numbers. One of their crucial properties, is that they are additive: For τ a quadratic irrationality and $r, s, t \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{R})$ three elements not lying on the geodesic associated to τ, we have:

$$\delta_{r,s}(\tau) + \delta_{s,t}(\tau) = \delta_{r,t}(\tau).$$

This is because the geodesic from r to s composed with the geodesic from s to t is homotopic to the geodesic from r to t (see Figure 2.1).

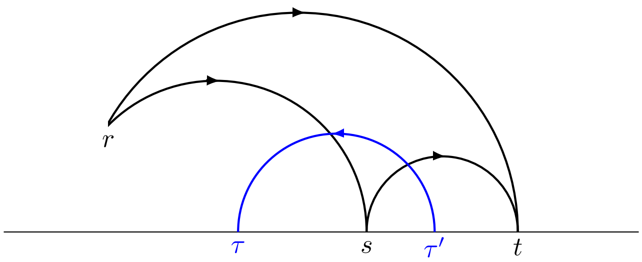


Figure 2.1: $\delta_{r,s}(\tau) + \delta_{s,t}(\tau) = \delta_{r,t}(\tau) = 0$

The second crucial property is $\mathrm{SL}_2(\mathbb{Z})$-equivariance: for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it holds that

$$\delta_{\gamma r, \gamma s}(\gamma \tau) = \delta_{r,s}(\tau).$$

We can now prove that the sum occuring in (2.1) is finite.

Lemma 2.3. *Let τ be a real quadratic irrationality, and let $r, s \in \mathbb{P}^1(\mathbb{Q})$. Then we have that the set $\{w \in \mathrm{SL}_2(\mathbb{Z})\tau \colon \delta_{r,s}(w) \neq 0\}$ is finite. Furthermore,*

$$\sum_{w \in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{r,s}(w) = 0. \tag{2.4}$$

Proof. First we will prove the statement for $r = \infty$ and $s = 0$. Let w be an element of $\{w \in \mathrm{SL}_2(\mathbb{Z})\tau \colon \delta_{r,s}(w) \neq 0\}$. Let $Q_w(z) = az^2 + bz + c$ be its associated primitive polynomial with discriminant Δ. Then $\delta_{\infty,0}(w) \neq 0$ is equivalent to $c/a = ww' < 0$. Every element of $\mathrm{SL}_2(\mathbb{Z})\tau$ has the same discriminant, and there exist only finitely many triples (a, b, c) such that $b^2 - 4ac = \Delta$ and $ac < 0$, so we find that this set is indeed finite.

Let S, T be the two standard generators of $\mathrm{SL}_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It follows from equivariance that applying the matrix S to any w in this set gives you an element of this set with opposite sign, which shows that

$$\sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{\infty,0}(w) = 0.$$

To extend this result to general $r, s \in \mathbb{P}^1(\mathbb{Q})$, we will use the following. Given $r, s, t \in \mathbb{P}^1(\mathbb{Q})$, we have

$$\{w \in \mathrm{SL}_2(\mathbb{Z})\tau\colon \delta_{r,t}(w) \neq 0\} = \{w \in \mathrm{SL}_2(\mathbb{Z})\tau\colon \delta_{r,s}(w) \neq 0\} \triangle \{w \in \mathrm{SL}_2(\mathbb{Z})\tau\colon \delta_{s,t}(w) \neq 0\}, \tag{2.5}$$

where $\triangle$ denotes the symmetric difference, and

$$\sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{r,t}(w) = \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{r,s}(w) + \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \delta_{s,t}(w). \tag{2.6}$$

By using equivariance, we may without loss of generality assume that $r = \infty$. Let γ_s be a matrix such that $s = \gamma_s\infty$. By writing γ_s as a product of matrices of the form T^nS for $n \in \mathbb{Z}$, we can split up the path $\infty \to \gamma_s\infty$ into finitely many paths of the form $\gamma\infty \to \gamma 0$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Alongside (2.5), (2.6) this proves the statement. □

Proof of Proposition 2.2. Well-definedness follows from Lemma 2.3, so we only need to prove the cocycle relation

$$\mathrm{kn}_\tau(\sigma\gamma) = \mathrm{kn}_\tau(\sigma) + \sigma \star_2 \mathrm{kn}_\tau(\gamma).$$

We use the additivity and the equivariance of δ to find

$$\begin{aligned}
\mathrm{kn}_\tau(\sigma\gamma) &= \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\sigma\infty}(w)}{z-w} + \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\sigma\infty,\sigma\gamma\infty}(w)}{z-w} \\
&= \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\sigma\infty}(w)}{z-w} + \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\gamma\infty}(w)}{z-\sigma w}.
\end{aligned}$$

One can make a tedious computation to see that for $w_1, w_2 \in \mathbb{R}$ there is an equality

$$\frac{1}{z-\sigma w_1} - \frac{1}{z-\sigma w_2} = \sigma \star_2 \left(\frac{1}{z-w_1} - \frac{1}{z-w_2'}\right).$$

Note that there are also more insightful ways of seeing this, e.g [Von, Lemmas 3.7 and 3.13]. Using Lemma 2.3 we now find

$$\sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\gamma\infty}(w)}{z-\sigma w} = \sigma \star_2 \sum_{w\in \mathrm{SL}_2(\mathbb{Z})\tau} \frac{\delta_{\infty,\gamma\infty}(w)}{z-w}.$$ □

Notice that the value of kn_τ in $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is trivial. Cocycles with this property are called *parabolic*. A parabolic cocycle is determined by its value in S, which is called a *rational period function* if it takes values in $\mathbb{C}(z)$ endowed with the weight two action of $\mathrm{SL}_2(\mathbb{Z})$. These were first studied by Knopp in [Kno78]. The group of rational period functions, and hence the group of parabolic cocycles taking values in $\mathbb{C}(z)$, were completely determined in [Kno78], [Ash89],

[CZ93]. The subgroup of parabolic cocycles whose values lie in the image of dlog is generated by the additive Knopp cocycles and the cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \mapsto \frac{c}{cz+d}. \tag{2.7}$$

This cocycle is the logarithmic derivative of the multiplicative cocycle $J \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_0^\times)$ given by

$$J : \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \mapsto cz + d \tag{2.8}$$

that occurs in the definition of modular forms.

We will now construct the lift Kn_τ of kn_τ. We take a different approach than in [DV21], where the authors use cohomological arguments to prove the existence of such a multiplicative lift. Instead, we take a more direct approach. We compute what the possible values of such a multiplicative lift can be in the generators S and $U := TS$. To prove that we can define a cocycle by giving the values in the generators in this way, and to find what relations they have to satisfy, we give a characterisation of 1-cocycles as homomorphisms.

Lemma 2.9. *Let G be a group and M a G-module. Let π_G be the projection map $G \ltimes M \to G$, and π_M the projection map $G \ltimes M \to M$. Then there is an isomorphism*

$$\begin{aligned} Z^1(G, M) &\to \{f \in \mathrm{Hom}(G, G \ltimes M) : \pi_G \circ f = \mathrm{id}_G\} \\ \varphi &\mapsto (\mathrm{id}_G, \varphi), \end{aligned} \tag{2.10}$$

with inverse $\pi_M \circ -$.

Proof. This proof is a trivial check of relations. □

Proposition 2.11. *Let G be a finitely presented group and M a G-module with action denoted $\star$. Then (by definition) there exist k, n and a short exact sequence*

$$0 \to F_k \xrightarrow{i} F_n \xrightarrow{p} G \to 0, \tag{2.12}$$

where F_k and F_n denote the free groups in respectively k, n generators. Let $(x_i)_{i=1}^n \subset F_n$ be a set of generators of F_n, $(y_j)_{j=1}^k \subset F_k$ a set of generators of F_k, and $(m_i)_{i=1}^k \subset M$ a collection of elements. Consider the action of F_n on M induced by p. If for any word $x_{i_1} \dots x_{i_k}$ that is the image of one the y_j, the $(m_i)_i$ satisfy

$$m_{i_1} + x_{i_1} \star m_{i_2} + (x_{i_1} x_{i_2}) \star m_{i_3} + \cdots + (x_{i_1} \dots x_{i_{k-1}}) \star m_{i_k} = 0, \tag{2.13}$$

then there exists a unique 1-cocycle φ for G taking values in M with $\varphi(p(x_i)) = m_i$ for all i.

Proof. By Lemma 2.9, there is an equivalence between 1-cocycles for G taking values in M and morphisms $f : G \to G \ltimes M$ satisfying $\pi_G \circ f = \mathrm{id}_G$. Under this equivalence, the cocycle φ would correspond to a homomorphism $f : G \to G \ltimes M$ such that $f(p(x_i)) = (p(x_i), m_i)$. Since $G \cong F_n / i(F_k)$ and F_n is free, such a homomorphism exists if and only if the induced homomorphism $F_k \to G \ltimes M$ is trivial. This property is equivalent to (2.13) being true for all images of the y_j. □

We are now able to explicitly write down the lifts of kn_τ under the logarithmic derivative using the following computation.

Lemma 2.14 (Lemma 2.1 from [DV21]). *Define*

$$h_\tau(z) := \prod_{w \in \mathrm{SL}_2(\mathbb{Z})\tau} (z-w)^{\delta_{\infty,0}(\tau)}. \tag{2.15}$$

Then we have

$$(1 \times S) \star_0 h_\tau = \xi_\tau^2, \tag{2.16}$$
$$(1 \times U \times U^2) \star_0 h_\tau = \xi_\tau^3 \varepsilon^{-3}, \tag{2.17}$$

where ξ_τ is given by

$$\xi_\tau = \prod_{\substack{w \in \mathrm{SL}_2(\mathbb{Z})\tau \\ w' < 0 < w}} w,$$

and

$$\varepsilon = \prod_{\substack{ww' < 0 \\ |w| < 1 < |w'|}} |w|^{-1} > 1.$$

is a fundamental unit of norm one in the order associated to τ.

The proof of this lemma mostly consists of a direct computation. Proving that ε is fundamental unit of norm one can be done using the theory of continued fractions, as in [Zag75a, Equation (6.4)]. These continued fractions are also introduced in the appendix.

Theorem 2.18. *There are 12 cocycles $\Phi \in Z^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}(z)^\times)$ such that $\mathrm{dlog}(\Phi) = \mathrm{kn}_\tau$, each differing by a cocycle taking values in the group of 12-th roots of unity μ_{12}. Precicely two of these cocycles take values in $K(z)^\times$, and these by the non-trivial cocycle taking values in $\{\pm 1\}$.*

Proof. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices S and $U = TS$. The relations in $\mathrm{SL}_2(\mathbb{Z})$ are generated by $S^4 = 1$, and $S^2U^3 = 1$. By Proposition 2.11, a cocycle Φ with $\mathrm{dlog}(\Phi) = \mathrm{kn}_\tau$ corresponds to a pair $(f, g) \in (\mathbb{C}(z)^\times)^2$ satisfying $\mathrm{dlog}(f) = \mathrm{dlog}(g) = \mathrm{kn}_\tau(S)$ and

$$\begin{cases} (1 \times S \times S^2 \times S^3) \star f = 1 \\ (1 \times S) \star f \times (-1 \times -U \times -U^2) \star g = 1, \end{cases} \tag{2.19}$$

where $\times$ denotes multiplication in $\mathbb{C}(z)^\times$. The equality $\mathrm{dlog}(f) = \mathrm{dlog}(g) = \mathrm{kn}_\tau(S)$ implies that f, g are equal to h (defined in (2.15)) up to multiplication with some constant. Using Lemma 2.14 we then compute

$$(1 \times S \times S^2 \times S^3) \star h = ((1 \times S) \star h)^2 = \xi_\tau^4.$$

Therefore the possible values for f are equal to $h/\xi_\tau \cdot i^k$ for some $k \in \{0, 1, 2, 3\}$. Again using Lemma 2.14 we find

$$(1 \times S) \star \frac{hi^k}{\xi_\tau} \times (-1 \times -U \times -U^2) \star h = \pm(-1)^k \xi_\tau^3 \varepsilon^{-3},$$

where ε is the fundamental unit > 1 of norm 1 in the order associated to τ. Therefore there are three options for g, namely $\pm(-1)^k \zeta_3^\ell \cdot h \cdot \varepsilon \xi_\tau^{-1}$ for $\ell \in \{0, 1, 2\}$. The cocycles with values in $K(z)^\times$ correspond to the cases where k is even and $\ell = 0$. □

Let Φ be a multiplicative lift of kn_τ. Since S, U generate $\mathrm{SL}_2(\mathbb{Z})$, the values of Φ in a matrix γ can be computed by writing γ as a word in S and U and using the cocycle relation. For example, since $T^{-1} = SU$ we find

$$\Phi(T^{-1}) = i^k \xi_\tau^{-1} h \times S \star \pm(-1)^k \zeta_3^\ell \xi_\tau^{-1} \varepsilon h = (-i)^k \zeta_3^\ell \times \varepsilon.$$

Therefore there exists exactly one multiplicative lift Φ with $\Phi(T) = \varepsilon^{-1} > 0$ corresponding to $k = \ell = 0$.

Definition 2.20. We call the unique cocycle $\mathrm{Kn}_\tau \in Z^1(\mathrm{SL}_2(\mathbb{Z}), K(z)^\times)$ such that $\mathrm{dlog}(\mathrm{Kn}_\tau) = \mathrm{kn}_\tau$ and $\mathrm{Kn}_\tau(T) = \varepsilon^{-1} > 0$, the *(multiplicative) Knopp cocycle.*

Computing the values of the Knopp cocycle using the cocycle relation gives us little theoretical insight and it would be nice to have an explicit formula. The methods of Chapter 4 are able to give an explicit formula for Kn_τ^2, but it is as of yet not in a useful form. Darmon and Vonk are able to give an explicit and useful formula for the reduction of $\mathrm{Kn}_\tau \bmod \varepsilon^{\mathbb{Z}}$.

Before giving this closed form we recall that the automorph of τ is a generator of the free part of the stabiliser of τ under the action of $\mathrm{SL}_2(\mathbb{Z})$. It is chosen such that it has eigenvalues $\varepsilon, \varepsilon^{-1}$ corresponding the the eigenvectors $\binom{\tau}{1}, \binom{\tau'}{1}$.

Lemma 2.21. *(Lemma 2.4 in [DV21]) Let γ_τ be the automorph of τ. Then*

$$\mathrm{Kn}_\tau(\gamma)(z) \equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\gamma\infty}(g\tau)} \bmod \varepsilon^{\mathbb{Z}}. \tag{2.22}$$

Proof. Note that this expression is (only) well defined up to $\varepsilon^{\mathbb{Z}}$ as $\gamma_\tau\binom{\tau}{1} = \varepsilon\binom{\tau}{1}$. Using the additivity of δ we find

$$\prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\sigma\gamma\infty}(g\tau)} \equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\sigma\infty}(g\tau)+\delta_{\sigma\infty,\sigma\gamma\infty}(g\tau)}.$$

By the equivariance of δ and Lemma 2.3 we compute

$$\begin{aligned}\prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\sigma\infty,\sigma\gamma\infty}(g\tau)} &\equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} z \\ 1 \end{pmatrix}, \sigma g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\gamma\infty}(g\tau)} \\ &\equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\sigma^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\gamma\infty}(g\tau)} \\ &\equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \det\left[\begin{pmatrix} \sigma^{-1}z \\ 1 \end{pmatrix}, g\begin{pmatrix} \tau \\ 1 \end{pmatrix}\right]^{\delta_{\infty,\gamma\infty}(g\tau)},\end{aligned}$$

to see that the expression in (2.22) is a cocycle. It is clearly a lift of kn_τ and its value in T is positive, so the proof is completed. □

Since $\delta_{r,s}(\tau) = -\delta_{r,s}(\tau')$ for all $r, s \in \mathbb{P}^1(\mathbb{Q})$, the cocycle $\mathrm{Kn}_{\tau'}$ is equal to the inverse of the Galois conjugate of Kn_τ. The $\varepsilon^{\mathbb{Z}}$ ambiguity of Lemma 2.21 disappears when taking the norm, which gives us the following expression.

Corollary 2.23. *Let γ_τ be the automorph of τ. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$(\mathrm{Kn}_\tau \div \mathrm{Kn}_{\tau'})(\gamma) = \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_\tau^{\mathbb{Z}}} \left(\det \left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right] \det \left[\begin{pmatrix} z \\ 1 \end{pmatrix}, g \begin{pmatrix} \tau' \\ 1 \end{pmatrix} \right] \right)^{\delta_{\infty,\gamma\infty}(g\tau)}.$$

2.2 Evaluating the Knopp cocycle

In this section we will study the evaluation of the Knopp cocycle in a second real quadratic irrationality.

Definition 2.24. Let τ_1, τ_2 be two real quadratic irrationalities that are not equivalent under $\mathrm{SL}_2(\mathbb{Z})$. Let γ_2 be the automorph of τ_2. We define

$$\Theta(\tau_1, \tau_2) := \mathrm{Kn}_{\tau_1}(\gamma_2)(\tau_2).$$

We continue by observing some of its properties.

Proposition 2.25. *Let τ_1, τ_2 be two real quadratic irrationalities in the fields $K_1 = \mathbb{Q}(\tau_1), K_2 = \mathbb{Q}(\tau_2)$. Write τ_1', τ_2' for the Galois conjugates of τ_1, τ_2. Let σ_1, σ_2 be the automorphisms of $K_1K_2/\mathbb{Q}$ such that $\sigma_1(\tau_1) = \tau_1', \sigma_1(\tau_2) = \tau_2$, and $\sigma_2(\tau_1) = \tau_1, \sigma_2(\tau_2) = \tau_2'$. Then we have the relations*

- *(i)* $\Theta(\sigma\tau_1, \tau_2) = \Theta(\tau_1, \sigma\tau_2) = \Theta(\tau_1, \tau_2)$ *for all* $\sigma \in \mathrm{SL}_2(\mathbb{Z})$;
- *(ii)* $\Theta(\tau_1', \tau_2) = \sigma_1(\Theta(\tau_1, \tau_2))^{-1}$;
- *(iii)* $\Theta(\tau_1, \tau_2') = \sigma_2(\Theta(\tau_1, \tau_2))^{-1}$;
- *(iv)* $\Theta(-\tau_1, -\tau_2) = \Theta(\tau_1, \tau_2)^{-1}$.

Proof. *(i)* Invariance in the first argument follows from $\mathrm{Kn}_{\sigma\tau_1} = \mathrm{Kn}_{\tau_1}$. Let γ_2 be the automorph of τ_2. Then invariance in the second argument follows from an easy computation using the cocycle relation and the fact that $\sigma\tau_2$ has automorph $\sigma\gamma_2\sigma_2^{-1}$.

(ii) We extend the automorphism σ_1 to the set $K_1K_2(z)$ such that it leaves z invariant. Since $\delta_{r,s}(w) = -\delta_{r,s}(w')$ holds for all $w \in \mathrm{SL}_2(\mathbb{Z})\tau_1$, we have the equality $\mathrm{Kn}_{\tau_1'} = \sigma_1(\mathrm{Kn}_{\tau_1})^{-1}$.

(iii) The conjugate τ_2' has automorph γ_2^{-1}. We compute using the cocycle property

$$\mathrm{Kn}_{\tau_1}(\gamma_2^{-1})(\tau_2') = \gamma_2 \star \mathrm{Kn}_{\tau_1}(\gamma_2)(\tau_2')^{-1} = \mathrm{Kn}_{\tau_1}(\gamma_2)(\tau_2')^{-1}.$$

(iv) For $w \in \mathrm{SL}_2(\mathbb{Z})\tau$ we have $\delta_{\infty,0}(-w) = -\delta_{\infty,0}(w)$, which implies $\mathrm{kn}_{-\tau}(S)(z) = -\mathrm{kn}_\tau(S)(-z)$. Since $S(-z) = -(Sz)$ and $T(-z) = -(T^{-1}z)$, consider the automorphism of $\mathrm{SL}_2(\mathbb{Z})$ given by conjugation with the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which maps S to $-S = S^{-1}$ and T to T^{-1}. From these relations it follows that $\varphi(\gamma)(z) := \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))(-z)^{-1}$ is a cocycle: we check

$$\begin{aligned}
\varphi(S\gamma)(z) &= \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)S\gamma\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))(-z)^{-1} \\
&= \mathrm{Kn}_{\tau_1}(-S)(-z)^{-1} \times (-S) \star_0 \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))(-z)^{-1} \\
&= \mathrm{Kn}_{\tau_1}(-S)(-z)^{-1} \times \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))(z^{-1})^{-1} \\
&= \varphi(S)(z) \times \varphi(\gamma)(-z^{-1}),
\end{aligned}$$

and

$$\begin{aligned}\varphi(T\gamma)(z) &= \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)T\gamma\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right))(-z)^{-1}\\ &= \mathrm{Kn}_{\tau_1}(T^{-1})(-z)^{-1}\times\left[T^{-1}\star_0\mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)\gamma\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right))\right](-z)^{-1}\\ &= \mathrm{Kn}_{\tau_1}(T^{-1})(-z)^{-1}\times\mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)\gamma\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right))(-z+1)^{-1}\\ &= \varphi(T)(z)\times\varphi(\gamma)(z-1).\end{aligned}$$

Since $\mathrm{dlog}(\varphi(S)) = \mathrm{kn}_{-\tau_1}(S)$ and $\varphi(T) = \mathrm{Kn}_{\tau_1}(T^{-1})^{-1} = \varepsilon^{-1} > 0$ we then find

$$\mathrm{Kn}_{-\tau_1}(\gamma)(z) = \mathrm{Kn}_{\tau_1}(\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)\gamma\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right))(-z)^{-1}.$$

The statement now follows once we note that the automorph of $-\tau_2$ is equal to the automorph of τ_2 conjugated by $\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$. □

The invariance of Θ under the action of $\mathrm{SL}_2(\mathbb{Z})$ on either argument leads to an interesting observation, for which we need some theory on binary quadratic forms.

For every real quadratic irrationality τ there is a unique primitive binary quadratic form $Q = aX^2 + bXY + cY^2$ with discriminant $\Delta = b^2 - 4ac > 0$ such that $\tau = \frac{-b+\sqrt{\Delta}}{2a}$. Conversely given a binary quadratic form Q, we write τ_Q for this principal root. The binary quadratic form Q is the homogenisation of the primitive quadratic polynomial introduced in Definition 1.1. From now on we will speak interchangeably about binary quadratic forms and primitive polynomials associated to (real) quadratic irrationalities.

There is a right action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of binary quadratic forms via

$$Q(X,Y)\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix} = Q(aX+bY, cX+dY).$$

Let $\mathcal{Q}_\Delta$ be the set of primitive binary quadratic forms of discriminant $\Delta > 0$. It is a well-known result that there is a bijection between $\mathcal{Q}_\Delta/\,\mathrm{SL}_2(\mathbb{Z})$ and the narrow class group of the unique order $\mathcal{O}_\tau := \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$ of discriminant Δ. It is given by

$$\begin{aligned}\mathcal{Q}_\Delta/\,\mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{Pic}^+(\mathcal{O}_\tau)\\ [aX^2+bXY+cY^2] &\mapsto \left[\left(a, \tfrac{-b+\sqrt{\Delta}}{2}\right)\right].\end{aligned}$$

Endowing the set binary quadratic forms $\mathcal{Q}_\Delta$ with the composition law of Gauss, this furthermore becomes an isomorphism of groups.

Therefore for real quadratic orders $\mathcal{O}_1, \mathcal{O}_2$ in respective number fields K_1, K_2, the map Θ induces a map

$$\begin{aligned}\Theta : \mathrm{Pic}^+(\mathcal{O}_1)\times\mathrm{Pic}^+(\mathcal{O}_2) &\to K_1K_2\\ ([Q_1],[Q_2]) &\mapsto \Theta(\tau_{Q_1},\tau_{Q_2}).\end{aligned}\tag{2.26}$$

By combining the relations of Lemma 2.24 you find that all Galois conjugates of $\Theta(Q_1, Q_2)$ appear in the image of (2.26). For example by combining relations *(ii), (iv)* of Proposition 2.25 you find

$$\sigma_1(\Theta(\tau_1,\tau_2)) = \Theta(\tau_1',\tau_2)^{-1} = \Theta(-\tau_1',-\tau_2).$$

When any of the associated elements in the narrow class group coincide, then it is possible that $\Theta(\tau_1, \tau_2)$ is equal to some of its Galois conjugates or its inverses. This can be detected by looking at the minimal polynomial. It will be palindromic or of degree lower than 4 if this is the case, which it very often is for small discriminants.

We continue with some examples that were computed using Sage [Ste+25]. See the appendix for the code that was used to generate these examples and for explanations of the algorithms used, for example on how to compute the automorph of real quadratic irrationality.

Example 2.27. Consider the smallest pair of discriminants $(\Delta_1, \Delta_2) = (5, 8)$. Both of their respective orders have narrow class number one, and hence all conjugates and their inverses are equal. Indeed we compute

$$\Theta(\tfrac{1+\sqrt{5}}{2}, \sqrt{2}) = 1.$$

Example 2.28. The smallest non-trivial example is obtained from the pair of discriminants $(\Delta_1, \Delta_2) = (5, 12)$. The order of discriminant 12 has narrow class number 2. The two distinct elements are for example represented by the binary quadratic forms associated to $\pm\sqrt{3}$. The image of Θ is given by the two roots of

$$11x^2 + 147 + 11,$$

which has discriminant $5^3 \cdot 13^2$.

Example 2.29. For the pair $(\Delta_1, \Delta_2) = (12, 21)$, both of the associated orders have narrow class number 2. The image of Θ contains the roots of

$$511031x^4 - 915810078500x^3 - 5198875539126x^2 - 915810078500x + 511031.$$

The constant coefficient factors as

$$511031 = 47 \cdot 83 \cdot 131.$$

Example 2.30. So far all of the polynomials have been palindromic. The smallest example where this is not the case occurs for the pair of discriminants $(12, 148)$. The orders of the respective class groups are equal to 2 and 3. A set of representatives of the class group of discriminant 12 is equal to $\{X^2 + 2XY - 2Y^2, -X^2 + 2XY + 2Y^2\}$. The corresponding real quadratic irrationalities are $\sqrt{-3} - 1$ and $1 - \sqrt{3}$. A set of representatives of the class group of discriminant 148 is equal to $\{X^2 + 12XY - 1Y^2, 4X^2 + 6XY - 7Y^2, 7X^2 + 6XY - 4Y^2\}$. We have computed the minimal polynomial of all six pairs of quadratic irrationalities.

$$\Theta(\sqrt{3}, -6+\sqrt{37}) \text{ and } \Theta(-\sqrt{3}, -6+\sqrt{37}) \text{ satisfy } 945131x^2 + 12127274x + 945131,$$
$$\Theta(\sqrt{3}, \tfrac{-3+\sqrt{37}}{4}) \text{ and } \Theta(-\sqrt{3}, \tfrac{-3+\sqrt{37}}{7}) \text{ satisfy } 636851927x^2 - 7354672441006x + 5440463831,$$
$$\Theta(-\sqrt{3}, \tfrac{-3+\sqrt{37}}{4}) \text{ and } \Theta(\sqrt{3}, \tfrac{-3+\sqrt{37}}{7}) \text{ satisfy } 5440463831x^2 - 7354672441006x + 636851927.$$

The leading coefficients factor as

$$945131 = 11^2 \cdot 73 \cdot 107, \quad 636851927 = 47 \cdot 73 \cdot 419 \cdot 443 \quad \text{and} \quad 5440463831 = 11^6 \cdot 37 \cdot 83.$$

Example 2.31. The smallest example of a minimal polynomial that is not palindromic and of degree 4 is already quite large. This is because there is only one relation allowed in either class group, which is very restrictive. Consider the discriminants $(\Delta_1, \Delta_2) = (148, 316)$ with respective narrow class numbers 3 and 6. We compute the minimal polynomial of $\Theta(\frac{-3+\sqrt{37}}{4}, \frac{-3+\sqrt{79}}{7})$ to be

$$
\begin{aligned}
&193240409517090703026435905772939782700872686875905204195145135198856189246270869\,7\ x^4 \\
-\ &430643525838691421803622688407370296938634027537264459799537823070057493187032319317202518\,8\ x^3 \\
-\ &1433527099068786175625435736316989160151220911840287207120598918279644834855788896901746238294232701\,8\ x^2 \\
+\ &167318569477589416801094674925848787436864051023713929899133665282498312401926855500589481\,2\ x \\
+\ &62611282959971165236114513553945184977289328158699272876446643264567685445748697.
\end{aligned}
$$

Its leading coefficient is factored as

$$
\begin{aligned}
&7^2 \cdot 47^8 \cdot 61^2 \cdot 71^5 \cdot 73 \cdot 101 \cdot 107 \cdot 211 \cdot 619 \cdot 733 \cdot 739 \cdot 941^2 \\
&\cdot 953 \cdot 1163 \cdot 1259 \cdot 1669 \cdot 2347 \cdot 2657 \cdot 3881 \cdot 7723 \cdot 11467
\end{aligned}
$$

and its constant coefficient as

$$
\begin{aligned}
&3^3 \cdot 7^2 \cdot 17^2 \cdot 19^2 \cdot 29^2 \cdot 73^3 \cdot 101 \cdot 107 \cdot 139 \cdot 181^2 \cdot 269 \cdot 271 \cdot 379^2 \cdot 419^2 \\
&\cdot 433 \cdot 761 \cdot 1163 \cdot 1381 \cdot 1489 \cdot 1601 \cdot 3617 \cdot 3889 \cdot 8443 \cdot 9091 \cdot 11251.
\end{aligned}
$$

From these examples it is obvious that the primes $\mathfrak{p}$ of K_1K_2 dividing Θ are all small. With closer inspection you might see that if the arguments of $\Theta(\tau_1, \tau_2)$ have discriminants Δ_1, Δ_2, then these primes also divide a positive integer of the form

$$\frac{\Delta_1\Delta_2 - x^2}{4}.$$

This is also conjectured to be the case for the p-adic versions of Θ [DV21, Conjecture 2]. We will prove this to be true using an explicit expression of Θ modulo units.

Proposition 2.32. *(Proposition 2.5 in [DV21]) Let τ_1, τ_2 be real quadratic irrationalities with automorphs γ_1, γ_2, and let $\mathcal{O}_1, \mathcal{O}_2$ be their associated orders. Then we have*

$$\Theta(\tau_1, \tau_2) \equiv \prod_{g \in \gamma_2^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \det\left[g\begin{pmatrix}\tau_1\\1\end{pmatrix}, \begin{pmatrix}\tau_2\\1\end{pmatrix}\right]^{\delta(g\tau_1,\tau_2)} \bmod \mathcal{O}_1^\times\mathcal{O}_2^\times. \tag{2.33}$$

Proof. Applying Lemma 2.21 we find

$$\Theta(\tau_1, \tau_2) \equiv \prod_{g \in \mathrm{PSL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \det\left[\begin{pmatrix}\tau_1\\1\end{pmatrix}, g^{-1}\begin{pmatrix}\tau_2\\1\end{pmatrix}\right]^{\delta_{\infty,\gamma_2\infty}(g\tau_1)} \bmod \mathcal{O}_1^\times\mathcal{O}_2^\times$$

Since $g^{-1}\gamma_2^{-n}\left(\begin{smallmatrix}\tau_2\\1\end{smallmatrix}\right)$ and $g^{-1}\left(\begin{smallmatrix}\tau_2\\1\end{smallmatrix}\right)$ differ by a power of ε_2, we may group together all terms in the product that differ by a power of γ_2. The roots τ_2, τ_2' are respectively the stable and unstable fixed points of γ_2 acting on $\mathbb{P}^1(\mathbb{R})$, so we find for $g \in \gamma_2^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}$:

$$\sum_{n \in \mathbb{Z}} \delta_{\gamma_2^{-n}\infty, \gamma_2^{-n+1}}(g\tau_{Q_1}) = \delta(\tau_2', g\tau_1) = \delta(g\tau_1, \tau_2),$$

where $\delta(\tau, \tilde{\tau})$ denotes the signed intersection number of the geodesics connecting τ, τ' and $\tilde{\tau}, \tilde{\tau}'$. We now obtain the desired expression for $\Theta(\tau_1, \tau_2)$. ☐

Since there are no primes dividing units, the expression in (2.33) has the same factorisation as $\Theta(\tau_1, \tau_2)$. Using this expression we are able to prove our observation about the primes dividing $\Theta(\tau_1, \tau_2)$. We will make use of the embeddings of the number fields $\mathbb{Q}(\tau_1)$ and $\mathbb{Q}(\tau_2)$ into the quaternion algebra $M_2(\mathbb{Q})$. Let $Q = aX^2 + bXY + cY^2$ be the binary quadratic forms associated to the real quadratic irrationality τ with discriminant $\Delta = b^2 - 4ac$ not a square. Then we define

$$\begin{aligned} \alpha_\tau : K = \mathbb{Q}(\tau) &\to M_2(\mathbb{Q}) \\ \sqrt{\Delta} &\mapsto \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix}. \end{aligned} \tag{2.34}$$

The matrices in the image of α_τ all have the eigenvectors $(\tau, 1)^T, (\tau', 1)^T$. The quaternion algebra $M_2(\mathbb{Q})$ has the involution $\dagger$ given by

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}^\dagger = \begin{pmatrix} u & -s \\ -t & r \end{pmatrix}$$

(also known as adjugation) satisfying $M + M^\dagger = \mathrm{Tr}(M)I$ and $MM^\dagger = M^\dagger M = \det(M)I$ for $M \in M_2(\mathbb{Q})$. Notice that the matrices invariant under $\dagger$ are exactly the multiples of the identity matrix. The characteristic polynomial of any $M \in M_2(\mathbb{Q})$ is given by $X^2 - (M + M^\dagger)X + (MM^\dagger)$. For example, the image e of $\sqrt{\Delta}$ under α satisfies $e + e^\dagger = 0$ and $ee^\dagger = -\Delta$, and so e indeed satisfies $X^2 - \Delta$. (Note that we have abused language and identified the rational number Δ with its image $\Delta \cdot I_2$, as we will often do in the proof of the following theorem.) Lastly, we note that the image of the order $\mathbb{Z}[\frac{\Delta + \sqrt{\Delta}}{2}]$ is contained in $M_2(\mathbb{Z})$.

Theorem 2.35. *Let τ_1, τ_2 be two quadratic irrationalities in the number fields K_1, K_2. Let Q_1, Q_2 be the binary quadratic forms associated to τ_1, τ_2 with discriminants Δ_1, Δ_2. Write $L = K_1K_2$. Suppose $\mathfrak{p}$ is a prime of L dividing $\Theta(\tau_1, \tau_2)$. Let p be the rational prime contained in $\mathfrak{p}$. Then p divides a positive integer of the form*

$$\frac{\Delta_1\Delta_2 - x^2}{4}.$$

Proof. This proof is inspired by [GKZ87, Section I.3]. By Proposition 2.32, we find that $\mathfrak{p}$ divides $\det[g\binom{\tau_1}{1}, \binom{\tau_2}{1}]$ for some $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $\delta(g\tau_1, \tau_2) \neq 0$. Therefore the vectors $g\binom{\tau_1}{1}, \binom{\tau_2}{1}$ are linearly dependent mod $\mathfrak{p}$ and hence $g\tau_1 \equiv \tau_2 \bmod \mathfrak{p}$. Without loss of generality, we may assume that.

We define $\alpha_1 = \alpha_{\tau_1}, \alpha_2 = \alpha_{\tau_2}$ and $e_i = \alpha_i(\sqrt{\Delta_i}), E_i = \alpha_i(\frac{\Delta_i + \sqrt{\Delta_i}}{2})$ for $i \in \{1, 2\}$. Also let $Q_1 = a_1X^2 + b_1XY + c_1Y^2, Q_2 = a_2X^2 + b_2XY + c_2Y^2$ be the binary quadratic forms associated to τ_1, τ_2. We have $(e_1e_2 + e_2e_1)^\dagger = e_1e_2 + e_2e_1$ and hence we see that $e_1e_2 + e_2e_1$ is the image of integer, explicitly:

$$e_1e_2 + e_2e_1 = (2b_1b_2 - 4a_1c_2 - 4a_2c_1) \cdot I_2.$$

Let $x = b_1b_2 - 2a_1c_2 - 2a_2c_1$. Then we can also compute

$$(E_1E_2 - E_2E_1)^2 = \left(\frac{e_1e_2 - e_2e_1}{4}\right)^2 = \left(\frac{(e_1e_2 + e_2e_1)^2 - 2(e_1e_2e_2e_1 + e_2e_1e_1e_2)}{16}\right) = \frac{x^2 - \Delta_1\Delta_2}{4}.$$

Note that this is an integer because $E_1E_2 - E_2E_1 \in M_2(\mathbb{Z})$. Since $(\tau_1, 1)^T, (\tau_2, 1)^T$ are linearly dependent mod $\mathfrak{p}$, they are eigenvectors of both E_1, E_2 mod $\mathfrak{p}$. Therefore $E_1E_2 - E_2E_1$ mod $\mathfrak{p}$ has eigenvalue 0 corresponding to this eigenvector. We therefore see $\frac{1}{4}(\Delta_1\Delta_2 - x^2) = \det(E_1E_2 - E_2E_1) \equiv 0 \bmod \mathfrak{p}$.

We will now prove that this integer is positive. We know that $\delta(\tau_1, \tau_2) \neq 0$, which is equivalent to the statement that the cross ratio

$$\frac{\tau_1 - \tau_2}{\tau_1' - \tau_2} \frac{\tau_1' - \tau_2'}{\tau_1 - \tau_2'}$$

is negative. This cross ratio has the same sign as the norm of $\tau_1 - \tau_2$:

$$N_{K_1K_2/\mathbb{Q}}(\tau_1 - \tau_2) = \frac{a_1^2c_2^2 + a_2^2c_1^2 - a_1b_1b_2c_2 - a_2b_1b_2c_1 + a_1b_2^2c_1 + a_2b_1^2c_2 - 2a_1a_2c_1c_2}{a_1^2a_2^2},$$

which is equal to $\frac{1}{4}(x^2 - \Delta_1\Delta_2)$. Therefore $\frac{1}{4}(\Delta_1\Delta_2 - x^2)$ is positive. □

With notation as in the proof above, when p is inert in K_1 and K_2, then $p^k \mid \tau_1 - \tau_2$ is equivalent to the statementthat the minimal polynomials of τ_1, τ_2 are equal $\bmod p^k$. In that case $E_1E_2 - E_2E_1 \equiv 0 \bmod p^k$. The converse is also true and hence the multiplicity of p in $\tau_1 - \tau_2$ is the same as in $\det(E_1E_2 - E_2E_1)$. Defining $[\tau_1 \cdot \tau_2]_p = \mathrm{ord}_p(E_1E_2 - E_2E_1)$, one then finds the following exact formula for the multiplicity of p dividing $\Theta(\tau_1, \tau_2)$.

Proposition 2.36. *(Proposition 2.7 from [DV21]) Let p be a rational prime that is inert in K_1 and K_2. Then*

$$\mathrm{ord}_p(\Theta(\tau_1, \tau_2)) = \sum_{g \in \gamma_2^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} [g\tau_1 \cdot \tau_2]_p \cdot \delta(g\tau_1, \tau_2).$$

Using the explicit expression found in Proposition 2.32, we are also able to find the following.

Corollary 2.37. *(Proposition 2.5 from [DV21]) Let $\mathcal{O}_1, \mathcal{O}_2$ be the rings of integers of K_1, K_2 respectively. Then we have*

$$\Theta(Q_1, Q_2) \equiv \Theta(Q_2, Q_1)^{-1} \bmod \mathcal{O}_1^\times \mathcal{O}_2^\times.$$

Proof. Replacing g with g^{-1} in the product in (2.33) we get

$$\begin{aligned}\Theta(Q_1, Q_2) &\equiv \prod_{g \in \gamma_{Q_2}^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_{Q_1}^{\mathbb{Z}}} \det\left[g\begin{pmatrix}\tau_{Q_1}\\1\end{pmatrix}, \begin{pmatrix}\tau_{Q_2}\\1\end{pmatrix}\right]^{\delta(g\tau_{Q_1}, \tau_{Q_2})} \\ &\equiv \prod_{g \in \gamma_{Q_1}^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_{Q_2}^{\mathbb{Z}}} \det\left[\begin{pmatrix}\tau_{Q_1}\\1\end{pmatrix}, g\begin{pmatrix}\tau_{Q_2}\\1\end{pmatrix}\right]^{\delta(\tau_{Q_1}, g\tau_{Q_2})} \equiv \Theta(Q_2, Q_1)^{-1}.\end{aligned}$$

□

In any example computed using Sage, the relation of Corollary 2.37 also holds without the projection modulo units. The relation

$$\Theta(\tau_1, \tau_2) \times \Theta(\tau_2, \tau_1) = 1$$

is verified to hold for all pairs of distinct τ_1, τ_2 with corresponding discriminants lower than 160. These are 2211 relations, with some duplicates. Therefore we propose the following.

Conjecture. *The map Θ is antisymmetric.*

To prove this conjecture, one would likely need an explicit expression for Θ, and hence also an explicit expression for the multiplicative Knopp cocycle. Some methods related to Chapter 4 show promise and therefore the author hopes to prove this conjecture soon.

Remark 2.38. After finishing this thesis, the author has been made aware of other methods to prove this Corollary, covered in the PhD thesis of Sören Sprehe [Spr25].

Chapter 3

Cycle integrals of the j-function

We now switch to looking at an analytic attempt to generalising singular moduli to real quadratic fields: *cycle integrals*. Cycle integrals of modular forms have for example already appeared in [GKZ87]. Cycle integrals of the j-function have gained attention in the last decades because of the observations made by Kaneko [Kan09] and their interesting similarities to CM singular moduli when gathered in traces, which were mostly proved by Duke, Imamoḡlu, Tóth in [DIT11]. We will discuss these observations and results and define a possible real quadratic analogue of differences of singular moduli.

3.1 Evaluating the j-function in real quadratics

For convenience we recall the definition of 'evaluating' a modular function $f \in \mathbb{C}[j]$ in a real quadratic irrationality.

Definition 1.1. Let τ be a real quadratic irrationality and let $f \in \mathbb{C}[j]$ be a modular function that is holomorphic on $\mathcal{H}$. Let $Q(z) = az^2 + bz + c$ be the unique primitive quadratic polynomial with discriminant $\Delta = b^2 - 4ac$ such that τ is its *principal* root, i.e.

$$\tau = \frac{-b + \sqrt{\Delta}}{2a}.$$

Furthermore let γ_τ be the automorph of τ. Then we define the *value* of f in τ to be

$$f[\tau] := \int_{z_0}^{\gamma_\tau z_0} f(z) \frac{\sqrt{\Delta} dz}{Q(z)}$$

for any z_0 on the geodesic in $\mathcal{H}$ connecting τ and τ'.

We show that this definition is independent of z_0. First, we note that the hyperbolic arc-length $\frac{dz}{Q(z)}$ transforms as

$$\frac{d(\gamma z)}{Q(\gamma z)} = \frac{dz}{(Q \cdot \gamma)(z)}, \tag{3.1}$$

where $(Q \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(z) = (cz + d)^2 Q((az + b)/(cz + d))$, which is equal to the corresponding action on binary quadratic forms. Therefore, for $z_0, z_0' \in \mathcal{H}$, we have

$$\int_{z_0}^{\gamma_Q z_0} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} - \int_{z_0'}^{\gamma_Q z_0'} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} = \int_{z_0}^{z_0'} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} - \int_{\gamma_\tau z_0}^{\gamma_\tau z_0'} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} = 0,$$

where the last equality can be seen using the coordinate transformation $z \to \gamma_\tau^{-1} z$. (Note that $\gamma_\tau preserves Q$.) We see that Definition 1.1 is indeed independent of the choice of z_0. It can even be chosen to be any element of the upper half-plane. For convenience we will write C_Q for any path from z_0 to $\gamma_\tau z_0$.

Kaneko gives a slightly different definition of $j[\tau]$, dividing by $\int_{C_Q} \frac{\sqrt{\Delta} dz}{Q(z)} = 2 \log \varepsilon$. We will denote this normalised value as $j^{nor}(Q)$, as in [BI19]. Kaneko proves some simple relations and then studies the values of j numerically.

Proposition 3.2. *[Kan09] Let τ be a real quadratic irrationality with associated primitive polynomial Q and conjugate τ'. Then we have for $f \in \mathbb{C}[j]$*

(i) $f[\tau] = f[\sigma\tau]$ *for all* $\sigma \in \mathrm{SL}_2(\mathbb{Z})$*;*

(ii) $f[\tau] = f[\tau']$*;*

(iii) $f[-\tau] = \overline{f[\tau]}$.

Proof. *(i)* Let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$. Let γ_τ be the automorph of τ. The primitive polynomial associated to $\sigma\tau$ is equal to $Q \cdot \sigma^{-1}$. We make the coordinate transformation $z = \sigma^{-1} w$. By (3.1) we have

$$\int_{z_0}^{\gamma_\tau z_0} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} = \int_{\sigma^{-1} z_0}^{(\sigma\gamma_\tau\sigma^{-1})\sigma z_0} f(\sigma w) \frac{\sqrt{\Delta} dw}{(Q \cdot \sigma^{-1})(w)} = \int_{C_{Q\cdot\sigma^{-1}}} f(w) \frac{\sqrt{\Delta} dw}{(Q \cdot \sigma^{-1})(w)}.$$

(ii) If γ_τ is the automorph of τ, then τ' has automorph γ_τ^{-1}. The primitive polynomial associated to τ' is $-Q$. By the independence of the starting point we have that C_Q is inverse to C_{-Q}. Therefore we have

$$f(\tau) = \int_{C_Q} f(z) \frac{\sqrt{\Delta} dz}{Q(z)} = \int_{C_{-Q}} f(z) \frac{\sqrt{\Delta} dz}{-Q(z)} = f(\tau').$$

(iii) If $Q = a(z - \tau)(z - \tau')$, then the primitive polynomial associated to $-\tau$ is equal to $w(Q) := -a(z + \tau)(z + \tau')$. There is an equality $w(Q)(-\overline{z}) = -\overline{Q(z)}$. Let $r : [0, 1] \to C_Q$ be a parametrisation of C_Q. Then $-\overline{r}$ is a parametrisation of $C_{w(Q)}$. Therefore we have

$$f(\tau') = \int_0^1 \frac{f\big(-\overline{r(t)}\big)\sqrt{\Delta}}{w(Q)\big(-\overline{r(t)}\big)} \cdot -\overline{r'(t)} dt = \int_0^1 \frac{\overline{f(r(t))}\sqrt{\Delta}}{-\overline{Q(r(t))}} \cdot -\overline{r'(t)} dt = \overline{f(\tau)}.$$ □

Alongside stating the above proposition, Kaneko conjectures that the real parts of $j^{nor}(\tau)$ lie in the interval $[706.324\ldots, 744]$, with the lower bound being $706.324\ldots = j^{nor}(\frac{1+\sqrt{5}}{2})$, and that the imaginary parts lie in $(-1, 1)$. He also notices that the real part of $j^{nor}(\tau)$ is larger when τ can be closely approximated by a rational number. Therefore the values of $j^{nor}(\tau)$ seem to be especially interesting at Markoff irrationalities, which are real quadratic irrationalities that cannot be approximated well by rationals. Bengoechea and Imamoğlu investigate further in [BI19] [BI20] [Ben21] and [Ben22], where they prove parts of the conjectures of Kaneko. There is also more known about the traces of cycle integrals, see [DIT10b] [Mas12] [DFI11].

Although these are interesting observations and theorems, they do not seem to be of use in generating singular moduli for algebraic applications to Hilbert's 12th problem or the Birch and Swinnerton-Dyer conjecture, since no algebraic values have yet been recognised in them, despite reported vigorous efforts to do so by several authors mentioned above.

3.2 Differences of cycle integrals

In our context of trying to generalise differences of singular moduli, it becomes natural to study differences of values of the j-function in two real quadratics:

$$j[\tau_1] - j[\tau_2] = \int_{C_{Q_1}} j(z)\frac{\sqrt{\Delta_1}dz}{Q_1(z)} - \int_{C_{Q_2}} j(z)\frac{\sqrt{\Delta_2}dz}{Q_2(z)}. \tag{3.3}$$

Just like the differences of CM-singular moduli, they are invariant under the action on either argument. This means that it induces a map on the narrow class groups.

Definition 3.4. Let $\Delta_1, \Delta_2 > 0$ be two square-free discriminants, with associated orders $\mathcal{O}_1, \mathcal{O}_2$. We define the map

$$\begin{aligned} \Phi : \mathrm{Pic}^+(\mathcal{O}_1) \times \mathrm{Pic}^+(\mathcal{O}_2) &\to \mathbb{C} \\ ([Q_1], [Q_2]) &\mapsto j[\tau_{Q_1}] - j[\tau_{Q_2}] \end{aligned}$$

If Q_1, Q_2 have roots τ_1, τ_2, we will also write $\Phi(\tau_1, \tau_2)$ for $\Phi([Q_1], [Q_2])$.

By Proposition 2.25 and Proposition 3.2, the functions $\log N_{K_1K_2/\mathbb{Q}}(\Theta)$ and $\mathrm{Im}\,\Phi$ are both antisymmetric, invariant under $\mathrm{SL}_2(\mathbb{Z})$, $\tau \mapsto \tau'$, and they both flip sign when τ_1, τ_2 both flip sign. Moreover, Φ is also additive: for τ_1, τ_2, τ_3 three real quadratic irrationalities, we have

$$\Phi(\tau_1, \tau_2) + \Phi(\tau_2, \tau_3) = \Phi(\tau_1, \tau_3).$$

However, Θ is not additive.

Example 3.5. Consider the smallest pair of distinct discriminants $(\Delta_1, \Delta_2) = (5, 8)$, which both have class number 1. For any τ_5, τ_8 of discriminant $5, 8$ we have

$$\Phi(\tau_5, \tau_8) = -1143.15596424359.$$

Example 3.6. The smallest discriminant with a non-trivial class group is 12, which has class number 2. However, the relations of Proposition 3.2 still ensure that the values of Φ in any pair τ_5, τ_{12} of real quadratic irrationalities with discriminants $5, 12$ is the same:

$$\Phi(\tau_5, \tau_{12}) = -509.965894309611.$$

Example 3.7. For the pair of discriminants $(\Delta_1, \Delta_2) = (12, 148)$ with class numbers $2, 3$ there are two different values of Φ.

$$\begin{aligned} \Phi(\sqrt{3}, \tfrac{-3+\sqrt{37}}{4}) &= -5222.69226673213, \\ \Phi(\sqrt{3}, \sqrt{37}) &= -5354.17205090473. \end{aligned}$$

Example 3.8. For the pair of discriminants $(\Delta_1, \Delta_2) = (148, 316)$ of class numbers $3, 6$ we have up to conjugation four different values of Φ:

$$\begin{aligned} \Phi(\tfrac{-3+\sqrt{37}}{7}, \tfrac{-3+\sqrt{79}}{10}) &= -141.46027203105 - 3.303461418083i, \\ \Phi(\tfrac{-3+\sqrt{37}}{7}, \tfrac{-7+\sqrt{79}}{2}) &= -255.7402879002, \\ \Phi(\sqrt{37}, \tfrac{-3+\sqrt{79}}{10}) &= -9.98048785845 - 3.303461418083i, \\ \Phi(\sqrt{37}, \tfrac{-7+\sqrt{79}}{2}) &= -124.2605037276. \end{aligned}$$

We have not recognised any algebraic values in Φ.

3.3 Traces of singular moduli

The most important aspect of singular moduli is the fact that they are algebraic. Even though cycle integrals of the j-function have not been recognised to have this property, they do share many similarities with CM singular moduli when gathered in traces.

After Zagier's joint work with Gross on norms of differences of singular moduli, he also published a paper about their traces [Zag02]. He showed that they appear as the Fourier coefficients of a modular form of weight 3/2, and gave several relations between these coefficients. In [DIT11], Duke, Imamoğlu and Tóth extend these results to the real quadratic setting by introducing cycle integrals. They provide a natural real quadratic analogue of singular moduli. We will give an overview of the results of Zagier and Duke, Imamoğlu and Tóth and how they relate.

For d a discriminant, let $\mathcal{Q}_d$ be the set of binary quadratic forms of discriminant d. The set of binary quadratic forms is acted upon by $\mathrm{PSL}_2(\mathbb{Z})$ via $Q(X,Y)\cdot\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = Q(aX+bY, cX+dY)$ and the stabiliser of $Q \in \mathcal{Q}_d$ is denoted Γ_Q. The number of equivalence classes of primitive forms in $\mathcal{Q}_d$ is denoted $h(d)$ and also called the (narrow) class number of d. If $d<0$, then $j(\tau_Q)$ is an algebraic integer of degree $h(d)$. Its conjugates correspond to the values of j in the other primitive equivalence classes. Therefore, the trace of $j(\tau_Q)$ is equal to

$$\sum_{\substack{q\in\mathcal{Q}_d/\,\mathrm{SL}_2(\mathbb{Z})\\ q \text{ primitive}}} j(\tau_q).$$

Zagier considers a slight adjustment of these traces, namely the values

$$\mathrm{Tr}_d(j_1) := \sum_{Q\in\mathcal{Q}_d/\,\mathrm{SL}_2(\mathbb{Z})} |\Gamma_Q|^{-1} j_1(\tau_Q), \qquad \text{where } j_1 = j - 744.$$

The sum is taken over all equivalence classes, not just the primitive ones. He shows that they appear as coefficients of the weakly holomorphic form of weight 3/2

$$\mathbf{T}_- := -\theta_1(z)\frac{E_4(4z)}{\eta(4z)^6} = q^{-1} - 2 + \sum_{0>d\equiv 0,1 \bmod 4} \mathrm{Tr}_d(j_1) q^{|d|} \in M^!_{3/2},$$

where θ_1, E_4, η are given by

$$\theta_1(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \qquad E_4(z) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \qquad \eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Let $M^!_{k+1/2}$ be the space of weakly holomorphic forms of weight $k+1/2$ for $\Gamma_0(4)$ whose n-th coefficient vanish unless $(-1)^k n \equiv 0,1 \bmod 4$. Zagier shows that $M^!_{3/2}$ has a unique basis given by $\{g_d : 0<d\equiv 0,1 \bmod 4\}$ such that g_d has Fourier expansion

$$g_d = -q^{-d} + \sum_{0\geqslant n\equiv 0,1 \bmod 4} a(d,n) q^{|n|}.$$

The form $\mathbf{T}_-$ is the first element of this basis. The coefficients $a(d,n)$ of these modular forms also appear in the basis $\{f_d : d\le 0\}$ of $M^!_{1/2}$ as described by Borcherds [Bor95]. The f_d are given by

$$f_d = q^d + \sum_{n>0} a(n,d) q^n. \tag{3.9}$$

In [DIT11], it is shown that this basis extends naturally to a basis for the space $\mathbb{M}_{1/2} \supset M^!_{1/2}$ of *mock modular forms* of weight 1/2. Mock modular forms of weight 1/2, first understood by Zwegers [Zwe08][Zag09], are functions that can be completed to be modular by adding a certain non-modular function g^*, known as the *shadow* of the mock modular form. This shadow is associated to a weakly holomorphic form g of weight 3/2. Duke, Imamoğlu and Tóth show that for a positive discriminant d, there is a unique mock modular form f_d with shadow g_d. Define $a(n,d)$ for positive discriminants d such that

$$f_d = \sum_{n>0} a(n,d)q^n. \tag{3.10}$$

Then the Fourier coefficients satisfy $a(n,d) = a(d,n)$. The set $\{f_d : d \text{ any discriminant}\}$ forms a basis for $\mathbb{M}_{1/2}$. The element f_1 with shadow $g_1 = \mathbf{T}_-$ is equal to

$$\mathbf{T}_+ := \sum_{d>0} \mathrm{Tr}_d(j_1)q^d \in \mathbb{M}_{1/2},$$

with $\mathrm{Tr}_d(j_1)$ defined as

$$\mathrm{Tr}_d(j_1) := \frac{1}{2\pi} \sum_{Q\in\mathcal{Q}_d/\,\mathrm{SL}_2(\mathbb{Z})} \int_{C_Q} j_1(z)\tfrac{dz}{Q(z,1)}$$

for $d > 0$ not a square.

With this extended definition of the $a(n,d)$, Duke, Imamoğlu and Tóth also generalise a well known result of Zagier [Zag02, (25)] to include positive discriminants. This result involves certain 'twisted' traces. Let d be a negative or positive discriminant, and let $D > 0$ be a fundamental discriminant. If dD is not a square, define the twisted trace

$$\mathrm{Tr}_{d,D}(j_m) = \begin{cases} \frac{1}{\sqrt{D}} \sum \chi(Q)|\Gamma_Q|^{-1} j_m(\tau_Q) & \text{if } dD < 0, \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_m(z)\frac{dz}{Q(z,1)} & \text{if } dD > 0, \end{cases}$$

where the sums are taken over $Q \in \mathcal{Q}_d/\,\mathrm{SL}_2(\mathbb{Z})$, j_m is the unique function in $\mathbb{C}[j]$ with q-expansion $q^{-m} + O(q)$, and $\chi : \mathcal{Q}_{dD} \to \{-1,1\}$ is a certain function that restricts to a group character on the primitive classes. Then it is shown for $m \geqslant 1$ that

$$\mathrm{Tr}_{d,D}(j_m) = \sum_{n|m} \left(\frac{D}{m/n}\right) na(n^2D, d). \tag{3.11}$$

In particular for $m = 1$, this gives

$$a(D,d) = \mathrm{Tr}_{d,D}(j_1) = \begin{cases} \frac{1}{\sqrt{D}} \sum \chi(Q)|\Gamma_Q|^{-1} j_1(\tau_Q) & \text{if } dD < 0, \\ \frac{1}{2\pi} \sum \chi(Q) \int_{C_Q} j_1(z)\frac{dz}{Q(z,)} & \text{if } dD > 0. \end{cases}$$

In the case $m = 0$, Duke, Imamoğlu and Tóth consider the function

$$\mathbf{Z}_+ = \sum_{d>0} \mathrm{Tr}_d(1)q^d,$$

where $\mathrm{Tr}_d(1) = \pi^{-1}d^{-1/2}h(d)\log\varepsilon_d$. It relates to a function studied by Zagier in [Zag75b]

$$\mathbf{Z}_- = \sum_{d\leq 0} \mathrm{Tr}_d(1)q^{|d|},$$

where $\operatorname{Tr}_d(1) = H(|d|)$ for $d \leq 0$ is the Hurwitz class number. Zagier showed that this function can be completed to be modular of weight 3/2. Duke, Imamoğlu and Tóth give a similar completion of $\mathbf{Z}_+$, which has weight 1/2 for $\Gamma_0(4)$. They state that the automorphic nature of this completion gives reason to hope that there might be a connection between the cycle integrals of j and abelian extensions of real quadratic fields.

Lastly in [DIT11], Duke, Imamoğlu and Tóth give a real quadratic analogue of a result of Borcherds [Bor95, Theorem 14.1]. For all negative discriminants, Borcherds associates to $f_d = q^d + \sum_{n>0} a(n,d)q^n$ the infinite product

$$\Psi_d = q^{-\operatorname{Tr}_d(1)} \prod_{m>0} (1-q^m)^{a(m^2,d)}.$$

It is a meromorphic modular form of weight 0 whose zeros and poles are either cusps or imaginary quadratic irrationalities. Conversely, every such modular form also has an associated weakly modular form of weight 1/2. This can be used to give a product formula for the modular polynomial $\prod(j - j(\tau_Q))$, with the product running over $\mathrm{SL}_2(\mathbb{Z})$-representatives of primitive binary quadratic forms. The (scaled) logarithmic derivative of this infinite product is equal to

$$F_d = -\operatorname{Tr}_d(1) - \sum_{m>0} \Big(\sum_{n|m} n a(n^2,d)\Big) q^m \tag{3.12}$$

It is a meromorphic form of weight 2 with corresponding properties. Duke, Imamoğlu and Tóth consider these functions also for $d > 0$, and then the F_d satisfy a transformation property involving traces of additive Knopp cocycles. We will explore this further in Chapter 4.

Chapter 4

An analytic expression of the Knopp cocycle

In the previous two chapters we have discussed two attempts at generalising the theory of singular moduli to real quadratic fields. In this chapter, we show a connection between these two attempts. This connection is based on the appearance of the Knopp cocycle in the transformation formula for a function related to Borcherds products.

4.1 Real quadratic Borcherds products

We will start by introducing regular Borcherds products, which are inherently imaginary quadratic in nature. Afterwards, we will try to generalise these Borcherds products to also be 'real quadratic'. Recall that $M^!_{1/2}$ is the space of nearly holomorphic moudular forms $f = \sum a_n q^n$ of weight $1/2$ for $\Gamma_0(4)$ whose integer coefficients vanish unless $n \equiv 0, 1 \bmod 4$. Also define $M'_\times$ to be the multiplicative group of meromorphic modular forms of integral weight for some character of $\mathrm{SL}_2(\mathbb{Z})$, whose zeros and poles are either cusps or imaginary quadratic irrationals.

Theorem 4.1. *(Theorem 14.1 of [Bor95]) There is an isomorphism*

$$\begin{aligned} M^!_{1/2} &\longrightarrow M'_\times \\ f = \sum_{n \gg -\infty} a(n) q^n &\longmapsto \Psi_f = q^{-h} \prod_{n>0} (1 - q^n)^{a(n^2)}, \end{aligned} \tag{4.2}$$

where h is the constant term of $f \sum_n H(n) q^n$ for $H(n) = \sum_{Q \in \mathcal{Q}_{-n}/\,\mathrm{PSL}_2(\mathbb{Z})} \frac{1}{\#\,\mathrm{PSL}_2(\mathbb{Z})_Q}$ the Hurwitz class number of discriminant $-n$. The weight of Ψ_f is equal to $a(0)$ and the multiplicity of a zero of Ψ_f at an imaginary quadratic irrationality τ of discriminant $D < 0$ is $\sum_{d>0} a(Dd^2)$.

A product formula for a modular form as in (4.2) is called a *Borcherds product.* We recall from the previous chapter that Borcherds [Bor95], and later Zagier [Zag02] proved, that $M^!_{1/2}$ admits a basis $\{f_d : d \leq 0 \text{ a discriminant}\}$, where f_d is the unique modular form in $M^!_{1/2}$ having Fourier expansion such that

$$f_d = q^d + \sum_{n>0} a(n,d) q^n.$$

Looking at $d = 0$, we find that f_0 is equal to the θ-function

$$\theta = 1 + 2\sum_{n>0} q^{n^2}.$$

Applying Theorem 4.1 to $f_0 = \theta$ one finds

$$\Psi_\theta = q^{1/12}\prod_{n>0}(1-q^n)^2 = \Delta^{1/12}.$$

The coboundary of $\Delta^{1/12}$ is equal to the cocycle $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \mapsto \Delta(\gamma z)^{1/12}/\Delta(z)^{1/12} = cz + d$ from equation (2.8). The logarithmic derivative of $\Delta^{1/12}$ is equal to $\frac{1}{12}E_2$ when scaled by $(2\pi i)^{-1}$, which satisfies

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \star_2 \tfrac{1}{12}E_2\right)(z) = E_2(z) + \frac{1}{2\pi i}\frac{c}{cz+d},$$

which is the cocycle from (2.7). For $d < 0$, the modular form Ψ_{f_d} has weight 0 and hence its coboundary is trivial.

Also recall that Duke, Imamoḡlu and Tóth extended the basis $\{f_d : d < 0 \text{ a discriminant}\}$ of $M^!_{1/2}$ to a basis $\{f_d : d \text{ any discriminant}\}$ of the space $\mathbb{M}_{1/2}$ of mock modular forms of weight 1/2, where the f_d are as in (3.10). A natural question is now what happens when we apply the map $f \mapsto \Psi_f$ defined in (4.2) to f_d when d is a positive discriminant. We then obtain the function

$$\Psi_d := \Psi_{f_d} = q^{-\operatorname{Tr}_d(1)}\prod_{n>0}(1-q^n)^{-a(n^2,d)}. \tag{4.3}$$

Note that $\operatorname{Tr}_d(1)$ is not a rational power. We will prove that Ψ_d satisfies the transformation formula

$$\frac{\Psi_d(\gamma^{-1}z)}{\Psi_d(z)} = \prod_{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})} \operatorname{Kn}_{\tau_Q}(\gamma)(z)^{2i/\sqrt{d}} = \prod_{Q\in\mathcal{Q}_d}(z-\tau_Q)^{\delta_{\infty,\gamma\infty}(\tau_Q)\cdot 2i/\sqrt{d}}. \tag{4.4}$$

We will prove this by looking at the (scaled) logarithmic derivative

$$F_d := \frac{1}{2\pi i}\operatorname{dlog}(\Psi_d) = -\operatorname{Tr}_d(1) - \sum_{m\geqslant 1}\Big(\sum_{n|m} na(n^2,d)\Big)q^m = -\sum_{m\geqslant 0}\operatorname{Tr}_d(j_m)q^m \tag{4.5}$$

of Ψ_d, where the last equality follows from (3.11). For $d = 0$, we have seen that this gives $\frac{1}{12}E_2$. For $d > 0$, it is shown in [DIT11] that F_d satisfies

$$(\gamma - 1)\star_2 F_d(z) = \frac{1}{2\pi\sqrt{d}}\sum_{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})}\operatorname{kn}_{\tau_Q}(\gamma).$$

Definition 4.6. Let φ be a parabolic cocycle for $\operatorname{SL}_2(\mathbb{Z})$ taking values in $\mathcal{M}_2$. A *modular integral* F of φ is a holomorphic function on $\mathcal{H}^*$ satisfying

$$(\gamma - 1)\star_2 F = \varphi(\gamma)$$

for all $\gamma \in \operatorname{SL}_2(\mathbb{Z})$. Conversely, φ is called the *period* of F.

Since $M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, a modular integral is unique if it exists. In the outgrowth paper [DIT10a] of [DIT11] Duke Imamoğlu and Tóth also give a (new) construction for the modular integral F_τ of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$. Its existence was already proved by Knopp [Kno78] using certain Poincaré series.

The symmetrised multiplicative Knopp cocycle $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$ then also appears as the period of the lift Ψ_τ of F_τ under the logarithmic derivative

$$\frac{\Psi_\tau(\gamma^{-1}z)}{\Psi_\tau(z)} = (\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'})(\gamma)(z). \tag{4.7}$$

Closely following the construction of Duke, Imamoğlu and Tóth we are able to give a closed from expression for F_τ, which we can apply to (4.7) to obtain an explicit expression for $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$. We can also use this to obtain an explicit expression for Θ.

4.2 The modular integral of the symmetrised additive Knopp cocycle

Let τ be a real quadratic irrationality. We will follow the construction of [DIT11] [DIT10a] of the modular integral of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$. We will give an explicit expression for this modular integral.

Theorem 4.8. *(A specific case of [DIT10a, Theorem 3]) Let τ be a real quadratic irrationality with the associated binary quadratic form Q with discriminant $\Delta > 0$. Choose C_Q such that it lies on the geodesic connecting τ' to τ. Then $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$ has a modular integral given by*

$$F_\tau(z) = -\int_{C_Q} \frac{j'(z)}{j(u) - j(z)} \frac{\sqrt{\Delta}du}{Q(u)} + \sum_{q \sim Q} \delta_{\infty,z}(\tau_q) \frac{\sqrt{\Delta}}{q(z)}, \tag{4.9}$$

outside the set $\mathrm{SL}_2(\mathbb{Z})C_Q$. At ∞, it has q-expansion

$$F_\tau(z) = -\sum_{m=0}^{\infty} j_m[\tau] q^m, \qquad \text{where} \qquad j_m[\tau] = \int_{C_Q} j_m(u) \frac{\sqrt{\Delta}du}{Q(u)}. \tag{4.10}$$

The cycle integral in (4.9) is analytic outside $\mathrm{SL}_2(\mathbb{Z})C_Q$, with jumps along $\mathrm{SL}_2(\mathbb{Z})C_Q$. The second term is added to get rid of these jumps. Before proving this theorem, we state some lemmas.

Lemma 4.11. *(Specific case of [AKN97, Corollary 4]) Let $j' = \frac{1}{2\pi i}\frac{dj}{dz} = \frac{E_{14}}{\Delta}$. Then for fixed $s \in \mathcal{H}$ we have*

$$\frac{j'(z)}{j(u) - j(z)} = \sum_{m \geqslant 0} j_m(u) q(z)^m.$$

Note that this function is equal to the logarithmic derivative of $j(u) - j(z)$ with respect to z, scaled by $-(2\pi i)^{-1}$.

Lemma 4.12. *For fixed z not a zero of j' and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,*

$$\mathrm{res}_{u=\gamma z} \left(\frac{j'(z)}{j(u) - j(z)} \frac{\sqrt{\Delta}}{Q(u)} \right) = \frac{1}{2\pi i} \frac{\sqrt{\Delta}}{(Q \cdot \gamma)(z)}$$

Proof. We calculate

$$\operatorname{res}_{u=\gamma z}\left(\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta}}{Q(u)}\right) = \lim_{u\to\gamma z}\frac{(u-\gamma z)j'(z)\sqrt{\Delta}}{(j(u)-j(z))Q(u)} = \lim_{u\to\gamma z}\frac{j'(z)\sqrt{\Delta}}{\frac{dj}{du}(u)Q(u)} = \frac{1}{2\pi i}\frac{j'(z)}{j'(\gamma z)}\frac{\sqrt{\Delta}}{Q(\gamma z)}$$
$$= \frac{1}{2\pi i}\frac{1}{(cz+d)^2}\frac{\sqrt{\Delta}}{Q(\gamma z)},$$

which is equal to the desired expression. □

Proof. For a given $z \in \mathcal{H}$, the function

$$u \mapsto -\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta}}{Q(u)}$$

is meromorphic on $\mathcal{H}$, with poles at the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z. Therefore the function

$$z \mapsto -\int_{C_Q}\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta}du}{Q(u)}$$

is holomorphic on the set $\mathcal{H}^* \setminus \mathrm{SL}_2(\mathbb{Z})C_Q$. The function F_τ is obtained by taking the analytic continuation at the cusp. □

Proof of Theorem 4.8. For a given $z \in \mathcal{H}$, the function

$$u \mapsto -\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta}}{Q(u)}$$

is meromorphic on $\mathcal{H}$, with poles at the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z. Therefore the function

$$z \mapsto -\int_{C_Q}\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta}du}{Q(u)}$$

is holomorphic on the set $\mathcal{H}^* \setminus \mathrm{SL}_2(\mathbb{Z})C_Q$. The function F_τ is obtained by taking the analytic continuation at the cusp. By Lemma 4.11 we have

$$F_\tau(z) = -\int_{C_Q}\sum_{m\geqslant 0} j_m(u)q(z)^m\frac{\sqrt{\Delta}du}{Q(u)} = -\sum_{m\geqslant 0}\left(\int_{C_Q} j_m(u)\frac{\sqrt{\Delta}du}{Q(u)}\right)q(z)^m = -\sum_{m\geqslant 0} j_m[\tau]q^m$$

locally at ∞. Note that this q-expansion converges everywhere, but the swapping of the integral and the sum is only allowed sufficiently close to ∞, where $j'(z)/(j(u)-j(z))$ has no poles. We will first show that this analytic continuation is given by (4.9), and afterwards that this function has $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$ as its period.

To prove that F_τ is given by (4.9), assume first that $Q \not\sim -Q$. Let X_∞ be the connexted component of ∞ in $\mathcal{H}^* \setminus \mathrm{SL}_2(\mathbb{Z})C_Q$. Suppose $z \in \mathcal{H}$ lies in a connected component adjacent to X_∞. Then there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that the connected component of z is equal to γX_∞. Adjust the path C_Q to C_Q' such that z lies in the same connected component as ∞ in

$\mathcal{H}^* \setminus \mathrm{SL}_2(\mathbb{Z})C'_Q$. The path $C_Q - C'_Q$ is a loop that contains exactly one point in the orbit of z, namely γz, looping around it once with orientation $-\delta_{\infty,z}(\tau_{Q\cdot\gamma})$. Therefore we find

$$\begin{aligned} F_\tau(z) &= -\int_{C'_Q} \frac{j'(z)}{j(u)-j(z)} \frac{\sqrt{\Delta}du}{Q(u)} \\ &= -\int_{C_Q} \frac{j'(z)}{j(u)-j(z)} \frac{\sqrt{\Delta}du}{Q(u)} - 2\pi i \cdot \delta_{\infty,z}(\tau_{Q\cdot\gamma}) \cdot \mathrm{res}_{w=\gamma z} \frac{j'(z)}{j(w)-j(z)} \frac{\sqrt{\Delta}}{Q(w)} \\ &= -\int_{C_Q} \frac{j'(z)}{j(u)-j(z)} \frac{\sqrt{\Delta}du}{Q(u)} - \delta_{\infty,z}(\tau_{Q\cdot\gamma}) \cdot \frac{\sqrt{\Delta}}{(Q\cdot\gamma)(z)}, \end{aligned}$$

where we have used the Residue Theorem and Lemma 4.12. Note that $Q \cdot \gamma$ is the only binary quadratic form q in the orbit of Q such that $\delta_{\infty,z}(\tau_q) \neq 0$. Therefore $\frac{\delta_{\infty,z}(\tau_{Q\cdot\gamma})}{(Q\cdot\gamma)(z)} = \sum_{q\sim Q} \delta_{\infty,z}(\tau_q)\frac{\sqrt{\Delta}}{q(z)}$. For general $z \in \mathcal{H} \setminus \mathrm{SL}_2(\mathbb{Z})C_Q$, we may repeat this argument for every time a path from ∞ to z crosses $\mathrm{SL}_2(\mathbb{Z})C_Q$ to obtain (4.9) for all z.

If $Q \sim -Q$, then the geodesics connecting τ to τ' and τ' to τ overlap, and hence z crosses two geodesics simultaneously when crossing $\mathrm{SL}_2(\mathbb{Z})C_Q$. To fix this we can choose z_0 outside the geodesic connecting τ, τ', carry out the same argument and then let z_0 approach this geodesic.

We will now verify the transformation formula for F_τ. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \in \mathrm{SL}_2(\mathbb{Z})$, and let $z \in \mathcal{H} \setminus \mathrm{SL}_2(\mathbb{Z})C_Q$ be in the connected component of ∞. We compute

$$\begin{aligned} (\gamma \star_2 F_\tau)(z) &= \frac{1}{(cz+d)^2}\left(-\int_{C_Q} \frac{j'(\gamma^{-1}z)}{j(u)-j(\gamma^{-1}z)} \frac{\sqrt{\Delta}du}{Q(u)} - \sum_{q\sim Q} \delta_{\infty,\gamma^{-1}z}(\tau_q) \frac{\sqrt{\Delta}}{q(\gamma^{-1}z)}\right) \\ &= -\int_{C_Q} \frac{j'(z)}{j(u)-j(z)} \frac{\sqrt{\Delta}du}{Q(u)} - \sum_{q\sim Q} \delta_{\gamma\infty,z}(\tau_{q\cdot\gamma^{-1}}) \frac{\sqrt{\Delta}}{(q\cdot\gamma^{-1})(z)} \\ &= F_\tau(z) + \sum_{q\sim Q} \delta_{\infty,\gamma\infty}(\tau_q) \frac{\sqrt{\Delta}}{q(z)} \\ &= F_\tau(z) + \sum_{q\sim Q} \delta_{\infty,\gamma\infty}(\tau_q) \left(\frac{1}{z-\tau_q} - \frac{1}{z-\tau'_q}\right). \end{aligned}$$

Therefore the transformation property $(\gamma - 1) \star F_\tau = (\mathrm{kn}_\tau + \mathrm{kn}_{\tau'})(\gamma)$ holds in a neighbourhood of ∞. We conclude that it also holds on the entirety of $\mathcal{H}$ by analytic continuation. □

4.3 Explicit lift of the Knopp cocycle

Having constructed the modular integral of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$, and more importantly having found an explicit expression for it, will allow us to give an explicit expression for the symmetrised Knopp cocycle $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$. Evaluating this in another real quadratic irrationality will allow us to derive an explicit expression for a symmetrised Θ using analytic methods. This explicit expression of $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$ is obtained using the transformation formula of a lift of F under the logarithmic derivative. The following theorem is inspired by [DIT17, Theorem 2.1].

Theorem 4.13. *Let F_τ be the modular integral of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$. Then by definition F_τ is a holomorphic function on $\mathcal{H} \cup \{\infty\}$ satisfying*

$$F_\tau(\gamma^{-1}z) - F_\tau(z) = (\mathrm{kn}_\tau + \mathrm{kn}_{\tau'})(\gamma)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Let Ψ_τ be a lift of F_τ under the logarithmic derivative. Then Ψ_τ satisfies the transformation formula

$$\frac{\Psi_\tau(\gamma^{-1}z)}{\Psi_\tau(z)} = \exp \int_z^{\gamma^{-1}z} F_\tau(s)ds = (\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'})(\gamma)(z).$$

Proof. Let G_τ be an antiderivative of F_τ. Let $\Phi(\gamma)(z) = G_\tau(\gamma^{-1}z)/G_\tau(z)$ be the coboundary of G_τ. Then clearly Φ is a cocycle. It satisfies $\mathrm{dlog}(\Phi) = \mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$, and therefore Φ is equal to one of the twelve lifts of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$ under dlog. The theorem therefore follows if we show that $\Phi(T) > 0$.

Let $\tilde{F}$ be a primitive of F. By Theorem 4.8, $\tilde{F}(z)$ is equal to

$$c + j_0[\tau]z + \frac{1}{2\pi i}\sum_{m\geqslant 1}\frac{j_m[\tau]}{m}q^m$$

for some $c \in \mathbb{C}$ and $\mathrm{Im}\, z$ large enough. Up to a constant, G_τ is equal to $\exp \tilde{F}$. Note that Φ is independent of this constant. Using this we find $\Phi(T) = \exp j_0[\tau]$. Note that $j_0 = 1$. A simple calculation shows that $1[\tau] = \varepsilon^{-2}$, where ε is the fundamental unit of norm 1 in the order associated to τ with $\varepsilon > 0$. ☐

This theorem proves the transformation formula (4.4) by noting that

$$\begin{aligned} F_\Delta &= -\sum_{m\geqslant 0} \mathrm{Tr}_\Delta(j_m)q^m = -\frac{1}{2\pi\sqrt{\Delta}}\sum_{Q\in\mathcal{Q}_\Delta/\,\mathrm{SL}_2(\mathbb{Z})}\int_{C_Q}\sum_{m\geqslant 0} j_m(u)q^m\frac{\sqrt{\Delta}du}{Q(u)} \\ &= \frac{1}{\pi\sqrt{\Delta}}\sum_{Q\in\mathcal{Q}_\Delta/\,\mathrm{SL}_2(\mathbb{Z})} F_{\tau_Q} \end{aligned}$$

and

$$F_\Delta = \frac{1}{2\pi i}\,\mathrm{dlog}\,\Psi_\Delta.$$

Since we have an explicit formula for F_τ, we can use it to find an explicit expression for the antiderivative of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$, and use this to find an explicit expression for $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$.

Corollary 4.14. *Let τ be a real quadratic irrationality with the associated binary quadratic form Q of discriminant $\Delta > 0$. Let $\tilde{\varphi}$ be the cocycle with $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$ as its derivative (i.e. the coboundary of the modular integral F_τ). Then for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$\tilde{\varphi}(\gamma)(z) = -\int_z^{\gamma^{-1}z}\int_{C_Q}\frac{j'(s)}{j(u)-j(s)}\frac{\sqrt{\Delta}du}{Q(u)} + \sum_{q\sim Q}\delta_{\infty,s}(\tau_q)\frac{\sqrt{\Delta}ds}{q(s)} \tag{4.15}$$

and $\exp(\tilde{\varphi}) = \mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$.

The lift $\tilde{\varphi}$ of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$ under the derivative contains more information than the lift $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'} = \exp\tilde{\varphi}$ under dlog, since any multiple of $2\pi i$ gets lost when applying exp. The imaginary part of $\tilde{\varphi}$ was studied by Duke, Imamoḡlu and Tóth in [DIT17]. They show that there is an equality

$$-\frac{2}{\pi}\lim_{y\to\infty}\mathrm{Im}\,\tilde{\varphi}(\gamma, iy) = \#\{q \sim Q : \delta_{\infty,\gamma\infty}(\tau_q) \neq 0\} \in 2\mathbb{Z}.$$

By a Theorem of Birkhoff [Bir17], this quantity is closely related to the intersection number between modular geodesics associated to real quadratic irrationalities τ_1, τ_2.

There is also no ambiguity under this lifting construction. The other eleven lifts under the logarithmic derivative can be found by adding a cocycle in $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}/(2\pi i)^{\mathbb{Z}})$ before exponentiating, which is of course unnatural. Therefore, Corollary 4.14 gives us a canonical choice for the lift of $\mathrm{kn}_\tau + \mathrm{kn}_{\tau'}$. This choice is consistent with our previous choice of demanding Kn_τ to take values in $\mathbb{Q}(\tau)(z)$. However it cannot distinguish between the two lifts with this property. The map $\frac{d}{dz} : H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_0) \to H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_2)$ is not surjective, so not all rational cocycles can be lifted this way.

We now move on to an application of Corollary 2.23. We can evaluate the explicit formula we found for $\mathrm{Kn}_\tau \times \mathrm{Kn}_{\tau'}$ in a second real quadratic irrationality to obtain an explicit expression of a symmetrised version of Θ.

Theorem 4.16. *Let τ_1, τ_2 two real quadratic irrationalities in distinct $\mathrm{SL}_2(\mathbb{Z})$-orbits. Let Q_1, Δ_1 be the binary quadratic form and the discriminant associated to τ_1, and let γ_1, γ_2 be the automorphs of τ_1, τ_2. Recall from (2.8) that we have defined $J((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}))(z) = -cz + a$. We have*

$$\Theta(\tau_1, \tau_2)\Theta(\tau_1', \tau_2) = \exp \int_{\infty}^{\gamma_2\infty} \int_{u_0}^{\gamma_1 u_0} \frac{j'(z)}{j(u) - j(z)} \frac{\sqrt{\Delta_1} du}{Q_1(u)} dz \times \prod_{q \sim Q_1} \left[\left(\frac{\tau_q - \tau_2}{\tau_q' - \tau_2} \right) \times \sqrt{\left| \frac{J(\gamma_2)(\tau_q)}{J(\gamma_2)(\tau_q')} \right|} \right]^{\delta_{\infty, \gamma_2\infty}(\tau_q)} \tag{4.17}$$

for any $u_0 \in \mathcal{H}$, and

$$\Theta(\tau_1, \tau_2)\Theta(\tau_1', \tau_2)\Theta(\tau_1, \tau_2')\Theta(\tau_1', \tau_2') = \prod_{g \in \gamma_2^{\mathbb{Z}} \backslash \mathrm{PSL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \left(\frac{g\tau_1 - \tau_2}{g\tau_1' - \tau_2} \frac{g\tau_1' - \tau_2'}{g\tau_1 - \tau_2'} \right)^{\delta(g\tau_1, \tau_2)}. \tag{4.18}$$

Proof. We have $(\mathrm{Kn}_{\tau_1} \times \mathrm{Kn}_{\tau_1'})(\gamma_2)(z) = \exp \Phi(\gamma_2)(z)$ for $z \in \mathcal{H}$. To evaluate it in τ_2, we use the equality $\tau_2 = \lim_{n\to\infty} \gamma_2^n z_0$ for $z_0 \in \mathcal{H}$ and the continuity of $(\mathrm{Kn}_{\tau_1} \times \mathrm{Kn}_{\tau_1'})(\gamma_2)$. We obtain

$$\begin{aligned} \log \Theta(\tau_1, \tau_2)\Theta(\tau_1', \tau_2) &= \lim_{n\to\infty} - \int_{\gamma_2^n z_0}^{\gamma_2^{n-1} z_0} \int_{u_0}^{\gamma_1 u_0} \frac{j'(s)}{j(u) - j(s)} \frac{\sqrt{\Delta_1} du}{Q_1(u)} + \sum_{q \sim Q} \delta_{\infty, s}(\tau_q) \frac{\sqrt{\Delta_1}}{q(s)} ds \\ &= \lim_{n\to\infty} \int_{z_0}^{\gamma_2 z_0} \int_{u_0}^{\gamma_1 u_0} \frac{j'(z)}{j(u) - j(z)} \frac{\sqrt{\Delta_1} du}{Q_1(u)} + \sum_{q \sim Q} \delta_{\gamma_2^{-n+1}\infty, z}(\tau_q) \frac{\sqrt{\Delta_1}}{q(z)} dz, \end{aligned} \tag{4.19}$$

where we have made the coordinate transformation $z = \gamma_2^{-n+1} s$. Since this expression is independent of z_0, we may take the limit $z_0 \to i\infty$. We use the additivity of δ to obtain the equality $\delta_{\gamma_2^{-n+1}\infty, z}(\tau_q) = \delta_{\gamma_2^{-n+1}\infty, \infty}(\tau_q) + \delta_{\infty, z}(\tau_q)$. We split up the integral above into three terms and work the second and third out separately. For the second integral we make the coordinate transformation $z' = \gamma_2^{m+1} z$ to get

$$\begin{aligned} \lim_{z_0 \to i\infty} \lim_{n\to\infty} \int_{z_0}^{\gamma_2 z_0} \sum_{q \sim Q} \delta_{\gamma_2^{-n+1}\infty, \infty}(\tau_q) \frac{\sqrt{\Delta_1}}{q(z)} dz &= \sum_{m=0}^{\infty} \int_{\infty}^{\gamma_2\infty} \sum_{q \sim Q} \delta_{\gamma_2^{-m-1}\infty, \gamma^{-m}\infty}(\tau_q) \frac{\sqrt{\Delta_1}}{q(z)} dz \\ &= \sum_{q \sim Q} \delta_{\infty, \gamma_2\infty}(\tau_q) \int_{\infty}^{\tau_2} \frac{\sqrt{\Delta_1}}{q(z')} dz' = \sum_{q \sim Q} \delta_{\infty, \gamma_2\infty}(\tau_q) \log \left(\frac{\tau_q - \tau_2}{\tau_q' - \tau_2} \right). \end{aligned} \tag{4.20}$$

For $q = aX^2 + bXY + cY^2$ a binary quadratic form with $\delta_{\infty,\gamma_2\infty}(\tau_q) \neq 0$, let r_q be the intersection point of the geodesic connecting τ_q, τ_q' and the vertical line at $\gamma_2\infty$. Using elementary geometry you find the equality

$$\frac{\tau_q - r_q}{\tau_q' - r_q} = -i \cdot \sqrt{\left|\frac{\tau_q - \gamma_2\infty}{\tau_q' - \gamma_2\infty}\right|}.$$

Therefore we find for the third integral

$$\begin{aligned}\int_\infty^{\gamma_2\infty} \sum_{q\sim Q} \delta_{\infty,z}(\tau_q)\frac{\sqrt{\Delta_1}}{q(z)}ds &= \sum_{q\sim Q} \delta_{\infty,\gamma_2\infty}(\tau_q)\int_{r_q}^{\gamma_2\infty} \frac{\sqrt{\Delta_1}}{q(z)}ds \\ &= \sum_{q\sim Q} \delta_{\infty,\gamma_2\infty}(\tau_q)\left[\log\left(\frac{\tau_q - \gamma_2\infty}{\tau_q' - \gamma_2\infty}\right) - \log\left(\frac{\tau_q - r_q}{\tau_q' - r_q}\right)\right] \\ &= \sum_{q\sim Q} \delta_{\infty,\gamma_2\infty}(\tau_q)\left[\log\left|\frac{\tau_q - \gamma_2\infty}{\tau_q' - \gamma_2\infty}\right| + \pi i - \log\left|\frac{\tau_q - r_q}{\tau_q' - r_q}\right| + \frac{1}{2}\pi i\right] \\ &= \sum_{q\sim Q} \tfrac{1}{2}\delta_{\infty,\gamma_2\infty}(\tau_q)\log\left|\frac{\tau_q - \gamma_2\infty}{\tau_q' - \gamma_2\infty}\right|.\end{aligned}$$

Note that if $\gamma_2 = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, then $z - \gamma_2\infty = \frac{-1}{c}J(\gamma_2)(z)$. The imaginary parts of the logarithm are taken to lie in $(-\pi, \pi]$, though they cancel out mod $2\pi i$ because of Lemma 2.3. Adding the three integrals again and applying the exponential function leaves us with exactly (4.17).

We will now prove (4.18). As in (4.19) we find by making coordinate transformations

$$\begin{aligned}&\log\Theta(\tau_1,\tau_2)\Theta(\tau_1',\tau_2)\Theta(\tau_1,\tau_2')\Theta(\tau_1',\tau_2') \\ &= \lim_{n\to\infty}\int_{z_0}^{\gamma_2 z_0}\int_{u_0}^{\gamma_1 u_0} \frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta_1}du}{Q_1(u)} + \sum_{q\sim Q}\delta_{\gamma_2^{-n+1}\infty,z}(\tau_q)\frac{\sqrt{\Delta_1}}{q(z)}dz \\ &+ \lim_{n\to\infty}\int_{\gamma_2 z_0}^{z_0}\int_{u_0}^{\gamma_1 u_0} \frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta_1}du}{Q_1(u)} + \sum_{q\sim Q}\delta_{\gamma_2^{n}\infty,z}(\tau_q)\frac{\sqrt{\Delta_1}}{q(z)}dz,\end{aligned}$$

The cycle integrals cancel and we use the additivity of intersection numbers to find that this is equal to

$$\lim_{n\to\infty}\int_{z_0}^{\gamma_2 z_0}\sum_{q\sim Q}\delta_{\gamma_2^{-n+1}\infty,\gamma^n\infty}(\tau_q)\frac{\sqrt{\Delta_1}}{q(z)}dz,$$

which can be rewritten as

$$\sum_{q\sim Q}\delta_{\infty,\gamma_2\infty}(\tau_q)\log\left(\frac{\tau_2-\tau_q}{\tau_2-\tau_q}\frac{\tau_2'-\tau_q'}{\tau_2-\tau_q'}\right),$$

(cf. (4.20)). We note that

$$\frac{\tau_2-\tau_q}{\tau_2-\tau_q}\frac{\tau_2'-\tau_q'}{\tau_2-\tau_q'}$$

is independent of replacing τ_q with $\gamma_2^m\tau_q$ for some $m \in \mathbb{Z}$. We may therefore group together these terms. Now (4.18) follows after noting that

$$\sum_{n\in\mathbb{Z}}\delta_{\infty,\gamma_2\infty}(\gamma_2^n\tau_q) = \sum_{n\in\mathbb{Z}}\delta_{\gamma_2^{-n}\infty,\gamma_2^{1-n}\infty}(\tau_q) = \delta(\tau_q,\tau_2). \qquad \square$$

Note that the expression (4.18) for $\Theta(\tau_1,\tau_2)\Theta(\tau_1',\tau_2)\Theta(\tau_1,\tau_2')\Theta(\tau_1',\tau_2')$ is antisymmetric.

We may also express $\Theta(\tau_1,\tau_2)$ in terms of Ψ_τ. We have

$$\Theta(\tau_1,\tau_2)\times\Theta(\tau_1',\tau_2)=\lim_{n\to\infty}\frac{\Psi_{\tau_1}(\gamma_2^{n+1}z_0)}{\Psi_{\tau_1}(\gamma_2^{n}z_0)}$$

and

$$\Theta(\tau_1,\tau_2)\Theta(\tau_1',\tau_2)\Theta(\tau_1,\tau_2')\Theta(\tau_1',\tau_2')=\lim_{n\to\infty}\frac{\Psi_{\tau_1}(\gamma_2^{n+1}z_0)}{\Psi_{\tau_1}(\gamma_2^{n}z_0)}\frac{\Psi_{\tau_1}(\gamma_2^{-n-1}z_0)}{\Psi_{\tau_1}(\gamma_2^{-n}z_0)}.$$

The antisymmetry of $\Theta(\tau_1,\tau_2)\Theta(\tau_1',\tau_2)\Theta(\tau_1,\tau_2')\Theta(\tau_1',\tau_2')$ then gives us the peculiar equality

$$\lim_{n\to\infty}\frac{\Psi_{\tau_1}(\gamma_2^{n+1}z_0)}{\Psi_{\tau_1}(\gamma_2^{n}z_0)}\frac{\Psi_{\tau_1}(\gamma_2^{-n-1}z_0)}{\Psi_{\tau_1}(\gamma_2^{-n}z_0)}=\lim_{n\to\infty}\frac{\Psi_{\tau_2}(\gamma_1^{n}z_0)}{\Psi_{\tau_2}(\gamma_1^{n+1}z_0)}\frac{\Psi_{\tau_2}(\gamma_1^{-n}z_0)}{\Psi_{\tau_2}(\gamma_1^{-n-1}z_0)}.$$

4.4 Further study

There are several loose ends left. First of all, from Theorem 4.16 we can deduce that

$$\exp\int_{\infty}^{\gamma_2\infty}\int_{u_0}^{\gamma_1 u_0}\frac{j'(z)}{j(u)-j(z)}\frac{\sqrt{\Delta_1}du}{Q_1(u)}dz$$

is algebraic. An explicit algebraic expression for this integral would shed more light on the quantity $\Theta(\tau_1,\tau_2)\times\Theta(\tau_1',\tau_2)$. Combined with the explicit expression for $\Theta(\tau_1,\tau_2)/\Theta(\tau_1',\tau_2)$ that can be obtained from Corollary 2.23, this could lead to a proof that $\Theta(\tau_1,\tau_2)^2$ is antisymmetric.

Secondly, the methods in the proof of Theorem 4.16 are very similar to the argument in [DIT17] to show that the antiderivative of the Knopp cocycle is related to linking numbers of modular geodesics associated to any two real quadratic irrationalities τ_1 and τ_2. Explicitly they prove that

$$-\frac{2}{\pi}\lim_{n\to\infty}\lim_{z_0\to i\infty}\frac{\operatorname{Im}\tilde{\varphi}_{\tau_1}(\gamma_{\tau_2}^n,z_0)}{n}$$

is equal to the linking number of the two geodesics in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ associated to τ_1 and τ_2. Here $\tilde{\varphi}_{\tau_1}$ is the antiderivative of $\mathrm{kn}_{\tau_1}+\mathrm{kn}_{\tau_1'}$, which according to Corollary 4.14 is equal to

$$\tilde{\varphi}(\gamma)(z)=-\int_z^{\gamma^{-1}z}\int_{C_Q}\frac{j'(s)}{j(u)-j(s)}\frac{\sqrt{\Delta}du}{Q(u)}+\sum_{q\sim Q}\delta_{\infty,s}(\tau_q)\frac{\sqrt{\Delta}ds}{q(s)}.$$

In Theorem 4.16, we have instead looked at the real part. We have showed that

$$\log(\Theta(\tau_1,\tau_2)\times\Theta(\tau_1',\tau_2))=\lim_{n\to\infty}\lim_{z_0\to\infty}\operatorname{Re}\tilde{\varphi}_{\tau_1}(\gamma_{\tau_1}^n,z_0)$$

and used this to give an explicit expression for $\log(\Theta(\tau_1,\tau_2)\times\Theta(\tau_1',\tau_2))$. We see therefore that $\Theta(\tau_1,\tau_2)$ is closely related to linking numbers. This relation could be further studied.

Thirdly, one might wonder whether the p-adic limit of the Knopp cocycle also becomes a coboundary when looking at a bigger ambient space. Perhaps this might also lead to a better understanding of $\Theta_p(\tau_1,\tau_2)$ as defined in the introduction.

Lastly, our use of the logarithmic derivative seems convoluted. Summarising our current approach for finding an explicit formula for Θ: we take the logarithmic derivative of the difference of two j-functions $j(z_1) - j(z_2)$ and take the cycle integral over the other variable. We take the analytic continuation of this function before integrating with respect to the first variable again and exponentiating.

A more natural approach seems to be to not introduce the logarithmic derivative, but just work with the functions Ψ_τ. Perhaps these functions also come from certain mock modular forms, which would give us an extrinsic definition of the Ψ_τ without considering the logarithmic derivative. We can then also wonder whether there is a direct proof of the transformation formula of Ψ_τ.

A more naive approach is to take the double cycle integral of $j(z_1) - j(z_2)$ with respect to the primitive binary quadratic forms Q_1, Q_2 with positive non-square discriminant, without taking the logarithmic derivative first. In that case you obtain the quantity

$$\int_{C_{Q_1}} \int_{C_{Q_2}} j(z_1) - j(z_2) \frac{\sqrt{\Delta_1} dz_1}{Q_1(z_1)} \frac{\sqrt{\Delta_1} dz_1}{Q_1(z_1)} = \log(\varepsilon_2) j[\tau_1] - \log(\varepsilon_1) j[\tau_2].$$

We have also computed many values of this map, but we have not recognised them to be algebraic.

In conclusion: we have not been able to find an analytic construction of real quadratic differences of singular moduli. We have however found a connection between the p-adic work of Darmon and Vonk on real quadratic singular moduli and the complex analytic work of Kaneko and Duke, Imamoḡlu and Tóth on real quadratic singular moduli, which will hopefully lay the groundwork of such a construction in the future.

Appendices

Appendix A

Computing Θ

In this chapter we will discuss the algorithms used to compute values of Θ. We have implemented this algorithm in Sage, and refer the interested reador to GitHub for the source code. The general strategy of the algorithm is as follows. Let τ_1, τ_2 be two real quadratic numbers.

1. We compute the value of the Knopp cocycle in S:

$$\mathrm{Kn}_{\tau_1}(S) = \prod_{w \in \mathrm{SL}_2(\mathbb{Z})\tau} (z-w)^{\delta_{\infty,0}(w)} \times \prod_{\substack{w \in \mathrm{SL}_2(\mathbb{Z})\tau \\ w'<0<w}} w^{-1} \tag{A.1}$$

   and its value in T: $\mathrm{Kn}_{\tau_1}(T) = \varepsilon_1^{-1}$, the fundamental unit of norm 1 in the order associated to τ_1 lying in the interval $(0,1)$ [DV21, Lemma 2.2, Corollary 2.3].

2. We compute the automorph γ_2 of τ_2, and write it as a word in S and T.

3. We use the cocycle relation to compute the value of $\mathrm{Kn}_\tau(\gamma_2)$ and evaluate it in τ_2.

The automorph of τ_1 and the set $\{w \in \mathrm{SL}_2(\mathbb{Z})\tau_1 : \delta_{\infty,0}(w) \neq 0\}$ can be computed using the same algorithm. Since the automorph of τ_1 has eigenvalue ε_1, this allows us to compute both $\mathrm{Kn}_{\tau_1}(S)$ and $\mathrm{Kn}_{\tau_1}(T)$. We will discuss this algorithm.

Definition A.2. Let τ be a real quadratic irrationality with conjugate τ'. We call τ *nearly reduced* if $\tau\tau' < 0$, and *reduced* if it also satisfies $|\tau| < 1 < |\tau'|$.

This definition can also be formulated in terms of binary quadratic forms. See [Bue89] and [BV07]. To compute (A.1), we want to compute the set of nearly reduced numbers in the orbit of τ_1. It will be useful to reformulate this definition in the context of continued fraction expansions. For $r \in \mathbb{R}$ a real number, its continued fraction expansion is a sequence of integers $(a_n)_{n \geqslant 0}$ such that

$$r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0, a_1, a_2, \dots].$$

This expansion is made unique by choosing a_i to be positive for $i > 0$. It can be shown that r is a real quadratic irrationality if and only if its continued fraction expansion is *eventually periodic*, i.e. if there exist k, N such that $a_i = a_{i+N}$ for all $i \geqslant k$ [Bue89, Theorem 3.15]. We will denote such a sequence by $[a_0, \dots, a_{k-1}, \overline{a_k, \dots, a_{k+N}}]$. A real quadratic irrationality is said to be *purely periodic* if it satisfies this condition with $k = 0$.

Theorem A.3. *(Theorems 4.1 and 4.2 from [Old78]) Let τ be a quadratic irrationality with conjugate τ'. Then τ has a purely periodic expansion if and only if $\tau > 1$ and $-1 < \tau' < 0$. Moreover if $\tau = [\overline{a_0, a_1, \dots, a_n}]$, then $-(\tau')^{-1} = [\overline{a_n, \dots, a_1, a_0}]$.*

We see that a real quadratic irrationality is reduced if and only $|\tau|^{-1}$ has a purely periodic expansion. Suppose $|\tau|$ has periodic fraction expansion $[a_0, a_1, \dots]$. The matrices S, T have the following effect on the periodic fraction expansion.

$$|T\tau| = [a_0 + \operatorname{sgn}(\tau), a_1, \dots] \qquad \text{and} \qquad |S\tau| = \begin{cases} [0, a_0, a_1, \dots] & \text{if } a_0 \neq 0, \\ [a_1, a_2, \dots] & \text{if } a_0 = 0. \end{cases}$$

From these relations it can be seen that $\mathrm{SL}_2(\mathbb{Z})$ can arbitrarily change, add and remove finitely many terms of the fraction expansion of $|\tau|$, but it leaves the periodic tail invariant. The matrix $ST^{-\operatorname{sgn}(\tau)a_0}$ has the effect of removing the first coefficient of the fraction expansion of $|\tau|$. By repeatedly removing this first coefficient and finally applying S, one can find a reduced real quadratic irrationality in the orbit of τ. There are also more efficient algorithms to do this, cf. [BV07]. You can continue to remove more coefficients to find more reduced numbers in the orbit of τ. The following theorem tells us that we will eventually reach all reduced forms in this way.

Theorem A.4. *Given a reduced real quadratic irrationality $\tau = \pm[0, \overline{a_1, a_2, \dots, a_n}]$, define*

$$\sigma(\tau) = (T^{\operatorname{sgn}(\tau)a_1} S)\tau.$$

Then the image $\sigma(\tau)$ is also reduced, and σ defines a bijection on the set of reduced real quadratic numbers. The orbit of τ under σ contains exactly the reduced numbers that are $\mathrm{SL}_2(\mathbb{Z})$-equivalent to τ. The map σ has period $\operatorname{lcm}(2, n)$ on this orbit. Furthermore, the automorph of τ is equal to

$$\gamma = \left((T^{\operatorname{sgn}(\tau)a_n} S) \cdots (T^{\operatorname{sgn}(\tau)a_1} S)\right)^{-\operatorname{lcm}(2,n)/n}. \tag{A.5}$$

Proof. We have $\sigma(\tau) = \mp[0, \overline{a_2, \dots, a_n, a_1}]$, which is reduced by Theorem A.3. Any $\mathrm{SL}_2(\mathbb{Z})$-transformation changes the fraction of $|\tau|$ by only finitely many terms. Therefore the only possible reduced forms in the orbit of τ are given by $\pm[0, \overline{a_i, \dots, a_{i+n}}]$ for some i. When n is odd, these are all reached by powers of σ. When n is even, it can be shown using [Bue89, Proposition 3.19] that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, any continued fraction expansion of $\gamma\tau$ such that the coefficients are eventually positive, has an odd number of coefficients preceding the periodic part $\overline{a_1, \dots, a_n}$. However, the continued fraction expansion of $-\tau = [-1, 1, -1, 0, \overline{a_1, \dots, a_n}]$ has an even number of coefficients preceding this periodic part. Therefore τ is equivalent to $-\tau$ if and only if n is odd, and the orbit of τ under σ contains all equivalent reduced forms. It is now evident that the orbit $\{\sigma^i(\tau) : i \geqslant 0\}$ of τ under σ contains exactly the reduced forms equivalent to τ and that σ has period $\operatorname{lcm}(2, n)$ on this orbit.

Applying $\sigma^{\operatorname{lcm}(2,n)}$ to τ gives us the equality $\gamma^{-1}\tau = \tau$. There is no non-trivial word in S, T^n shorter than (A.5) that stabilises τ, so γ generates the stabiliser of τ up to sign. Recall that we have defined the automorph of τ such that the eigenvalue corresponding to τ is greater than 1. We verify that this is indeed the case. Since τ is reduced we have $|\tau| < 1$. If $\tau = w_1/w_2$, then the bottom entry of $T^{\operatorname{sgn}(\tau)a_1} S\left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$ is smaller than w_2 and of opposite sign. Inductively we find that the bottom entry of $\gamma^{-1}\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)$, which is the eigenvalue of γ^{-1} corresponding to τ, is a positive number smaller than 1. This completes the proof ☐

Theorem A.4 not only gives us an algorithm to compute the set of reduced forms in the orbit of a real quadratic irrationality τ, it also allows us to compute its automorph efficiently.

Let w be a nearly reduced real quadratic number with conjugate w'. If $|w - w'| > 1$, then w can be translated to be reduced. If $|w - w'| < 1$, then we have $| - w^{-1} + w'^{-1}| > 1$ and hence $-w^{-1}$ is a translate of a reduced form. Therefore the set of nearly reduced forms can be computed simultaneously with the computation of the set of reduced forms, by keeping track of all the nearly reduced translates.

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