Black holes in modified gravity
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Black holes in modified gravity

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Abstract

In this thesis we study spherically symmetric gravitational collapse of a massless scalar field in a minimally modified gravity theory called VCDM. VCDM is a scalar-tensor theory, in which the role of Λ in ΛCDM is replaced by a free function $V(\phi)$ of a non-propagating auxiliary scalar field ϕ. This theory propagates only two local physical degrees of freedom and is supplemented by a so-called shadowy mode. Unlike general relativity, the theory does not have 4D diffeomorphism. We add a matter action to the gravitational action of VCDM, and derive the equations of motion as is done in Jalali et al. 2024. Subsequently, we restrict our study to a subcase of the equations. We find that in this sub-case, the massless scalar field is either non-dynamical or does not allow for spatial inhomogeneities. Furthermore, the conditions for an apparent horizon cannot be satisfied within this case. We numerically evolve the system and confirm that no apparent horizon forms during the spherical scalar collapse in this sub-case.

Acronyms

ADM Arnowitt, Deser and Misner

AH Apparent Horizon

DoF Degree of freedom

BH Black Hole

ELE Euler-Lagrange Equations

EoM Equations of Motion

GR General Relativity

MG Modified Gravity

MMG Minimially Modified Gravity

GW Gravitational Waves

MOTS Marginally Outer Trapped Surface

UV Ultraviolet

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1 Introduction and motivation

Whether you are a physicist or not, it is hard have never heard about the most successful theory of gravity to date: general relativity (GR). Developed by Albert Einstein over a century ago (Einstein 1916), GR revolutionized our understanding of gravity, replacing the Newtonian framework that had dominated science for more than 200 years. While Newton's laws provide excellent approximations in everyday situations, they began to show cracks by the 19th century. For example, the unexplained precession of Mercury's orbit (Leverrier 1859) and the growing tension between Newtonian gravity and the observed behavior of light in gravitational fields hinted at deeper inconsistencies. These anomalies motivated a profound shift in how we understand space, time, and motion, culminating in Einstein's general theory of relativity. Crucially, GR not only explained Mercury's motion but also correctly predicted the deflection of starlight by the Sun's gravity, an effect confirmed during the 1919 solar eclipse (Eddington et al. 1920). In the modern era, GR has passed even more extreme tests. The detection of gravitational waves from merging black holes by the LIGO and Virgo collaborations provided direct evidence for both the existence of gravitational waves and black holes as predicted by GR (Abbott et al. 2016). Even more recently, the Event Horizon Telescope produced the first image of a black hole's shadow, offering an unprecedented look at the warped spacetime near an event horizon (Akiyama et al. 2019).

Despite its remarkable successes, general relativity is not without significant conceptual and theoretical challenges. On the cosmological scale, GR requires the existence of unseen components, namely dark matter and dark energy, to account for a wide range of observations, including galaxy rotation curves and the accelerated expansion of the universe (Planck Collaboration 2020). Even when the accelerated expansion is attributed to a cosmological constant Λ consistent with Einstein's field equations, the value inferred from observations is astonishingly small compared to estimates from quantum field theory (Weinberg 1989). These unknown substances have yet to be directly detected, raising questions about the completeness of GR as a fundamental theory of gravity. Moreover, GR's incompatibility with quantum field theory leaves the problem of gravity's unification with other fundamental forces unresolved (Clifton et al. 2012). Such limitations have inspired the exploration of alternative or extended theories of gravity. These theories aim to reproduce GR's well-tested predictions while potentially explaining cosmological phenomena without invoking dark components, and providing a consistent framework for quantum gravity (Joyce et al. 2015). The ongoing quest to reconcile GR's success with its theoretical shortcomings drives much of modern gravitational research.

Over the past century, numerous attempts have been made to generalize or extend Einstein's theory of gravity. The earliest and perhaps most natural extensions considered is adding additional degrees of freedom, such as scalar fields. One of the earliest scalar-tensor theories, the Brans-Dicke theory (Brans & Dicke 1961), was motivated by Mach's principle and introduced a varying gravitational constant mediated by a scalar field. This theory remained consistent with many weak-field tests while deviating from GR in cosmological and strong-field regimes.

Subsequent developments expanded upon this idea, leading to a broad class of scalar-tensor theories aimed at explaining cosmological phenomena such as inflation, dark energy, or late-time acceleration. In the 1970s, efforts to identify the most general scalar-tensor theory with second-order equations of motion led to the construction of Horndeski gravity (Horndeski 1974).

This theory includes a scalar field coupled to gravity in such a way that avoids instabilities, ensuring that the equations of motion remain of second order. For decades, Horndeski's theory was largely overlooked, but it resurfaced in the context of modern cosmology and gravitational wave physics, particularly after the discovery of cosmic acceleration. More recently, the gravitational wave event GW170817 and its associated electromagnetic counterpart placed stringent constraints on the speed of gravitational waves (Abbott et al. 2017), ruling out large portions of the Horndeski parameter space and motivating more refined models (Creminelli & Vernizzi 2017).

Degenerate Higher-Order Scalar-Tensor (DHOST) theories were developed to extend Horndeski gravity by allowing higher-order derivatives in the action while carefully constructing the theory to remain free of ghost-like instabilities through the enforcement of degeneracy conditions (Langlois & Noui 2016; Crisostomi et al. 2016). These degeneracy conditions ensure that despite the presence of higher derivatives, the theory propagates only the desired physical degrees of freedom, typically one scalar mode alongside the two tensor modes of general relativity. However, it was later realized that a broader class of theories, known as Unitary Degenerate Higher-Order Scalar-Tensor (U-DHOST) theories, can exhibit degeneracy only in the unitary gauge. In these theories, an additional non-propagating mode, often called the shadowy mode, emerges. This mode satisfies an elliptic differential equation on spatial hypersurfaces, meaning it does not correspond to a dynamical degree of freedom but can nonetheless influence the theory's structure and solutions (Felice et al. 2021).

Another proposed modified gravity theory, called VCDM, is also supplemented by such a shadowy mode. The theory is constructed such that it only propagates two local physical degrees of freedom just like in GR (the so called minimally modified gravity theories). From observations of gravitational waves it is preferred to find a theory that agrees with only two local physical propagating degrees of freedom, while also addressing some of the key shortcomings of GR. In Felice et al. 2021, they showed that the Hubble tension is reduced within VCDM. From the evidence of black holes, we often require a new theory of gravity to admit black holes. Hence, black holes (Felice et al. 2021) and spherical collapse (Felice et al. 2022; Felice et al. 2023; Jalali et al. 2024) in VCDM were also carefully studied.
In this thesis, we extend this study by also considering spherically symmetric scalar collapse in VCDM as was done in Jalali et al. 2024. We study a specific case of the study, namely the case where the shift is zero and perform a numerical study to see if an apparent horizon forms.

The outline of this thesis is as follows. We start with the theoretical background in Section 2., in which we discuss the ΛCDM model, general relativity in both the Lagrangian and Hamiltonian formalism, and black holes in GR. We then turn to modified gravity and minimally modified gravity theories. We end the section by discussing the construction of VCDM, the theory of interest, and black holes and spherically symmetric collapse within this theory. In Section 3. we discuss the the setup of our spherically symmetric scalar collapse and in Section 4. the numerical integration of this collapse. Then in Section 5. we present our results. We end this thesis with a summary and discussion in Section 6. and discuss possible future research within VCDM as well as in modified gravity in general. Additionally, we provide three appendices containing the derivation of the apparent horizon condition, the equations of motion, and the use of integrability conditions for these EoM.

2 Theoretical background

In this section we first cover a recap on the ΛCDM model and General Relativity (GR). We discuss GR in both the Lagrangian and Hamiltonian formulation, and discuss black holes in GR. We then turn to modified gravity and minimally modified gravity. Subsequently, we discuss the theory of our interest, VCDM. We go through the construction of the theory and discuss black holes and spherical collapse within this framework.

2.1 ΛCDM

The ΛCDM model is our best current understanding of the universe. It assumes GR as the theory of gravity, explains the Universe's accelerated expansion as being caused by dark energy, represented by the cosmological constant (Λ), and postulates the existence of dark matter. The evidence for this accelerated expansion first came from observations of distant Type Ia supernovae (Riess et al. 1998) appearing dimmer than expected, indicating that the universe is not only expanding, but doing so at an increasing rate. Later, detailed measurements of the Cosmic Microwave Background (CMB) radiation, especially the statistical properties of its temperature fluctuations captured in its power spectrum, confirmed this picture with stunning precision (Aghanim et al. 2020).

Despite this success, there is a profound mystery hidden in the tiny size of Λ. The cosmological constant represents the energy density of empty space, i.e. the vacuum energy. Theoretical predictions from quantum field theory estimate this vacuum energy to be much larger than the observed values for Λ. This mismatch between theory and observation is known as the *cosmological constant problem* (Silvestri et al. 2009; Adler et al. 1995; Martin 2012). This bridges to an even larger problem. General relativity, the assumed theory of gravity in ΛCDM, is incompatible with quantum field theory. This problem, together with the fact that we have not directly detected dark matter yet, which makes up about 85% of the universe's matter (Kisslinger & Das 2019), has motivated scientists to consider alternatives to the ΛCDM model.

In short, while ΛCDM fits a vast range of data, the tiny value of Λ and the undetected dark components keep us searching for deeper explanations and alternative theories.

2.2 General relativity

To explore theories that go beyond general relativity, it's important to first have a solid grasp of general relativity itself. In this section, we will review the key ideas from general relativity that are most relevant to this project.

2.2.1 The Lagrange Formulation

General relativity can be formulated via the principle of stationary action. The dynamical variable is the spacetime metric $g_{\mu\nu}$, and the action is given by the *Einstein-Hilbert action*:

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int R\sqrt{-g}\, d^4x, \tag{1}$$

where R is the Ricci scalar curvature, i.e. the double contraction of the Riemann tensor, $g = \det(g_{\mu\nu})$, G the gravitational constant and we used unites $c = 1$. To obtain the field equations, we vary the action with respect to the inverse metric $g^{\mu\nu}$. Taking the variation of the action yields

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R)\, \delta g^{\mu\nu} \sqrt{-g}\, d^4x,$$

up to a boundary term which vanishes if the variation $\delta g^{\mu\nu}$ is zero on the boundary. Varying S_{EH} with respect to $g_{\mu\nu}$ results in the *Einstein's Field Equations* (EFE) in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

The left-hand side defines the *Einstein tensor*

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

We will now take into account vacuum energy into the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\, d^4x, \tag{2}$$

where Λ is the *cosmological constant*. To include matter, we add a matter action S_M to the total action

$$S = S_{\mathrm{EH}} + S_M. \tag{3}$$

The variation of the matter action with respect to the metric defines the energy-momentum tensor

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{4}$$

Requiring that the total action is stationary under arbitrary metric variations yields the full Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{5}$$

This formulation provides a powerful geometric description of gravitation: matter tells spacetime how to curve through $T_{\mu\nu}$, and spacetime tells matter how to move via the curvature encoded in $G_{\mu\nu}$. Solutions to the EFE are metrics of spacetime. These solutions vary depending on the physical situation, ranging from the flat Minkowski metric of empty spacetime, to the Schwarzschild solution describing the exterior of a static spherical mass, to more extreme cases like the Kerr or Reissner-Nordström metrics for rotating or charged black holes.

2.3 Hamiltonian formulation

Although Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ fully captures the dynamics of general relativity, it is often both convenient and insightful to reformulate the theory using the Lagrangian and Hamiltonian frameworks, even within a purely classical context. These alternative formulations not only offer a different mathematical perspective but also help illuminate the structure and evolution of spacetime in meaningful ways. For instance, Einstein's equation can be elegantly derived from a simple Lagrangian. The Hamiltonian formulation, on the other hand, brings clarity to the dynamic nature of general relativity by emphasizing how spatial geometry evolves with time. Yet, perhaps the most compelling reason to study general relativity in these forms is their essential role in the pursuit of a quantum theory of gravity. While the EFE are sufficient for classical physics, most approaches to quantum field theory rely fundamentally on having either a Lagrangian or a Hamiltonian formulation of the underlying classical theory. In particular, the path integral approach to quantization depends on a classical action principle, which requires a Lagrangian, and canonical quantization requires a Hamiltonian framework. Therefore, recasting general relativity in these frameworks is not merely a matter of convenience. It is a necessary step for any serious attempt at quantizing gravity. In this way, the Lagrangian and Hamiltonian an important role in the attempt to obtain a theory of quantum gravity.

The Lagrangian formulation is "spacetime covariant", whereas the Hamiltonian formulation requires a decomposition of space and time. Before we continue on the Hamiltonian formulation, we will first discuss this decomposition.

2.3.1 3+1 decomposition

In most theories of classical physics we have a spacetime background and our goal is to determine the time evolution of quantities in the background from their initial values and time derivatives. However, in GR we are solving for the spacetime itself. This raises the question as to what initial data must be specified in order to determine the spacetime structure. To answer this question we must view GR as describing the time evolution of some quantity. We consider a globally hyperbolic spacetime $(M, g_{\mu\nu})$. We consider only *globally hyperbolic spacetimes*, meaning that the spacetime has a Cauchy surface. We can foliate this spacetime into Cauchy surfaces Σ_t (theorem 8.3.14 in Wald 1984), parametrized by a global time function t. Let n^a be the normal vector field to the hypersurfaces Σ_t. The spatial metric induced by $g_{\mu\nu}$ is given by

$$h_{ab} = g_{ab} + n_a n_b. \tag{6}$$

We let t^a be a vector field on M satisfying $t^a \nabla_a t = 1$. We can decompose t^a in terms of its normal and tangential parts

$$N = -t^a n_a = \frac{1}{n^a \nabla_a t} \tag{7}$$

$$N_a = h_{ab} t^b, \tag{8}$$

defining the *lapse function* N and the *shift vector* N_a, respectively. The lapse function N measures the flow of proper time with respect to t. The shift vector N^a measure the shift tangential to the hypersurface contained in t^a. In Fig. (1) an illustrative example is shown of two hypersurfaces Σ_t and $\Sigma_{t+\Delta t}$. We can interpret moving 'forward in time' from t to $t + \Delta t$ as the spatial metric changing in a 3D manifold from $h_{ab}(t)$ to $h_{ab}(t + \Delta t)$, suggesting that we consider the spatial metric on the hypersurface as the dynamical variable of GR. Hence the initial data that needs to be prescribed is a spatial metric and its 'time derivative' on the 3D manifold.

We can express the normal vector n^a in terms of the lapse function, shift vector and vector field t^a

$$n^a = \frac{1}{N}\left(t^a - N^a\right). \tag{9}$$

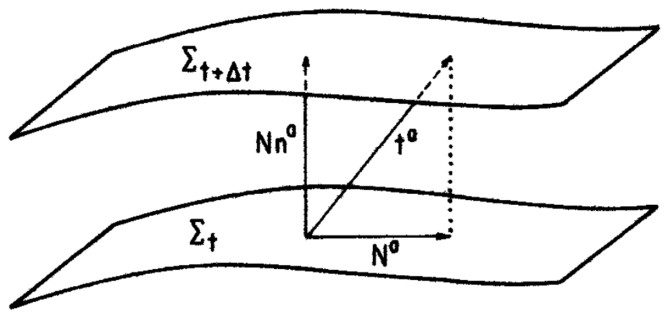

Figure 1: A spacetime diagram showing the definition of the lapse function N and the shift vector N^a (Wald 1984).

We define the *extrinsic curvature* K_{ab} as

$$K_{ab} = h_a^c \nabla_c n_b \tag{10}$$
$$= \frac{1}{2} \pounds_n h_{ab} \tag{11}$$

For the interpretation of the extrinsic curvature see the illustrative example in Fig (2).

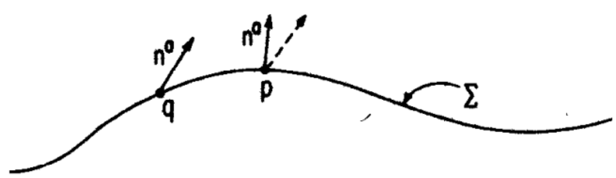

Figure 2: A spacetime diagram showing the extrinsic curvature of a hypersurface Σ (Wald 1984). The failure of the vector n^a to be parallel transported from point q to p relates to the bending of Σ in the spacetime in which it is embedded. This can also be seen from equation (10).

Furthermore, we can derive the following expressions relating the curvature ${}^3R^d_{abc}$ on the hypersurface, to the spacetime curvature $R_{abc}{}^d$ (Wald 1984)

$${}^{(3)}R_{abc}{}^d = h_a{}^f h_b{}^g h_c{}^h h^d{}_j R_{fgh}{}^j - K_{ac}K_b{}^d + K_{bc}K_a{}^d \tag{12}$$
$$D_a K^a{}_b - D_b K^a{}_a = R_{cd} n^d h^c{}_b, \tag{13}$$

where equation (12) is known as Gauss's equation and equation (13) as Codazzi's equation. Surprisingly, together they are called the *Gauss-Codazzi equations*. By taking appropriate traces we obtain the hypersurface curvature scalar

$$R = {}^{(3)}R + K^2 - K_{ab}K^{ab} - 2\nabla_a \left(n^b \nabla_b n^a - n^a \nabla_b n^b \right) \tag{14}$$

where the last term $-2\nabla_a \left(n^b\nabla_b n^a - n^a\nabla_b n^b\right)$ is the total divergence. Rewriting the inverse metric using the normal vector in equation (9) then yields

$$g^{ab} = h^{ab} - n^a n^b \tag{15}$$

$$= h^{ab} - \frac{1}{N^2}\left(t^a - N^a\right)\left(t^b - N^b\right). \tag{16}$$

Choosing a set of coordinates (t, x^i) with $i = 1, 2, 3$ we can now write the line element as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ab}(N^a dt + dx^a)(N^b dt + dx^b). \tag{17}$$

The term $\sqrt{-g}$ reads

$$\sqrt{-g} = N\sqrt{h}. \tag{18}$$

We are now able to rewrite the Lagrangian density in equation (1), up to boundary terms, as

$$\mathcal{L}_G = \sqrt{h}N\left[{}^3R + K_{ab}K^{ab} - K^2\right]. \tag{19}$$

2.3.2 Hamiltonian density

With the 3+1 deocomposition we will now develop the Hamiltonian formulation.
The extrinsic curvature K_{ab} is related to the 'time derivative' $\dot{h}_{ab} \equiv h_a^b h_b^d \pounds_t h_{cd}$ of the induced spatial metric h_{ab} via

$$\begin{aligned}
K_{ab} &= \frac{1}{2}\mathcal{L}_n h_{ab} = \frac{1}{2}\left[n^c\nabla_c h_{ab} + h_{ac}\nabla_b n^c + h_{cb}\nabla_a n^c\right] \\
&= \frac{1}{2}N^{-1}\left[Nn^c\nabla_c h_{ab} + Nh_{ac}\nabla_b n^c + Nh_{cb}\nabla_a n^c\right] \\
&= \frac{1}{2}N^{-1}\left[N\cdot\frac{1}{N}(t^c - N^c)\nabla_c h_{ab} + N\cdot\frac{1}{N}h_{ac}\nabla_b(t^c - N^c) + N\cdot\frac{1}{N}h_{cb}\nabla_a(t^c - N^c)\right] \\
&= \frac{1}{2}N^{-1}\left[(t^c - N^c)\nabla_c h_{ab} + h_{ac}\nabla_b(t^c - N^c) + h_{cb}\nabla_a(t^c - N^c)\right] \\
&= \frac{1}{2}N^{-1}\left[\dot{h}_{ab} - D_a N_b - D_b N_a\right]
\end{aligned} \tag{20}$$

in which we used equation (9). The momentum conjugate of h_{ab} is

$$\pi^{ab} = \frac{\partial\mathcal{L}_{\mathcal{G}}}{\partial\dot{h}_{ab}} = \sqrt{h}(K^{ab} - Kh^{ab}) \tag{21}$$

From taking the trace of π^{ab} we find K^{ab} in terms of π^{ab}

$$h_{ab}\pi^{ab} = -2\sqrt{h}K \Rightarrow K = \frac{-\pi}{2\sqrt{h}} \tag{22}$$

$$K^{ab} = \frac{1}{\sqrt{h}}\left(\pi^{ab} - \frac{1}{2}\pi h^{ab}\right), \tag{23}$$

where $\pi = \pi^a_a$. These expressions can then be substituted into $\mathcal{L}_G$ in equation (19) to get the density in the ADM form (Arnowitt, Deser and Misner 1962). We define the Hamiltonian density by

$$\mathcal{H}_G = \pi^{ab}\dot{h}_{ab} - \mathcal{L}_G \tag{24}$$

$$= \sqrt{h}\left[N\Big[-{}^3R + \frac{1}{h}\pi^{ab}\pi_{ab} - \frac{1}{2h}\pi^2\Big] - 2N_b\left[D_a\left(\frac{1}{\sqrt{h}\pi^{ab}}\right)\right] + 2D_a\left(\frac{1}{\sqrt{h}}N_b\pi^{ab}\right)\right] \tag{25}$$

where the last term in equation (25) contributes only a boundary term and will be dropped. Varying $\mathcal{H}_G$ with respect to the lapse function N and shift vector N^a yields

$$H_0 \equiv \frac{\partial\mathcal{H}_G}{\partial N} = -{}^3R + \frac{1}{h}\pi^{ab}\pi_{ab} - \frac{1}{2h}\pi^2 \tag{26}$$

$$H_b \equiv \frac{\partial\mathcal{H}_G}{\partial N^b} = -2D_a\left(\frac{1}{\sqrt{h}}\pi^{ab}\right). \tag{27}$$

The four constraints are referred to as the *Hamiltonian constraint* and the *momentum constraint*, respectively. This allows us to write the Hamiltonian density as

$$\mathcal{H}_G = \sqrt{h}\left(N\mathcal{H}_0 + N^b\mathcal{H}_b\right). \tag{28}$$

We derive the dynamical equations from the Hamiltonian density in (25) (Wald 1984),

$$\dot{h}_{ab} = \frac{\delta H_G}{\delta\pi^{ab}} = 2h^{-1/2}N\left(\pi_{ab} - \frac{1}{2}h_{ab}\pi\right) + 2D_{(a}N_{b)}, \tag{29}$$

$$\begin{aligned}\dot{\pi}^{ab} = -\frac{\delta H_G}{\delta h_{ab}} &= -Nh^{1/2}\left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}Rh^{ab}\right)\\ &+ \frac{1}{2}Nh^{-1/2}h^{ab}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right)\\ &- 2Nh^{-1/2}\left(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi\pi^{ab}\right)\\ &+ h^{1/2}\left(D^aD^bN - h^{ab}D^cD_cN\right)\\ &+ h^{1/2}D_c\left(h^{-1/2}Nc^{(a}D_cN^{b)}\right),\end{aligned} \tag{30}$$

where equations (26),(27), (29) and (30) are equivalent to the EFE in vacuum. Thus we have successfully given a Hamiltonian formulation of the theory.

2.4 Black holes and the Schwarzschild solution

An important class of exact solutions to the EFE arises in the absence of matter, i.e., in vacuum where $T_{\mu\nu} = 0$. Under the additional assumption of *spherical symmetry*, the EFE admit a unique static solution: the *Schwarzschild metric*. This result is encapsulated in *Birkhoff's theorem* (Birkhoff 1923), which states that any spherically symmetric solution of the vacuum Einstein equations must be static and asymptotically flat. Consequently, the exterior gravitational field of a spherically symmetric mass distribution, whether it be a star, compact object, or black hole, is described by the Schwarzschild geometry. The Schwarzschild metric in standard coordinates (t, r, θ, ϕ) is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{31}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on the 2-sphere, and M is the total mass of the central object. This solution represents the spacetime outside a static, non-rotating, uncharged mass. When the mass is concentrated within its Schwarzschild radius $r_s = 2GM$, the geometry corresponds to a black hole with an event horizon at $r = r_s$. The *event horizon* is defined as the boundary beyond which events cannot influence an outside observer.

In contrast, *apparent horizons*, i.e. *marginally outer trapped surfaces* (MOTS), provide conditions that can be checked locally on a given spatial slice of spacetime, making them useful for identifying black holes in dynamical or evolving scenarios. A *trapped surface* (Penrose 1965) is a closed two-dimensional surface on which both ingoing and outgoing light rays are converging, indicating that even outward-directed light cannot escape. Penrose's idea of a trapped surface gives us a practical way to identify the location of a black hole without needing to predict its entire future. Consider a two-dimensional surface, Σ, in spacetime. From every point on Σ, imagine sending out a light ray, and then define Σ_t as the surface formed by following these light rays for a parameter t, so Σ_t represents a "shell of light" expanding (or contracting) from Σ. It's important to note that this parameter t is not really time in the usual sense, since light itself does not experience time passing. Normally, if the light rays are directed inward, we expect the area of Σ_t to shrink over t, and if they are directed outward, the area usually grows, as it does for a standard sphere in flat (Minkowski) spacetime. But in regions of very strong gravity, even the outward-directed light rays can produce a family of light shells whose area decreases with t at every point on Σ. When this happens, Σ is called an (outer) trapped surface. This idea can be seen in Fig. (3).

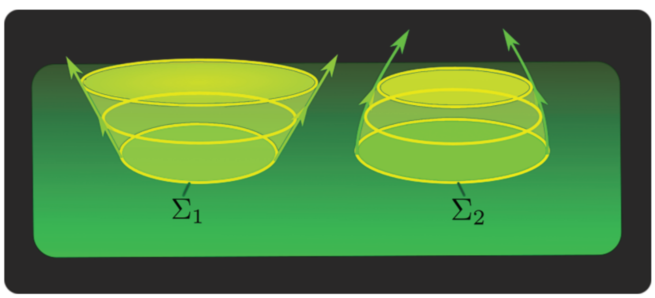

Figure 3: Typically, the outgoing 'light shell' would have increasing area form as the light emits from the surface as is seen on the left for Σ_1. However, for a trapped surface this area is decreasing as can be seen on the right for Σ_2 (Huang & Lee).

A trapped surface S is considered a *marginally outer trapped surface* if, at every point on S, the outward-directed null geodesics (light rays) is exactly 0. It is the outer boundary of the trapped surfaces, also called the *apparent horizon*. In the context of general relativity, the apparent horizon must lie inside or coincide with the event horizon. However, in modified theories of gravity it is even possible that the AH lies outside of the event horizon. The absence of an apparent horizon does not imply the absence of a black hole (Shapiro & Baumgarte 2010). For example, there exists slicings of Schwarzschild geometry where there is no apparent horizon (Wald & Iyer 1991).
As an example of a marginally outer trapped surface, consider spherically symmetric gravitational collapse, with the exterior spacetime described by the Schwarzschild metric in ingoing

Eddington-Finkelstein coordinates in Fig. (4). Any light ray or massive test particle starting at $r < 2m$ will inevitably crash into the singulatiry $r = 0$. The surfaces defined by constant v and r are trapped surfaces when $r < 2m$ (S^2 in the figure) and MOTS when $r = 2m$.

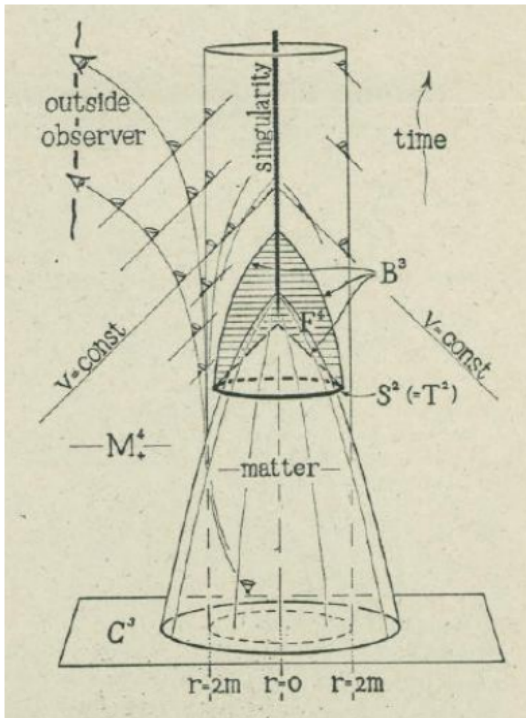

FIG. 1. Spherically symmetrical collapse (one space dimension surpressed). The diagram essentially also serves for the discussion of the asymmetrical case.

Figure 4: Spherically symmetric collapse in the Schwarzschild metric in the ingoing Eddington-Finkelstein coordinates (Penrose 1965).

In this work, finding an apparent horizon by solving for a marginally outer trapped surface during a spherically symmetric scalar collapse plays a central role.

2.5 Modified gravity

In this section we motivate as to why one would modify gravity, what theories are equivalent to GR encoded in Lovelock's theorem, and ways to break this theorem, i.e. to develop a theory that is truely different from GR.

2.5.1 Motivation

There are strong reasons, both observational and theoretical, to consider modifications of general relativity, the prevailing theory of gravity. While GR has been remarkably successful in explaining a wide range of phenomena, from planetary orbits to gravitational waves, several fundamental puzzles remain unresolved. These suggest that GR, elegant as it is, might not be the final word on gravity.

First, a number of mysteries in cosmology challenge the explanatory power of GR when combined with the Standard Model of particle physics. Dark energy, which appears to be responsible for the accelerated expansion of the Universe (Riess et al. 1998), and dark matter, which makes up most of the gravitational mass in galaxies and clusters yet remains undetected through non-gravitational means (Rubin et al. 1970), both demand new physics. Although it is possible to account for these observations by introducing new forms of matter and energy, an alternative and equally compelling approach is to modify the gravitational sector itself. For instance, some modified gravity theories, such as $f(R)$ gravity or scalar-tensor theories, attempt to reproduce dark energy-like behavior without invoking a cosmological constant Nojiri et al. 2007. Likewise, alternatives to particle dark matter, such as MOND and its relativistic generalizations, seek to explain galactic dynamics through deviations from Newtonian gravity Milgrom 1983. Additionally, inflation and the initial Big Bang singularity suggest that GR might not hold in the most extreme regimes (Guth 1981).

Second, strong theoretical motivations drive the search for modifications of GR, particularly in the context of quantum gravity. GR, as a classical theory, becomes inadequate at very short distances where quantum effects dominate. Straightforward quantization of the Einstein field equations leads to a non-renormalizable theory, indicating the need for new conceptual frameworks. Moreover, the cosmological constant problem (as discussed in Section (2.1)), which is the enormous discrepancy between the observed value of the vacuum energy density and the much larger value predicted by quantum field theory, highlights a profound conflict between gravity and quantum mechanics (Weinberg 1989). Approaches such as string theory (Polchinski 1998), loop quantum gravity (Rovelli et al. 2004), Hořava-Lifshitz gravity (Horava 2009), and ghost-free nonlocal gravity (Deser and Woodard 2007) all involve deviations from GR, especially in the ultraviolet (UV) regime. These theories often introduce higher-order curvature corrections, new fields, or modified symmetries. In this light, modifying gravity is a necessary step toward unifying general relativity with quantum mechanics.

Third, exploring modifications of gravity can enhance our understanding of GR itself. Even if future experiments confirm GR's predictions to high precision, it is essential to test these predictions against viable alternatives. This process strengthens the empirical foundation of GR and delineates the boundaries of its applicability. Modified gravity theories serve as counterexamples, against which GR can be evaluated and constrained. From a theoretical perspective, deforming a theory and analyzing how its properties change often reveals its core principles. In this sense, attempting to "break" GR may, paradoxically, allow us to better understand why it works so well.

In summary, the motivation to modify gravity stems from the ambition to explain the full range of observed cosmic phenomena, to reconcile gravity with quantum quantum field thoery, and to enhence our understanding of GR itselfs.

2.5.2 Lovelock's theorem

The desire for new gravity theories raises the question how one modifies gravity, and when do we consider a modified gravity theory to be non-equivalent to GR. Hence we must first describe what we consider to be equivalent to GR, which is encoded in Lovelock's theorem. Hence we first must get a notion to when a theory is equivalent to GR.

Consider the left hand side of the Einstein Field Equations (EFE) $A_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu}$ with $G_{\mu\nu}$ the Einstein tensor. It has the following three properties:

1) It is symmetric as it is the sum of the symmetric tensors $G_{\mu\nu}$ and $g_{\mu\nu}$.

2) It is composed of the metric tensor $g_{\mu\nu}$, its first and second derivatives.

3) It is divergence free. This follows from the twice contracted Bianchi Identity, which is consistent with the conservation of the energy momentum tensor $\nabla_\mu T^{\mu\nu}$.

Lovelock's theorem states that in four dimensional spacetime, any tensor that satisfies the three conditions above, must be of the form

$$A_{\mu\nu} = \alpha G_{\mu\nu} + \lambda g_{\mu\nu}. \tag{32}$$

This powerful theorem means that if we try to create a gravitational theory in four dimensions, the only possible symmetric and divergence free equations of motion as obtained from $\mathcal{L}(g_{\mu\nu})$ of second order are the EFE.

As a consequence, one of the three conditions must be broken in order to construct a gravity theory that is truly different from GR. There are several ways to break the conditions of Lovelock's theorem to obtain gravitational actions that are not equivalent to the Einstein-Hilbert action. We discuss a few of them below.

2.5.2.1 Higher order curvature terms

One way to break Lovelock's theorem is by allowing for higher order derivatives of curvature, the so called $f(R)$ *theories*. In the most general form

$$S = \int d^4x\sqrt{-g}f(R), \tag{33}$$

where $f(R)$ is a function of the Ricci scalar R. An advantage of e.g. R^2 gravity is that it is renormalizable (Stelle 1977).

2.5.2.2 Higher dimensional theories

Lovelock's theorem no longer holds when spacetime has more than four dimensions. This opens a pathway to the construction of modified gravity theories by extending the dimensionality of spacetime beyond the familiar 3+1 framework.

From a mathematical perspective, Riemannian geometry is inherently compatible with spacetimes of arbitrary dimension D, so we have the tools to study higher dimensional modified gravity theories. Physically, the idea is strongly motivated by attempts at unification. The Kaluza-Klein (KK) framework, for instance, originally aimed to unify gravity and electromagnetism by postulating a fifth dimension. (Kaluza 1921; Kaluza 2018). Another example of a higher-dimensional theory is superstring theory (Green et al. 2012), which tries to unify quantum theory with gravity. Although it is more common to study extra spatial dimensions in these higher dimensional theories, adding an extra time dimension has also been studied in Shtanov et al. 2003.

The challenge of higher-dimensional theories lies in phenomenology (Clifton et al. 2012). Gravity does not seem to behave like a higher dimensional theory. Consider for example the stability of planetary orbits. The Newtonian gravitational potential of a point mass, scaling as $1/r^{D-3}$. For a spacetime dimension of e.g. 10 in superstring theory, we cannot have stable planetary orbits, hence a higher dimensional theory should not *appear* as 10 dimensional on those scales.

2.5.2.3 Non-locality

Other modifications deal with abandoning the *locality* in GR (Capozziello et al. 2022) by introducing non-local terms into the action. This is referred to as dynamical non-locality.

2.5.2.4 Additional fields

Another way to break Lovelock's theorem is by introducing additional fields to the action, e.g. scalar, vector and tensor fields. The modified gravity theories with an additional scalar field, the ones of interest in this thesis, are referred to as *scalar-tensor theories*. Scalar-tensor theories have a long history. Originally proposed by Pascaul Jordan in the 1950's and later formalized by Brans and Dicke (Brans and Dicke 1961), who formulated a theory where the scalar field mediates gravitational interactions alongside the metric tensor, offering an alternative to general relativity with a varying effective gravitational coupling. Since then, scalar-tensor theories have been extensively studied and generalized. Horndeski's theory (Horndeski 1974) represents the most general scalar-tensor theory yielding second-order field equations. Following this, Higher-Order Scalar-Tensor (HOST) theories were introduced to allow higher-order derivatives in the equations of motion while avoiding ghosts via degeneracy conditions. More recently, beyond-Horndeski theories (Gleyzes et al. 2015) and Degenerate Higher-Order Scalar-Tensor (DHOST) theories (Langlois & Noui 2016) have been formulated, relaxing some restrictions while maintaining the absence of ghosts through carefully imposed degeneracy conditions. These frameworks broaden the landscape of viable scalar-tensor models, allowing richer phenomenology that can still be free of pathological degrees of freedom. An even more recent development is the class of Unitary Degenerate Higher Order Scalar-Tensor (U-DHOST) theories (Felice et al. 2018), which are only degenerate in the unitary gauge.

Scalar-tensor theories and their modern extensions thus constitute an active field of research, offering promising alternatives and generalizations to general relativity that can be tested with cosmological and astrophysical observations.

2.5.3 Ghosts and instabilities

One of the problems that can arise when modifying gravity is *ghosts*, i.e. fields with quanta having negative energy or negative norm indicating an instability in the field theory (Joyce et al. 2015). In most cases, ghost instabilities appear as scalar fields with a wrong-sign kinetic term. For instance, consider the Lagrangian

$$\mathcal{L}_{\text{ghost}} = \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 \tag{34}$$

At first glance, this may seem harmless, especially since the sign of the kinetic term can depend on the chosen metric signature. However, the issue becomes critical when such a field couples to other healthy fields with the correct sign. For example:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 + \lambda\phi^2\chi^2 \tag{35}$$

Here, ϕ is a standard scalar field with the usual kinetic term, while χ has the wrong sign. This mismatch leads to a severe instability: the vacuum becomes unstable to spontaneous production of $\phi\phi + \chi\chi$ pairs at no energy cost. Because the energy gained from producing the ϕ pair is exactly canceled by the negative energy of the ghostly χ pair, such events are not suppressed. As a result, the vacuum can decay into arbitrarily many pairs of normal and ghost particles at no energy cost, occurring with an unbounded (infinite) rate. This signals that the theory is ill-defined.

To construct viable field theories, it is crucial to avoid such ghost modes. In practice, ghosts frequently arise when higher-derivative terms are included in the Lagrangian. A key result known as *Ostrogradsky's theorem* (Ostrogradsky 1850), states that for a nondegenerate Lagrangian which depends on higher derivatives, the Hamiltonian is necessarily unbounded. We will show this through an example (Creminelli et al. 2005). We consider a higher-derivative theory with the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\psi)^2 + \frac{1}{2\Lambda^2}(\Box\psi)^2 - V(\psi) \tag{36}$$

where $\Box = \partial^\mu\partial_\mu$ is the D'Alembertian and Λ is the cutoff scale of the effective field theory. At energies well below the cutoff scale Λ, the higher-derivative term in the Lagrangian, $\frac{1}{2\Lambda^2}(\Box\psi)^2$, is suppressed because the derivatives acting on the fields are small compared to Λ. However, as the energies approach Λ, the term $\frac{1}{2\Lambda^2}(\Box\psi)^2$ is no longer negligible, and the higher-order corrections that were previously ignored become important. Notably, the equations of motion derived from equation (36) are fourth order, signaling the hidden presence of a ghost.

To make this explicit, we can introduce an auxiliary field χ to rewrite the theory

$$\mathcal{L} = -\frac{1}{2}(\partial\psi)^2 + \chi\psi - \frac{\Lambda^2}{2}\chi^2 - V(\psi) \tag{37}$$

These theories are equivalent, which can be seen from solving the equation of motion for χ yields $\chi = \psi/\Lambda^2$, and substituting this back into the Lagrangian, from which equation (36) is recovered. We now want to reduce the Lagrangian to a two-derivative kinetic Lagrangian, from which we can read if there is a ghost. To decouple the kinetic mixing, we redefine $\psi = \phi - \chi$. After integration by parts, the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{\Lambda^2}{2}\chi^2 - V(\phi,\chi) \tag{38}$$

We now see that the original higher-derivative theory is equivalent to a theory with one healthy scalar field ϕ and a ghostly scalar field χ. Importantly, the presence of a ghost does not automatically render a theory unviable. If the mass of the ghost lies above the cutoff Λ, we can treat it as an artifact of truncating the effective theory. In this case, the ghost does not affect low-energy physics, and we may hope that a UV-complete theory resolves the instability.

However, if a ghost appears *within* the energy range where the effective field theory is supposed to be valid, then the theory becomes unpredictive and must be reconsidered or abandoned.

2.5.3.1 Shadowy mode

While ghosts correspond to extra dynamical degrees of freedom that are associated with negative kinetic energy and signal a true instability in the theory, there exists a phenomenon known as the *shadowy mode*, which mimics a propagating degree of freedom under certain coordinate choices but does not correspond to an actual dynamical dynamical degree of freedom (Felice et al. 2018, Felice et al. 2021).

The idea of a *generalized instantaneous mode*, or *shadowy mode*, can be illustrated with a simple example, as presented in Felice et al. 2018, of a field described by a Lagrangian that does not contain time derivatives. Consider the following Lagrangian for a scalar field ψ in flat Minkowski spacetime

$$L[\psi] = \frac{1}{2}\psi\Delta\psi. \tag{39}$$

Here, $\Delta = \nabla^2$ is the standard Euclidean Laplacian acting on the spatial coordinates. This Lagrangian leads to the Laplace equation

$$\Delta\psi = 0, \tag{40}$$

which is purely elliptic and contains no time derivatives, implying that ψ is non-dynamical and determined instantaneously across space. Now suppose we perform a change of coordinates, introducing a new time variable $t' = t + vx$, where $v \neq 0$, while keeping the spatial coordinates unchanged. In these new coordinates (t', x', y', z'), the same Lagrangian becomes:

$$L[\psi] = -\frac{1}{2}\left[(v\,\partial_{t'}\psi + \partial_{x'}\psi)^2 + (\partial_{y'}\psi)^2 + (\partial_{z'}\psi)^2\right] \ni -\frac{v^2}{2}(\partial_{t'}\psi)^2 \tag{41}$$

Now, the Lagrangian appears to contain a negative kinetic term for ψ. This gives the illusion that ψ has become a propagating degree of freedom in this frame. However, this apparent dynamical behavior is deceptive: the time derivative arises due to coordinate mixing, and the underlying equation of motion is still elliptic. The seemingly dynamical DoF is referred to as the shadowy mode.

In Felice et al. 2018, this idea was further explored in the context of a specific class of higher-order scalar-tensor theories known as U-DHOST theories, which are degenerate only in the *unitary gauge*. In this gauge the coordinates are chosen such that the scalar field is spatially uniform and is restricted to configurations where the gradient of the scalar field is timelike. This can be either achieved by choosing the scalar function to be a function of time only, or by introducing a gauge fixing term. In such models, an apparent extra degree of freedom seems to emerge when the scalar field is described in a generic (non-unitary) coordinate system. At first glance, this may appear to signal an Ostrogradsky instability due to the presence of higher-order equations of motion. However, by studying a simplified toy model in which gravity was ignored, it was demonstrated that the apparent extra mode does not actually propagate but is instead governed by an elliptic equation. It is not dynamical, but is instead determined instantaneously across space at each moment in time by the configuration of the other fields. The shadowy mode is defined in a coordinate-independent manner as a mode living on a spacelike hypersurface, exactly like the shadows produced by people on a beach. While in the unitary gauge this elliptic character of the EoM is manifest, in other gauges it becomes obscured by the mixing of temporal and spatial coordinates. The discrepancy in apparent degrees of freedom between gauges is thus resolved by recognizing that boundary conditions, such as regularity at spatial infinity, are implemented implicitly in the unitary gauge, whereas they must be enforced explicitly in non-unitary gauges. As such, the system contains the same number of physical degrees of freedom in any coordinate frame, and the shadowy mode does not indicate a genuine instability.

Our theory of interest, VCDM, is also supplemented by a shadowy mode. In Section 3.3

we will see how it is also governed by an elliptic equation and can be eliminated by imposing appropriate boundary conditions, namely regularity at infinity.

2.6 Minimally Modified Gravity

2.6.1 Motivation

The number of local physical degrees of freedom in GR is two, corresponding to the two transverse, traceless tensor polarizations of gravitational waves (GWs). This fundamental aspect has been robustly confirmed by gravitational wave observations (Abbott et al. 2016). The Advanced LIGO-Virgo network achieved the first direct constraints on the polarization of gravitational waves. The signal GW170814, detected by both Advanced LIGO and Virgo, showed a strong preference for a model with purely tensor polarization modes over purely vector or tensor polarizations (Isi & Weinstein 2017; Abbott et al. 2017). The the multi-messenger event GW170817, which provided stringent tests of the polarization content of GWs and showed consistency with only the two tensor polarizations predicted by GR, favoring two local physical DoF over additional scalar or vector modes (Abbott et al. 2019). However, the current LIGO and Virgo detectors have limited sensitivity to discern the polarization content of gravitational-wave transients, the addition of new detectors such as KAGRA (Aso et al. 2013) and LIGO-India (Iyer et al. 2011) enable more precise measurements of gravitational-wave polarizations and break existing degeneracies. Given these results, any viable modification of GR should preserve this minimal number of propagating degrees of freedom. This motivates the study of Minimally Modified Gravity (MMG) theories, which alter gravitational dynamics without introducing extra local degrees of freedom beyond the two standard tensorial modes.

The motivation to explore MMG arises from the desire to extend GR in a way that maintains its core observational successes, especially the nature of GWs, while potentially addressing open theoretical challenges such as the cosmological constant problem, dark energy, or quantum gravity effects. By ensuring only two propagating modes, MMG theories represent the most conservative class of modifications, balancing theoretical innovation with observational viability. This approach enables us to investigate new physics while respecting the stringent constraints from gravitational wave experiments. We divide MMG theories into the two types (Aoki et al. 2019; Felice et al. 2020) which we will discuss in the next section.

2.6.2 Type-I and type-II minimally modified gravity

Before we discuss the two types we first get a better understanding of how matter couples in modified gravity theories. Let us study a scalar-tensor theory as is discussed in Aoki et al. 2019. In the *Jordan frame* (also called the matter frame), the matter fields are directly coupled to the metric, and the action takes the form

$$I = \frac{1}{2}\int d^4x\sqrt{-g^J}\left[\Omega^2(\phi)R[g^J] + \cdots\right] + I_{\text{matter}}[g^J_{\mu\nu};\text{matter}], \tag{2.1}$$

where the "$\cdots$" represents terms like the kinetic energy of the scalar field ϕ, and the matter part of the action is minimally coupled to $g^J_{\mu\nu}$. Changing to the *Einstein frame*, performing a conformal transformation of the metric $g^E_{\mu\nu} = \Omega^2(\phi)g^J_{\mu\nu}$, the action becomes

$$I = \frac{1}{2}\int d^4x\sqrt{-g^E}\left[R[g^E] + \cdots\right] + I_{\text{matter}}[\Omega^{-2}(\phi)g^E_{\mu\nu};\text{matter}], \tag{2.2}$$

In this setup, the gravitational part of the action looks just like that of GR. It is the Einstein-Hilbert term, but the matter now couples to both the scalar field ϕ and the Einstein-frame metric $g^E_{\mu\nu}$ through the combination $\Omega^{-2}(\phi)g^E_{\mu\nu}$. So, while the gravitational dynamics mimic those of GR, the matter, gravity is still modified due to the non-trivial matter coupling. Because the theory can be expressed as GR plus additional scalar and matter fields, this kind of theory is categorized as *type-I*. On the other hand, there are more general modified gravity theories where no Einstein frame can be defined. These are labeled as *type-II* theories. In summary:

- Type-I: This is a theory that propagates two local physical degrees of freedom (i.e. the two polarizations of transverse-traceless gravitational waves) in which there exist an Einstein frame. This means a frame in which all GR solutions are solutions of the theory under consideration at least locally if the matter is minimally coupled to the metric. By a change of variables these theories can recast as GR plus matter fields.

- Type-II: Theories with two local gravitational degrees of freedom that do not have an Einstein frame. By definition, by a change of variables it cannot be recast as GR plus extra degree(s) of freedom and matter fields.

A systematic study of type-I theories are presented in Aoki et al. 2018. Examples of type-II theories are Cuscuton (Afshordi et al. 2007), minimal theory of massive gravity (Felice & Mukohyama 2016) and our theory of interest for this thesis, VCDM.

2.7 VCDM

The type-II minimally modified gravity called VCDM is introduced in Felice et al. 2020. Its construction and degrees of freedom will be discussed in this subsection.

2.7.1 Construction

The theory is constructed through the following steps:

1. We start from the Hamiltonian of GR in the ADM formalism.
2. We perform a canonical transformation to a new frame via a generating functional that depends on new variables and old momenta.
3. We introduce a cosmological constant term in the new frame.
4. We add a gauge fixing term, in order to keep the theory minimal, i.e. with only two local physical degrees of freedom.
5. We perform the inverse canonical transformation to go back to the original frame.

We will discuss the steps in more detail below, but for the full construction the reader is referred to Felice et al. 2020. We begin with the Hamiltonian in GR in the ADM formalism. We start with the metric

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -N^2dt^2 + \gamma_{ij}(N^idt + dx^i)(N^jdt + dx^j). \tag{42}$$

where N the lapse function, N^i the shift vector and γ_{ij} is the induced 3D spatial metric on the t-constant hypersurface, and the Hamiltonian

$$H_{\mathrm{tot}} = \int d^3x \left[N\mathcal{H}_0(\gamma,\pi) + N^i\mathcal{H}_i(\gamma,\pi) + \lambda\pi_N + \lambda^i\pi_i\right], \tag{43}$$

with λ and λ_i Lagrange multipliers, where

$$\mathcal{H}_0 = \frac{2}{M_P^2\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} - \frac{M_P^2\sqrt{\gamma}}{2}R(\gamma) \tag{44}$$

$$\mathcal{H}_i = -2\sqrt{\gamma}\gamma_{ij}D_k\left(\frac{\pi^{jk}}{\sqrt{\gamma}}\right) \tag{45}$$

with D_k the spatial covariant derivative compatible with γ_{ij}. For step 2. a generating functional is introduced

$$F = F\left(\mathfrak{N}, \mathfrak{N}^i, \Gamma_{ij}, \pi_{\mathfrak{N}}, \pi_i, \pi^{ij}\right), \tag{46}$$

depending on the new variables $\mathfrak{N}, \mathfrak{N}^i, \Gamma_{ij}$, and the old momenta. Schematically, it depends on new variables q and old momenta p, such that $q = -\frac{\partial F}{\partial p}$ and $\tilde{p} = -\frac{\partial F}{\partial \tilde{q}}$. The form of the generating functional was, for simplicity, chosen to be

$$F = -\int d^3x \left[M_P^2\sqrt{\Gamma}f(\phi,\psi) + \mathfrak{N}^i\pi_i\right] \tag{47}$$

where

$$\phi = \frac{1}{M_P^2\sqrt{\Gamma}}\pi^{ij}\Gamma_{ij}, \quad \psi = \frac{1}{M_P^2\sqrt{\Gamma}}\pi_N\mathfrak{N}. \tag{48}$$

The Hamiltonian is then canonically transformed by computing the functional derivatives that establish the relations between old and new variables, which are then used to rewrite the Hamiltonian in terms of the transformed variables. We promoted ψ and ϕ to two independent three dimensional scalar fields. Furthermore, a cosmological constant term (step 3.) and a gauge-fixing (step 4.) term were added. The resulting Hamiltonian is given by

$$H_{\text{tot}} = \int d^3x \Big[\mathfrak{N} f_\psi \mathcal{H}_0(\Gamma, \Pi, \phi, \psi) + \mathfrak{N}^i \mathcal{H}_i(\Gamma, \Pi, \phi, \psi) + \tilde{\lambda} \Pi_{\mathfrak{N}} + \lambda^i \Pi_i + f_\phi^{3/2} \sqrt{\Gamma} \lambda_C C(\Gamma, \Pi, \phi, \psi) + f_\phi^{3/2} \sqrt{\Gamma} \lambda_D D(\Gamma, \Pi, \phi, \psi) + \lambda_\phi \pi_\phi + \lambda_\psi \pi_\psi + f_\phi^{3/2} \sqrt{\Gamma} \lambda_g^i f \partial_i \phi + M_P^2 \mathfrak{N} \sqrt{\Gamma} \tilde{\Lambda} \Big] \tag{49}$$

where

$$C = \phi - \frac{1}{M_P^2 \sqrt{\Gamma}} \Gamma^{ij} \frac{1}{f_\phi} \left(\Pi_{ij} - \frac{M_P^2}{2\sqrt{\Gamma}} \Gamma_{ij} \left(f - f_\phi \phi - f_\psi \psi \right) \right), \tag{50}$$

$$D = \psi - \frac{1}{M_P^2 \sqrt{\Gamma}} \frac{\Pi_{\mathfrak{N}}}{f_\psi \mathfrak{N}}. \tag{51}$$

and $f_\phi = \frac{\partial f}{\partial \phi}$ and $f_\psi = \frac{\partial f}{\partial \psi}$. Then in step 5., the inverse canonical transformation is applied to this modified Hamiltonian in order to return to the original frame. The resulting theory deviates from GR but continues to propagate only two physical degrees of freedom. The full Hamiltonian is given by:

$$\begin{aligned} H_{\text{tot}} = \int d^3x \Bigg[& N \mathcal{H}_0(\gamma, \pi) + N^i \mathcal{H}_i(\gamma, \pi) + \lambda \pi_N + \lambda^i \pi_i + \sqrt{\gamma}\, \lambda_C \left(\phi - \frac{f_\phi^{1/2}}{M_P^2} \frac{\pi^{ij}}{\sqrt{\gamma}} \gamma_{ij} \right) \\ & + \sqrt{\gamma}\, \lambda_D \left(\psi - \frac{f_\phi^{3/2}}{M_P^2 f_\psi} \frac{\pi_N}{\sqrt{\gamma}} N \right) + \lambda_\phi \pi_\phi + \lambda_\psi \pi_\psi \\ & + \sqrt{\gamma}\, \lambda_{gf}^i\, \partial_i \phi + \frac{M_P^2}{f_\psi f_\phi^{3/2}} N \sqrt{\gamma} \tilde{\Lambda} \Bigg]. \end{aligned} \tag{52}$$

An analysis of the system's dynamical constraints shows that terms of second or higher order in ψ within $f(\phi, \psi)$ do not contribute. Hence, we can write the function $f(\phi, \psi)$ at first order in ψ as:

$$f(\phi, \psi) = \bar{f}(\phi) + f_1(\phi)\psi, \tag{53}$$

where $\bar{f}(\phi)$ is the leading-order term and $f_1(\phi)$ captures the linear contribution.

This leads to the final simplified form of the Hamiltonian:

$$\begin{aligned} H_{\text{tot}} = \int d^3x \Bigg[& N \mathcal{H}_0(\gamma, \pi) + N^i \mathcal{H}_i(\gamma, \pi) + \lambda \pi_N + \lambda^i \pi_i \\ & + \sqrt{\gamma}\, \lambda_C \left(\phi - \frac{f_0^{1/2}}{M_P^2} \frac{\pi^{ij}}{\sqrt{\gamma}} \gamma_{ij} \right) + \lambda_\phi \pi_\phi \\ & + \sqrt{\gamma}\, \lambda_{gf}^i\, \partial_i \phi + \frac{N \sqrt{\gamma}}{f_1 f_0^{3/2} M_P^2} \tilde{\Lambda} \Bigg], \end{aligned} \tag{54}$$

with $f_0 \equiv \frac{d\bar{f}}{d\bar{\phi}}$ and $\mathcal{H}_0$ and $\mathcal{H}_i$ as defined in equations (26)-(27). At this stage, assuming we are interested in exploring the regime where the general relativity limit corresponds to $C \to 0$, we introduce a field redefinition as follows

$$\phi = f_0^{1/2}\,\bar{\phi}, \tag{55}$$

$$\frac{f_0^{1/2}}{M_P^2}\lambda_C = \bar{\lambda}_C, \tag{56}$$

$$\frac{M_P^2\tilde{\Lambda}}{f_1 f_0^{3/2}} = M_P^2 V(\bar{\phi}). \tag{57}$$

Next, redefining λ_ϕ and $\lambda_{\rm gf}$, and dropping the bars for simplicity, we arrive at the total Hamiltonian:

$$\begin{aligned} H_{\rm tot} = \int d^3x \Bigg[& N\mathcal{H}_0(\gamma,\pi) + N^i\mathcal{H}_i(\gamma,\pi) + \sqrt{\gamma}\lambda_C\left(M_P^2\phi - \frac{\pi^{ij}}{\sqrt{\gamma}}\gamma_{ij}\right) \\ & + \lambda_\phi\pi_\phi + \sqrt{\gamma}M_P^2\lambda^i_{\rm gf}\,\partial_i\phi + N\sqrt{\gamma}M_P^2 V(\phi)\Bigg], \end{aligned} \tag{58}$$

where N and N^i act as Lagrange multipliers. In the limit $V \to$ constant, we retrieve general relativity (Felice et al. 2020).
Finally, performing a Legendre transformation and relabeling the Lagrange multipliers, the Lagrangian for the theory becomes

$$\mathcal{L} = \frac{M_P^2}{2}N\sqrt{\gamma}\left[R + K^{ij}K_{ij} - K^2 - 2V(\phi) - \frac{2}{N}\lambda^i_{gf}\,\partial_i\phi - \frac{3}{2}\lambda^2 - 2\lambda(K+\phi)\right]. \tag{59}$$

By working in the so-called unitary gauge, the gravitational action of VCDM can explicitly be written

$$I_g = M_{\rm Pl}^2\int dtd^3x\, N\sqrt{\gamma}\left\{\frac{1}{2}\left[R + K_{ij}K^{ij} - K^2 - 2V(\phi)\right] - \frac{\lambda^i_{\rm gf}}{N}\partial_i\phi - \frac{3}{4}\lambda^2 - \lambda(K+\phi)\right\} \tag{60}$$

where $R = {}^3R$ is the three dimensional Ricci scalar, $M_{\rm Pl} = 1/\sqrt{8\pi G_N}$ which we will set to unity, the extrinsic curvature K_{ij}, its inverse K^{ij} and the trace K are given by

$$K_{ij} = \frac{1}{2N}\left(\partial_t\gamma_{ij} - D_iN_j - D_jN_i\right) \tag{61}$$

$$K_{ij} = \gamma_{ik}\gamma_{jl}K^{kl} \tag{62}$$

$$K = \gamma^{ij}K_{ij} \tag{63}$$

where D_i is the three dimensional covariant derivative and N_i is given by $N_i = \gamma_{ij}N^j$. The first part of the action,

$$I_{g,\rm GR} = \frac{1}{2}\int dtd^3x\sqrt{\gamma}\,N\left(R + K_{ij}K^{ij} - K^2\right), \tag{64}$$

is just the gravitational action from GR in the ADM formalism. In contrast to GR, VCDM additionally also depends on the scalar field ϕ and a free potential function $V(\phi)$, and two Lagrange multipliers. The lagrange multipliers λ^i_{gf} and λ constrain the scalar field ϕ so that the theory only propagates two local physical DoF.

Since we are interested in spherical collapse, we add a matter action. As a matter source, we consider a canonical massless scalar field ψ described by the action

$$I_m = \frac{1}{2}\int dtd^3x\, N\sqrt{\gamma}\left[(\partial_\perp\psi)^2 - \gamma^{ij}\partial_i\psi\partial_j\psi\right], \quad \partial_\perp \equiv \frac{1}{N}(\partial_t - N^i\partial_i), \tag{65}$$

which is just the action of a scalar field in GR in ADM formalism. Starting with the standard scalar field action in GR:

$$I_{m,\mathrm{GR}} = -\frac{1}{2}\int d^4x\sqrt{-g}\, g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \tag{66}$$

In ADM formalism:

$$\sqrt{-g} = N\sqrt{\gamma}, \qquad g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = \frac{N^i}{N^2}, \qquad g^{ij} = \gamma^{ij} - \frac{N^iN^j}{N^2} \tag{67}$$

Then the kinetic term becomes:

$$g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi = -\frac{1}{N^2}(\partial_t\psi - N^i\partial_i\psi)^2 + \gamma^{ij}\partial_i\psi\partial_j\psi \tag{68}$$

Define the derivative along the normal vector:

$$\partial_\perp\psi \equiv \frac{1}{N}(\partial_t\psi - N^i\partial_i\psi) \quad \Rightarrow \quad (\partial_\perp\psi)^2 = \frac{1}{N^2}(\partial_t\psi - N^i\partial_i\psi)^2 \tag{69}$$

So the scalar field action becomes:

$$I_{m,\mathrm{GR}} = -\frac{1}{2}\int dt\, d^3x\, N\sqrt{\gamma}\left[-(\partial_\perp\psi)^2 + \gamma^{ij}\partial_i\psi\partial_j\psi\right] \tag{70}$$

$$= \frac{1}{2}\int dt\, d^3x\, N\sqrt{\gamma}\left[(\partial_\perp\psi)^2 - \gamma^{ij}\partial_i\psi\partial_j\psi\right], \tag{71}$$

which is indeed the matter action I_m in equation (65). The total action is

$$\begin{aligned} S &= I_g + I_m \\ &= \int dt\, d^3x\, N\sqrt{\gamma}\Bigg\{\frac{1}{2}\left[R + K_{ij}K^{ij} - K^2 - 2V(\phi)\right] - \frac{\lambda^i_{gf}}{N}\partial_i\phi - \frac{3}{4}\lambda^2 - \lambda(K+\phi) \\ &\quad + \frac{1}{2}\left[(\partial_\perp\psi)^2 - \gamma^{ij}\partial_i\psi\partial_j\psi\right]\Bigg\} \end{aligned} \tag{72}$$

2.7.2 Black holes and spherical collapse in VCDM

2.7.3 Motivation

Black holes are no longer just theoretical predictions of general relativity, they are firmly established astrophysical objects. The first direct image of a black hole shadow, captured by the

Event Horizon Telescope (Akiyama et al. 2019), provided striking visual confirmation of the existence of event horizons. In parallel, gravitational wave detections from LIGO and Virgo, starting with the landmark observation GW150914 (Abbott et al. 2016), have revealed the dynamics of binary black hole mergers with precision that matches general relativity's predictions. These observations have moved black holes from speculative objects to essential testing grounds for any theory of gravity.

Because of this, any modified theory of gravity that hopes to be taken seriously must at least admit black hole solutions that are physically realistic and consistent with these observations.

Studying spherically symmetric collapse in such theories serves two purposes. First, it allows us to check whether standard black hole-like solutions are still possible, or if modifications lead to qualitatively new outcomes. Second, it helps identify the regimes where the theory remains predictive, or where a UV completion becomes necessary. If a theory like VCDM is to remain viable on cosmic scales while also accounting for compact objects, it must reproduce the known features of black holes, or at least provide testable alternatives.

2.7.4 Spherically symmetric vacuum solutions and collapse

In Felice et al. 2021, spherically symmetric vacuum solutions were studied to explore implications and predictions of VCDM for strong gravitational systems, i.e. black holes. They showed that, even though VCDM propagates only two local physical degrees of freedom, the vacuum solutions are generally distinct from those in GR, leading to the breakdown of Birkhoff's theorem. The solutions in this theory is determined not just by the cosmological constant and the mass, but also by several free functions time. These functions are fixed by imposing appropriate asymptotic conditions at spatial infinity or by setting boundary conditions at a finite radius. This behavior implies the presence of non-propagating modes governed by elliptic rather than hyperbolic equations, commonly referred to as instantaneous or shadowy modes, whose evolution is shaped by boundary conditions rather than initial conditions (see Section 2.5.3.1).

In the static limit, the solutions reduce to either the Schwarzschild or Schwarzschild-(A)dS solutions. The *Schwarzschild-(A)dS* solution describes a static, spherically symmetric black hole in a universe with a positive (de Sitter) or negative (Anti-de Sitter) cosmological constant, generalizing the usual Schwarzschild solution by incorporating large-scale curvature of spacetime. The effective cosmological constant inferred from the curvature of this static solution is influenced not only by the gravitational action itself but also by how spacetime is foliated into constant-t slices, an effect stemming from the breaking of temporal diffeomorphism. However, after imposing appropriate asymptotic conditions such that the solutions represent compact objects in the corresponding cosmological setup, the standard Schwarzschild or Schwarzschild-(A)dS metric was recovered and the effective cosmological constant was consistent with the one inferred from the action.

Static, spherically symmetric configurations of stars composed of barotropic perfect fluids are examined within VCDM in Felice et al. 2022. By imposing physically reasonable boundary conditions on the Misner-Sharp mass, they found that the resulting solutions exactly match those in GR and satisfy the standard Tolman-Oppenheimer-Volkoff (TOV) equation.

In order to better understand the structure of compact objects in the strong gravity regime, gravitational collapse in VCDM must be studied. In Felice et al. 2023, the gravitational collapse of a cloud of dust in VCDM was studied. They found that the collapse corresponds to a particular foliation of the Oppenheimer-Snyder solution in GR with a constant trace of the extrinsic curvature with respect to the constant time foliation t. The final state of the collapse

in this solution leads to a static configuration with the lapse function vanishing at a radius inside the apparent horizon. This is reached within an infinite time-t with t the cosmological time (i.e. an observer located far away from the collapse). The vanishing lapse implies the necessity for a UV completion to describe the physics inside the black hole.
In Jalali et al. 2024, the spherically symmetric collapse of a massless scalar field was studied in VCDM. They integrated out the so called shadowy mode by imposing asymptotically flat spacetime in the Minkowski slicing. This reduced the equations of motion to those of GR with the *maximal slicing* (the trace of the extrinsic curvature being zero). VCDM however, does not have time diffeomorphism so the time slicing cannot be changed without changing the physical meaning. They then numerically evolved the system to see if an apparent horizon formed. Depending on the amplitude of the massless scalar field, a black hole would form. For small amplitudes, there would not be an apparent horizon formation, nor would there be a singularity or breakdown of time slicing. However, there would be an apparent horizon formation for sufficiently large amplitudes of the scalar field. After the black hole formation, the solution is described by a static configuration, i.e. Schwarzschild geometry with a finite and time-dependent lapse function. The singularity at the center of the spherical symmetry is never reached, and hence here a UV completion is also needed.

In this thesis we study a special case of the equations of motion in Jalali et al. 2024, namely the case where the shift is zero. We study if an apparent horizon would form and if we would also recover a Schwarzschild geometry outside of the horizon if one were to form.

3 Setup

In this section we derive (for full derivations see Appendix (B)) the equations of motion, assuming a spherically symmetric ansatz.

3.1 Equations of motion

Since we are interested in spherically symmetric scalar collapse, we adopt the spherically symmetric ansatz

$$N = \alpha(t,r), \quad N_i dx^i = \beta(t,r)dr, \quad \gamma_{ij}dx^i dx^j = dr^2 + \Phi(t,r)d\Omega_2^2, \quad \psi = \psi(t,r) \tag{73}$$

where α relates to the lapse, β to the shift, Φ is the areal radius and $d\Omega_2^2$ the metric of the unit 2-sphere. Furthermore we write

$$\phi = \phi(t,r), \quad \lambda = \lambda(t,r) \quad \lambda^i_{gf}\partial_i = \tilde{\lambda}(t,r)\partial_r \tag{74}$$

To simplify our equations of motion we will use the trace of the extrinsic curvature which is given by (for derivation see Appendix B, equation 215)

$$K = 2\partial_\perp \ln\Phi - \frac{1}{\alpha}\partial_r\beta, \tag{75}$$

and furthermore introduce the following variables

$$Q \equiv K^r_r - \frac{1}{3}K = -\frac{2}{3}(\partial_\perp \ln\Phi + \partial_r\beta) \tag{76}$$

$$P \equiv \partial_\perp\psi \tag{77}$$

$$a \equiv \partial_r \ln\alpha, \tag{78}$$

where Q is the traceless part of the extrinsic curvature, P is the derivative of the field ψ in the direction normal to the hypersurfaces and a is related to the spatial derivative of the lapse. Within the adopted ansatz, the position of the apparent horizon (AH) $r = r_{\mathrm{AH}}$ can be found by solving

$$g^{\Phi\Phi}|_{r=r_{\mathrm{AH}}} = 0 \tag{79}$$

where $g^{\Phi\Phi}$ is given by

$$g^{\Phi\Phi} = -(\partial_\perp\Phi)^2 + (\partial_r\Phi)^2 = -\left(-\frac{1}{2}Q + \frac{1}{3}K\right)^2\Phi^2 + (\partial_r\Phi)^2 \tag{80}$$

Note the difference in sign in front of Q with Jalaili et al. 2024. The derivation of equation (79) and (80) can be found in Appendix A.

We introduced the variables P, Q, a, and K into the action and imposed their defining relations via constraints with Lagrange multipliers. The Lagrange multipliers were then solved for by varying the action with respect to P, Q, a, and K (see Appendix B). The total action on the spherical background then yields

$$\begin{aligned} S^{\text{spher}} = \int d\theta d\phi dr dt \sin\theta \Bigg[&\alpha\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \alpha\Phi^2 \Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) \\ &- V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi - \frac{3}{4}\lambda^2 - \lambda(K + \phi) + \frac{1}{2}\left[P^2 - (\partial_r\psi)^2\right] + P(\partial_\perp\psi - P) \\ &- \left(\frac{2}{3}K + \lambda\right)\left(2\partial_\perp \ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right) - Q\left(\partial_\perp \ln\Phi + \frac{1}{\alpha}\partial_r\beta + \frac{3}{2}Q\right)\Big]\Bigg] \end{aligned} \tag{81}$$

The constraint equations are

$$\frac{\partial_r^2\Phi}{\Phi} = \frac{1 - (\partial_r\Phi)}{2\Phi^2} - \frac{3}{8}Q^2 - \frac{1}{4}P^2 - \frac{1}{4}(\partial_r\psi)^2 - \frac{1}{2}V(\phi) + \frac{1}{6}\phi^2 \tag{82}$$

$$\partial_r Q = P\partial_r\psi - 3Q\partial_r \ln\Phi \tag{83}$$

$$\partial_r\phi = 0 \tag{84}$$

where equation (82) comes from taking the variation of S^{spher} with respect to the lapse α, equation (83) from taking the variation with respect to the shift β and equation (84) from taking the variation of S^{spher} with respect to $\tilde{\lambda}$. The dynamical equations are

$$\partial_t\psi = \alpha P + \beta\partial_r\psi \tag{85}$$

$$\partial_t P = \alpha\Big[- KP + (a + 2\partial_r \ln\Phi)\,\partial_r\psi + \partial_r^2\Psi\Big] + \beta\partial_r P \tag{86}$$

$$\partial_t\Phi = \alpha\left(\frac{1}{3}K - \frac{1}{2}Q\right)\Phi + \beta\partial_r\Phi \tag{87}$$

$$\begin{aligned} \partial_t Q = \alpha\Bigg[&- KQ + 2\partial_r a + 2a^2 + 2a\partial_r \ln\Phi + \frac{1 - (\partial_r\Phi)^2}{\Phi^2} - \frac{9}{4}Q^2 - \frac{3}{2}P^2 - \frac{1}{3}\phi^2 + V(\phi) \\ &+ \frac{1}{2}(\partial_r\psi)^2\Bigg] + \frac{4}{3}\partial_t\phi + \beta\partial_r Q \end{aligned} \tag{88}$$

The first spatial derivative of the lapse and shift follow from the definitions of a, Q and K and are given by

$$\partial_r \ln\alpha = a \tag{89}$$

$$\partial_r\beta = -\alpha\left(Q + \frac{1}{3}K\right) \tag{90}$$

The equations that determine the Lagrangia multipliers are

$$\lambda = -\frac{2}{3}(K + \phi) \tag{91}$$

$$2\tilde{\lambda}\partial_r \ln\Phi + \partial_r\tilde{\lambda} = -\alpha\left[\frac{2}{3}(K + \phi) - V'(\phi)\right] \tag{92}$$

The dynamical equations of Q and ϕ are coupled. We will adopt integrability conditions to decouple these equations and obtain the remaining non-dynamical equations.

3.2 Integrability conditions

Additional conditions are needed to find the independent dynamical equations for Q and ϕ. We will apply so-called integrability conditions (C. J. Papachristou 2019). Considering a system of partial differential equations (PDE)

$$\frac{\partial u}{\partial x} = P(x,y), \qquad \frac{\partial u}{\partial y} = Q(x,y), \tag{93}$$

the system (93) only has a solution for u, i.e. the system is only *integrable*, if the equations are *compatible* with each other. The *compatibility condition* or *integrability condition* is found by equating the partial derivative of $P(x,y)$ with respect to y, with the partial derivative with respect to x of $Q(x,y)$,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{94}$$

If the integrability condition (94) is not satisfied, the system of partial differential equations (93) has no solution for u. Integrability conditions can be used to find the independent equations of a system of differential equations. Consider for example the system

$$\begin{cases} u_x + u_y + y u_z = 0 \\ u_y = 0. \end{cases} \tag{95}$$

By taking the derivative with respect to y and applying the integrability conditions $u_{xy} = u_{yx}$ and $u_{yz} = u_{zy}$, the we find

$$u_{xy} + u_{yy} + u_z + y u_{zy} = 0, \tag{96}$$

and because of the integrability conditions

$$u_{xy} = u_{yx} \tag{97}$$
$$u_{zy} = u_{yz} \tag{98}$$

we find

$$u_z = 0. \tag{99}$$

Substituting $u_y = 0$ and $u_z = 0$ into the first equation of the system

$$u_x + 0 + y \cdot 0 = 0 \Rightarrow u_x = 0. \tag{100}$$

Hence we have found two new equations from using integrability conditions. From $u_x = 0$ and $u_z = 0$ we find that our system reduces to just one independent equation, namely

$$\begin{cases} u_y = 0. \end{cases} \tag{101}$$

We apply the same concept to our system of EoM, to find the independent equations. For our system to have a solution for Q and ϕ, the following compatibility/integrability conditions must hold

$$\boxed{\partial_r \partial_t Q = \partial_t \partial_r Q} \text{ and } \boxed{\partial_r \partial_t \phi = \partial_t \partial_r \phi}. \tag{102}$$

We first define the following expressions to simplify our obtained equations after the applying of the integrability conditions

$$\begin{aligned} \mathrm{eq}(103) = \alpha\Big[- KQ - \frac{1}{4}Q^2 + \frac{2}{3}(a^2 + \partial_r a - a\partial_r \ln\Phi) + \frac{1-(\partial_r\Phi)^2}{\Phi^2} - \frac{1}{6}P^2 + \frac{1}{2}(\partial_r\psi)^2 \\ + \frac{1}{9}\phi^2 - \frac{1}{3}V(\phi)\Big] + \beta(P\partial_r\psi - 3Q\partial_r\ln\Phi) \end{aligned} \tag{103}$$

$$\begin{aligned} \mathrm{eq}(104) = -3a\partial_r a + a\left[\frac{(\partial_r\Phi)^2 - 1}{\Phi^2} + \frac{9}{4}Q^2 + \frac{3}{2}P^2 + \frac{1}{2}(\partial_r\psi)^2\right] - 2a^2\partial_r\ln\Phi - a^3 \\ + P(3Q\partial_r\psi + 2\partial_r P) - 9Q^2\partial_r\ln\Phi + 2(a\partial_r\ln\Phi - \partial_r a)\partial_r\ln\Phi. \end{aligned} \tag{104}$$

Using these expressions for simplification, we find the dynamical equations for Q now reads

$$\partial_t Q = \mathrm{eq}(103) + \frac{2}{3}\frac{\alpha\Phi}{\partial_r\Phi}\left[\mathrm{eq}(104) - \partial_r^2 a\right] \tag{105}$$

Substituting $\partial_t Q$ into the coupled equation (88) we find an expression for $\partial_t\phi$. First we introduce the following expression

$$\mathrm{eq}(106) = \alpha\left[-a^2 - \partial_r a - 2\partial_r\ln\Phi + \frac{3}{2}Q^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi)\right]. \tag{106}$$

We compute $\partial_t\phi$ (for derivation see Appendix (C))

$$\partial_t\phi = \mathrm{eq}(106) + \frac{\alpha\Phi}{2\partial_r\Phi}\left[\mathrm{eq}(104) - \partial_r^2 a\right] \tag{107}$$

The equations of $\partial_t Q$ and $\partial_t\phi$ are still coupled through $\partial_r^2 a$. To decouple these equations we apply the additional integrability condition

$$\boxed{\partial_t\partial_r^2\Phi = \partial_r^2\partial_t\Phi} \tag{108}$$

from which we find a complex expression for $\partial_r^2 a$, still dependent on $\partial_r^2 K$. For the decoupled equations of $\partial_r^2 K$ and $\partial_r^2 a$ we refer to equations (24) and (25) in Jalali et al. 2024, in which

$$\begin{aligned} \partial_r^2 K = -2\left(a + \partial_r\ln\Phi\right)\partial_r K \\ - \left[\frac{a^2}{2} + \partial_r a + 2a\partial_r\ln\Phi - \frac{3}{2}Q^2 - P^2 - \frac{1}{3}\varphi^2 + V(\varphi)\right]\left[K + \varphi - \frac{3}{2}V'(\varphi)\right], \end{aligned} \tag{109}$$

$$\begin{aligned} \partial_r^2 a = -3a\partial_r a + a\Big[(\partial_r\Phi)^2 - \frac{1}{\Phi^2} + \frac{9}{4}Q^2 + \frac{3}{2}P^2 + \frac{1}{2}(\partial_r\psi)^2\Big] - 2a^2\partial_r\ln\Phi - a^3 \\ + P(3Q\partial_r\psi + 2\partial_r P) - 9Q^2\partial_r\ln\Phi + 2(a\partial_r\ln\Phi - \partial_r a)\partial_r\ln\Phi. \end{aligned} \tag{110}$$

The expression (110) reduces the equations (105) and (286) to

$$\begin{aligned} \partial_t Q = \alpha\Big[- KQ - \frac{1}{4}Q^2 + \frac{2}{3}\left(a^2 + \partial_r a - a\,\partial_r\ln\Phi\right) + \frac{1-(\partial_r\Phi)^2}{\Phi^2} - \frac{1}{6}P^2 + \frac{1}{2}(\partial_r\psi)^2 \\ + \frac{1}{9}\phi^2 - \frac{1}{3}V(\phi)\Big] + \beta\left(P\,\partial_r\psi - 3Q\,\partial_r\ln\Phi\right), \end{aligned} \tag{111}$$

$$\partial_t\phi = \alpha\left[-a^2 - \partial_r a - 2a\,\partial_r\ln\Phi + \frac{3}{2}Q^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi)\right], \tag{112}$$

which are the same as equations (20) and (21) in the paper, respectively. In the paper Jalali et al. 2024, they started from an ansatz containing an additional function R^2 in the ansatz

$$\gamma_{ij}dx^i dx^j = R^2(t,r)dr^2 + \Phi(t,r)^2 d\Omega_2^2 \tag{113}$$

which is not showing in their equation (8) nor was it mentioned in the paper. However, it was used in order to derive the equations of motion and later it was set to unity. With this extra function being present, $\partial_t\phi$ and $\partial_t Q$ were directly obtained (without integrability conditions) as presented in equations (111)-(112). From the integrability conditions in equations (102) and (108) the (decoupled) equations (109)-(110) were obtained. The obtained constraint equations (82)-(84) and dynamical equations (85)-(87) remain the same with the correct ansatz and setting $R(t,r)^2$ to unity.

In summary, we have the following constraint equations

$$\frac{\partial_r^2\Phi}{\Phi} = \frac{1-(\partial_r\Phi)}{2\Phi^2} - \frac{3}{8}Q^2 - \frac{1}{4}P^2 - \frac{1}{4}(\partial_r\psi)^2 - \frac{1}{2}V(\phi) + \frac{1}{6}\phi^2 \tag{114}$$

$$\partial_r Q = P\partial_r\psi - 3Q\partial_r \ln\Phi \tag{115}$$

$$\partial_r\phi = 0, \tag{116}$$

the following dynamical equations

$$\partial_t\psi = \alpha P + \beta\partial_r\psi \tag{117}$$

$$\partial_t P = \alpha\Big[- KP + (a + 2\partial_r \ln\Phi)\,\partial_r\psi + \partial_r^2\Psi\Big] + \beta\partial_r P \tag{118}$$

$$\partial_t\Phi = \alpha\left(\frac{1}{3}K - \frac{1}{2}Q\right)\Phi + \beta\partial_r\Phi \tag{119}$$

$$\begin{aligned}\partial_t Q = \alpha\Big[& - KQ - \frac{1}{4}Q^2 + \frac{2}{3}\left(a^2 + \partial_r a - a\,\partial_r\ln\Phi\right) + \frac{1-(\partial_r\Phi)^2}{\Phi^2} - \frac{1}{6}P^2 + \frac{1}{2}(\partial_r\psi)^2 \\ &+ \frac{1}{9}\phi^2 - \frac{1}{3}V(\phi)\Big] + \beta\left(P\,\partial_r\psi - 3Q\,\partial_r\ln\Phi\right),\end{aligned} \tag{120}$$

$$\partial_t\phi = \alpha\left[-a^2 - \partial_r a - 2a\,\partial_r\ln\Phi + \frac{3}{2}Q^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi)\right], \tag{121}$$

the non-dynamical equations

$$\partial_r\ln\alpha = a \tag{122}$$

$$\partial_r\beta = -\alpha\left(Q + \frac{1}{3}K\right) \tag{123}$$

$$\begin{aligned}\partial_r^2 K =& -2\left(a + \partial_r\ln\Phi\right)\partial_r K \\ &- \left[\frac{a^2}{2} + \partial_r a + 2a\partial_r\ln\Phi - \frac{3}{2}Q^2 - P^2 - \frac{1}{3}\varphi^2 + V(\varphi)\right]\left[K + \varphi - \frac{3}{2}V'(\varphi)\right],\end{aligned} \tag{124}$$

$$\begin{aligned}\partial_r^2 a =& -3a\partial_r a + a\Big[(\partial_r\Phi)^2 - \frac{1}{\Phi^2} + \frac{9}{4}Q^2 + \frac{3}{2}P^2 + \frac{1}{2}(\partial_r\psi)^2\Big] - 2a^2\partial_r\ln\Phi - a^3 \\ &+ P(3Q\partial_r\psi + 2\partial_r P) - 9Q^2\partial_r\ln\Phi + 2(a\partial_r\ln\Phi - \partial_r a)\partial_r\ln\Phi.\end{aligned} \tag{125}$$

The equations that determine the Lagrange multipliers are

$$\lambda = -\frac{2}{3}(K+\phi) \tag{126}$$

$$2\tilde{\lambda}\partial_r \ln\Phi + \partial_r\tilde{\lambda} = -\alpha\left[\frac{2}{3}(K+\phi) - V'(\phi)\right] \tag{127}$$

3.3 Integrating out the shadowy mode

In numerical simulations, we restrict ourselves to considering asymptotic flatness at spatial infinity without excitations of the scalar field ψ. In vacuum the theory allows for a flat spacetime in the standard Minkowski slicing without spontaneous breaking of the spatial diffeomorphism invariance when

$$\lim_{r\to\infty} K = 0, \qquad \lim_{r\to\infty} K_{ij}K^{ij} \to 0, \qquad \lim_{r\to\infty} R \to 0 \qquad \lim_{r\to\infty} \tilde{\lambda} = 0, \qquad \lim_{r\to\infty} \dot{\phi} = 0. \tag{128}$$

These conditions are satisfied when $\phi = \phi_0$, so both constant in space and time, and

$$V'(\phi_0) = \frac{2}{3}\phi_0 \tag{129}$$

$$V(\phi_0) = \frac{1}{3}\phi_0^2 \tag{130}$$

The equations of motion simplify using these assumptions. Since the dynamical equation of ϕ is zero we can derive a simplified non-dynamical equation for a

$$\partial_r a = -a^2 - 2a\partial_r \ln\Phi + \frac{3}{2}Q^2 + P^2. \tag{131}$$

The non-dynamical equation for K also simplifies

$$\partial_r^2 K = -2\left(a + \partial_r \ln\Phi\right)\partial_r K. \tag{132}$$

Integrating equation (132) once gives

$$\partial_r K(t,r) = \frac{f(t)}{(\alpha(t,r)\Phi(t,r))^2} \tag{133}$$

where $f(t)$ is an arbitrary function of t (note the exponent difference of the denominator with Jalali et al. 2024). At the center of symmetry where the areal radius $\Phi(t,r)$ is zero, the regularity of the solution requires that $\alpha(t,r=0) \neq 0$ and that $\partial_r K(t,r=0) = 0$. For this reason $f(t) = 0$ and thus $\partial_r K(t,r) = 0$. We now have that $K(r,t) = g(t)$ is an arbitrary function of t. From our imposed boundary conditions in (128) it follows that $K(r,t) = 0$. The equation (80) reduces to

$$g^{\Phi\Phi} = -\frac{1}{4}Q^2\Phi^2 + (\partial_r\Phi)^2 \tag{134}$$

Since K and ϕ are constants, it follows from equation (126) that λ is also constant. Accordingly, after this simplification our EoM reduce to solutions of GR in a particular slicing.
One might ask why we would want to study this system if its equations match GR in a particular slicing. The key is that VCDM breaks time-diffeomorphism symmetry, so the choice of time slicing is physically meaningful. We focus on whether an apparent horizon (AH) forms during collapse before the time foliation breaks down or a singularity forms. Unlike in GR, where all slicings are equivalent, in VCDM different slicings represent physically distinct scenarios.

3.4 Non-dynamical Q

For the purpose of numerical simulation, the variable Q can be simplified by treating it as non-dynamical. This means we no longer evolve Q using its dynamical equation (120), and instead determine Q from the constraint equation (115). Together with the simplifications from the previous subsection, this leads us to a final set of equations where the system is fully specified by one constraint and a set of evolution and auxiliary relations.
The (unchanged) constraint equation is

$$\frac{\partial_r^2\Phi}{\Phi} = \frac{1-(\partial_r\Phi)^2}{2\Phi^2} - \frac{3}{8}Q^2 - \frac{1}{4}P^2 - \frac{1}{4}(\partial_r\psi)^2. \tag{135}$$

The dynamical evolution equations are

$$\partial_t\psi = \alpha P + \beta\partial_r\psi, \tag{136}$$

$$\partial_t P = \alpha\left[\partial_r^2\psi + (a + 2\partial_r \ln\Phi)\,\partial_r\psi\right] + \beta\partial_r P, \tag{137}$$

$$\partial_t\Phi = -\frac{1}{2}\alpha Q\Phi + \beta\partial_r\Phi. \tag{138}$$

And the remaining non-dynamical relations, which act as constraints or auxiliary definitions, are

$$\partial_r \ln\alpha = a, \tag{139}$$

$$\partial_r\beta = -\alpha Q, \tag{140}$$

$$\partial_r a = -a^2 - 2a\partial_r\ln\Phi + \frac{3}{2}Q^2 + P^2, \tag{141}$$

$$\partial_r Q = -3Q\partial_r\ln\Phi + P\partial_r\psi. \tag{142}$$

3.5 A special case: $\beta = 0$

As a simplification, we will study a sub-case of the equations, namely the case where the shift $\beta = 0$ (motivated by the shape of common metrics e.g. Schwarzschild metric in equation (31)). This reduces equations (136)-(142), starting with the dynamical equations to

$$\partial_t\psi = \alpha P \tag{143}$$

$$\partial_t P = 0 \tag{144}$$

$$\partial_t\Phi = 0 \tag{145}$$

and the non-dynamical equations to

$$\partial_r\ln\alpha = a \tag{146}$$

$$\partial_r\beta = 0 \tag{147}$$

$$\partial_r a = -a^2 - 2a\partial_r\ln\Phi + P^2 \tag{148}$$

$$\partial_r Q = 0 \tag{149}$$

and the constraint equation

$$\frac{\partial_r^2\Phi}{\Phi} = \frac{1-(\partial_r\Phi)^2}{2\Phi^2} - \frac{1}{4}P^2. \tag{150}$$

Note that instead of $\partial_r \psi = 0$, another option would have been $P = 0$. That results in our matter source being non-dynamical, and hence no collapse of the matter field ψ would happen. From the integrability condition

$$\boxed{\partial_r \partial_t \psi = \partial_t \partial_r \psi} \tag{151}$$

we find that equation (148) can be written as

$$\partial_r^2 \ln P = (\partial_r \ln P)^2 - 2\partial_r \ln P \partial_r \Phi - P^2. \tag{152}$$

With the simplifications, $g^{\Phi\Phi}$ in equation (80) is given by

$$g^{\Phi\Phi} = (\partial_r \Phi)^2. \tag{153}$$

Hence the apparent horizon condition now reads

$$(\partial_r \Phi)^2|_{r=r_{\mathrm{AH}}} = 0 \tag{154}$$

The equations that we will numerically evolve are equation (150) and equation (152). In the next section we discuss the boundary conditions for these equations.

3.6 Boundary conditions

For the numerical simulation we define the following boundary conditions for evolving the equations (150) and (152). We set the center of spherical symmetry, i.e. where Φ is zero at $r = 0$ and we consider the region $r \geq 0$

$$\Phi(t, r = 0) = 0 \tag{155}$$

Equation (150) implies that

$$\partial_r \Phi(t, r = 0) = 1 \tag{156}$$

Furthermore, we introduce an outer boundary r_b, so that the region we consider is $0 \leq r \leq r_b$. For sufficiently large r_b we can set

$$P(t, r = r_b) = 0 \tag{157}$$

$$\partial_r P(t, r = 0) = 0 \tag{158}$$

Since the equations of interest are non-dynamical we do not have to set initial conditions.

4 Numerical integration

We numerically integrate the coupled second-order differential equations (150) and (152), using the boundary conditions provided in Section 3.6. The goal is to determine the functions $\Phi(r)$ and $P(r)$, which encode essential aspects of the system's geometry and matter content. From this we can check wether the horizon condition in equation (154) is satisfied at $r > 0$. We solve these equations using Python's `scipy.integrate.solve_bvp`, a solver for boundary value problems (BVPs) for ordinary differential equations.

4.1 Reformulation of the System

The original system involves second order derivatives of the functions $\Phi(r)$ and $P(r)$. We define

$$\ln\Phi(r) = y_0(r), \quad \frac{d\ln\Phi}{dr} = y_1(r), \quad \ln P(r) = y_2(r), \quad \frac{d\ln P}{dr} = y_3(r) \tag{159}$$

This converts the system of second-order equations into a first-order system in the 4-vector $\mathbf{y}(r) = [y_0, y_1, y_2, y_3]$. The equations become:

$$\frac{dy_0}{dr} = y_1 \tag{160}$$

$$\frac{dy_1}{dr} = -\frac{3}{2}y_1^2 + \frac{1}{2}e^{-2y_0} - \frac{1}{4}e^{2y_2} \tag{161}$$

$$\frac{dy_2}{dr} = y_3 \tag{162}$$

$$\frac{dy_3}{dr} = y_3^2 - 2y_1y_3 - e^{2y_2} \tag{163}$$

These equations are implemented in the function `ode(r, y)`.

4.2 Boundary Conditions

We considered the domain $r \in [0, r_b]$. An outer boundary r_b of the system is placed far from the origin $r = 0$, to prevent it from disturbing the collapsing process and to best simulate an asymptotically flat spacetime. The boundary conditions become

- At $r = 0$:

$$\Phi(0) = \varepsilon \Rightarrow y_0(0) = \ln\varepsilon \tag{164}$$

$$\frac{d\ln\Phi}{dr}(0) = 0 \Rightarrow y_1(0) = 0 \tag{165}$$

$$\frac{d\ln P}{dr}(0) = 0 \Rightarrow y_3(0) = 0 \tag{166}$$

- At $r = 80$:

$$P(80) = \varepsilon \Rightarrow y_2(80) = \ln\varepsilon \tag{167}$$

Here, $\varepsilon = 10^{-12}$ is a small regularization parameter used to avoid singular behavior at $r = 0$. Furthermore, $r_b = 80$ is arbitrary in the sense that we get the same (stable) result for $r_b \in [40, 130]$. These conditions are implemented in the function `bc(ya, yb)`.

4.3 Solving the System

The problem is solved using `solve_bvp`, which iteratively adjusts the initial guess (a linear interpolation between their boundary values) to satisfy both the ODEs and boundary conditions within a tolerance of 10^{-6}.

After solving, the physical functions are recovered via:

$$\Phi(r) = e^{y_0(r)}, \quad P(r) = e^{y_2(r)} \tag{168}$$

These are plotted to visualize their behavior across the domain. From this we can study if the boundary conditions are met and get insight if an horizon is forming before the areal radius is zero. We plot $(\partial_r \Phi)^2$ as function of r, which provides insight into whether the horizon condition is met. For this we use a threshold of where $(\partial_r \Phi)^2 = 10^{-30}$.

5 Results

We want to study if an apparent horizon forms during the collapse, i.e. if the condition in equation (154) is satisfied before $\Phi = 0$. If we look at equations (143)-(152) we would not expect an apparent horizon formation since

$$\partial_r \psi = 0, \tag{169}$$

i.e. no inhomogeneities in the matter source ψ are allowed. In Fig. (5) we indeed observe that P (remember that P is the derivative of ψ in the normal direction) stays approximately constant, as expected. Furthermore, $\partial_r \Phi$ does not become zero during the collapse.

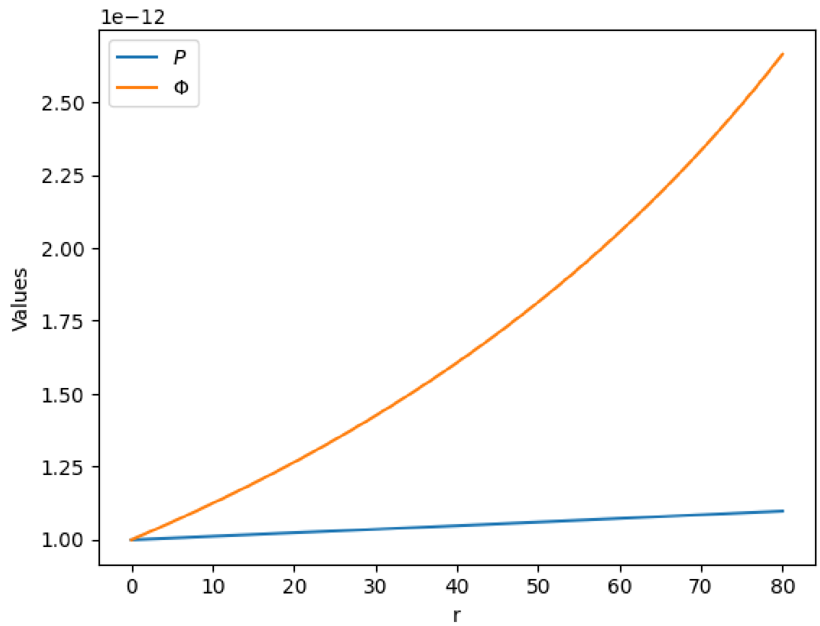

Figure 5: The areal radius Φ and P during the collapse.

To get a better view on this we plotted $(\partial_r \Phi)^2$ as function of Φ to confirm this. In Fig. (6) we see that the horizon condition is not satisfied until $\Phi = 0$.

From Jalali et al. 2024 it was clear that in certain cases there would be no AH formation. From Fig. 6 of Jalali et al. 2024, it followed that if the initial condition for for the Gaussian wave packet of ψ_0 was zero, either by the amplitude A being zero, or for large values of the variance s, there was no AH formation, i.e. when $\partial_r \psi$ was zero or approaches to be spatially constant (as initial condition). Let us study the outgoing and ingoing expansion scalars in (188) and (195), which has now reduced to

$$\theta_{\text{out}} = \frac{2}{\Phi} \partial_r \Phi \tag{170}$$

$$\theta_{\text{in}} = -\frac{2}{\Phi} \partial_r \Phi \tag{171}$$

The expansion of the outgoing null curves being zero and the expansion of the ingoing null curves to being negative cannot be simultaneously satisfied (i.e. a marginally outer trapped surface cannot form). Furthermore it is also not possible to obtain $\theta_{\text{in}} < 0$ and $\theta_{\text{out}} < 0$, i.e. a trapped surface. This seems like a gauge artifact and hints towards a vanishing shift being a poor choice.

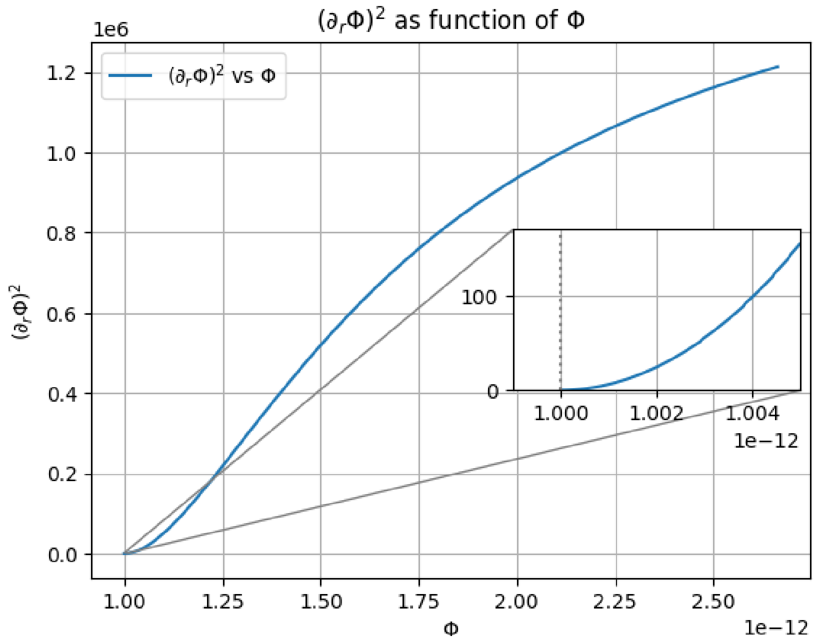

Figure 6: $(\partial_r\Phi)^2$ during the collapse.

If we consider the equations of motion in the case of $\beta = \beta(t)$, we see that again we would have to branches of solutions. Either $\partial_r\psi = 0$ or $P = 0$ (since the traceless part of the extrinsic curvature Q would still be zero). This time however, $P = 0$ does not lead to a vanishing $\partial_t\psi$;

$$\partial_t\psi = \beta(t)\partial_r\psi \tag{172}$$

The expansion scalars read

$$\theta_{\text{out}} = \frac{2}{\Phi}\left(-\frac{\beta(t)\partial_r\Phi}{\alpha(t,r)} + \partial_r\Phi\right) \tag{173}$$

$$\theta_{\text{in}} = \frac{2}{\Phi}\left(-\frac{\beta(t)\partial_r\Phi}{\alpha(t,r)} - \partial_r\Phi\right) \tag{174}$$

If $\alpha = \beta(t)$ for some $\Phi > 0$ then

$$\theta_{\text{out}} = 0 \tag{175}$$

$$\theta_{\text{in}} = -\frac{4}{\Phi}\partial_r\Phi. \tag{176}$$

If $(\partial_r \Phi) > 0$ then $\theta_{\text{in}} < 0$ and $\theta_{\text{out}} = 0$. Hence, this sub-case might lead to an apparent horizon formation since inhomogeneities are allowed (although it should be noted that ψ is not truly dynamical in this case). It would be interesting to numerically evolve the system with $\beta = \beta(t)$ and see if the apparent horizon condition is satisfied in that subcase.
Hence, in the spherically symmetric ansatz, $\beta \neq 0$ is necessary to allow for inhomogeneities in the field ψ, for ψ to be dynamical, and for the expansion scalars to possibly satisfy $\theta_{\text{out}} = 0 \ \wedge \ \theta_{\text{in}} < 0$, possibly leading to an apparent horizon formation during the collapse.

6 Summary and Discussion

In this thesis we studied spherical collapse of a massless scalar field in the minimally modified gravity theory called VCDM, in which we followed the paper Jalali et al. 2024 closely. VCDM is a type-II minimally modified gravity, meaning that it has no Einstein frame and propagates only two local physical degrees of freedom. Furthermore, it is also supplemented by a so called shadowy mode governed by an elliptic equation.
In this work, we showed the main steps on how the theory is constructed. We started from the Hamiltonian of GR in the ADM formalism. A canonical transformation is performed to a new frame, after which a cosmological constant term and a gauge fixing term are added. Subsequently, an inverse canonical transformation is performed to go back to the original frame, but now it is a theory different from GR. After performing a Legendre transformation, the Lagrangian is obtained. Additionally, to the gravitational action, we also add a matter source ψ, a massless scalar field, since we are interested in studying gravitational scalar collapse.
We then assume a spherically symmetric ansatz and derive the equations of motion from the total action, in which we used integrability conditions to get all the independent equations.
We integrate out the shadowy mode by imposing appropriate boundary conditions, namely an asymptotically flat spacetime in the standard Minkowski time slicing. As a result, the EoM reduced to those of GR with the maximal slicing. However, VCDM does not have 4D diffeomorphism, and hence it is still interesting to see if an apparent horizon forms.
For numerical purposes we downgrade the field Q, the traceless part of the extrinsic curvature, to a non-dynamical one. This simplified our EoM.
As a simplification, we then turn to studying a sub-case $\beta = 0$, where β is the r-component of the shift vector in our spherically symmetric ansatz. This choice was motivated by the shape of the well-known Schwarzschild metric. With the shift being zero, either the matter field ψ is non-dynamical, or no inhomogeneities are allowed. Studying the expansion scalars of a congruence of radially ingoing and outgoing null curves, we find that $\theta_{\text{out}} = 0$ and $\theta_{\text{in}} < 0$ cannot be simultaneously satisfied for the choice of $\beta = 0$. After simplifying our equations of motion, we numerically integrate our system and confirm that there is no apparent horizon formation as expected. From Jalali et al. 2024 the solution outside of the horizon was described by Schwarzschild geometry, and it was not trivial that the case $\beta = 0$ would be pathological to solve for a marginally outer trapped surface. We briefly discuss the equations of motion considering the case of $\beta = \beta(t)$. In this case, the time derivative of the matter field ψ does not vanish and ψ is allowed to have spatial inhomogeneities. Studying the expansion scalar we find that within this sub-case it might be possible to have zero expansion for a congruence of radially outgoing null curves and negative expansion for the radially ingoing null curves. It could be explored if this sub-case leads to an apparent horizon. It should be noted however that ψ is not truly dynamical, also in this sub-case.
Hence, we conclude that $\beta \neq 0$ is necessary in order to study an apparent horizon formation, and that the sub-case $\beta(t)$ might light to an apparent horizon formation.

7 Future work

A natural extension would be to numerically evolve the system in the case of $\beta = \beta(t)$, since in that sub-case the matter field has time dependence and allows for spatial inhomogeneities, and the expansion scalars are such that it could possibly lead to $\theta_{\text{in}} < 0 \ \wedge \ \theta_{\text{out}} = 0$.
Present investigations of the theory VCDM are mostly confined to spherically symmetric setups, where solutions often resemble those found in general relativity in a particular slicing. However, to properly evaluate the full potential of VCDM, it is important to explore its behavior in more general spacetimes. One could study for example the non-symmetrical dynamics of rotating black holes and black hole and neutron star binaries, which could lead to ϕ not being constant in time.

In this work we studied the gravitational collapse of a massless scalar field, and a natural extension of this would be to study a gravitational collapse in systems composed of more realistic forms of matter, such as fluids, or considering stars with additional physical properties like electric charge or angular momentum. These additions may lead to a time-dependent scalar field, producing solutions that depart significantly from GR. Such non-GR dynamics must be examined if VCDM is to be tested meaningfully.

Furthermore, is also interesting to explore the gravitational wave phenomenology of VCDM. Understanding the types of waveforms this model produces in events like neutron star or black hole mergers may reveal observable deviations from GR predictions and offer insight into the role of scalar fields in strong gravity.

Beyond VCDM, the study of modified gravity theories remains essential for deepening our understanding of gravity and the foundations of GR. Gravitational collapse offers a powerful setting for such studies, as it raises important questions about the formation and nature of black holes. Modified theories may predict different end states than GR, some of which could be observable. Scenarios like this highlight the need for continued theoretical and phenomenological investigation of models like VCDM. Such studies may not only clarify the nature of dark matter and dark energy but also contribute to a more complete understanding of gravity itself.

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A Apparent horizon

In this appendix we derive the equations (79) and (80). We start from the spherically symmetric ansatz in the ADM formalism (equation (42)),

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -\alpha(r,t)^2 dt^2 + (dr + \beta(r,t)dt)^2 + \Phi(r,t)^2 d\Omega^2 \tag{177}$$

with $\alpha, \beta > 0$ and $\Phi(r,t) \geq 0$. By performing the coordinate transformation $r \to \Phi(r,t)$:

$$\begin{aligned} g^{\Phi\Phi} &= \frac{\partial\Phi}{\partial x^{\mu}}\frac{\partial\Phi}{\partial x^{\nu}} g^{\mu\nu} \\ &= \left(\frac{\partial\Phi}{\partial t}\right) g^{tt} + 2\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial r} g^{tr} + \left(\frac{\partial\Phi}{\partial r}\right)^2 g^{rr} \end{aligned}$$

By computing the inverse metric tensor we find

$$g^{tt} = -\frac{1}{\alpha^2}, \quad g^{rt} = \frac{\beta}{\alpha^2}, \quad g^{rr} = 1 - \frac{\beta^2}{\alpha^2} \tag{178}$$

and hence for $g^{\Phi\Phi}$

$$\begin{aligned} g^{\Phi\Phi} &= -\frac{(\partial_t\Phi)^2}{\alpha^2} + 2\partial_t\Phi\partial_r\Phi\frac{\beta}{\alpha^2} + (\partial_r\Phi)^2\left(1 - \frac{\beta^2}{\alpha^2}\right) \\ &= -(\partial_{\perp}\Phi)^2 + (\partial_r\Phi)^2 \end{aligned} \tag{179}$$

where $\partial_{\perp} = (\partial_t - \beta\partial_r)/\alpha$ is the derivative in the direction perpendicular to the t-constant hypersurface. We construct two null vectors fields

$$k^{\mu}\partial_{\mu} = \partial_{\perp} + \partial_r \tag{180}$$
$$l^{\mu}\partial_{\mu} = \partial_{\perp} - \partial_r \tag{181}$$

where k is tangent to a congruence of radially outgoing null curves and l is tangent to a congruence of radially ingoing null curves. Note that the only non-zero components of k and l are $k^t = 1/\alpha$, $k^r = (1 - \beta/\alpha)$, $l^t = 1/\alpha$ and $l^r = -1 - \beta/\alpha$. We have the following relations

$$\begin{aligned} k_{\mu}k^{\mu} &= g_{tt}k^t k^t + 2g_{tr}k^t k^r + g_{rr}k^r k^r \\ &= (\beta^2 - \alpha^2)\frac{1}{\alpha^2} + 2\beta\frac{1}{\alpha}\left(1 - \frac{\beta}{\alpha}\right) + \left(1 - \frac{\beta}{\alpha}\right)^2 \\ &= 0, \end{aligned} \tag{182}$$

$$\begin{aligned} l_{\mu}l^{\mu} &= g_{tt}l^t l^t + 2g_{tr}l^t l^r + g_{rr}l^r l^r \\ &= (\beta^2 - \alpha^2)\frac{1}{\alpha^2} + 2\beta\frac{1}{\alpha}\left(-1 - \frac{\beta}{\alpha}\right) + \left(-1 - \frac{\beta}{\alpha}\right)^2 \\ &= 0. \end{aligned} \tag{183}$$

Furthermore we have the relations

$$\begin{aligned}
k^\mu l_\mu &= k^t g_{t\mu} l^\mu + k^r g_{r\mu} l^\mu \\
&= k^t g_{tt} l^t + k^t g_{tr} l^r + k^r g_{rr} l^r + k^r g_{rt} l^t \\
&= \frac{1}{\alpha}\left(\beta^2 - \alpha^2\right)\frac{1}{\alpha} + \frac{1}{\alpha}\left(\beta\left(-1 - \frac{\beta}{\alpha}\right)\right) + \left(1 - \frac{\beta}{\alpha}\right)\left(-1 - \frac{\beta}{\alpha}\right) + \left(1 - \frac{\beta}{\alpha}\right)\beta\frac{1}{\alpha} \\
&= -2, \qquad (184)
\end{aligned}$$

$$\begin{aligned}
l^\mu k_\mu &= l^t g_{t\mu} k^\mu + l^r g_{r\mu} k^\mu \\
&= l^t g_{tt} k^t + l^t g_{tr} k^r + l^r g_{rr} k^r + l^r g_{rt} k^t \\
&= \frac{1}{\alpha}\left(\beta^2 - \alpha^2\right)\frac{1}{\alpha} + \frac{1}{\alpha}\left(\beta\left(1 - \frac{\beta}{\alpha}\right)\right) + \left(-1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\beta}{\alpha}\right) + \left(-1 - \frac{\beta}{\alpha}\right)\beta\frac{1}{\alpha} \\
&= -2 \qquad (185)
\end{aligned}$$

The tensor $\sigma^{\mu\nu}$ represents the projection of $g^{\mu\nu}$ onto the 2-dimensional surfaces that are transverse to the radial null directions k and l. Explicitly, it is defined by

$$\sigma^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}(k^\mu l^\nu + l^\mu k^\nu) \qquad (186)$$

The relations (182), (183),(184) and (185) ensure that $\sigma^{\mu\nu} k_\mu = \sigma^{\mu\nu} l_\nu = 0$. Consequently, $\sigma^{\mu\nu}$ captures only the angular part of the inverse metric $g^{\mu\nu}$. In the spherically symmetric ansatz considered, we have $\sigma^{tt} = \sigma^{tr} = \sigma^{rt} = \sigma^{rr} = 0$.
The apparent horizon r_{AH} is found by solving for a marginally outer trapped surface (MOTS), i.e. if the expansion of the radially outgoing null geodesics is equal to 0. The expansion scalar is θ_{out}

$$\theta_{\text{out}} = \sigma^{\mu\nu} \nabla_\nu k_\mu \qquad (187)$$

is given by the trace of the covariant derivative of the outgoing null vector projected onto the 2-sphere orthogonal to the radial and temporal directions. This projection isolates the angular part of the divergence of the null geodesics, which determines how the cross-sectional area of the outgoing light front changes. In particular, the projection ensures that only the transverse (spherical) directions contribute, as those span the surface whose expansion we are measuring.

We compute the expansion scalar

$$\begin{aligned}
\theta_{\text{out}} &= \sigma^{\theta\theta}\nabla_\theta k_\theta + \sigma^{\phi\phi}\nabla_\phi k_\phi \\
&= \frac{1}{\Phi^2}\left(\partial_\theta k_\theta - \Gamma^\lambda_{\theta\theta}k_\lambda\right) + \frac{1}{\sin^2\theta\Phi^2}\left(\partial_\phi k_\phi - \Gamma^\lambda_{\phi\phi}k_\lambda\right) \\
&= \frac{1}{\Phi^2}\left(-\Gamma^t_{\theta\theta}k_t - \Gamma^r_{\theta\theta}k_r\right) + \frac{1}{\sin^2\theta\Phi^2}\left(-\Gamma^t_{\phi\phi}k_t - \Gamma^r_{\phi\phi}k_r\right) \\
&= \frac{1}{\Phi^2}\left(-\left(\frac{1}{2}g^{t\sigma}(\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\sigma\theta} - \partial_\sigma g_{\theta\theta})\right)k_t - \left(\frac{1}{2}g^{r\sigma}(\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\sigma\theta} - \partial_\sigma g_{\theta\theta})\right)k_r\right) \\
&\quad + \frac{1}{\sin^2\theta\Phi^2}\left(-\left(\frac{1}{2}g^{t\sigma}(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\sigma\phi} - \partial_\sigma g_{\phi\phi})\right)k_t - \left(\frac{1}{2}g^{r\sigma}(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\sigma\phi} - \partial_\sigma g_{\phi\phi})\right)k_r\right) \\
&= \frac{1}{\Phi^2}\left(-\frac{1}{\alpha^2}\Phi\partial_t\Phi(\beta-\alpha) + \frac{\beta}{\alpha^2}\Phi\partial_r\Phi(\beta-\alpha) + \left(1-\frac{\beta^2}{\alpha^2}\right)\Phi\partial_r\Phi + \frac{\beta}{\alpha^2}\Phi\partial_t\Phi\right) \\
&\quad + \frac{1}{\sin^2\theta\Phi^2}\Bigg(-\frac{1}{\alpha^2}\sin^2\theta\Phi\partial_t\Phi(\beta-\alpha) + \frac{\beta}{\alpha^2}\sin^2\theta\Phi\partial_r\Phi(\beta-\alpha) + \frac{\beta}{\alpha^2}\sin^2\theta\Phi\partial_t\Phi \\
&\quad + \left(1-\frac{\beta^2}{\alpha^2}\sin^2\theta\Phi\partial_r\Phi\right)\Bigg) \\
&= \frac{2}{\Phi}\left(\frac{1}{\alpha}\partial_t\Phi - \frac{\beta}{\alpha}\partial_r\Phi + \partial_r\Phi\right) \\
&= \frac{2}{\Phi}\left(\partial_\perp\Phi + \partial_r\Phi\right),
\end{aligned} \tag{188}$$

where we used that

$$\begin{aligned}
k_t &= g_{\mu t}k^\mu \\
&= g_{tt}k^t + g_{rt}k^r \\
&= \left(\beta^2-\alpha^2\right)\frac{1}{\alpha} + \beta\left(1-\frac{\beta}{\alpha}\right) \\
&= \frac{\beta^2}{\alpha} - \alpha + \beta - \frac{\beta^2}{\alpha} \\
&= \beta - \alpha
\end{aligned} \tag{189}$$

and

$$\begin{aligned}
k_r &= g_{\mu r}k^\mu \\
&= g_{tr}k^t + g_{rr}k^r \\
&= \beta\frac{1}{\alpha} + \left(1-\frac{\beta}{\alpha}\right) \\
&= 1.
\end{aligned} \tag{190}$$

To find the marginally outer trapped surface we set the expansion scalar to zero, i.e. equation (188), $\theta_{\text{out}}|_{r=r_{\text{AH}}} = 0$

$$\begin{aligned} \frac{2}{\Phi}\left(\partial_\perp \Phi + \partial_r \Phi\right) &= 0 \\ &\Rightarrow \\ \left(\partial_\perp \Phi + \partial_r \Phi\right)|_{r_{\text{AH}}} &= 0 \end{aligned} \tag{191}$$

This implies that at the apparent horizon $\partial_\perp \Phi = -\partial_r \Phi$ and hence

$$g^{\Phi\Phi}|_{r=r_{\text{AH}}} = 0 \tag{192}$$

Let us now turn to the expansion scalar of the ingoing congruence defined by

$$\theta_{\text{in}} \equiv \sigma^{\mu\nu}\nabla_\nu l_\mu. \tag{193}$$

Computing θ_{in} using we find

$$\begin{aligned}
\theta_{\text{in}} &= \sigma^{\theta\theta}\nabla_\theta l_\theta + \sigma^{\phi\phi}\nabla_\phi l_\phi \\
&= \frac{1}{\Phi^2}\left(\partial_\theta l_\theta - \Gamma^\lambda_{\theta\theta} l_\lambda\right) + \frac{1}{\sin^2\theta\Phi^2}\left(\partial_\phi l_\phi - \Gamma^\lambda_{\phi\phi} l_\lambda\right) \\
&= \frac{1}{\Phi^2}\left(-\Gamma^t_{\theta\theta} l_t - \Gamma^r_{\theta\theta} l_r\right) + \frac{1}{\sin^2\theta\Phi^2}\left(-\Gamma^t_{\phi\phi} l_t - \Gamma^r_{\phi\phi} l_r\right) \\
&= \frac{1}{\Phi^2}\left(-\left(\frac{1}{2}g^{t\sigma}(\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\sigma\theta} - \partial_\sigma g_{\theta\theta})\right) l_t - \left(\frac{1}{2}g^{r\sigma}(\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\sigma\theta} - \partial_\sigma g_{\theta\theta})\right) l_r\right) \\
&\quad + \frac{1}{\sin^2\theta\Phi^2}\left(-\left(\frac{1}{2}g^{t\sigma}(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\sigma\phi} - \partial_\sigma g_{\phi\phi})\right) l_t - \left(\frac{1}{2}g^{r\sigma}(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\sigma\phi} - \partial_\sigma g_{\phi\phi})\right) l_r\right) \\
&= \frac{1}{\Phi^2}\left[\left(-\frac{1}{2}g^{tt}(\partial_t g_{\theta\theta}) - \frac{1}{2}g^{tr}(-\partial_r g_{\theta\theta})\right) l_t + \left(-\frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta} - \frac{1}{2}g^{rt}(\partial_t g_{\theta\theta})\right) l_r\right] \\
&\quad + \frac{1}{\sin^2\theta\Phi^2}\left[\left(-\frac{1}{2}g^{tt}(\partial_t g_{\phi\phi}) - \frac{1}{2}g^{tr}(-\partial_r g_{\phi\phi})\right) l_t + \left(-\frac{1}{2}g^{rr}(-\partial_r g_{\phi\phi}) - \frac{1}{2}g^{rt}(-\partial_t g_{\phi\phi})\right) l_r\right] \\
&= \frac{1}{\Phi^2}\left[\left(-\frac{1}{2}\left(-\frac{1}{\alpha^2}\right)(-2\Phi\ \partial_t\Phi) - \frac{1}{2}\left(\frac{\beta}{\alpha^2}\right)(-2\Phi\partial_r\Phi)\right)(-\beta-\alpha)\right. \\
&\quad \left. + \left(-\frac{1}{2}\left(1 - \frac{\beta^2}{\alpha^2}\right)(-2\Phi\partial_r\Phi) - \frac{1}{2}\left(\frac{\beta}{\alpha^2}\right)(-2\Phi\partial_t\Phi)\right)(-1)\right] \\
&\quad + \frac{1}{\sin^2\theta\Phi^2}\left[\left(-\frac{1}{2}\left(-\frac{1}{\alpha^2}\right)\left(-\sin^2\theta 2\Phi\partial_t\Phi\right) - \frac{1}{2}\left(\frac{\beta}{\alpha^2}\right)\left(-\sin^2\theta 2\Phi\partial_r\Phi\right)\right)(-\beta-\alpha)\right. \\
&\quad \left. + \left(-\frac{1}{2}\left(1 - \frac{\beta^2}{\alpha^2}\right)\left(-\sin^2\theta 2\Phi\partial_r\Phi\right) - \frac{1}{2}\left(\frac{\beta}{\alpha^2}\right)\left(-\sin^2\theta 2\Phi\partial_t\Phi\right)\right)(-1)\right]
\end{aligned}$$

Simplifying gives

$$\theta_{\text{in}} = \frac{1}{\Phi}\left[\frac{\beta}{\alpha^2}\partial_t\Phi + \frac{1}{\alpha}\partial_t\Phi - \frac{\beta^2}{\alpha^2}\partial_r\Phi - \frac{\beta}{\alpha}\partial_r\Phi - \partial_r\Phi + \frac{\beta^2}{\alpha^2}\partial_r\Phi - \frac{\beta}{\alpha^2}\partial_t\Phi \right. \tag{194}$$

$$\left. + \frac{\beta}{\alpha^2}\partial_t\Phi + \frac{1}{\alpha}\partial_t\Phi - \frac{\beta^2}{\alpha^2}\partial_r\Phi - \frac{1}{\alpha}\partial_r\Phi - \partial_r\Phi + \frac{\beta^2}{\alpha^2}\partial_r\Phi - \frac{\beta}{\alpha^2}\partial_t\Phi\right]$$

$$= \frac{2}{\Phi}\left(\partial_\perp\Phi - \partial_r\Phi\right) \tag{195}$$

where we used

$$\begin{aligned} l_t &= g_{\mu t} l^\mu \\ &= g_{rt} l^t + g_{tt} l^t \\ &= \beta\left(-1 - \frac{\beta}{\alpha}\right) + \left(\beta^2 - \alpha^2\right)\frac{1}{\alpha} \\ &= -\beta - \frac{\beta^2}{\alpha} + \frac{\beta^2}{\alpha} - \alpha \\ &= -\beta - \alpha \end{aligned} \tag{196}$$

and

$$\begin{aligned} l_r &= g_{\mu r} l^\mu \\ &= g_{rr} l^r + g_{tr} l^t \\ &= -1 - \frac{\beta}{\alpha} + \beta\frac{1}{\alpha} \\ &= -1 \end{aligned} \tag{197}$$

For a marginally outer trapped surface $\theta_{\text{out}} = 0$ and $\theta_{\text{in}} < 0$, i.e. a congruence of ingoing null curves converges where the expansion of outgoing null curves is exactly zero.

B Equations of motion

The non-zero metric elements are:

$$\gamma_{rr} = 1 \tag{198}$$
$$\gamma_{\theta\theta} = \Phi^2(r,t) \tag{199}$$
$$\gamma_{\phi\phi} = \Phi^2(r,t)\sin^2\theta \tag{200}$$

and its inverses:

$$\gamma^{rr} = 1 \tag{201}$$
$$\gamma^{\theta\theta} = \frac{1}{\Phi^2(r,t)} \tag{202}$$
$$\gamma^{\phi\phi} = \frac{1}{\Phi^2 \sin^2\theta} \tag{203}$$

Furthermore, the determinant γ of γ_{ij} is given by:

$$\begin{aligned}\gamma &= \gamma_{rr}\gamma_{\theta\theta}\gamma_{\phi\phi}\\ &= \Phi^4 \sin^2\theta\end{aligned} \tag{204}$$

From $N_i dx^i = \beta(r,t)dr$ we note that the elements of the shift vector are:

$$N_r = \beta(r,t) \tag{205}$$
$$N_\theta = 0 \tag{206}$$
$$N_\phi = 0 \tag{207}$$

We compute the trace of the extrinsic curvature:

$$K = \gamma^{ij}K_{ij} \tag{208}$$
$$= \gamma^{rr}K_{rr} + \gamma^{\theta\theta}K_{\theta\theta} + \gamma^{\phi\phi}K_{\phi\phi} \tag{209}$$

We start by computing K_{rr} by using the defintion of K_{ij} :

$$K_{ij} = \frac{1}{2N}(\partial_t\gamma_{ij} - D_iN_j - D_jN_i) \tag{210}$$

from which follows:

$$\begin{aligned}K_{rr} &= \frac{1}{2\alpha}(\partial_t(1) - 2D_rN_r)\\ &= \frac{1}{2\alpha}(-2(\partial_rN_r - \Gamma^\sigma_{rr}N_\sigma))\\ &= \frac{1}{2\alpha}(-2(\partial_r\beta))\\ &= -\frac{1}{\alpha}\partial_r\beta\end{aligned} \tag{211}$$

For $K_{\theta\theta}$ we find:

$$
\begin{aligned}
K_{\theta\theta} &= \frac{1}{2N}(\partial_t \gamma_{\theta\theta} - 2D_\theta N_\theta) \\
&= \frac{1}{2\alpha}(2\Phi\partial_t\Phi - 2(\partial_\theta N_\theta - \Gamma^\sigma_{\theta\theta}N_\sigma)) \\
&= \frac{1}{2\alpha}\left(2\Phi\partial_t\Phi - 2\Phi\partial_r\Phi\beta\right) \\
&= \frac{1}{\alpha}\left(\Phi\partial_t\Phi - \Phi\partial_r\Phi\beta\right)
\end{aligned}
\tag{212}
$$

and $K_{\phi\phi}$:

$$
\begin{aligned}
K_{\phi\phi} &= \frac{1}{2\alpha}(\partial_t \gamma_{\phi\phi} - 2D_\phi N_\phi) \\
&= \frac{1}{2\alpha}(\partial_t(\Phi^2 \sin^2\Phi) - 2(\partial_\phi N_\phi - \Gamma^\sigma_{\phi\phi}N_\sigma)) \\
&= \frac{1}{2\alpha}\big(2\Phi\partial_t\Phi\sin^2\theta - 2\Phi\sin^2\theta\partial_r\Phi\beta\big) \\
&= \frac{1}{\alpha}\big(\Phi\partial_t\Phi\sin^2\theta - \Phi\sin^2\theta\partial_r\Phi\beta\big)
\end{aligned}
\tag{213}
$$

and hence for the trace of the extrinsic curvature we find:

$$
\begin{aligned}
K &= \gamma^{rr}K_{rr} + \gamma^{\theta\theta}K_{\theta\theta} + \gamma^{\phi\phi}K_{\phi\phi} \\
&= -\frac{1}{\alpha}\partial_r\beta + \frac{1}{\Phi^2}\left(\frac{1}{\alpha}\left(\Phi\partial_t\Phi - \Phi\partial_r\Phi\beta\right)\right) + \frac{1}{\Phi^2\sin^2\theta}\left(\frac{1}{\alpha}\big(\Phi\partial_t\Phi\sin^2\theta - \Phi\sin^2\theta\partial_r\Phi\beta\big)\right) \\
&= -\frac{1}{\alpha}\partial_r\beta + \frac{1}{\alpha}\frac{\partial_t\Phi}{\Phi} - \frac{1}{\alpha}\frac{\partial_r\Phi}{\Phi}\beta + \frac{1}{\alpha}\frac{\partial_t\Phi}{\Phi} - \frac{1}{\alpha}\frac{\partial_r\Phi}{\Phi}\beta \\
&= -\frac{1}{\alpha}\partial_r\beta + 2\frac{1}{\alpha}\frac{\partial_t\Phi}{\Phi} - 2\frac{1}{\alpha}\frac{\partial_r\Phi}{\Phi}\beta \\
&= -\frac{1}{\alpha}\partial_r\beta + 2\frac{1}{\alpha}\partial_t\ln\Phi - 2\frac{1}{\alpha}\partial_r\ln\Phi\beta
\end{aligned}
\tag{214}
$$

and finally

$$
\boxed{K = -\frac{1}{\alpha}\partial_r\beta + 2\partial_\perp\ln\Phi}
\tag{215}
$$

The K_{ij} elements for $i \neq j$:

$$\begin{aligned}
K_{r\theta} &= \frac{1}{2\alpha}(\partial_t \gamma_{r\theta} - D_r N_\theta - D_\theta N_r) \\
&= \frac{1}{2\alpha}(-\partial_r N_\theta + \Gamma^\sigma_{r\theta} N_\sigma - \partial_\theta N_r + \Gamma^\sigma_{\theta r} N_\sigma) \\
&= \frac{1}{2\alpha}(\Gamma^\theta_{r\theta} N_\theta + \Gamma^\theta_{\theta r} N_\theta) \\
&= \frac{1}{2\alpha}(0) \\
&= 0 && (216) \\
K_{\theta r} &= 0 && (217) \\
K_{r\phi} &= \frac{1}{2\alpha}(\partial_t \gamma_{r\phi} - D_r N_\phi - D_\phi N_r) \\
&= \frac{1}{2\alpha}(-\partial_r N_\phi + \Gamma^\sigma_{r\phi} N_\sigma - \partial_\phi N_r + \Gamma^\sigma_{\phi r} N_\sigma) \\
&= \frac{1}{2\alpha}(\Gamma^\phi_{r\phi} N_\theta + \Gamma^\phi_{\phi r} N_\phi) \\
&= \frac{1}{2\alpha}(0) \\
&= 0 && (218) \\
K_{\phi r} &= 0 && (219) \\
K_{\theta\phi} &= \frac{1}{2\alpha}(\partial_t \gamma_{\theta\phi} - D_\theta N_\phi - D_\phi N_\theta) \\
&= \frac{1}{2\alpha}(-\partial_\theta N_\phi + \Gamma^\sigma_{\theta\phi} N_\sigma - \partial_\phi N_\theta + \Gamma^\sigma_{\phi\theta} N_\sigma) \\
&= \frac{1}{2\alpha}(\Gamma^\phi_{\theta\phi} N_\phi + \Gamma^\phi_{\phi\theta} N_\phi) \\
&= \frac{1}{2\alpha}(0) \\
&= 0 && (220) \\
K_{\phi\theta} &= 0 && (221)
\end{aligned}$$

Next we compute K^{rr}, $K^{\theta\theta}$ and $K^{\phi\phi}$:

$$\begin{aligned}
K^{rr} &= \gamma^{rk}\gamma^{rl} K_{kl} \\
&= \gamma^{rr}\gamma^{rr} K_{rr} \\
&= K_{rr} \\
&= -\frac{1}{\alpha}\partial_r \beta && (222) \\
K^{\theta\theta} &= \gamma^{\theta k}\gamma^{\theta l} K_{kl} \\
&= \gamma^{\theta\theta}\gamma^{\theta\theta} K_{\theta\theta} \\
&= \frac{1}{\Phi^2}\frac{1}{\Phi^2}\frac{1}{\alpha}(\Psi \partial_t \Phi - \Phi \partial_r \Phi \beta) \\
&= \frac{1}{\alpha}\frac{1}{\Phi^3}(\partial_t \Phi - \partial_r \Phi \beta) && (223)
\end{aligned}$$

and $K^{\phi\phi}$:

$$\begin{aligned}
K^{\phi\phi} &= \gamma^{\phi k}\gamma^{\phi l}K_{kl} \\
&= \gamma^{\phi\phi}\gamma^{\phi\phi}K_{\phi\phi} \\
&= \frac{1}{\Phi^2\sin^2\Phi}\frac{1}{\Phi^2\sin^2\Phi}\frac{1}{\alpha}(\Phi\partial_t\Phi\sin^2\theta - \Phi\sin^2\theta\partial_r\beta) \\
&= \frac{1}{\alpha}\frac{1}{\Phi^3\sin^2\theta}(\partial_t\Phi - \partial_r\Phi\beta) \qquad (224)
\end{aligned}$$

We compute $K_{ij}K^{ij}$ appearing in the gravitation action:

$$\begin{aligned}
K_{rr}K^{rr} &= K_{rr}^2 \\
&= \frac{1}{\alpha^2}(\partial_r\beta)^2 \qquad (225) \\
K_{\theta\theta}K^{\theta\theta} &= \frac{\Phi}{\alpha}(\partial_t\Phi - \partial_r\Phi\beta)\frac{1}{\alpha}\frac{1}{\Phi^3}(\partial_t\Phi - \partial_r\Phi\beta) \\
&= \frac{1}{\alpha^2}\frac{1}{\Phi^2}(\partial_t\Phi - \partial_r\Phi\beta)^2 \qquad (226) \\
K_{\phi\phi}K^{\phi\phi} &= \frac{1}{\alpha}(\Phi\partial_t\Phi\sin^2\theta - \Phi\sin^3\theta\partial_r\Phi\beta)\frac{1}{\alpha}\frac{1}{\alpha}\frac{1}{\Phi^3\sin^2\theta}(\partial_t\Phi - \partial_r\Phi\beta) \\
&= \frac{1}{\alpha^2}\frac{1}{\Phi^2}(\partial_t\Phi - \partial_r\Phi\beta)^2 \qquad (227) \\
K_{ij}K^{ij} &= \frac{1}{\alpha^2}(\partial_r\beta)^2 + \frac{2}{\alpha^2}\frac{1}{\Phi^2}(\partial_t\Phi - \partial_r\Phi\beta)^2 \\
&= \frac{1}{\alpha^2}\left((\partial_r\beta)^2 + \frac{2}{\Phi^2}(\partial_t\Phi - \partial_r\Phi\beta)^2\right) \qquad (228)
\end{aligned}$$

The Christoffel symbols are:

$$
\begin{aligned}
\Gamma^r_{rr} &= \tfrac{1}{2}\gamma^{rr}(\partial_r\gamma^{rr} + \partial_r\gamma_r r - \partial_r\gamma_r r)\\
&= 0\\
\Gamma^r_{\phi r} &= \Gamma^r_{r\phi} = \tfrac{1}{2}\gamma^{rr}(\partial_\phi\gamma_{rr} + \partial_r\gamma_{r\phi} - \partial_r\gamma_{\phi r})\\
&= 0\\
\Gamma^r_{\theta\theta} &= \tfrac{1}{2}\gamma^{rr}(\partial_\theta\gamma_{rr} + \partial_\theta\gamma_{rr} - \partial_r\gamma_{\theta\theta})\\
&= \tfrac{1}{2}(-\partial_r\Phi^2)\\
&= -\tfrac{1}{2}(2\Phi\partial_r\Phi)\\
&= -\Phi\partial_r\Phi\\
\Gamma^r_{\phi\phi} &= \tfrac{1}{2}(-\partial_r\gamma_{\phi\phi})\\
&= -\tfrac{1}{2}\partial_r(\Phi^2\sin^2\theta)\\
&= -\tfrac{1}{2}2\Phi\partial_r\Phi\sin^2\theta\\
&= -\phi\partial_r\Phi\sin^2\theta\\
\Gamma^r_{\theta r} &= \Gamma^r_{r\theta}\\
&= 0\\
\Gamma^r_{\theta\phi} &= \Gamma^r_{\phi\theta}\\
&= 0\\
\Gamma^\theta_{\theta\phi} &= \Gamma^\theta_{\phi\theta}\\
&= \tfrac{1}{2}\gamma^{\theta\theta}(\partial_\phi\gamma_{\theta\theta})\\
&= 0\\
\Gamma^\theta_{\theta\theta} &= 0\\
\Gamma^\theta_{rr} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\theta_{\phi\phi} &= \tfrac{1}{2}(-\partial_\theta\gamma_{\phi\phi})\\
&= \tfrac{1}{2}\gamma^{\theta\theta}(-\partial_\theta\gamma_{\phi\phi})\\
&= \tfrac{1}{2\Phi^2}(-\partial_\theta\Phi^2\sin^2\theta)\\
&= -\tfrac{1}{2\Phi^2}\Phi^2 2\sin\theta\cos\theta\\
&= -\sin\theta\cos\theta\\
\Gamma^\phi_{\theta\theta} &= \tfrac{1}{2}\gamma^{\phi\phi}(\partial_\phi\gamma_{\theta\theta}) = 0\\
\Gamma^\theta_{r\theta} &= \Gamma^r_{\theta r}\\
&= \tfrac{1}{2}\gamma^{\theta\theta}(\partial_r\gamma_{\theta\theta})\\
&= \tfrac{1}{2}\tfrac{1}{\Phi^2}(2\Phi\partial_r\Phi)\\
&= \tfrac{\partial_r\Phi}{\Phi}\\
\Gamma^\phi_{\phi r} &= \Gamma^\phi_{r\phi}\\
&= \tfrac{1}{2}\gamma^{\phi\phi}(\partial_r\gamma_{\phi\phi})\\
&= \tfrac{1}{2}\tfrac{1}{\Phi^2\sin^2\theta}\partial_r\Phi^2\sin^2\theta\\
&= \tfrac{1}{2}\tfrac{2\Phi\partial_r\Phi}{\Phi^2}\\
&= \tfrac{\partial_r\Phi}{\Phi}\\
\Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta}\\
&= \tfrac{1}{2}\gamma^{\phi\phi}(\partial_\theta\gamma_{\phi\phi})\\
&= \tfrac{1}{2}\tfrac{1}{\Phi^2\sin^2\theta}(\partial_\theta\Phi^2\sin^2\theta)\\
&= \tfrac{1}{2}\tfrac{1}{\sin^2\theta}2\sin\theta\cos\theta\\
&= \tfrac{\cos\theta}{\sin\theta}\\
&= \cot\theta
\end{aligned}
$$

We compute the Ricci scalar using the Christoffel symbols and the (inverse) metric elements given by

$$
\begin{aligned}
R &= \gamma^{ij}R_{ij} \qquad (229)\\
&= \gamma^{rr}R_{rr} + \gamma^{\theta\theta}R_{\theta\theta} + \gamma^{\phi\phi}R_{\phi\phi} \qquad (230)
\end{aligned}
$$

We start by computing the components of R_{rr}:

$$
\begin{aligned}
R_{rr} &= \partial_\lambda\Gamma^\lambda_{rr} - \partial_r\Gamma^\lambda_{\lambda r} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{rr} - \Gamma^\lambda_{r\rho}\Gamma^\rho_{\lambda}r \qquad (231)\\
& \qquad (232)\\
\partial_\lambda\Gamma^\lambda_{rr} &= \partial_r\Gamma^r_{rr} + \partial_\theta\Gamma^\theta_{rr} + \partial_\phi\Gamma^\phi_{rr} \qquad (233)\\
\partial_r\Gamma^r_{rr} &= \partial_r\frac{1}{2}\gamma^{rr}(\partial_r\gamma_{rr} + \partial_r\gamma_{rr} - \partial_r\gamma_{rr}) \qquad (234)\\
&= 0 \qquad (235)\\
\partial_\theta\Gamma^\theta_{rr} &= \partial_\theta\frac{1}{2}\gamma^{\theta\theta}(\partial_r\gamma_{r\theta} + \partial_r\gamma_{\theta r} - \partial_\theta\gamma_{rr}) \qquad (236)\\
&= 0 \qquad (237)\\
\partial_\phi\Gamma^\phi_{rr} &= \partial_\phi\frac{1}{2}\gamma^{\phi\phi}(\partial_r\gamma_{r\phi} + \partial_r\gamma_{\phi r} - \partial_\phi\gamma_{rr}) \qquad (238)\\
&= 0 \qquad (239)\\
\Rightarrow \partial_\lambda\Gamma^\lambda_{rr} &= 0 \qquad (240)
\end{aligned}
$$

The second component of R_{rr}:

$$\begin{aligned}
\partial_r \Gamma^\lambda_{\lambda r} &= \partial_r \frac{1}{2}\gamma^{\lambda\lambda}(\partial_\lambda \gamma_{r\lambda} + \partial_r \gamma_{\lambda\lambda} - \partial_\lambda \gamma_{\lambda r}) \\
&= \partial_r \frac{1}{2}(\gamma^{\theta\theta}\partial_r \gamma_{\theta\theta} + \gamma^{\phi\phi}\partial_r \gamma_{\phi\phi}) \\
&= \partial_r \frac{1}{2}\left(\frac{1}{\Phi^2}\partial_r \Phi^2 + \frac{1}{\Phi^2 \sin^2\theta}\partial_r \Phi^2 \sin^2\theta\right) \\
&= \frac{1}{2}\partial_r \left(\frac{2\Phi\partial_r\Phi}{\Phi^2} + \frac{2\Phi\partial_r\Phi}{\Phi^2}\right) \\
&= 2\partial_r \frac{\partial_r \Phi}{\Phi} \\
&= 2\left(\frac{\partial_r^2\Phi}{\Phi} - \frac{\partial_r\Phi}{\Phi^2}\partial_r\Phi\right) \\
&= 2\frac{\partial_r^2\Phi}{\Phi} - 2\frac{(\partial_r\Phi)^2}{\Phi^2} \qquad (241)
\end{aligned}$$

The third component of R_{rr}, i.e. $\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{rr}$, equals 0. The last term of R:

$$\begin{aligned}
\Gamma^\lambda_{r\rho}\Gamma^\rho_{\lambda r} &= \Gamma^\theta_{r\theta}\Gamma^\theta_{\theta r} + \Gamma^\phi_{r\phi}\Gamma^\phi_{\phi r} \\
&= \left(\frac{\partial_r\Phi}{\Phi}\right)^2 + \left(\frac{\partial_r\Phi}{\Phi}\right)^2 \\
&= 2\left(\frac{\partial_r\Phi}{\Phi}\right)^2 \qquad (242)
\end{aligned}$$

Putting all the terms together:

$$\begin{aligned}
R_{rr} &= -2\frac{\partial_r^2\Phi}{\Phi} + 2\frac{(\partial_r\Phi)^2}{\Phi^2} - 2\left(\frac{\partial_r\Phi}{\Phi}\right)^2 \\
&= -2\frac{\partial_r^2\Phi}{\Phi} \qquad (243)
\end{aligned}$$

Similarly for $R_{\theta\theta}$:

$$R_{\theta\theta} = \partial_\lambda \Gamma^\lambda_{\theta\theta} - \partial_\theta \Gamma^\lambda_{\lambda\theta} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\theta\theta} - \Gamma^\lambda_{\theta\rho}\Gamma^\rho_\lambda \theta \qquad (244)$$

The first component is given by:

$$\begin{aligned}
\partial_\lambda \Gamma^\lambda_{\theta\theta} &= \partial_r \Gamma^r_{\theta\theta} + \partial_\theta \Gamma^\theta_{\theta\theta} + \partial_\phi \Gamma^\phi_{\theta\theta} \\
&= \partial_r(-\Phi\partial_r\Phi) \\
&= -(\partial_r\Phi)^2 - \Phi\partial_r^2\Phi \qquad (245)
\end{aligned}$$

The second component:

$$\begin{aligned}
\partial_\theta \Gamma^\lambda_{\theta\lambda} &= \partial_\theta \Gamma^\phi_{\theta\phi} \\
&= \partial_\theta \cot\theta \qquad (246) \\
&= -\csc^2\theta \qquad (247)
\end{aligned}$$

The third component of $R_{\theta\theta}$:

$$\begin{aligned}\Gamma^{\lambda}_{\theta\theta}\Gamma^{\sigma}_{\lambda\sigma} &= \Gamma^{r}_{\theta\theta}\Gamma^{\sigma}_{r\sigma} \\ &= \Gamma^{r}_{\theta\theta}\Gamma^{\phi}_{r\phi} + \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{r\theta} \\ &= -\Phi\partial_r\Phi\frac{\partial_r}{\Phi} - \Phi\partial_r\Phi\frac{\partial_r\Phi}{\Phi} \\ &= -2(\partial_r\Phi)^2 \end{aligned} \tag{248}$$

and last but not least:

$$\begin{aligned}\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma} &= 2\Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\theta\phi} \\ &= -2\Phi\partial_r\Phi\frac{\partial_r\Phi}{\Phi} + \cot^2\theta \\ &= -2(\partial_r\Phi)^2 + 2\cot^2\theta \end{aligned} \tag{249}$$

Combining the different terms of $R_{\theta\theta}$:

$$\begin{aligned}R_{\theta\theta} &= -(\partial_r\Phi)^2 - \Phi\partial_r^2\Phi + \csc^2\theta - 2(\partial_r\Phi)^2 + 2(\partial_r\Phi)^2 - 2\cot^2\theta \\ &= -(\partial_r\Phi)^2 - \Phi\partial_r^2\Phi + 1. \end{aligned} \tag{250}$$

Finally we compute the last term of the Ricci scalar:

$$R_{\phi\phi} = \partial_\lambda\Gamma^{\lambda}_{\phi\phi} - \partial_\phi\Gamma^{\lambda}_{\lambda\phi} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\phi\phi} - \Gamma^{\lambda}_{\phi\rho}\Gamma^{\rho}_{\lambda\phi} \tag{251}$$

We again start by computing the different components of $R_{\phi\phi}$:

$$\begin{aligned}\partial_\lambda\Gamma^{\lambda}_{\phi\phi} &= \partial_r\Gamma^{r}_{\phi\phi} + \partial_\theta\Gamma^{\theta}_{\phi\phi} + \partial_\phi\Gamma^{\phi}_{\phi\phi} \\ &= \partial_r(-\Phi\sin^2\theta\partial_r\Phi) + \partial_\theta(-\sin\theta\cos\theta) \\ &= -\sin^2\theta\partial_r\Phi\partial_r\Phi - \Phi\sin^2\theta\partial_r^2\Phi + \sin^2\theta - \cos^2\theta \\ &= -\sin^2\theta(\partial_r\Phi)^2 - \Phi\sin^2\theta\partial_r^2\Phi + \sin^2\theta - \cos^2\theta \end{aligned} \tag{252}$$

The second term $\partial_\phi\Gamma^{\lambda}_{\phi\lambda} = 0$. The third term:

$$\begin{aligned}\Gamma^{\lambda}_{\phi\phi}\Gamma^{\rho}_{\lambda\rho} &= \Gamma^{r}_{\phi\phi}\Gamma^{\rho}_{r\rho} + \Gamma^{\rho}_{\phi\phi}\Gamma^{\rho}_{\sigma\rho} \\ &= \Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{r\phi} + \Gamma^{r}_{\phi\phi}\Gamma^{\theta}_{r\theta} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} \\ &= (-\Phi\sin^2\theta\partial_r\Phi)\left(\frac{\partial_r\Phi}{\Phi}\right) + (-\Phi\sin^2\theta\partial_r\Phi)\left(\frac{\partial_r\Phi}{\Phi}\right) + (-\sin\theta\cos\theta)(\cot\theta) \\ &= -\sin^2\theta(\partial_r\Phi)^2 - \sin^2\theta(\partial_r\Phi)^2 - \sin\theta\cos\theta\cot\theta \\ &= -2\sin^2\theta(\partial_r\Phi)^2 - \cos^2\theta \end{aligned} \tag{253}$$

The last term of $R_{\phi\phi}$ is given by:

$$\begin{aligned}\Gamma^{\rho}_{\phi\lambda}\Gamma^{\lambda}_{\phi\rho} &= 2\Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{\phi r} + 2\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} \\ &= 2(-\Phi\sin^2\theta\partial_r\Phi)\left(\frac{\partial_r\Phi}{\Phi}\right) + 2(-\sin\theta\cos\theta)\cot\theta \\ &= -2\sin^2\theta(\partial_r\Phi)^2 - 2\sin\theta\cos\theta\cot\theta \\ &= -2\sin^2\theta(\partial_r\Phi)^2 - 2\cos^2\theta \end{aligned} \tag{254}$$

Hence $R_{\phi\phi}$ is given by:

$$\begin{aligned}
R_{\phi\phi} &= -\sin^2\theta(\partial_r\Phi)^2 - \Phi\sin^2\theta\partial_r^2\Phi + \sin^2\theta - \cos^2\theta - 2\sin^2\theta(\partial_r\Phi)^2 - \cos^2\theta + 2\sin^2\theta(\partial_r\Phi)^2 + 2\cos^2\theta \\
&= -\sin^2\theta(\partial_r\Phi)^2 - \Phi\sin^2\theta\partial_r^2\Phi + \sin^2\theta \qquad (255)
\end{aligned}$$

Then finally the Ricci scalar is given by:

$$\begin{aligned}
R &= \gamma^{rr}R_{rr} + \gamma^{\theta\theta}R_{\theta\theta} + \gamma^{\phi\phi}R_{\phi\phi} \\
&= -2\frac{\partial_r^2\Phi}{\Phi} + \frac{1}{\Phi^2}(-(\partial_r\Phi)^2 - \Phi\partial_r^2\Phi + 1) + \frac{1}{\Phi^2\sin^\theta}(-\sin^2\theta(\partial_r\Phi)^2 - \Phi\sin^2\theta\partial_r^2\Phi + \sin^2\theta) \\
&= -2\frac{\partial_r^2\Phi}{\Phi} - \frac{(\partial_r\Phi)^2}{\Phi^2} - \frac{\partial_r^2\Phi}{\Phi} + \frac{1}{\Phi^2} - \frac{(\partial_r\Phi)^2}{\Phi^2} - \frac{\partial_r^2\Phi}{\Phi} + \frac{1}{\Phi^2} \\
&= -2\frac{\Phi\partial_r^2\Phi}{\Phi^2} - 2\frac{(\partial_r\Phi)^2}{\Phi^2} - 2\frac{\Phi\partial_r^2\Phi}{\Phi^2} + \frac{2}{\Phi^2} \\
&= -4\frac{\Phi\partial_r\Phi}{\Phi^2} - 2\frac{(\partial_r\Phi)^2}{\Phi^2} + \frac{2}{\Phi^2} \\
&= \frac{2}{\Phi^2}[1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2] \qquad (256)
\end{aligned}$$

We introduce the following variables:

$$Q \equiv K_r^r - \frac{1}{3}K = \frac{2}{3}(\partial_\perp \ln\Phi - \partial_r\beta) \qquad (257)$$

$$P \equiv \partial_\perp\psi \qquad (258)$$

$$a \equiv \partial_r \ln\alpha \qquad (259)$$

Combining all terms we find the Langrangian density to be:

$$\begin{aligned}
\mathcal{L} = \sin\theta\Bigg[&\alpha\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \alpha\Phi^2\Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi \\
&- \frac{3}{4}\lambda^2 - \lambda(K + \phi) + \frac{1}{2}\big[P^2 - (\partial_r\psi)^2\big]\Big]\Bigg] \qquad (260)
\end{aligned}$$

where we used:

$$\begin{aligned}
\partial_\perp \ln\Phi &= -\frac{1}{2}Q + \frac{1}{3}K \qquad (261) \\
&= -\frac{1}{2}\left(-\frac{2}{3}(\partial_\perp\ln\Phi + \frac{1}{\alpha}\partial_r\beta\right) + \frac{1}{3}(2\partial_\perp\ln\Phi - \frac{1}{\alpha}\partial_r\beta) \\
&= \frac{1}{3}\partial_\perp\ln\Phi + \frac{1}{3}\frac{1}{\alpha}\partial_r\beta + \frac{2}{3}\partial_\perp\ln\Phi - \frac{1}{3}\frac{1}{\alpha}\partial_r\beta \\
&= \partial_\perp\ln\Phi \\
\partial_r\beta &= -\alpha\left(Q + \frac{1}{3}K\right) \qquad (262) \\
\partial_\perp\Phi &= \left(\frac{1}{3}K - \frac{1}{2}Q\right)\Phi \qquad (263)
\end{aligned}$$

The action on the spherical background is:

$$
\begin{aligned}
S^{\text{spher}} = \int d\theta d\phi dr dt \sin\theta \Bigg[& \alpha\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \alpha\Phi^2 \Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi \\
& - \frac{3}{4}\lambda^2 - \lambda(K+\phi) + \frac{1}{2}\left[P^2 - (\partial_r\psi)^2\right]\Big] + \mu_1(\partial_\perp\psi - P) + \mu_2\left(2\partial_\perp \ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right) \\
& + \mu_3(\partial_r \ln\alpha - a) + \mu_4\left(-\frac{2}{3}\left(\partial_\perp \ln\Phi + \frac{1}{\alpha}\partial_r\beta\right) - Q\right)\Bigg]
\end{aligned}
\tag{264}
$$

Taking the variation with respect to the auxilary fields P, K, a and Q we find the corresponding Lagrangian multipliers μ_1, μ_2,μ_3 and μ_4:

$$
\begin{aligned}
\frac{\delta S}{\delta P} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial P} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\dot{P})} + \mu_1\frac{\partial(\partial_\perp\psi - P)}{\partial P} &= 0 \\
\alpha\Phi^2 P - \mu_1 &= 0 \\
\mu_1 &= \alpha\Phi^2 P
\end{aligned}
\tag{265}
$$

$$
\begin{aligned}
\frac{\delta S}{\delta K} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial K} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\dot{K})} + \mu_2\frac{\partial\left(2\partial_\perp\ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right)}{\partial K} &= 0 \\
-\frac{2}{3}\alpha\Phi^2 K - \alpha\Phi^2\lambda - \mu_2 &= 0 \\
\mu_2 &= -\alpha\Phi^2\left(\frac{2}{3}K + \lambda\right)
\end{aligned}
\tag{266}
$$

$$
\begin{aligned}
\frac{\delta S}{\delta a} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial a} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\dot{a})} + \mu_3\frac{\partial(\partial_r\ln\alpha - a)}{\partial a} &= 0 \\
\mu_3 &= 0
\end{aligned}
\tag{267}
$$

$$
\begin{aligned}
\frac{\delta S}{\delta Q} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial Q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\dot{Q})} + \mu_4\frac{\partial\left(-\frac{2}{3}(\partial_\perp\ln\Phi + \frac{1}{\alpha}\partial_r\beta) - Q\right)}{\partial Q} &= 0 \\
\alpha\Phi^2\frac{3}{2} - \mu_4 &= 0 \\
\mu_4 &= \alpha\Phi^2\frac{3}{2}Q
\end{aligned}
\tag{268}
$$

Hence the action is given by:

$$
\boxed{
\begin{aligned}
S^{\text{spher}} = \int d\theta d\phi dr dt \sin\theta \Bigg[& \alpha\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \alpha\Phi^2 \Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) \\
& - V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi - \frac{3}{4}\lambda^2 - \lambda(K+\phi) + \frac{1}{2}\left[P^2 - (\partial_r\psi)^2\right] + P(\partial_\perp\psi - P) \\
& - \left(\frac{2}{3}K + \lambda\right)\left(2\partial_\perp\ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right) - Q\left(\partial_\perp\ln\Phi + \frac{1}{\alpha}\partial_r\beta + \frac{3}{2}Q\right)\Big]\Bigg]
\end{aligned}
}
\tag{269}
$$

Taking the variation with respect to α:

$$\frac{\delta S}{\delta \alpha} = 0$$
$$\Rightarrow$$
$$\frac{\partial \mathcal{L}}{\partial \alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t \alpha)} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial(\partial_r \alpha)} = 0$$

$$\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \Phi^2\Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{3}{4}\lambda^2 - \lambda\,(K+\phi) + \frac{1}{2}P^2 - \frac{1}{2}(\partial_r\psi)^2 - P^2$$
$$+ \left(\frac{2}{3}K + \lambda\right)K - \frac{3}{2}Q^2\Big] = 0$$
$$\left(1 - 2\Phi\partial_r^2\Phi - (\partial_r\Phi)^2\right) + \Phi^2\Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{3}{4}\left(-\frac{2}{3}K - \frac{2}{3}\phi\right)^2 - \left(-\frac{2}{3}K - \frac{2}{3}\phi\right)(K+\phi)$$
$$+\frac{1}{2}P^2 - \frac{1}{2}(\partial_r\psi)^2 - P^2 + \left(\frac{2}{3}K + \left(-\frac{2}{3}K - \frac{2}{3}\phi\right)\right)K - \frac{3}{2}Q^2\Big] = 0$$

Rewriting gives:

$$\frac{1}{\Phi^2} - 2\frac{\partial_r^2\Phi}{\Phi} - \frac{(\partial_r\Phi)^2}{\Phi^2} + \frac{3}{4}Q^2 - \frac{1}{3}K^2 - V(\phi) - \frac{1}{3}K^2 - \frac{2}{3}K\phi - \frac{1}{3}\phi^2 + \frac{2}{3}K^2 + \frac{4}{3}K\phi + \frac{2}{3}\phi^2 - \frac{1}{2}P^2$$
$$- \frac{1}{2}(\partial_r\psi)^2 - \frac{2}{3}K\phi - \frac{3}{2}Q^2 = 0$$
$$\boxed{\frac{\partial_r^2\Phi}{\Phi} = \frac{1 - (\partial_r\Phi)}{2\Phi^2} - \frac{3}{8}Q^2 - \frac{1}{4}P^2 - \frac{1}{4}(\partial_r\psi)^2 - \frac{1}{2}V(\phi) + \frac{1}{6}\phi^2} \tag{270}$$

Taking the variation with respect to β:

$$\frac{\delta S}{\delta \beta} = 0$$
$$\Rightarrow$$
$$\frac{\partial \mathcal{L}}{\partial \beta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t \beta)} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial(\partial_r \beta)} = 0$$

$$-\Phi^2 P\partial_r\psi - \Phi^2\left(\frac{2}{3}K + \lambda\right)(-2\frac{\partial_r\Phi}{\Phi}) + \Phi^2 Q\frac{\partial_r\Phi}{\Phi} - \frac{d}{dr}\Big[\Phi^2\left(\frac{2}{3}K + \lambda\right) - \Phi^2 Q\Big] = 0$$
$$-\Phi^2 P\partial_r\psi - \Phi^2\left(\frac{2}{3}K + \lambda\right)(-2\frac{\partial_r\Phi}{\Phi}) + \Phi^2 Q\frac{\partial_r\Phi}{\Phi} - 2\Phi\partial_r\Phi(-\frac{2}{3}\phi) + 2\Phi\partial_r\Phi Q + \Phi^2\partial_r Q = 0$$
$$-P\partial_r\psi - \frac{4}{3}\phi\frac{\partial_r\Phi}{\Phi} + Q\frac{\partial_r\Phi}{\Phi} + \frac{4}{3}\frac{\partial_r\Phi}{\Phi}\phi + 2\frac{\partial_r\Phi}{\Phi}Q + \partial_r Q = 0$$
$$-P\partial_r\psi + 3\frac{\partial_r\Phi}{\Phi}Q = -\partial_r Q$$
$$\boxed{\partial_r Q = P\partial_r\psi - 3Q\partial_r\ln\Phi} \tag{271}$$

Taking the variation of the action with respect to Q:

$$\begin{aligned}\frac{\delta S}{\delta Q} &= 0 \\ &\Rightarrow \\ \frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t Q)} - \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_r Q)} &= 0\end{aligned}$$

Leading to:

$$\begin{aligned}\alpha\Phi^2\left(\frac{3}{2}Q - \partial_\perp \ln\Phi - \frac{1}{\alpha}\partial_r\beta - \frac{6}{2}Q\right) &= 0 \\ \alpha\Phi^2\left(-\frac{3}{2}Q - \left(\frac{1}{\alpha}\frac{\partial_t\Phi}{\Phi} - \frac{\beta}{\alpha}\frac{\partial_r\Phi}{\Phi}\right) - \frac{1}{\alpha}\partial_r\beta\right) &= 0 \\ \alpha\Phi\left(-\frac{3}{2}\left(-\frac{2}{3}\left(\frac{\partial_t\Phi}{\alpha\Phi} - \frac{\beta}{\alpha}\frac{\partial_r\Phi}{\Phi} + \frac{1}{\alpha}\partial_r\beta\right)\Phi\right) - \frac{1}{\alpha}(\partial_t\Phi - \beta\partial_r\Phi) - \frac{1}{\alpha}\partial_r\beta\Phi\right) &= 0 \\ \alpha\Phi\left(\frac{1}{\alpha}(\partial_t\Phi - \beta\partial_r\Phi) + \frac{1}{\alpha}\partial_r\beta\Phi - \partial_\perp\Phi - \frac{1}{\alpha}\partial_r\beta\Phi\right) &= 0 \\ \Phi\left(\partial_t\Phi - \beta\partial_r\Phi - \alpha\partial_\perp\Phi\right) &= 0\end{aligned}$$

Plugging in our expression for $\partial_\perp\Phi$ we find our dynamical equation for Φ:

$$\boxed{\partial_t\Phi = \alpha\left(\frac{1}{3}K - \frac{1}{2}Q\right)\Phi + \beta\partial_r\Phi} \tag{272}$$

Taking the variation with respect to $\tilde{\lambda}$:

$$\begin{aligned}\frac{\delta S}{\delta\tilde{\lambda}} &= 0 \\ &\Rightarrow \\ \frac{\partial \mathcal{L}}{\partial\tilde{\lambda}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t\tilde{\lambda})} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial(\partial_r\tilde{\lambda})} &= 0 \\ -\frac{1}{\alpha}\partial_r\Phi &= 0\end{aligned}$$

$$\boxed{\partial_r\phi = 0} \tag{273}$$

Taking variation with respect to ψ we find:

$$\begin{aligned}\frac{\delta S}{\delta\psi} &= 0 \\ &\Rightarrow \\ \frac{\partial \mathcal{L}}{\partial\psi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t\psi)} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial(\partial_r\psi)} &= 0 \\ -\frac{d}{dt}\left(\Phi^2 P\right) - \frac{d}{dr}\left(-\alpha\Phi^2(\partial_r\Phi - \Phi^2\beta P)\right) &= 0\end{aligned}$$

Leading to:

$$-2\Phi\partial_t\Phi P - \Phi^2\partial_t P + \partial_r\alpha\Phi^2\partial_r\Phi + 2\alpha\Phi\partial_r\Phi\partial_r\psi + \alpha\Phi^2\partial_r^2\psi + 2\Phi\partial_r\Phi\beta P + \Phi^2\partial_r\beta P + \Phi^2\beta\partial_r P = 0$$

Rewriting:

$$\begin{aligned}
\Phi^2\partial_t P &= -2\Phi\partial_t\Phi P + \partial_r\alpha\Phi^2\partial_r\psi + 2\alpha\Phi\partial_r\Phi\partial_r\psi + \alpha\Phi^2\partial_r^2\psi + 2\Phi\partial_r\Phi\beta P + \Phi^2\partial_r\beta P + \Phi^2\beta\partial_r P\\
\partial_t P &= -\frac{2}{\Phi}\partial_t\Phi P + \partial_r\alpha\partial_r\psi + \alpha\frac{2}{\Phi}\partial_r\Phi\partial_r\psi + \alpha\partial_r^2\psi + \frac{2}{\Phi}\partial_r\Phi\beta P + \partial_r\beta P + \beta\partial_r P\\
&= -2\partial_t\ln\Phi P + \partial_r\alpha\partial_r\psi + 2\alpha\partial_r\ln\Phi\partial_r\psi + \alpha\partial_r^2\psi + 2\partial_r\ln\Phi\beta P + P\partial_r\beta + \beta\partial_r P\\
&= \alpha\Big[-\frac{2}{\alpha}\partial_t\ln\Phi P + \frac{2}{\alpha}\partial_r\ln\Phi\beta P + \frac{1}{\alpha}\partial_r\alpha\partial_r\psi + 2\partial_r\ln\Phi\partial_r\psi + \partial_r^2\psi\Big] + \beta\partial_r P\\
&= \alpha\Big[-2\partial_\perp\ln\Phi P + \frac{1}{\alpha}P\partial_r\beta + \frac{1}{\alpha}\partial_r\alpha\partial_r\psi + 2\partial_r\ln\Phi\partial_r\psi + \partial_r^2\psi\Big] + \beta\partial_r P\\
&= \alpha\Big[-KP + \partial_r\ln\alpha\partial_r\psi + 2\partial_r\ln\Phi\partial_r\psi + \partial_r^2\psi\Big] + \beta\partial_r P
\end{aligned}$$

And finally

$$\boxed{\partial_t P = \alpha\Big[-KP + (a + 2\partial_r\ln\Phi)\,\partial_r\psi + \partial_r^2\psi\Big] + \beta\partial_r P} \tag{274}$$

Taking variation with respect to P we find:

$$\begin{aligned}
\frac{\delta S}{\delta P} &= 0\\
&\Rightarrow\\
\frac{\partial\mathcal{L}}{\partial P} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\partial_t P)} - \partial_r\frac{\partial\mathcal{L}}{\partial(\partial_r P)} &= 0\\
P + \partial_\perp\psi - 2P &= 0\\
P &= \partial_\perp\psi
\end{aligned}$$

$$\boxed{\partial_t\psi = \alpha P + \beta\partial_r\psi} \tag{275}$$

To obtain the dynamical equation of Q and ϕ we take the variation with respect to Φ:

$$\begin{aligned}
\frac{\delta S}{\delta\Phi} &= 0\\
&\Rightarrow\\
\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\Phi)} - \partial_r\frac{\partial\mathcal{L}}{\partial(\partial_r\Phi)} - \partial_r^2\frac{\partial\mathcal{L}}{\partial(\partial_r^2\Phi)} &= 0
\end{aligned}$$

Leading to

$$
\begin{aligned}
&-2\alpha\partial_r^2\Phi + 2\alpha\Phi\Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi - \frac{3}{4}\lambda^2 - \lambda\left(K+\phi\right) + \frac{1}{2}\big[P^2 - (\partial_r\psi)^2\big]\\
&+ P\left(\partial_\perp\psi - P\right) - \left(\frac{2}{3}K + \lambda\right)\left(2\partial_\perp\ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right) - Q\left(\partial_\perp\ln\Phi + \frac{1}{\alpha}\partial_r\beta + \frac{3}{2}Q\right)\Big]\\
&- \frac{4}{3}\phi\Phi\alpha\partial_\perp\ln\Phi + Q\Phi\alpha\partial_\perp\ln\Phi\partial_t Q\Phi - \frac{4}{3}\phi\partial_t\Phi - \frac{4}{3}\partial_t\phi\Phi + Q\partial_t\Phi + 2\alpha\partial_r^2\Phi + 2\partial_r\alpha\partial_r\Phi\\
&+ \frac{4}{3}\partial_r\Phi\beta\phi + \frac{4}{3}\phi\Phi\partial_r\beta - \partial_r Q\beta\Phi - Q\partial_r\beta\Phi - Q\beta\partial_r\Phi + \partial_r\left(-2\partial_r\alpha\Phi - 2\alpha\partial_r\Phi\right) + \partial_r^2(-2\alpha\Phi) = 0
\end{aligned}
$$

$$
\begin{aligned}
&2\alpha\Phi\Big[\left(\frac{3}{4}Q^2 - \frac{1}{3}K^2\right) - V(\phi) - \frac{1}{\alpha}\tilde{\lambda}\partial_r\phi - \frac{3}{4}\lambda^2 - \lambda\left(K+\phi\right) + \frac{1}{2}\big[P^2 - (\partial_r\psi)^2\big] + P\left(\partial_\perp\psi - P\right)\\
&- \left(\frac{2}{3}K + \lambda\right)\left(2\partial_\perp\ln\Phi - \frac{1}{\alpha}\partial_r\beta - K\right) - Q\left(\partial_\perp\ln\Phi + \frac{1}{\alpha}\partial_r\beta + \frac{3}{2}Q\right)\Big] - \frac{8}{3}\phi\Phi\alpha\partial_\perp\ln\Phi\\
&+ Q\Phi\alpha\partial_\perp\ln\Phi - \frac{4}{3}\partial_t\phi\Phi + \partial_t Q\Phi + Q\partial_t\Phi + \frac{4}{3}\phi\Phi\partial_r\beta - \partial_r Q\beta\Phi - Q\partial_r\beta\Phi - Q\beta\partial_r\Phi\\
&- 2\partial_r^2\alpha\Phi - 2\partial_r\alpha\partial_r\Phi - 2\alpha\partial_r^2\Phi = 0
\end{aligned}
$$

$$
\begin{aligned}
-\partial_t Q = {}& 2\alpha\Big[\frac{3}{4}Q^2 - V(\phi) + \frac{2}{3}K\phi + \frac{1}{3}\phi^2 + \frac{1}{2}[P^2 - (\partial_r\psi)^2]\Big] - \frac{4}{3}K\phi\alpha + Q\alpha\partial_\perp\ln\Phi - \frac{4}{3}\partial_t\phi\\
&+ Q\frac{\partial_t\Phi}{\Phi} - \partial_r Q\beta - Q\partial_r\beta - Q\beta\frac{\partial_r\Phi}{\Phi} - 2\alpha\partial_r a - 2\partial_r\alpha\frac{\partial_r\Phi}{\Phi} - 2\alpha\frac{\partial_r^2\Phi}{\Phi} - 2\alpha a^2
\end{aligned}
$$

$$
\begin{aligned}
-\partial_t Q = {}& 2\alpha\Big[\frac{3}{4}Q^2 - V(\phi) + \frac{1}{3}\phi^2 + \frac{1}{2}[P^2 - (\partial_r\psi)^2]\Big] - \partial_r Q\beta - Q\partial_r\beta + Q\alpha\partial_\perp\ln\Phi - \frac{4}{3}\partial_t\phi\\
&+ Q\alpha\partial_\perp\ln\Phi - 2\alpha\partial_r a - 2\partial_r\alpha\frac{\partial_r\Phi}{\Phi} - 2\alpha\frac{\partial_r^2\Phi}{\Phi} - 2\alpha a^2
\end{aligned}
$$

$$
\begin{aligned}
-\partial_t Q = {}& \alpha\Big[\frac{3}{2}Q^2 - 2V(\phi) + \frac{2}{3}\phi^2 + P^2 - (\partial_r\psi)^2\Big] - \partial_r Q + KQ\alpha - \frac{4}{3}\partial_t\phi - 2\alpha\partial_r a\\
&- 2\partial_r\alpha\partial_r\ln\Phi - 2\alpha\frac{\partial_r^2\Phi}{\Phi} - 2\alpha a^2
\end{aligned}
$$

$$
\begin{aligned}
\partial_t Q = {}& \alpha\Big[-KQ + 2\partial_r a + 2a\partial_r\ln\Phi + 2\frac{\partial_r^2\Phi}{\Phi} - \frac{3}{2}Q^2 + 2V(\phi) + 2a^2 - \frac{2}{3}\phi^2 - P^2 + (\partial_r\psi)^2\Big]\\
&+ \frac{4}{3}\partial_t\phi + \partial_r Q\beta
\end{aligned}
$$

Substituting the equations for $\frac{\partial_r^2\Phi}{\Phi}$ and $\partial_r Q$ we find

$$
\boxed{
\begin{aligned}
\partial_t Q = {}&\alpha\Big[-KQ + 2\partial_r a + 2a^2 + 2a\partial_r\ln\Phi + \frac{1-(\partial_r\Phi)^2}{\Phi^2} - \frac{9}{4}Q^2 - \frac{3}{2}P^2 - \frac{1}{3}\phi^2\\
&+ V(\phi) + \frac{1}{2}(\partial_r\psi)^2\Big] + \frac{4}{3}\partial_t\phi + \beta(P\partial_r\psi - 3Q\partial_r\ln\Phi)
\end{aligned}
}
\tag{276}
$$

From the definitions of a and Q in equation (257) and that of K we find the following dynamical equations

$$\boxed{\partial_r \ln \alpha = a} \tag{277}$$

and

$$\boxed{\partial_r \beta = -\alpha(Q + \frac{1}{3}K)} \tag{278}$$

Taking the variation with respect to λ:

$$\begin{aligned}
\frac{\delta S}{\delta \lambda} &= 0 \\
&\Rightarrow \\
\frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t \lambda)} - \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_r \lambda)} &= 0 \\
-\frac{3}{2}\lambda - K - \phi - \left(2\partial_\perp \ln \Phi - \frac{1}{\alpha}\partial_r \beta - K\right) &= 0 \\
-\frac{3}{2}\lambda - \phi - 2\partial_\perp \ln \Phi + \frac{1}{\alpha}\partial_r \beta &= 0 \\
-\frac{3}{2}\lambda - \phi - K &= 0
\end{aligned}$$

$$\boxed{\lambda = -\frac{2}{3}(K + \phi)} \tag{279}$$

The other equation that determines the Lagrange multiplier follows from taking the variation with respect to ϕ:

$$\begin{aligned}
\frac{\delta S}{\delta \phi} &= 0 \\
&\Rightarrow \\
\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - \partial_r \frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} &= 0 \\
\alpha \Phi^2 \left[-V'(\phi) - \lambda\right] - \frac{d}{dr}\left(-\alpha \Phi^2 \frac{1}{\alpha}\tilde{\lambda}\right) &= 0 \\
\alpha \Phi^2 \left[-V'(\phi) - \lambda\right] + 2\Phi \partial_r \Phi \tilde{\lambda} + \Phi^2 \partial_r \tilde{\lambda} &= 0 \\
\alpha \left[\frac{2}{3}(K + \phi) - V'(\phi)\right] + 2\frac{\partial_r \Phi}{\Phi}\tilde{\lambda} + \partial_r \tilde{\lambda} &= 0
\end{aligned}$$

$$\boxed{2\tilde{\lambda}\partial_r \ln \Phi + \partial_r \tilde{\lambda} = -\alpha \left[\frac{2}{3}(K + \phi) - V'(\phi)\right]} \tag{280}$$

C Integrability Conditions

To decouple the dynamical equations of Q and ϕ we apply the integrability conditions

$$\boxed{\partial_r\partial_t Q = \partial_t\partial_r Q} \text{ and } \boxed{\partial_r\partial_t\phi = \partial_r\partial_t\phi} \tag{281}$$

from which we obtain

$$\begin{aligned}
\partial_t Q = \alpha\Big[&-\frac{2}{3}a^2 - \frac{1}{6}P^2 + \frac{1}{9}\phi^2 + \frac{1}{\Phi^2} - KQ - \frac{19}{4}Q^2 - \frac{1}{3}V(\phi) - \frac{2}{3}\partial_r a - \frac{2}{3}a\frac{1}{\Phi\partial_r\Phi} \\
&- \frac{2}{3}a^3\frac{\Phi}{\partial_r\Phi} + \frac{aP^2\Phi}{\partial_r\Phi} + \frac{3}{2}\frac{a\Phi Q^2}{\partial_r\Phi} - \frac{3}{2}\partial_r\ln\Phi\frac{\Phi}{\partial_r\Phi}Q^2 - \frac{2a\Phi\partial_r a}{\partial_r\Phi} + \frac{4}{3}\frac{P\Phi\partial_r P}{\partial_r\Phi} \\
&+ \frac{4}{3}\frac{a\partial_r\Phi}{\Phi} - \frac{3\beta Q\partial_r\Phi}{\alpha\Phi} - \frac{(\partial_r\Phi)^2}{\Phi^2} + \beta P\partial_r\psi\frac{1}{\alpha} + \frac{2P\Phi Q\partial_r\psi}{\partial_r\Phi} + \frac{1}{2}(\partial_r\psi)^2 \\
&+ \frac{1}{3}\frac{a\Phi(\partial_r\psi)^2}{\partial_r\Phi} - \frac{2}{3}\frac{\Phi\partial_r^2 a}{\partial_r\Phi}\Big] \\
= \alpha\Big[&-\frac{2}{3}a^2 - \frac{1}{6}P^2 + \frac{1}{9}\phi^2 + \frac{1}{\Phi^2} - KQ - \frac{19}{4}Q^2 - \frac{1}{3}V(\phi) - \frac{2}{3}\partial_r a - \frac{2}{3}a\frac{1}{\Phi\partial_r\Phi} \\
&- \frac{2}{3}a^3\frac{\Phi}{\partial_r\Phi} + \frac{aP^2\Phi}{\partial_r\Phi} + \frac{3}{2}\frac{a\Phi Q^2}{\partial_r\Phi} - \frac{3}{2}Q^2 - \frac{2a\Phi\partial_r a}{\partial_r\Phi} + \frac{4}{3}\frac{P\Phi\partial_r\Phi}{\partial_r\Phi} + \frac{1}{2}(\partial_r\psi)^2 \\
&+ \frac{1}{3}\frac{a\Phi(\partial_r\psi)^2}{\partial_r\Phi} - \frac{2}{3}\frac{\Phi\partial_r^2 a}{\partial_r\Phi} + \frac{4}{3}a\partial_r\ln\Phi - 3\beta Q\partial_r\ln\Phi\frac{1}{\alpha} - \frac{(\partial_r\Phi)^2}{\Phi^2} + \beta P\partial_r\psi\frac{1}{\alpha} \\
&+ \frac{2PQ\Phi\partial_r\psi}{\partial_r\Phi}\Big] \\
= \alpha\Big[&-KQ - \frac{1}{6}P^2 - \frac{1}{3}V(\phi) - \frac{1}{9}\phi^2 - \frac{1}{4}Q^2 + \frac{1}{\Phi^2} + \frac{1}{2}(\partial_r\psi)^2 + \frac{2}{3}a^2 \\
&+ \frac{2}{3}\partial_r a - \frac{2}{3}a\partial_r\ln\Phi - \frac{(\partial_r\Phi)^2}{\Phi^2}\Big] + \beta(P\partial_r\psi - 3Q\partial_r\ln\Phi) \\
&+ \alpha\Big[-6Q^2 - \frac{4}{3}a^2 - \frac{4}{3}\partial_r a - \frac{2}{3}a\frac{1}{\Phi\partial_r\Phi} - \frac{2}{3}a^3\frac{\Phi}{\partial_r\Phi} + \frac{aP^2\Phi}{\partial_r\Phi} + \frac{3}{2}a\frac{\Phi Q^2}{\partial_r\Phi} \\
&- 2a\partial_r a\frac{\Phi}{\partial_r\Phi} + \frac{4}{3}\frac{P\partial_r P\Phi}{\partial_r\Phi} + \frac{1}{3}a(\partial_r\psi)^2\frac{\Phi}{\partial_r\Phi} - \frac{2}{3}\frac{\Phi\partial_r^2 a}{\partial_r\Phi} + 2a\partial_r\ln\Phi + \frac{2PQ\partial_r\psi}{\partial_r\Phi}\Big]
\end{aligned}$$

We define the following expressions

$$\begin{aligned}
\text{eq}(282) = \alpha\Big[&-KQ - \frac{1}{4}Q^2 + \frac{2}{3}(a^2 + \partial_r a - a\partial_r\ln\Phi) + \frac{1-(\partial_r\Phi)^2}{\Phi^2} - \frac{1}{6}P^2 + \frac{1}{2}(\partial_r\psi)^2 \\
&+ \frac{1}{9}\phi^2 - \frac{1}{3}V(\phi)\Big] + \beta(P\partial_r\psi - 3Q\partial_r\ln\Phi)
\end{aligned} \tag{282}$$

$$\begin{aligned}
\text{eq}(283) = &-3a\partial_r a + a\left[\frac{(\partial_r\Phi)^2 - 1}{\Phi^2} + \frac{9}{4}Q^2 + \frac{3}{2}P^2 + \frac{1}{2}(\partial_r\psi)^2\right] - 2a^2\partial_r\ln\Phi - a^3 \\
&+ P(3Q\partial_r\psi + 2\partial_r P) - 9Q^2\partial_r\ln\Phi + 2(a\partial_r\ln\Phi - \partial_r a)\partial_r\ln\Phi.
\end{aligned} \tag{283}$$

We continue to simplify $\partial_t Q$ using these expressions

$$\partial_t Q = \text{eq}(282) + \frac{2\alpha\Phi}{3\partial_r\Phi}\Big[-9Q^2\partial_r\ln\Phi - 2a^2\partial_r\ln\Phi - 2\partial_r a\partial_r\ln\Phi - a\frac{1}{\Phi^2} - a^3 + \frac{3}{2}aP^2 + \frac{9}{4}aQ^2 - 3a\partial_r a + 2P\partial_r P + \frac{1}{2}a(\partial_r\psi)^2 + 3a(\partial_r\ln\Phi)^2 + 3PQ\partial_r\psi - \partial_r^2 a\Big]$$
$$= \text{eq}(282) + \frac{2\alpha\Phi}{3\partial_r\Phi}\Big[-9Q^2\partial_r\ln\Phi - 2a^2\partial_r\ln\Phi - 2\partial_r a\partial_r\ln\Phi - a\frac{1}{\Phi^2} - a^3 + \frac{3}{2}aP^2 + \frac{9}{4}aQ^2 - 3a\partial_r a + 2P\partial_r P + \frac{1}{2}a(\partial_r\psi)^2 + 2a(\partial_r\ln\Phi)^2 + a\frac{(\partial_r\Phi)^2}{\Phi^2} + 3PQ\partial_r\psi - \partial_r^2 a\Big]$$

and finally we obtain

$$\boxed{\partial_t Q = \text{eq}(282) + \frac{2\alpha\Phi}{3\partial_r\Phi}\Big[\text{eq}(283) - \partial_r^2 a\Big]} \tag{284}$$

By setting the coupled dynamical equation for Q, equation (88), equal to the decoupled dynamical equation of Q, equation (282), we find $\partial_t\phi$. First, we additionally define the following expression

$$\text{eq}(285) = \alpha\left[-a^2 - \partial_r a - 2\partial_r\ln\Phi + \frac{3}{2}Q^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi)\right] \tag{285}$$

which we use in simplifying $\partial_t\phi$

$$\partial_t\phi = \alpha\Big[-2a^2 + P^2 + \frac{1}{3}\phi^2 - \frac{15}{8}Q^2 - V(\phi) - 2\partial_r a - \frac{a}{2\Phi\partial_r\Phi} - \frac{a^3\Phi}{2\partial_r\Phi} + \frac{3aP^2\Phi}{4\partial_r\Phi} + \frac{9a\Phi Q^2}{8\partial_r\Phi} - \frac{9\Phi\partial_r\ln\Phi Q^2}{8\partial_r\Phi} - \frac{3}{2}\frac{a\Phi\partial_r a}{\partial_r\Phi} + \frac{P\partial_r P\Phi}{\partial_r\Phi} - \frac{a\partial_r\Phi}{2\Phi} + \frac{3}{2}\frac{3PQ\Phi\partial_r\psi}{2\partial_r\Phi} + \frac{a\Phi(\partial_r\psi)^2}{4\partial_r\Phi} - \frac{\partial_r^2 a}{2\partial_r\Phi}\Big]$$
$$= \alpha\Big[-a^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi) + \frac{3}{2}Q^2 - \frac{27}{8}Q^2 - \partial_r a - \partial_r a - a^2 - 2a\partial_r\ln\Phi - \frac{a}{2\Phi\partial_r\Phi} - \frac{a^3\Phi}{2\partial_r\Phi} + \frac{3aP^2\Phi}{4\partial_r\Phi} + \frac{9}{8}\frac{a\Phi Q^2}{\partial_r\Phi} - \frac{9}{8}Q^2 - \frac{3}{2}\frac{a\Phi\partial_r a}{\partial_r\Phi} + \frac{P\partial_r P\Phi}{\partial_r\Phi} + \frac{3}{2}a\partial_r\ln\Phi\Phi + \frac{3PQ\partial_r\psi\Phi}{2\partial_r\Phi} + \frac{a\Phi(\partial_r\psi)^2}{4\partial_r\Phi} - \frac{\partial_r^2 a}{2\partial_r\Phi}\Big]$$
$$= \text{eq}(285) + \alpha\Big[-\frac{9}{2}Q^2 - \partial_r a - a^2 - \frac{a}{2\Phi\partial_r\Phi} - \frac{a^3\Phi}{2\partial_r\Phi} + \frac{3aP^2\Phi}{4\partial_r\Phi} + \frac{9}{8}\frac{a\Phi Q^2}{\partial_r\Phi} - \frac{3}{2}\frac{a\Phi\partial_r a}{\partial_r\Phi} + \frac{P\partial_r P\Phi}{\partial_r\Phi} + \frac{3}{2}a\partial_r\ln\Phi + \frac{3}{2}\frac{PQ\partial_r\psi\Phi}{\partial_r\Phi} + \frac{a\Phi(\partial_r\psi)^2}{4\partial_r\Phi} - \frac{\partial_r^2 a\Phi}{2\partial_r\Phi}\Big]$$
$$= \text{eq}(285) + \frac{\alpha\Phi}{2\partial_r\Phi}\Big[-9Q^2\partial_r\ln\Phi - 2\partial_r a\partial_r\ln\Phi - 2a^2\partial_r\ln\Phi - \frac{a}{\Phi^2} - a^3 + \frac{3}{2}aP^2 + \frac{9}{4}aQ^2 - 3a\partial_r a\Big] + 2P\partial_r P + 3a(\partial_r\ln\Phi)^2 + 3PQ\partial_r\psi + \frac{a}{2}(\partial_r\psi)^2 - \partial_r^2 a$$

and finally

$$\boxed{\partial_t\phi = \text{eq}(285) + \frac{\alpha\Phi}{2\partial_r\Phi}\Big[\text{eq}(283) - \partial_r^2 a\Big]} \tag{286}$$

By applying the second integrability condition of equations (281), a complex expression is obtained for $\partial_r^3 a$.

The equations (284) and (286) are still coupled through $\partial_r^2 a$. To obtain an expression for $\partial_r^2 a$ we apply the following integrability condition

$$\boxed{\partial_t \partial_r^2 \Phi = \partial_r^2 \partial_t \Phi} \tag{287}$$

from which we obtain a complex expression for $\partial_r^2 a$ as function $\partial_r^2 K$. For the decoupled equations of $\partial_r^2 K$ and $\partial_r^2 a$ we refer to equations (24) and (25) in Jalali et al. 2024, in which

$$\begin{aligned} \partial_r^2 K = & -2\left(a + \partial_r \ln \Phi\right) \partial_r K \\ & - \left[\frac{a^2}{2} + \partial_r a + 2a\partial_r \ln \Phi - \frac{3}{2} Q^2 - P^2 - \frac{1}{3}\varphi^2 + V(\varphi)\right]\left[K + \varphi - \frac{3}{2} V'(\varphi)\right], \end{aligned} \tag{288}$$

$$\begin{aligned} \partial_r^2 a = & -3a\partial_r a + a\Big[(\partial_r \Phi)^2 - \frac{1}{\Phi^2} + \frac{9}{4} Q^2 + \frac{3}{2} P^2 + \frac{1}{2}(\partial_r \psi)^2\Big] - 2a^2 \partial_r \ln \Phi - a^3 \\ & + P(3Q\partial_r \psi + 2\partial_r P) - 9Q^2 \partial_r \ln \Phi + 2(a\partial_r \ln \Phi - \partial_r a)\partial_r \ln \Phi, \end{aligned} \tag{289}$$

i.e. $\partial_r^2 a =$ eq(283). The expression (289) reduces the equations (284) and (286) to

$$\begin{aligned} \partial_t Q = \alpha\Big[& - KQ - \frac{1}{4} Q^2 + \frac{2}{3}\left(a^2 + \partial_r a - a\,\partial_r \ln \Phi\right) + \frac{1 - (\partial_r \Phi)^2}{\Phi^2} - \frac{1}{6} P^2 + \frac{1}{2}(\partial_r \psi)^2 \\ & + \frac{1}{9}\phi^2 - \frac{1}{3} V(\phi)\Big] + \beta\left(P\,\partial_r \psi - 3Q\,\partial_r \ln \Phi\right), \end{aligned} \tag{290}$$

$$\partial_t \phi = \alpha\left[-a^2 - \partial_r a - 2a\,\partial_r \ln \Phi + \frac{3}{2} Q^2 + P^2 + \frac{1}{3}\phi^2 - V(\phi)\right], \tag{291}$$

i.e. $\partial_t Q =$ eq(282). In the paper Jalali et al. 2024, they started from an ansatz containing an additional function R^2 in $\gamma_{ij} dx^i dx^j = R^2(t,r) dr^2 + \Phi(t,r)^2 d\Omega_2^2$ which is not showing in their equation (8) as it was later set to unity. However, it was used in order to derive the equations of motion. With this extra function being present, $\partial_t \phi$ and $\partial_t Q$ were directly obtained (without integrability conditions) as presented in equations (290)-(291). From the integrability conditions in equations (281) and (287) the (decoupled) equations (288)-(289) were obtained.

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