Towards the Clemens-Schmid Exact Sequence

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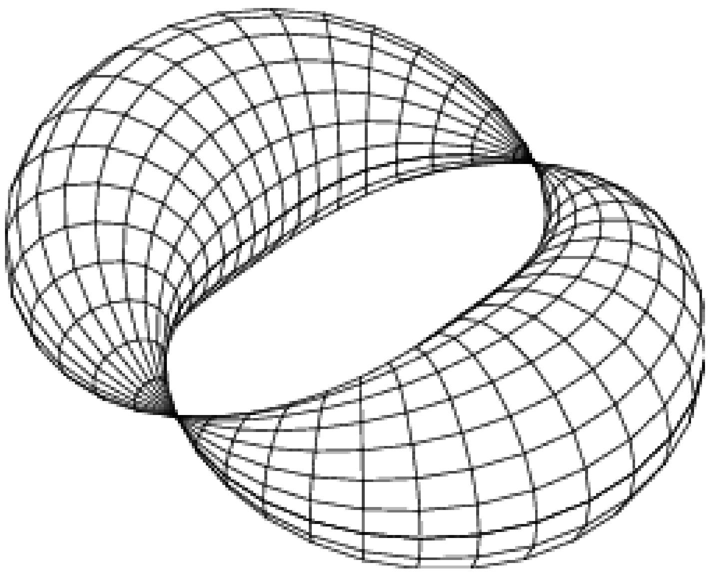

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Abstract

This thesis explores the behavior of cohomology under degeneration, with the main goal of understanding the Clemens–Schmid exact sequence, which relates the cohomology of the general fiber in a semistable degeneration to that of the special fiber. The sequence is a fundamental tool in Hodge theory for analyzing how the topology and Hodge structures of a smooth fiber change as it degenerates. In particular, it imposes strong constraints on the monodromy action and shows how information about the cohomology and monodromy of the smooth fiber can be extracted from the central fiber alone.

In Chapter 1, we begin with the necessary background on spectral sequences. We introduce filtered and double complexes and explain how spectral sequences arise from them. These tools are used throughout the thesis to compute and organize cohomological information in degenerations.

Chapter 2 introduces Kähler manifolds, the Hodge decomposition and the theory of Hodge structures. We discuss differential forms, Hermitian metrics, and how the Hodge decomposition arises naturally in the setting of compact Kähler manifolds.

In Chapter 3, we study normal crossing divisors and their cohomology. We define the de Rham and Čech double complexes, and use them to compute the cohomology of a normal crossing divisor in a Kähler manifold. We also construct the spectral weight filtration on the cohomology of the divisor and introduce the dual complex, which captures the combinatorics of the intersection pattern.

Chapter 4 builds towards the Clemens–Schmid exact sequence, which is the central result of the thesis. We introduce semistable degenerations and the semistable reduction theorem, and study the behavior of the monodromy operator on cohomology. Although we do not prove the Clemens–Schmid sequence, we use it to obtain interesting results and to analyze examples.

In Chapter 5, we turn to degenerations of smooth curves. We focus on the case where the central fiber is a nodal curve and generalize the consequences of the Clemens–Schmid sequence in this setting. We give explicit formulas for the genus and Betti numbers of the general fiber, in terms of topological data of the central fiber.

Finally, in Chapter 6, we study degenerations of surfaces, focusing on projective $K3$ surfaces and abelian surfaces, and examine how the monodromy weight filtration appears in these cases.

Acknowledgements

I want to sincerely thank my supervisor, Dr. Robin de Jong, for his continuous support and guidance during my thesis. His patience, understanding and helpful advice helped me overcome many challenges along the way. I really appreciate his encouragement and the constructive feedback that pushed me to improve and think more deeply about my work. Working with him has been a great learning experience, and I'm very grateful for everything he has done to help me grow as a researcher.

Contents

Chapter 1

Spectral Sequences

In this chapter, we introduce spectral sequences and the basic algebraic setup behind them. We'll start by defining graded and bigraded modules, and then move on to differential structures and exact couples. From there, we'll see how spectral sequences naturally arise and how to work with them. The goal is to understand the definitions and constructions well enough to use spectral sequences as a tool in later chapters, where they'll help us compute and organize cohomological information. Some references for this chapter are [3, Chapter 3, Section 14], [9] and [15].

Throughout this section, let R be a commutative ring with unity. All modules are R-modules.

1.1 Cohomology Spectral Sequences

Definition 1.1.1. A *graded module* A is a collection of modules $\{A^p\}_{p\in\mathbb{Z}}$.
For A, B graded modules, a *map of graded modules of degree* s, is a collection of module homomorphisms $(f^p : A^p \to B^{p+s})_{p\in\mathbb{Z}}$.

A *differential graded module* A is a graded module equipped with a map of graded modules $d : A \to A^{\bullet+s}$ such that $d^2 = 0$. We call d the *differential* and s the *degree* of d.

Remark 1.1.2. A *chain complex* is a differential graded module with d of degree -1.
A *cochain complex* is a differential graded module with d of degree 1.

Definition 1.1.3. A *bigraded module* E is a collection of modules $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$. For D, E bigraded modules, a *map of bigraded modules of bidegree* (s,t) with $s,t \in \mathbb{Z}$, is a collection of module homomorphisms $(f^{p,q} : D^{p,q} \to E^{p+s,q+t})_{p,q\in\mathbb{Z}}$.

A *differential bigraded module* is a bigraded module E equipped with a map of bigraded modules $d : E^{\bullet,\bullet} \to E^{\bullet+s,\bullet+t}$ such that $d^2 = 0$. We call d the *differential* and (s,t) the *bidegree* of d.

Definition 1.1.4. Fix r_0 a nonnegative integer. A *cohomology spectral sequence* is a family of differential bigraded modules $\{E_r, d_r\}_{r\geq r_0}$, where each differential

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

has bidegree $(r, -r+1)$. These pages are linked by the condition that the $(r+1)$-st page is the cohomology of the r-th page:

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

We say the spectral sequence is *first quadrant* if $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$, for all $r \geq r_0$.

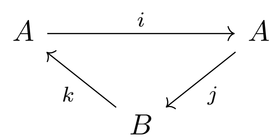
Remark 1.1.5. For each $r \geq r_0$, the collection $E_r = \{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ is referred to as the *r-th page* of the spectral sequence.

We say that a spectral sequence *converges* if, for some sufficiently large r_1, the pages stabilize: $E_r \cong E_{r_1}$ for all $r \geq r_1$. In this case, we say that the spectral sequence *degenerates* at E_{r_1}, we denote the limiting page by E_∞ and refer to it as the *limit* or *last page* of the spectral sequence.

1.2 Exact Couples

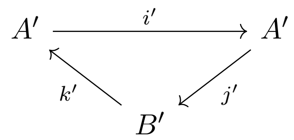
Spectral sequences often arise from structures called *exact couples*, which encode a recursive relationship between modules and allow for the construction of the pages of a spectral sequence step by step.

Definition 1.2.1. An *exact couple* (A, B, i, j, k) is a diagram of modules and linear maps:



such that the sequence is exact at each object: the image of one map equals the kernel of the next.

From an exact couple (A, B, i, j, k), one can define a differential $d := jk : B \to B$, with $d^2 = j(kj)k = 0$. This allows us to construct a new *derived exact couple* (A', B', i', j', k') as follows:



- $A' = i(A)$ (the image of i),
- $B' = H(B, d) = \ker(d)/\operatorname{im}(d)$,
- $i' : A' \to A'$ is induced by i,
- $j' : A' \to B'$ is defined by $i(x) \mapsto [j(x)]$,
- $k' : B' \to A'$ is defined by $[y] \mapsto k(y)$.

Proposition 1.2.2. *The derived couple (A', B', i', j', k') is an exact couple.*

Proof. We check first that the maps j' and k' are well-defined:

- Since $kj = 0$, observe that for any $i(x) \in A$, $jkj(x) = 0$ and thus $j(x) \in \ker d$. If $i(x) = i(x')$ then $x - x' \in \ker i = \operatorname{im} k$ and thus, there is a $y \in B$ with $k(y) = x - x'$. Now, $jk(y) = j(x) - j(x')$ and so, $[j(x)] = [j(x') + jk(y)] = [j(x')]$ in $B' = H(B, d)$ and j' is well-defined.

- If $[y] = [y']$ in B' then $y = y' + jk(z)$ for some $z \in B$ and $k(y) = k(y') + kjk(z) = k(y')$. Thus, k' is well-defined.

Now we will prove exactness.

- At the left A': Take $x \in A$. If $i(x) \in \ker i'$ then $i^2(x) = 0$. That means, there is a $y \in B$ such that $k(y) = i(x)$. This implies $jk(y) = ji(x) = 0$ and $y \in \ker d$. Hence, $k'([y]) = k(y) = i(x)$ and $i(x) \in \operatorname{im} k'$. Conversely, if $i(x) \in \operatorname{im} k'$, for $x \in A$, there exists a $y \in \ker d$ such that $k(y) = i(x)$. But then $i^2(x) = ik(y) = 0$ and thus $i(x) \in \ker i'$. We conclude $\ker i' = \operatorname{im} k'$.

- At the right A': Take $i^2(x) \in \operatorname{im} i'$ for $x \in A$. Of course, $j'(i^2(x)) = [ji(x)] = [0]$ and thus $i^2(x) \in \ker j'$. Conversely, if $j'(i(x)) = [j(x)] = [0]$ for $x \in A$, then there exists a $y \in B$ such that $j(x) = jk(y)$. That means $x - k(y) \in \ker j = \operatorname{im} i$ and thus there is a $z \in A$ with $i(z) = x - k(y)$. But then, $i(x) = i^2(z) + ik(y) = i^2(z)$ and thus $i(x) \in \operatorname{im} i'$. We conclude $\ker j' = \operatorname{im} i'$.

- At B': Given $i(x) \in A'$ we have $k'j'(i(x)) = k([j(x)]) = kj(x) = 0$. Conversely, take $k'([y]) = k(y) = 0$ for $y \in \ker d \subset B$. Since $\ker k = \operatorname{im} j$, there is a $x \in A$ such that $y = j(x)$, and thus, $[y] = [j(x)] = j'(i(x)) \in \operatorname{im} j'$. We conclude $\ker k' = \operatorname{im} j'$.

□

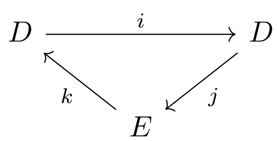
By iterating this process, starting with an exact couple we can generate a sequence of exact couples

$$(A_n, B_n, i_n, j_n, k_n)$$

where the n-th exact couple is derived from the previous one.

The notion of an exact couple naturally extends to the setting of (bi)graded modules. When A and B are (bi)graded modules, we say that (A, B, i, j, k) is an exact couple if the maps i, j, and k are morphisms of (bi)graded modules of certain (bi)degrees.

Theorem 1.2.3. *Let D and E be bigraded modules equipped with maps of bigraded modules i, j, k of bidegree $(-1, 1), (0, 0), (1, 0)$ respectively such that they form an exact couple:*



Then we have a cohomology spectral sequence $\{E_r, d_r\}_{r \geq 1}$, where $E_1 = E$, $(D_r, E_r, i_r, j_r, k_r)$ for $r > 1$ is the $(r-1)$-th derived exact couple and the differentials d_r are given by the composition $d_r = j_r k_r$.

Proof. Since by construction $E_r = H(E_{r-1}, d_{r-1})$, it suffices to verify that the differential d_r has bidegree $(r, 1-r)$. We will prove by induction that the bidegrees of i_r, j_r and k_r are $(-1,1)$, $(r-1, 1-r)$ and $(1,0)$ respectively.
The base case $r = 1$ is clear. Fix $r \geq 2$ and assume that the bidegrees of i_{r-1}, j_{r-1}, and k_{r-1} are $(-1,1)$, $(r-2, 2-r)$, and $(1,0)$, respectively.

Since i_r is induced by i_{r-1}, and $k_r([y]) = k_{r-1}(y)$, it follows that the bidegrees of i_r and k_r remain $(-1,1)$ and $(1,0)$, respectively.

Now, because i_{r-1} has bidegree $(-1,1)$, we have

$$D_r^{p,q} = i_{r-1}(D_{r-1}^{p+1,q-1}).$$

For any $x \in D_{r-1}^{p+1,q-1}$, we have

$$j_r(i_{r-1}(x)) = [j_{r-1}(x)]_r,$$

where $[\,\cdot\,]_r$ denotes the class in $E_r = \ker d_{r-1}/\operatorname{im} d_{r-1}$.

By the inductive hypothesis, $j_{r-1}(x) \in E_{r-1}^{p+r-1,q-r+1}$. Since $E_r^{p,q}$ is a subquotient of $E_{r-1}^{p,q}$, it follows that the bidegree of j_r is $(r-1, 1-r)$. Therefore, the differential $d_r = j_r k_r$ has bidegree $(r-1, 1-r) + (1,0) = (r, 1-r)$ as required. □

Example 1.2.4 (The Bockstein Spectral Sequence - Cohomology version)**.** Let C be a cochain complex of torsion-free abelian groups and p a prime number. Then there is a short exact sequence

$$0 \to C \xrightarrow{\cdot p} C \xrightarrow{\text{mod } p} C \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to 0$$

Taking cohomology, we get an exact couple of bigraded modules

$$\begin{array}{ccc} H^*(C) & \xrightarrow{\quad i=\cdot p \quad} & H^*(C) \\ & {}_{k=\delta}\nwarrow \qquad \swarrow_{j=\ \text{mod } p} & \\ & H^*(C \otimes \mathbb{Z}/p\mathbb{Z}) & \end{array}$$

where $(H^*(C))^{p,q} = H^{p+q}(C)$ and $(H^*(C \otimes \mathbb{Z}/p\mathbb{Z}))^{p,q} = H^{p+q}(C \otimes \mathbb{Z}/p\mathbb{Z})$. We can regard i, j, k of bidegrees $(-1,1)$, $(0,0)$, $(1,0)$ respectively. By Theorem 1.2.3, we get a spectral sequence, with first page $E_1^{p,q} = H^{p+q}(C \otimes \mathbb{Z}/p\mathbb{Z})$.

1.3 Filtered Complexes

Filtered chain complexes give rise to exact couples and therefore to spectral sequences. This is one of the most basic sources of spectral sequences.

Definition 1.3.1. A decreasing *filtration* $F_\bullet$ on a differential graded module (A, d), d of degree s, is a collection of submodules

$$\cdots \supset F_{p-1}A^n \supset F_pA^n \supset F_{p+1}A^n \supset \cdots$$

for each integer n, satisfying the following properties:

1. $d(F_pA^n) \subseteq F_pA^{n+s}$ (compatible with the differential)
2. $\bigcup_p F_pA^n = A^n$ for every n. (exhaustive)
3. $\bigcap_p F_pA^n = 0$ for every n. (Hausdorff)

The filtration is said to be *bounded* if we can write it as

$$A^n = F_0A^n \supset F_1A^n \supset \ldots \supset F_mA^n \supsetneq F_{m+1}A^n = 0$$

for some m.

Remark 1.3.2. Similarly, an increasing filtration on a differential graded module, is the same as a decreasing filtration with the subset symbols reversed.

Definition 1.3.3. A *filtered differential graded module* $(A, d, F_\bullet)$ is a differential graded module (A, d) equipped with a decreasing filtration $F_\bullet$.

Remark 1.3.4. From property (1) of the definition of filtration, the differential d induces a well-defined differential $d : F_pA^n/F_{p+1}A^n \to F_pA^{n+s}/F_{p+1}A^{n+s}$, and makes $\{F_pA/F_{p+1}A, d\}$ into a differential graded module. We denote $\mathrm{Gr}_p^F A = F_pA/F_{p+1}A$ the *graded pieces* of such filtration.

We now turn our attention to filtered cochain complexes. Cohomology spectral sequences arise naturally from decreasing filtrations of complexes as follows.

Given a filtered cochain complex $\{A, d\}$, the filtration on A induces a filtration on the cohomology of A, $H^*(A)$:

$$F_pH^*(A) = \mathrm{im}\,(H^*(F_pA) \to H^*(A))$$

For each level p of the filtration, we have a short exact sequence of cochain complexes

$$0 \to F_{p+1}A \to F_pA \to F_pA/F_{p+1}A \to 0$$

which induces a long exact sequence in cohomology

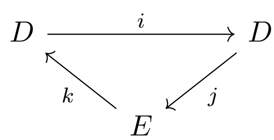
$$\cdots \xrightarrow{k} H^{p+q}(F_{p+1}A) \xrightarrow{i} H^{p+q}(F_pA) \xrightarrow{j} H^{p+q}(F_pA/F_{p+1}A) \xrightarrow{k} H^{p+q+1}(F_{p+1}A) \to \ldots$$

where i is induced by the inclusion, j by the quotient map and k is the connecting map.

Define $E^{p,q} = H^{p+q}(F_pA/F_{p+1}A) = H^{p+q}(\mathrm{Gr}_p^F A)$ and $D^{p,q} = H^{p+q}(F_pA)$. Then we have a long exact sequence of graded modules

$$\cdots \to D^{p+1,q-1} \xrightarrow{i} D^{p,q} \xrightarrow{j} E^{p,q} \xrightarrow{k} D^{p+1,q} \to \ldots$$

with i of bidegree $(-1, 1)$, j of bidegree $(0, 0)$ and k of bidegree $(1, 0)$. This gives an exact couple



By Theorem 1.2.3 this yields a spectral sequence $E_r^{p,q}$ with cohomology type.

Theorem 1.3.5. *If the filtration on A is bounded, then $E_r^{p,q}$ converges and*

$$E_\infty^{p,q} = \mathrm{Gr}_p^F H^{p+q}(A) = F_p H^{p+q}(A)/F_{p+1}H^{p+q}(A).$$

Remark 1.3.6. The result of the theorem is often written $E_r^{p,q}(A) \Rightarrow H^{p+q}(A)$ and E^r is said to converge to $H^*(A)$.

Proof. A proof can be found in [9, Chapter 3, Theorem 3.2]. □

Remark 1.3.7. Let k be a field. Given a filtered graded k-module H, i.e., a collection of filtered k-vector spaces, we can reconstruct H from the graded pieces $\mathrm{Gr}^F H$ if the filtration $F_\bullet$ is bounded. In this case

$$H^n = \bigoplus_{p+q=n} \mathrm{Gr}_p^F H^{p+q}.$$

1.4 Double Complexes

Definition 1.4.1. A *double complex* (A, d, δ) is a bigraded module A equipped with two differentials:

- the *horizontal differential* $d : A^{p,q} \to A^{p+1,q}$,
- the *vertical differential* $\delta : A^{p,q} \to A^{p,q+1}$,

such that the following conditions hold:

$$d^2 = 0, \qquad \delta^2 = 0, \qquad d\delta + \delta d = 0.$$

We say that the double complex is *first quadrant* if $A^{p,q} = 0$ whenever $p < 0$ or $q < 0$.

Let (A, d, δ) be a first quadrant double complex. We define its total complex $\mathrm{Tot}(A)^\bullet$ by

$$\mathrm{Tot}(A)^n = \bigoplus_{p+q=n} A^{p,q}, \quad \text{with differential } D = d + \delta.$$

There are two standard filtrations on $\mathrm{Tot}(A)^\bullet$:

- The *column filtration*:
$$\widetilde{W}_p = \bigoplus_{i \geq p} \bigoplus_{j \geq 0} A^{i,j},$$
- The *row filtration*:
$$\widetilde{W}_q = \bigoplus_{i \geq 0} \bigoplus_{j \geq q} A^{i,j}.$$

We observe that these filtrations are bounded and so we can apply Theorem 1.3.5. Different choices of filtration yield different initial pages of the spectral sequence associated to the filtered complex $\mathrm{Tot}(A)^\bullet$. Using the column filtration, the spectral sequence has:

$$\begin{aligned} E_0^{p,q} &= A^{p,q}, & d_0 &= \delta, \\ E_1^{p,q} &= H^q(A^{p,\bullet}, \delta), & d_1 &= d, \\ E_2^{p,q} &= H^p(H^q(A^{\bullet,\bullet}, \delta), d), & d_2 &= ? \end{aligned}$$

While with the row filtration:

$$\begin{aligned} E_0^{p,q} &= A^{p,q}, & d_0 &= d, \\ E_1^{p,q} &= H^p(A^{\bullet,q}, d), & d_1 &= \delta, \\ E_2^{p,q} &= H^q(H^p(A^{\bullet,\bullet}, d), \delta), & d_2 &= ? \end{aligned}$$

The remarkable fact is that both spectral sequences converge to the same object:

Theorem 1.4.2. *Let (A, d, δ) be a first quadrant double complex and $(\mathrm{Tot}(A), D = d + \delta)$ its associated total complex. The column filtration and row filtration give two different spectral sequences which both converge to the cohomology of the total complex.*

$$E_r^{p,q} \Rightarrow H^{p+q}(\mathrm{Tot}(A), D).$$

Proof. A proof for this theorem can be found in [3, Theorem 14.14]. □

Remark 1.4.3. Although the spectral sequences arising from the row and column filtrations both converge to the cohomology of the total complex, they may degenerate at different pages.

Definition 1.4.4. We say that an element $a \in A^{p,q}$ *survives* to E_r if it represents a cohomology class in $E_r^{p,q}$. In this case, we denote the class of a in E_r by $[a]_r$.

Each $E_r^{p,q}$ is a subquotient of $A^{p,q}$, so the definition makes sense. The following proposition will be useful later.

Proposition 1.4.5. *If $a \in A^{p,q}$ survives to E_r, it can be extended to a 'zig-zag'; there exist elements $c_i \in A^{p-i,q+i}$, $1 \leq i \leq r-1$, such that*

$$\begin{aligned} 0 &= da, \\ \delta a &= dc_1, \\ \delta c_1 &= dc_2, \\ &\vdots \\ \delta c_{r-2} &= dc_{r-1}, \end{aligned}$$

and such that, setting $c_0 = a$, the action of the differentials d_i on E_i is given by $d_i([a]_i) = [\delta c_{i-1}]_i$, $i \leq r$.

$$
\begin{array}{ccccccccccccc}
c_{r-1} & \xrightarrow{d} & & & & & & & & & & & \\
 & & \uparrow{\scriptstyle\delta} & & & & & & & & & & \\
 & & c_{r-2} & \xrightarrow{d} & & & & & & & & & \\
 & & & & \uparrow{\scriptstyle\delta} & & & & & & & & \\
 & & & & \ddots & \xrightarrow{d} & & & & & & & \\
 & & & & & & \uparrow{\scriptstyle\delta} & & & & & & \\
 & & & & & & c_2 & \xrightarrow{d} & & & & & \\
 & & & & & & & & \uparrow{\scriptstyle\delta} & & & & \\
 & & & & & & & & c_1 & \xrightarrow{d} & & & \\
 & & & & & & & & & & \uparrow{\scriptstyle\delta} & & \\
 & & & & & & & & & & a & \xrightarrow{d} & 0
\end{array}
$$

Proof. A proof can be found in [3, Chapter 3, Section 14]. □

Remark 1.4.6. The c_i's are not unique. However, the class of δc_{i-1} in E_i does not depend on the choice of c_{i-1} provided that the difference is d-exact. In other words, if $dc_{i-1} = dc'_{i-1}$ then $[\delta c_{i-1}]_i = [\delta c'_{i-1}]_i$.

Chapter 2

Kähler Manifolds and Hodge Decomposition

In this chapter, we briefly review some foundational results on Kähler manifolds and Hodge theory that will be used throughout the rest of the thesis. We focus in particular on the structure of differential forms, the Hodge decomposition, and the cohomological properties characteristic of compact Kähler manifolds. Some references for this chapter are [3], [12] and [14].

For this chapter, X is a complex manifold of dimension n.

2.1 Differential Forms and the Dolbeault Operators

We denote the *(real) tangent bundle* (smooth vector bundle of rank $2n$) of X by TX and its complexified dual, called the *cotangent bundle* by $T^*X = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{C})$. A *smooth differential $\mathbb{C}$-valued k-form* is a smooth section of the k-th exterior power of the cotangent bundle: $\bigwedge^k T^*X$. We call k the *degree* of the form. We denote by $\mathcal{A}_X^k$ the *sheaf of smooth differential k-forms on X*; for an open subset U of X,

$$\mathcal{A}_X^k(U) := C^\infty(U, \bigwedge^k T^*X).$$

Locally on a complex manifold of complex dimension n with local holomorphic coordinates $(z_1, \dots, z_n)$, a smooth k-form can be written as a linear combination of wedge products of the local coordinate 1-forms, dz^i, and their complex conjugates, $d\bar{z}^i$:

$$\alpha = \sum_{|I|+|J|=k} \alpha_{I,J}\, dz^I \wedge d\bar{z}^J = \sum_{|I|+|J|=k} \alpha_{I,J}\, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

where $I = (i_1 < \cdots < i_p)$, $J = (j_1 < \cdots < j_q)$, $p + q = k$ and $\alpha_{I,J} \in C^\infty(U)$ are smooth complex-valued functions. We call such a k-form, a *form of type* (p, q). We denote by $\mathcal{A}_X^{p,q}$ the space of (p, q)-forms.

We can define the Dolbeault operators

$$\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q} \quad \text{and} \quad \overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$$

as follows. Let $\alpha = \sum_{|I|+|J|=k} \alpha_{I,J}\, dz^I \wedge d\bar{z}^J$ then

$$\partial\alpha = \sum_{|I|+|J|=k} \sum_{l} \frac{\partial \alpha_{I,J}}{\partial z^l} dz^l \wedge dz^I \wedge d\bar{z}^J,$$

and

$$\overline{\partial}\alpha = \sum_{|I|+|J|=k} \sum_{l} \frac{\partial \alpha_{I,J}}{\partial \bar{z}^l} d\bar{z}^l \wedge dz^I \wedge d\bar{z}^J.$$

One can check that they satisfy the Leibniz rule:

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge \partial\beta,$$
$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge \overline{\partial}\beta,$$

and that

$$\partial^2 = \overline{\partial}^2 = \overline{\partial}\partial + \partial\overline{\partial} = 0.$$

We define $d = \partial + \overline{\partial} : \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ the *exterior derivative*. Clearly,

$$d^2 = (\partial + \overline{\partial})^2 = \partial^2 + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\partial}^2 = 0$$

and d satisfies the Leibniz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge d\beta.$$

Definition 2.1.1. We call a k-form α *closed* if $d\alpha = 0$. We call a k-form α *exact* if $\alpha = d\beta$ for some $(k-1)$-form β. Since $d^2 = 0$, every exact form is closed.

Remark 2.1.2. If a (p, q)-form α is d-closed, then it is automatically ∂- and $\overline{\partial}$-closed. Indeed, since $d = \partial + \overline{\partial}$, we have

$$0 = d\alpha = \partial\alpha + \overline{\partial}\alpha.$$

But $\partial\alpha \in \mathcal{A}_X^{p+1,q}$ and $\overline{\partial}\alpha \in \mathcal{A}_X^{p,q+1}$, which lie in different bidegrees and are therefore linearly independent. It follows that both components must vanish:

$$\partial\alpha = 0 \quad \text{and} \quad \overline{\partial}\alpha = 0.$$

Theorem 2.1.3 (de Rham's Theorem)**.** *Let X be a complex manifold, and let $\mathcal{A}_X^k$ denote the sheaf of $\mathbb{C}$-valued smooth differential k-forms. Let d be the exterior derivative on $\mathcal{A}_X = \bigoplus_k \mathcal{A}_X^k$. Then the k-th cohomology of X with complex coefficients, is given by the space of closed k-forms, modulo the exact k-forms:*

$$H^k(X, \mathbb{C}) \cong H^k_{dR}(X) := \frac{\ker\Big(d : \mathcal{A}_X^k(X) \to \mathcal{A}_X^{k+1}(X)\Big)}{\operatorname{im}\Big(d : \mathcal{A}_X^{k-1}(X) \to \mathcal{A}_X^k(X)\Big)}.$$

2.2 Holomorphic Forms

Let X be a complex manifold. The tangent bundle as a smooth vector bundle is a real rank $2n$ vector bundle TX on X. An *almost complex structure* J is an endomorphism $J : TX \to TX$ with the property that $J^2 = -\operatorname{Id}$. The complex structure on X equips TX with a natural almost complex structure given by multiplication by i in local holomorphic coordinates.

After complexifying the tangent bundle to $TX \otimes_{\mathbb{R}} \mathbb{C} \to X$, the endomorphism J extends $\mathbb{C}$-linearly to an endomorphism of $TX \otimes \mathbb{C}$, defined by

$$J(u + iv) = J(u) + iJ(v), \quad \text{for } u, v \in TX.$$

Since $J^2 = -\operatorname{Id}$, this extended J has eigenvalues i and $-i$, and the complexified tangent bundle splits as

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

where $T^{1,0}X$ is the i-eigenbundle and $T^{0,1}X$ the $-i$-eigenbundle. We call $T^{1,0}X$ the *holomorphic tangent bundle* and $T^{0,1}X$ the *anti-holomorphic tangent bundle*.

The *holomorphic cotangent bundle* is the dual bundle $T^*_{1,0}X := (T^{1,0}X)^*$, and similarly the *anti-holomorphic cotangent bundle* is $T^*_{0,1}X := (T^{0,1}X)^*$. Complex conjugation interchanges these bundles, giving a natural $\mathbb{R}$-linear (but in general not $\mathbb{C}$-linear) isomorphism $T^{1,0}X \cong T^{0,1}X$.

Locally, in a system of holomorphic coordinates $(z_1, \dots, z_n)$ on X, the holomorphic cotangent bundle $T^*_{1,0}X$ is spanned by the differentials $\{dz^1, \dots, dz^n\}$, and the anti-holomorphic cotangent bundle $T^*_{0,1}X$ is spanned by $\{d\bar{z}^1, \dots, d\bar{z}^n\}$.

This decomposition extends to the exterior powers:

$$\bigwedge^k (TX \otimes \mathbb{C})^* \cong \bigoplus_{p+q=k} \left(\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X \right)$$

Smooth sections of the summand bundles on the RHS are the smooth differential forms of type (p, q) on X, i.e., sections of $\mathcal{A}^{p,q}_X$.

The *holomorphic* p-forms on X are defined as holomorphic sections of $\bigwedge^p T^*_{1,0}X$. Equivalently, a smooth $(p, 0)$-form α is holomorphic if and only if $\bar{\partial}\alpha = 0$. Therefore, Ω^p_X can be defined as the sheaf of holomorphic sections of $\bigwedge^p T^*_{1,0}X$ or equivalently, $\Omega^p_X = \ker\left(\bar{\partial} : \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X\right)$.

The *canonical line bundle* of X is defined as the top exterior power of the holomorphic cotangent bundle:

$$K_X := \bigwedge^n T^*_{1,0}X$$

Theorem 2.2.1 (Dolbeault's Theorem)**.** *Let X be a complex manifold, let $\mathcal{A}^k_X$ denote the sheaf of $\mathbb{C}$-valued smooth differential k-forms and $\Omega^p_X = \ker(\bar{\partial} : \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X)$ the sheaf of $\mathbb{C}$-valued holomorphic p-forms. Then the sheaf cohomology of Ω^p_X is isomorphic to the Dolbeault cohomology group:*

$$H^q(X, \Omega^p_X) = H^{p,q}_{\bar{\partial}}(X) = \frac{\ker\left(\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}\right)}{\operatorname{im}\left(\bar{\partial} : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q}\right)}.$$

2.3 Hermitian Metric

Definition 2.3.1. Let V be a complex vector space of dimension n. A hermitian form h on V is an $\mathbb{R}$-bilinear map $h : V \times V \to \mathbb{C}$ such that

1. it is $\mathbb{C}$ linear with respect to the first argument and $\mathbb{C}$-antilinear with respect to the second one, i.e., $h(\lambda u, v) = \lambda h(u, v)$ and $h(u, \lambda v) = \bar{\lambda} h(u, v)$, where $u, v \in V$ and $\lambda \in \mathbb{C}$.
2. $h(v, u) = \overline{h(u, v)}$

We say that a hermitian form h is *positive definite*, if in addition $h(u, u) > 0$, for every $u \in V$, $u \neq 0$.

Write $\langle\ ,\ \rangle := \Re(h)$ and $\omega = -\Im(h)$ such that $h = \langle\ ,\ \rangle - i\omega$. We have the following proposition:

Proposition 2.3.2. *Let $h = \langle \cdot, \cdot \rangle - i\,\omega$ be a positive definite Hermitian form on a complex vector space V. Then:*

- *$\langle \cdot, \cdot \rangle$ is a real scalar product on the underlying real vector space of V.*
- *ω is a form with the following properties:*
  - *it is of type $(1, 1)$, i.e., $\omega(iu, iv) = \omega(u, v)$.*
  - *it is skew-symmetric, i.e., $\omega(u, v) = -\omega(v, u)$.*
  - *it is non-degenerate, i.e., $\omega(u, v) = 0$ for all v, implies $u = 0$.*
  - *it is positive, i.e., $\omega(u, iu) > 0$ for all $u \neq 0$.*
- *Moreover, $\langle \cdot, \cdot \rangle$ determines ω uniquely and vice versa.*

Proof. By definition of a Hermitian form, we see that $\langle \cdot, \cdot \rangle = \mathrm{Re}(h(\cdot, \cdot))$ is an $\mathbb{R}$-bilinear form. It is also symmetric; for all $u, v \in V$,

$$\langle u, v \rangle = \mathrm{Re}(h(u, v)) = \mathrm{Re}(\overline{h(v, u)}) = \mathrm{Re}(h(v, u)) = \langle v, u \rangle,$$

Positivity of h implies $\langle u, u \rangle > 0$ for $u \neq 0$, so $\langle \cdot, \cdot \rangle$ is a positive definite scalar product on the real vector space underlying V.

Next, $\omega = -\mathrm{Im}(h(\cdot, \cdot))$ is a real-valued bilinear form. From the Hermitian symmetry of h,

$$h(v, u) = \overline{h(u, v)} \implies \mathrm{Im}(h(v, u)) = -\mathrm{Im}(h(u, v)),$$

so ω is skew-symmetric: $\omega(u, v) = -\omega(v, u)$.

Now, ω is a 2-form and decomposes into form type as

$$\omega = \omega_{2,0} + \omega_{1,1} + \omega_{0,2}$$

Recall that by definition of type, a (p, q)-form, is complex linear in each of its p (type $(1, 0)$) holomorphic arguments and complex anti-linear in each of its q (type $(0, 1)$) anti-holomorphic arguments. By the definition of h,

$$h(iu, iv) = -i^2 h(u, v) = h(u, v)$$

and thus, $\omega(iu, iv) = \omega(u, v)$. This forces $\omega_{2,0} = \omega_{0,2} = 0$ and thus, ω is of type $(1, 1)$.

To prove non-degeneracy, assume $\omega(u, v) = 0$ for all $v \in V$. Then

$$0 = \omega(u, v) = -\mathrm{Im}(h(u, v)) \implies h(u, v) \in \mathbb{R} \quad \forall v.$$

Similarly, evaluating at iv,

$$h(u, iv) = i \cdot h(u, v),$$

which is purely imaginary unless $h(u, v) = 0$. Since $h(u, iv)$ must be real as well , it forces $h(u, v) = 0$ for all v. By positive-definiteness and non-degeneracy of h, this implies $u = 0$. Hence, ω is non-degenerate.

We see:

$$\omega(u, v) = -\Im(h(u, v)) = \Im(i^2 h(u, v)) = \Im(ih(iu, v)) = \langle iu, v\rangle$$

Hence, ω determines $\langle\ ,\ \rangle$ and conversely, $\langle\ ,\ \rangle$ determines ω. In addition, for $u \neq 0$ and $v = iu$ on the identity above, we get

$$\omega(u, iu) = \langle iu, iu\rangle = h(iu, iu) > 0$$

and ω is positive. □

Definition 2.3.3. A *hermitian metric* over a smooth manifold X is a smooth global section h of the complex vector bundle $(T^{1,0}X \otimes_{\mathbb{C}} T^{0,1}X)^*$, where $T^{1,0}X$ is the holomorphic tangent bundle of X and $T^{0,1}X = \overline{T^{1,0}X}$ its complex conjugate, such that for every $p \in X$, the map $h_p : T_p^{1,0}X \otimes_{\mathbb{C}} T_p^{0,1}X \to \mathbb{C}$ is a positive definite hermitian form on $T_p^{1,0}X$.

Proposition 2.3.4. *Every complex manifold admits a Hermitian metric.*

Definition 2.3.5. A *Hermitian manifold* (X, h) is a complex manifold X with a choice of a hermitian metric h.

An immediate consequence of Proposition 2.3.2 is the following:

Theorem 2.3.6. *If (X, h) is a hermitian manifold, then $\omega = -\Im(h)$ is a non-degenerate, skew-symmetric, positive $(1, 1)$-form. Conversely, if a $(1, 1)$-form ω on a complex manifold X is non-degenerate, skew-symmetric and positive, it arises as $\omega = -\Im(h)$ for the hermitian metric*

$$h(u, v) = -\omega(iu, v) - i\omega(u, v).$$

Definition 2.3.7. A *Kähler manifold* is a complex manifold that admits a hermitian metric h with $\omega = -\Im(h)$ closed, i.e., $d\omega = 0$.

Examples. 1. The complex vector space $\mathbb{C}^n$ with the standard Hermitian metric is a Kähler manifold (but not compact).

2. The complex projective space $\mathbb{CP}^n$ with the Fubini–Study metric is a compact Kähler manifold.

3. A smooth subvariety of $\mathbb{CP}^n$, or equivalently, a complex manifold embedded in projective space, equipped with the restriction of the Fubini–Study metric, is a compact Kähler manifold.

4. A submanifold of a Kähler manifold is naturally a Kähler manifold, with the induced Kähler form ω.

5. A Riemann surface (a complex manifold of complex dimension 1), endowed with a Hermitian metric h, is Kähler. The associated 2-form ω is defined on a real 2-dimensional manifold, so it is automatically closed: $d\omega = 0$.

2.4 The Laplacian and Decomposition of Forms

Theorem 2.4.1. *Let X be a compact complex manifold of complex dimension n. A choice of Hermitian metric $h = \langle\ ,\ \rangle - i\omega$ on X determines:*

1. *A Hermitian L^2-inner product $\langle\alpha,\beta\rangle_{L^2} = \int_X \langle\alpha,\beta\rangle\, dV$ on the space of global smooth differential forms $\mathcal{A}_X(X)$, where $\mathcal{A}_X = \bigoplus_k \mathcal{A}_X^k$.*

2. *A* Hodge star operator $*\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{n-p,n-q}$ *defined by*
$$\langle\alpha,\beta\rangle_{L^2} = \int_X \alpha \wedge *\overline{\beta},$$
satisfying $*^2 = (-1)^{p+q}\mathrm{Id}$ on (p,q)-forms.

3. *An adjoint operator $d^* = -*d*\colon \mathcal{A}_X^{k+1} \to \mathcal{A}_X^k$ to the exterior derivative $d\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$, characterized by*
$$\langle d\alpha,\beta\rangle_{L^2} = \langle\alpha, d^*\beta\rangle_{L^2}\,,$$
and satisfying $(d^*)^2 = 0$.

4. *Similarly, adjoints $\partial^* = -*\partial*\colon \mathcal{A}_X^{p+1,q} \to \mathcal{A}_X^{p,q}$ and $\overline{\partial}^* = -*\overline{\partial}*\colon \mathcal{A}_X^{p,q+1} \to \mathcal{A}_X^{p,q}$, satisfying $(\partial^*)^2 = 0$, $(\overline{\partial}^*)^2 = 0$, and $d^* = \partial^* + \overline{\partial}^*$.*

Note that we take X to be compact so the hermitian product is well defined. With a choice of a hermitian metric h on X, we define the *Laplacian*
$$\Delta_d = dd^* + d^*d : \mathcal{A}_X^k \to \mathcal{A}_X^k.$$

The kernel
$$\mathcal{H}_d^k = \ker\{\Delta_d : \mathcal{A}_X^k \to \mathcal{A}_X^k\}$$
is the space of *d-harmonic k-forms.*

Of course $d\alpha = d^*\alpha = 0$ implies $\Delta_d \alpha = 0$. The other direction can be seen via the equality

$$\langle \Delta_d \alpha, \alpha \rangle_{L^2} = \langle dd^* \alpha, \alpha \rangle_{L^2} + \langle d^* d, \alpha \rangle_{L^2} = \langle d^* \alpha, d^* \alpha \rangle_{L^2} + \langle d\alpha, d\alpha \rangle_{L^2} = \|d^*\alpha\|^2 + \|d\alpha\|^2$$

and hence, $\Delta_d \alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$.

Theorem 2.4.2 ([14], Sections 5.2.2 & 5.2.3)**.** *We have a decomposition of k-forms, orthogonal with respect to the hermitian product on $\mathcal{A}_X(X)$:*

$$\mathcal{A}_X^k(X) \cong \mathcal{H}_d^k(X) \oplus d\mathcal{A}_X^{k-1}(X) \oplus d^* \mathcal{A}_X^{k+1}(X)$$

Corollary 2.4.3. *The d-closed forms are given by*

$$\ker d = \mathcal{H}_d^k(X) \oplus d\mathcal{A}_X^{k-1}(X).$$

This implies $\mathcal{H}_d^k(X) \cong H_{dR}^k(X) \cong H^k(X, \mathbb{C})$.

Proof. Using Theorem 2.4.2 we can express a k-form α as

$$\alpha = \alpha_0 + d\beta + d^*\beta'$$

with α_0 d-harmonic, and the summands on the RHS are pairwise orthogonal. Now, if α is d-closed,

$$\left\langle d^*\beta', d^*\beta' \right\rangle_{L^2} = \left\langle \alpha - \alpha_0 - d\beta, d^*\beta' \right\rangle_{L^2} = \left\langle \alpha, d^*\beta' \right\rangle_{L^2} = \left\langle d\alpha, \beta' \right\rangle_{L^2} = 0$$

and hence $d^*\beta' = 0$ as desired. □

Similarly, we can construct

$$\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q} \qquad \text{and} \qquad \Delta_{\partial} = \partial\partial^* + \partial^*\,\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q}$$

whose kernels give the spaces of $\overline{\partial}$-*harmonic* and ∂-*harmonic* (p,q)-forms:

$$\mathcal{H}_{\overline{\partial}}^{p,q} = \ker\{\Delta_{\overline{\partial}} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}\} \qquad \text{and} \qquad \mathcal{H}_{\partial}^{p,q} = \ker\{\Delta_{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}\}$$

Using the same trick as before, we can show that

$$\Delta_{\overline{\partial}}\alpha = 0 \iff \overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$$

and

$$\Delta_{\partial}\alpha = 0 \iff \partial\alpha = \partial^*\alpha = 0$$

Theorem 2.4.4. *We have a decomposition of (p,q)-forms, orthogonal with respect to the hermitian product on $\mathcal{A}_X(X)$:*

$$\mathcal{A}_X^{p,q}(X) \cong \mathcal{H}_{\overline{\partial}}^{p,q}(X) \oplus \overline{\partial}\mathcal{A}_X^{p,q-1}(X) \oplus \overline{\partial}^* \mathcal{A}_X^{p,q+1}(X)$$
$$\mathcal{A}_X^{p,q}(X) \cong \mathcal{H}_{\partial}^{p,q}(X) \oplus \partial\mathcal{A}_X^{p,q-1}(X) \oplus \partial^* \mathcal{A}_X^{p,q+1}(X)$$

Corollary 2.4.5. *The ∂-closed forms are given by*

$$\ker \partial = \mathcal{H}^{p,q}_{\partial}(X) \oplus \partial \mathcal{A}^{p,q-1}_X(X)$$

Similarly, the $\overline{\partial}$-closed forms are given by

$$\ker \overline{\partial} = \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial} \mathcal{A}^{p,q-1}_X(X)$$

This implies $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong H^{p,q}_{\overline{\partial}}(X) \cong H^q(X, \Omega^p_X)$.

2.5 Hodge Decomposition on a Kähler manifold

In general, Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ are not related. However, if the manifold X is Kähler , we will see in Theorem 2.5.1 that

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial$$

It follows that the corresponding spaces of harmonic (p,q)-forms coincide: $\mathcal{H}^{p,q}_{\overline{\partial}} = \mathcal{H}^{p,q}_{\partial}$.

To simplify notation, we will write $\mathcal{H}^k = \mathcal{H}^k_d$ for the space of d-harmonic k-forms, and $\mathcal{H}^{p,q} := \mathcal{H}^{p,q}_{\partial}$ for the space of ∂-harmonic forms of type (p,q).

Given a k-form α, we can decompose α into its (p,q)-components with $p+q=k$.

$$\alpha = \sum_{p+q=k} \alpha^{p,q}$$

If in addition α is d-harmonic, i.e., $\Delta_d \alpha = 0$, then every component $\alpha^{p,q}$ must be d-harmonic since

$$0 = \Delta_d \alpha = \sum_{p+q=k} \Delta_d \alpha^{p,q}$$

and every component on the RHS has different bidegree. In particular, each $\alpha^{p,q}$ is ∂-harmonic; $\Delta_d \alpha^{p,q} = 0$ implies $d\alpha^{p,q} = d^* \alpha^{p,q} = 0$ which forces $\partial \alpha^{p,q} = \overline{\partial} \alpha^{p,q} = \partial^* \alpha^{p,q} = \overline{\partial}^* \alpha^{p,q} = 0$.

The decomposition of a harmonic k-form into its harmonic (p,q)-components with $p+q=k$ and the relation of the Laplacians on a compact Kähler manifold gives the decomposition:

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \qquad \text{and} \qquad \overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$$

From Theorem 2.1.3 and Theorem 2.4.2 we have that $\mathcal{H}^k \cong H^k(X, \mathbb{C})$. We will see that this decomposition induces decomposition on the cohomology of X.

Theorem 2.5.1. *On a Kähler manifold X with Kähler form ω, the Dolbeault operators, the Laplacian, the Hodge star operator, and the Lefschetz operator*

$$L \colon \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q+1}_X, \quad \alpha \mapsto \omega \wedge \alpha,$$

together with their adjoints, satisfy a collection of commutation relations known as the Kähler identities*:*

- $[L^*, \partial] = i\bar{\partial}^*$
- $[L^*, \bar{\partial}] = -i\partial^*$
- $[L^*, \partial^*] = -i\bar{\partial}$
- $[L^*, \bar{\partial}^*] = i\partial$
- $[L, \partial] = 0$
- $[L, \bar{\partial}] = 0$
- $[L, \partial^*] = 0$
- $[L, \bar{\partial}^*] = 0$
- $[\Delta_d, L] = 0$
- $[\Delta_d, L^*] = 0$
- $[\Delta_d, \partial] = 0$
- $[\Delta_d, \bar{\partial}] = 0$
- $[\Delta_d, *] = 0$
- $\overline{\partial}\partial^* + \partial^*\overline{\partial} = 0$
- $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$
- $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$

where $[A, B] := AB - BA$ is the commutator operator.

Proof. Proof of the Kähler identities can be found in [14, Chapter 6]. □

Lemma 2.5.2 ($\partial\overline{\partial}$-lemma)**.** *Let X be a Kähler manifold and $\alpha \in \mathcal{A}_X^{p,q}(X)$ be a smooth complex differential (p, q)-form on X with $p, q \geq 1$ which is d-closed, i.e., $d\alpha = 0$. Then, the following are equivalent:*

(1) α is $\partial\overline{\partial}$ exact, i.e., $\alpha = \partial\overline{\partial}\beta$ for some $\beta \in \mathcal{A}_X^{p-1,q-1}(X)$.

(2) α is ∂ exact, i.e., $\alpha = \partial\gamma$ for some $\gamma \in \mathcal{A}_X^{p-1,q}(X)$

(2') α is $\overline{\partial}$ exact, i.e., $\alpha = \overline{\partial}\gamma'$ for some $\gamma' \in \mathcal{A}_X^{p,q-1}(X)$

(2") α is d-exact, i.e., $\alpha = d\gamma''$ for some $\gamma'' \in \mathcal{A}_X^{p+q-1}(X)$.

(3) α is orthogonal to $\mathcal{H}^{p,q}(X) \subset \mathcal{H}^{p+q}(X)$

Proof. Using the identities $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$ and $d = \partial + \overline{\partial}$ we see that (1) implies (2), $(2')$ and $(2'')$:

$$\alpha = \partial\overline{\partial}\beta = \partial(\overline{\partial}\beta) = \overline{\partial}(\partial(-\beta)) = d(\partial(-\beta)).$$

Moreover, any of the conditions (2), $(2')$ or $(2'')$ implies (3), by Theorem 2.4.2 and Theorem 2.4.4.
We will prove (3) $\implies$ (1). Take α orthogonal to $\mathcal{H}^{p,q}(X)$. Using the orthogonal decomposition of Theorem 2.4.4 we have that

$$\alpha = \overline{\partial}\gamma + \overline{\partial}^*\gamma'$$

for some $\gamma \in \mathcal{A}_X^{p,q-1}(X)$ and $\gamma' \in \mathcal{A}_X^{p,q+1}(X)$ with

$$\left\langle \overline{\partial}\gamma, \overline{\partial}^*\gamma' \right\rangle_{L^2} = 0$$

Since α is d closed, α must be both ∂ and $\overline{\partial}$ closed (see Remark 2.1.2). This gives

$$\left\langle \overline{\partial}^*\gamma', \overline{\partial}^*\gamma' \right\rangle_{L^2} = \left\langle \alpha - \overline{\partial}\gamma, \overline{\partial}^*\gamma' \right\rangle_{L^2} = \left\langle \alpha, \overline{\partial}^*\gamma' \right\rangle_{L^2} = \langle \overline{\partial}\alpha, \gamma' \rangle_{L^2} = 0$$

and thus $\overline{\partial}^*\gamma' = 0$, and $\alpha = \overline{\partial}\gamma$. Now, we apply the Hodge decomposition on γ for ∂ to get

$$\gamma = \gamma_0 + \partial\eta + \partial^*\eta'$$

with γ_0 ∂-harmonic, $\eta \in \mathcal{A}_X^{p-1,q-1}(X)$ and $\eta' \in \mathcal{A}_X^{p+1,q-1}(X)$.
Since $\Delta_\partial = \Delta_{\overline{\partial}}$, γ_0 is also $\overline{\partial}$-harmornic, and thus $\overline{\partial}\gamma_0 = \overline{\partial}^*\gamma_0 = 0$. This implies

$$\alpha = \overline{\partial}\partial\eta + \overline{\partial}\partial^*\eta'$$

We use the Kähler identity $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$ and the fact that

$$\langle \partial\eta, \partial^*\eta' \rangle_{L^2} = 0$$

to get similarly

$$\langle \overline{\partial}\partial^*\eta', \overline{\partial}\partial^*\eta' \rangle_{L^2} = \langle \alpha, \overline{\partial}\partial^*\eta' \rangle_{L^2} = -\langle \alpha, \partial^*\overline{\partial}\eta' \rangle_{L^2} = \langle \partial\alpha, \overline{\partial}\eta' \rangle_{L^2} = 0$$

and hence $\overline{\partial}\partial^*\eta' = 0$, and $\alpha = \overline{\partial}\partial\eta = -\partial\overline{\partial}\eta$ as desired. □

Theorem 2.5.3 (Hodge Decomposition)**.** *Let X be a compact Kähler manifold. Then the cohomology of X with complex coefficients admits a decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \qquad \textit{with} \qquad \overline{H^{p,q}(X)} = H^{q,p}(X)$$

where $H^{p,q}(X)$ is the subspace of de Rham cohomology classes which are representable by a closed (p,q)-form. In particular, the decomposition does not depend on the choice of the hermitian metric.

Proof. The isomorphism $\mathcal{H}_d^k \xrightarrow{\sim} H^k(X,\mathbb{C})$ is given by sending a d-harmonic form α to the class $[\alpha]$. Let $K^{p,q} \subset H^{p+q}(X,\mathbb{C})$ be the subspace of the de Rham cohomology corresponding to $\mathcal{H}^{p,q}$ under this isomorphism.
We will show that $K^{p,q} = H^{p,q}(X)$. That is, a cohomology class admits a closed (p,q)-form representative if and only if it admits a harmonic (p,q)-form representative. The inclusion $K^{p,q} \subset H^{p,q}(X)$ is clear. Now for the inverse inclusion take a class in $H^{p,q}(X)$ represented by a d-closed, and hence ∂-closed, (p,q)-form ω. By corollary 2.4.5 we can write

$$\omega = \alpha + \partial\beta$$

where α is ∂-harmonic and $\beta \in \mathcal{A}_X^{p,q-1}(X)$. Now, we apply Lemma 2.5.2 to write $\omega = \alpha + d\gamma$ for some γ $(p+q-1)$-form. We conclude that ω and α represent the same class in $H^{p,q}(X)$ and since α is harmonic, $H^{p,q}(X) \subset K^{p,q}$. □

We have seen in Corollary 2.4.5 that

$$\mathcal{H}^{p,q}(X) \cong H^q(X,\Omega_X^p)$$

So we get an alternative description of the decomposition of a compact Kähler manifold:

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X,\Omega_X^p)$$

Definition 2.5.4. Let X be a compact complex manifold. The numbers $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ are called the *Hodge numbers* of X and the numbers $b_k = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ are called the *Betti numbers* of X.

Remark 2.5.5. We assume that X is compact so that the cohomology groups $H^k(X, \mathbb{C})$ and $H^q(X, \Omega_X^p)$ are finite-dimensional. This ensures that the Hodge numbers $h^{p,q}$ and the Betti numbers b_k are well-defined.

Theorem 2.5.6. *Let X be a compact Kähler manifold. Then, there is an isomorphism*

$$* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$$

Hence, the Hodge numbers of X satisfy

$$h^{p,q} = h^{q,p} = h^{n-p,n-q}$$

and the Betti numbers are given by

$$b_k = \sum_{p+q=k} h^{p,q}.$$

Proof. Since $\Delta_d = 2\Delta_{\bar\partial}$, we can see from the Kähler identities (2.5.1) that $\Delta_{\bar\partial}$ commutes with the Hodge star operator

$$* : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{n-p,n-q}.$$

This induces an isomorphism $\mathcal{H}^{p,q} \xrightarrow{\sim} \mathcal{H}^{n-p,n-q}$. Combining with Theorem 2.5.3 the result follows. ☐

The Hodge numbers can be arranged in a symmetric pattern known as the *Hodge diamond*. For example, for a compact Kähler manifold of complex dimension 2, the Hodge diamond looks like

$$\begin{array}{ccccc|l}
 & & h^{0,0} & & & b_0 \\
 & h^{1,0} & & h^{0,1} & & b_1 \\
h^{2,0} & & h^{1,1} & & h^{0,2} & b_2 \\
 & h^{2,1} & & h^{1,2} & & b_3 = b_1 \\
 & & h^{2,2} & & & b_4 = b_0
\end{array}$$

while for a compact Kähler manifold of complex dimension 3, the Hodge diamond takes the form:

$$\begin{array}{ccccccc|l}
 & & & h^{0,0} & & & & b_0 \\
 & & h^{1,0} & & h^{0,1} & & & b_1 \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & & b_2 \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & b_3 \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & & b_4 = b_2 \\
 & & h^{3,2} & & h^{2,3} & & & b_5 = b_1 \\
 & & & h^{3,3} & & & & b_6 = b_0
\end{array}$$

Theorem 2.5.6 implies that this diagram is symmetric across the vertical axis and the center.

Related to the Hodge numbers, for a connected compact complex manifold X we define the following:

1. **Holomorphic Euler characteristic.** The *Euler characteristic* of the *structure sheaf* $\mathcal{O}_X = \Omega_X^0$, sheaf of complex holomorphic functions, is defined by:

$$\chi(\mathcal{O}_X) := \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{O}_X) = \sum_{i=0}^{n} (-1)^i h^{0,i}.$$

2. **Topological Euler characteristic.** The *Euler number* (or topological Euler characteristic) of X is given by:

$$e(X) := \sum_{i=0}^{2n} (-1)^i b_i(X),$$

where $b_i(X) = \dim H^i(X, \mathbb{C})$ are the Betti numbers.

3. **Geometric genus.** The *geometric genus* of X is the Hodge number

$$p_g(X) := \dim H^0(X, \Omega_X^n) = h^{n,0}.$$

It is a birational invariant of complex manifolds and algebraic varieties.

4. **Arithmetic genus.** The *arithmetic genus* of X is defined as:

$$p_a(X) := (-1)^{\dim X} \left(\chi(\mathcal{O}_X) - 1\right)$$

And now we present some results from Hodge theory on compact Kähler manifolds.

Proposition 2.5.7. *Let X be a compact Kähler manifold and let $\alpha \in \mathcal{A}_X^{p,q}$ be a smooth differential form that is d-exact. Then there exist forms $\beta' \in \mathcal{A}_X^{p-1,q}$ and $\beta'' \in \mathcal{A}_X^{p,q-1}$ such that $\alpha = d\beta' = d\beta''$.*

Proof. Since α is d-exact, it is in particular d-closed. By Lemma 2.5.2, α is $\partial\overline{\partial}$-exact: there exists a form $\beta \in \mathcal{A}_X^{p-1,q-1}$ such that $\alpha = \partial\overline{\partial}\beta$.
Define $\beta' := -\partial\beta \in \mathcal{A}_X^{p,q-1}$ and $\beta'' := \overline{\partial}\beta \in \mathcal{A}_X^{p-1,q}$. Then,

$$d\beta' = -(\partial + \overline{\partial})\partial\beta = -\overline{\partial}\partial\beta = \partial\overline{\partial}\beta = \alpha,$$

and similarly,

$$d\beta'' = (\partial + \overline{\partial})\overline{\partial}\beta = \partial\overline{\partial}\beta = \alpha.$$

This proves the claim. □

Proposition 2.5.8 (Principle of two types)**.** *Let X be a compact Kähler manifold and let $[\beta] \in H^n(X, \mathbb{C})$ be a cohomology class that can be represented by both $\beta \in \mathcal{A}_X^{p,q}$ and $\beta' \in \mathcal{A}_X^{p',q'}$ with $p + q = p' + q' = n$ and $p \neq p'$. Then $[\beta] = 0$.*

Proof. The result follows from the Hodge decomposition on $H^n(X, \mathbb{C})$:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where the sum is direct. If the cohomology class $[\beta] \in H^n(X, \mathbb{C})$ can be represented by both a (p, q)-form and a (p', q')-form with $p \neq p'$, then $[\beta]$ lies in two distinct components of this decomposition. Since the decomposition is direct, this forces $[\beta] = 0$. □

Proposition 2.5.9. *The odd Betti numbers $b_{2k+1} = \dim_{\mathbb{C}} H^{2k+1}(X, \mathbb{C})$ of a compact Kähler manifold X are even.*

Proof. Theorem 2.5.6 gives

$$b_{2k+1} = \sum_{p+q=2k+1} h^{p,q}.$$

Since $p+q = 2k+1$ is odd, there are no terms with $p = q$. By symmetry of the Hodge numbers, the summands come in complex conjugate pairs: $h^{p,q} = h^{q,p}$ with $(p, q) \neq (q, p)$. Therefore, the dimension is even:

$$b_{2k+1} = \sum_{p+q=2k+1} h^{p,q} = 2 \sum_{p=0}^{k} h^{p,2k+1-p}.$$

□

Theorem 2.5.10. *The even Betti numbers $b_{2k} = \dim_{\mathbb{C}} H^{2k}(X, \mathbb{C})$, $1 \leq k \leq \dim_{\mathbb{C}} X$ of a compact Kähler manifold X are positive.*

Proof. Let $\dim_{\mathbb{C}} X = n$. It suffices to show that $h^{k,k} > 0$ for all $k = 1, \ldots, n$. Let ω be the Kähler form of X. We can see from the Kähler identities (Theorem 2.5.1) that the Lefschetz operator $L : \alpha \mapsto \omega \wedge \alpha$ commutes with Δ_d:

$$[L, \Delta_d] = 0 \iff L\Delta_d - \Delta_d L = 0$$

This shows that wedging any harmonic form with ω will again produce a harmonic form. In particular, $\omega = \omega \wedge 1$ is harmonic and $\omega^k = \omega \wedge \cdots \wedge \omega$ (taken k times) are harmonic (k, k)-forms for all $1 \leq k \leq n$. □

2.6 Hodge Structures

Definition 2.6.1. Let n be an integer. A *pure Hodge structure of weight n* consists of an abelian group $H_{\mathbb{Z}}$ and a decomposition of its complexification $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ into a direct sum of complex subspaces $H^{p,q}$, where $p + q = n$, with the property that the complex conjugate of $H^{p,q}$ is $H^{q,p}$:

$$H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q} \qquad \text{and} \qquad \overline{H^{p,q}} = H^{q,p}$$

Remark 2.6.2. One can also speak of rational (resp. real) pure Hodge structures, obtained by replacing the group $H_{\mathbb{Z}}$ with a rational (resp. real) vector space.

Examples. (a) If X is a compact Kähler manifold, then the cohomology group $H^n(X, \mathbb{C})$ is obtained from the integral cohomology $H^n(X, \mathbb{Z})$ by extension of scalars:

$$H^n(X, \mathbb{C}) = H^n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

By Theorem 2.5.3, this complex cohomology group admits a decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X, \mathbb{C}),$$

This decomposition endows $H^n(X, \mathbb{Z})$ with a pure Hodge structure of weight n.

(b) Let $n = 2k$ be even. Defining $H = H^{k,k}$ and $H^{p,q} = 0$ for $(p, q) \neq (k, k)$ we obtain the *trivial pure Hodge structure of weight* $2k$.

(c) Define $\mathbb{Z}(1) := 2\pi i\mathbb{Z} \subset \mathbb{C}$ as a subgroup of $\mathbb{C}$ and set $H = \mathbb{Z}(1) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C} = H^{-1,-1}$. This is a Hodge structure of weight -2 and it is the unique one dimensional pure Hodge structure of weight -2 up to isomorphism. This Hodge structure is called the *Tate-Hodge structure.*

(d) Define $\mathbb{Z}(n) := (2\pi i)^n\mathbb{Z}$ the $n-$th tensor power of $\mathbb{Z}(1)$. It is pure Hodge structure of weight $-2n$.

2.7 Smooth Curves over $\mathbb{C}$

By a *smooth curve over* $\mathbb{C}$, we mean a smooth, connected, one-dimensional complex projective variety. There are two important notions of genus associated with a smooth curve C over $\mathbb{C}$:

- The *geometric genus* $p_g(C)$ is the topological genus of C, viewed as a compact, orientable surface of real dimension 2; that is, the number of "holes" in the surface. Algebraically, it is defined as
$$p_g(C) = h^{1,0}(C) = \dim H^0(C, \Omega^1_C).$$
- The *arithmetic genus* $p_a(C)$ is defined algebraically as
$$p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 - h^{0,0}(C) + h^{0,1}(C) = h^{0,1}(C).$$

Since on a compact Kähler manifold we have $h^{0,1}(C) = h^{1,0}(C)$, the two notions of genus agree. We refer to their common value simply as the *genus* of C, denoted by $g(C)$.

A smooth complex curve is, up to isomorphism, the same as a connected compact Riemann surface. The Hodge numbers of such a curve of genus g are given by the following Hodge diamond:

$$\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

Definition 2.7.1. An *elliptic curve over* $\mathbb{C}$ is a smooth complex projective curve of genus 1.

2.8 Smooth Surfaces over $\mathbb{C}$

By a *smooth surface over* $\mathbb{C}$, we mean a smooth, connected, two-dimensional complex projective variety. As before, such a variety is naturally a compact Kähler manifold.
Let S be a smooth projective surface. We define the *irregularity* $q(S)$ of S by

$$q(S) := h^{1,0}(S).$$

We say that a surface S is *regular*, if $q(S) = 0$.

We also have *Noether's formula*:

$$\chi(\mathcal{O}_S) = \frac{K_S^2 + e(S)}{12},$$

where K_S is the canonical divisor on S, and $K_S^2 := K_S \cdot K_S$ denotes its self-intersection number.

Example 2.8.1. A *ruled surface* over a smooth connected curve C is a smooth, connected projective surface S together with a surjective morphism $\varphi : S \to C$ such that the fiber over each point $y \in C$ is isomorphic to $\mathbb{P}^1$, and such that φ admits a section. It has irregularity $q(S) = g(C)$ and Hodge numbers $h^{2,0} = 0$ and $h^{1,1} = 2$.

Example 2.8.2. A *rational surface* is any surface birationally equivalent to $\mathbb{P}^2$. It is regular, i.e., $q(S) = 0$.

2.9 $K3$ Surfaces

Definition 2.9.1. A $K3$ *surface* is a compact, connected complex manifold S of dimension 2 such that the canonical bundle $K_S = \bigwedge^2 T^*S$ is trivial and the irregularity vanishes, i.e., $q(S) = h^{1,0}(S) = 0$.

Theorem 2.9.2. *Every $K3$ surface is a compact Kähler manifold.*

Let S be a $K3$ surface. Since S is regular, we have $h^{0,1}(S) = h^{1,0}(S) = 0$. Furthermore, since $\Omega_S^2 \cong \mathcal{O}_S$, we have

$$h^{0,2} = h^{2,0} = \dim H^0(S, \Omega_S^2) = \dim H^0(S, \mathcal{O}_S) = h^{0,0} = 1.$$

We find

$$\chi(\mathcal{O}_S) = 2, \quad \text{and} \quad e(S) = 4 + h^{1,1}.$$

Noether's formula then gives $h^{1,1} = 20$. The Hodge diamond of a $K3$ surface is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

2.10 Complex Tori and Abelian Varieties

In this section, we introduce only the basic background on complex tori and abelian varieties needed for the rest of the thesis. We omit many interesting aspects of the theory focusing on definitions and results that will be used in later chapters. The main reference for this material is [2].

Definition 2.10.1. Let g be a positive integer. A *complex torus* $X = V/\Lambda$ of dimension g is the quotient of a complex vector space $V \cong \mathbb{C}^g$ by a full rank lattice $\Lambda \subset V$; that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2g}$.

For the rest of the chapter, we fix a complex vector space V of dimension g, and let $\Lambda \subset V$ be a full lattice. We denote by $X = V/\Lambda$ the corresponding complex torus.

The complex structure on V descends to the quotient, and the standard flat Hermitian metric on $\mathbb{C}^g$ induces a Kähler metric on X, making it a compact Kähler manifold.

Since V is contractible, it serves as the universal cover of X, and the fundamental group $\pi_1(X)$ is naturally identified with the lattice $\Lambda \subset V$. As a consequence, since Λ is abelian, the first homology group of X is

$$H_1(X, \mathbb{Z}) \cong \Lambda.$$

Moreover, by the universal coefficient theorem, the first cohomology group satisfies

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(X, \mathbb{Z}), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}).$$

In the case of a complex torus $X = V/\Lambda$, the cohomology ring $H^\bullet(X, \mathbb{Z})$ is particularly well-behaved:

Proposition 2.10.2 ([2], Lemma 1.3.1)**.** *Let $X = V/\Lambda$ be a complex torus of dimension g. Then we have an isomorphism*

$$\bigwedge^n H^1(X, \mathbb{Z}) \xrightarrow{\sim} H^n(X, \mathbb{Z})$$

for all $0 \leq n \leq 2g$.

Denoting

$$\mathrm{Alt}^n(\Lambda, \mathbb{Z}) := \bigwedge^n \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$$

as the group of $\mathbb{Z}$-valued alternating n-forms on Λ, Proposition 2.10.2 yields

$$H^n(X, \mathbb{Z}) \cong \mathrm{Alt}^n(\Lambda, \mathbb{Z})$$

for every $n \geq 0$. Now let $\mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$ denote the group of $\mathbb{C}$-valued, $\mathbb{R}$-linear alternating n-forms on V. The canonical identification

$$\mathrm{Alt}^n(\Lambda, \mathbb{Z}) \otimes \mathbb{C} \cong \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$$

implies

$$H^n(X, \mathbb{C}) \cong \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \bigwedge^n H^1(X, \mathbb{C}).$$

The Hodge decomposition on $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ induces a corresponding decomposition on $H^n(X,\mathbb{C})$:

$$H^n(X,\mathbb{C}) \cong \bigwedge^n H^1(X,\mathbb{C}) \cong \bigoplus_{p+q=n} \left(\bigwedge^p H^{1,0}(X) \otimes \bigwedge^q H^{0,1}(X)\right).$$

This gives the identification

$$H^{p,q}(X) \cong \bigwedge^p H^{1,0}(X) \otimes \bigwedge^q H^{0,1}(X).$$

Topologically, X is a real torus of dimension $2g$, so $b_1(X) = 2g$, and therefore $h^{1,0}(X) = g$. It follows that

$$h^{p,q}(X) = \binom{g}{p}\binom{g}{q}.$$

In particular, the Hodge diamond of a complex torus of dimension $g = 2$ is:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

The set of isomorphism classes of holomorphic line bundles on X can be naturally identified with $H^1(X,\mathcal{O}_X^*)$, by interpreting holomorphic line bundles in terms of their transition functions. We define the *first Chern class* of a holomorphic line bundle $\mathcal{L}$ on X as follows. Consider the short exact sequence of sheaves on X:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}_X^* \to 1,$$

where ι is the inclusion of the constant sheaf $\mathbb{Z}$ into the sheaf of holomorphic functions $\mathcal{O}_X$, and the map

$$\exp(2\pi i\cdot) : \mathcal{O}_X \to \mathcal{O}_X^*, \quad f \mapsto e^{2\pi i f}$$

is the exponential map. This sequence induces a long exact sequence on cohomology, in particular:

$$\cdots \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to \cdots$$

We define the *first Chern class* of a holomorphic line bundle $\mathcal{L} \in H^1(X,\mathcal{O}_X^*)$ to be $c_1(\mathcal{L})$, i.e., its image under the map c_1. Since

$$H^2(X,\mathbb{Z}) \cong \mathrm{Alt}^2(\Lambda,\mathbb{Z}),$$

we may regard $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$ as an alternating $\mathbb{Z}$-valued form $E_{\mathcal{L}}$ on the lattice Λ.

We can extend $E_{\mathcal{L}}$ to V by tensoring with $\mathbb{R}$:

$$E_{\mathcal{L}} \otimes \mathbb{R} : V \times V \to \mathbb{R}$$

to obtain a real alternating form on V. We denote this form again with $E_{\mathcal{L}}$.

Conversely, it can be shown that a real alternating form E on V represents the Chern class of a holomorphic line bundle $\mathcal{L}$ on X if and only if $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all $v, w \in V$. (See [2, Proposition 2.1.6].)

Mimicking the proof of Proposition 2.3.2, we can show that a real alternating form E on V satisfying $E(iv, iw) = E(v, w)$ corresponds to a Hermitian form $H : V \times V \to \mathbb{C}$, via

$$E = \Im(H) \qquad \text{and} \qquad H(v, w) = E(iv, w) + iE(v, w)$$

for all $v, w \in V$.

Putting everything together, we have the following:

Theorem 2.10.3. *Let $X = V/\Lambda$ be a complex torus. There is a one-to-one correspondence between the following data:*

1. *Chern classes of holomorphic line bundles on X.*
2. *Real alternating forms $E : V \times V \to \mathbb{R}$ satisfying:*

$$E(\Lambda, \Lambda) \subset \mathbb{Z} \quad \textit{and} \quad E(iv, iw) = E(v, w) \textit{ for all } v, w \in V.$$

3. *Hermitian forms $H : V \times V \to \mathbb{C}$ such that the imaginary part satisfies:*

$$\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}.$$

Definition 2.10.4. Let $\mathcal{L}$ be a holomorphic line bundle on a complex torus $X = V/\Lambda$. We say that $\mathcal{L}$ is *positive definite* if the corresponding Hermitian form $c_1(\mathcal{L}) = H$ on V is positive definite.

Definition 2.10.5. A *polarization* on a complex torus $X = V/\Lambda$ is a definite Hermitian form H on V, whose imaginary part $E = \Im H$ satisfies $E(\Lambda, \Lambda) \subset \mathbb{Z}$.

With these definitions and the previous results, we conclude that any positive definite holomorphic line bundle $\mathcal{L}$ on $X = V/\Lambda$ defines a polarization $H = c_1(\mathcal{L})$ on X. Thus, by abuse of notation, we sometimes consider the line bundle $\mathcal{L}$ itself as a polarization.

Definition 2.10.6. An *abelian variety* is a complex torus that admits a polarization, i.e., a positive definite holomorphic line bundle. A *polarized abelian variety* $(X, \mathcal{L})$ is an abelian variety X equipped with a polarization $\mathcal{L}$.

Example 2.10.7. Suppose $X = \mathbb{C}/\Lambda$ is an elliptic curve. Without loss of generality we may assume that $\Lambda = \mathbb{Z} \oplus z\mathbb{Z}$ for some $z \in \mathbb{C}$ with $\Im(z) > 0$. Define

$$H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad H(v, w) = \frac{v\overline{w}}{\Im(z)}$$

It is easy to check that H is a hermitian form with $\Im(H)(\Lambda, \Lambda) \subset \mathbb{Z}$. Since H is positive definite, X is an abelian variety. So every elliptic curve is an abelian variety.

Chapter 3

Normal Crossing Divisors

In this chapter, we study subspaces of a Kähler manifold with normal crossing singularities. We begin with the definition of a normal crossing divisor and its stratification, and then construct two key double complexes: the de Rham complex and the Čech complex. Finally, we introduce the dual complex. The main reference for this chapter is [10].

3.1 Definition and Stratification

Definition 3.1.1 (Normal Crossing Divisor)**.** Let $\mathcal{X}$ be a Kähler manifold of dimension $n+1$. A closed analytic subset $X \subset \mathcal{X}$ is called a *normal crossing divisor* if it is a reduced divisor of $\mathcal{X}$ with normal crossings. In other words,

1. X is of *pure codimension* 1 in $\mathcal{X}$; that is, every irreducible component of X has complex dimension n.

2. for every point $x \in X$, there exists a holomorphic coordinate chart $(U; z_1, \ldots, z_{n+1})$ of $\mathcal{X}$ centered at x such that $X \cap U$ is defined by

$$z_1 \cdots z_r = 0$$

   for some $1 \leq r \leq n+1$. In particular, $X \subset \mathcal{X}$ locally consists of coordinate hyperplanes intersecting transversely.

X is a *simple* normal crossing if its irreducible components X_i are all smooth.

Remark 3.1.2. When $\mathcal{X} = \mathbb{P}^{n+1}$, we call X *normal crossing variety.*

For the remainder of this chapter, we fix a Kähler manifold $\mathcal{X}$ of dimension $n+1$, and let $X = \bigcup_i X_i$ be a *compact* simple normal crossing divisor in $\mathcal{X}$, with X_i its irreducible components.

For any finite index set I, denote $X_I = \bigcap_{i \in I} X_i$, and define the *codimension-q stratum* of X as

$$X^{(q)} = \bigsqcup_{|I|=q+1} X_I.$$

and the 'gluing' maps $i_q \colon X^{(q)} \to X$ induced by inclusions. As a matter of convention, we assume that all index sets I are increasingly ordered.

Observe that each stratum of X arises as a finite transverse intersection of the X_i's, and is therefore itself a smooth submanifold of $\mathcal{X}$. Since X is compact and each stratum is closed in X, it follows that the strata are compact as well. Being closed complex submanifolds of the ambient Kähler manifold $\mathcal{X}$, they inherit Kähler structures from $\mathcal{X}$, and hence, each stratum is a compact Kähler manifold. This allows us to apply the results developed in Chapter 2 to the strata $X^{(q)}$.

Example 3.1.3. Figure 3.1 shows the stratification of $X = X_0 \cup X_1 \cup X_2 \subset \mathbb{P}^2$ with $X_i \cong \mathbb{P}^1$. We have $X^{(0)} = X_0 \sqcup X_1 \sqcup X_2$, $X^{(1)} = \{X_{01}, X_{02}, X_{12}\}$ and $X^{(q)} = \emptyset$ for $q > 1$.

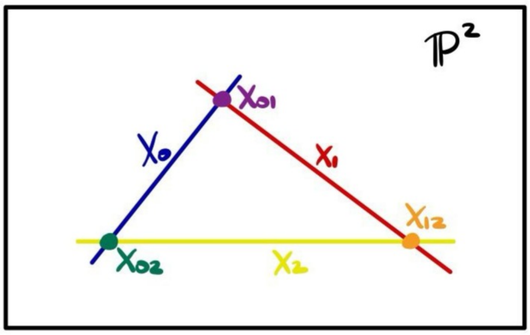


Figure 3.1: Lines intersecting in $\mathbb{P}^2$

Example 3.1.4. Let $X = \bigcup X_i$ be the union of l distinct, connected, smooth projective curves in $\mathbb{P}^2$, where each X_i has degree d_i and genus g_i. Then,

$$X^{(0)} = \bigsqcup_{i=0}^{l-1} X_i$$

is a disjoint union of l smooth complex curves of total geometric genus $g = \sum_i g_i$. The stratum $X^{(1)}$ consists of the (finitely many) points where pairs of distinct curves intersect. The union X is a simple normal crossing divisor when all intersections are transverse, meaning that at most two curves meet at any given point, and the intersection is transverse in the usual sense. In this case, by Bézout's theorem, the number of intersection points is

$$N := \#X^{(1)} = \sum_{i<j} d_i d_j.$$

We have natural inclusions $j_k : X^{(1)} \to X^{(0)}$ for $k = 0, 1$, where j_0 is induced by $X_{ij} \hookrightarrow X_j$ and j_1 by $X_{ij} \hookrightarrow X_i$. These induce morphisms of sheaves of locally constant $\mathbb{Q}$-valued functions:

$$j_k^* : \mathbb{Q}_{X^{(0)}} \to \mathbb{Q}_{X^{(1)}},$$

and we define the map $\delta = j_0^* - j_1^*$. Explicitly, if f is a locally constant function on $X^{(0)}$, then

$$(\delta f)(X_{ij}) = f(X_j) - f(X_i).$$

This leads to the short exact sequence of sheaves:

$$0 \to \mathbb{Q}_X \xrightarrow{\gamma} \bigoplus_i \mathbb{Q}_{X_i} \xrightarrow{\delta} \bigoplus_{i<j} \mathbb{Q}_{X_{ij}} \to 0,$$

where γ is the natural restriction map.

Remark 3.1.5. Strictly speaking, the sheaves $\bigoplus_i \mathbb{Q}_{X_i}$ and $\bigoplus_{i<j} \mathbb{Q}_{X_{ij}}$ appearing in the sequence should be understood as the pushforwards of the constant sheaves on $X^{(0)}$ and $X^{(1)}$, respectively, under the gluing maps i_0 and i_1 into X. For simplicity and readability, we omit the pushforward notation and write them directly as sheaves on X.

This induces a long exact sequence in cohomology:

$$0 \to H^0(X) \xrightarrow{\gamma} \bigoplus_i H^0(X_i) \xrightarrow{\delta} \bigoplus_{i<j} H^0(X_{ij}) \xrightarrow{\alpha} H^1(X) \xrightarrow{\gamma} \bigoplus_i H^1(X_i) \to 0,$$
$$0 \to H^2(X) \to \bigoplus_i H^2(X_i) \to 0.$$

Since each X_i is connected, we have $H^0(X_i) \cong \mathbb{Q}$, so $\bigoplus_i H^0(X_i) \cong \mathbb{Q}^l$. Likewise, each X_{ij} is a finite set of points, so $H^0(X_{ij}) \cong \mathbb{Q}^{d_i d_j}$, and thus:

$$\bigoplus_{i<j} H^0(X_{ij}) \cong \mathbb{Q}^N.$$

Also, $H^1(X_i) \cong \mathbb{Q}^{2g_i}$ for a connected smooth curve of genus g_i, so $\bigoplus_i H^1(X_i) \cong \mathbb{Q}^{2g}$ where $g = \sum_i g_i$. Finally, $H^2(X_i) \cong \mathbb{Q}$ for each curve. Therefore, the long exact sequence becomes:

$$0 \to \mathbb{Q} \to \mathbb{Q}^l \xrightarrow{\delta} \mathbb{Q}^N \xrightarrow{\alpha} H^1(X) \xrightarrow{\gamma} \mathbb{Q}^{2g} \to 0,$$
$$0 \to H^2(X) \to \mathbb{Q}^l \to 0.$$

From this, we conclude:

$$b_2(X) = \dim_{\mathbb{Q}} H^2(X, \mathbb{Q}) = l,$$

and the dimension of $H^1(X, \mathbb{Q})$ is:

$$b_1(X) = \dim_{\mathbb{Q}} H^1(X, \mathbb{Q}) = 2g + N - l + 1.$$

We also obtain a natural *weight filtration* on $H^1(X, \mathbb{Q})$, arising from the exact sequence:

$$0 \subset W_0 = \operatorname{im}(\alpha) = \operatorname{coker}(\delta) \subset W_1 = H^1(X, \mathbb{Q}),$$

with

$$\mathrm{Gr}_1 = W_1/W_0 = H^1(X, \mathbb{Q})/\operatorname{im}(\alpha) \cong H^1(X, \mathbb{Q})/\ker(\gamma) \cong \operatorname{im}(\gamma) \cong \bigoplus_i H^1(X_i)$$

We will see how this process can be generalized to any compact simple normal crossing divisor in a Kähler manifold in Section 3.3.

3.2 The de Rham Double Complex

Let $\mathcal{K}^{p,q} = (i_q)_*\mathcal{A}^p_{X^{(q)}}$ be the pushforward-to-X sheaf of smooth differential p-forms on $X^{(q)}$. It comes naturally equipped with two differentials: the horizontal differential $d : \mathcal{K}^{p,q} \to \mathcal{K}^{p+1,q}$, the *exterior derivative* on p-forms, and the vertical differential $\delta : \mathcal{K}^{p,q} \to \mathcal{K}^{p,q+1}$, called the *combinatorial differential*, defined as follows.

Let

$$j_k : X^{(q+1)} \to X^{(q)}$$

be the map induced by the natural inclusions $X_{i_0\cdots i_{q+1}} \hookrightarrow X_{i_0\cdots \widehat{i_k}\cdots i_{q+1}}$ for $0 \le k \le q+1$. These induce restriction maps

$$j_k^* : \mathcal{K}^{p,q} \to \mathcal{K}^{p,q+1}.$$

Define

$$\delta = (-1)^p \sum_{k=0}^{q+1} (-1)^k j_k^*.$$

We already know that $d^2 = 0$, and it is straightforward to verify by direct computation that $\delta^2 = 0$ as well. Moreover, due to the sign factor $(-1)^p$ in the definition of δ, d and δ anticommute:

$$d\delta + \delta d = 0.$$

This double complex of sheaves gives a total complex of sheaves $\mathcal{K}^\bullet = \mathrm{Tot}(\mathcal{K})^\bullet$, with

$$\mathcal{K}^\ell = \bigoplus_{p+q=\ell} \mathcal{K}^{p,q}, \quad D = d + \delta.$$

Given an open set $U \subseteq X$, we can define a double complex of $\mathbb{C}$-vector spaces $K^{p,q} = \mathcal{K}^{p,q}(U)$. If we filter the total complex $K^\bullet = \mathcal{K}^\bullet(U)$ by rows, the corresponding spectral sequence has:

$$\begin{aligned} &E_0^{p,q} = K^{p,q}, && d_0 = d, \\ &E_1^{p,q} = H^p(K^{\bullet,q}, d), && d_1 = \delta, \\ &E_2^{p,q} = H^q(E_1^{p,\bullet}, \delta), && d_2 \text{ unknown in general.} \end{aligned}$$

From the total complex $\mathcal{K}^\bullet$, one can extract information about the cohomology of X:

Proposition 3.2.1. *The sequence*

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{K}^0 \xrightarrow{D} \mathcal{K}^1 \xrightarrow{D} \mathcal{K}^2 \longrightarrow \cdots$$

where the first arrow is the natural inclusion, is a resolution of the constant sheaf $\mathbb{C}_X$ by Γ-acyclic sheaves. Hence, the cohomology of X with complex coefficients is computed by the global sections of this complex:

$$H^*(X,\mathbb{C}) \cong H^*(\Gamma(X,\mathcal{K}^\bullet)).$$

Proof. The sheaves $\mathcal{K}^{p,q}$ are sheaves of smooth differential forms on smooth manifolds, so they are fine ([13, Prop. 12.3]), and therefore Γ-acyclic. Since each $\mathcal{K}^\ell$ is a direct sum of sheaves of smooth differential forms on the strata $X^{(q)}$, it follows that $\mathcal{K}^\ell$ is also a Γ-acyclic sheaf.

To prove exactness, it suffices to check the sequence is exact locally. Let U be an open subset of the ambient space $\mathcal{X}$ such that $X \cap U$ is diffeomorphic to the union of k coordinate hyperplanes in $\mathbb{C}^{n+1}$ (where $n+1 = \dim_{\mathbb{C}} \mathcal{X}$):

$$X \cap U \cong \{z_1 \cdots z_k = 0\}.$$

Thus, the irreducible components of $X \cap U$ are given by $\{z_i = 0\}$ for $i = 1, \ldots, k$. For an ordered index set $I \subset \{1, \ldots, k\}$, we define:

$$X_I \cap U \cong \bigcap_{i \in I} \{z_i = 0\}, \quad \text{and} \quad U^{(q)} := X^{(q)} \cap U \cong \bigsqcup_{|I|=q+1} X_I \cap U.$$

Note that $U^{(q)} = \emptyset$ for $q \geq k$.

Set $K^\ell := \mathcal{K}^\ell(U)$ for $\ell \geq 0$, and consider the complex $K^\bullet$:

$$0 \to K^0 \xrightarrow{D} K^1 \xrightarrow{D} K^2 \xrightarrow{D} \cdots .$$

We show that this complex is exact at all terms except the first by analyzing the spectral sequence arising from the row filtration of the double complex $K^{p,q} = \mathcal{K}^{p,q}(U)$. On the E_0 page, the q-th row is:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}^0_{X^{(q)}}(U^{(q)}) & \xrightarrow{d} & \mathcal{A}^1_{X^{(q)}}(U^{(q)}) & \xrightarrow{d} & \mathcal{A}^2_{X^{(q)}}(U^{(q)}) & \xrightarrow{d} & \cdots \\ p=-1 & & p=0 & & p=1 & & p=2 & & \end{array}$$

Taking cohomology with respect to d (the horizontal differential), the E_1 page is given by the de Rham cohomology of $U^{(q)}$. Since each connected component of $U^{(q)}$ is contractible, we obtain:

$$E_1^{p,q} = H^p(U^{(q)}, \mathbb{C}) = \begin{cases} \mathbb{C}^{\binom{k}{q+1}} & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

The E_2 page is then the cohomology of the first column with respect to the combinatorial differential δ, acting on the terms $E_1^{0,\bullet}$. This is precisely the Čech cohomology of the $(k-1)$-simplex, which corresponds to the *dual complex* associated to $X \cap U = \{z_1 \cdots z_k = 0\}$ (see Section 3.5 and Proposition 3.5.3). The dual complex encodes the intersection pattern of the components $\{z_i = 0\}$: it has one vertex for each component, and a q-simplex for each nonempty intersection of $q+1$ components. Thus, the terms $E_1^{0,q} \cong \mathbb{C}^{\binom{k}{q+1}}$ match the dimension of the space of q-cochains on the simplex, and the differential δ corresponds to the Čech differential computing its simplicial cohomology with constant coefficients. Hence:

$$E_2^{p,q} = \begin{cases} \mathbb{C} & \text{if } (p,q) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

and the spectral sequence therefore degenerates at the E_2 page. By Theorem 1.4.2 it converges to the cohomology of the total complex. Therefore,

$$H^*(K^\bullet, D) = \begin{cases} \mathbb{C} & \text{in degree } 0, \\ 0 & \text{in higher degrees.} \end{cases}$$

This shows that the complex $K^\bullet$ is exact in positive degrees, and the kernel at the first term is precisely the constant sheaf $\mathbb{C}_X$. We conclude that the sequence in the statement of the proposition is indeed a Γ-acyclic resolution of the constant sheaf $\mathbb{C}_X$. □

In what follows, we focus on the *de Rham double complex* obtained by taking global sections:

$$K^{p,q} := \Gamma(X, \mathcal{K}^{p,q}) = \mathcal{K}^{p,q}(X).$$

The associated total complex is given by

$$K^\ell = \Gamma(X, \mathcal{K}^\ell) = \mathcal{K}^\ell(X).$$

As in the proof of Proposition 3.2.1, the first page of the spectral sequence associated with the row filtration is given by

$$E_1^{p,q} = H^p(X^{(q)}, \mathbb{C}).$$

We call this spectral sequence the *de Rham spectral sequence*. In the global setting we do not have a straightforward way to compute the second page E_2 explicitly, as we did in the local case, since we are no longer working with contractible open sets.

Nevertheless, by Theorem 1.4.2, this spectral sequence converges to the cohomology of the total complex $K^\bullet$, which, by Proposition 3.2.1, computes the cohomology of X with complex coefficients:

$$E_1^{p,q} = H^p(X^{(q)}, \mathbb{C}) \quad \Longrightarrow \quad H^{p+q}(K^\bullet, D) \cong H^{p+q}(X, \mathbb{C}).$$

At this point, it is important to highlight a special feature of the objects appearing in the spectral sequence. As previously noted, the strata $X^{(q)}$ are smooth, compact Kähler manifolds. Smoothness follows directly from the definition of the strata, while compactness is inherited from X, and the Kähler structure is induced from the ambient space $\mathcal{X}$.

As we have discussed in Chapter 2, the complex cohomology groups $H^p(X^{(q)}, \mathbb{C})$ carry a natural pure Hodge structure of weight p. Since the first page of the spectral sequence is given by

$$E_1^{p,q} = H^p(X^{(q)}, \mathbb{C}),$$

each $E_1^{p,q}$ carries this pure Hodge structure.

Furthermore, for $r > 1$, the terms $E_r^{p,q}$ are obtained as subquotients of $E_1^{p,q}$ through the successive differentials of the spectral sequence. As a result, they inherit a pure Hodge structure of weight p. This structure is preserved throughout the spectral sequence and remains compatible with the differentials.

We use this fact to prove the following:

Theorem 3.2.2. *The de Rham spectral sequence arising from the double complex $K^{p,q}$ of global differential forms on the strata of X, with the filtration by rows, degenerates at the E_2-page.*

Proof. It suffices to show that $d_r = 0$ for all $r \geq 2$.

Let $a \in E_r^{p,q}$ be a class that survives to the E_r-page. By Proposition 1.4.5, the class $[a]_r$ can be extended to a zig-zag: there exist $c_i \in K^{p-i,q+i}$ for $i = 1, \ldots, r-1$, satisfying

$$dc_i = \delta c_{i-1} \quad (\text{where } c_0 = a), \qquad \text{and} \qquad d_r([a]_r) = [\delta c_{r-1}]_r.$$

Without loss of generality, suppose that δc_{r-2} is a form of type (s,t). Since $\delta c_{r-2} = dc_{r-1}$ is d-exact, we may apply Proposition 2.5.7 to write

$$\delta c_{r-2} = d\beta' = d\beta'',$$

where β' and β'' are forms of type $(s-1,t)$ and $(s,t-1)$, respectively.

By Remark 1.4.6, the cohomology class $[\delta c_{r-1}]_r$ is equal to both $[\delta\beta']_r$ and $[\delta\beta'']_r$. However, since the combinatorial differential δ preserves type, $\delta\beta'$ and $\delta\beta''$ live in distinct bidegrees:

$$\delta\beta' \in \mathcal{A}^{s-1,t}_{X^{(q+r)}}, \quad \delta\beta'' \in \mathcal{A}^{s,t-1}_{X^{(q+r)}}.$$

Now, $E_r^{p-r+1,q+r}$ is a subquotient of $E_1^{p-r+1,q+r} = H^{p-r+1}(X^{(q+r)}, \mathbb{C})$, which, by Hodge theory, carries a pure Hodge structure of weight $p-r+1$. Since $\delta\beta'$ and $\delta\beta''$ have different types but are equal in cohomology, the principle of two types (Proposition 2.5.8) implies that their class is trivial:

$$[\delta\beta']_r = [\delta\beta'']_r = 0.$$

Therefore,

$$d_r([a]_r) = [\delta c_{r-1}]_r = 0.$$

This proves that all higher differentials vanish, and hence the spectral sequence degenerates at the E_2-page. □

3.3 The Čech Double Complex

We now construct a different double complex that will allow us to compute the cohomology of X with rational coefficients.

Definition 3.3.1. An open cover $\{U_\alpha\}$ of a topological space is called a *good cover* if each U_α is contractible, and all finite intersections

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$$

are either empty or contractible.

Proposition 3.3.2. *Let $\mathcal{X}$ be a Kähler manifold and $X \subset \mathcal{X}$ a simple normal crossing divisor. Then there exists a good open cover $\{U_\alpha\}$ of $\mathcal{X}$ such that the induced cover $\mathcal{U} = \{U_\alpha \cap X\}$ is a good cover of X.*

Sketch of Proof. It is a fact that every smooth manifold admits a good cover; around each point of $\mathcal{X}$, we can choose a coordinate chart diffeomorphic to an open ball in $\mathbb{R}^{2\dim\mathcal{X}}$, which is contractible. By paracompactness, any open cover can be refined to a good cover.

Since $X \subset \mathcal{X}$ is a simple normal crossing divisor, it is locally defined by the vanishing of a product of coordinate functions, say $z_1 \cdots z_k = 0$ in local coordinates. By taking the good cover $\{U_\alpha\}$ of $\mathcal{X}$ sufficiently fine, we can ensure that each $U_\alpha \cap X$ is either empty or is a union of coordinate hyperplanes intersecting transversely, and hence is also contractible. The same holds for finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \cap X$. Thus, the restriction $\{U_\alpha \cap X\}$ defines a good cover of X. □

Let A be an abelian group. When a topological space Y admits a good open cover $\mathcal{U}$, the Čech cohomology $\check{H}^*(\mathcal{U}, A)$ computed with respect to this cover agrees with the sheaf cohomology $H^*(Y, A_Y)$, where A_Y denotes the constant sheaf with values in A. Moreover, if Y is Hausdorff, locally contractible, and every open subset is paracompact (for instance, if Y is a topological manifold), then the singular cohomology $H^*(Y, A)$ coincides with the sheaf cohomology $H^*(Y, A_Y)$. In our setting, the space X and each stratum $X^{(q)}$ are topological manifolds satisfying these conditions. Consequently, for these spaces, the singular cohomology, Čech cohomology, and sheaf cohomology for the constant sheaf all agree.

By choosing a good cover as in the proposition above, we ensure that the Čech cohomology of X with rational coefficients agrees with its singular (or sheaf) cohomology:

$$\check{H}^*(\mathcal{U}, \mathbb{Q}_X) \cong H^*(X, \mathbb{Q}).$$

Moreover, the restriction of the open cover $\{U_\alpha\}$ to each stratum $X^{(q)}$, given by

$$\mathcal{U}_q = \{U_\alpha \cap X^{(q)}\},$$

defines a good cover of $X^{(q)}$ as well. This follows from the fact each $X_I \cap U_\alpha$ is either empty or contractible (as in the proof of Proposition 3.2.1). Hence, for each q, we similarly have:

$$\check{H}^*(\mathcal{U}_q, \mathbb{Q}_{X^{(q)}}) \cong H^*(X^{(q)}, \mathbb{Q}).$$

For notational simplicity, we will omit subscripts on constant sheaves and write $\mathbb{Q}$ instead of $\mathbb{Q}_X$ or $\mathbb{Q}_{X^{(q)}}$ in what follows.

We define the *Čech double complex* $C^{p,q} = C^p(\mathcal{U}_q, \mathbb{Q})$ as the Čech cochains of degree p, associated to the open cover $\mathcal{U}_q$ of the q-stratum of X, with coefficients in $\mathbb{Q}$. This complex is equipped with two differentials: the horizontal differential $d : C^{p,q} \to C^{p+1,q}$, the *Čech coboundary*, and the vertical differential $\delta : C^{p,q} \to C^{p,q+1}$, the *combinatorial differential* defined similarly to before: For each $U \in \mathcal{U}$, and $0 \leq k \leq q+1$, we have maps

$$j_k^U : U \cap X^{(q+1)} \to U \cap X^{(q)}$$

induced by the natural inclusions $X_{i_0 \cdots i_{q+1}} \hookrightarrow X_{i_0 \cdots \widehat{i_k} \cdots i_{q+1}}$. These give rise to restriction maps on sections of the constant sheaf:

$$j_k^{U*} : \mathbb{Q}(U \cap X^{(q)}) \to \mathbb{Q}(U \cap X^{(q+1)}),$$

which in turn induce maps on the Čech cochains:

$$j_k^* : C^p(\mathcal{U}_q, \mathbb{Q}) \to C^p(\mathcal{U}_{q+1}, \mathbb{Q}), \qquad j_k^* = \prod_{|I|=p+1} j_k^{U_I*}$$

The differential δ is then defined as

$$\delta = (-1)^p \sum_{k=0}^{q+1} (-1)^k j_k^*.$$

We begin by filtering the double complex $C^{p,q}$ by rows. With this choice of filtration, the first page E_1 of the associated spectral sequence is obtained by computing cohomology with respect to the horizontal differential, that is, the Čech coboundary. Consequently, the E_1-page consists of Čech cohomology groups. Under the assumptions on the open cover introduced above, this yields

$$E_1^{p,q} = H^p(X^{(q)}, \mathbb{Q}).$$

We call this spectral sequence, the *Čech spectral sequence.* Tensoring with $\mathbb{C}$, we recover the first page of the de Rham spectral sequence:

$$E_1^{p,q} \otimes \mathbb{C} = H^p(X^{(q)}, \mathbb{C}) = {}^{\mathrm{DR}}E_1^{p,q}.$$

Since the differential d_1 on the E_1-pages in both the Čech and de Rham spectral sequences is defined via the same combinatorial formula, we obtain an embedding of the Čech complex into the de Rham complex. As the de Rham spectral sequence degenerates at the E_2-page, the same follows for the Čech spectral sequence.

It is difficult to describe the second page of the Čech spectral sequence. Instead, we consider the column filtration of the double complex $C^{p,q}$, which yields a different spectral sequence. By Theorem 1.4.2, both sequences must converge to the same object.

In this case, the first page is computed by taking cohomology with respect to the vertical differential δ. To make this more transparent, we observe that for fixed p, the E_0-page consists of columns of the form

$$\begin{array}{ccccccccc} 0 & \to & C^p(\mathcal{U}_0, \mathbb{Q}) & \xrightarrow{\delta} & C^p(\mathcal{U}_1, \mathbb{Q}) & \xrightarrow{\delta} & C^p(\mathcal{U}_2, \mathbb{Q}) & \xrightarrow{\delta} & \cdots \\ q=-1 & & q=0 & & q=1 & & q=2 & & \end{array}$$

Taking cohomology with respect to δ, we obtain

$$E_1^{p,q} = \begin{cases} C^p(\mathcal{U}, \mathbb{Q}) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

To see this, we examine more closely the kernel

$$\ker\left(\delta : \prod_V \mathbb{Q}(V \cap X^{(0)}) \longrightarrow \prod_V \mathbb{Q}(V \cap X^{(1)})\right)$$

where the product ranges over all V, the intersections of p elements of the open cover $\mathcal{U}$, as in the definition of Čech cochains of degree p. A locally constant function on the irreducible components of X assigns a value to each component, and the differential δ computes the differences of these values on the pairwise intersections $X_{ij} = X_i \cap X_j$, namely $(\delta f)(X_{ij}) = f(X_j) - f(X_i)$. Being in the kernel of δ is therefore equivalent to assigning the same value to all irreducible components, and thus to all of X. Hence, on degree 0, $\ker \delta = C^p(\mathcal{U}, \mathbb{Q})$.

For degrees $q > 0$, we have the identity:

$$\ker\left(\prod_V \mathbb{Q}(V \cap X^{(q)}) \xrightarrow{\delta} \prod_V \mathbb{Q}(V \cap X^{(q+1)})\right) = \operatorname{im}\left(\prod_V \mathbb{Q}(V \cap X^{(q-1)}) \xrightarrow{\delta} \prod_V \mathbb{Q}(V \cap X^{(q)})\right)$$

Recall that by the choice of the open cover $\mathcal{U}$, any intersection V of p elements of $\mathcal{U}$ satisfies the property that, for every multi-index I with $|I| = q + 1$, the intersection $V \cap X_I$ is contracible and hence, connected. In particular, the connected components of $V \cap X^{(q)}$ are precisely the sets $V \cap X_I$ as I ranges over multi-indices of length $q + 1$.
Fix such a V, and for a set of indices I, write $Y_I := V \cap X_I$. Then we have a disjoint union:

$$Y^{(q)} := V \cap X^{(q)} = \bigsqcup_{|I|=q+1} Y_I.$$

Suppose X has $l + 1$ irreducible components, indexed by $\{0, 1, \ldots, l\}$, and let the indices in each I be ordered: $i_0 < i_1 < \cdots < i_q$. A locally constant function f on $Y^{(q)}$ in the kernel of δ amounts to giving numbers $x_{i_0 \ldots i_q} \in \mathbb{Q}$ satisfying the cocycle condition:

$$\sum_{k=0}^{q+1} (-1)^k x_{i_0 \ldots \hat{i_k} \ldots i_{q+1}} = 0.$$

To find a locally constant function g on $Y^{(q-1)}$ such that $\delta g = f$ means solving the system of linear equations:

$$x_{i_0 \ldots i_q} = \sum_{k=0}^{q} (-1)^k x_{i_0 \ldots \hat{i_k} \ldots i_q}$$

for all strictly increasing index tuples $i_0 < \cdots < i_q$. This system always has a solution. One such solution is given by:

$$x_{i_1 \ldots i_q} = \begin{cases} 0 & \text{if } i_1 = 0, \\ x_{0 i_1 \ldots i_q} & \text{if } i_1 > 0. \end{cases}$$

Alternatively, one can interpret the data $x_{i_0 \ldots i_q}$ as values on q-simplices of the standard l-simplex, with δ being the usual combinatorial differential. Since the standard simplex is contractible, its cohomology vanishes in degrees > 0, so every cocycle is a coboundary. This guarantees the existence of such a function g whenever $f \in \ker \delta$.

Consequently, the second page of the spectral sequence is given by

$$E_2^{p,q} = \begin{cases} H^p(X, \mathbb{Q}) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}.$$

Since only the first row survives, the sequence degenerates at E_2 and thus converges to the cohomology of X with rational coefficients. This shows that the spectral sequence arising from the row filtration, i.e., the Čech spectral sequence, also converges to the same target:

$$E_1^{p,q} = H^p(X^{(q)}, \mathbb{Q}) \quad \Longrightarrow \quad H^{p+q}(X, \mathbb{Q}).$$

3.4 The Weight Filtration on $H^m(X, \mathbb{Q})$

To obtain the Čech spectral sequence we have filtered the total complex $\mathrm{Tot}(C)^\bullet$ by rows:

$$\widetilde{W}_k \,\mathrm{Tot}(C)^\bullet = \bigoplus_{p \geq 0} \bigoplus_{q \geq k} C^{p,q}.$$

This filtration induces a (decreasing) filtration on the cohomology of $\mathrm{Tot}(C)^\bullet$, and consequently on the cohomology of X, $H^m(X, \mathbb{Q})$, which we denote again by $\widetilde{W}_k$:

$$\widetilde{W}_k H^m(X, \mathbb{Q}) = \bigoplus_{\substack{p+q=m \\ q \geq k}} \mathrm{Gr}_q^{\widetilde{W}} H^m(X, \mathbb{Q}) = \bigoplus_{\substack{p+q=m \\ q \geq k}} E_\infty^{p,q} = \bigoplus_{\substack{p+q=m \\ q \geq k}} E_2^{p,q}$$

We adjust this filtration to obtain an increasing filtration, referred to as the *spectral weight filtration*, defined by

$$W_k H^m(X, \mathbb{Q}) := \widetilde{W}_{m-k} H^m(X, \mathbb{Q}) = \bigoplus_{\substack{p+q=m \\ q \geq m-k}} E_2^{p,q} = \bigoplus_{\substack{p+q=m \\ p \leq k}} E_2^{p,q}$$

One must make sure that the graded pieces of this filtration agree with the graded pieces of the old one, so the convergence of the spectral sequence still holds. We have

$$\mathrm{Gr}_p^{\widetilde{W}} H^{p+q} = \frac{\widetilde{W}_p H^{p+q}}{\widetilde{W}_{p+1} H^{p+q}} = \frac{W_q H^{p+q}}{W_{q-1} H^{p+q}} = \mathrm{Gr}_q^W H^{p+q} = \mathrm{Gr}_p^W H^{p+q}$$

where the last equality comes from Theorem 1.4.2 and the fact that the E_0 pages of the two spectral sequences differ by interchaning the roles of p and q.

The following two corollaries follow immediately:

Corollary 3.4.1. *The spectral weight filtration on $H^m := H^m(X, \mathbb{Q})$ satisfies $W_m H^m = H^m$. Hence, the filtration is of the form*

$$0 \subset W_0 H^m \subset W_1 H^m \subset \cdots \subset W_m H^m = H^m$$

Corollary 3.4.2. *If $X^{(k)} = \emptyset$ then $W_{m-k}H^m = 0$.*

Proof. If $X^{(k)} = \emptyset$ then $E_1^{p,q} = 0$ for all $p \geq 0$ and $q \geq k$. But then $E_2^{p,q} = 0$ as well for $p \geq 0$ and $q \geq k$ and hence $W_{m-k}H^m = \widetilde{W}_k H^m = 0$. □

Example 3.4.3. Let X be the union of l connected smooth curves as in Example 3.1.4. We will compute the cohomology of X with the theory we developed in this chapter. To save space, all the cohomology groups are with rational coefficients. The first page of the Čech spectral sequence on X is:

$$E_1 : \begin{array}{c|ccc} q=2 & 0 & 0 & 0 \\ q=1 & \bigoplus_{i<j} H^0(X_{ij}) & \bigoplus_{i<j} H^1(X_{ij}) & \bigoplus_{i<j} H^2(X_{ij}) \\ q=0 & \bigoplus_i H^0(X_i) & \bigoplus_i H^1(X_i) & \bigoplus_i H^2(X_i) \\ \hline & p=0 & p=1 & p=2 \end{array}$$

which becomes

$$E_1 : \begin{array}{c|ccc} q=2 & 0 & 0 & 0 \\ q=1 & \mathbb{Q}^N & 0 & 0 \\ q=0 & \mathbb{Q}^l & \mathbb{Q}^{2g} & \mathbb{Q}^l \\ \hline & p=0 & p=1 & p=2 \end{array}$$

We obtain the second page by taking cohomology with respect to the vertical differential δ (going upwards). Hence, the second page is:

$$E_2 : \begin{array}{c|ccc} q=2 & 0 & 0 & 0 \\ q=1 & \mathbb{Q}^N / \operatorname{im}\delta & 0 & 0 \\ q=0 & \ker\delta & \mathbb{Q}^{2g} & \mathbb{Q}^l \\ \hline & p=0 & p=1 & p=2 \end{array}$$

Recall that each entry $E_2^{p,q}$ on the second page is the p-graded piece of the cohomology of X:

$$E_2^{p,q} \cong \operatorname{Gr}_p H^{p+q}(X, \mathbb{Q})$$

We reconstruct $H^{p+q}(X, \mathbb{Q})$ by summing along the appropriate diagonal:

$$\begin{aligned} H^0(X, \mathbb{Q}) &\cong \ker\delta \\ H^1(X, \mathbb{Q}) &\cong (\mathbb{Q}^N / \operatorname{im}\delta) \oplus \mathbb{Q}^{2g} \\ H^2(X, \mathbb{Q}) &\cong \mathbb{Q}^l \end{aligned}$$

Now, since X is connected, we have that $\dim_{\mathbb{Q}} \ker\delta = 1$ and $\dim_{\mathbb{Q}} \operatorname{im}\delta = l - 1$. This gives us the Betti numbers:

$$b_0(X) = 1, \quad b_1(X) = 2g + N - l + 1, \quad b_2(X) = l$$

as we already found in Example 3.1.4.

To determine the induced weight filtration on $H^m(X, \mathbb{Q})$, we focus on the diagonal $p+q=m$ on E_2. The weight filtration is obtained by summing contributions from the columns up to a given index. For example:

$$W_0 H^1(X, \mathbb{Q}) = \frac{\mathbb{Q}^N}{\operatorname{im}(\delta)} \cong \mathbb{Q}^{N-l+1}, \quad W_0 H^2(X, \mathbb{Q}) = W_1 H^2(X, \mathbb{Q}) = 0.$$

3.5 The Dual Complex Γ

Definition 3.5.1. Define the *dual complex* Γ of X to be a simplicial complex with a vertex P_i for each irreducible component X_i of X, and the simplex $[P_{i_0}, \ldots, P_{i_p}]$ belongs to Γ if and only if $X_{i_0,\ldots,i_p}$ is nonempty.

Example 3.5.2. In Figure 3.2, X is a normal crossing variety and Γ is its dual complex.

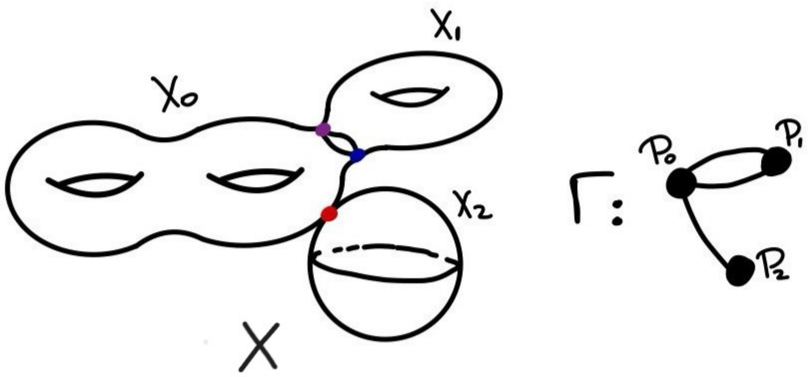


Figure 3.2

Proposition 3.5.3. *With the above definition of the dual complex Γ, we have*

$$\operatorname{Gr}_0 H^m(X, \mathbb{Q}) \cong H^m(|\Gamma|, \mathbb{Q}).$$

Proof. By the construction of the Čech spectral sequence computing the cohomology of X, we have

$$W_0 H^m(X, \mathbb{Q}) = \operatorname{Gr}_0 H^m(X, \mathbb{Q}) \cong H^m_\delta(H^0(X^{(\bullet)})),$$

where H^m_δ denotes the cohomology of the complex formed by the zeroth cohomology of the strata:

$$0 \longrightarrow H^0(X^{(0)}) \xrightarrow{\delta} H^0(X^{(1)}) \xrightarrow{\delta} H^0(X^{(2)}) \xrightarrow{\delta} \cdots.$$

This complex is precisely the simplicial cochain complex associated to the dual complex Γ. Therefore,

$$\operatorname{Gr}_0 H^m(X, \mathbb{Q}) \cong H^m(|\Gamma|, \mathbb{Q}).$$

□

Chapter 4

The Clemens-Schmid Exact Sequence

In the study of degenerations of algebraic varieties, particularly over the complex unit disk, it's important to understand how the topology and Hodge structures of the fibers behave near the central fiber. When the degeneration is semistable, that is, when the central fiber is a normal crossing divisor, the geometry becomes easier to work with, and both topological and algebraic tools can be used together. In this chapter, we look at what happens to cohomology in such situations, focusing on the Clemens-Schmid exact sequence. Our approach uses semistable reduction, monodromy around the central fiber, and the weight spectral sequence that comes from the normal crossing structure. Some references for this chapter are [7, Chapter 5], [10] and [12]

4.1 Semistable Degenerations

Definition 4.1.1 (Semistable Degeneration)**.** Let Δ be the unit open disk around 0 in the complex plane and let $\mathcal{X}$ be a Kähler manifold. A *degeneration* is a surjective, proper, flat, holomorphic map $\pi : \mathcal{X} \to \Delta$ of relative dimension n, such that each *generic fiber* $\mathcal{X}_t := \pi^{-1}(t)$ for $t \neq 0$ is a smooth complex variety. The degeneration is *semistable* if, in addition, the *central fiber* $\mathcal{X}_0$ is a normal crossing divisor, that is, π in a neighborhood of each point $x \in \mathcal{X}_0$ is defined by

$$x_1 x_2 \cdots x_k = t$$

for some k with $1 \leq k \leq n+1$. Moreover, the degeneration is *strictly semistable* if, in addition, the central fiber has smooth components, i.e., it is a simple normal crossing divisor.

Remark 4.1.2. Properness implies that each fiber $\mathcal{X}_t$, $t \in \Delta^* = \Delta \setminus \{0\}$, is a compact complex submanifold of $\mathcal{X}$. In a strictly semistable degeneration, $\mathcal{X}_0$ is a compact simple normal crossing divisor in $\mathcal{X}$. This allows us to apply the results from Chapter 3 to $\mathcal{X}_0$.

Remark 4.1.3. A degeneration restricted to the punctured disk, $\pi^* : \mathcal{X}^* = \pi^{-1}(\Delta^*) \to \Delta^*$ is automatically *smooth of relative dimension n*. Smooth in this context means flat with all the fibers smooth. ([5, Chapter III, Theorem 10.2]). Also, π^* is a submersion as a map between smooth manifolds. ([5, Chapter III, Proposition 10.4]).

Definition 4.1.4. Let $\pi : \mathcal{X} \to \Delta$ be a degeneration. A degeneration $\psi : \mathcal{Y} \to \Delta$ is called a *modification* of π, if there exists a map $f : \mathcal{Y} \to \mathcal{X}$, biholomorphic outside of the central fiber, and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad f \quad} & \mathcal{X} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Delta & \end{array}$$

By *Hironaka's theorem* (also known as the *desingularization of varieties*), any degeneration can be modified into one of the form $\pi \colon \mathcal{X} \to \Delta$, where the central fiber $\mathcal{X}_0$ is a divisor with normal crossings. That is, locally near each point $x \in \mathcal{X}_0$, the map π can be written as

$$x_1^{a_1} x_2^{a_2} \cdots x_{n+1}^{a_k} = t,$$

for some integers $a_i \geq 0$. In our context, a degeneration is called *semistable* if $a_i \leq 1$ for all i - that is, if $\mathcal{X}_0$ is a *reduced* divisor with normal crossings, i.e., a *normal crossing divisor.*

However, not every degeneration can be directly modified into a semistable one. Nonetheless, Mumford showed in [6, Chapter 2] that, after a suitable base change, such a semistable model can always be obtained.

Given a degeneration $\pi : \mathcal{X} \to \Delta$ and a base change map $b : \Delta \to \Delta$, $t \mapsto t^a$ for some positive integer a, we can construct a new degeneration $\pi_b := \mathcal{X}_b := \mathcal{X} \times_\Delta \Delta \to \Delta$ via the fiber product:

$$\begin{array}{ccc} \mathcal{X}_b & \longrightarrow & \mathcal{X} \\ \Big\downarrow{\scriptstyle\pi_b} & & \Big\downarrow{\scriptstyle\pi} \\ \Delta & \xrightarrow{\ b\ } & \Delta \end{array}$$

This corresponds to pulling back the family π along the map b, thereby replacing the base disc with a ramified cover.

Theorem 4.1.5 (Semistable Reduction Theorem)**.** *Given a degeneration $\pi : \mathcal{X} \to \Delta$, there exists a basis change $b : \Delta \to \Delta$, $t \mapsto t^a$, for some a, a semistable degeneration $\psi : \mathcal{Y} \to \Delta$ which is a modification of $\pi_b : \mathcal{X}_b \to \Delta$ and a commutative diagram*

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\ f\ } & \mathcal{X}_b & \longrightarrow & \mathcal{X} \\ & {\scriptstyle\psi}\searrow & \Big\downarrow{\scriptstyle\pi_b} & & \Big\downarrow{\scriptstyle\pi} \\ & & \Delta & \xrightarrow{\ b\ } & \Delta \end{array}$$

Moreover, $f : \mathcal{Y} \to \mathcal{X}_b$ is obtained by blowing up and blowing down subvarieties of the central fiber.

This theorem allows us to reduce to the semistable case, provided we work within a framework that is invariant under blow-ups and blow-downs. Once a semistable model has been obtained, we may resolve the singularities on the components of the central fiber to achieve a strictly semistable degeneration.

Example 4.1.6 (Degeneration of 0-manifolds)**.** Consider the case where k distinct points in $\mathbb{C}$ degenerate to a single point, that is, they merge into one. This family is given by the subvariety

$$\mathcal{X} = \{(x,t) \in \mathbb{C} \times \Delta \mid x^k = t\} \subset \mathbb{C} \times \Delta,$$

with projection to Δ via the second coordinate.

The generic fiber $\mathcal{X}_t$ for $t \neq 0$ consists of k distinct points, while the central fiber is a single point of multiplicity k. Hence, the central fiber is a divisor with normal crossings, but it is not reduced.

To obtain a semistable model, we first perform the base change $b\colon \Delta \to \Delta$, $t \mapsto t^k$, and form the fiber product

$$\mathcal{X}_b := \mathcal{X} \times_\Delta \Delta = \{(x,t) \in \mathbb{C} \times \Delta \mid x^k = t^k\}.$$

This can be written as the union of k smooth components:

$$\mathcal{X}_b = \bigcup_{j=0}^{k-1} \left\{(x,t) \in \mathbb{C} \times \Delta \mid x = \zeta_k^j t\right\},$$

where ζ_k is a primitive k-th root of unity. Each component is smooth and maps isomorphically to Δ, but they intersect at the origin $(x,t) = (0,0)$, so the central fiber remains non-reduced.

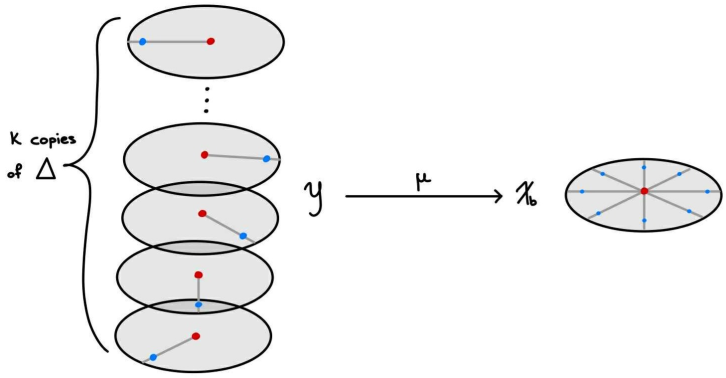
To separate these branches and obtain a reduced central fiber, we normalize $\mathcal{X}_b$. Define

$$\mathcal{Y} := \bigsqcup_{j=0}^{k-1} \Delta,$$

as k copies of Δ, and let

$$\mu\colon \mathcal{Y} \to \mathcal{X}_b$$

be the normalization map, with each copy of Δ mapping isomorphically to the component $x = \zeta_k^j t$.



The family $\mathcal{Y} \to \Delta$ is the trivial family of k disjoint sections. In particular, its central fiber is a disjoint union of k points. Thus, after base change and normalization, the degeneration becomes semistable and, in fact, trivial.

The following result will allow us to work cohomologically with $\mathcal{X}_0$ in place of the total space $\mathcal{X}$ which is often more manageable due to the stratified normal crossing structure of $\mathcal{X}_0$.

Proposition 4.1.7. *In the case of a semistable degeneration, the central fiber $\mathcal{X}_0$ is a deformation retract of the total space $\mathcal{X}$. As a consequence,*

$$H_m(\mathcal{X}_0, \mathbb{Q}) \cong H_m(\mathcal{X}, \mathbb{Q}) \qquad \textit{and} \qquad H^m(\mathcal{X}_0, \mathbb{Q}) \cong H^m(\mathcal{X}, \mathbb{Q})$$

The proof is technical. A detailed construction in the case of families of curves can be found in [1, p.146-148]. The construction of the deformation retraction can be generalized to arbitrary dimension.

For the rest of the chapter, fix a generic smooth fiber X_t and denote

- $H^m(\mathcal{X}, \mathbb{Q}) \cong H^m(\mathcal{X}_0, \mathbb{Q})$ by H^m
- $H_m(\mathcal{X}, \mathbb{Q}) \cong H_m(\mathcal{X}_0, \mathbb{Q})$ by H_m and
- $H^m(\mathcal{X}_t, \mathbb{Q})$ by H^m_{lim}.

4.2 The Monodromy Weight Filtration

Let $\pi : \mathcal{X} \to \Delta$ be a degeneration, and let $\pi^* : \mathcal{X}^* \to \Delta^*$ denote the restriction to the punctured disk. Fix an integer m, and choose a generic fiber $\mathcal{X}_t$ for some $t \in \Delta^*$.

Ehresmann's lemma states that if $f : M \to N$ is a smooth, proper, surjective submersion between smooth manifolds, then f is a locally trivial fibration. In our setting, π^* satisfies the hypotheses of Ehresmann's lemma, and hence is a locally trivial fibration. It follows that the fibers of π^* are all diffeomorphic (and in particular, homotopy equivalent), and that a loop γ in Δ^* based at t determines, up to homotopy, a monodromy homeomorphism

$$T_\gamma : \mathcal{X}_t \to \mathcal{X}_t$$

of the fiber over t. This homeomorphism induces automorphisms on the homology and cohomology of the generic fiber:

$$T_{\gamma *} : H_*(\mathcal{X}_t, \mathbb{Z}) \to H_*(\mathcal{X}_t, \mathbb{Z}), \qquad T_\gamma^* : H^*(\mathcal{X}_t, \mathbb{Z}) \to H^*(\mathcal{X}_t, \mathbb{Z}).$$

The action of a canonical generator of $\pi_1(\Delta^*) \cong \mathbb{Z}$ on the limiting cohomology group defines an automorphism

$$T : H^m_{\text{lim}} \to H^m_{\text{lim}},$$

called the *Picard-Lefschetz transformation.*

Theorem 4.2.1 (Monodromy Theorem)**.** *For a degeneration $\pi : \mathcal{X} \to \Delta$, the Picard-Lefschetz transformation acting on H^m_{lim} is quasi-unipotent, with index of unipotency at most $m+1$; that is, there exists a positive integer a such that*

$$(T^a - I)^{m+1} = 0.$$

In particular, after a finite base change $t \mapsto t^a$, the monodromy becomes unipotent.

By taking the base change $t \mapsto t^a$, we may assume that the monodromy T is unipotent. This allows us to define the logarithm of the monodromy,

$$N = \log T = (T - I) - \frac{1}{2}(T - I)^2 + \frac{1}{3}(T - I)^3 - \cdots$$

acting on H^m_{lim}. N is nilpotent, and the index of unipotency of T coincides with the index of nilpotency of N. In particular, $N^{m+1} = 0$, and $N = 0$ if and only if $T = 1$.

We define a weight filtration on the vector space H^m_{lim}

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m-1} \subset W_{2m} = H^m_{\text{lim}}$$

called the *monodromy weight filtration*:

Proposition 4.2.2. *Let H be a finite-dimensional $\mathbb{Q}$-vector space and $N : H \to H$ a nilpotent linear map, i.e., $N^{m+1} = 0$ for some $m > 0$. Then there exists a unique increasing filtration*

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m} = H$$

such that $N(W_k) \subset W_{k-2}$ for all k, and the induced maps

$$N^k : \operatorname{Gr}^W_{m+k} H \longrightarrow \operatorname{Gr}^W_{m-k} H \tag{4.1}$$

are isomorphisms for all $0 \le k \le m$, where $\operatorname{Gr}^W_\ell H := W_\ell / W_{\ell-1}$.

Proof. Set $W_j := 0$ for all $j < 0$. The isomorphism condition (4.1) for $k = m$, implies $\operatorname{im} N^m \subseteq W_0$ and $W_{2m-1} \subseteq \ker N^m$. Therefore, we have

$$\begin{aligned} \dim_{\mathbb{Q}} H - \dim_{\mathbb{Q}} \ker N^m = \dim_{\mathbb{Q}} \operatorname{im} N^m \le \dim_{\mathbb{Q}} W_0 &= \dim_{\mathbb{Q}} H - \dim_{\mathbb{Q}} W_{2m-1} \\ &\le \dim_{\mathbb{Q}} H - \dim_{\mathbb{Q}} \ker N^m \end{aligned}$$

so all inequalities are equalities, which forces

$$W_0 = \operatorname{im} N^m, \qquad W_{2m-1} = \ker N^m.$$

Now set $H_1 := W_{2m-1}/W_0$, on which N descends to a well-defined nilpotent operator with $N^m|_{H_1} = 0$. The condition (4.1) for $k = m - 1$ reads:

$$N^{m-1} : \frac{W_{2m-1}}{W_{2m-2}} = \frac{H_1}{W_{2m-2}/W_0} \xrightarrow{\sim} \frac{W_1}{W_0}.$$

This implies

$$\frac{W_1}{W_0} = \operatorname{im}(N^{m-1}|_{H_1}), \qquad \frac{W_{2m-2}}{W_0} = \ker(N^{m-1}|_{H_1}),$$

and we define W_1, $W_{2m-2} \subset H$ as the inverse images under the projection $H \to H/W_0$.

Proceeding inductively, assume that for some $1 \le k < m$, we have constructed

$$0 \subset W_0 \subset \cdots \subset W_{k-1} \subset W_{2m-k} \subset \cdots \subset W_{2m} = H$$

such that:

(1) $N(W_{2m-k}) \subset W_{2m-k}$,

(2) $N^{m-k+1}(W_{2m-k}) \subset W_{k-2}$,

(3) $W_{k-1} \subset N^{m-k}(W_{2m-k})$.

These hold for $k = 1$ by construction. From (2) and (3), it follows that $N(W_{k-1}) \subset W_{k-1}$. Set $H_k := W_{2m-k}/W_{k-1}$, so N induces a well-defined nilpotent map on H_k with $N|_{H_k}^{m-k+1} = 0$. Then condition (4.1) for $m - k$ forces

$$\frac{W_k}{W_{k-1}} = \operatorname{im}(N^{m-k}|_{H_k}), \qquad \frac{W_{2m-k-1}}{W_{k-1}} = \ker(N^{m-k}|_{H_k}).$$

Since $W_k/W_{k-1} \subset W_{2m-k-1}/W_{k-1}$, we have $W_k \subset W_{2m-k-1}$.

More explicitly property (3) allows us to write:

$$W_k = N^{m-k}(W_{2m-k}), \qquad W_{2m-k-1} = N^{-(m-k)}(W_{k-1}).$$

One verifies easily that this construction satisfies the inductive hypotheses. In particular, we have:

$$N(W_k) = N^{m-k+1}(W_{2m-k}) \subset W_{k-2}.$$

This completes the construction. □

Example 4.2.3. When $N^2 = 0$ and $N \neq 0$, the monodromy weight filtration is of the form

$$0 \subset W_0 = \operatorname{im} N \subset W_1 = \ker N \subset W_2 = H$$

Example 4.2.4. Consider the map $N : \mathbb{Q}^{m+1} \to \mathbb{Q}^{m+1}$, $(x_1, \ldots, x_{m+1}) \mapsto (0, x_1, \ldots, x_m)$ that shifts by one entry and adds a zero in the beginning. It is clear that $N^{m+1} = 0$ and $N^m \neq 0$. The weight filtration has

$$0 \subset W_0 = W_1 \cong \mathbb{Q} \subset W_2 = W_3 \cong \mathbb{Q}^2 \subset \cdots \subset W_{2m-2} = W_{2m-1} \cong \mathbb{Q}^m \subset W_{2m} = \mathbb{Q}^{m+1}$$

with graded pieces $\operatorname{Gr}_k = \mathbb{Q}$ for k even and $\operatorname{Gr}_k = 0$ for k odd.

We now list some fundamental properties of this filtration, stated specifically for the logarithm of the Picard-Lefschetz monodromy operator. Sketches of the proofs for these properties can be found in [4, p. 255].

Proposition 4.2.5. *Let $N = \log T : H^m_{\lim} \to H^m_{\lim}$ be the logarithm of the Picard-Lefschetz monodromy operator and $\{W_k H^m_{\lim}\}$ the monodromy weight filtration constructed as above. Denote*

- $Gr_k H^m_{\lim} = W_k H^m_{\lim}/W_{k-1} H^m_{\lim}$
- $K^m = \ker(N : H^m_{\lim} \to H^m_{\lim})$
- $W_k K^m = W_k H^m_{\lim} \cap K^m$

- $\mathrm{Gr}_k\, K^m = W_k K^m / W_{k-1} K^m$.

Then the following hold:

1. $N(W_k H^m_{\mathrm{lim}}) = (\mathrm{im}\, N) \cap W_{k-2} H^m_{\mathrm{lim}}$

2. *for* $k \leq m$

$$\mathrm{Gr}_k H^m_{\mathrm{lim}} \cong \bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{Gr}_{k-2a} K^m$$

3. *for* $0 \leq k \leq m$*,* $N^k : H^m_{\mathrm{lim}} \to H^m_{\mathrm{lim}}$ *is the zero map if and only if* $W_{m-k} H^m_{\mathrm{lim}} = 0$*; if and only if* $W_{m-k} K^m = 0$

4. $\mathrm{coker}(\mathrm{Gr}_{m+2}\, H^m_{\mathrm{lim}} \xrightarrow{N} \mathrm{Gr}_m\, H^m_{\mathrm{lim}}) = \mathrm{Gr}_m\, H^m_{\mathrm{lim}} / \,\mathrm{im}(\mathrm{Gr}_{m+2}\, H^m_{\mathrm{lim}} \xrightarrow{N} \mathrm{Gr}_m\, H^m_{\mathrm{lim}}) \cong \mathrm{Gr}_m\, K^m$

4.3 The Spectral Weight Filtration

As discussed in Section 3.4, in the case of a strictly semistable degeneration, the *spectral weight filtration* on $H^m := H^m(\mathcal{X}_0, \mathbb{Q})$ takes the form:

$$0 \subset W_0 H^m \subset \cdots \subset W_m H^m = H^m.$$

This filtration induces, via duality, a weight filtration on the homology group $H_m := H_m(\mathcal{X}_0, \mathbb{Q})$, defined by:

$$W_{-k} H_m := \mathrm{Ann}(W_{k-1} H^m) = \{h \in H_m \mid (h, W_{k-1} H^m) = 0\}.$$

Then, the associated graded pieces satisfy:

$$\mathrm{Gr}_{-k}\, H_m \cong (\mathrm{Gr}_k\, H^m)^* \,.$$

With this definition, the graded pieces vanish outside the expected range:

$$\mathrm{Gr}_k\, H^m = 0 \quad \text{for } k < 0 \text{ or } k > m,$$

$$\mathrm{Gr}_k\, H_m = 0 \quad \text{for } k > 0 \text{ or } k < -m.$$

4.4 Weighted Vector Spaces

Definition 4.4.1 (Weighted Vector Space)**.** A *weighted vector space* is a $\mathbb{Q}$-vector space H equipped with an increasing filtration by $\mathbb{Q}$-subspaces:

$$0 \subset W_0 H \subset \cdots \subset W_k H \subset W_{k+1} H \subset \cdots \subset H,$$

called the *weight filtration*.

A *morphism of weighted vector spaces of type* r is a linear map $\phi : H \to H'$ such that

$$\phi(W_k H) = W_{k+2r} H' \cap \mathrm{Im}(\phi).$$

The following spaces carry natural weight filtrations making them weighted vector spaces:

- The cohomology $H^m := H^m(\mathcal{X}_0, \mathbb{Q})$, with the *spectral weight filtration*:

$$0 \subset W_0 H^m \subset \cdots \subset W_m H^m = H^m.$$

- The limit cohomology $H^m_{\text{lim}} := H^m(\mathcal{X}_t, \mathbb{Q})$, with the *monodromy weight filtration*:

$$0 \subset W_0 H^m_{\text{lim}} \subset \cdots \subset W_{2m} H^m_{\text{lim}} = H^m_{\text{lim}}.$$

Remark 4.4.2. By Proposition 4.2.5(1), the logarithm of the monodromy operator $N = \log T$, where T is the Picard–Lefschetz transformation, defines a morphism of weighted vector spaces of type -1 acting on H^m_{lim}.

4.5 The Clemens–Schmid Exact Sequence

To define the Clemens-Schmid exact sequence for a semistable degeneration $\pi : \mathcal{X} \to \Delta$, we begin by constructing several natural maps between (co)homology groups.

First, recall the Lefschetz duality:

Theorem 4.5.1 (Lefschetz Duality)**.** *Let M be an orientable, connected n-dimensional manifold with boundary ∂M, and suppose that the homology and cohomology groups of M are finitely generated. Then, for any abelian group G, there is a natural isomorphism*

$$H_{n-m}(M; G) \cong H^m(M, \partial M; G),$$

for all $0 \leq m \leq n$.

Lefschetz duality applies to the total space $\mathcal{X}$: every complex manifold is orientable, and since the homology and cohomology groups of $\mathcal{X}$ coincide with those of the compact central fiber $\mathcal{X}_0$ (see Proposition 4.1.7), they are finitely generated. Therefore, we obtain an isomorphism between the relative cohomology and homology groups of $\mathcal{X}$, which has complex dimension $n+1$:

$$H_{2n+2-m}(\mathcal{X}; \mathbb{Q}) \cong H^m(\mathcal{X}, \partial\mathcal{X}; \mathbb{Q})$$

By composing with the map to absolute cohomology, appearing in the long exact sequence of the pair $(\mathcal{X}, \partial\mathcal{X})$, we obtain the map:

$$\alpha : H_{2n+2-m}(\mathcal{X}; \mathbb{Q}) \xrightarrow{\text{LD}} H^m(\mathcal{X}, \partial\mathcal{X}; \mathbb{Q}) \to H^m(\mathcal{X}; \mathbb{Q}).$$

The map

$$\beta : H^m(\mathcal{X}_t; \mathbb{Q}) \to H_{2n-m}(\mathcal{X}_t; \mathbb{Q}) \xrightarrow{i_*} H_{2n-m}(\mathcal{X}; \mathbb{Q})$$

is obtained by applying first Poincaré duality on the smooth compact fiber $\mathcal{X}_t$ to obtain an isomorphism

$$H^m(\mathcal{X}_t; \mathbb{Q}) \cong H_{2n-m}(\mathcal{X}_t; \mathbb{Q}),$$

and then composing with the pushforward map in homology i_* induced by the inclusion.

We also have the pullback

$$i^* : H^m(\mathcal{X}, \mathbb{Q}) \to H^m(\mathcal{X}_t, \mathbb{Q})$$

induced by the inclusion.

Theorem 4.5.2. *The maps α, i^*, N, and β are morphisms of weighted vector spaces. In particular,*

- *α is of type $n+1$,*
- *i^* is of type 0,*
- *N is of type -1,*
- *β is of type $-n$.*

Theorem 4.5.3 (Clemens–Schmid Exact Sequence)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration. Then the following sequence of vector spaces is exact:*

$$\cdots \xrightarrow{\beta} H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i^*} H^m_{\text{lim}} \xrightarrow{N} H^m_{\text{lim}} \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \xrightarrow{i^*} \cdots$$

Moreover, if π is strictly semistable, then the sequence is an exact sequence of weighted vector spaces. Consequently, this induces exact sequences on the filtered and graded pieces with respect to the respective weight filtrations.

4.6 Some Consequences

A number of results follow from the Clemens–Schmid exact sequence:

Theorem 4.6.1 (Local Invariant Cycle Theorem)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration. Then the sequence*

$$H^m(\mathcal{X}, \mathbb{Q}) \xrightarrow{i^*} H^m(\mathcal{X}_t, \mathbb{Q}) \xrightarrow{N} H^m(\mathcal{X}_t, \mathbb{Q})$$

is exact, where $i : \mathcal{X}_t \hookrightarrow \mathcal{X}$ denotes the natural inclusion. In other words, every monodromy-invariant class in $H^m(\mathcal{X}_t, \mathbb{Q})$ arises as the restriction of a class in $H^m(\mathcal{X}, \mathbb{Q})$.

Proof. Immediate from Theorem 4.5.3. □

Proposition 4.6.2. *In a semistable degeneration, the number of connected components of the central fiber coincides with that of the generic fiber. In particular, the central fiber is connected if and only if the generic fiber is.*

Proof. Since the central fiber has complex dimension n, $H_{2n+2} = 0$. On the other hand, Theorem 4.2.1 implies that the monodromy transformation T acts trivially on H^0_{lim}, so N vanishes on H^0_{lim}. Applying the Clemens–Schmid exact sequence in degree 0, we obtain:

$$0 \to H^0 \to H^0_{\text{lim}} \to 0,$$

which implies that $H^0 \cong H^0_{\text{lim}}$. □

Proposition 4.6.3. *Let* K^m *denote* $\ker(N : H^m_{\lim} \to H^m_{\lim})$*. Then,* $W_k H^m \cong W_k K^m$ *for* $k < m$*.*

Proof. We apply Theorem 4.5.2 to decompose the Clemens-Schmid exact sequence according to the weight filtration:

$$\cdots \xrightarrow{\beta} W_{k-2n-2}H_{2n+2-m} \xrightarrow{\alpha} W_k H^m \xrightarrow{i^*} W_k H^m_{\lim} \xrightarrow{N} W_{k-2}H^m_{\lim} \xrightarrow{\beta} \cdots$$

By the construction of the filtration on the graded pieces $\mathrm{Gr}_l\, H_s$ we have

$$\mathrm{Gr}_l\, H_s = 0 \quad \text{for } l > 0 \text{ or } l < -s.$$

Since $k < m$ implies $k - 2n - 2 < -(2n + 2 - m)$ we have $\mathrm{Gr}_{k-2n-2}\, H_{2n+2-m} = 0$, and thus, $W_{k-2n-2}H_{2n+2-m} = 0$. This makes the sequence

$$0 \to W_k H^m \xrightarrow{i^*} W_k H^m_{\lim} \xrightarrow{N} W_{k-2}H^m_{\lim}$$

exact. This gives $W_k K^m \cong W_k H^m$. □

Proposition 4.6.4. *For* $k > 0$*,* $N^k : H^m_{\lim} \to H^m_{\lim}$ *is the zero map if and only if* $W_{m-k}H^m = 0$*. In particular,* $N^m = 0$ *if and only if* $H^m(|\Gamma|) = 0$*.*

Proof. Property (3) of the monodromy weight filtration (4.2.5) and the previous proposition imply

$$N^k = 0 \iff W_{m-k}K^m = 0 \iff W_{m-k}H^m = 0$$

The last assertion follows from Proposition 3.5.3. □

Proposition 4.6.5. *The following sequence is exact*

$$0 \to \mathrm{Gr}_{m-2}\, K^{m-2} \to \mathrm{Gr}_{m-2n-2}\, H_{2n+2-m} \to \mathrm{Gr}_m\, H^m \to \mathrm{Gr}_m\, K^m \to 0$$

Proof. We decompose the Clemens-Schmid exact sequence according to the graded pieces:

$$\cdots \xrightarrow{i^*} \mathrm{Gr}_m\, H^{m-2}_{\lim} \xrightarrow{N} \mathrm{Gr}_{m-2}\, H^{m-2}_{\lim} \xrightarrow{\beta} \mathrm{Gr}_{m-2n-2}\, H_{2n+2-m} \xrightarrow{\alpha} \mathrm{Gr}_m\, H^m \xrightarrow{i^*}$$

$$\xrightarrow{i^*} \mathrm{Gr}_m\, H^m_{\lim} \xrightarrow{N} \mathrm{Gr}_{m-2}\, H^m_{\lim} \xrightarrow{\beta} \cdots$$

which simplifies to:

$$0 \to \operatorname{coker} N \xrightarrow{\beta} \mathrm{Gr}_{m-2n-2}\, H_{2n+2-m} \xrightarrow{\alpha} \mathrm{Gr}_m\, H^m \xrightarrow{i^*} \operatorname{im} i^* \to 0.$$

But by property (4) of the monodromy weight filtration $\operatorname{coker} N \cong \mathrm{Gr}_{m-2}\, K^{m-2}$ and

$$\operatorname{im} i^* = \ker(N : \mathrm{Gr}_m\, H^m_{\lim} \to \mathrm{Gr}_{m-2}\, H^m_{\lim}) \cong \mathrm{Gr}_m\, K^m$$

□

4.7 First Cohomology Groups

Let $\pi : \mathcal{X} \to \Delta$ be a strictly semistable degeneration. The fibers are of complex dimension n, and thus have real dimension $2n$. It follows that $H_{2n+1} = 0$ and therefore, the Clemens-Schmid exact sequence in degree 1 simplifies to

$$0 \to H^1 \xrightarrow{i^*} H^1_{\text{lim}} \xrightarrow{N} H^1_{\text{lim}}$$

being exact. In particular, this implies that $K^1 = \ker\left(N : H^1_{\text{lim}} \to H^1_{\text{lim}}\right) \cong H^1$.

Now, applying Proposition 4.2.5 (2), we obtain

$$\operatorname{Gr}_2 H^1_{\text{lim}} \cong \operatorname{Gr}_0 H^1_{\text{lim}} \cong \operatorname{Gr}_0 K^1 \cong \operatorname{Gr}_0 H^1 \cong H^1(|\Gamma|)$$

where $|\Gamma|$ denotes the dual graph of $\mathcal{X}_0$.

Recalling that the weight-graded pieces satisfy $\operatorname{Gr}_k H^m \cong E_2^{k,m-k}$, we also find

$$\operatorname{Gr}_1 H^1_{\text{lim}} \cong \operatorname{Gr}_1 K^1 \cong \operatorname{Gr}_1 H^1 = E_2^{1,0} \cong \ker\left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)})\right)$$

Knowing the dimensions of the graded pieces allows us to reconstruct the total dimension of H^1_{lim}. Letting

$$\Phi := \dim_{\mathbb{Q}} \operatorname{Gr}_1 H^1 = \dim_{\mathbb{Q}} \ker\left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)})\right),$$

we obtain

$$b_1(\mathcal{X}_t) = \dim_{\mathbb{Q}} H^1_{\text{lim}} = \sum_{k=0}^{2} \dim_{\mathbb{Q}} \operatorname{Gr}_k H^1_{\text{lim}} = 2b_1(|\Gamma|) + \Phi$$

Moreover, by Theorem 4.6.4, we get the monodromy criterion:

$$N = 0 \text{ on } H^1_{\text{lim}} \quad \Longleftrightarrow \quad b_1(|\Gamma|) = 0 \quad \Longleftrightarrow \quad b_1(\mathcal{X}_t) = \Phi.$$

Chapter 5

Degeneration of Curves

In this chapter, we aim to provide intuitive descriptions, visual interpretations, and key results rather than detailed proofs or technical developments. The main reference for this chapter is [1, Chapter X].

5.1 Basic Constructions

We now consider the case when the smooth fibers are connected complex projective curves, i.e., connected compact Riemann surfaces. In this case, we see what happens when the central fiber is a *nodal curve*, that is, a connected algebraic curve such that every one of its points is either smooth or a *node*: locally complex-analytically isomorphic to a neighborhood of the origin in the locus with equation $xy = 0$ in $\mathbb{C}^2$.

Example 5.1.1. Figure 5.1 shows a degeneration of a torus to a pinched one.

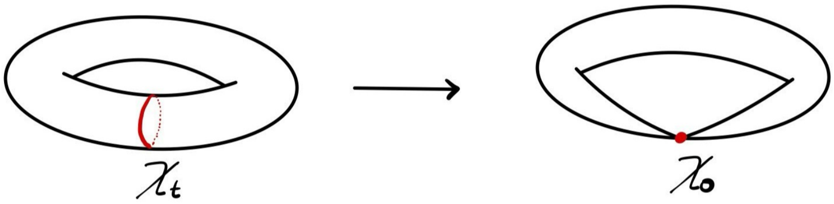


Figure 5.1: Degeneration of a torus

Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of curves. To achieve strict semistability, one resolves the nodes caused by self-intersections of the components via successive blow-ups. Each blow-up essentially replaces a node with an exceptional curve isomorphic to $\mathbb{P}^1$.

Figure 5.2

By generalizing our techniques from the strictly semistable case, we can describe the monodromy T and its logarithm N for semistable degenerations that allow self-intersections within irreducible components.

Let E' denote the finite set of nodes on $\mathcal{X}_0$ that arise from self-intersections of individual irreducible components, and let $E = \{p_1, \ldots, p_e\}$ be the set of all nodes of $\mathcal{X}_0$. Denote by

$$\mu : \mathrm{Bl}_{E'}(\mathcal{X}) \to \mathcal{X}$$

the blow-up of $\mathcal{X}$ at the points in E', and by

$$\nu : \widetilde{\mathcal{X}_0} \to \mathcal{X}_0$$

the *normalization map.* Write $\{r_i, s_i\} = \nu^{-1}(p_i)$ for the pair of points lying over each node p_i. Geometrically, this amounts to constructing a new space $\widetilde{\mathcal{X}_0}$ from $\mathcal{X}_0$ in which the nodes have been "untangled": instead of two branches meeting at a single point, each node is replaced by two distinct smooth points, one on each branch (see Figure 5.7). The normalization decomposes as $\widetilde{\mathcal{X}_0} = \bigsqcup \widetilde{X}_i$, where each $\widetilde{X}_i$ is the normalization of the irreducible component X_i.

The central fiber $\mathcal{X}_0$ can be reconstructed by starting with the disjoint union of the normalizations $\widetilde{X}_j$ of its irreducible components, for $j = 1, \ldots, v$, and then identifying pairs of points $r_{i_k} \sim s_{i_k}$ resulting into self-intersections within the components.

The definition of the dual graph Γ must be adjusted: each vertex corresponds to an irreducible component of $\mathcal{X}_0$, and each edge corresponds to a node, regardless of whether the node arises from the intersection of two distinct components or from a self-intersection of a single component. Observe that the dual graph is connected, since $\mathcal{X}_0$ is.

Example 5.1.2. Figure 5.3 shows a nodal curve and its dual graph.

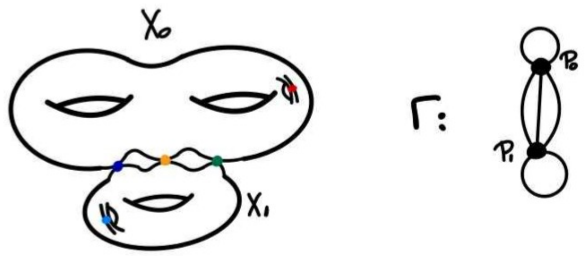


Figure 5.3: The dual graph of a nodal curve.

Proposition 5.1.3. *With the notation above, let* Γ_{Bl} *be the dual graph of* $\mathrm{Bl}_{E'}(\mathcal{X})$ *and* Γ *be the generalized dual graph of* $\mathcal{X}_0$*. Then,* μ *induces a homeomorphism*

$$|\Gamma| \xrightarrow{\sim} |\Gamma_{\mathrm{Bl}}|$$

Proof. We will show that a blow-up at a node arising from a self-intersection of a component of the central fiber does not change the topology of the dual graph. In particular, we will show that a self loop in the dual graph becomes a cycle with two vertices and two edges.

Let $p \in E'$ be a node on a component X of the central fiber $\mathcal{X}_0 \subset \mathcal{X}$, and consider the blow-up of $\mathcal{X}$ at p,

$$\mu : \mathrm{Bl}_p(\mathcal{X}) \to \mathcal{X}.$$

This process replaces the point p with a copy of $\mathbb{P}^1$, known as the *exceptional divisor*, denoted by $S = \mu^{-1}(p) \cong \mathbb{P}^1$. Intuitively, the blow-up separates the tangent directions at p, resolving the self-intersection. The map μ is a biholomorphism away from the exceptional divisor:

$$\mu : \mathrm{Bl}_p(\mathcal{X}) \setminus S \xrightarrow{\sim} \mathcal{X} \setminus \{p\}$$

so the geometry of $\mathcal{X}$ is unchanged outside of p.

Now, consider the *strict transform* X' of the component X under the blow-up. It is obtained by taking the closure of $X \setminus \{p\}$ under the inverse image of μ. The strict transform intersects the exceptional divisor S in two distinct points, say Q and R, which correspond to the two branches of X meeting at p prior to the blow-up. These points reflect the distinct tangent directions at the node. See Figures 5.2 and 5.4 for a visual reference.

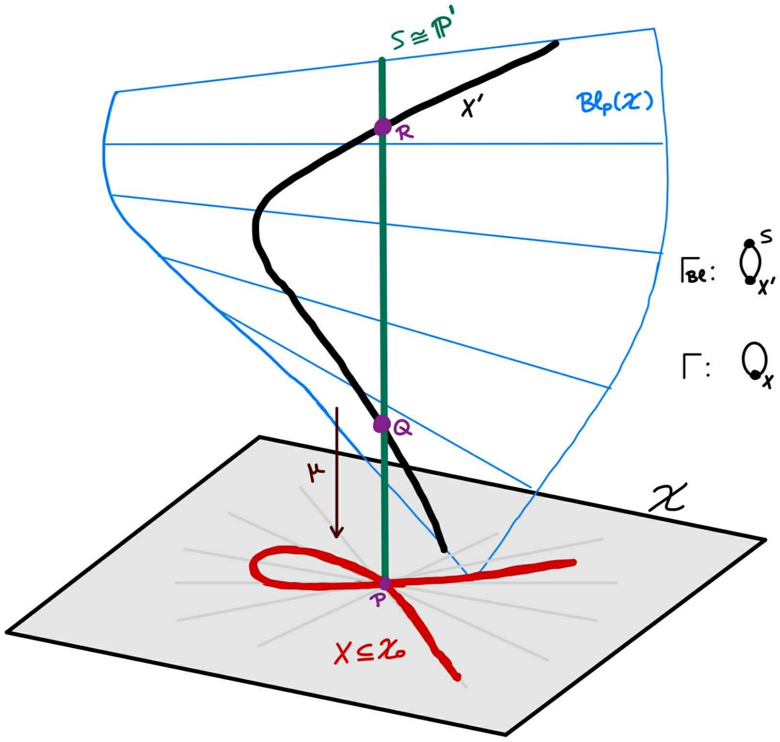


Figure 5.4: The blow up of $\mathcal{X}$ at p

Topologically, this means that in the dual graph of $\mathrm{Bl}_p(\mathcal{X})$, the single vertex corresponding to X (with a self-loop edge at p) is replaced by two vertices: one for X' and one for S, connected by two edges corresponding to the intersections at Q and R. The resulting graph is a 2-cycle: two vertices joined by two edges. Its topological realization is still a circle, hence homeomorphic to a self-loop.

Repeating this for each $p \in E'$ shows that every self-loop in Γ is replaced in Γ_{Bl} by a 2-cycle. Since the rest of the graph remains unchanged, the topological realizations of Γ and Γ_{Bl} are homeomorphic. □

5.2 The Dehn Twist and the Picard-Lefschetz Transformation

We now introduce the notion of a *Dehn twist*. Informally, the Dehn twist δ_c about a smooth, simple closed curve c on an oriented smooth surface S is a homeomorphism constructed as follows: first, orient c and cut the surface S along c; then rotate the right edge of c by 180° counterclockwise and the left edge by 180° clockwise; finally, glue the edges back together. Precise construction of the Dehn Twist can be found in [1, pp. 145-146].

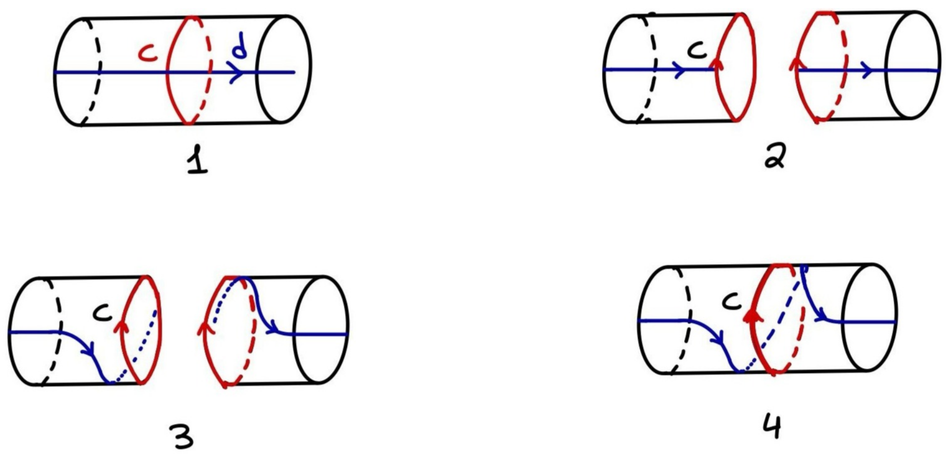


Figure 5.5: The Dehn Twist

It is easy to see that on homology, the induced map $\delta_{c*} : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$ is given by

$$\delta_{c*}(d) = d + (d \cdot c)c,$$

where $d \cdot c$ denotes the intersection number of the cycles d and c.

Proposition 4.1.7 says that the central fiber $\mathcal{X}_0$ is a deformation retract of the total space $\mathcal{X}$. Composing this deformation retraction with the inclusion of the nearby smooth fiber $\mathcal{X}_t \hookrightarrow \mathcal{X}$, we obtain a continuous map

$$r_t : \mathcal{X}_t \to \mathcal{X}_0.$$

By choosing local analytic coordinates near a node of $\mathcal{X}_0$, the total space $\mathcal{X}$ can be locally modeled by the family of curves $\{xy = t\} \subset \mathbb{C}^2 \times \Delta$, whose central fiber at $t = 0$ is the nodal curve $xy = 0$. For small t, the fiber $\mathcal{X}_t$ is smooth and topologically a surface with a small neck, while $\mathcal{X}_0$ has a node at the origin (see 5.6).

In this model, the deformation retraction collapses the vanishing neck in $\mathcal{X}_t$ to the node in $\mathcal{X}_0$, and the map r_t can be constructed so that it is a homeomorphism away from a single embedded circle $c \subset \mathcal{X}_t$, which is collapsed to the node. This circle c is known as the *vanishing cycle*. Its name reflects the fact that it "vanishes" in the limit as $t \to 0$.

It can be shown that the Picard-Lefschetz representation

$$T_* : \pi_1(\Delta^*, t) \to \mathrm{Aut}(H_1(\mathcal{X}_t, \mathbb{Z}))$$

sends a generator of $\pi_1(\Delta^*, t) \cong \mathbb{Z}$ to the automorphism induced by the Dehn twist δ_c along the vanishing cycle $c \subset \mathcal{X}_t$. A detailed proof of this fact can be found in [1, pp. 148-149].

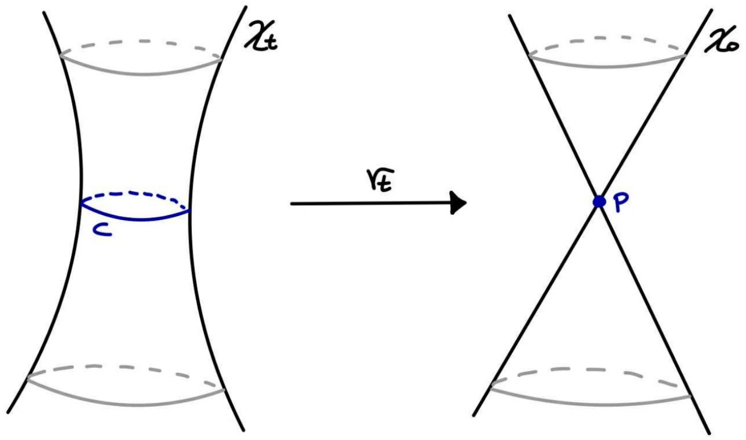


Figure 5.6: The family $\{xy = t\} \subset \mathbb{C}^2 \times \Delta$

Globally, the smooth fiber $\mathcal{X}_t$ may acquire multiple nodes in the central fiber $\mathcal{X}_0$, and each of these corresponds to a distinct vanishing cycle $c_i \subset \mathcal{X}_t$. These cycles can be realized as smoothly embedded circles that collapse to the nodes as $t \to 0$, and they generate a distinguished set of homology classes in $H_1(\mathcal{X}_t, \mathbb{Z})$. In this case, the Dehn twist can be generalized:

Theorem 5.2.1 (Picard-Lefschetz Formula)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of curves and let $\{c_i\} \subset H_1(\mathcal{X}_t, \mathbb{Z})$ be the set of vanishing cycles associated with π. Then for any class $d \in H_1(\mathcal{X}_t, \mathbb{Z})$, the monodromy action is given by*

$$T(d) = d + \sum_i (d \cdot c_i) c_i,$$

where $d \cdot c_i$ denotes the topological intersection number.

This formula shows that the monodromy acts trivially on cycles disjoint from all vanishing cycles, and twists those intersecting c_i by adding integer multiples of c_i, analogous to the effect of a right-handed Dehn twist in surface topology.

Since $(T - I)^2 = 0$ on the dual space H^1_{lim}, the logarithm of the monodromy, which now simplifies as $N = T - I$, satisfies

$$N(d^*) = \sum_{i=1}^{k} (d \cdot \gamma_i) \gamma_i^*.$$

For simplicity, we omit the star and denote cohomology classes by the same symbol.

Example 5.2.2. Consider a torus X, with $H_1(X, \mathbb{Z}) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. Let α be the loop that is contracted like Figure 5.1. Then, under the logarithm N of the Picard-Lefschetz transformation on the dual space H^1_{lim} we have:

$$\alpha \mapsto 0, \qquad \beta \mapsto \alpha.$$

5.3 Constructing the Generic Fiber

Since the degeneration acts by 'collapsing' certain embedded circles - namely, the vanishing cycles c_i - to nodes p_i in the central fiber, one can heuristically think of the generic fiber $\mathcal{X}_t$ as being obtained by reversing this process. It is made explicit in [1, pp. 149-152].

We begin by performing the *real oriented blow-up* of $\widetilde{\mathcal{X}_0}$ along the set

$$\widetilde{E} = \{r_1, s_1, \ldots, r_e, s_e\},$$

which consists of the preimages of the nodes of $\mathcal{X}_0$ under the normalization map ν. This yields a surface $\mathrm{Bl}_{\widetilde{E}}(\widetilde{\mathcal{X}_0})$, and we denote the blow-up map by

$$\tau : \mathrm{Bl}_{\widetilde{E}}(\widetilde{\mathcal{X}_0}) \to \widetilde{\mathcal{X}_0}.$$

The blow-up replaces each point of $\widetilde{E}$ with a copy of S^1, resulting in a (possibly disconnected) Riemann surface with boundary.

Next, we construct a new surface Σ by gluing each pair of boundary circles $\tau^{-1}(r_i)$ and $\tau^{-1}(s_i)$ for $i = 1, \ldots, e$, to form a family of embedded circles γ_i. To ensure that the resulting surface is oriented, each gluing is performed so that the orientation of $\tau^{-1}(r_i)$ is identified with the opposite orientation of $\tau^{-1}(s_i)$. The outcome is a connected, compact Riemann surface.

Let

$$h : \mathrm{Bl}_{\widetilde{E}}(\widetilde{\mathcal{X}_0}) \to \Sigma$$

denote the quotient map, and let

$$\xi : \Sigma \to \mathcal{X}_0$$

be the map that collapses each γ_i to the corresponding node p_i. One can regard Σ as the generic fiber $\mathcal{X}_t$, and the circles $\gamma_i \subset \Sigma$ as the vanishing cycles of the degeneration.

Example 5.3.1. Figure 5.7 shows the normalization, the real blow up, and the associated curve Σ for a specific example of a central fiber.

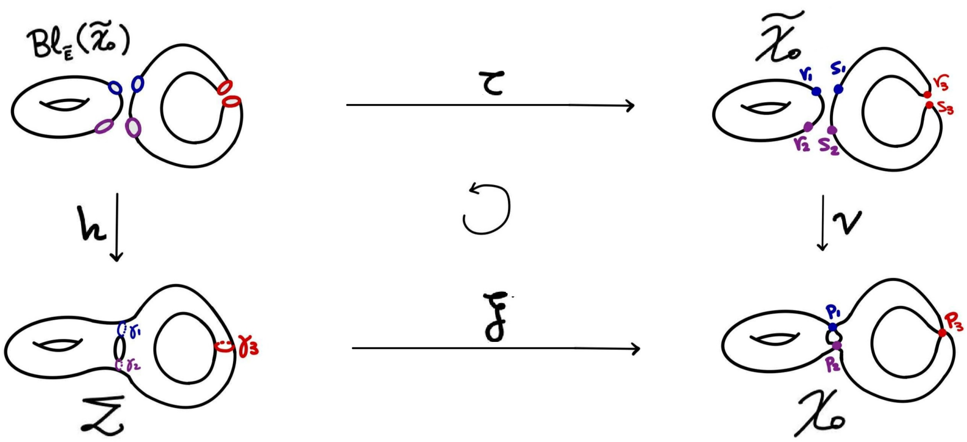


Figure 5.7: Construction of Σ

5.4 Computing the (Co)homology of $\mathcal{X}_0$

In this section, we present key results that will help us compute the (co)homology of the central fiber, using the tools developed in the previous sections. For this section, all (co)homology groups are taken with rational coefficients, unless otherwise specified.

As before, let $\nu\colon \widetilde{\mathcal{X}_0} \to \mathcal{X}_0$ denote the normalization of $\mathcal{X}_0$. Let V, with $\#V = v$, be the set of irreducible components of $\mathcal{X}_0$, and let $E = \{p_1, \dots, p_e\}$ be the set of nodes of $\mathcal{X}_0$. Let $\widetilde{E} = \{r_1, s_1, \dots, r_e, s_e\}$ denote the preimages of the nodes under ν, so that $\nu^{-1}(p_i) = \{r_i, s_i\}$. Recall that the restriction $\nu\colon \widetilde{\mathcal{X}_0} \setminus \widetilde{E} \xrightarrow{\sim} \mathcal{X}_0 \setminus E$ is a biholomorphism. From comparing the long exact sequences in homology for the pairs $(\mathcal{X}_0, E)$ and $(\widetilde{\mathcal{X}_0}, \widetilde{E})$:

$$\begin{array}{ccccccccccc} 0 \longrightarrow & H_1(\mathcal{X}_0) & \longrightarrow & H_1(\mathcal{X}_0, E) & \longrightarrow & H_0(E) & \longrightarrow & H_0(\mathcal{X}_0) & \longrightarrow & H_0(\mathcal{X}_0, E) & \longrightarrow 0 \\ & & & \| & & & & & & \| & \\ 0 \longrightarrow & H_1(\widetilde{\mathcal{X}_0}) & \longrightarrow & H_1(\widetilde{\mathcal{X}_0}, \widetilde{E}) & \longrightarrow & H_0(\widetilde{E}) & \longrightarrow & H_0(\widetilde{\mathcal{X}_0}) & \longrightarrow & H_0(\widetilde{\mathcal{X}_0}, \widetilde{E}) & \longrightarrow 0 \end{array}$$

one can get

$$b_1(\mathcal{X}_0) + e - 1 = b_1(\widetilde{\mathcal{X}_0}) + 2e - v$$

or

$$b_1(\mathcal{X}_0) = b_1(\widetilde{\mathcal{X}_0}) + e - v + 1 = b_1(\widetilde{\mathcal{X}_0}) + b_1(|\Gamma|).$$

In fact, something stronger holds:

Proposition 5.4.1. *With the previous notation, the following short exact sequence holds:*

$$0 \longrightarrow H^1(|\Gamma|, \mathbb{Z}) \longrightarrow H^1(\mathcal{X}_0, \mathbb{Z}) \xrightarrow{\nu^*} H^1(\widetilde{\mathcal{X}_0}, \mathbb{Z}) \longrightarrow 0.$$

Proof. Let $\mathbb{Z}_{X_0}$ and $\mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ be the constant sheaves with values in $\mathbb{Z}$ on $\mathcal{X}_0$ and $\widetilde{\mathcal{X}_0}$ respectively. There is a natural map of sheaves $\mathbb{Z}_{X_0} \to \nu_* \mathbb{Z}_{\widetilde{\mathcal{X}_0}}$, and we claim that this fits into a short exact sequence of sheaves on X_0:

$$0 \longrightarrow \mathbb{Z}_{X_0} \longrightarrow \nu_* \mathbb{Z}_{\widetilde{\mathcal{X}_0}} \longrightarrow Q \longrightarrow 0, \tag{5.1}$$

where Q is a skyscraper sheaf supported on E, with stalks isomorphic to $\mathbb{Z}$.

We check exactness on stalks: For a point $p \in X_0$, the stalk of $\nu_* \mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ at p is the direct sum of the stalks of $\mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ at the preimages $\nu^{-1}(p)$. If $p \in X_0$ is a smooth point (i.e., not a node), then ν is a local isomorphism near p, and $\nu^{-1}(p)$ consists of a single point. Hence in this case the map $\mathbb{Z}_{X_0,p} \to (\nu_* \mathbb{Z}_{\widetilde{\mathcal{X}_0}})_p$ is an isomorphism, and the cokernel vanishes. If $p \in X_0$ is a node, then $\nu^{-1}(p) = \{r, s\}$ and the sequence becomes isomorphic to the sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

where the first nonzero arrow is given by $\mathbb{Z} \ni a \mapsto (a, a)$ and the second nonzero arrow by $\mathbb{Z}^2 \ni (a, b) \mapsto b - a$.

We now pass to cohomology. Since skyscraper sheaves have vanishing higher cohomology, the long exact sequence in cohomology associated to (5.1) is

$$0 \longrightarrow H^0(X_0, \mathbb{Z}_{\mathcal{X}_0}) \longrightarrow H^0(\mathcal{X}_0, \nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \longrightarrow H^0(Q) \longrightarrow H^1(X_0, \mathbb{Z}_{\mathcal{X}_0}) \longrightarrow H^1(\widetilde{\mathcal{X}_0}, \nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \longrightarrow 0.$$

We claim that

$$H^i(\mathcal{X}_0, \nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \cong H^i(\widetilde{\mathcal{X}_0}, \mathbb{Z}),$$

for $i = 0, 1$. To prove this, we introduce the *Leray spectral sequence* for sheaf cohomology:

$$E_2^{p,q} = H^p(\mathcal{X}_0, R^q\nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \Longrightarrow H^{p+q}(\widetilde{\mathcal{X}_0}, \mathbb{Z}_{\widetilde{\mathcal{X}_0}}).$$

The constant sheaf $\mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ is flasque and the direct image of a flasque sheaf under a continuous map remains flasque. Hence, $\nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ is also a flasque sheaf on $\mathcal{X}_0$, and so all higher direct images $R^q\nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}$ vanish for $q > 0$. Therefore, the only nonzero terms on the E_2-page of the Leray spectral sequence lie in the $q = 0$ row, and we obtain

$$H^p(\mathcal{X}_0, \nu_*\mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \cong H^p(\widetilde{\mathcal{X}_0}, \mathbb{Z}) \quad \text{for all } p \geq 0.$$

We now identify each term:

- $H^0(\mathcal{X}_0, \mathbb{Z}_{\mathcal{X}_0}) \cong \mathbb{Z}$, since $\mathcal{X}_0$ is connected.
- $H^0(\widetilde{\mathcal{X}_0}, \mathbb{Z}_{\widetilde{\mathcal{X}_0}}) \cong \mathrm{Hom}(V, \mathbb{Z}) = \mathbb{Z}^V$.
- $H^0(Q) \cong \mathrm{Hom}(E, \mathbb{Z}) = \mathbb{Z}^E$.

Thus we arrive at the exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^V \xrightarrow{\delta} \mathbb{Z}^E \longrightarrow H^1(X_0, \mathbb{Z}) \longrightarrow H^1(\widetilde{\mathcal{X}_0}, \mathbb{Z}) \longrightarrow 0. \tag{5.2}$$

We now show that $\mathrm{coker}(\delta) \cong H^1(|\Gamma|, \mathbb{Z})$. Let $C_0(\Gamma)$ and $C_1(\Gamma)$ denote the sets of vertices and edges of the dual graph Γ, respectively. As before, we identify the set of vertices with the set V of irreducible components of $\mathcal{X}_0$, and the set of edges with the set E of nodes.

To describe the map $\delta : \mathbb{Z}^V \to \mathbb{Z}^E$ more explicitly, we give an orientation to Γ, that is, we assign a direction to each edge $e \in E$, specifying a source vertex $e^- \in V$ and a target vertex $e^+ \in V$, such that the map δ sends a function $f : V \to \mathbb{Z}$ to the function $\delta(f) : E \to \mathbb{Z}$ defined by

$$\delta(f)(e) = f(e^+) - f(e^-).$$

In other words, δ is the dual of the boundary map $\partial : C_1(\Gamma) \longrightarrow C_0(\Gamma)$, which sends each edge e to $\partial(e) = e^+ - e^-$. The cochain complex $\mathbb{Z}^V \xrightarrow{\delta} \mathbb{Z}^E$ then computes the simplicial cohomology of the dual graph. In particular, $\ker(\delta) \cong H^0(|\Gamma|, \mathbb{Z}) \cong \mathbb{Z}$, and $\mathrm{coker}(\delta) \cong H^1(|\Gamma|, \mathbb{Z})$. □

Another useful result is the following:

Proposition 5.4.2 ([1, pp. 160]). *The sequence*

$$0 \to H^1(|\Gamma|, \mathbb{Z}) \to H_1(\mathcal{X}_t, \mathbb{Z}) \xrightarrow{r_*} H_1(\mathcal{X}_0, \mathbb{Z}) \to 0,$$

with r_* induced by the inclusion $\mathcal{X}_t \hookrightarrow \mathcal{X}$ followed by the retraction $\mathcal{X} \to \mathcal{X}_0$, is exact.

Combining the previous two propositions we get the following:

Theorem 5.4.3. *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of curves and let $\widetilde{\mathcal{X}_0} = \bigsqcup_i \widetilde{X}_i$ be the normalization of the central fiber $\mathcal{X}_0$, where each $\widetilde{X}_i$ is the normalization of the irreducible component X_i of $\mathcal{X}_0$. Then*

$$g(\mathcal{X}_t) = b_1(|\Gamma|) + g(\widetilde{\mathcal{X}_0}) = b_1(|\Gamma|) + \sum_i g(\widetilde{X}_i)$$

Remark 5.4.4. In a strictly semistable degeneration of curves, the components of the central fiber are smooth and thus $\widetilde{X_i} \cong X_i$. Also, since $H^1(\mathcal{X}_0^{(1)}) = 0$,

$$\Phi = \dim_{\mathbb{Q}} H^1(\mathcal{X}_0^{(0)}) = 2\sum_i g(X_i).$$

In this case, using the results in Section 4.7, we have that

$$g(\mathcal{X}_t) = b_1(\Gamma) + \sum_i g(X_i)$$

Theorem 5.4.3 is a generalization of this fact.

Corollary 5.4.5. *We obtain the monodromy criterion for a semistable degeneration of curves:*

$$N = 0 \text{ on } H^1_{\text{lim}} \iff b_1(|\Gamma|) = 0 \iff g(\mathcal{X}_t) = \sum_j g(\widetilde{X}_j).$$

Example 5.4.6. Consider the degeneration with central fiber as in Figure 5.8.

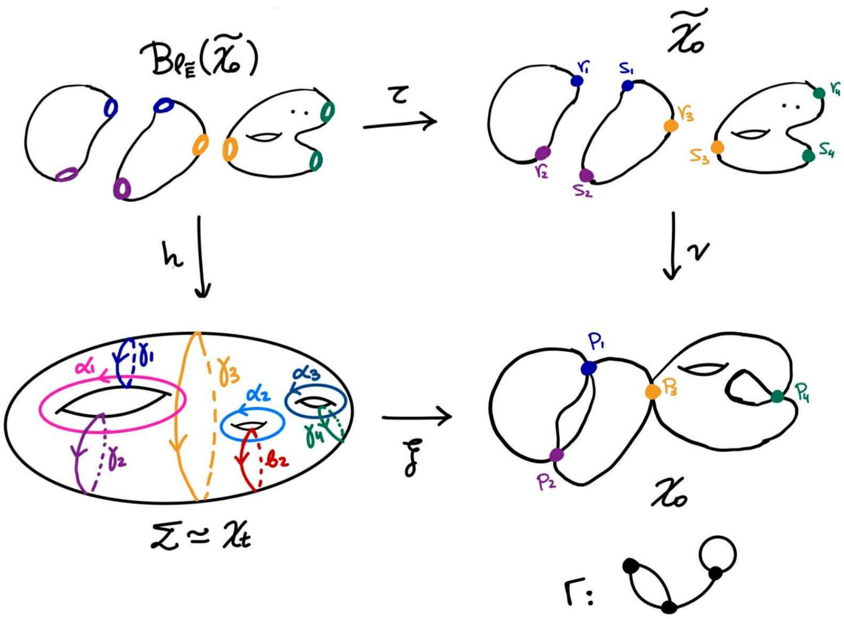


Figure 5.8

From the figure, we find $b_1(\mathcal{X}_t) = 6$, $b_1(|\Gamma|) = 2$ and $b_1(\widetilde{\mathcal{X}_0}) = 2$. Either of Theorem 5.4.1 or Theorem 5.4.2 give $b_1(\mathcal{X}_0) = 4$.

We will explicity compute N. Take $\alpha_1, \gamma_1, \alpha_2, \beta_2, \alpha_3, \gamma_4$ as the generators of $H_1(\mathcal{X}_t)$. The vanishing cycles are $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. As elements in $H^1_{\lim}$ we have the relations

$$\gamma_2 = -\gamma_1 \qquad \text{and} \qquad \gamma_3 = 0$$

From Theorem 5.2.1, the Picard-Lefschetz transformation acts on the generators as

- $\alpha_1 \mapsto \alpha_1 - \gamma_1 + \gamma_2 = \alpha_1 - 2\gamma_1$
- $\gamma_1 \mapsto \gamma_1$
- $\alpha_2 \mapsto \alpha_2$
- $\beta_2 \mapsto \beta_2$
- $\alpha_3 \mapsto \alpha_3 + \gamma_4$
- $\gamma_4 \mapsto \gamma_4$

And thus $\dim_{\mathbb{Q}} K^1 = \dim_{\mathbb{Q}} \ker N = \dim_{\mathbb{Q}} \ker (T - I) = 4$ as expected from Proposition 4.6.1. The monodromy weight filtration of $H^1_{\lim} \cong \mathbb{Q}^6$ is

$$0 \subset W_0 = \mathbb{Q}^2 \subset W_1 = \mathbb{Q}^4 \subset W_2 = \mathbb{Q}^6$$

Proposition 5.4.7. *Let $\pi\colon \mathcal{X} \to \Delta$ be a semistable degeneration curves. Then the generic fiber $\mathcal{X}_t$ is isomorphic to $\mathbb{P}^1$ if and only if the central fiber is a* tree of $\mathbb{P}^1$'s, *that is:*

- *each irreducible component of $\mathcal{X}_0$ is isomorphic to $\mathbb{P}^1$,*
- *the dual graph Γ of $\mathcal{X}_0$ is connected and acyclic (i.e., a tree: no cycles or self-loops).*

Proof. To see this, first note that $\mathcal{X}_t$ is connected if and only if $\mathcal{X}_0$ is connected, which in turn holds if and only if the dual graph Γ is connected (Proposition 4.6.2).

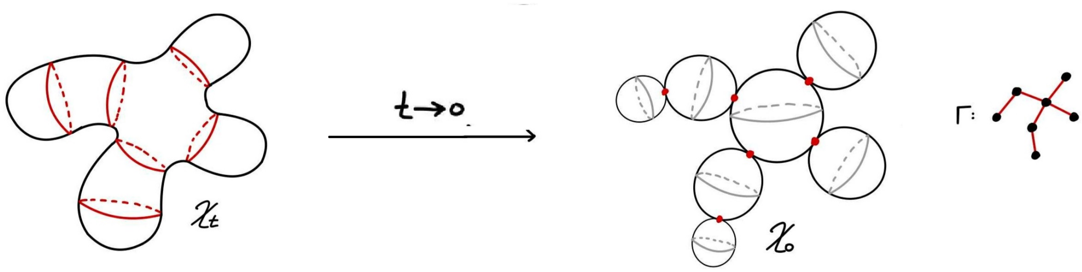
Second, recall that a connected graph with v vertices is a tree if and only if it has exactly $v - 1$ edges, which is equivalent to having first Betti number $b_1(|\Gamma|) = 0$.

Third, if the dual graph has no self-loops (as is the case for a tree), then the components of $\mathcal{X}_0$ are smooth, and hence each component is isomorphic to its own normalization: $\widetilde{X}_i \cong X_i$.

Applying Theorem 5.4.3, we compute the genus of the generic fiber:

$$g(\mathcal{X}_t) = b_1(|\Gamma|) + \sum_i g(\widetilde{X}_i).$$

If $\mathcal{X}_t \cong \mathbb{P}^1$, then $g(\mathcal{X}_t) = 0$, which forces both $b_1(|\Gamma|) = 0$ and $g(\widetilde{X}_i) = 0$ for all i. In other words, Γ is a tree, and each $\widetilde{X}_i \cong X_i \cong \mathbb{P}^1$, since smooth connected compact curves of genus zero are isomorphic to $\mathbb{P}^1$. Conversely, if $\mathcal{X}_0$ is a tree of $\mathbb{P}^1$'s, then the same formula shows that $g(\mathcal{X}_t) = 0$. With the same argument as before, $\mathcal{X}_t \cong \mathbb{P}^1$.



□

Chapter 6

Degenerations of Surfaces

In this chapter, we study semistable degenerations of complex surfaces. We start with some general computations using the Clemens-Schmid exact sequence, which help us express the graded pieces of H^2_{lim} in terms of the geometry of the central fiber and its dual complex. From this, we extract some useful criteria for when the monodromy is zero. To see these ideas in action, we look at degenerations of $K3$ surfaces, and abelian surfaces.

6.1 First Computations

Recall that $\mathcal{X}_0^{(0)}$ is a disjoint union of complex surfaces (see 2.8), and $\mathcal{X}_0^{(1)}$ is a disjoint union of complex curves (see 2.7) called the *double curves*.

We define

- $\Phi = \dim_{\mathbb{Q}} \ker \left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)}) \right) = \dim_{\mathbb{Q}} \operatorname{Gr}_1 H^1$.
- $q = \frac{1}{2} b_1(\mathcal{X}_0^{(0)}) = \sum_i q(X_i)$, the sum of the irregularities of the components.
- $g = \frac{1}{2} b_1(\mathcal{X}_0^{(1)}) = \sum_{i<j} g(X_{ij})$, the sum of the genera of the double curves.

Proposition 6.1.1. *In a semistable degeneration of surfaces, the dimensions of the graded pieces of H^2_{lim} are given by:*

- $\dim_{\mathbb{Q}} \operatorname{Gr}_0 H^2_{\text{lim}} = \dim_{\mathbb{Q}} \operatorname{Gr}_4 H^2_{\text{lim}} = b_2(|\Gamma|)$
- $\dim_{\mathbb{Q}} \operatorname{Gr}_1 H^2_{\text{lim}} = \dim_{\mathbb{Q}} \operatorname{Gr}_3 H^2_{\text{lim}} = \Phi - 2q + 2g$

Consequently,

$$\dim_{\mathbb{Q}} \operatorname{Gr}_2 H^2_{\text{lim}} = b_2(\mathcal{X}_t) - 2\Phi - 4g + 4q - 2b_2(|\Gamma|).$$

Independently of the Betti number of the generic fiber,

$$\dim_{\mathbb{Q}} \operatorname{Gr}_2 H^2_{\text{lim}} = b_0(|\Gamma|) + b_2(|\Gamma|) - \#\{X_i\} + \dim_{\mathbb{Q}} \ker \left(H^2(\mathcal{X}_0^{(0)}) \to H^2(\mathcal{X}_0^{(1)}) \right).$$

Proof. By Propositions 4.2.5(2) and 4.6.3, we have

$$\operatorname{Gr}_0 H^2_{\text{lim}} \cong \operatorname{Gr}_0 K^2 \cong \operatorname{Gr}_0 H^2 \cong H^2(|\Gamma|),$$

so that

$$\dim_{\mathbb{Q}} \mathrm{Gr}_0 H^2_{\lim} = b_2(|\Gamma|).$$

Similarly, the first graded piece is given by

$$\mathrm{Gr}_1 H^2_{\lim} \cong \mathrm{Gr}_1 K^2 \cong \mathrm{Gr}_1 H^2 \cong E_2^{1,1} = \frac{H^1(\mathcal{X}_0^{(1)})}{\mathrm{im}\left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)})\right)}.$$

Using the identity

$$\dim_{\mathbb{Q}} \mathrm{im}\left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)})\right) = \dim_{\mathbb{Q}} H^1(\mathcal{X}_0^{(0)}) - \dim_{\mathbb{Q}} \ker\left(H^1(\mathcal{X}_0^{(0)}) \to H^1(\mathcal{X}_0^{(1)})\right),$$

we deduce:

$$\dim_{\mathbb{Q}} \mathrm{Gr}_1 H^2_{\lim} = b_1(\mathcal{X}_0^{(1)}) - \left(b_1(\mathcal{X}_0^{(0)}) - \Phi\right) = \Phi - 2q + 2g.$$

To compute $\mathrm{Gr}_2 H^2_{\lim}$, we use the identity:

$$b_2(\mathcal{X}_t) = \dim \mathrm{Gr}_2 H^2_{\lim} + 2 \dim \mathrm{Gr}_1 H^2_{\lim} + 2 \dim \mathrm{Gr}_0 H^2_{\lim},$$

so that

$$\dim_{\mathbb{Q}} \mathrm{Gr}_2 H^2_{\lim} = b_2(\mathcal{X}_t) - 2(\Phi - 2q + 2g) - 2b_2(|\Gamma|).$$

We now derive a formula for $\mathrm{Gr}_2 H^2_{\lim}$ independent of $b_2(\mathcal{X}_t)$. By Poincaré duality, we have:

$$\mathrm{Gr}_{-4} H_4 = (\mathrm{Gr}_4 H^4)^* = (E_2^{4,0})^* = H^4(\mathcal{X}_0^{(0)})^* \cong H_0(\mathcal{X}_0^{(0)})^*,$$

so

$$\dim_{\mathbb{Q}} \mathrm{Gr}_{-4} H_4 = \#\{X_i\},$$

the number of irreducible components. We also have:

$$\mathrm{Gr}_2 H^2 = E_2^{2,0} = \ker\left(H^2(\mathcal{X}_0^{(0)}) \to H^2(\mathcal{X}_0^{(1)})\right).$$

From Proposition 4.6.5, we have the exact sequence:

$$0 \to \mathrm{Gr}_0 K^0 \to \mathrm{Gr}_{-4} H_4 \to \mathrm{Gr}_2 H^2 \to \mathrm{Gr}_2 K^2 \to 0.$$

Taking dimensions and using $\dim_{\mathbb{Q}} \mathrm{Gr}_0 K^0 = b_0(|\Gamma|)$, we get:

$$\dim_{\mathbb{Q}} \mathrm{Gr}_2 K^2 = b_0(|\Gamma|) - \#\{X_i\} + \dim_{\mathbb{Q}} \ker\left(H^2(\mathcal{X}_0^{(0)}) \to H^2(\mathcal{X}_0^{(1)})\right).$$

Finally, since $\mathrm{Gr}_2 H^2_{\lim} = \mathrm{Gr}_2 K^2 + \mathrm{Gr}_0 K^2$ by Proposition 4.2.5(2), and $\mathrm{Gr}_0 K^2 = H^2(|\Gamma|)$, we conclude:

$$\dim_{\mathbb{Q}} \mathrm{Gr}_2 H^2_{\lim} = b_0(|\Gamma|) + b_2(|\Gamma|) - \#\{X_i\} + \dim_{\mathbb{Q}} \ker\left(H^2(\mathcal{X}_0^{(0)}) \to H^2(\mathcal{X}_0^{(1)})\right).$$

□

Theorem 6.1.2 (Monodromy Criteria for Surfaces)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of surfaces. Then:*

$$N = 0 \text{ on } H^1_{\lim} \iff b_1(|\Gamma|) = 0$$

$$N^2 = 0 \text{ on } H^2_{\lim} \iff b_2(|\Gamma|) = 0$$

$$N = 0 \text{ on } H^2_{\lim} \iff b_2(|\Gamma|) = 0 \text{ and } \Phi + 2g = 2q$$

Proof. The first two statements are immediate from Proposition 4.6.4. For the last one, recall that the monodromy weight filtration on $H^2_{\lim}$ is

$$0 \subset W_0 = \operatorname{im} N^2 \subset W_1 = N(W_3) \subset W_2 = N^{-1}(W_0) \subset W_3 = \ker N^2 \subset H^2_{\lim}$$

This gives $N = 0$ on $H^2_{\lim}$ if and only if $N^2 = 0$ and $\dim_{\mathbb{Q}} \mathrm{Gr}_1 = 0$. □

6.2 $K3$ Surfaces

We now apply the preceding results to study the monodromy weight filtration of a degeneration of projective $K3$ surfaces, making use of the results established in Sections 2.8 and 2.9.

The following classification theorem will be fundamental:

Theorem 6.2.1 ([8], Theorem II)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of projective $K3$ surfaces. Then π is a modification of a semistable degeneration whose central fiber $\mathcal{X}_0$ is of one of the following types:*

Type (1) *$\mathcal{X}_0$ is a $K3$ surface.*

Type (2) *$\mathcal{X}_0 = X_0 \cup X_1 \cup \cdots \cup X_{k+1}$, where:*

- *X_0 and X_{k+1} are smooth rational surfaces,*
- *For $1 \le i \le k$, each X_i is a ruled surface over an elliptic curve,*
- *For $1 \le i \le k$, each X_i intersects only $X_{i\pm1}$ in a smooth elliptic curve, a section of the ruling.*

Type (3) *$\mathcal{X}_0 = \bigcup_i X_i$, where:*

- *Each component X_i is a smooth rational surface,*
- *The intersections $X_i \cap X_j$ are isomorphic to $\mathbb{P}^1$,*
- *The dual complex of $\mathcal{X}_0$ is homeomorphic to a sphere S^2.*

Applying the monodromy criteria developed in Section 4.7 we analyze each type

Type (1): The central fiber $\mathcal{X}_0$ is a $K3$ surface, so $h^1(\mathcal{X}_0) = 0$, implying $q = \Phi = 0$. Since $\mathcal{X}_0^{(1)}$ is empty, we also have $g = 0$. The dual graph $|\Gamma|$ is just a point, hence $h^1(|\Gamma|) = h^2(|\Gamma|) = 0$. Therefore, the logarithm of the monodromy satisfies $N = 0$ on $H^2_{\lim}$.

Type (2): The dual complex $|\Gamma|$ is homeomorphic to the interval $[0,1]$, so again $h^1(|\Gamma|) = h^2(|\Gamma|) = 0$ and $N^2 = 0$ on H^2_{lim}. Since $h^1(\mathcal{X}_t) = 0$, it follows that $\Phi = 0$. The components X_0 and X_{k+1} are regular, while the intermediate components X_i for $i = 1, \dots, k$ each have irregularity 1, giving $q = k$. Each of the $k+1$ double curves has genus 1, so $g = k+1$. Then,

$$\Phi - 2q + 2g = 0 - 2k + 2(k+1) = 2 \neq 0,$$

and thus $N \neq 0$. The monodromy weight filtration on $H^2_{\text{lim}} \cong \mathbb{Q}^{22}$ is:

$$0 = W_0 \subset W_1 = \mathbb{Q}^2 \subset W_2 = \mathbb{Q}^{20} \subset W_3 = W_4 = \mathbb{Q}^{22}.$$

Type (3): The dual graph $|\Gamma|$ has $h^2(|\Gamma|) = 1$, so $N^2 \neq 0$ on H^2_{lim}. Since $h^1(\mathcal{X}_t) = 0$ and $h^1(|\Gamma|) = 0$, we obtain $\Phi = 0$. All components are rational surfaces, so $q = 0$, and the double curves are also rational, implying $g = 0$. Hence $\dim_{\mathbb{Q}} \mathrm{Gr}_1 H^2_{\text{lim}} = 0$. The monodromy weight filtration on $H^2_{\text{lim}} \cong \mathbb{Q}^{22}$ is:

$$0 \subset W_0 = W_1 = \mathbb{Q}^1 \subset W_2 = W_3 = \mathbb{Q}^{21} \subset W_4 = \mathbb{Q}^{22}.$$

6.3 Abelian Surfaces

We now study the monodromy filtration in a degeneration of abelian surfaces, making use of the results established in Section 2.10.

We have the following classification theorem; more details can be found in [11, Chapter II].

Theorem 6.3.1. *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of abelian surfaces. Then π is a modification of a semistable degeneration whose central fiber $\mathcal{X}_0$ is of one of the following types:*

Type (1) *$\mathcal{X}_0$ is an abelian surface.*

Type (2) *$\mathcal{X}_0 = X_0 \cup X_1 \cup \cdots \cup X_{k-1}$, where:*

- *Each component X_i is a ruled surface over an elliptic curve,*
- *The components are glued together in a cyclic pattern. In particular, each X_i meets only X_{i-1} and X_{i+1} (indices modulo k), and the intersections $X_i \cap X_{i\pm 1}$ are smooth elliptic curves, which are sections of the ruling on X_i.*

Type (3) *$\mathcal{X}_0 = \bigcup_i X_i$, where:*

- *Each component X_i is a smooth rational surface,*
- *The intersections $X_i \cap X_j$ are isomorphic to $\mathbb{P}^1$,*
- *The dual complex of $\mathcal{X}_0$ is homeomorphic to a topological 2-torus $S^1 \times S^1$.*

Applying the monodromy criteria developed in Section 4.7 we analyze each type:

Type (1): The central fiber $\mathcal{X}_0$ is an abelian surface, so $\Phi = \dim_{\mathbb{Q}} H^1(\mathcal{X}_0, \mathbb{Q}) = 4$, $g = 0$ and $q = h^{1,0}(\mathcal{X}_0) = 2$. Also, $b_1(\Gamma) = b_2(\Gamma) = 0$. Since $\Phi + 2g - 2q = 0$, we conclude that N is zero both on H^1_{lim} and H^2_{lim}.

Type (2): The dual complex $|\Gamma|$ is homeomorphic to S^1, so $b_1(|\Gamma|) = 1$ and thus $N \neq 0$ on H^1_{lim}, but $b_2(|\Gamma|) = 0$ and $N^2 = 0$ on H^2_{lim}. Since $b_1(\mathcal{X}_t) = 4$, we have $\Phi = b_1(\mathcal{X}_t) - 2b_1(\Gamma) = 2$. The components X_i each have irregularity 1, giving $q = k$. Each of the k double curves has genus 1, so $g = k$. Then,

$$\Phi - 2q + 2g = 2 - 2k + 2k = 2 \neq 0,$$

and thus $N \neq 0$ on H^2_{lim}. The monodromy weight filtration on $H^1_{\text{lim}} \cong \mathbb{Q}^4$ is:

$$0 \subset W_0 = \mathbb{Q}^1 \subset W_1 = \mathbb{Q}^3 \subset W_2 = \mathbb{Q}^4,$$

while the monodromy weight filtration on $H^2_{\text{lim}} \cong \mathbb{Q}^6$ is:

$$0 = W_0 \subset W_1 = \mathbb{Q}^2 \subset W_2 = \mathbb{Q}^4 \subset W_3 = W_4 = \mathbb{Q}^6.$$

Type (3): The dual complex $|\Gamma|$ has $b_2(|\Gamma|) = 1$, so $N^2 \neq 0$ on H^2_{lim}. Since $b_1(\mathcal{X}_t) = 4$ and $b_1(|\Gamma|) = 2$, we obtain $\Phi = 0$. All components are rational surfaces, so $q = 0$, and the double curves are isomorphic to $\mathbb{P}^1$, implying $g = 0$. The monodromy weight filtration on $H^1_{\text{lim}} \cong \mathbb{Q}^4$ is:

$$0 \subset W_0 = W_1 = \mathbb{Q}^2 \subset W_2 = \mathbb{Q}^4,$$

while the monodromy weight filtration on $H^2_{\text{lim}} \cong \mathbb{Q}^6$ is:

$$0 \subset W_0 = W_1 = \mathbb{Q}^1 \subset W_2 = W_3 = \mathbb{Q}^5 \subset W_4 = \mathbb{Q}^6.$$

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