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Powers of commutators in infinite groups

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Powers of commutators in infinite groups

Master thesis

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1 Introduction

A *Honda* group is a group in which every generator of every subgroup generated by a commutator is itself also a commutator. In algebraic terms we say that a group G is Honda if for every integer r and all $x, u, z \in G$ one has

$$[x, u] = z, \langle z \rangle = \langle z^r \rangle \Rightarrow \exists v, w \in G : [v, w] = z^r.$$

Here we use the convention that $[x, u] = xux^{-1}u^{-1}$. In 1953, K. Honda [4] proved that every finite group is Honda (Proposition 2.5 below). A natural question to ask is if *every* group is Honda. However, in 1977 ([8], page 488, Result (C)) S.J. Pride gave a family of groups that he showed were not Honda. On the other hand, B. Martin proved in [6] that many linear algebraic groups are Honda.

We write \sim for “is conjugate to”. With the same proof that K. Honda used to show that every finite group is Honda, one can also show that every finite group G has the following property: for every integer r and all $x, u, z \in G$ one has

$$[x, u] = z, \langle x \rangle = \langle x^r \rangle, \langle z \rangle = \langle z^r \rangle \Rightarrow \exists w \in G : [x^r, w] \sim z^r.$$

We call such a group a *quasi-Honda* group. It is easy to see that every quasi-Honda torsion group is Honda (Proposition 2.3). The groups given by S.J. Pride are quasi-Honda (Proposition 2.6), so in general quasi-Honda does not imply Honda. One may now wonder if *every* group is quasi-Honda.

To prove that every finite group is Honda, K. Honda applied an argument that W. Burnside had used in 1911 to prove that every finite group G has the following property: for every integer r and all $x, y, z \in G$ one has

$$xy = z, \langle x \rangle = \langle x^r \rangle, \langle y \rangle = \langle y^r \rangle, \langle z \rangle = \langle z^r \rangle \Rightarrow \exists x', y', z' \in G : x'y' = z', x' \sim x^r, y' \sim y^r, z' \sim z^r$$

([1], Chapter XV, Theorem VII). We will call a group with this property a *quasi-Burnside* group. Since it is easy to prove that every quasi-Burnside group is quasi-Honda (Proposition 2.4.(b)), Honda’s theorem is a consequence of Burnside’s result. Just as in the case of the quasi-Honda property one may wonder if every group is quasi-Burnside. The answer to both of these questions is negative, but we were able to answer a more refined question, which we proceed to formulate.

If, in the definitions of quasi-Honda and quasi-Burnside, one has $r = -1$, then one can give explicit formulas for w and for x', y', z' ; e.g. $w = xu$ and $x' = x^{-1}$, $y' = xy^{-1}x^{-1}$, $z' = z^{-1}$. However, for general r it is only meaningful to ask for such formulas if one guarantees that the conditions $\langle x \rangle = \langle x^r \rangle$, $\langle z \rangle = \langle z^r \rangle$, and (in the quasi-Burnside case) $\langle y \rangle = \langle y^r \rangle$ are satisfied. If $r \notin \{\pm 1\}$, then these conditions are equivalent to the existence of positive integers k, l, m that are coprime to r for which one has $x^k = 1$, ($y^l = 1$), and $z^m = 1$. This leads to the following definitions. Given a ring R , we write R^* for its unit group. Let $k, l, m \geq 2$ be integers and let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$. A group G is (k, l, m, r) -*quasi-Burnside* if for all $x, y, z \in G$ one has

$$xy = z, x^k = y^l = z^m = 1 \Rightarrow \exists x', y', z' \in G : x'y' = z', x' \sim x^r, y' \sim y^r, z' \sim z^r.$$

Now let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$. A group G is (k, m, r) -*quasi-Honda* if for all $x, u, z \in G$ one has

$$[x, u] = z, x^k = z^m = 1 \Rightarrow \exists w \in G : [x^r, w] \sim z^r.$$

Clearly, a group G is quasi-Burnside if and only if for all k, l, m and every $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ the group G is (k, l, m, r) -quasi-Burnside. Analogously, a group G is quasi-Honda if and only if for all k, m and every $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ the group G is (k, m, r) -quasi-Honda.

In this thesis we classify for which values of k, l, m, r respectively k, m, r every group is (k, l, m, r) -quasi-Burnside respectively (k, m, r) -quasi-Honda. Our results are summarized in the two theorems below. If $\gcd(k, m) \leq 2$, and we are given $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$, then we write r^* for the unique element in $(\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ with $r^* \equiv r \pmod{k}$ and $r^* \equiv -r \pmod{m}$. This element r^* exists by Lemma 2.9.

Theorem 1.1. *Let $k, m \geq 2$ be integers and let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$. Then the following are equivalent.*

- (1) *Every group is (k, m, r) -quasi-Honda.*
- (2) *Every group is (k, k, m, r) -quasi-Burnside, or both $\gcd(k, m) \leq 2$ holds and every group is (k, k, m, r^*) -quasi-Burnside.*
- (3) *We have $\frac{2}{k} + \frac{1}{m} \geq 1$, or $r \in \{\pm 1\}$, or both $\gcd(k, m) \leq 2$ and $r^* \in \{\pm 1\}$.*

The equivalence (1) \Leftrightarrow (2) of Theorem 1.1 is proved at the end of §6, and the equivalence (2) \Leftrightarrow (3) of Theorem 1.1 is a direct consequence of Theorem 1.2, which we shall prove first. We write $\text{PSL}_2(\mathbb{R})$ for the projective special linear group of degree 2 over the real numbers.

Theorem 1.2. *Let $k, l, m \geq 2$ be integers and let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$. Then the following are equivalent.*

- (1) *Every group is (k, l, m, r) -quasi-Burnside.*
- (2) *The group $\text{PSL}_2(\mathbb{R})$ is (k, l, m, r) -quasi-Burnside.*
- (3) *We have $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \geq 1$ or $r \in \{\pm 1\}$.*

The implication (1) \Rightarrow (2) of Theorem 1.2 is trivial, the implication (2) \Rightarrow (3) is proved in §3, just above Proposition 3.8, and the implication (3) \Rightarrow (1) is proved at the end of §2.

The proof of Theorem 1.2 uses facts about products of triples of conjugacy classes in $\text{PSL}_2(\mathbb{R})$, and the proof of Theorem 1.1 uses a special notion of a reduced word in the free product of a group and an infinite cyclic group.

For the entirety of this thesis we fix three integers $k, l, m \geq 2$.

In §2 we prove some basic facts about (k, l, m, r) -quasi-Burnside and (k, m, r) -quasi-Honda groups. We show that every group being (k, l, m, r) -quasi-Burnside is equivalent to the existence of $g, h \in B_{k, l, m} := \langle a, c \mid a^k = (a^{-1}c)^l = c^m = 1 \rangle$ such that $a^r \cdot g(a^{-1}c)^r g^{-1} = hc^r h^{-1}$ (Proposition 2.11.(a)), and we show that every group being (k, m, r) -quasi-Honda is equivalent to the existence of $d, e \in H_{k, m} := \langle a, b, c \mid [a, b] = c, a^k = c^m = 1 \rangle$ such that $[a^r, d] = ec^r e^{-1}$ (Proposition 2.11.(b)). Such elements g, h and d, e , if they exist, are essentially the formulas that we asked for. The group $B_{k, l, m}$ is known as a *von Dyck group* or a *triangle group* in the literature. The von Dyck groups are usually studied via the group $\text{PSL}_2(\mathbb{R})$, so it is not surprising that $\text{PSL}_2(\mathbb{R})$ plays an important role in this thesis.

As stated in Proposition 2.7, one readily checks that the elements r in $(\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ respectively $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ for which a given group G is (k, m, r) -quasi-Honda respectively (k, l, m, r) -quasi-Burnside form a subgroup. In the quasi-Burnside case, we denote that subgroup of $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ by M_G . In §3 we determine M_G for $G = \text{PSL}_2(\mathbb{R})$ (Proposition 3.1.(b)), allowing us to prove the implication (2) \Rightarrow (3) of Theorem 1.2. Then, in §4, we examine the structure of the groups $H_{k, m}, B_{k, l, m}$ and show that $B_{k, k, m}$ may be viewed as a subgroup of $H_{k, m}$ (Proposition 4.1.(c)), which is the key to the proof of the equivalence (1) \Leftrightarrow (2) of Theorem 1.1. In §5 we define the

special notion of a reduced word mentioned above, and prove some of its properties. In §6 this notion enables us to prove Proposition 6.2, which we then use, together with the theory from §4, to prove the equivalence (1) \Leftrightarrow (2) of Theorem 1.1.

2 Quasi-Burnside and quasi-Honda groups

As mentioned in the introduction, we fix integers $k, l, m \geq 2$ for the entirety of this thesis.

In this section we prove some basic facts about (k, l, m, r) -quasi-Burnside and (k, m, r) -quasi-Honda groups, and at the end we prove the implication (3) \Rightarrow (1) of Theorem 1.2.

Let x, y be two elements of a group G . The notation $x \sim y$ means “ x is conjugate to y ”. We write $\text{ord}(x)$ for the order of x , and we write ${}^y x$ for yxy^{-1} . Given an integer n , we have $(yxy^{-1})^n = yx^n y^{-1}$, hence the notation ${}^y x^n$ is unambiguous.

Proposition 2.1. *Let G be a group, let $x, y, z, u \in G$, and let r be an integer.*

- (a) *Suppose $\text{ord}(x) = \infty$ or $\text{ord}(z) = \infty$. If $[x, u] = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle z \rangle = \langle z^r \rangle$, then there exists $w \in G$ such that $[x^r, w] = z^r$.*
- (b) *Suppose $\text{ord}(x) = \infty$ or $\text{ord}(y) = \infty$ or $\text{ord}(z) = \infty$. If $xy = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle y \rangle = \langle y^r \rangle$, $\langle z \rangle = \langle z^r \rangle$, then there exist $x', y', z' \in G$ such that $x'y' = z'$, $x' \sim x^r$, $y' \sim y^r$, $z' \sim z^r$.*

Proof. The proofs are analogous, so we only prove (a). Suppose $[x, u] = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle z \rangle = \langle z^r \rangle$. If $\text{ord}(x) = \infty$, then $r \in \{\pm 1\}$ since $\langle x \rangle = \langle x^r \rangle$. Analogously we have $r \in \{\pm 1\}$ if $\text{ord}(z) = \infty$. As shown in the introduction, if $r \in \{\pm 1\}$, then we can give an explicit formula for w as above. \square

Lemma 2.2. *Let r and n be coprime integers, and let S be a finite set of prime numbers. Then there exists a positive integer r' with $r' \equiv r \pmod{n}$, that is not divisible by any element of S .*

Proof. Define $T := \{p \in S : p \nmid r\}$ and $q := \prod_{p \in T} p$. Then $r' = r + nq$ will do. \square

Proposition 2.3. *If a group is quasi-Honda and torsion, then it is Honda.*

Proof. Let G be a group that is quasi-Honda and torsion. Let r be an integer, $x, u, z \in G$ with $[x, u] = z$, $\langle z \rangle = \langle z^r \rangle$. Let S be the set of prime numbers that divide $\text{ord}(x)\text{ord}(z)$. We have $\gcd(r, \text{ord}(z)) = 1$, so by Lemma 2.2 there exists a positive integer r' with $r' \equiv r \pmod{\text{ord}(z)}$, that is coprime to $\text{ord}(x)\text{ord}(z)$. Choose such an r' . Then $\langle z \rangle = \langle z^{r'} \rangle$ and $\langle x \rangle = \langle x^{r'} \rangle$, so there exists $w \in G$ such that $[x^{r'}, w] \sim z^{r'} = z^r$. Thus z^r is a commutator. \square

Proposition 2.4. (a) *Let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$. If a group is (k, k, m, r) -quasi-Burnside, then it is (k, m, r) -quasi-Honda.*

- (b) *If a group is quasi-Burnside, then it is quasi-Honda.*

Proof. It suffices to only prove (a), since a group is quasi-Burnside respectively quasi-Honda if and only if it is (k, k, m, r) -quasi-Burnside respectively (k, m, r) -quasi-Honda for all k, m, r . Let G be a

group. Given $x, z \in G$, we have

$$\begin{aligned} \exists w \in G : [x^r, w] \sim z^r &\Leftrightarrow \exists w \in G : x^r \cdot {}^w x^{-r} \sim z^r \\ &\Leftrightarrow \exists a, b, w \in G : {}^a x^r \cdot {}^{aw} x^{-r} = {}^b z^r \\ &\Leftrightarrow \exists x', y', z' \in G : x' y' = z', x' \sim x^r, y' \sim x^{-r}, z' \sim z^r. \end{aligned}$$

Thus G is (k, m, r) -quasi-Honda if and only if for all $x, u, z \in G$ we have that if $[x, u] = z$, $x^k = z^m = 1$, then there exist $x', y', z' \in G$ such that $x' y' = z'$, $x' \sim x^r$, $y' \sim x^{-r}$, $z' \sim z^{-r}$. This is the definition of a (k, k, m, r) -quasi-Burnside group restricted to the case where $y = {}^u x^{-1}$ for some $u \in G$. \square

Proposition 2.5. *Every finite group is quasi-Burnside, quasi-Honda and Honda.*

Proof. Given a conjugacy class C of a group, and an integer s , we write C^s for the conjugacy class $\{c^s : c \in C\}$. We write $\#$ for the cardinality of a set. Theorem VII from Chapter XV of [1] by Burnside states the following. Let G be a finite group and let s be an integer that is coprime to the exponent of G . Let C, D, E be conjugacy classes of G . Then for all $z \in E$ and all $z' \in E^s$ one has

$$\#\{(x, y) \in C \times D : xy = z\} = \#\{(x', y') \in C^s \times D^s : x' y' = z'\}.$$

Now let G be a finite group, r an integer. Let $x, y, z \in G$ such that $xy = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle y \rangle = \langle y^r \rangle$, $\langle z \rangle = \langle z^r \rangle$ and let $n := \text{ord}(x)\text{ord}(y)\text{ord}(z)$. Notice that $\gcd(r, n) = 1$. Let S be the set of prime numbers that divide the exponent of G . By Lemma 2.2 there exists a positive integer r' that is coprime to the exponent of G and that satisfies $r' \equiv r \pmod{n}$. Choose such r' . Applying Burnside's Theorem VII with C, D, E equal to the conjugacy classes of respectively x, y, z , and with $s = r'$, we find that there exist $x', y', z' \in G$ such that $x' y' = z'$, $x' \sim x^{r'} = x^r$, $y' \sim y^{r'} = y^r$, $z' \sim z^{r'} = z^r$. Thus every group is quasi-Burnside. By Proposition 2.4.(b) every finite group is quasi-Honda, and then by Proposition 2.3 every finite group is Honda. \square

Given two groups G, H , we denote by $G * H$ the free product of G and H . Result (C) on page 488 of [8] states that given an integer $n \geq 2$, the group $P_n := \langle a, b, c \mid [a, b] = c, c^n = 1 \rangle$ is not Honda. It turns out that P_n is quasi-Burnside and quasi-Honda.

Proposition 2.6. *Let $n \geq 2$ be an integer. Then P_n is quasi-Burnside and quasi-Honda.*

Proof. Let $x, y, z \in P_n$ and let r be an integer such that $xy = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle y \rangle = \langle y^r \rangle$, $\langle z \rangle = \langle z^r \rangle$. We will show that there exist $x', y', z' \in P_n$ with $x' y' = z'$, $x' \sim x^r$, $y' \sim y^r$, $z' \sim z^r$. By Proposition 2.1.(b) we may assume that x, y, z have finite order. Let $\langle a \rangle, \langle b \rangle$ be infinite cyclic groups, let $A_n = \langle c_i \mid c_i^n = 1 \mid i \in \mathbb{Z} \rangle * \langle b \rangle$, and define $A_n \rtimes \langle a \rangle$ by $ac_i a^{-1} = c_{i+1}$ for all i and $aba^{-1} = c_0 b$. One readily shows that there is an isomorphism $A_n \rtimes \langle a \rangle \xrightarrow{\sim} P_n : a \mapsto a, b \mapsto b, c_i \mapsto a^i c a^{-i}$. Any element of finite order in $A_n \rtimes \langle a \rangle$ belongs to the kernel of the canonical quotient map $A_n \rtimes \langle a \rangle \rightarrow (A_n \rtimes \langle a \rangle) / A_n \cong \mathbb{Z}$, so to A_n . Note that A_n is the free product of $\langle b \rangle$ and a countably infinite number of cyclic groups of order n , generated by the c_i 's. Corollary 1 of Proposition 2 in Section I.1.3 of [9] states that every element of finite order in a free product is conjugate to an element of one of the groups in the free product. Thus any element of finite order in $A_n \rtimes \langle a \rangle$ is conjugate in A_n to a power of c_i for some i . View x, y, z as elements of $A_n \rtimes \langle a \rangle$ and let s, t, u be integers such that x, y, z are conjugate in A_n to respectively c_h^s, c_i^t, c_j^u for some integers h, i, j . Let $\langle \beta \rangle, \langle \gamma \rangle$ be cyclic groups of respective orders ∞, n , and let $\sigma : A_n \rightarrow \langle \beta \rangle \times \langle \gamma \rangle$ be the homomorphism that sends b to β and sends c_i to γ for all i . Since $\langle \beta \rangle \times \langle \gamma \rangle$ is abelian, we have

$\gamma^u = \sigma(z) = \sigma(x)\sigma(y) = \gamma^{s+t}$. Thus $u \equiv s + t \pmod{n}$, so $c^{sr} \cdot c^{tr} = c^{(s+t)r} = c^{ur}$. In P_n we have $c^{sr} \sim x^r$, $c^{tr} \sim y^r$, $c^{ur} \sim z^r$, so P_n is quasi-Burnside.

Now Proposition 2.4.(b) gives that P_n is quasi-Honda. \square

Proposition 2.7. *Let G be a group. Then $\{r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^* \mid G \text{ is } (k, m, r)\text{-quasi-Honda}\}$ and $\{r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* \mid G \text{ is } (k, l, m, r)\text{-quasi-Burnside}\}$ are subgroups of $(\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ respectively $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ containing -1 . \square*

Recall that $k, l, m \geq 2$ are still fixed integers.

Let G be a group and let n be an integer. We write $G[n]$ for $\{g \in G : g^n = 1\}$ and $G[n]^0$ for $G[n] \setminus \{1\}$. We write G/\sim for the set of conjugacy classes of G . Given $C, D, E \in G/\sim$, we write CDE for $\{cde : c \in C, d \in D, e \in E\} \subseteq G$ and write C^n for $\{c^n : c \in C\} \in G/\sim$. Define

$$B_G := \{(C, D, E) \in (G/\sim)^3 : 1 \in CDE, C \subseteq G[k]^0, D \subseteq G[l]^0, E \subseteq G[m]^0\}.$$

Note that given a representative $[r] \in \mathbb{Z}$ of some $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ and $C \subseteq G[k]$, the conjugacy class $C^r := C^{[r]}$ is well-defined because $C^{[r]} = C^{[r] + \text{lcm}(k, l, m)}$. Given $D \subseteq G[l]$ and $E \subseteq G[m]$, the same is true for D^r and E^r . Finally we define

$$M_G := \{r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* \mid \forall (C, D, E) \in B_G : (C^r, D^r, E^r) \in B_G\}.$$

In Lemma 2.8, we use the following equivalent definition of a (k, l, m, r) -quasi-Burnside group. Let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$. A group G is (k, l, m, r) -quasi-Burnside if for all $c, d, e \in G$ one has

$$cde = c^k = d^l = e^m = 1 \Rightarrow \exists c', d', e' \in G : c'd'e' = 1, c' \sim c^r, d' \sim d^r, e' \sim e^r.$$

Lemma 2.8. *Let G be a group and let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$. Then G is (k, l, m, r) -quasi-Burnside if and only if $r \in M_G$. Moreover, M_G is a subgroup of $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ containing -1 .*

Proof. The group G is (k, l, m, r) quasi-Burnside if and only if for all $c, d, e \in G$ with $c^k = d^l = e^m = cde = 1$, there exist $c', d', e' \in G$ such that $c'd'e' = 1, c' \sim c^r, d' \sim d^r, e' \sim e^r$, so if and only if for all $(C, D, E) \in (G/\sim)^3$ with $C \subseteq G[k], D \subseteq G[l], E \subseteq G[m], 1 \in CDE$ we have $1 \in C^r D^r E^r$. Clearly, if $(C, D, E) \in (G/\sim)^3$ with $C \subseteq G[k], D \subseteq G[l], E \subseteq G[m], 1 \in CDE$ and at least one of C, D, E is equal to $\{1\}$, then $1 \in C^r D^r E^r$. Thus every group is (k, l, m, r) quasi-Burnside if and only if for all $(C, D, E) \in (G/\sim)^3$ with $C \subseteq G[k]^0, D \subseteq G[l]^0, E \subseteq G[m]^0, 1 \in CDE$ we have $1 \in C^r D^r E^r$, which is equivalent to $r \in M_G$. The second statement then follows from Proposition 2.7. \square

Lemma 2.9. *The map $(\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^* \rightarrow (\mathbb{Z}/k\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z})^*, a \mapsto (a \pmod{k}, a \pmod{m})$ is an isomorphism of groups if and only if $\gcd(k, m) \leq 2$.*

Proof. Denote by f the map from the lemma and by φ the Euler totient function. Clearly f is an injective group homomorphism. Since $\varphi(\text{lcm}(k, m)) \cdot \varphi(\gcd(k, m)) = \varphi(k) \cdot \varphi(m)$, we have that f is surjective if and only if $\varphi(\gcd(k, m)) = 1$, which is equivalent to $\gcd(k, m) \leq 2$. \square

If $\gcd(k, m) \leq 2$ and $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$, then write r^* for the unique element of $(\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ with $r^* \equiv r \pmod{k}$ and $r^* \equiv -r \pmod{m}$. Clearly $r^* = 1^* \cdot r$ and $(r^*)^* = r$. Also note that $(-r)^* = -(r^*)$, so the notation $-r^*$ is unambiguous.

Proposition 2.10. *Let $r \in \{\pm 1\} \subseteq (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$. Then every group is (k, m, r) -quasi-Honda. Moreover, if $\gcd(k, m) \leq 2$, then every group is (k, m, r^*) -quasi-Honda.*

Proof. The first statement was explicitly proved in the introduction. Suppose $\gcd(k, m) \leq 2$. If for elements x, u, z of some group G we have $[x, u] = z$, $x^k = z^m = 1$, then $[x^{1^*}, u^{-1}] = [x, u^{-1}] = u^{-1}z^{-1} = u^{-1}z^{1^*}$. Proposition 2.7 then gives that G is $(k, m, -1^*)$ -quasi-Honda. \square

Define $B_{k,l,m} = \langle a, c \mid a^k = (a^{-1}c)^l = c^m = 1 \rangle$ and $H_{k,m} = \langle a, b, c \mid [a, b] = c, a^k = c^m = 1 \rangle$. These groups have certain universal properties, which are described in Proposition 2.11. In §4, we will determine the structure of these groups, and prove some of their properties.

Proposition 2.11. *Let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$.*

- (a) *Every group is (k, l, m, r) -quasi-Burnside if and only if there exist $g, h \in B_{k,l,m}$ such that $a^r \cdot {}^g(a^{-1}c)^r = {}^h c^r$.*
- (b) *Every group is (k, m, r) -quasi-Honda if and only if there exist $d, e \in H_{k,m}$ such that $[a^r, d] = {}^e c^r$.*

Proof. The proofs are similar, so we only prove (a). “ \Rightarrow ” Trivial. “ \Leftarrow ” Let $g, h \in B_{k,l,m}$ such that $a^r \cdot {}^g(a^{-1}c)^r = {}^h c^r$. If x, y, z are elements of a group G with $xy = z$, $x^k = y^l = z^m = 1$, then there exists a homomorphism $f : B_{k,l,m} \rightarrow G : a \mapsto x, c \mapsto z$. Applying f to $a^r \cdot {}^g(a^{-1}c)^r = {}^h c^r$ gives $f(h)z^r = x^r \cdot f(g)(x^{-1}z)^r = x^r \cdot f(g)y^r$. \square

Proof of the implication (3) \Rightarrow (1) of Theorem 1.2. As shown in the introduction, if $r \in \{\pm 1\}$, then every group is (k, l, m, r) -quasi-Burnside. If $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1$, then up to permutation we have that $(k, l, m) \in \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\}$, thus $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* = \{\pm 1\}$, so every group is (k, l, m, r) -quasi-Burnside.

Suppose $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$. Then it is a well-known fact that the von Dyck group $B_{k,l,m}$ is finite. (See for example §15 of [5].) By Proposition 2.5, every finite group is (k, l, m, r) -quasi-Burnside, so there exist $f, g, h \in B_{k,l,m}$ such that $f a^r \cdot {}^g(a^{-1}c^{-1})^r \cdot {}^h c^r = 1$. For such f, g, h we have $a^r \cdot f^{-1}g(a^{-1}c^{-1})^r \cdot f^{-1}h c^r = 1$, so Proposition 2.11.(a) gives that every group is (k, l, m, r) -quasi-Burnside. \square

3 Projective special linear group

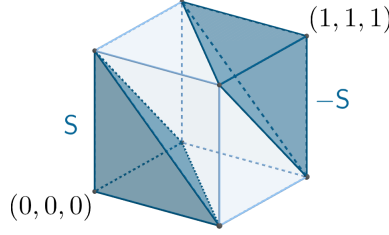
The *special linear group of degree 2 over the real numbers* $\text{SL}_2(\mathbb{R})$ is the group of real 2 by 2 matrices with determinant 1. Denote by I the identity element of $\text{SL}_2(\mathbb{R})$. The *projective special linear group of degree 2 over the real numbers* $\text{PSL}_2(\mathbb{R})$ is the quotient group $\text{SL}_2(\mathbb{R})/\langle -I \rangle$.

In this section, we describe B_G and M_G , as defined just before Lemma 2.8, for $G = \text{PSL}_2(\mathbb{R})$. This allows us to prove the implication (2) \Rightarrow (3) of Theorem 1.2 (just above Proposition 3.8), and Proposition 3.8, which we use in §4 towards the proof of the equivalence (1) \Leftrightarrow (2) of Theorem 1.1.

Let $H = (\frac{1}{k}\mathbb{Z}/\mathbb{Z}) \oplus (\frac{1}{l}\mathbb{Z}/\mathbb{Z}) \oplus (\frac{1}{m}\mathbb{Z}/\mathbb{Z})$ and regard it as a subgroup of $(\mathbb{R}/\mathbb{Z})^3$. If $r \in \mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z}$ and $(\frac{a}{k}, \frac{b}{l}, \frac{c}{m}) \in H$, then we define $r \cdot (\frac{a}{k}, \frac{b}{l}, \frac{c}{m}) = (\frac{ra}{k}, \frac{rb}{l}, \frac{rc}{m})$, making H into a $\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z}$ -module. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0, x + y + z < 1\}$ and view it as a subset of $(\mathbb{R}/\mathbb{Z})^3$ via the canonical quotient map $\mathbb{R}^3 \rightarrow (\mathbb{R}/\mathbb{Z})^3$. Define

$$M = \{r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* : r \cdot (H \cap (S \cup -S)) = H \cap (S \cup -S)\}.$$

See below a picture of S and $-S$ inside $(\mathbb{R}/\mathbb{Z})^3$.



Proposition 3.1. (a) We have $M_{\text{PSL}_2(\mathbb{R})} = M$.

(b) If $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$, then $M_{\text{PSL}_2(\mathbb{R})} = \{\pm 1\}$.

We prove Proposition 3.1.(a) just below Lemma 3.4, and prove Proposition 3.1.(b) just below Lemma 3.7.

Given $a \in \mathbb{R}/\mathbb{Z}$, we write σ_a for $\begin{pmatrix} \cos[a]\pi & -\sin[a]\pi \\ \sin[a]\pi & \cos[a]\pi \end{pmatrix} \cdot \langle -I \rangle \in \text{PSL}_2(\mathbb{R})$, where $[a] \in \mathbb{R}$ is some representative of a , and we write C_a for the conjugacy class of σ_a . One readily shows that the definitions of σ_a and C_a do not depend on the choice of representative. Write $\text{PSL}_2(\mathbb{R})^{\text{Tor}}$ for the set of elements of finite order in $\text{PSL}_2(\mathbb{R})$, and given a subset $F \subseteq \text{PSL}_2(\mathbb{R})$ that is stable under conjugation in $\text{PSL}_2(\mathbb{R})$, denote by F/\sim the quotient set of F by conjugation in $\text{PSL}_2(\mathbb{R})$. There is a $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ -action on the set $X = (\text{PSL}_2(\mathbb{R})[k]/\sim) \times (\text{PSL}_2(\mathbb{R})[l]/\sim) \times (\text{PSL}_2(\mathbb{R})[m]/\sim)$, defined by mapping $(r, (C_a, C_b, C_c)) \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* \times X$ to $(C_a^r, C_b^r, C_c^r) \in X$. Write $*$ for this action, and note that for such r and (C_a, C_b, C_c) we have $r * (C_a, C_b, C_c) = (C_a^r, C_b^r, C_c^r) = (C_{ra}, C_{rb}, C_{rc})$.

Lemma 3.2. (a) The map $\mathbb{Q}/\mathbb{Z} \rightarrow \text{PSL}_2(\mathbb{R})^{\text{Tor}}/\sim$ sending a to C_a is a bijection. Moreover, if n is an integer and $a \in \mathbb{Q}/\mathbb{Z}$, then $n \cdot a = 0$ if and only if $C_a^n = \{1\}$.

(b) The map $\psi : H \rightarrow \text{PSL}_2(\mathbb{R})[k]/\sim \times \text{PSL}_2(\mathbb{R})[l]/\sim \times \text{PSL}_2(\mathbb{R})[m]/\sim$ that sends (a, b, c) to (C_a, C_b, C_c) is a bijection that respects the $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ -actions, i.e., for all $(a, b, c) \in H$ and all $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ we have $r * \psi(a, b, c) = \psi(r(a, b, c))$.

Proof. We first prove statement (a). Write φ for the map from statement (a), and note that $\text{im}(\varphi) \subseteq \text{PSL}_2(\mathbb{R})^{\text{Tor}}/\sim$, so it is well-defined. Given $c \in \mathbb{R}$, we define $s_c := \begin{pmatrix} \cos c\pi & -\sin c\pi \\ \sin c\pi & \cos c\pi \end{pmatrix}$. It is a well-known fact that if $A \in \text{SL}_2(\mathbb{R})$ is of finite order, then for some $c \in \mathbb{Q}$ we have $A \sim s_c$. The pre-image under the canonical map $\text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R})$ of a torsion element is a torsion element, so the elements of finite order in $\text{PSL}_2(\mathbb{R})$ are conjugate to some σ_a with $a \in \mathbb{Q}/\mathbb{Z}$. Thus the map from the lemma is surjective.

To show that φ is injective, suppose $C_a = C_b$ with $a, b \in \mathbb{Q}/\mathbb{Z}$. If $a = 0$ or $b = 0$, then $a = 0 = b$, so we may assume a, b to be nonzero. Let $[a], [b] \in (0, 1) \cap \mathbb{Q}$ be representatives of respectively a, b . Then $s_{[a]} \sim s_{[b]}$ or $s_{[a]} \sim -s_{[b]} = s_{-1+[b]}$. It is a well-known fact that if for an element $A \in \text{SL}_2(\mathbb{R})$ we have $|\text{tr}(A)| < 2$, then A is conjugate to a unique matrix of the form s_c with $c \in (-1, 1) \setminus \{0\}$. (See for example Theorem 3.1 of [2].) For all $c \in ((-1, 1) \setminus \{0\}) \cap \mathbb{Q}$ we have $|\text{tr}(s_c)| < 2$, so $s_{[a]} = s_{[b]}$ or $s_{[a]} = s_{-1+[b]}$. Clearly $s_{[a]} = s_{-1+[b]}$ is not possible, so $s_{[a]} = s_{[b]}$, and thus $a = b$. This proves the first part of statement (a). One easily shows that the second part follows from the first.

Now we prove statement (b). By statement (a), we have that ψ restricted to $\frac{1}{k}\mathbb{Z}/\mathbb{Z}$ bijectively maps to $\text{PSL}_2(\mathbb{R})[k]/\sim$. The same is true when we replace k by l or m , so ψ is a bijection. This bijection

respects the $(\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ -actions, since for all $(a, b, c) \in H$ and $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ we have $(C_a^r, C_b^r, C_c^r) = (C_{ra}, C_{rb}, C_{rc})$. \square

Lemma 3.3. *Let $a, b, c \in (\mathbb{R}/\mathbb{Z}) \setminus \{0\}$ and let $[a], [b], [c] \in (0, 1)$ be their respective representatives. Then $1 \in C_a C_b C_c$ if and only if $[a] + [b] + [c] \notin (1, 2)$.*

Proof. This is a reformulation of a part of Corollary 3.3 (b) of [7]. \square

In §8 of [3], Lemma 3.3 was deduced from a correspondence between solutions $(x, u, z) \in C_a \times C_b \times C_c$ to $xuz = 1$ and hyperbolic triangles with angles $[a]\pi, [b]\pi, [c]\pi$, where $[a], [b], [c] \in (0, 1)$ are respective representatives of a, b, c , and the well-known fact that there exists a hyperbolic triangle with angles $[a]\pi, [b]\pi, [c]\pi$ if and only if $[a] + [b] + [c] < 1$.

Lemma 3.4. *We have $B_{\text{PSL}_2(\mathbb{R})} = \{(C_a, C_b, C_c) \in (\text{PSL}_2(\mathbb{R})^{\text{Tor}}/\sim)^3 \mid 1 \in C_a C_b C_c, C_a^k = C_b^l = C_c^m = \{1\}, a, b, c \in (\mathbb{Q}/\mathbb{Z}) \setminus \{0\}\}$. Moreover, let $T = \{(a, b, c) \in ((\mathbb{Q}/\mathbb{Z}) \setminus \{0\})^3 \mid a + b + c = 0\}$. Then there is a bijection $B_{\text{PSL}_2(\mathbb{R})} \rightarrow H \cap (S \cup -S \cup T)$ sending (C_a, C_b, C_c) to (a, b, c) .*

Proof. Let $G = \text{PSL}_2(\mathbb{R})$. By Lemma 3.2.(a) we have that the group B_G consists of the elements (C_a, C_b, C_c) with $1 \in C_a C_b C_c, C_a^k = C_b^l = C_c^m = \{1\}, a, b, c \in (\mathbb{Q}/\mathbb{Z}) \setminus \{0\}$. By Lemma 3.3, these elements of B_G are exactly the elements $(C_a, C_b, C_c) \in (G^{\text{Tor}}/\sim)^3$ with $(a, b, c) \in H \cap (S \cup -S \cup T)$. Hence, the map from the lemma exists and is a bijection. \square

Proof of Proposition 3.1.(a). By definition of $M_{\text{PSL}_2(\mathbb{R})}$ we have that $r \in M_{\text{PSL}_2(\mathbb{R})}$ is equivalent to $r * B_{\text{PSL}_2(\mathbb{R})} \subseteq B_{\text{PSL}_2(\mathbb{R})}$. By Lemmas 3.2.(b) and 3.4, this is equivalent to $r \cdot (H \cap (S \cup -S \cup T)) \subseteq H \cap (S \cup -S \cup T)$. If $(a, b, c) \in H$, then clearly $a + b + c = 0$ if and only if $r(a + b + c) = 0$, so multiplication by r is a permutation of $H \cap T$. So $r \cdot (H \cap (S \cup -S \cup T)) \subseteq H \cap (S \cup -S \cup T)$ is equivalent to $r(H \cap (S \cup -S)) \subseteq H \cap (S \cup -S)$, which is equivalent to $r \in M$ since multiplication by r on $H \cap (S \cup -S)$ is injective and hence bijective. \square

We will now prove some lemmas so we can find $M_{\text{PSL}_2(\mathbb{R})}$ in Proposition 3.1.(b). Recall that we defined $S = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0, x + y + z < 1\}$, and that we view it as a subset of $(\mathbb{R}/\mathbb{Z})^3$.

Lemma 3.5. *Let $r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^*$ and let a, b be integers such that $0 < a < k, 0 < b < l, \frac{a}{k} + \frac{b}{l} < 1$. Then $\#\{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{a}{k}, \frac{b}{l}, z) \in S\} = \lceil m - \frac{am}{k} - \frac{bm}{l} \rceil - 1$.*

Proof. Given $z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ such that $(\frac{a}{k}, \frac{b}{l}, z) \in S$, its representative $[z] \in (0, 1)$ satisfies $0 < [z] < 1 - \frac{a}{k} - \frac{b}{l}$, and there is a unique integer $0 < c < m - \frac{am}{k} - \frac{bm}{l}$ with $[z] = \frac{c}{m}$. Given such c , the class z of $\frac{c}{m}$ in $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$ satisfies $(\frac{a}{k}, \frac{b}{l}, z) \in S$. Thus $\#\{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{a}{k}, \frac{b}{l}, z) \in S\}$ is equal to the number of integers strictly between 0 and $m - \frac{am}{k} - \frac{bm}{l}$, which is $\lceil m - \frac{am}{k} - \frac{bm}{l} \rceil - 1$. \square

Recall that $M = \{r \in (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* : r \cdot (H \cap (S \cup -S)) = H \cap (S \cup -S)\}$.

Lemma 3.6. *Let $r \in M$ be such that for all elements in $H \cap S$ of the form $(\frac{1}{k}, \frac{1}{l}, z)$ we have $(\frac{r}{k}, \frac{r}{l}, rz) \in H \cap S$. Write r_k respectively r_l for the smallest positive representative of r modulo k respectively l . Then $\lceil m - \frac{m}{k} - \frac{m}{l} \rceil = \lceil m - \frac{r_k m}{k} - \frac{r_l m}{l} \rceil$. Moreover, if $m \geq \max\{k, l\}$, then $r \equiv 1 \pmod{\text{lcm}(k, l)}$.*

Proof. Clearly, $\lceil m - \frac{m}{k} - \frac{m}{l} \rceil \geq \lceil m - \frac{r_k m}{k} - \frac{r_l m}{l} \rceil$. As for all elements in $H \cap S$ of the form $(\frac{1}{k}, \frac{1}{l}, z)$ we have $(\frac{r}{k}, \frac{r}{l}, rz) \in H \cap S$, the map $\{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{1}{k}, \frac{1}{l}, z) \in S\} \rightarrow \{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{r}{k}, \frac{r}{l}, z) \in S\}$, $z \mapsto rz$ is well-defined. Since multiplication by r is a bijection on H , this map is injective, so

$$\# \left\{ z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid \left(\frac{1}{k}, \frac{1}{l}, z\right) \in S \right\} \leq \# \left\{ z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid \left(\frac{r}{k}, \frac{r}{l}, z\right) \in S \right\}.$$

Lemma 3.5 then gives $\lceil m - \frac{m}{k} - \frac{m}{l} \rceil \leq \lceil m - \frac{r_k m}{k} - \frac{r_l m}{l} \rceil$.

Now suppose $m \geq \max\{k, l\}$. If $r_k = r_l = 1$ we are done, so suppose r_k, r_l are not both 1. Without loss of generality we assume $r_k \geq 2$. Let $t = m - \frac{m}{k} - \frac{m}{l}$, $u = m - \frac{r_k m}{k} - \frac{r_l m}{l}$. Then $\lceil t \rceil = \lceil u \rceil$, so

$$1 > t - u = m \left(1 - \frac{1}{k} - \frac{1}{l} \right) - m \left(1 - \frac{r_k}{k} - \frac{r_l}{l} \right) = m \left(\frac{r_k - 1}{k} + \frac{r_l - 1}{l} \right) \geq \frac{m}{k} \geq 1.$$

Contradiction. So $r_k = r_l = 1$. □

In the lemma below, given two sets of integers A, B , we write $A \equiv B \pmod{m}$ if for all $a \in A$, there is some $b \in B$ such that $a \equiv b \pmod{m}$, and vice versa.

Lemma 3.7. *Suppose $m \geq 3$, let $r \in (\mathbb{Z}/m\mathbb{Z})^*$ and let c be an integer such that $1 \leq c \leq m - 2$. If $\{1, 2, \dots, c\} \equiv \{r, 2r, \dots, cr\} \pmod{m}$, then $r = 1$.*

Proof. Suppose $\{1, \dots, c\} \equiv \{r, 2r, \dots, cr\} \pmod{m}$. Note that $c < \frac{m}{2}$ or $m - (c + 1) < \frac{m}{2}$. Choose $c' \in \{c, m - (c + 1)\}$ so that $c' < \frac{m}{2}$. Claim: $\{1, 2, \dots, c'\} \equiv \{r, 2r, \dots, c'r\} \pmod{m}$. Proof claim: we already know this is true for $c' = c$, so it remains to be shown that $\{1, 2, \dots, m - (c + 1)\} \equiv \{r, 2r, \dots, (m - (c + 1))r\} \pmod{m}$. Since $\{1, \dots, c\} \equiv \{r, 2r, \dots, cr\} \pmod{m}$, we also have $\{c + 1, \dots, m - 1\} \equiv \{(c + 1)r, \dots, (m - 1)r\} \pmod{m}$, thus

$$\begin{aligned} \{1, 2, \dots, m - (c + 1)\} &\equiv \{-(m - 1), -(m - 2), \dots, -(c + 1)\} \\ &\equiv \{-(m - 1)r, -(m - 2)r, \dots, -(c + 1)r\} \\ &\equiv \{r, 2r, \dots, (m - (c + 1))r\} \pmod{m}, \end{aligned}$$

proving the claim. Suppose $r \neq 1$ and assume without loss of generality that r is an integer and $0 < r < m$. Define $d := \lfloor \frac{c'}{r} \rfloor$. We will get a contradiction with $\{1, 2, \dots, c'\} = \{r, 2r, \dots, c'r\}$ by showing that $d + 1 \leq c'$ and $c' < (d + 1)r < m$. Since $r \neq 1$ we have $d = \lfloor \frac{c'}{r} \rfloor \leq \frac{c'}{2} < c'$, so $d + 1 \leq c'$. Since $d > \frac{c'}{r} - 1$, we also have $dr > (c' - r) = c' - r$, so $(d + 1)r = dr + r > c'$. Finally, since $\{1, 2, \dots, c'\} = \{r, 2r, \dots, c'r\}$, we have $r \leq c'$, so $(d + 1)r = \lfloor \frac{c'}{r} \rfloor r + r \leq 2c' < m$. □

Proof of Proposition 3.1(b). Proposition 3.1(a) gives that $M = M_{\text{PSL}_2(\mathbb{R})}$, so it suffices to show that if $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$, then $M = \{\pm 1\}$. By Lemma 2.8 and Proposition 3.1(a) we have that $\{\pm 1\} \subseteq M$.

Suppose $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$. Clearly M does not change if we permute k, l, m , so we may assume without loss of generality that $m \geq \max\{k, l\}$. Then $m \geq 4$. Let $r \in M$. Claim: at least one of the sets $T := \{(\frac{r}{k}, \frac{r}{l}, z) : z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}\} \cap S$ and $T^* := \{(\frac{r}{k}, \frac{r}{l}, z) : z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}\} \cap -S$ is empty. Proof claim: define $\pi : (\mathbb{R}/\mathbb{Z})^3 \rightarrow (\mathbb{R}/\mathbb{Z})^2$, $(x, y, z) \mapsto (x, y)$. Notice that $\pi(T) \cap \pi(T^*) \subseteq \pi(S) \cap \pi(-S) = \emptyset$, so $(\frac{r}{k}, \frac{r}{l}) \notin \pi(T) \cap \pi(T^*)$, proving the claim.

So since $r \in M$, either for all elements in $H \cap S$ of the form $(\frac{1}{k}, \frac{1}{l}, z)$ we have $(\frac{r}{k}, \frac{r}{l}, rz) \in H \cap S$ or for all such elements we have $(\frac{r}{k}, \frac{r}{l}, rz) \in -(H \cap S)$. Choose $r' \in \{r, -r\}$ so that for all elements

in $H \cap S$ of the form $(\frac{1}{k}, \frac{1}{l}, z)$ we have $(\frac{r'}{k}, \frac{r'}{l}, r'z) \in H \cap S$. Note that $r' \in M$ by Lemma 2.8 and Proposition 3.1.(a). Now Lemma 3.6 gives that $r' \equiv 1 \pmod{\text{lcm}(k, l)}$.

If $(\frac{1}{k}, \frac{1}{l}, z) \in H \cap S$, then $(\frac{1}{k}, \frac{1}{l}, r'z) = (\frac{r'}{k}, \frac{r'}{l}, r'z) \in H \cap S$. This, combined with the fact that multiplication by r' is a bijection on $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$, gives that multiplication by r' permutes the set of elements in $H \cap S$ of the form $(\frac{1}{k}, \frac{1}{l}, z)$. Thus $\{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{1}{k}, \frac{1}{l}, (r')^{-1}z) \in S\} = \{z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid (\frac{1}{k}, \frac{1}{l}, z) \in S\}$. Let $c := \lceil m(1 - \frac{1}{k} - \frac{1}{l}) \rceil - 1$ and let $q : (\mathbb{Z}/\text{lcm}(k, l, m)\mathbb{Z})^* \rightarrow (\mathbb{Z}/m\mathbb{Z})^*$ be the homomorphism that sends x to $x \pmod{m}$. We want to apply Lemma 3.7 to m , $q(r')$ and c . Lemma 3.5 gives

$$\begin{aligned} \left\{ \frac{1}{m}, \frac{2}{m}, \dots, \frac{c}{m} \right\} &= \left\{ z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid \left(\frac{1}{k}, \frac{1}{l}, z \right) \in S \right\} \\ &= \left\{ z \in \frac{1}{m}\mathbb{Z}/\mathbb{Z} \mid \left(\frac{1}{k}, \frac{1}{l}, (r')^{-1}z \right) \in S \right\} \\ &= \left\{ \frac{r'}{m}, \frac{2r'}{m}, \dots, \frac{cr'}{m} \right\}. \end{aligned}$$

Since $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ we have $\frac{1}{m} < 1 - \frac{1}{k} - \frac{1}{l}$, so $1 < m - \frac{m}{k} - \frac{m}{l}$. Thus $2 \leq \lceil m - \frac{m}{k} - \frac{m}{l} \rceil$ and hence $1 \leq c$. Since $m \geq \max\{k, l\}$, we have $\frac{1}{k} + \frac{1}{l} > \frac{1}{m}$, so $1 - \frac{1}{k} - \frac{1}{l} < 1 - \frac{1}{m} = \frac{m-1}{m}$. Therefore $c < m(1 - \frac{1}{k} - \frac{1}{l}) < m-1$. For all integers d we have $\frac{dr'}{m} = \frac{dq(r')}{m}$, so we may apply Lemma 3.7 to m , c and $q(r')$. This gives us $q(r') \equiv 1 \pmod{m}$, so $r' \equiv 1 \pmod{m}$, and thus $r \in \{\pm r'\} = \{\pm 1\}$. \square

Proof of the implication (2) \Rightarrow (3) of Theorem 1.2. Suppose $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ and $r \notin \{\pm 1\}$. Then $r \notin M_{\text{PSL}_2(\mathbb{R})}$ by Proposition 3.1.(b), so $\text{PSL}_2(\mathbb{R})$ is not (k, l, m, r) -quasi-Burnside by Lemma 2.8. \square

The proposition below will be used in §4.

Proposition 3.8. *Suppose $k = l$ and let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$. If $\frac{2}{k} + \frac{1}{m} \leq 1$, then there are $x, y, z \in \text{PSL}_2(\mathbb{R})$ with $xy = z$, $x^k = y^k = z^m = 1$ such that there do not exist $x', x'', z' \in \text{PSL}_2(\mathbb{R})$ with $x'x'' = z'$, $x' \sim x^{-r}$, $x'' \sim x^r$, $z' \sim z^r$.*

Proof. Given a group G , recall that $B_G := \{(C, D, E) \in (G/\sim)^3 : 1 \in CDE, C \subseteq G[k]^0, D \subseteq G[k]^0, E \subseteq G[m]^0\}$. Analogous to the definition of M_G in §2, we define

$$N_G := \{r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^* \mid \forall (C, D, E) \in B_G : (C^{-r}, C^r, E^r) \in B_G\}.$$

Given a group G , one can show analogous to Lemma 2.8, that $r \in N_G$ if and only if for all $x, y, z \in G$ we have

$$xy = z, x^k = y^k = z^m = 1 \Rightarrow \exists x', x'', z' \in G : x'x'' = z', x' \sim x^{-r}, x'' \sim x^r, z' \sim z^r.$$

So it suffices to show that if $\frac{2}{k} + \frac{1}{m} \leq 1$, then $r \notin N_{\text{PSL}_2(\mathbb{R})}$.

Note that since $k = l$, we have $H = (\frac{1}{k}\mathbb{Z}/\mathbb{Z}) \oplus (\frac{1}{k}\mathbb{Z}/\mathbb{Z}) \oplus (\frac{1}{m}\mathbb{Z}/\mathbb{Z})$. By definition of $N_{\text{PSL}_2(\mathbb{R})}$ we have that $r \in N_{\text{PSL}_2(\mathbb{R})}$ if and only if for all $(C_a, C_b, C_c) \in B_{\text{PSL}_2(\mathbb{R})}$ we have $(C_a^{-r}, C_a^r, C_c^r) \in B_{\text{PSL}_2(\mathbb{R})}$. Since $(C_a^{-r}, C_a^r, C_c^r) = (C_{-ra}, C_{ra}, C_{rc})$, Lemmas 3.2.(b) and 3.4 give that this is equivalent to the statement that for all $(a, b, c) \in H \cap (S \cup -S \cup T)$ we have $(-ra, ra, rc) \in H \cap (S \cup -S \cup T)$. Given $s \in \mathbb{Q}/\mathbb{Z}$, we write $[s]$ for the smallest non-negative representative of s modulo 1. Note that given

$(a, b, c) \in H \cap (S \cup -S \cup T)$ we have

$$\begin{aligned} (-ra, ra, rc) \in H \cap (S \cup -S \cup T) &\Leftrightarrow [-ra] + [ra] + [rc] \leq 1 \text{ or } [-ra] + [ra] + [rc] \geq 2 \\ &\Leftrightarrow 1 - [ra] + [ra] + [rc] \leq 1 \text{ or } 1 - [ra] + [ra] + [rc] \geq 2 \\ &\Leftrightarrow [rc] \leq 0 \text{ or } [rc] \geq 1 \Leftrightarrow rc = 0 \Leftrightarrow c = 0, \end{aligned}$$

but such c is never equal to 0. Thus $r \in N_{\text{PSL}_2(\mathbb{R})}$ if and only if $H \cap (S \cup -S \cup T) = \emptyset$. If $\frac{2}{k} + \frac{1}{m} \leq 1$, then $(\frac{1}{k}, \frac{1}{k}, \frac{1}{m}) \in H \cap (S \cup -S \cup T)$, so then $r \notin N_{\text{PSL}_2(\mathbb{R})}$. \square

Proposition 3.9. *The group $\text{PSL}_2(\mathbb{R})$ is Honda and quasi-Honda.*

Proof. Theorem 3 of [10] states that if K is a field with more than 3 elements, then every element of $\text{PSL}_n(K)$ is a commutator. Thus $\text{PSL}_n(K)$ is Honda for such K , so in particular $\text{PSL}_2(\mathbb{R})$ is Honda.

Let r be an integer, $x, y, z \in \text{PSL}_2(\mathbb{R})$ and suppose that $[x, y] = z$, $\langle x \rangle = \langle x^r \rangle$, $\langle z \rangle = \langle z^r \rangle$. We will show that there exists $w \in \text{PSL}_2(\mathbb{R})$ with $[x, w] = z^r$. By Proposition 2.1.(a) we may assume that x and z have finite order. By Lemma 3.2.(a), the non-identity elements in $\text{PSL}_2(\mathbb{R})^{\text{Tor}}$ are exactly the elements conjugate to some σ_a where $a \in (\mathbb{Q}/\mathbb{Z}) \setminus \{0\}$. Suppose $z \neq 1$. Then $x \neq 1$, so $x = {}^g\sigma_a$, $z = {}^h\sigma_c$ with $g, h \in \text{PSL}_2(\mathbb{R})$ and $a, c \in (\mathbb{Q}/\mathbb{Z}) \setminus \{0\}$. We have $1 = x(yx^{-1})z^{-1} = ({}^g\sigma_a)({}^{yg}\sigma_{1-a})({}^h\sigma_{1-c})$, so if we write $[a], [1-a], [1-c] \in (0, 1)$ for the respective representatives of $a, 1-a, 1-c$, then Lemma 3.3 gives $2 - [c] = [a] + [1-a] + [1-c] \notin (1, 2)$. Contradiction, so $z = 1$. Thus $[x, y] = z = z^r$. \square

4 Universal groups

Recall that $B_{k,l,m} = \langle a, c \mid a^k = (a^{-1}c)^l = c^m = 1 \rangle$ and $H_{k,m} = \langle a, b, c \mid [a, b] = c, a^k = c^m = 1 \rangle$.

We will now determine the structure of $H_{k,m}$ and $B_{k,k,m}$. Define

$$G_{k,m} := \langle c_i \ (i \in \mathbb{Z}/k\mathbb{Z}) \mid c_i^m = 1 \ (i \in \mathbb{Z}/k\mathbb{Z}), c_{k-1} \cdots c_1 c_0 = 1 \rangle.$$

One easily checks that there exists a unique $\sigma \in \text{Aut}(G_{k,m})$ such that for all $i \in \mathbb{Z}/k\mathbb{Z}$ one has $\sigma(c_i) = c_{i+1}$. This σ defines a semidirect product $G_{k,m} \rtimes \langle a \rangle$ where a has order k , and the inner action of a on $G_{k,m}$ is given by σ . Given this σ , one readily shows that there exists a unique $\tau \in \text{Aut}(G_{k,m} * \langle b \rangle)$, where b has infinite order, with $\tau|_{G_{k,m}} = \sigma$, $\tau(b) = c_0 b$. Notice that $\tau^k(b) = c_{k-1} \cdots c_1 c_0 b = b$, thus $\tau^k = \text{id}$. So τ defines a semidirect product $(G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle$ where a has order k , and the inner action of a on $G_{k,m} * \langle b \rangle$ is given by τ . We deliberately use the same letter a as in $B_{k,k,m}$ and $H_{k,m}$, and the same letter b as in $H_{k,m}$. Proposition 4.1 justifies this. In this proposition, $H_{k,m}$ and $B_{k,k,m}$ are written in terms of free and semidirect products, allowing us to easily embed $B_{k,k,m}$ in $H_{k,m}$.

Proposition 4.1. *Let $\langle a \rangle, \langle b \rangle$ be cyclic groups of orders k, ∞ respectively.*

- (a) *There exists an isomorphism of groups $G_{k,m} \rtimes \langle a \rangle \xrightarrow{\sim} B_{k,k,m} : a \mapsto a, c_i \mapsto a^i c a^{-i}$.*
- (b) *There exists an isomorphism of groups $(G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle \xrightarrow{\sim} H_{k,m} : a \mapsto a, b \mapsto b, c_i \mapsto a^i c a^{-i}$. Moreover, $(G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle = \langle a, b, c_i \ (i \in \mathbb{Z}/k\mathbb{Z}) \mid c_i^m = 1 \ (i \in \mathbb{Z}/k\mathbb{Z}), a^k = 1, {}^a b = c_0 b, {}^a c_i = c_{i+1} \ (i \in \mathbb{Z}/k\mathbb{Z}) \rangle$.*
- (c) *There exists an embedding $B_{k,k,m} \hookrightarrow H_{k,m} : a \mapsto a, c \mapsto c$.*

Proof. The proof of (a) is similar to the proof of the first statement of (b), so we omit it. It is not hard to see that $(G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle \rightarrow H_{k,m} : a \mapsto a, b \mapsto b, c_i \mapsto a^i c a^{-i}$ and $H_{k,m} \rightarrow (G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle : a \mapsto a, b \mapsto b, c \mapsto c_0$ extend to inverse homomorphisms, which proves the first statement of (b). One easily checks that the relations $a^k = 1$, ${}^a b = c_0 b$ and ${}^a c_i = c_{i+1}$ (for all $i \in \mathbb{Z}/k\mathbb{Z}$) imply that $c_{k-1} \cdots c_1 c_0 = 1$, proving the second statement of (b). Since $H_{k,m} \cong (G_{k,m} * \langle b \rangle) \rtimes \langle a \rangle$, $B_{k,k,m} \cong G_{k,m} \rtimes \langle a \rangle$ and the action of $\langle a \rangle$ on $G_{k,m}$ is identical in both groups, we may view $B_{k,k,m}$ as a subgroup of $H_{k,m}$. \square

Note that given $s \in \mathbb{Z}/k\mathbb{Z}$, the automorphisms σ^s and τ^s are well-defined, as $\sigma^k = \text{id}$ and $\tau^k = \text{id}$. If $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ and $[r]$ is the smallest positive representative of r modulo $\text{lcm}(k, m)$, then we define $c_{r-1} \cdots c_1 c_0 := c_{[r]-1} \cdots c_1 c_0$.

Lemma 4.2. *In $G_{k,m} \rtimes \langle a \rangle$ we have $c_{r-1} \cdots c_1 c_0 = a^r \cdot (a^{-1} c_0)^r$.*

Proof. For all $s \in \mathbb{Z}/k\mathbb{Z}$ we have $a^s c_0 a^{-s} = \sigma^s(c_0) = c_s$, so

$$(a^{-1} c_0)^r = a^{-r} a^{r-1} c_0 a^{-(r-1)} a^{r-2} c_0 \cdots a^{-2} a c_0 a^{-1} c_0 = a^{-r} c_{r-1} \cdots c_1 c_0. \quad \square$$

The following proposition and lemma will be used in §6 to prove the equivalence (1) \Leftrightarrow (2) of Theorem 1.1. Recall that if $\gcd(k, m) \leq 2$, then given $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ there exists a unique element $r^* \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ with $r^* \equiv r \pmod{k}$ and $r^* \equiv -r \pmod{m}$ (Lemma 2.9).

Proposition 4.3. *Let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$.*

- (a) *Every group is (k, k, m, r) -quasi-Burnside if and only if there exist $x, y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$.*
- (b) *Suppose $\gcd(k, m) \leq 2$. Then every group is (k, k, m, r^*) -quasi-Burnside if and only if there exist $x, y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x) x^{-1} = {}^y c_i^r$.*
- (c) *Every group is (k, m, r) -quasi-Honda if and only if there exist $v, w \in G_{k,m} * \langle b \rangle$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\tau^r(v) v^{-1} = {}^w c_i^r$.*

Proof. The proofs of (a) and (c) are similar, so we omit the proof of (c). Suppose every group is (k, k, m, r) -quasi-Burnside. Then by Proposition 2.11.(a), there exist $g, h \in B_{k,k,m}$ such that $a^r \cdot g(a^{-1} c)^r = {}^h c^r$, so by 4.1.(a) there exist $g, h \in G_{k,m} \rtimes \langle a \rangle$ such that $a^r \cdot g(a^{-1} c_0)^r = {}^h c_0^r$. Choose such g, h , and write $g = a^j \cdot x$ with $x \in G_{k,m}$ and $j \in \mathbb{Z}/k\mathbb{Z}$. Then $a^r \cdot a^j \cdot x(a^{-1} c_0)^r = {}^h c_0^r$. By conjugating both sides by a^{-j} , and writing $a^{-j} \cdot h = y \cdot a^i$ with $y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$, we find $a^r \cdot x(a^{-1} c_0)^r = a^{-j} {}^h c_0^r = {}^{y a^i} c_0^r$. By Lemma 4.2 we have $a^r \cdot x(a^{-r} c_{r-1} \cdots c_1 c_0) = {}^{y a^i} c_0^r$, so $\sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = {}^{y a^i} c_0^r = {}^y c_i^r$. The entire argument is reversible, proving statement (a).

Suppose $\gcd(k, m) \leq 2$. By (a), every group being (k, k, m, r^*) -quasi-Burnside is equivalent to the existence of $x, y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$. Choose such x, y, i . Inverting both sides of the equation, and then conjugating by x^{-1} gives $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x^{-1}) x = x^{-1} {}^y c_i^r$, so for $x' = x^{-1}$, $y' = x^{-1} y$ we have $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x') (x')^{-1} = {}^{y'} c_i^r$. This proves one implication of (b). By reading the argument backwards, one proves the other implication. \square

Lemma 4.4. *Let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$ and suppose $\frac{2}{k} + \frac{1}{m} \leq 1$. Then there do not exist $x, y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\sigma^r(x) x^{-1} = {}^y c_i^r$ or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$.*

Proof. Analogously to Proposition 2.11, one can show that every group G has the property that for all $x, y, z \in G$ we have

$$xy = z, x^k = y^k = z^m = 1 \Rightarrow \exists x', x'', z' \in G : x'x'' = z', x' \sim x^{-r}, x'' \sim x^r, z' \sim z^r$$

if and only if there exist $g, h \in B_{k,k,m}$ such that $a^{-r} \cdot {}^g a^r = {}^h c^r$. By Proposition 3.8, the group $\mathrm{PSL}_2(\mathbb{R})$ does not have the property above, so such g, h do not exist. If $x, y \in G_{k,m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ satisfy $\sigma^r(x)x^{-1} = {}^y c_i^r$, then $a^r x a^{-r} x^{-1} = {}^{y a^i} c_0^r$. Conjugating both sides by $x^{-1} a^{-r}$ then gives $a^{-r} \cdot x^{-1} a^r = x^{-1} a^{-r} y a^i c_0^r$. Then by Proposition 4.1.(a) there exist g, h as above, giving a contradiction and proving the first statement of the lemma.

Let $x, y \in G_{k,m}$, $i \in \mathbb{Z}/k\mathbb{Z}$ such that $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$. By Lemma 4.2 we have ${}^y c_i^r = (c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x) c_{r-1} \cdots c_1 c_0 x^{-1} = ((a^{-1} c_0)^{-r} a^{-r}) a^r x a^{-r} (a^r (a^{-1} c_0)^r) x^{-1}$. Thus ${}^{y a^i} c_0^r = {}^y c_i^r = (a^{-1} c_0)^{-r} x (a^{-1} c_0)^r x^{-1}$. By Proposition 4.1.(a) there now exist $v, w \in B_{k,k,m}$ such that $(a^{-1} c)^{-r} \cdot {}^v (a^{-1} c)^r = {}^w c^r$. Define $\alpha : B_{k,k,m} \rightarrow B_{k,k,m}$, $a \mapsto a^{-1} c$, $c \mapsto a^{-1} c a$. One easily shows that α is a homomorphism and that α^2 is the identity, so α is an automorphism of order 2 that sends $a^{-1} c$ to a . Applying α to our formula gives $a^{-r} \cdot \alpha(v) a^r = \alpha(w) a^{-1} c^r$, but in the beginning of the proof we showed that such elements $g := \alpha(v)$, $h := \alpha(w) a^{-1}$ do not exist. \square

5 Reduced words

In this section we define a special notion of a reduced word in a free product of a group and an infinite cyclic group. This special notion has the advantage that under certain automorphisms of this free product, the “lengths” of these reduced words are preserved. (Proposition 6.1.) We will use this special notion to prove a combinatorial group theoretic result (Proposition 6.2), with which we can prove the equivalence (1) \Leftrightarrow (2) of Theorem 1.1.

Throughout this section, G will be a group and $\langle b \rangle$ will be an infinite cyclic group.

Definition 5.1. Let s be a non-negative integer. A reduced word of length s in $G * \langle b \rangle$ is a sequence $(u_0, u_1, \dots, u_{2s})$ with $u_{2i} \in G$ for $0 \leq i \leq s$ and $u_{2i+1} \in \{b, b^{-1}\}$ for $0 \leq i < s$ such that there does not exist i with $0 < i < s$, $u_{2i} = 1$ and $u_{2i-1} = u_{2i+1}^{-1}$.

Note that the length of a reduced word is the number of its letters that are equal to b or b^{-1} .

Proposition 5.2. For each non-negative integer s , let Y_s be the set of reduced words of length s in $G * \langle b \rangle$. Then for every $g \in G * \langle b \rangle$ there is a unique non-negative integer s and a unique $(u_0, \dots, u_{2s}) \in Y_s$ such that $g = u_0 u_1 \cdots u_{2s}$. \square

We say that (u_0, \dots, u_{2s}) is the reduced word of g .

Proof. We follow a method that J.-P. Serre used in the proof of Theorem 1 in section 1.2 of [9], originally invented by B.L. van der Waerden [11].

Let $C = G * \langle b \rangle$ and let $Y = \bigsqcup_{s \in \mathbb{Z}_{\geq 0}} Y_s$ be the disjoint union of all Y_s 's. We will let C act on Y by letting G and $\langle b \rangle$ act on Y separately. For every non-negative integer s , we define the action of G on Y by the maps

$$\alpha_s : G \times Y_s \rightarrow Y_s : (v, (u_0, \dots, u_{2s})) \mapsto (v u_0, \dots, u_{2s}).$$

For all $\epsilon \in \{\pm 1\}$ we define the map

$$\beta_\epsilon : Y \rightarrow Y : (u_0, \dots, u_{2s}) \mapsto \begin{cases} (u_2, \dots, u_{2s}), & \text{if } s \geq 1, u_0 = 1, u_1 = b^{-\epsilon}, \\ (1, b^\epsilon, u_0, \dots, u_{2s}), & \text{else.} \end{cases}$$

One easily checks that $\beta_{-1}\beta_1 = \beta_1\beta_{-1} = \text{id}$, thus β_1 is a permutation of Y . Hence we may define the action of $\langle b \rangle$ by the group homomorphism $\langle b \rangle \rightarrow \text{Sym}(Y)$ sending b to β_1 .

We finish the proof by showing that the map $\theta : Y \rightarrow C$ sending (u_0, \dots, u_{2s}) to $u_0 \cdots u_{2s}$ is a bijection. For all $w \in Y$ and all $h \in G \cup \{b, b^{-1}\}$ we have $\theta(hw) = h\theta(w)$, so $\theta(gw) = g\theta(w)$ for all $g \in C$. Thus for all $g \in C$ we have $g = g \cdot \theta((1)) = \theta(g \cdot (1))$, where $(1) \in Y$ is the word consisting of only $1 \in G$, so θ is surjective. One can show, by induction on the length of w , that for all $w \in Y$ one has $\theta(w) \cdot (1) = w$. So if $\theta(w) = \theta(w')$, then $w = \theta(w) \cdot (1) = \theta(w') \cdot (1) = w'$, so θ is injective. \square

Lemma 5.3. *Let $(u_i)_{i=0}^{2s}, (v_i)_{i=0}^{2s'}$ be reduced words in $G * \langle b \rangle$ with s, s' non-negative integers. If $(u_0, \dots, u_{2s-1}, u_{2s}v_0, v_1, \dots, v_{2s'})$ is not a reduced word in $G * \langle b \rangle$, then $s \geq 1, s' \geq 1, u_{2s-1} = v_1^{-1}$, and $u_{2s}v_0 = 1$. \square*

Proof. Suppose that $w := (u_0, \dots, u_{2s-1}, u_{2s}v_0, v_1, \dots, v_{2s'})$ is not a reduced word. Then there exists a triplet (w_1, w_2, w_3) of consecutive elements of w for which one has $w_2 = 1$ and $w_1 = w_3^{-1} \in \{b, b^{-1}\}$. Since $(u_i)_{i=0}^{2s}, (v_i)_{i=0}^{2s'}$ are reduced words, w_1, w_2, w_3 are not all three elements of $(u_i)_{i=0}^{2s}$, and not all three elements of $(v_i)_{i=0}^{2s'}$. So $s \geq 1, s' \geq 1, w_1 = u_{2s-1}, w_2 = u_{2s}v_0$ and $w_3 = v_1$. \square

Definition 5.4. We define $\text{len} : G * \langle b \rangle \rightarrow \mathbb{Z}_{\geq 0}$ to be the function that sends an element of $G * \langle b \rangle$ to the length of its unique reduced word given by Proposition 5.2.

Proposition 5.5. *For all $u \in G * \langle b \rangle$ we have $\text{len}(u) = \text{len}(u^{-1})$. More precisely, if $(u_i)_{i=0}^{2s}$ is the reduced word of some $u \in G * \langle b \rangle$ with s a non-negative integer, then $(u_i^{-1})_{i=0}^{2s}$ is the reduced word of u^{-1} . \square*

Lemma 5.6. *Let $u, v \in G * \langle b \rangle$ and let $(u_i)_{i=0}^{2s}, (v_i)_{i=0}^{2s'}$ be their respective reduced words with s, s' non-negative integers. Then there exists an integer $0 \leq n \leq \min\{s, s'\}$ such that the sequence $w_n := (u_0, \dots, u_{2s-2n-1}, u_{2s-2n}v_{2s'-2n}^{-1}, v_{2s'-2n-1}^{-1}, \dots, v_0^{-1})$ is a reduced word. Moreover, for the smallest such integer n , we have that w_n is the reduced word of uv^{-1} , and $\text{len}(uv^{-1}) = s + s' - 2n$.*

Note that $2n$ is the number of pairs of letters that cancel each other in uv^{-1} .

Proof. Let $0 \leq n \leq \min\{s, s'\}$ be an integer. Proposition 5.5 gives that $(v_i^{-1})_{i=0}^{2s'}$ is a reduced word, so if $n = \min\{s, s'\}$, then Lemma 5.3 gives that w_n is a reduced word. Suppose n is the smallest integer such that w_n is a reduced word. Then for all i with $0 \leq i < n$ we have that $(u_0, \dots, u_{2s-2i-1}, u_{2s-2i}v_{2s'-2i}^{-1}, v_{2s'-2i-1}^{-1}, \dots, v_0^{-1})$ is not a reduced word, so then Lemma 5.3 gives $u_{2s-2i-1}u_{2s-2i}v_{2s'-2i}^{-1}v_{2s'-2i-1}^{-1} = 1$. Thus $uv^{-1} = u_0 \cdots u_{2s-2n-1} \cdot u_{2s-2n}v_{2s'-2n}^{-1} \cdot v_{2s'-2n-1}^{-1} \cdots v_0^{-1}$, so w_n is the reduced word of uv^{-1} . Now $\text{len}(uv^{-1}) = s + s' - 2n$. \square

6 Passing to a subgroup

In this section, we use the special notion of a reduced word from the last section to prove that there exists a solution of a certain equation in the free product of some group G with an infinite cyclic group $\langle b \rangle$ if and only if there exists a solution of at least one of four equations in G (Proposition

6.2). These equations are very similar to those in the theory about when certain equations in $G_{k,m}$ and $G_{k,m} * \langle b \rangle$ have solutions (Proposition 4.3 and Lemma 4.4). This allows us to finish the proof of Theorem 1.1, by applying Proposition 6.2 to our group $G_{k,m}$, and then using the theory from §4.

Analogously to the existence of τ just before Proposition 4.1 we have that given a group G , an infinite cyclic group $\langle b \rangle$, $\varphi \in \text{Aut}(G)$, and $p \in G$, there exists a unique $\psi \in \text{Aut}(G * \langle b \rangle)$ with $\psi|_G = \varphi$, $\psi(b) = pb$.

Proposition 6.1. *Let $\varphi \in \text{Aut}(G)$, $p \in G$, and let ψ be the unique automorphism of $G * \langle b \rangle$ with $\psi|_G = \varphi$, $\psi(b) = pb$. Then for all $v \in G * \langle b \rangle$ we have $\text{len}(v) = \text{len}(\psi(v))$. More precisely, if $(v_i)_{i=0}^{2s}$ is the reduced word of v with s a non-negative integer, define for all $0 \leq i < s$ the number $\epsilon_{2i+1} \in \{\pm 1\}$ to be such that $v_{2i+1} = b^{\epsilon_{2i+1}}$, define*

$$p_\epsilon = \begin{cases} p & \text{if } \epsilon = 1, \\ 1 & \text{if } \epsilon = -1, \end{cases}$$

and define

$$\begin{cases} u_0 = \varphi(v_0)p_{\epsilon_1}, \\ u_{2i} = (p_{-\epsilon_{2i-1}})^{-1}\varphi(v_{2i})p_{\epsilon_{2i+1}} & \text{if } 0 < i < s, \\ u_{2i+1} = v_{2i+1} & \text{if } 0 \leq i < s, \\ u_{2s} = (p_{-\epsilon_{2s-1}})^{-1}\varphi(v_{2s}). \end{cases}$$

Then $(u_i)_{i=0}^{2s}$ is the reduced word of $\psi(v)$.

Proof. One straightforwardly checks that $\psi(v) = u_0 \cdots u_{2s}$, and that $(u_i)_{i=0}^{2s}$ is a reduced word. \square

Proposition 6.2. *Let G be a group, $\langle b \rangle$ an infinite cyclic group, $\varphi \in \text{Aut}(G)$, $p, q \in G$, and let ψ be the unique automorphism of $G * \langle b \rangle$ with $\psi|_G = \varphi$, $\psi(b) = pb$. Then there exist $v, w \in G * \langle b \rangle$ such that $\psi(v)v^{-1} = {}^wq$ if and only if there exist $x, y \in G$ such that*

$$\varphi(x)x^{-1} = {}^yq, \text{ or } \varphi(x)px^{-1} = {}^yq, \text{ or } p^{-1}\varphi(x)x^{-1} = {}^yq, \text{ or } p^{-1}\varphi(x)px^{-1} = {}^yq.$$

Proof. To prove the implication \Leftarrow , choose $v = x$, $w = y$ in the first case, $v = xb$, $w = y$ in the second case, $v = b^{-1}x$, $w = b^{-1}y$ in the third case and $v = b^{-1}xb$, $w = b^{-1}y$ in the fourth case.

As for the other implication: if $q = 1$, then any $y \in G$ satisfies $\varphi(1)1^{-1} = {}^yq$, so we may assume that $q \neq 1$. Let $v, w \in G * \langle b \rangle$ be such that $\psi(v)v^{-1} = wqw^{-1}$. Note that $\text{len}(\psi(v)) = \text{len}(v)$ by Proposition 6.1. Let $(u_i)_{i=0}^{2s}$, $(v_i)_{i=0}^{2s}$, $(w_i)_{i=0}^{2t}$ be the reduced words of respectively $\psi(v)$, v , w . As $q \neq 1$, we have $w_{2t}qw_{2t}^{-1} \neq 1$. Clearly $(w_0, \dots, w_{2t-1}, w_{2t}q)$ is a reduced word, and by Proposition 5.5 the sequence $(w_i^{-1})_{i=2t}^0$ is a reduced word. Lemma 5.3 then gives that $(w_0, \dots, w_{2t}qw_{2t}^{-1}, \dots, w_0^{-1})$ is the reduced word of wqw^{-1} .

We want to apply Lemma 5.6 to $\psi(v)$ and v to find the reduced word of $\psi(v)v^{-1}$. Since $\psi(v)v^{-1} = wqw^{-1}$, we have $\text{len}(\psi(v)v^{-1}) = \text{len}(wqw^{-1}) = 2t$. Thus n as in Lemma 5.6 is equal to $s - t$. We have $u_{2s-2(s-t)} = u_{2t}$ and $v_{2s-2(s-t)} = v_{2t}$, so Lemma 5.6 gives that $(u_0, \dots, u_{2t-1}, u_{2t}v_{2t}^{-1}, v_{2t-1}^{-1}, \dots, v_0^{-1})$ is the reduced word of $\psi(v)v^{-1}$. As $\psi(v)v^{-1} = wqw^{-1}$, the middle letters of their reduced words are equal, so $u_{2t}v_{2t}^{-1} = w_{2t}qw_{2t}^{-1}$. Proposition 6.1 gives $u_{2t} \in \{\varphi(v_{2t}), \varphi(v_{2t})p, p^{-1}\varphi(v_{2t}), p^{-1}\varphi(v_{2t})p\}$, so

$$w_{2t}qw_{2t}^{-1} = u_{2t}v_{2t}^{-1} \in \{\varphi(v_{2t})v_{2t}^{-1}, \varphi(v_{2t})pv_{2t}^{-1}, p^{-1}\varphi(v_{2t})v_{2t}^{-1}, p^{-1}\varphi(v_{2t})pv_{2t}^{-1}\}. \quad \square$$

Lemma 6.3. *Let $r \in (\mathbb{Z}/\text{lcm}(k, m)\mathbb{Z})^*$, let $i \in \mathbb{Z}/k\mathbb{Z}$, let $\langle b \rangle$ be an infinite cyclic group, and let $G_{k, m}$, σ , τ , $c_{r-1} \cdots c_1 c_0$ be defined as in §4. Suppose $\frac{2}{k} + \frac{1}{m} \leq 1$.*

- (a) *Then there exist $v, w \in G_{k, m} * \langle b \rangle$ such that $\tau^r(v)v^{-1} = {}^w c_i^r$ if and only if there exist $x, y \in G_{k, m}$ such that $\sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$ or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)x^{-1} = {}^y c_i^r$.*
- (b) *If $\gcd(k, m) > 2$, then there exist $v, w \in G_{k, m} * \langle b \rangle$ such that $\tau^r(v)v^{-1} = {}^w c_i^r$ if and only if there exist $x, y \in G_{k, m}$ such that $\sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$.*

Proof. Applying Proposition 6.2 to $G = G_{k, m}$, $\varphi = \sigma^r$, $p = c_{r-1} \cdots c_1 c_0$, $q = c_i^r$, $\psi = \tau^r$ gives that there exist $v, w \in G_{k, m} * \langle b \rangle$ such that $\tau^r(v)v^{-1} = {}^w c_i^r$ if and only if there exist $x, y \in G_{k, m}$ such that $\sigma^r(x)x^{-1} = {}^y c_i^r$, or $\sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$, or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)x^{-1} = {}^y c_i^r$, or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$. By Lemma 4.4, there do not exist $x, y \in G_{k, m}$ such that $\sigma^r(x)x^{-1} = {}^y c_i^r$ or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$, proving (a).

Let $\langle \alpha \rangle$ be a cyclic group of order $\gcd(k, m)$, and define the group homomorphism $\chi : G_{k, m} \rightarrow \langle \alpha \rangle$ so that for all $i \in \mathbb{Z}/k\mathbb{Z}$ one has $\chi(c_i) = \alpha$. Notice that $\chi \circ \sigma^r = \chi$. Suppose $\gcd(k, m) > 2$. If there are $x, y \in G_{k, m}$ such that $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)x^{-1} = {}^y c_i^r$, then applying χ would give $\alpha^{-r} = \alpha^r$, so then $\alpha^{2r} = 1$. This contradicts $\gcd(k, m) > 2$, proving (b). \square

Proof of the equivalence (1) \Leftrightarrow (2) of Theorem 1.1. If $\frac{2}{k} + \frac{1}{m} \geq 1$, then by Theorem 1.2 every group is (k, k, m, r) -quasi-Burnside, so by Proposition 2.4.(a) every group is also (k, m, r) -quasi-Honda.

Suppose that $\frac{2}{k} + \frac{1}{m} < 1$. We prove the equivalence in the case where $\gcd(k, m) > 2$ and in the case where $\gcd(k, m) \leq 2$. By Proposition 4.3.(c), every group is (k, m, r) -quasi-Honda if and only if there exist $v, w \in G_{k, m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\tau^r(v)v^{-1} = {}^w c_i^r$.

If $\gcd(k, m) > 2$, then Lemma 6.3.(b) gives that the existence of such v, w, i is equivalent to the existence of $x, y \in G_{k, m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$. By Proposition 4.3.(a) this is equivalent to every group being (k, k, m, r) -quasi-Burnside. If $\gcd(k, m) \leq 2$, then Lemma 6.3.(a) gives the existence of v, w, i as above is equivalent to the existence of $x, y \in G_{k, m}$ and $i \in \mathbb{Z}/k\mathbb{Z}$ such that $\sigma^r(x)c_{r-1} \cdots c_1 c_0 x^{-1} = {}^y c_i^r$ or $(c_{r-1} \cdots c_1 c_0)^{-1} \sigma^r(x)x^{-1} = {}^y c_i^r$. Propositions 4.3.(a) and 4.3.(b) give that this is equivalent to every group being (k, k, m, r) -quasi-Burnside or every group being (k, k, m, r^*) -quasi-Burnside. \square

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