Symmetry Breaking and Topological Defects: A Theoretical Framework

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Symmetry Breaking and Topological Defects: A Theoretical Framework

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Abstract

Topological defects are field excitations that are stabilised by topological obstructions. They emerge in symmetry breaking phase transitions. Such transitions are mathematically described by quotients of Lie groups and result in a vacuum manifold. Topological defects can be characterised by the topology of the vacuum manifold using homotopy theory. As such, the study of topological defects presents a surprising application of the theoretical field of algebraic topology.

This thesis provides the mathematical background of Lie groups and homotopy theory as well as the physical background of field theory and symmetry breaking needed to understand topological defects. We look at some examples of topological defects and the grand unified theory of $SU(5)$. We will conclude that strings cannot form as a result of symmetry breaking phase transitions of the grand unified theory of $SU(5)$.

Contents

Chapter 1

Introduction

"Topological defects are ubiquitous in physics and also appear in cosmology" [1]. Topological defects turn-up in the discovery of gauge bosons in the standard model in particle physics, which describes the elementary particles widely thought of as the building blocks of our universe. They also play a major role in grand unified theories. Grand unified theories explore the possibility of a merger of the electromagnetic interactions with the nuclear interactions.

The study of topological defects is an alluring subject because of the extensive mathematics involved. The mathematics used to describe topological defects is an exquisite study on it's own. What makes this study of topological defects especially captivating is the unexpected application of the, at first sight, mere theoretical algebraic topology in physics. The direct physical relevance of homotopy theory is remarkable.

Topological defects typically arise during symmetry breaking phase transitions. When the symmetry of a system breaks, the newly defined ground states are incoherently chosen, which may result in structures that preserve excitations of the field. These are topological defects. The type of topological defects formed is determined by the topology of the vacuum manifold.

In many papers and researches the rigorous maths is swept under the rug, for more accessible papers. However, more rigorous maths also has it's beauty, therefore this thesis will cover the lion's share of the mathematics needed to describe topological defects.

The mathematical subjects included in this thesis revolves around Lie groups, Lie algebras, homogeneous manifolds and homotopy theory. Lie groups and Lie algebras are inescapable in mathematics, describing all sorts of transformations. As for the physics, we will study field theory, gauge theory and symmetry breaking. The goal of this thesis is to explain topological defects at an undergraduate level, including the mathematical splendour.

This thesis assumes a graduate level of mathematics and physics as prior knowledge, specifically group theory and differentiable manifolds are important.

In chapter 2 we will start our study of Lie groups and algebras which, among other things, form the mathematical frame work of symmetries. We extend our mathematical framework to homogeneous manifolds and we will learn to characterise such manifolds as quotients of Lie groups.

Next, in chapter 3, we will be introducing homotopy groups, which allow us to distinguish between topological spaces such as homogeneous manifolds. In order to calculate homotopy groups, we will acquaint ourselves with two long exact sequences of homotopy groups.

In chapter 4 we will commence the physical background needed to describe topological defects, with an introduction to field theory. A specialisation area in field theory is that of gauge theory, which we discuss with a passage along Yang-Mills theory.

We begin chapter 6 with an explanation of symmetry breaking, which describes the forming mechanism of topological defects. Subsequently, we will explain what topological defects are and look at the examples of a domain wall, a string and a monopole. Finally, we will study the grand unified theory of $SU(5)$, and determine the topological defects that may arise as a result of the symmetry breaking pattern.

Chapter 2

Lie groups and algebras

In this chapter we will lay the ground works for the rest of this thesis. We will consider Lie groups and Lie algebras and some important properties they have. We will focus on matrix Lie groups and algebras as those will be paramount in what follows. We will be using the definitions of manifolds, smoothness, differentials etc. as described in [2].

2.1 Lie groups

Lie groups are a special set of differentiable manifolds. They are differentiable manifolds that also have a group structure. This group structure is compatible with the structure of the differentiable manifold.

The theory in this section is based on chapter three of [3].

Let us begin by defining a Lie group.

Definition 2.1.1. *A* ***Lie group*** *G is a differentiable manifold, with a group structure, such that the map:*

$$\psi : G \times G \to G, \quad (\sigma, \tau) \mapsto \sigma\tau^{-1} \tag{2.1}$$

is smooth.

This definition is equivalent to demanding that both the group operation (multiplication) and the map sending each element to it's inverse are smooth maps. Let us examine a few example of Lie groups.

- The simplest example is euclidean space $\mathbb{R}^n$ where we choose the operation to be vector addition. It is easy to check that vector addition is a smooth group operation with a smooth inverse.

- The non-zero complex numbers $\mathbb{C}^*$ form a Lie group under multiplication. Note that omitting the zero is necessary to ensure smoothness of the inverse map and to ensure that the group structure is well-defined.

- The unit circle S^1 as a closed subgroup of $\mathbb{C}^*$ becomes a Lie group by Cartan's theorem, see Theorem 2.1.2.

- The product of two Lie groups is again a Lie group, with the product manifold structure and the direct product chosen as group structure, see [2].

Cartan's theorem is a useful statement which we will use to identify Lie groups as subgroups of known Lie groups.

Theorem 2.1.2 (Closed subgroup theorem, Cartan)**.** *Suppose G is a Lie group, and let H be a closed subgroup of G, that is subgroup of G that is also a topologically closed subset of G, then H is a Lie group with group structure from G.*

Proof. See Theorem 20.12 in [2]. □

The matrix group $\mathrm{Mat}(n \times n, \mathbb{R})$ can be given a differentiable manifold structure, through the identification of each argument of the matrix with a coordinate in $\mathbb{R}^{n^2}$. With this identification we can define a topology on $\mathrm{Mat}(n \times n, \mathbb{R})$ and this turns the set into a differentiable manifold. The general linear group is a subgroup of the matrix group that deserves a little more attention. Let us define it.

Definition 2.1.3. *Let* $\det : \mathrm{Mat}(n \times n, \mathbb{R}) \to \mathbb{R}$ *be the determinant map.* ***The real general linear group of dimension*** n*,* $GL(n, \mathbb{R})$*, is the matrix subgroup of* $\mathrm{Mat}(n \times n, \mathbb{R})$*, defined as follows:*

$$GL(n, \mathbb{R}) = \det{}^{-1}(\mathbb{R} \setminus \{0\}).$$

Equivalently, it is the set of invertible real $n \times n$ matrices.
We can also define ***the complex general linear group****,* $GL(n, \mathbb{C})$*, as the set of invertible complex $n \times n$ matrices, which is a subgroup of* $\mathrm{Mat}(n \times n, \mathbb{C})$*. If* $\det_{\mathbb{C}} : \mathrm{Mat}(n \times n, \mathbb{C}) \to \mathbb{C}$ *is the determinant map for complex matrices we have,*

$$GL(n, \mathbb{C}) = \det{}_{\mathbb{C}}^{-1}(\mathbb{C} \setminus \{0\}).$$

When we talk about the general linear group we may refer to both the real and the complex general linear group, $GL(n, \mathbb{R}/\mathbb{C})$. Which is meant will be clear from context, often the distinction is redundant.

Trough the restriction of charts, $GL(n, \mathbb{R}/\mathbb{C}) \subseteq \text{Mat}(n \times n, \mathbb{R})$ becomes a differentiable manifold, as explained in example 1.26 in [2]. In fact, the general linear group of dimension n, $GL(n, \mathbb{R}/\mathbb{C})$, is a Lie group under matrix multiplication.

Lemma 2.1.4. *The general linear group $GL(n, \mathbb{R}/\mathbb{C})$ is a Lie group.*

Proof. We will prove this for $\mathbb{R}$ only, but the proof for $\mathbb{C}$ is completely analogous.

All matrices in the general linear group are invertible, so we can define a group structure on $GL(n, \mathbb{R})$ with matrix multiplication. We need to check that matrix multiplication and taking the inverse are smooth maps.

For multiplication, we simply note that, written as a map between $\mathbb{R}^{n^2}$, all component maps of the multiplication are smooth as polynomials in the coefficients of our matrices.

Smoothness of inverses is a bit harder to prove. We use Cramers rule (Proposition 10.22 [4]) to write inverses of matrices as a scalar multiplication combined with polynomials. For $A \in GL(n, \mathbb{R})$ we define A_{ij} as the matrix A with the i^{th} row and the j^{th} column removed, and $\tilde{A} = (a_{ij})$ with $a_{ij} = (-1)^{i+j} \det(A_{ji})$ then we have:

$$A^{-1} = \frac{1}{\det A} \tilde{A}. \tag{2.2}$$

Here $\det A \neq 0$ and every entry of $\tilde{A}$ is a polynomial in the entries of matrix A. This is clearly a smooth map, so the general linear group of dimension n is a Lie group. □

Some useful subgroups of the general linear group are the following:

- The special linear group, both complex and real,

$$SL(n) = \{A \in GL(n, \mathbb{R}/\mathbb{C}) \,|\, \det(A) = 1\}, \tag{2.3}$$

- The orthogonal group

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^\top = A^{-1}\}, \tag{2.4}$$

- The special orthogonal group

$$SO(n) = \{A \in SL(n, \mathbb{R}) | A^\top = A^{-1}\}, \tag{2.5}$$

- The unitary group

$$U(n) = \{A \in GL(n, \mathbb{C}) | A^\dagger = A^{-1}\}, \tag{2.6}$$

where $A^\dagger$ is the complex conjugate transposed, $A^\dagger = (A^*)^\top$, and

- The special unitary group

$$SU(n) = \{A \in SL(n, \mathbb{C}) | A^\dagger = A^{-1}\}. \tag{2.7}$$

With the closed subgroup theorem of Cartan, Theorem 2.1.2, we can conclude that these subgroups are all Lie groups. We will be using the general linear group, both complex and real, and their Lie subgroups, repeatedly in the rest of this thesis.

2.2 Lie algebras

Now that we have introduced Lie groups, it is time to consider Lie algebras. Both Lie groups and Lie algebras will be important in describing gauge theories and field symmetries later in this thesis. This section is based on Chapter 8 of [2].

First we discuss some basic theory of vector fields. Recall that a tangent vector on a manifold M is a derivation of a smooth map on M.

Definition 2.2.1. *A linear map $v : C^\infty(M) \to \mathbb{R}$ is a **derivation** at p, if*

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M). \tag{2.8}$$

*The set of all derivations at p, T_pM, is the **tangent space** of M at p and $v \in T_pM$ is a **tangent vector** at p. The **tangent bundle** of M, TM, is the union over $p \in M$ of all tangent spaces T_pM.*

Now let us define a vector field.

Definition 2.2.2. *Let M be a smooth manifold, and TM its tangent bundle. A **vector field** is a continuous map $X : M \to TM$, given by $p \to X_p$ such that for all $p \in M$ we have $X_p \in T_pM$.*
*In other words, a **vector field** is a section of the projection map $\pi : TM \to M$, that sends any tangent vector to its base.*

We define a rough vector field similarly, only without the requirement of continuity. A vector field X is called **smooth** if it is a smooth map between the manifolds M and TM. We introduce the notation $\mathfrak{X}(M)$ for the set of all smooth vector fields on M.

Lemma 2.2.3. *The set $\mathfrak{X}(M)$ is a vector space under the following pointwise addition and scalar multiplication: $(aX + bY)_p = aX_p + bY_p$.*
Furthermore, if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$ then fX is a vector field defined by $(fX)_p = f(p) \cdot X_p$.

Proof. See page 177 and Proposition 8.8 in [2]. □

It is important to note the difference between fX and Xf. For $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, fX as defined above, is a vector field on M. On the other hand, Xf is a smooth function, defined by the action of the vector field X on the smooth map f. In other words, we have $Xf \in C^\infty(M)$ defined as $(Xf)(p) = X_p(f)$.

Next, we introduce a relation between two vector fields. Recall that the differential $dF_p : T_pM \to T_{F(p)}N$ of a map $F : M \to N$ at point p is the map that we define through its evaluation in a function $f \in C^\infty(N)$:

$$dF_p(v)(f) = v(f \circ F). \tag{2.9}$$

See definition 1.22 of [3] .

For a smooth map $F : M \to N$ and a vector field X on M, we can find tangent vectors of N with the differential of F, $dF_p(X_p) \in T_{F(p)}N$. The set of vectors defined as such does not always form a vector field on N. For example, if F is not surjective, we cannot identify a tangent vector at each point in N. Still, there may exist vector fields on N containing the vectors $dF_p(X_p) \in T_{F(p)}N$.

Definition 2.2.4. *Let $F : M \to N$ be a smooth map of manifolds, X a vector field on M and Y a vector field Y on N. X and Y are called F-related if $\forall p \in M$ the relation $dF_p(X_p) = Y_{F(p)} \in T_{F(p)}N$ holds.*

A useful characterisation of two vector fields being F-related is given in the following proposition.

Proposition 2.2.5. *Suppose $F : M \to N$ is a smooth map of manifolds, and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are F-**related** if and only if for every open subset $U \subseteq N$ and every $f \in C^\infty(U, \mathbb{R})$ we have:*

$$X(f \circ F) = (Yf) \circ F. \tag{2.10}$$

Proof. Let $p \in M$ be any point and $f \in C^\infty(U, \mathbb{R})$ be any smooth real-valued function defined on an open subset $U \ni F(p)$ of N. Then we have

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)f \tag{2.11}$$

and

$$(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)}f. \tag{2.12}$$

These are equal for all $f \in C^{\infty}(U, \mathbb{R})$ if and only if $dF_p(X_p) = Y_F(p)$ for all $p \in M$, and thus X and Y are F-related. □

For a given function $F : M \to N$ and a smooth vector X field on M, there might not exist any vector field on N that is F-related to X. In some cases however, we do know that such a field exists and may even be unique. As could be expected, a diffeomorphism gives a one-to-one correspondence between a vector field in its domain and in its codomain.

Proposition 2.2.6. *Suppose M and N are smooth manifolds and $F : M \to N$ is a diffeomorphism. Then for every $X \in \mathfrak{X}(M)$, there exists a unique smooth vector field on N that is F-related to X.*

Proof. See Proposition 8.19 in [2]. □

F-relations can also be defined when F is an endomorphism. This will be important in the construction of vector fields that are homogeneous, that is, the vector field behaves similarly at every point on the manifold.

We will now define the Lie bracket which will be important for Lie algebras. The Lie bracket is a way to construct smooth vector fields given two smooth vector fields.

Definition 2.2.7. *The **Lie bracket** of the vector fields X and Y is defined by*

$$\begin{aligned} [X, Y] : C^{\infty}(M) &\to C^{\infty}(M), \\ [X, Y]f &= XYf - YXf. \end{aligned} \tag{2.13}$$

The notation XYf, indicates that first Y acts on f and consequently X acts on Yf.

Lemma 2.2.8. *The Lie bracket of a pair of smooth vector fields is a smooth vector field.*

Proof. See Lemma 8.25 of [2]. □

The Lie bracket has some nice properties.

Proposition 2.2.9. *For all $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in C^{\infty}(M)$ and $a, b \in \mathbb{R}$, the Lie bracket satisfies the following identities:*

(a) *Bilinearity:*

$$\begin{aligned}[aX + bY, Z] &= a[X, Z] + b[Y, Z] \\ [X, aY + bZ] &= a[X, Y] + b[X, Z]\end{aligned} \tag{2.14}$$

(b) *Antisymetry:*

$$[X, Y] = -[Y, X] \tag{2.15}$$

(c) *The Jacobi identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{2.16}$$

(d) *Product with smooth maps:*

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X \tag{2.17}$$

Proof. See Proposition 8.28 in [2]. □

Another important property of the Lie bracket is that it preserves F-relations.

Proposition 2.2.10. *If $F : M \to N$ is a smooth map between manifolds and $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are vector fields such that X_i and Y_i are F-related, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof. See Proposition 8.30 in [2]. □

The commutator operator and the map F commute for F-related vector fields. The preservation of F-relations will allow us to define the set of left-invariant vector fields. This will be important in the construction of Lie algebras of Lie groups. First we define a Lie algebra.

Definition 2.2.11. *A **Lie algebra** $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ with a bilinear operator $[\quad,\quad] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that is anti symmetric and satisfies the Jacobi identity, Equation 2.16.*

To get some feeling for Lie algebras we present a few examples.

1. The vector space $\mathrm{Mat}(n \times n, \mathbb{R})$ of real $n \times n$ matrices under the commutator bracket, $[A, B] = AB - BA$, becomes a Lie algebra of dimension n^2 denoted by $\mathfrak{gl}(n, \mathbb{R})$.

2. Similarly $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$ dimensional Lie algebra of $\mathrm{Mat}(n, \mathbb{C})$ with the commutator bracket.

3. If we have a vector space V and on the vector space of endomorphisms on V we define the commutator bracket similar to the matrix commutator $[A, B] = A \circ B - B \circ A$, we find a Lie algebra on $\mathrm{End}(V)$, denoted by $\mathfrak{gl}(V)$.

4. The space of all smooth vector fields $\mathfrak{X}(M)$ on a manifold M, under the Lie bracket is a Lie algebra.

5. We can define a bracket such that all entries map to zero. Under this definition any vector space becomes a Lie algebra. We call such Lie algebras abelian, as they relate to abelian Lie groups, see Corollary 3.50(b) of [3]. We will not go into this in this thesis.

Since Lie algebras are vector spaces, we can use all techniques of linear algebra to find their characteristic properties. We want to use this simplicity to better understand Lie groups, therefore we want to find a Lie algebra distinctive to a Lie group. For this, we define an important property of vector fields.

Definition 2.2.12. *A vector field $X \in \mathfrak{X}(G)$ on a Lie group G is called **left-invariant** if it is invariant under all left translations. In other words, X is left invariant if it is L_g-related to itself for every left-multiplication L_g. That is:*

$$d(L_g)_h(X_h) = X_{gh}, \quad \forall g, h \in G \tag{2.18}$$

The set of all left-invariant vector fields is a linear subspace of $\mathfrak{X}(G)$ and it is closed under the Lie bracket (Proposition 8.33 in [2]). These two properties make the set into a Lie algebra. We will call the set of left-invariant vector fields on a Lie group G, 'the' Lie algebra of G and denote it by $\mathfrak{g}$. The notation $\mathrm{Lie}(G)$ may be used to emphasise the Lie group G.

Definition 2.2.13. *Let G be a Lie group. **The Lie algebra** of G, often denoted by $\mathfrak{g}$, is the set of left-invariant vector fields on G.*

Let us take a look at the dimension of the Lie algebra $\mathfrak{g}$ of a Lie group G. The dimension of $\mathfrak{g}$ reflects the profound connection between the Lie algebra and the group, as a matter of fact, the dimensions of the algebra and the group are equal. Let us prove this.

Theorem 2.2.14. *Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. Then the evaluation map $\varepsilon : \mathfrak{g} \to T_eG$, $X \to X_e$ is a vector space isomorphism. We thus find* $\dim \mathfrak{g} = \dim T_eG = \dim G$.

Proof. We want to prove that ε is a bijective linear map. Note that by Lemma 2.2.3 linearity of ε is ensured.

For injectivity, suppose $X_e = 0$ for some $X \in \mathfrak{g}$. Then by left-invariance we find that $X_g = d(L_g)_e(X_e) = 0$, so X has to be the constant zero vector field and ε is injective.

To prove surjectivity, take $v \in T_eG$. We define a rough vector field

$$v^L|_g = d(L_g)_e(v). \tag{2.19}$$

Note that any left-invariant vector field X satisfying $X_e = v$ has to abide the equation above. Hence, if there is any left-invariant vector field on G that maps to v it is equal to v^L. We now prove that v^L is smooth and left-invariant.

We will use Proposition 8.14 in [2], which states that a vector field X on G is smooth if and only if for all $f \in C^\infty(G)$ the map Xf is a smooth map on G. Accordingly, we will prove that $v^L f$ is smooth for all maps $f \in C^\infty(G)$.

Define a curve $\gamma : (-1,1) \to G$, such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then for $g \in G$ we find

$$\begin{aligned} (v^L f)(g) &= v^L|_g f \\ &= d(L_g)_e(v) f. \end{aligned} \tag{2.20}$$

Here we used the definition of a vector field acting on a function, and the definition of v^L. Using the derivative and implementing the curve γ we find

$$\begin{aligned} d(L_g)_e(v) f &= v(f \circ L_g) \\ &= \gamma'(0)(f \circ L_g) \\ &= \frac{d}{dt}\Big|_{t=t0} (f \circ L_g \circ \gamma)(t). \end{aligned} \tag{2.21}$$

We rewrite this using $\varphi : (-1,1) \times G \to \mathbb{R}$, $\varphi(t,g) = f(g\gamma(t))$. Then φ is a smooth map as a composition of group multiplication and the smooth maps f and γ. Therefore, we conclude that $(v^L(f))(g) = \partial\varphi/\partial t(0,g)$ is a smooth map in g for all smooth maps $f \in C^\infty(G)$.

We now prove that v^L is left invariant, that is $d(L_h)_g(v^L|_g) = v^L|_{hg}$ for all $g, h \in G$. Take $g, h \in G$, then we have

$$\begin{aligned} d(L_h)_g(v^L|_g) &= d(L_h)_g \circ d(L_g)_e(v) = d(L_h \circ L_g)(v) \\ &= d(L_{hg})_e(v) = v^L|_{hg}. \end{aligned} \tag{2.22}$$

So we have found $v^L \in \mathfrak{g}$ and since $\varepsilon(v^L) = v^L|_e = d(L_e)_e(v) = id(v) = v$, the map ε is surjective. □

This theorem states something a lot stronger than only the equality of dimensions. First of all, we have that the Lie algebra of a Lie group G is isomorphic to the tangent space of G in the identity. This is useful for finding Lie algebras, because the space of left-invariant vector fields is generally harder to find than the tangent space. Further, the vector field v^L as defined in Equation 2.19 implies that any left-invariant vector field is in fact smooth and we are thus justified to omit the assumption of smoothness in Definition 2.2.13.

Lemma 2.2.15. *Any left-invariant rough vector field on a Lie group G is smooth.*

Proof. Suppose X is a left invariant vector field, and $X_e = v \in T_eG$. Then by Equation 2.19 we have that $X = v^L$ and thus X is smooth. □

Let us use Theorem 2.2.14 to find the Lie algebras of some Lie groups. First of all, some Lie algebras below Definition 2.2.11 are 'the' Lie algebras of well chosen Lie groups.

1. As the notation suggests, the Lie algebra of the general linear group, is $\mathfrak{gl}(n, \mathbb{R})$, in other words it is the vector space of $n \times n$ matrices. We know that for any element of a vector space, the tangent space of the vector space to this point is equal to the whole vector space. Using the identification between $\mathrm{Mat}(n \times n, \mathbb{R})$ and $\mathbb{R}^{n^2}$ it is clear that the tangent space of $\mathbb{R}^{n^2}$ in the identity element is equal to $\mathbb{R}^{n^2}$. Thus the Lie algebra of $GL(n, \mathbb{R})$ is defined as $\mathfrak{gl}(n, \mathbb{R}) := \mathrm{Mat}(n \times n, \mathbb{R})$.
2. Likewise, $\mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of $GL(n, \mathbb{C})$. This can be seen using the identification between $\mathrm{Mat}(n \times n, \mathbb{C})$ and $\mathbb{C}^{n^2}$.

Let us also take a look at the Lie algebras of some subgroups of the matrix groups. For this we use yet a different characterisation of the Lie algebra of a Lie group containing the exponential map. For general Lie groups the exponential map is a map from the Lie algebra to the Lie group. We will only use it for matrix Lie groups, in which case it is simply the exponential map we are used to. For A an $n \times n$ matrix we have:

$$\exp(A) = e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k, \tag{2.23}$$

which is a convergent sequence by Proposition 20.2 of [2].

For matrix Lie groups we can now define the Lie algebra as in [5].

Definition 2.2.16. *Let $G \subseteq Gl(n, \mathbb{R}/\mathbb{C})$. We can define the Lie algebra of G as*

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{R}) | \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}. \tag{2.24}$$

Using this characterisation we find the Lie algebras of $O(n)$ and $U(n)$.

- The orthogonal algebra,

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) |\, \exp(tX) \in O(n) \text{ for all } t \in \mathbb{R}\}. \tag{2.25}$$

  We know that $\exp(tX) \in O(n)$ if and only if

$$\exp(tX^\top) = (\exp(tX))^\top = (\exp(tX))^{-1} = \exp(-tX). \tag{2.26}$$

  Taking the derivative of the exponents with respect to t of the first and last expression and evaluating at $t = 0$, we obtain

$$\frac{d}{dt}\,|_{t=0} \exp(tX^\top) = X^\top = -X = \frac{d}{dt}\,|_{t=0} \exp(-tX). \tag{2.27}$$

  So we need $X^\top = -X$. The other direction is clear, since if $X^\top = -X$ we find $\exp(tX^\top) = \exp(-tX)$ and thus $\exp(tX) \in O(n)$. Hence, $\mathfrak{o}(n)$ is the set of skew-symmetric real matrices,

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) | X^\top = -X\}. \tag{2.28}$$

- The unitary algebra,

$$\begin{aligned} \mathfrak{u}(n) &= \{X \in \mathfrak{gl}(n, \mathbb{C}) |\, \exp(tX) \in U(n) \text{ for all } t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{gl}(n, \mathbb{C}) |\, \exp(tX)^\dagger = \exp(-tX) \text{ for all } t \in \mathbb{R}\}. \end{aligned} \tag{2.29}$$

  With the same differentiation as for $O(n)$, we obtain the condition $X^\dagger = -X$, which is also a sufficient condition. The acquire that $\mathfrak{u}(n)$ is the set of skew-hermitian complex matrices,

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X^\dagger = -X\}. \tag{2.30}$$

To find the Lie algebras of the special linear group, we use one more lemma.

Lemma 2.2.17. *For* $X \in \mathrm{Mat}(n \times n, \mathbb{R}/\mathbb{C})$, *we have*

$$\det(\exp(X)) = \exp(\mathrm{Tr} X). \tag{2.31}$$

Proof. See Lemma 4.13 of [5]. □

With this proposition, we can investigate the Lie algebras of the special linear group and therefore also of the subgroups $SO(N)$ and $SU(N)$.

- The special linear algebra, both in the complex and real case, requires $\det(\exp(tX)) = 1$. Using Lemma 2.2.17, we find

$$1 = \det(\exp(tX)) = \exp(t\mathrm{Tr}X). \tag{2.32}$$

  Since this need to hold for all $t \in \mathbb{R}$ we need $\mathrm{Tr}X = 0$. Clearly, if $\mathrm{Tr}X = 0$ we find $\det(\exp(tX)) = \exp(t\mathrm{Tr}X) = 1$, and therefore this is also a sufficient condition. We have,

$$\mathfrak{sl}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}/\mathbb{C}) | \mathrm{Tr}X = 0\}. \tag{2.33}$$

- The special orthogonal algebra is the Lie algebra of matrices X , for which $\exp(tX)$ is orthogonal and has unit determinant. This constitutes an intersection of $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{o}(n)$. So we obtain

$$\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) | X^\top = -X, \mathrm{Tr}X = 0\}. \tag{2.34}$$

- The special unitary algebra, follows with the same reasoning,

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X^\dagger = -X, \mathrm{Tr}X = 0\}. \tag{2.35}$$

2.2.1 The structure constants

We will describe the structure constants of Lie algebras following Section 3.2 of [6].

A Lie algebra is a vector space, this means that we can find a basis for the Lie algebra. We focus on matrix Lie groups G and the corresponding matrix Lie algebras $\mathfrak{g}$. In this case, the basis consists of $n = \dim G = \dim \mathfrak{g}$ generating matrices T_i. The commutator of the Lie algebra decomposes in terms of the generators.

Definition 2.2.18. *Let $\mathfrak{g}$ be a Lie algebra (or the Lie algebra of G) with generators T_i. The coefficients C_{ijk} such that*

$$[T_i, T_j] = \sum_{k=1}^{n} C_{ijk} T_k, \tag{2.36}$$

*are called the **structure constants** of the Lie algebra, or equivalently of the Lie group G.*

Structure constants are always anti-symmetric in the first two indices by anti-symmetry of the Lie bracket. Note that the structure constants are dependent on the choice of basis.

Let us take a look at an example.

$SU(2)$ is a matrix Lie group of dimension 3. The Lie algebra $\mathfrak{su}(3)$, as discussed before is the set of anti-Hermitian, traceless 2×2 matrices. We choose a basis $T_k = -\frac{i}{2}\tau_k$, where τ_k are the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.37}$$

We determine the structure constants. Direct calculation yields

$$[\tau_k, \tau_l] = 2i\varepsilon_{klm}\tau_m, \tag{2.38}$$

where ε_{klm} is the Levi-Civita symbol. The Levi-Civita symbol, or the permutation symbol, for three indices is defined as on page 172 of [7],

$$\varepsilon_{klm} = \begin{cases} 1 & \text{if } (klm) \text{ is an even permutation of } (123), \\ 0 & \text{if } \{k,l,m\} \neq \{1,2,3\}, \\ -1 & \text{if } (klm) \text{ is an odd permutation of } (123). \end{cases} \tag{2.39}$$

For $SU(3)$ with generators $-\frac{i}{2}\tau_k$, we find $C_{klm} = \varepsilon_{klm}$.

In physics it is customary to define the $\mathfrak{su}(2)$ algebra as the set of Hermitian, traceless 2×2 matrices, with basis $T_k = \frac{1}{2}\tau_k$. This can easily be realised by multiplying all matrices by a factor i. With the physics convention, we find purely imaginary structure constants $C_{klm} = i\varepsilon_{klm}$.

The generators of any Lie algebra can be chosen to form an orthonormal basis with normalisation according to

$$\mathrm{Tr}(t_i t_j) = -\frac{1}{2}\delta_{ij}. \tag{2.40}$$

When working in such a basis, the structure constants are not only anti-symmetric in the first two coordinates, but also in the last two, yielding fully anti-symmetric structure constants.

Lemma 2.2.19. *If (t_i) is an orthonormal basis of a Lie algebra, with normalisation according to Equation 2.40, then the structure constants of the Lie algebra are fully anti-symmetric.*

Proof. Using Theorem 2.2.18 we find, for fixed k,

$$[t_i, t_j]t_k = \sum_l C_{ijl} t_l t_k. \tag{2.41}$$

Using Equation 2.40 this yields,

$$\mathrm{Tr}([t_i, t_j]t_k) = \sum_l \mathrm{Tr}(C_{ijl}t_l t_k) = -\frac{1}{2}C_{ijk}. \tag{2.42}$$

Therefore we have

$$C_{ijk} = -2\mathrm{Tr}([t_i, t_j]t_k) = -2\left(\mathrm{Tr}(t_i t_j t_k) - \mathrm{Tr}(t_j t_i t_k)\right), \tag{2.43}$$

abd we conclude

$$\begin{aligned} C_{ikj} &= -2\left(\mathrm{Tr}(t_i t_k t_j) - \mathrm{Tr}(t_k t_i t_j)\right) \\ &= -2\left(\mathrm{Tr}(t_j t_i t_k) - \mathrm{Tr}(t_i t_j t_k)\right) \\ &= -C_{ijk}. \end{aligned} \tag{2.44}$$

For the second equality, we use that the trace is invariant under cyclic shifts, which follows from Lemma 9.24 in [4]. □

The structure constants will re-appear when we are looking for the general description of gauge fields of compact Lie groups.

2.2.2 The adjoint representation

When talking about Lie groups and Lie algebras, representations are often considered. In this thesis the adjoint representation of matrix Lie groups is a relevant representation, which we will therefore discuss. Let us first give a more general definition of a representation.

Definition 2.2.20. *A* ***representation of a Lie group*** *G in a linear space V is a group homomorphism of G to the space of automorphism on V,*

$$T : G \to Aut(V). \tag{2.45}$$

A ***representation of a Lie algebra*** *$\mathfrak{g}$ in a linear space V is a vector space homomorphism from $\mathfrak{g}$ to the space of endomorphism of V, where the homomorphisms respect the Lie bracket,*

$$T : \mathfrak{g} \to End(V). \tag{2.46}$$

We now specify to matrix Lie groups and algebras. We start with the adjoint representation of a matrix Lie group G, in the linear space of its own Lie algebra $\mathfrak{g}$.

Definition 2.2.21. *The* ***adjoint representation,*** Ad*, of a matrix Lie group G in its Lie algebra* $\mathfrak{g}$ *is given by,*

$$\begin{aligned} \mathrm{Ad} : G &\to Aut(\mathfrak{g}), \\ g &\mapsto (A \mapsto gAg^{-1}). \end{aligned} \tag{2.47}$$

Lemma 2.2.22. *The map Ad is a well-defined group homomorphism.*

Proof. We need to prove that for $g \in G$ and $A \in \mathfrak{g}$ we have $gAg^{-1} \in \mathfrak{g}$ and that the map Ad preserves the relations under the group operations and the commutator bracket.

Take $g \in G, A \in \mathfrak{g}$. Using the Definition 2.2.16, we find

$$gAg^{-1} \in \mathfrak{g} \iff \forall t \in \mathbb{R} \text{ , } \exp(tgAg^{-1}) \in G. \tag{2.48}$$

The equality $(gtAg^{-1})^k = g(tA)^k g^{-1}$ yields $\exp(gtAg^{-1}) = g\exp(tA)g^{-1}$, but we know that $A \in \mathfrak{g}$ and thus for all $t \in \mathbb{R}$ we have $\exp(tA) \in G$ and $g, g^{-1} \in G$, so $g\exp(tA)g^{-1} \in G$. As such, the adjoint map is well-defined.

We check the properties of a group homomorphism. For $g, g_1, g_2 \in G$ and $A \in \mathfrak{g}$, we have

$$\mathrm{Ad}(g_1 g_2)A = g_1 g_2 A g_2^{-1} g_1^{-1} = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2)A, \tag{2.49}$$

$$\mathrm{Ad}(g) \circ \mathrm{Ad}(g^{-1})A = gg^{-1}Agg^{-1} = \mathrm{Id}A \tag{2.50}$$

and thus Ad is a homomorphism. □

For matrix Lie groups we conclude that the adjoint representation is simply the conjugation map.

2.2.3 Compact Lie groups and algebras

In what follows we will mainly be working with compact and simple Lie groups and corresponding compact and simple Lie algebras. We will base this section on Section 2.6 of [6].

Sadly, the mathematical rigour of part of this section is beyond the scope of this thesis and we will not include all proofs or simply refer to more advanced sources. The statements can be found in Section 2.6 of [6] and their proofs follow from the theory presented in Chapter 4 of [8]. We do include these theorems, as they are important in later parts of the thesis when we consider non-abelian gauge fields.

Definition 2.2.23. ***A compact Lie group*** *is a Lie group that is compact as a topological space, that is to say, any open cover of the Lie group has a finite subcover.*

We will examine some earlier examples of matrix Lie groups. Since the topology on matrix Lie groups is defined by the topology on Euclidean space, any matrix Lie group is compact if and only if it is a closed an bounded subset of $\mathrm{Mat}(n \times n)$.

- $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$ are not compact, since they are not bounded. For example, the sequence

$$(A_k) = \begin{pmatrix} k & 0 \\ 0 & I_{n-1} \end{pmatrix}, k \in \mathbb{N} \tag{2.51}$$

  is contained in $GL(n, \mathbb{R}/\mathbb{C})$ and the norm of these matrices diverges.

- $SO(n)$ is compact.
  We can define a continuous map,

$$\begin{aligned} f_n : \ \mathrm{Mat}(n \times n, \mathbb{R}) &\to \mathrm{Mat}(n \times n, \mathbb{R}), \\ f_n(A) &= AA^\top. \end{aligned} \tag{2.52}$$

  Then $O(n) = f_n^{-1}(\{I_n\})$ and thus $O(n)$ is is closed in $\mathrm{Mat}(n \times n, \mathbb{R})$. We know that $SO(n) = (\det|_{O(n)})^{-1}(\{1\})$, so $SO(n)$ is a closed subgroup of $O(n)$ and thereby a closed subgroup of $\mathrm{Mat}(n \times n, \mathbb{R})$.

  Furthermore, the norm of orthonormal matrices is bounded by 1 and thus $SO(n)$ is compact.

- $SU(n)$ is compact.

  Similar to the case for $SO(n)$ we can define $SU(n)$ as the inverse of a closed set under a continuous map, and therefore $SU(n)$ is closed. It is bounded also bounded, because the norm of all unitary matrices is equal to 1. Hence we find that $SU(n)$ is compact.

- $U(1)$ is compact. This can be seen using the fact $U(1) \cong S^1$, which is a compact subset of $\mathbb{C}$. And thus $U(1)$ is a compact subset of $GL(1, \mathbb{C})$.

A compact Lie algebra is a Lie algebra corresponding to a compact Lie group. An important property of compact Lie algebras is given in Section 3.4 of [6].

Theorem 2.2.24. *If Lie algebra is compact, it has a positive semi-definitive scalar product which is invariant under the action of the adjoint representation of the group.*

Proof. A more general theory of Lie algebras discusses the Killing form. The Killing form is a symmetric bilinear from on a Lie algebra (page 13 of [8]). With this in mind:

Corollary 4.26 of [8] proves the existence of a negative semidefinite form on $\mathfrak{g}$, and thus also of a positive semidefinite form on $\mathfrak{g}$. Proposition 13.1 of [9] now proves that this form is also invariant under the adjoint representation of the Lie group. □

The existence of a positive definite bilinear form will be important in non-abelian gauge theory, therefore we will often be using compact Lie algebras in the rest of this thesis. To find such forms, the decomposition of compact Lie algebras into smaller Lie algebras will be important.

Definition 2.2.25. *A **(Lie) subalgebra** is a real vector space that is a subspace of a Lie algebra, which closed under the commutator operation.*

Definition 2.2.26. *Let $\mathfrak{A}, \mathfrak{B}$ be two Lie algebras of dimensions $n_{\mathfrak{A}}$ and $n_{\mathfrak{B}}$ respectively. The direct sum of $\mathfrak{A}$ and $\mathfrak{B}$ forms a vector space. We can define a bracket on this vector space pointwise, for $(A, B), (C, D) \in \mathfrak{A} + \mathfrak{B}$ we define:*

$$[(A, B), (C, D)]_{\mathfrak{A}+\mathfrak{B}} = ([A, C]_{\mathfrak{A}}, [B, D]_{\mathfrak{B}}). \tag{2.53}$$

*With this definition **the direct sum** $\mathfrak{A} + \mathfrak{B}$ becomes a Lie algebra of dimension $n_{\mathfrak{A}} + n_{\mathfrak{B}}$. The direct sum of N Lie algebras can be constructed inductively.*

Given a Lie algebra there are some especially interesting subalgebras called ideals, or invariant subalgebras, which are defined similarly to ideals in ring theory.

Definition 2.2.27. *Let $\mathfrak{A}$ be a Lie algebra and $\mathfrak{C}$ a Lie subalgebra, then $\mathfrak{C}$ is called an **ideal**, or an **invariant subalgebra** of $\mathfrak{A}$, if for all $c \in \mathfrak{C}$ and for all $a \in \mathfrak{A}$*

$$[c, a] \in \mathfrak{C}. \tag{2.54}$$

An eminent Lie (sub)algebra is $U(1)$. To see this, we take a look at the following theorem.

Theorem 2.2.28. *All abelian compact Lie algebras are direct sums of $U(1)$ algebras.*

Proof. See section 3.3 in [10] . □

Next we define a simple Lie algebra.

Definition 2.2.29. *A compact Lie algebra is said to be **simple** if it does not contain any ideals. A compact Lie group is said to be **semi-simple** if it does not contain an Abelian ideal.*

The matrix Lie groups $SU(n)$, $SO(n)$, $O(n)$ and $U(n)$ are all semi-simple, as can be seen in Example 1 on page 33 of [8].

Since simple Lie groups are relatively easy to work with it is convenient to look at a decomposition of compact Lie algebras in terms of simple Lie algebras. Fortunately, all compact Lie algebras can be decomposed into a unique sum of simple Lie algebras and $U(1)$ algebras.

Theorem 2.2.30. *Any compact Lie algebra $\mathfrak{A}$ can be uniquely represented as a direct sum of a finite number of $U(1)$ subalgebras and simple subalgebras.*

Proof. See page 55 of [6]. The proof uses Theorem 1.51 in [8] and 2.2.28. □

As pointed out earlier, compact Lie groups and algebras will be an important subject of study when considering non-abelian Gauge theory. For this purpose, there is one more statement that we will use: we want to know how many positive definite scalar products exist for a given compact Lie group. This is given on page 55 [6].

Theorem 2.2.31. *Suppose $\mathfrak{A}$ is a simple Lie algebra. Then there exists only one invariant (under the adjoint operator) positive-definite scalar product, up to multiplication with a positive number.*
If $\mathfrak{B}$ is a compact Lie group, then all invariant positive definite scalar products are given by positive linear combinations of invariant positive definite scalar of the simple Lie subalgebras as in Theorem 2.2.30. With positive linear combinations, we mean linear combinations in which all coefficients are positive.

For matrix Lie algebras this positive definite scalar product is the trace, see page 53 of [6]..

Corollary 2.2.32. *Let G be a matrix Lie group with Lie group $\mathfrak{g}$. If G is compact then the map*

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R}/\mathbb{C} \\ (A, B) &\mapsto -Tr(AB) \end{aligned} \tag{2.55}$$

is a positive definite bilinear form which is invariant under the adjoint representation. If G is not compact, no such map exists.

By now we have gathered enough information on Lie groups and Lie algebras to continue with homogeneous manifolds. This we will do in the next section.

2.3 Homogeneous manifolds

On to the next interesting topic: Homogeneous manifolds. As you might have guessed a homogeneous manifold is a differentiable manifold, but we endow it with some extra structure. To be more precise, we let a Lie group G act on our manifold in a smooth and transitive way.
This section is based on the theory discussed in Chapter 21 of [2].

Definition 2.3.1. *An action of a group G on a set X is called **transitive**, if for every $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = y$.*

Definition 2.3.2. *An action of a group G on a differentiable manifold M is called **smooth**, if for all $g \in G$ the map $\varphi_g : M \xrightarrow{\cong} M, p \mapsto g \cdot p$, is a diffeomorphism of M.*

Definition 2.3.3. *A **homogeneous manifold** is a differentiable manifold endowed with a smooth and transitive action by a Lie group G. This space is sometimes also called a homogeneous space, or a homogeneous G-space to emphasise the Lie group G.*
Note that in this definition smoothness refers to the smoothness of an action not of a mapping.

Let us again take a look at some examples.

- The action of $\mathbb{R}^n$ on itself. This is a homogeneous manifold, because $\mathbb{R}^n$ is a Lie group and a differentiable manifold and it acts smoothly and transitively on itself.

- The action of $O(n)$ on S^{n-1} for $n \geq 2$.
  This is a more interesting example and we will take a look at the transitivity and the smoothness of the action of $O(n)$ on S^{n-1}. First note that the natural action of $O(n)$ on $\mathbb{R}^n$ is smooth. The action of $O(n)$ on $\mathbb{R}^n$ is a linear map that preserves distances, thus a rotation, and since rotations of $\mathbb{R}^n$ correspond to diffeomorphisms of $\mathbb{R}^n$, we conclude that the action of $O(n)$ on $\mathbb{R}^n$ is a smooth action.
  We define the map $p : \mathbb{R}^n \to S^{n-1}$, $p(x) = \sqrt{x_1^2 + ... + x_n^2}$. This is a smooth map and we have $S^{n-1} = p^{-1}(1)$, where 1 is a regular value of the map p. Using the regular level set theorem, Corollary 8.10 in [2], we find that S^{n-1} is an embedded submanifold of $\mathbb{R}^n$. This is also explained in Example 8.11. Subsequently, the rotations of S^{n-1} are also smooth maps, and thus the action of $O(n)$ on S^{n-1} is smooth.

For transitivity we use that $O(n)$ is the set of matrices whose columns form an orthonormal basis of $\mathbb{R}^n$. If we set $p = (1, 0, ..., 0) \in S^{n-1}$ to be the north pole we are finished if for any $x \in S^{n-1}$ we can find a matrix $A_x \in O(n)$ such that $A_x p = x$, because then for $y \in S^{n-1}$ we use A_y such that $(A_x A_y^{-1})y = x$ with $A_x A_y^{-1} \in O(n)$.

Since x has unit norm, we can extend x to an orthonormal basis $(x, v_2, ..., v_n)$ of $\mathbb{R}^n$. Then we find that

$$A = \begin{pmatrix} | & | & & | \\ x & v_2 & ... & v_n \\ | & | & & | \end{pmatrix} \in O(n) \tag{2.56}$$

and $A_x p = x$. So $O(n)$ acts transitively on S^{n-1}.

- The action of $SO(n)$ on S^{n-1} for $n \geq 2$.
  The action of $SO(n)$ on S^{n-1} is smooth because $SO(n) \subseteq O(n)$. That the action is also transitive follows from the previous example and the fact that we can choose an orthonormal basis $(x, v_2, .., v_n)$ such that $\det A_x = +1$. If the determinant of A_x is equal to -1, we can simply replace v_2 with $-v_2$ to obtain $A'_x \in SO(n)$.

Homogeneous manifolds can be constructed from Lie groups and can even be characterised by them. Below we present and prove the homogeneous manifold construction theorem and the homogeneous manifold characterisation theorem. The first theorem shows how to construct homogeneous manifolds as a quotient of a Lie group G and a closed subgroup $H \subseteq G$ of the Lie group and the second asserts that we can characterise any homogeneous manifold by such a quotient.

Definition 2.3.4. *A smooth map $F : M \to N$ between manifolds, is called a **smooth submersion** if it's differential is surjective at each point.*

Theorem 2.3.5 (Homogeneous Manifold Construction Theorem)**.** *Let G be a Lie group and $H \subseteq G$ a closed subgroup. Then G/H is a topological manifold of dimension $dimG - dimH$ and has a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion. The regular left action of G on G/H, defined through:*

$$g_1 \cdot (g_2 H) = (g_1 \cdot g_2) H \,,$$

turns G/H into a homogeneous manifold over the Lie group G.

This is a very nice result: if only we know a Lie group and a closed subgroup, the theorem grants new homogeneous manifolds. This means that homogeneous manifolds are not a rare sight, in fact, any Lie group acting on itself is a homogeneous manifold over itself.

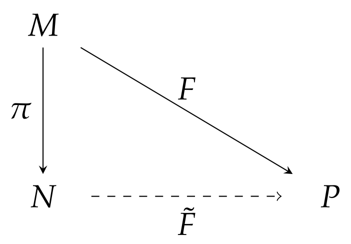
An important part of this theorem is to determine if the quotient G/H of a Lie group with a closed subgroup is in fact a differentiable manifold. It is not a priori clear that such a quotient will give us a Hausdorff and second countable space as we need. We use a theorem that is a little more general and states that the quotient M/G of a smooth manifold M with a Lie group G is a differentiable manifold, if G acts smoothly, freely and properly on M. It can be shown that we need G to act freely on M to ensure that M/G is Hausdorff and we need G to act properly on M to ensure that M/G is second countable.

Theorem 2.3.6 (Quotient manifold theorem). *Suppose G is a Lie group acting smoothly, freely and properly on a smooth manifold M. Then the quotient M/G is a topological manifold of dimension* $\dim M - \dim G$*, which has a unique smooth structure such that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Proof. See Theorem 21.10 of [2]. □

The next theorem will help us in proving the existence, smoothness and uniqueness of maps from quotient spaces that we need later on.

Theorem 2.3.7 (Passing smoothly to the quotient). *Let M and N be smooth manifolds and $\pi : M \to N$ a surjective smooth submersion. If P is a smooth manifold and $F : M \to P$ is a smooth map that is constant on the fibres of π, then there exists a unique smooth map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$. This gives us the following commutative diagram:*



Proof. See theorem 4.20 in [2]. □

Now we are ready to prove the homogeneous manifold construction theorem.

Proof (Homogeneous manifold construction theorem, Theorem 2.3.5). We want to use our previous theorems. First we prove that G/H is a differentiable manifold using Theorem 2.3.6. For this, note that the orbit space of the right action of H on G is equal to the left coset space G/H. We need to prove that G acts smoothly, freely and properly on M/G. The action of H on G is:

- **smooth**
  The action of H on G is a restriction of the left multiplication of G on itself, so it is smooth.

- **free**
  For $g \in G$ $h \in H$ we have: $gh = g \Rightarrow h = e$, because the action is the restriction of the left multiplication of G on itself, which is a free action.

- **proper**
  For this we use Proposition 21.5(b) of [2]. Let $(g_i) \subseteq G$ be a convergent sequence in G and $(h_i) \subseteq H$ a sequence in H such that $(g_i \cdot h_i) \subseteq G$ is a convergent subsequence in G. We want to prove that (h_i) contains a convergent subsequence. By continuity of left multiplication we know that $h_i = g_i^{-1} \cdot (g_i h_i)$ is a convergent sequence in G. We have chosen H to be closed, and $(h_i) \subseteq H$, so the sequence also converges in H. We have found a subsequence of (h_i) that converges in H, so 21.5(b) of [2] gives us that the action of H on G is proper.

Now we can use the quotient manifold theorem, Theorem 2.3.6, which states that G/H is a topological manifold with a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion. Hence G/H is a differentiable manifold.

Id_G and π are smooth submersions and because products of smooth submersions are again smooth submersions, we can construct a smooth submersion $Id_G \times \pi : G \times G \to G \times G/H$. Further, we define $m : G \times G \to G$ as the multiplication map and $\theta : G \times G/H \to G/H$ as the action of G on G/H. Now we consider the following diagram.

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ {\scriptstyle Id_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow[\;\theta\;]{} & G/H \end{array}$$

We want to use Theorem 2.3.7 to show that θ is smooth. For this we need to check if $\pi \circ m$ is constant on the fibres of π. Let $k_1, k_2 \in G$ and take $(g_1, g_2), (f_1, f_2) \in (Id_G \times \pi)^{-1}(k_1, k_2 H)$. Then we have $g_1 = k_1 = f_1$ by the identity, and we need $g_2 h = f_2$ for some $h \in H$. Plugging this into $m \circ \pi$ we get:

$$\pi \circ m(g_1, g_2) = \pi(g_1 g_2) = (g_1 g_2)H \tag{2.57}$$

$$\begin{aligned} \pi \circ m(f_1, f_2) &= \pi \circ m(g_1, g_2 h) = \pi(g_1 g_2 h) \\ &= (g_1 g_2 h)H = (g_1 g_2)H \end{aligned} \tag{2.58}$$

So by the uniqueness of the map in Theorem 2.3.7 we find that θ has to be a smooth map. Note that the above also proves that θ is well-defined. Per definition of θ we have:

$$\theta(g_1 \cdot g_2, g_3 H) = g_1 \cdot g_2 \cdot g_3 H = \theta(g_1, \theta(g_2, g_3 H)) \tag{2.59}$$

$$\theta(e, gH) = e \cdot gH = gH, \tag{2.60}$$

implying that θ defines a group action of G on G/H.

Finally we check that θ is transitive. Suppose $g_1 H, g_2 H \in G/H$. Define $\tilde{g} = g_2 g_1^{-1}$, then $\tilde{g} \in G$ and $\theta(\tilde{g}, g_1 H) = g_2 \cdot g_1^{-1} \cdot g_1 \cdot H = g_2 H$. □

The homogeneous manifold construction theorem yields a homogeneous manifold for any given a Lie group with a closed subgroup. Actually the relation between this quotient and a homogeneous manifold is even stronger: we can characterise every homogeneous G-space through such a quotient of Lie groups. We will take a look at the homogeneous manifold characterisation theorem to prove this, but we first need an extra definition.

Definition 2.3.8. *A map $F : M \to N$ between G-spaces M and N, is called* ***equivariant with respect to the*** *G**-actions**, if for all $g \in G$ we have:*

$$F(g \cdot p) = g \cdot F(p), \tag{2.61}$$

for left actions. For right actions we equivalently have: $F(p \cdot g) = F(p) \cdot g$.

Theorem 2.3.9 (Homogeneous Manifold Characterisation Theorem)**.** *Let M be a homogeneous manifold over Lie group G, and $p \in M$ any point on de manifold. Then the isotropy group $G_p := \{g \in G : g \cdot p = p\}$ is a closed subgroup of G and the map*

$$F : G/G_p \to M, \quad gG_p \mapsto g \cdot p \tag{2.62}$$

is an equivariant diffeomorphism.

This theorem states, that we can characterise any homogeneous manifold as a quotient of Lie groups. This is nice, because there is a lot of knowledge about quotients of groups which we can now use this to find properties of homogeneous manifolds.

For the proof we need one more theorem.

Theorem 2.3.10. *Let M and N be smooth manifolds, and let G be a Lie group. Suppose $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G-action on M and a smooth G-action on N. Then F has constant rank. If F is surjective it is a smooth submersion, if F is injective it is a smooth immersion and if F is bijective it is a diffeomorphism.*

Proof. See theorem 7.25 of [2]. □

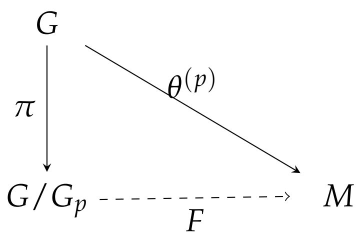
Proof (Homogeneous manifold characterisation theorem, Theorem 2.3.9). For $p \in M$, the orbit map $\theta^{(p)} : G \to M, g \mapsto g\cdot$ is a smooth map, proposition 7.26 of [2]. Therefore $G_p = (\theta^{(p)})^{-1}(\{p\})$ is a closed set as inverse image of a closed set under a continuous map. Cartan's theorem, 2.1.2, gives us that G_p is a Lie subgroup.

Now we prove that F is well-defined. Suppose $g_1H = g_2H$, then there exists an $h \in G_p$ such that $g_1h = g_2$. Then we find:

$$F(g_2H) = g_2 \cdot p = g_1 \cdot h \cdot p = g_1 \cdot p = F(g_1H) \tag{2.63}$$

Where we used that $h \cdot p = p$ for $h \in G_p$, so F is well-defined. Additionally, for $g, g' \in G$ we have $F(g' \cdot gH) = g'g \cdot p = g' \cdot F(gH)$, so F is equivariant.

We will prove that F is smooth. Firstly, the projection $\pi : G \to G/G_p$, $g \mapsto gG_p$ is a smooth submersion, Example 4.2 in [2]. Next, note that for two elements $g_1, g_2 \in \pi^{-1}(\tilde{g}G_p)$ in the same fibre of π, there exists an $h \in G_p$ such that $g_1h = g_2$, so $\theta^{(p)}(g_2) = g_1hp = g_1p = \theta^{(p)}(g_1)$. This means $\theta^{(p)}$ is constant on the fibres of π, so by Theorem 2.3.7 there exists a unique smooth map $\tilde{F}$ such that $\tilde{F} \circ \pi = \theta^{(p)}$. Since the map F we defined earlier is such a map we conclude $F = \tilde{F}$ and thus F is smooth.



Finally, F is also bijective. For injectivity, suppose $F(g_1G_p) = F(g_2G_p)$, so $g_1 \cdot p = g_2 \cdot p$. This means $g_2^{-1}g_1 \cdot p = p$, so $g_2^{-1}g_1 \in G_p$ and therefore

$g_1G_p = g_2G_p$. For surjectivity, let $q \in M$. Since G acts transitively on M, there exists a $g \in G$ such that $g \cdot p = q$, yielding $F(gG_p) = g \cdot p = q$.

Now Theorem 2.3.10 implies that F is an equivariant diffeomorphism. □

Let us examine an example of a homogeneous manifold as a quotient of Lie groups. We will consider the action of $SO(3)$ on the manifold S^2. We want to use the homogeneous manifold characterisation theorem, Theorem 2.3.9, to find the quotient to which S^2 is diffeomorphic.

We take the north pole $N = (0, 0, 1) \in S^2$ as fixed point on the sphere. We want to find the isotropy group $SO(3)_N$. Note that $SO(3)$ acts on S^2 by rotating it. If we fix the north pole, the south pole is fixed also and the only rotations that remain are rotations in the xy-plane. These rotations correspond exactly to the group $SO(2)$, so we find $SO(3)_N \cong SO(2)$. The homogeneous manifold characterisation theorem concludes $S^2 \cong SO(3)/SO(2)$.

This example can easily be generalised to higher dimensions where we find $S^n = SO(n+1)/SO(n)$ for $n \geq 1$. The specific example for $n = 2$ will re-appear when we study symmetry breaking patterns. In chapter 6 we will construct the vacuum manifold as a quotient of symmetry groups.

In the next chapter we will discuss homotopy theory, an important tool to characterise topological spaces and consequently homogeneous manifolds.

Chapter 3

Homotopy theory

Another important topic in the study of topological defects is that of homotopy theory. Homotopy theory can be used to characterise topological spaces. We will use this characterisation to better understand the types of topological defects.

The first order homotopy group is also known as the fundamental group $\pi_1(X, x_0)$. The fundamental group detects 'one dimensional holes', that is, holes in the space around which we cannot contract loops. We will define higher order homotopy groups and use this to find 'higher dimensional holes', holes around which we cannot contract n-spheres, in a topological space.

This section is based on the contents of Chapter 4 of Hatcher's Algebraic topology [11]. All maps in this chapter will be assumed to be continuous, unless otherwise stated.

3.1 Homotopy groups

To start with a definition of homotopy groups, we first need to look at a few notational conventions. First of all we have the n dimensional unit cube $I^n = [0,1]^n$. Then we define ∂I^n to be the boundary of I^n, this is the set of points in I^n for which at least one of the coordinates is equal to 0 or 1. Lastly for X, Y topological spaces and $A \subseteq X$ and $B \subseteq Y$ the map $f : (X, A) \to (Y, B)$, is a map $f : X \to Y$, such that $f(A) \subseteq B$. This notation can be extended to a map between any n-tuple, where each set is contained in the former. Now we can take a look at homotopy groups.

Definition 3.1.1. *Given a topological space X with a basepoint $x_0 \in X$, we define **the n^{th} homotopy group** of X in x_0 as the set of homotopy classes of* *maps $f : (I^n, \partial I^n) \to (X, x_0)$, where we only allow homotopies that satisfy $F(t, \partial I^n) = x_0$ for all $t \in [0,1]$. That is:*

$$\pi_n(X, x_0) := \{f : (I^n, \partial I^n) \to (X, x_0)\}/\sim, \tag{3.1}$$

where $f, g : (I^n, \partial I^n) \to (X, x_0)$ are equivalent if and only if there exists a homotopy $F : I \times (I^n, \partial I^n) \to (X, x_0)$ between f and g such that for all $t \in I$ we have $F(t, \partial I^n) = x_0$. Such a homotopy is said to be a homotopy relative to ∂I^n.

To turn this set into a group we define an addition. For (X, x_0) a pointed topological space and $f, g \in \pi_n(X, x_0)$, we have:

$$(f+g)(s_1, s_2, s_3, ..., s_n) := \begin{cases} f(2s_1, s_2, s_3, ..., s_n), & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, s_3, ..., s_n), & s_1 \in [\frac{1}{2}, 1]. \end{cases} \tag{3.2}$$

Note that this definition generalises the definition of the concatenation of paths in $\pi_1(X, x_0)$. The addition on $\pi_n(X, x)$ turns this set into a group with identity element the constant map $c_{x_0} : (I^n, \partial I^n) \to (X, x_0)$, $s \mapsto x_0$ for all $s \in I^n$ and inverses $-f(s_1, s_2, ..., s_n) = f(1 - s_1, s_2, ..., s_n)$. We use the additive notation, because for $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian. This can be seen inFigure 3.1.

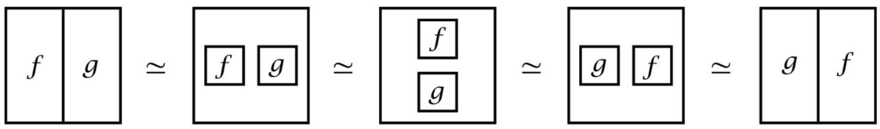


Figure 3.1: *With a homotopy as illustrated above, we can use the higher dimension change the order of two maps. Therefore $\pi_n(X, x_0)$ is abelian for $n \geq 2$. [11]*

We can also define homotopy groups in an alternative way.

Definition 3.1.2. *For a topological space X with basepoint $x_0 \in X$ and $s_0 \in S^n$ we define **the n^{th} homotopy group of** X in x_0 as the set of homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$, where we only allow homotopies that consistently map s_0 to x_0, $F(t, s_0) = x_0$ for all $t \in I$. That is:*

$$\pi_n(X, x_0) := \{f : (S^n, s_0) \to (X, x_0)\}/\sim$$

Where $f, g : (S^n, s_0) \to (X, x_0)$ are equivalent if and only if there exists a homotopy $F : I \times (S^n, s_0) \to (X, x_0)$ between f and g such that for all t we have $F(t, s_0) = x_0$.

Note that this definition is equivalent to the first definition, because $I^n/\partial I^n \cong S^n, \partial I^n/\partial I^n \cong s_0$ and also the homotopies that we allow can thus be converted from homotopies $F : I \times (I^n, \partial I^n) \to (X, x_0)$, to homotopies $F : I \times (S^n, s_0) \to (X, x_0)$.

Before we look at some examples, we simplify the notation for homotopy groups of path connected spaces. As for the fundamental group, homotopy groups of path connected spaces are independent of the chosen base point. In this case the base point is sometimes omitted from the notation.

Lemma 3.1.3. *If X is a path connected space and $x_0, x_1 \in X$ are two points, then the homotopy groups of X based in points x_0 or x_1 are isomorphic:*

$$\pi_n(X, x_0) \simeq \pi_n(X, x_1).$$

Proof. For this we chose a path $\gamma : I \to X$, with starting point $\gamma(0) = x_0$ and end point $\gamma(1) = x_1$. To each map $f : (I^n, \partial I^n) \to (X, x_1)$ we can now associate a new map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$. First we reduce the domain of f to a smaller concentric cube $i^n \subseteq I^n$, we can then insert the map γ on very segment between i^n and ∂I^n. This can be seen in Figure 3.2.

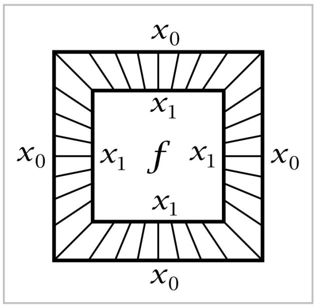


Figure 3.2: *If X is path-connected, we can move any map $f : (I^n, \partial I^n) \to (X, x_1)$ with basepoint x_1 to a new map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ with basepoint x_0 using a path $\gamma : I \to X$ from x_1 to x_0.*

A homotopy of f can thus be translated into a homotopy of γf. Notice that for $n = 1$ the notation γf is a little misleading, because in this case we have $f \in \pi_1(X, x_1)$ and $\gamma f = \gamma^{-1} \circ f \circ \gamma$. For $n \geq 2$ this is not a problem.

The action of paths γ, η on maps f, g has a few basic properties:

1. $\gamma(f + g) \sim \gamma f + \gamma g$
2. $(\gamma\eta)f \sim \gamma(\eta f)$
3. $1f \sim f$

An explicit formula for the first homotopy is:

$$H(t,s_1,...,s_n) = \begin{cases} \gamma(f+0)\left((2-t)s_1 \qquad ,s_2,...,s_n\right), s_1 \in [0,\frac{1}{2}] \\ \gamma(0+g)\left((2-t)s_1+t-1,s_2,...,s_n\right), s_1 \in [\frac{1}{2},1] \end{cases} \tag{3.3}$$

Consequently, we can define an isomorphism $\beta_\gamma : \pi_n(X,x_1) \to \pi_n(X,x_0)$, by $\beta_\gamma([f]) = [\gamma f]$, where the three properties above ensure that β_γ is indeed an isomorphism. □

Let us take a look at some examples. First of all we consider π_0. For this we extend the definitions of I^n and ∂I^n to $n = 0$ with $I^0 = \{0\}$ and $\partial I^0 = \emptyset$. The set of homotopy classes now indicates the path-components of X. Thus any space X with $n > 0$ path connected components yields, $\pi_0(X) \simeq \mathbb{Z}/n\mathbb{Z}$ and a path connected space has trivial zeroth homotopy group.

For π_1, we have the fundamental group. We will consider a circle S^1 and give a short explanation for its fundamental group. We can assign to each map $f : (S^1, s_0) \to (S^1, 1)$ a winding number, that is, the number of times $f(S^1)$ winds itself around the circle, were anti-clockwise rotations are assigned a positive value and clockwise windings a negative value. Using this assignment we note that any two maps with the same winding number are homotopic relative to the base points. Thus $\pi_1(S^1) \simeq \mathbb{Z}$. More rigorously this can be shown using lifts, see [12].

For S^2 we note that $\pi_1(S^2) = 0$, since any loop can be contracted to a point, and thus all loops are homotopy equivalent. The same argument holds for $n > 1$, it follows that $\pi_1(S^n) = 0$.

Another accessible example is that of contractible spaces. A contractible space X is path connected, so we have $\pi_0(X) = 0$. More than that, any map $f : (S^n, s_0) \to (X, x_0)$ can be contracted to a point, since the whole space can bee contracted to a point. As a consequence, any two maps $f, g : (S^n, s_0) \to (X, x_0)$ are homotopy equivalent via the constant map and there is only one class of maps in each homotopy group, $\pi_n(X) = 0$ for all $n \in \mathbb{N}$.

Homotopy groups are generally hard to calculate. We will take a look at a few theorems that will help us to determine some homotopy groups.

Theorem 3.1.4. *A covering space projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms*

$$p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0), \tag{3.4}$$

for all $n \geq 2$.

Proof. See Proposition 4.1 in [11]. □

Corollary 3.1.5. *$\pi_n(X, x_0) = 0$ for $n \geq 2$ if X has a contractible universal cover.*

Proof. Suppose X is a space with a contractible universal cover $\tilde{X}$. Then for $n \geq 2$ we find isomorphisms

$$p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0). \tag{3.5}$$

Since the universal cover is contractible, we have $\pi_n(\tilde{X}, \tilde{x}_0) = 0$ and thus the statement follows. □

Let us examine some examples. For S^1 we have a universal cover $\mathbb{R}$, so as before we indeed find, $\pi_n(S^1) = 0$ or $n \geq 2$.

Suppose X is a topological graph, see definition 6.5.1 of [13], then X admits a universal covering, and this universal covering is a topological tree, Proposition 6.5.3(b) of [13]. A topological tree is contractible and thus any topological graph has trivial homotopy groups for $n \geq 2$. Topological graphs include the circle S^1 and the figure of eight, $S^1 \wedge S^1$. We find $\pi_n(S^1 \wedge S^1) = 0$ for $n \geq 2$.

To be able to find more homotopy groups, we will examine homotopy groups of product spaces. Homotopy groups turn out to work nicely with product spaces.

Proposition 3.1.6. *Let $\Pi_\alpha X_\alpha$ be a product of path connected spaces X_α, then the homotopy groups of the product is isomorphic to the product of the homotopy groups:*

$$\pi_n(\Pi_\alpha X_\alpha) \approx \Pi_\alpha \pi_n(X_\alpha). \tag{3.6}$$

Proof. See Proposition 4.2 in [11]. □

This proposition allows us to determine the homotopy groups of, for example, the torus:

$$\pi_n(T^2) = \pi_n(S^1 \times S^1) = \pi_n(S^1) \times \pi_n(S^1) \tag{3.7}$$

$$= \begin{cases} \mathbb{Z}^2 & \text{if } n = 1, \\ 0 & \text{else .} \end{cases} \tag{3.8}$$

3.2 Relative homotopy groups

We will continue with a useful generalisation of homotopy groups, namely the relative homotopy groups. We will use relative homotopy groups to make a long exact sequence of homotopy groups, which will help us to calculate more homotopy groups.

As above, for $n > 1$, we define I^n to be the n dimensional cube and ∂I^n to be the boundary of I^n. We now also define I^{n-1} as the face of I^n where $s_n = 0$ and J^{n-1} as the closure of all other faces, $J^{n-1} = \overline{I^n \setminus I^{n-1}}$. For $n = 1$ we define $\partial I = \{0\}$ and $J^0 = \{1\}$. Now we can define relative homotopy groups.

Definition 3.2.1. *For a space X and $A \subseteq X$ a subset with basepoint $x_0 \in A$ and $n \geq 1$, we define the* ***relative homotopy groups*** *$\pi_n(X, A, x_0)$ as the set of homotopy classes of maps $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, where homotopies $F : (I^n, \partial I^n, J^{n-1}) \times I \to (X, A, x_0)$ need to satisfy both $F(\partial I^n \times I) \subseteq A$ and $F(J^{n-1} \times I) = x_0$.*
For $n = 1$ we obtain the homotopy classes of paths in X with starting point in A and endpoint $x_0 \in A$.

We do not define this for $n = 0$, because there is no evident way to do this. As for (absolute) homotopy groups, we can reformulate this definition using spheres instead of cubes. This is similar to Definition 3.1.2, where we now replace the triple $(I^n, \partial I^n, J^{n-1})$ with the triple (D^n, S^{n-1}, s_0), which we obtain by contracting J^{n-1} to a point.

We can define addition on the relative homotopy group completely analogous to the absolute case, except that the coordinate s_n is fixed, so we can only define the addition in Equation 3.2 for $n \geq 2$. This turns $\pi_n(X, A, x_0)$ into a group for $n \geq 2$ and for $n \geq 3$ this group is abelian, because, once more, we have a 'free' dimension to change the order of maps.

Let us introduce a useful formulation of what it means to be trivial as an element of $\pi_n(X, A, x_0)$.

Lemma 3.2.2 (Compression criterion)**.** *A map $f : D^n, S^{n-1}, x_0)$ corresponds to the identity element of $\pi_n(X, A, x_0)$ if and only if it is homotopic to a map whose image is contained in A.*

Proof. See page 343 in [11]. □

A continuous map $f : (X, A, x_0) \to (Y, B, y_0)$ induces maps $f_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$, which are homomorphisms for $n \geq 2$. These induced maps follow a few basic properties:

1. $(fg)_* = f_* g_*$,
2. $\mathbb{1}_* = \mathbb{1}$,
3. $f* = g_*$ iff $f \simeq g$ through a homotopy $F : (X, A, x_0) \times I \to (Y, B, y_0)$.

3.3 A long exact sequence

As the long exact sequence for homology groups, we want to find a long exact sequence for homotopy groups. This sequence will allow us to determine more homotopy groups. We will be using the relative homotopy groups to construct such a sequence.

Theorem 3.3.1. *Let X be a space and $A \subseteq X$ a subset with basepoint $x_0 \in A$. Then the following long sequence is exact.*

$$\begin{aligned} \cdots \to &\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \\ \xrightarrow{\partial} &\pi_{n-1}(A, x_0) \cdots \to \pi_0(X, x_0) \end{aligned} \tag{3.9}$$

Where i and j are the inclusion maps, and ∂ is called the boundary map. The boundary map ∂ arises from the restriction of maps $(I^n, \partial I^n, j^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or similarly from the the restriction of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to S^{n-1}. The boundary map ∂ is a homomorphism for $n > 1$.

Proof. Following the proof of Theorem 4.3 in [11], we proof that for $x_0 \in B \subseteq A \subseteq X$ the sequence

$$\begin{aligned} \cdots \to &\pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \\ \xrightarrow{\partial} &\pi_{n-1}(A, B, x_0) \cdots \to \pi_1(X, A, x_0) \end{aligned} \tag{3.10}$$

is exact. To prove the statement in the theorem we then take $B = \{x_0\}$ and we prove the last two steps separately. First, we prove exactness at each of the three sets of homotopy groups.

Exactness at $\pi_n(X, B, x_0)$. We need to prove that Im $i_* = \ker j_*$. For this, we first note that $j_* \circ i_* \equiv 0 : \pi_n(A, B, x_0) \to \pi_n(X, A, x_0)$ by the compression criterion (3.2.2). This implies Im$i_* \subseteq \ker j_*$.

For the second inclusion, we take a map $f : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$, such that $[f] \in \ker j_*$, that is $[f] = 0 \in \pi_n(X, A, x_0)$. Using the compression criterion, we find that f is homotopic to a map g for which $g(I^n) \subseteq A$ holds. This implies $i_*^{-1}(g) \in \pi_n(A, B, x_0)$, so we find that $[f] = [g] \in$ Im i_*, which gives us the inclusion $\ker j_* \subseteq$ Im i_*.

Exactness at $\pi_n(X, A, x_0)$. The first inclusion, Im $j_* \subseteq \ker \partial$ is again clear when we notice that $j_* \circ \partial \equiv 0$. For $h : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$, we have Im $h|_{I^{n-1}} \subseteq B$, which implies $[h] = 0 \in \pi_{n-1}(A, B, x_0)$, by Theorem 3.2.2.

The other direction is a bit harder. Let $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, with $[f|_{I^{n-1}}] = 0] \in \pi_n(A, B, x_0)$. Then $f|_{I^{n-1}} \simeq g$ with Im $g \subseteq B$. Suppose F is the homotopy relative to ∂I^n between f and g. Then we can combine the maps F and f, notation: $f * F$ as seen in Figure 3.3 producing a map $\tilde{f} : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$. We observe that $\tilde{f} \simeq f$, for example with the homotopy that tacks on an increasingly large part of the map F. This gives us that $[f] \in$ Im j_*.

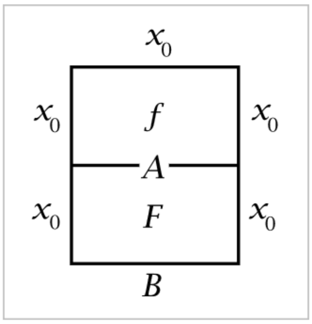


Figure 3.3: *Because f and F are both x_0 for $t = 0, 1$ and have one 'border' where they are equal to $f|_{I^{n-1}}$, we can stick them together to form a new map $\tilde{f} = f * F$.*

Exactness at $\pi_n(A, B, x_0)$. Again we notice that $\partial \circ i_* = 0$. Suppose $h : (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$. Then h provides a homotopy relative to I^n between $h|_{I^n}$ and the constant map c_{x_0}. Therefore $[f|_{I^n}] = 0 \in \pi_n(X, B, x_0)$ through Theorem 3.2.2.

For the other inclusion, we take a map $f : (I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$, $[f] \in \ker i_*$ and F a nullhomotopy of $f : (I^n, \partial I^n, J^{n-1}) \times I \to (A, B, x_0)$ relative to ∂I^n. Define $g = F|_{I^{n-1} \times I}$. We can illustrate the map F as seen on the left in Figure 3.4, where the last two coordinates are drawn. On the right you see a re-parametrisation of the last two coordinates, s_n and s_{n+1}, to find $F'|_{I^n \times \{0\}} = g * f$. This means that $g * f = F'|_{I^n \times \{0\}} \in$ Im ∂. Following the argumentation in the previous step find that $[f] = [g * f]$ yielding $[f] \in$ Im ∂.

It remains to prove exactness of the last two steps,

$$\pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0). \tag{3.11}$$

Exactness at $\pi_1(X, A, x_0)$. For $g : (\{0\}, \{1\}) \to (A, x_0)$ with $[g] \in \pi_0(A, x_0)$, we have $[g] = 0$ if and only if $g(0) \in U_{x_0}$, where U_{x_0} is the

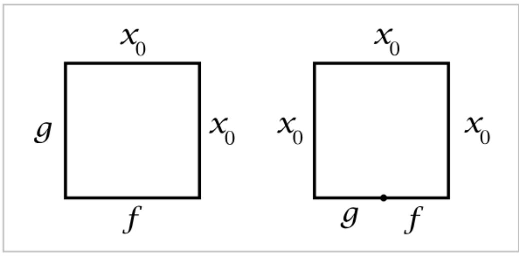


Figure 3.4: *Left: F is a map from $I^n \times I \to X$, where we see the image of F at the boundaries of $s_{n+1} = 0, 1$ and $s_n = 0, 1$.*
Right: We can re-parametrise F, to give g tacked onto f for $s_{n+1} = 0$.

path connected component of x_0 in A. Take $[f] \in \pi_1(X, x_0)$, with a representative $f : (I, \{0,1\}) \to (X, x_0)$. Mapping $[f]$ according to j_* and ∂ we find

$$\begin{aligned}(\partial \circ j_*)[f] &= \partial[\tilde{f} : (I, \{0\}, \{1\}) \to (X, A, x_0)] \\ &= [f' : (\{0\}, \{1\}) \to (A, x_0)],\end{aligned} \tag{3.12}$$

where, for a representative of $[f']$, we have $f'(0) = f(0) = x_0 \in U_{x_0}$ and thus $[f'] = 0 \in \pi_0(A, x_0)$. This proves $\mathrm{Im} j_* \subseteq \ker \partial$.

Now suppose $[f] \in \ker \partial$. Then $f : (I, \{0\}, \{1\}) \to (X, A, x_0)$ and $f(0) \in U_{x_0}$. We define a path $\gamma : [0,1] \to U_{x_0}$ such that $\gamma(0) = x_0$ and $\gamma(1) = f(0)$. We can concatenate γ and f and define $f^* = \gamma * f$. We obtain a map $f^* : [0,1] \to X$, with $f^*(0) = \gamma(0) = x_0$ and $f^*(1) = f(1) = x_0$, which implies that $[f^*] \in \pi_1(X, x_0)$. Further, the map $F : I \times I \to X$ which tacks on pieces of γ of increasing length in front of f is a homotopy between f and f^* relative to $\{0\}$, because $\gamma([0,1]) \subseteq U_{x_0}$. Hence, we have $j_*[f^*] = [f]$, and $\mathrm{Im} j_* \supseteq \ker \partial$.

Exactness at $\pi_1(A, x_0)$. Suppose $[f] \in \pi_1(X, A, x_0)$, then a representative of this class is a map $f : (I, \{0\}, \{1\}) \to (X, A, x_0)$, that is, it is a path in X between a point in A and x_0. Then $i_* \circ \partial[f]$ yields a class $[f']$ with $f' : (\{0\}, \{1\}) \to (X, x_0)$, where $f'(0) = f(0)$ and $f'(1) = f(1) = x_0$. Because f was a path, we know that $f(0)$ and x_0 are connected by a path, hence $f'(0)$ is an element of the same path component as x_0, thus $[f'] = 0 \in \pi_0(X, x_0)$ and $\mathrm{Im}\, \partial \subseteq \ker i_*$.

Define V_{x_0} as the path component of x_0 in X. We take $[f] \in \ker i_*$, this implies $f(0) \in V_{x_0}$ for all representatives of $[f]$. Since $f(0)$ and x_0 are in the same path connected component we can define a path $\gamma : [0,1] \to V_{x_0}$ such that $\gamma(0) = f(0) \in A \cap V_{x_0}$ and $\gamma(1) = x_0$. Therefore we find $[\gamma] \in \pi_1(X, A, x_0)$ and $\partial[\gamma] = [f]$. So we conclude $\mathrm{Im}\, \partial \supseteq \ker i_*$ and we are done. □

We will take a look at an example using this theorem. Suppose X is a path connected space and define the cone over X,

$$\begin{aligned} CX =& (X \times I)/\sim, \\ (x,t) \sim (y,s) \iff & (x = y \text{ and } t = s), \text{ or } t = s = 0. \end{aligned} \tag{3.13}$$

We can think of X as the subspace $X \times \{1\} \subseteq CX$ of the cone. We can now write down part of the long exact sequence of the theorem,

$$\pi_n(CX, x_0) \xrightarrow{j_*} \pi_n(CX, X, x_0) \xrightarrow{\partial} \pi_{n-1}(X, x_0) \xrightarrow{i_*} \pi_{n-1}(CX, x_0). \tag{3.14}$$

We also know that the cone CX is contractible and thus $\pi_n(CX) = 0$ for $n \geq 1$, which gives us

$$0 \to \pi_n(CX, X, x_0) \to \pi_{n-1}(X, x_0) \to 0, \tag{3.15}$$

for $n \geq 2$. We conclude that $\pi_n(CX, X, x_0) \simeq \pi_{n-1}(X, x_0)$ for all $n \geq 2$.

In section 6.2 we will be using homotopy groups to characterise topological defects and in section 6.3 we will look for the homotopy groups of a physically interesting homogeneous manifold to explain which topological defects might form. To find homotopy groups of a quotient, such as homogeneous manifold, we need another long exact sequence, which we discuss in the next section.

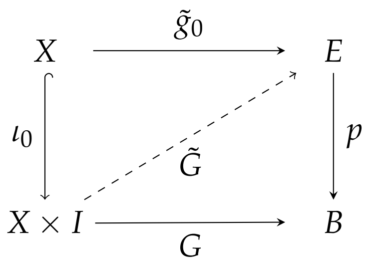
3.4 Fibrations and another long exact sequence

Earlier we discussed homogeneous manifolds. We will later find that such manifolds are at the foundation of topological defects. In order to find homotopy groups of homogeneous manifolds, we construct another long exact sequence. First, we need some new definitions. Let us start with a property of maps: the homotopy lifting property.

This section is based mainly on §4.2 of [11] and on parts of Chapter 2 in [14].

Definition 3.4.1. *Let E, B and X be topological spaces. A map $p : E \to B$ has* ***the homotopy lifting property*** *with respect to X if for any given homotopy $G : X \times I \to B$ and any lift $\tilde{g}_0 : X \to E$ of $g_0 = G|_{X \times \{0\}}$, there exists a homotopy $\tilde{G} : X \times I \to E$ lifting G, such that $\tilde{g}_0 = \tilde{G}|_{X \times \{0\}}$.*

That is, we see that p has the homotopy lifting property with respect to X, if for any G and $\tilde{g}_0$ as before, there exists a $\tilde{G}$ such that the diagram below commutes.



A simple example of a pair (p, X) that has the homotopy lifting property could be a projection map π with any space Y. Another example is that of a covering map $p : E \to B$, again with any space Y. Maps that satisfy the homotopy lifting property with respect to all spaces X are said to satisfy the homotopy lifting property. They are also called fibrations.

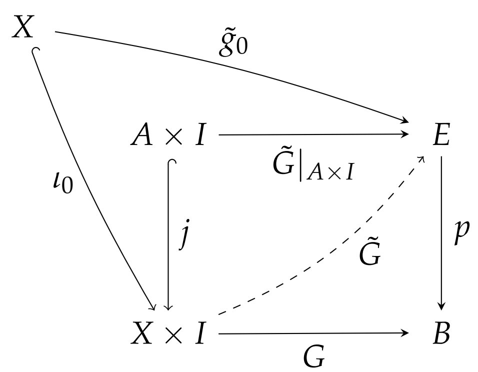
Definition 3.4.2. *A **fibration** $f : E \to B$ is a map that satisfies the homotopy lifting property with respect to all space X.*

Let us check that a projection map is indeed a fibration. Take a projection $\pi : B \times F \to B$, a homotopy $G : X \times I \to B$ and $\tilde{g}_0 : X \to B \times F$ the initial lift. We can write $\tilde{g}_0(x) = (G(x, 0), h(x))$ for some map $h : X \to F$, because $\tilde{g}_0$ is a lift of $G|_{X \times \{0\}}$. Then we can define the lift $\tilde{G} : X \times I \to B \times F$ as $\tilde{G}(x, t) = (G(x, t), h(x))$, so the projection map π satisfies the homotopy lifting property for any space X and is thus a fibration.

In example 2.2.1 and Theorem 2.3 in [14] it is shown that a covering map is also a fibration.

Another definition that we will use is the homotopy lifting property for a pair (X, A). This definition will be useful to prove the next theorem.

Definition 3.4.3. *A map $p : E \to B$ has the homotopy lifting property for a pair (X, A) if for any homotopy $G : X \times I \to B$, map $\tilde{g}_0 : X \to E$ and lift $\tilde{G}_A : A \times I \to E$ of $G|_{A \times I}$, we can extend $\tilde{G}_A : A \times I \to E$ to a lift of G, $\tilde{G} : X \times I \to E$, such that $\tilde{g}_0 = \tilde{G}|_{X \times \{0\}}$.*
In a commutative diagram this looks as follows.



We take a look at the case where $X = D^k$. For this, we first introduce one last property of maps: the lift extension property for a pair.

Definition 3.4.4. *Let Z be a set and $A \subset Z$ a subset. The map $p : E \to B$ has the **lift extension property** for a pair (Z, A) if every map $f : Z \to B$ has a lift $\tilde{f} : Z \to E$, which extends a given lift $\tilde{g} : A \to Z$ of $f|_A$.*

The homotopy lifting property for a set X and the homotopy lifting property for a pair (X, A) can both be seen as special cases of the lift extension property. The lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$, corresponds to the homotopy lifting property for the pair (X, A), and the lift extension property for the pair $(X \times I, X \times \{0\})$, corresponds to the homotopy lifting property with respect to X.

Lemma 3.4.5. *The homotopy lifting property with respect to D^k is equivalent to the homotopy lifting property for the pair $(D^k, \partial D^k)$.*

Proof. I will only give an outline of the proof, for the full proof see page 376 of [11].

With the correspondence illustrated above we have that the homotopy lifting property with respect to D^k, is the same as the lift extension property for the pair $(X \times I, X \times \{0\})$ and the homotopy lifting property of the pair $(D^k, \partial D^k)$ is the same as the lift extension property for the pair $(D^k \times I, D^k \times \{0\} \cup \partial D^k \times I)$. We see that $(D^k \times I, D^k \times \{0\})$ and $(D^k \times I, D^k \times \{0\} \cup \partial D^k \times I)$ are homeomorphic pairs. Using CW complexes, as explained in [11], equivalence of the two extension properties can be shown. □

Maps that satisfy the homotopy lifting property for disks will give us our next long exact sequence. Using the long exact sequence of Theorem 3.3.1 we will present another long exact sequence, based on maps satisfying the homotopy lifting property for lifts. We will be able to use this sequence to find homotopy groups of quotient spaces.

Theorem 3.4.6. *Suppose $p : E \to B$ has the homotopy lifting property with respect to all disks $D^k := \{x \in \mathbb{R}^k | \ \ ||x|| \leq 1\}$ with $k \geq 0$. Let $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$ be base points.*

Then the induced map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. For B path-connected this implies that there exists a long exact sequence

$$\begin{aligned} \cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \\ \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0. \end{aligned} \tag{3.16}$$

Proof. We need to show that p_* is bijective.

Surjectivity: Take $f : (I^n, \partial I^n) \to (B, b_0)$ with $[f] \in \pi_n(B, b_0)$. p has the relative homotopy lifting property for the pair $(D^k, \partial D^k) \cong (I^k, \partial I^k)$. Furthermore, we can lift $f|_{J^{n-1}}$ to E with the constant map c_{x_0}. Note that $I^{n-1} \subseteq J^{n-1}$, so we can choose coordinates such that $c_{x_0}|_{I^{n-1}}$ can be considered as the initial homotopy $\tilde{g}_0$. Now the homotopy lifting property for a pair gives us that we can extend c_{x_0} to a lift $\tilde{f} : I^n \to E$ of f.

Since $(p \circ \tilde{f})(\partial I^n) = f(\partial I^n) = \{b_0\}$, we have $\tilde{f}(\partial I^n) \subseteq p^{-1}(\{b_0\}) = F$. Therefore we know that $[\tilde{f}] \in \pi_n(E, F, x_0)$. Finally, per definition of p_* we have $p_*([\tilde{f}]) = [p \circ \tilde{f}] = [f]$, and thus p_* is surjective.

Injectivity: For $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1} \to (E, F, x_0)$ with $p_*([\tilde{f}_0] = p_*([\tilde{f}_1])$, we know that there exists a homotopy $G : (I^n \times I, \partial I^n \times I) \to (E, F, x_0)$ relative to ∂I^n between $p \circ \tilde{f}_0$ and $p \circ \tilde{f}_1$. We want to use the homotopy lifting property. We take as our initial lift $\tilde{g}_0 = \tilde{f}_0$ and give a partial lift of G,

$$\tilde{G}_A(x) = \begin{cases} \tilde{f}_0 & \text{if } x \in I^n \times \{0\}, \\ \tilde{f}_1 & \text{if } x \in I^n \times \{1\}, \\ c_{x_0} & \text{if } x \in J^{n-1} \times I. \end{cases} \tag{3.17}$$

Note that $\tilde{G}_A$ is a lift of $G|_A$ where G is taken as a map with domain I^{n+1}. We rewrite A as follows.

$$A = I^n \times \{0,1\} \cup J^{n-1} \times I \tag{3.18}$$
$$= I^n \times \{0,1\} \cup \overline{(\partial I^n \setminus I^{n-1})} \times I \tag{3.19}$$
$$\cong \overline{\partial I^{n+1} \setminus I^n} = J^n \tag{3.20}$$

Where we take the third step by permuting the last two coordinates. Now we can use the homotopy lifting property, to extend $\tilde{G}_A$ to a lift $\tilde{G}$ of G on I^{n+1}. Note that $\tilde{G}$ is a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$, so we find $[\tilde{f}_0] = [\tilde{f}_1]$ and thus p_* is injective.

To find the long exact sequence we use Theorem 3.3.1 for the pair (E, F). Using the isomorphism p_*, we replace $\pi_n(E, F, x_0)$ with $\pi_n(B, b_0)$ to find the sequence up to $\pi_0(E, x_0)$.
We can put a 0 at the end of the sequence, because $\pi_0(F, x_0) \to \pi_0(E, x_0)$ is a surjective map. B is path connected, so for any point $x \in E$ we can construct a path from x to F, as a lift of a path in B from $p(x)$ to b_0. Therefore there is an element $f_0 \in F$ contained in each path component of E and the class of the constant map $[c_{f_0}]$ will be mapped to corresponding path connected component in E. Thus $\pi_0(F, x_0) \to \pi_0(E, x_0)$ is a surjective map. □

Next we will introduce the concept of a fibre bundle. A fibre bundle is a topological space E with a projection map $p : E \to B$, a fibre F and a base space B, such that locally E is homeomorphic to the product space $B \times F$.

Definition 3.4.7. *Let F, B and E be topological spaces and $p : E \to B$ be a surjective continuous map such that for all $b \in B$ there exists an open neighbourhood $b \in U \subseteq B$ whose inverse image in E is homeomorphic to $U \times F$ via the map $h : p^{-1}(U) \to U \times F$, for which we have $p = \pi_1 \circ h$. Here π_1 is the projection onto the first coordinate.*

Then we call the space E together with the map p a ***fibre bundle*** *with* ***fibre*** *F. Further we call B the* ***base space****, E the* ***total space*** *and h the* ***local trivialisation*** *of the fibre bundle.*

A fibre bundle structure is determined by the projection map p, but it is often written as the short exact sequence $F \to E \to B$ or as the tuple (E, B, F, h). Fibre bundles help us to better understand the structure of homogeneous manifolds and will therefore be useful in understanding topological defects.

To get a better understanding of fibre bundles, we will discuss some examples. They can also be found on pages $90 - 91$ in [14] and on pages $377 - 379$ in [11].

1. The projection map $p : B \times F \to B$ induces a fibre bundle, known as the product bundle, $(B \times F, B, F, p)$.

2. A fibre bundle with discrete fibre, is a covering space. The converse is true for covering spaces whose fibres all have the same cardinality. For instance, covering spaces over a connected base space are fibre bundles.

3. A non-trivial example is that of the mobiüs band. We have a total space $E = I \times [-1, 1]/\sim$, where $(0, v) \sim (1, -v)$, that is, E is the mobiüs band. The projection map $p : E \to (I/\sim) \cong S^1$, with $0 \sim 1$ is defined as $\overline{p(t, v)} = \bar{t}$. We find fibres

$$p^{-1}(\bar{t}) = \overline{\{t\} \times [-1, 1]} \cong [-1, 1]. \tag{3.21}$$

   By removing the lines at $\overline{t_1} = \bar{1}$ or $\overline{t_2} = \overline{\tfrac{1}{2}}$ from the mobiüs band, we find two rectangles U_1 and U_2 that combined cover the whole mobiüs band. Further we define $l_i := (I/\sim) \setminus \{t_i\}$, to notice

$$U_i = p^{-1}(l_i) \cong l_i \times [-1, 1]. \tag{3.22}$$

   So the mobiüs band with the projection map p is indeed a fibre bundle over S^1 with fibre $[-1, 1]$.

4. We can combine two mobiüs bands to obtain a Klein bottle, a fibre bundle with base space S^1 and fibre S^1. We set $K = (E_1 \sqcup E_1)/\sim^*$, where the equivalence relation is defined such that for $\overline{(t,v)}_1 \in E_1$ and $\overline{(s,w)}_2 \in E_2$ we have $\overline{(t,1)}_1 \sim^* \overline{(t,-1)}_2$. We define the projection map $q : K \to S^1$, $q(\overline{(t,v)}) = \bar{t}$ and find fibres

$$q^{-1}(\{\bar{t}\}) = \overline{\{t\} \times [-1,1]} \cong [-1,1]/\sim^* \cong S^1. \tag{3.23}$$

With similar arguments as for the mobiüs band, this becomes a fibre bundle.

5. A differentiable n-manifold is the base space of a fibre bundle, namely that of the tangent bundle with the projection map $\pi : TM \to M$ and fibre $\mathbb{R}^n$.

6. A Lie group G, with closed subgroup H and the emergent homogeneous manifold G/H can be seen as the fibre bundle $(G, G/H, H, p)$, where p is the projection $p : G \to G/H$. Note that we indeed find $p^{-1}(gH) = gH \cong H$, where the first gH refers to an element of G/H while the second gH is seen as a subset of G.

7. Projective spaces also give rise to interesting fibre bundles. They are specific examples of fibre bundles of Lie groups as in the example above. Real projective spaces can be constructed as fibre bundles $S^n \to \mathbb{R}P^n$ with fibre S^0. In terms of Lie groups we have $G = S^n$ and $H = \{\pm 1\}$.

   Analogously we find complex projective spaces in the fibre bundles $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, where $\mathbb{C}P^n := S^{2n+1}/\sim$ and

$$(z_0, .., z_n) \sim \lambda(z_0, .., z_n) \text{ for al } \lambda \in S^1 \subseteq \mathbb{C}. \tag{3.24}$$

   Note that S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ with real dimension $2n + 1$. This means that we indeed have $n + 1$ complex coordinates to describe elements of S^{2n+1}. In terms of fibre bundles of Lie groups, this is an example where $G = S^{2n+1}$, $H = S^1$ and consequently the complex projective space is the homogeneous manifold G/H.

   To see that this is indeed a fibre bundle, we construct the homeomorphisms as in the definition. Define $U_i := \{[z_0, ..., z_n] \mid z_i \neq 0\} \subseteq \mathbb{C}P^n$, and homeomorphisms

$$\begin{aligned} h_i : p^{-1}(U_i) &\to U_i \times S^1 \\ (z_0, ..., z_n) &\mapsto ([z_0, ..., z_n], \frac{z_i}{|z_i|}). \end{aligned} \tag{3.25}$$

Note that h_i is indeed a homeomorphism with inverse

$$h_i^{-1}([z_0, ..., z_n], \lambda) = \lambda |z_i| z_i^{-1}(z_0, ..., z_n). \tag{3.26}$$

8. In the case of $n = 1$ this yields the Hopf bundle,

$$S^1 \to S^3 \to S^2. \tag{3.27}$$

The Hopf bundle is a very well known fibre bundle. More on the Hopf bundle and the similarly defined second and third Hopf bundles can be found in Examples 4.45, 4.46 and 4.47 of [11].

We will want to use the long exact sequence of Theorem 3.4.6 for the example of fibre bundles $H \to G \to G/H$. The following theorem tells us that we are warranted to do so. First we define paracompactness as on page 9 of [2].

Definition 3.4.8. *A collection of subsets $\mathcal{X}$ of M is said to be **locally finite** if for each $x \in M$ there is exists a neighbourhood that intersects at most finitely many sets in $\mathcal{X}$.*
*Given a cover $\mathcal{U}$ of a topological space M another cover $\mathcal{V}$ is called a **refinement** of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists an $U \in \mathcal{U}$ such that $V \subseteq U$.*
*A topological space M is called **paracompact** if every open cover of M admits an open locally finite refinement.*

Theorem 3.4.9. *Fibre bundles over manifolds are fibrations.*

Proof. Per definition of paracompactness, any open covering of a paracompact space has a numerable refinement. Given a map $p : E \to B$ and a numerable covering $\mathcal{U}$ of B such that for $U \in \mathcal{U}$ the map $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a fibration, then p is a fibration. This follows from Theorem 12 of [14]. Therefore, with Theorem 13 and corollary 14 in [14], we can conclude that all fibre bundles over paracompact and Hausdorff base spaces are fibrations. Since manifolds are paracompact by Theorem 1.15 of [11] and Hausdorff, fibre bundles over manifolds are fibrations. □

This concludes our discussion of homotopy theory and with that also the mathematical background needed to dive into the physics of topological defects.

Chapter 4

Field theory

With the mathematical background completed, it is time to move on to the physical background. We will begin with an introduction in classical field theory. This chapter is based on the contents of chapter 12 in [7] and chapters $1-5$ in [6].

4.1 Minkowski space and Lorentz invariance

In field theory we work in spacetime, that is the space around us with time as a fourth coordinate. In this thesis will will not work with curved space, so that mathematically the spacetime can can be considered as $\mathbb{R}^4$. We will be working with the Minkowski metric. This metric on $\mathbb{R}^4$ defines distances with respect to the four coordinates t, x, y and z which are often denoted by x^0, x^1, x^2 and x^3. The Minkowski metric is the metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.1}$$

Before we continue, we introduce some convenient notation: the Einstein summation convention. If in an expression the same index occurs twice or more often in the same term, we will assume a summation over this index. Unless otherwise stated, or clear from context, Roman indices, such as i, j, will refer to the three coordinates of physical space, (x, y, z), whereas Greek indices, such as μ, ν refer to all four coordinates of spacetime (t, x, y, z). We will also be using the convention of [6] to use only upper indices. For example, we have

$$x^i x^i = \sum_{i=1}^{3} (x^i)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{4.2}$$

and

$$x^\mu x^\mu = \sum_{\mu=0}^{3} (x^\mu)^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{4.3}$$

With this, we can concisely write down the spacetime interval between two events:

$$s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \,. \tag{4.4}$$

We call s^2 the distance. For convenience we will work with units $c = 1$, which yields

$$s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \,. \tag{4.5}$$

This flat construction of spacetime, $\mathbb{R}^4$ with the Minkowski metric, is called Minkowski space, M. When we describe physical systems or quantities, we want them to be invariant of the choice of coordinates in Minkowski space. This means that all the equations and quantities that we use, have to be invariant under coordinate transformations in Minkowski space. We will look for transformations that leave the spacetime interval in Equation 4.4 invariant. A simple transformation is translation, we can shift al coordinates along a constant vector $a \in M$:

$$x^{\mu'} = x^\mu + a^\mu. \tag{4.6}$$

A more interesting transformation is one where we multiply x^μ with a matrix $\Lambda \in \mathrm{Mat}(4 \times 4, \mathbb{R})$,

$$x' = \Lambda x. \tag{4.7}$$

Now we have to check, for which matrices Λ we find the same distances in the coordinate system of x as that of x'.

We need:

$$\begin{aligned} s^2 = (\Delta x)^\top \eta (\Delta x) &= (\Delta x')^\top \eta (\Delta x') \\ &= (\Delta x)^\top \Lambda^\top \eta \Lambda (\Delta x), \end{aligned} \tag{4.8}$$

which gives us

$$\eta = \Lambda^\top \eta \Lambda. \tag{4.9}$$

The matrices Λ that adhere to this relation are known as Lorentz matrices, which form the Lorentz. There are three main types of transformations in the Lorentz group.

- We have conventional rotations in space over an angle $\theta \in [0, 2\pi)$. For example the rotation in the xy-plane:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

  The rotational 2×2 sub-matrix can be in any pair of x, y, z coordinates.

- We also have boosts, which can be thought of as rotations between space and time, with parameter $\phi \in (-\infty, \infty)$. For example, the boost in the coordinates t and x:

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

  Again the boost can be in any pair of t with one of the spatial coordinates x, y, z.

- Furthermore, we have discrete transformations that reverse the direction of time of one or more spatial coordinates. For example, the reversing of the time coordinate:

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.12}$$

The Lorentz group is the group that consists of all transformations obtained by multiplications of matrices corresponding to rotations, boosts and coordinate reversals. It is a non-abelian group. The group containing all translations and Lorentz transformations is referred to as the Poincaré group, which is also a non-abelian group. We will often consider equations or quantities that are invariant under the Lorentz transformations. These are called Lorentz covariant. Scalars that are Lorentz covariant are Lorentz scalars [15].

With one last notational convection we wrap up the prior knowledge of field theory. We will be using the following notation for derivatives of functions:

$$\partial_\mu f = \frac{df}{dx^\mu}. \tag{4.13}$$

In case of repeated indices this yields,

$$\partial_\mu f \partial_\mu f = \sum_{\mu=0}^{3} (\frac{df}{dx^\mu})^2. \tag{4.14}$$

Again for Roman indices we only sum over the three coordinates of physical space.

4.2 From discrete to continuous

Field theories are mathematical models that describe physical concepts using fields. Fields are continuous maps from spacetime to some real or complex vector space. Examples of fields include the temperature, which is a real number given at each point in space time, the wind, which gives a real three vector at each point in space time, and the electromagnetic field.

The basics of classical field theory can be derived from classical mechanics. For example, a string of countably infinite particles at a distance a apart and connected through springs, becomes a continuous string in the limit $a \to 0$. This is the strategy we apply to find the first principles in classical field theory. In [7], a very complete derivation of many such principles is given. We will only be giving a very short overview of what is covered there.

To go from classical mechanics to field theory, one of the first things to change is the Lagrangian, which has to become a Lagrangian density. A Lagrangian density of a field in its most general form is a function of field variables η_ρ, first derivatives of the field variable $\frac{d\eta_\rho}{dx^\mu} = \partial_\mu \eta_\rho$ and the four parameters of spacetime x and t:

$$\mathscr{L} = \mathscr{L}(\eta_\rho, \partial_\mu \eta_\rho, x, t) = \mathscr{L}(\eta_\rho, \frac{d\eta_\rho}{dx}, \frac{d\eta_\rho}{dt}, x, t), \tag{4.15}$$

where ρ is an index that runs over all field variables. The field variables η_ρ can be thought of as generalised coordinates of the coordinates η_i in classical mechanics.

By complete analogy of classical mechanics, we can derive the Euler-Lagrange equations based on Hamilton's principle using the variation of the action with respect to the field variables. For the derivation we again refer to [7]. The Euler-Lagrange equations are as follows:

$$\frac{d}{dx^\nu}\left(\frac{\partial \mathscr{L}}{\partial \eta_{\rho,\nu}}\right) - \frac{\partial \mathscr{L}}{\partial \eta_\rho} = 0. \tag{4.16}$$

Using the Lagrangian density, also referred to simply as the Lagrangian, we can look for the field equations of η_ρ, the field analogue of the equations of motion.

4.2.1 The stress-energy tensor

Using the Lagrangian we can define the stress-energy tensor, also known as the stress-energy-momentum or energy-momentum tensor. It is a useful concept with a myriad of applications. In this section, we will explain what the stress-energy tensor is and what information it contains.

Given a Lagrangian density $\mathscr{L}$ we define the stress-energy tensor as

$$T_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \eta_{\rho,\nu}} \eta_{\rho,\nu} - \mathscr{L} g_{\mu\nu} \, . \tag{4.17}$$

In this case $g_{\mu\nu}$ is the metric tensor. As we are working with Minkowski space we will use $g_{\mu\nu} = \eta_{\mu\nu}$.

Let us take a look at some of the most prevalent information contained in the stress-energy tensor. For an extensive overview and the derivations of all identifications see Section $12-3$ of [7].

- The component T_{00} can be identified as the total energy density of the model, and thus the total energy, R_0, can be found by integration

$$R_\mu = \int T_{\mu 0} dV. \tag{4.18}$$

- R_μ, for $\mu = 0, 1, 2, 3$ as defined as above, are known as the conserved currents, because $\frac{dR_\mu}{dt} = 0$.

- The three vector with components T_{i0} can be identified with the field momentum density.

- The three vectors $\mathbf{T}_0$ with components T_{0j} represent the field energy current density, whereas the three vectors $\mathbf{T}_i$ with components T_{ij} represent the current density of the field momentum density .

The most important component for us is T_{00}, as we will try to find the energy contained in topological defects.

4.3 Examples of field theories

We will discuss two field theories. We will begin with the simplest example, that of a complex scalar field, and continue with the electromagnetic field. The example of the electromagnetic field will be our first encounter with gauge theory. Both examples are discussed in more detail in chapter 12 in [6] and chapters 1 and 2 in [6].

4.3.1 The complex scalar field

A complex scalar field is a function from Minkowski space to the complex numbers, $\varphi : M \to \mathbb{C}$. It can either be expressed in terms of the real part $\varphi_1(x) = \frac{1}{\sqrt{2}}\mathfrak{Re}\{\varphi(x)\}$ and the imaginary part $\varphi_2(x) = \frac{1}{\sqrt{2}}\mathfrak{Im}\{\varphi(x)\}$ or in terms of the field itself φ and its complex conjugate φ^*.

We will take a look at the following Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi^*\partial_\mu\varphi - \frac{1}{2}m^2\varphi^*\varphi. \tag{4.19}$$

This Lagrangian will give us only second-order differential equations for the field equations, which makes it comprehensible, and it is indeed a Lorentz scalar. With Noether's theorem, which we will discuss in section 4.4, this Lagrangian leads to an associated charge and current density. This makes it a physically interesting example to consider. In the Lagrangian we can identify the kinetic term

$$\mathscr{L}_T(\varphi) = \frac{1}{2}(\partial_\mu\varphi)^2, \tag{4.20}$$

and the potential term

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \tag{4.21}$$

Using the Euler-Lagrange equations for fields, Equation 4.16, we find the second-order field equations for φ and φ^*

$$\partial_\mu \partial_\mu \varphi + m^2 \varphi = 0, \tag{4.22}$$

$$\partial_\mu \partial_\mu \varphi^* + m^2 \varphi^* = 0. \tag{4.23}$$

These equations are known as the Klein-Gordon(-Fock) equations. Their solutions are wave equations as is derived in Section $12-6$ of [7].

The Lagrangian above contains terms in the field of at most order 2, $\varphi^\dagger \varphi$, and is thus quadratic in the field. If a Lagrangian is quadratic in the fields, we call the fields it describes free or linear, because we find linear equations for the fields. Higher order terms in the fields can occur in the Lagrangian and are referred to as interaction terms or non-linear terms. In this thesis, we will consider terms up to fourth order. Interacting fields still have to be invariant under translations and Lorentz transformations. This assumption is met when the Lagrangian of interaction (i.e. higher order terms of the Lagrangian) is a Lorentz scalar.

When we consider a set of N independent complex scalar fields, described using one Lagrangian, we will be using vector notation. For example

$$\mathscr{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi - m^2(\varphi^\dagger \varphi) - \lambda(\varphi^\dagger \varphi)^2 \tag{4.24}$$

describes a set of N complex scalar fields, where $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$ is a vector of N scalar complex fields and $\varphi^\dagger$ is the Hermitian transpose of φ,

$$\varphi^\dagger = (\varphi_1^*, \cdots, \varphi_N^*) \tag{4.25}$$

In the Lagrangian λ and m^2 are real numbers, and if $m^2 > 0$ we call $m > 0$ the mass of the field. The partial derivative $\partial_\mu \varphi$ acts on all components of φ separately.

4.3.2 The electromagnetic field

The electromagnetic field is probably the most familiar example of a field theory. The electric- and magnetic field are described by Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4.26}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi \mathbf{j}}{c}. \tag{4.27}$$

In these expressions ρ is the total charge and $\mathbf{j}$ is the total current. We can also express the electric and the magnetic field in the form of one antisymmetric field strength tensor $F_{\mu\nu}$, such that:

$$F_{\mu\mu} = 0, \tag{4.28}$$

$$F_{i0} = -F_{0i} = E_i, \tag{4.29}$$

$$F_{ij} = \varepsilon_{ijk} B_k. \tag{4.30}$$

Where we use the Einstein summation convention in the last line, and ε_{ijk} is the Levi-Civita symbol for three indices, Equation 2.39.

The field strength tensor, also known as the strength tensor or field tensor, is a tensor that we will run into more often once we discuss gauge theory. There, the electromagnetic example will be our guide line, which we will use to generalise gauge theory to gauge fields that transform under non-abelian Lie groups. In this process we will use the strength tensor as it can be fitted into the format of Lie groups and Lie algebras nicely.

The field variables of the electromagnetic field are truly given by the four-vector $\mathbf{A}_\mu$, where A_0 is a scalar field and $\mathbf{A}_i$ is a three dimensional vector potential of $\mathbf{B}$. Then we have:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}_i}{\partial t}. \tag{4.31}$$

In this context, the field tensor $F_{\mu\nu}$ can be defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We can write Maxwell's equations in a vacuum using the field tensor $F_{\mu\nu}$. For the homogeneous Maxwell equations (Equation 4.26) we note that they can be written as the following equations (four equations, since there is no summation convention)

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0. \tag{4.32}$$

Substitution shows that these equations are trivial if we write them in terms of the field variable A_μ. Therefore the homogeneous Maxwell equations should not be considered as the field equations. Equation 4.27 can also be written in terms of the field tensor. For this we define j_μ as the four-vector consisting of the charge $j_0 = \rho$ and the current $j_i = \mathbf{j}_i$. We then find the four equations

$$\partial_\nu F_{\mu\nu} = \frac{4\pi j_\mu}{c}. \tag{4.33}$$

In a vacuum, without interactions with charges or currents, this reduces to $\partial_\nu F_{\mu\nu} = 0$.

These equations are also obtained using the Euler-Lagrange equations for the following Lagrangian, which is thus known as the Lagrangian of electrodynamics,

$$\mathscr{L}_A = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu}. \tag{4.34}$$

The electromagnetic field does not have a unique description in terms of the field variables A_μ. For any function $\alpha : M \to \mathbb{C}$ depending on the spacetime coordinates, we can transform $\mathbf{A}_\mu$ as follows:

$$A_\mu(x) \to A'_\mu(x) = A_\mu + \partial_\mu \alpha \tag{4.35}$$

By equality of mixed partials, we find the exact same strength tensor $F_{\mu\nu}$ for A'_μ as for A_μ and therefore also the same action and energy. Thus the transformation of A_μ leaves all observable, and thus all physical, quantities invariant. Such a transformation is called an invariant transformation. An important distinction to make, is that between global transformations, which are independent of the spacetime coordinates, and gauge transformations, which are local transformations, and thus dependent on the spacetime coordinates. We will discuss both types of transformations in more detail in chapter 5.

4.4 Noether's theorem

An important theorem, that is most elegantly expressed in field theory, is Noether's theorem. It states that any symmetry or invariance implies the conservation of a quantity. An example is the time invariance of a Lagrangian, which implies energy conservation. Another is when there is no explicit dependence on a give space coordinates, which implies that the corresponding momentum is conserved, Section $12-7$ in [7]. In electrodynamics, conservation of charge and current can be derived from the invariance of field transformations which do not affect the spacetime coordinates, see Section 2.8 in [6].

We will not go into the details of Noether's theorem. For a comprehensive explanation and derivation see Section $12-7$ in [7] and for a more to the point explanation focussed on our applications, see Section 12.8 in [6]. For us it is sufficient to understand that symmetries of a system correspond to conservations of a quantity in the system.

Chapter 5

Gauge theory

In this chapter we will discuss symmetric models described by gauge theories. We will explain what gauges are and subsequently discuss some examples. An important sub field of gauge theory that we will come across is Yang-Milss theory. Yang-Mills theory is a small deviation from our principle goal of understanding topological defects, but it is very instructive and the two topics are deeply related. Again, electrodynamics will be an important example.

5.1 Global symmetries

We saw before that the description of the electromagnetic field is not unique. Analogously, the Lagrangian of any field can be invariant under a set of symmetries. What we mean by this is that if we transform the field according to the action of a certain group G, we want the Lagrangian to remain the same. That is, if

$$\varphi'(x) = g \cdot \varphi(x) \text{ for } g \in G, \tag{5.1}$$

then we want

$$\mathscr{L}(\varphi') = \mathscr{L}(\varphi). \tag{5.2}$$

We will later find that the group G needs to be a compact Lie group as discussed in chapter 2.

A simple example can be seen using the Lagrangian

$$\mathscr{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi - m^2(\varphi^\dagger \varphi) - \lambda(\varphi^\dagger \varphi)^2. \tag{5.3}$$

As before, $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$ is a vector of N scalar complex fields and $\varphi^\dagger$ is the Hermitian transpose of φ. The partial derivative with respect to x^μ, ∂_μ, acts separately on all components of φ. It is clear to see that this Lagrangian is invariant under transformations of the group $U(1)$. Take $g = e^{i\alpha}$ where $\alpha \in \mathbb{R}$, such that $\varphi'(x) = e^{i\alpha}\varphi(x)$ and correspondingly $(\varphi^\dagger)'(x) = e^{-i\alpha}\varphi^\dagger(x)$. Then we find, omitting the dependence on x,

$$(\varphi^\dagger\varphi)' = e^{-i\alpha}\varphi^\dagger e^{i\alpha}\varphi = \varphi^\dagger\varphi. \tag{5.4}$$

Note that for the kinetic term in the Lagrangian the same derivation holds, because $e^{i\alpha}$ is independent of the spacetime coordinate and thus it can be pulled in front of the derivative to yield the same conclusion. Thus we find that $\mathscr{L}$ is invariant under global transformations of the Lie group $U(1)$.

This Lagrangian, however, also has a more interesting symmetry, namely that of the special unitary group of $N \times N$ matrices $SU(N)$. We now take $g = \omega \in SU(N)$, such that

$$\varphi'(x) = \omega\varphi(x), \text{ and} \tag{5.5}$$
$$(\varphi^\dagger)'(x) = \varphi^\dagger\omega^\dagger(x). \tag{5.6}$$

Using the fact that ω is a unitary matrix we again find

$$(\varphi^\dagger\varphi)' = \varphi^\dagger\omega^\dagger\omega\varphi = \varphi^\dagger\varphi. \tag{5.7}$$

Analogously, the kinetic term in the Lagrangian is also invariant under $SU(N)$ transformations.

These are both examples where the Lagrangian is invariant under global transformations of a simple Lie group. Gauge theory is the theory that considers local transformations, that is transformations that are dependent on the spacetime. The important requirement in Gauge theory is that observable quantities, as well as actions and equations of motion should be gauge invariant, that is, they should be invariant under the gauge transformations. As in the examples above, invariant means that if we replace the field variables $\varphi(x)$ by transformed field variables $\varphi'(x) = g(x)\varphi(x)$ under an action of a certain group G, $g(x) \in G$ for all $x \in M$, we obtain the same quantity or equation. If the Lagrangian is gauge invariant, this gives us a gauge invariant action, strength-tensor and field equations.

Effectively, gauge invariance is a way of implementing invariance in the mathematical description of a system. For the same system one can use a set of descriptions, which are mathematically related via a Lie group, to find the same physical quantities corresponding to the system. Sometimes it is desirable to eliminate this mathematical freedom. In this case we can choose a gauge condition to set the gauge. A gauge condition is an extra condition we can put on our field variables, which is not gauge invariant. There are many ways to do this, depending on the gauge invariance considered. Some gauge conditions only eliminate part of the mathematical freedom, whereas others eliminate all gauge redundancy. In the case of the electromagnetic field there are a few commonly used gauge conditions that can be imposed on the field variables A_μ. These are the following

1. The coulomb gauge
$$\mathrm{div}\mathbf{A} = \partial_i A_i = 0,$$
2. The Lorents gauge
$$\partial_\mu A_\mu = 0,$$
3. The zero gauge
$$A_0 = 0.$$

5.2 Yang-Mills theory

The electromagnetic field is a well known example of a gauge theory. The symmetry group of the electromagnetic field is $U(1)$, an abelian group. Chen Ning Yang and Robert Mills generalised gauge theory to non-abelian groups [16]. This section aims to give a short introduction to Yang-Mills theory and provides some examples of non-abelian symmetries. Here our previous discussion of Lie groups and Lie algebras will be used. We will first discuss an example with gauge group $SU(N)$, subsequently we will look at a more general statement.

5.2.1 Gauge invariance with $SU(N)$

Previously we saw global $SU(N)$ invariance of the Lagrangian

$$\mathscr{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \lambda(\varphi^\dagger \varphi)^2. \tag{5.8}$$

We will now consider the gauge invariance of the special unitary group, meaning that we will consider local transformations,

$$\varphi'(x) = \omega(x)\varphi(x), \tag{5.9}$$

where $\omega(x) \in SU(N)$ for all $x \in M$ and the map $\omega : M \to SU(N)$ has to be C^1.

We note that all the potential terms in de Lagrangian are still invariant under this transformation by unity of the matrix $\omega(x)$. The kinetic term, however, is not invariant any more. We have:

$$\partial_\mu \varphi'(x) = \partial_\mu \left[\omega(x)\varphi(x)\right] = \partial_\mu \omega(x) \cdot \varphi(x) + \omega(x)\partial_\mu \varphi(x). \tag{5.10}$$

The derivative of ω gives us a problem, because it will not cancel with the derivative of $\omega^\dagger$. Luckily, there is a solution to this problem. We can introduce a covariant derivative D_μ which will replace our conventional derivative ∂_μ. This covariant derivative will be dependent on the spacetime coordinates and we can define it as

$$\partial_\mu \varphi(x) \to D_\mu(x)\varphi(x) := \left[(\partial_\mu + A_\mu(x)\right] \varphi(x). \tag{5.11}$$

Here A_μ is a vector field, that takes matrix values. A_μ is called the gauge-field, and in case of a non-abelian group, such as $SU(N)$, it is also known as the Yang-Mills field. With this definition the covariant derivative is an operator acting on the field φ.

The exact workings and a mathematical description of the covariant derivative can be given using principle bundles and connections. Sadly this is beyond the scope of this thesis, and we will simply accept the definition above. For those interested in the details of the covariant derivative, we refer to chapter 6 of [17].

We want the covariant derivative to solve the problem of the kinetic term of our Lagrangian, therefore it has to transform under our transformations as

$$(D_\mu \varphi)' = \omega D_\mu \varphi. \tag{5.12}$$

Note that we omitted the explicit dependence on x. In the rest of this section the dependence on spacetime coordinates of φ, ω, A_μ and D_μ will be assumed and will sometimes be implicit. Let us use the demanded transformation relation of the covariant derivative, to find the transformation relation of A_μ.

$$(D_\mu \varphi)' = \omega D_\mu \varphi \tag{5.13}$$
$$\partial_\mu(\omega\varphi) + A'_\mu \omega\varphi = \omega\partial_\mu\varphi + \omega A_\mu \varphi \tag{5.14}$$
$$\partial_\mu \omega \cdot \varphi + A'_\mu \omega\varphi = \omega A_\mu \varphi \tag{5.15}$$

Each term now contains a field φ on the right-hand side. Considering that we want this relation to hold for all fields, we can leave out the field φ. We then get an equality of operators. Using

$$\begin{aligned}\partial_\mu(\omega\omega^{-1}) &= \partial_\mu(I) = 0\\ &= \partial_\mu\omega\cdot\omega^{-1} + \omega\partial_\mu\omega^{-1},\end{aligned} \tag{5.16}$$

we find the transformation relation of the vector field A_μ:

$$A'_\mu = \omega A_\mu \omega^{-1} + \omega\partial_\mu\omega^{-1}. \tag{5.17}$$

This transformation relation will also tell us something about the structure of A_μ. As it turns out we need A_μ to be an element of the Lie algebra of $SU(N)$. For $N = 2$, we will prove that if A_μ takes values in the Lie algebra, $\mathfrak{su}(N)$, then A'_μ will also take values in $\mathfrak{su}(N)$. This means that A_μ taking values in $\mathfrak{su}(2)$ results in a well-defined transformation.

Lemma 5.2.1. *If $A_\mu(x) \in \mathfrak{su}(2)$ for all $x \in M$, then $A'_\mu(x) \in \mathfrak{su}(2)$ for all $x \in M$. Where M is Minkowski space, i.e. $\mathbb{R}^4$ with the Minkowski metric.*

Proof. Recall that for $\omega(x) \in SU(2)$ we can write $\omega = \begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}$ where $\alpha(x), \beta(x) \in \mathbb{C}$ and $|\alpha(x)|^2 + |\beta(x)|^2 = 1$. First we take a look at the $\omega\partial_\mu\omega^{-1}$ term. We find $\omega^{-1} = \frac{1}{\det(\omega)}\begin{pmatrix}\overline{\alpha} & \overline{\beta}\\ \beta & \alpha\end{pmatrix}$. Since $\omega \in SU(2)$, we have $\det(\omega) = 1$, consequently we find

$$\omega\partial_\mu\omega^{-1} = \begin{pmatrix}\alpha\partial_\mu\overline{\alpha} + \overline{\beta}\partial_\mu\beta & \alpha\partial_\mu\overline{\beta} - \overline{\beta}\partial_\mu\alpha\\ \beta\partial_\mu\overline{\alpha} - \overline{\alpha}\partial_\mu\beta & \beta\partial_\mu\overline{\beta} + \overline{\alpha}\partial_\mu\alpha\end{pmatrix}. \tag{5.18}$$

If we now use

$$\partial_\mu(\alpha\overline{\alpha} + \beta\overline{\beta}) = \partial_\mu 1 = 0, \tag{5.19}$$

and commutativity in $\mathbb{C}$, we see that

$$\alpha\partial_\mu\overline{\alpha} + \overline{\alpha}\partial_\mu\alpha + \beta\partial_\mu\overline{\beta} + \overline{\beta}\partial_\mu\beta = 0. \tag{5.20}$$

Accordingly $\mathrm{Tr}(\omega\partial_\mu\omega^{-1}) = 0$ and $(\omega\partial_\mu\omega^{-1})^\dagger = -\omega\partial_\mu\omega^{-1}$, so $\omega\partial_\mu\omega^{-1} \in \mathfrak{su}(2)$.

Next, let us take a look at the $\omega A_\mu\omega^{-1}$ term. Once again, one can do this as a linear algebra exercise. However, we can also take a look at the adjoint representation for matrix Lie groups, Definition 2.2.21. This yields $\omega A_\mu \omega^{-1} = \mathrm{Ad}(\omega)(A_\mu)$ and since the adjoint map maps elements $\omega \in SU(2)$ to endomorphisms, we know that $\omega A_\mu\omega^{-1} \in \mathfrak{su}(2)$. □

The term $\omega\partial_\mu\omega^{-1} \in \mathfrak{su}(2)$ foreshadows that many matrices in the Lie algebra are reached. In fact, using $SU(N)$ symmetries, the field A_μ takes on exactly the values in $\mathfrak{su}(N)$ [6].

We now know that the field A_μ takes on values in the Lie algebra of $SU(N)$, but we know nothing about its behaviour. That is why we will construct a Lagrangian for the Yang-Mills field A_μ. Analogous to electrodynamics we want the Lagrangian to be of the form $-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$. For this to be defined, we first need to take a look at what the field strength tensor looks like for general Lie groups. Motivated by electrodynamics we will look for a strength tensor with the term $\partial_\mu A_\nu - \partial_\nu A_\mu$. We want the strength tensor to transform under the gauge transformations as it does in electrodynamics, that is, according to the adjoint representation

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \omega(x)F_{\mu\nu}(x)\omega^{-1}(x). \tag{5.21}$$

Now we can check if $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a possibility, by simply checking if it is invariant under the adjoint representation. Using the transformation relation of A_μ we find

$$\begin{aligned} \partial_\nu A'_\mu = \partial_\nu(\omega)A_\mu\omega^{-1} + \omega\partial_\nu(A_\mu\omega^{-1}) + \omega A_\mu\partial_\nu\omega^{-1} \\ + \partial_\nu(\omega)\partial_\mu\omega^{-1} + \omega\partial_\nu\partial_\mu(\omega^{-1}), \end{aligned} \tag{5.22}$$

and thus

$$\begin{aligned} \partial_\mu A'_\nu - \partial_\nu A'_\mu = \omega(\partial_\mu A_\nu - \partial_\nu A_\mu)\omega^{-1} \\ \partial_\mu(\omega)A_\nu\omega^{-1} + \omega A_\nu\partial_\mu\omega^{-1} \\ - \partial_\nu(\omega)A_\mu\omega^{-1} - \omega A_\mu\partial_\nu\omega^{-1} \\ + \partial_\mu(\omega)\partial_\nu\omega^{-1} - \partial_\nu(\omega)\partial_\mu\omega^{-1}. \end{aligned} \tag{5.23}$$

This quantity is not invariant under the adjoint representation, so we will need to find a way to make it invariant. For this we use the fact that the field takes values in the Lie algebra and we can thus take a look at the bracket corresponding to matrix Lie algebras: the matrix commutator. Let us take a look at how the commutator transforms. To write it concisely, we use the following identities.

$$\omega\partial_\mu(\omega^{-1}) \cdot \omega = -\partial_\mu(\omega) \cdot \omega^{-1}\omega = -\partial_\mu(\omega) \tag{5.24}$$

$$\omega\partial_\mu(\omega^{-1}) \cdot \omega\partial_\nu(\omega^{-1}) = -\omega\omega^{-1}\partial_\mu(\omega)\partial_\nu(\omega^{-1}) = \partial_\mu(\omega)\partial_\nu(\omega^{-1}) \tag{5.25}$$

Then we find

$$\begin{aligned}[A'_\mu, A'_\nu] = &\omega[A_\mu, A_\nu]\omega^{-1} \\ &+ \omega A_\mu \partial_\nu(\omega^{-1}) + \partial_\nu(\omega) A_\mu \omega^{-1} \\ &- \omega A_\nu \partial_\mu(\omega^{-1}) - \partial_\mu(\omega) A_\nu \omega^{-1} \\ &- \partial_\mu(\omega)\partial_\nu(\omega^{-1}) + \partial_\nu(\omega)\partial_\mu(\omega^{-1}).\end{aligned} \tag{5.26}$$

Combining these two identities yields an invariant quantity that we define as the strength tensor, which then obeys Equation 5.21,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{5.27}$$

We want the Lagrangian to be gauge-invariant and quadratic. With Corollary 2.2.32 we find that the following Lagrangian fulfils these wishes:

$$\mathscr{L}_A = \frac{1}{2g^2}\mathrm{Tr} F_{\mu\nu} F_{\mu\nu}. \tag{5.28}$$

In this equation we have introduced the constant $g^2 > 0$ for convenience and we use a positive sign to ensure positive energy for fields with small energy. Using Theorem 2.2.31, we find that, up to scalar multiplication, this is the only possible Lagrangian that is quadratic, gauge-invariant, and that gives a positive energy for small fields.

It is important to note that A_μ is a self-interacting field, because the Lagrangian contains terms of third and fourth order in the field A_μ. These terms are of the form $\mathrm{Tr}(\partial_\mu A_\nu \cdot A_\mu A_\nu)$ and $\mathrm{Tr}(A_\mu A_\nu A_\mu A_\nu)$.

To explain the positive sign and the introduction of the factor g, it is useful to look at the decomposition of the field A_μ in terms of real fields. For simplicity we will work with $SU(2)$, instead of a general N. The general case is completely analogous, but requires more generators of the Lie algebra. In the case $SU(2)$, a Lie group of dimension 3, we obtain 3 real fields, one for each generator of the Lie algebra $\mathfrak{su}(2)$:

$$A_\mu(x) = -ig\frac{\tau^a}{2} A^a_\mu(x). \tag{5.29}$$

In this equation the three real fields $A^a_\mu(x)$ correspond to the three (Hermitian) generators $\frac{\tau^a}{2}$ of $\mathfrak{su}(2)$ and the same factor g as above is introduced for convenience. We also can rewrite the strength tensor in terms of this decomposition:

$$F_{\mu\nu}(x) = -ig\frac{\tau^a}{2} F^a_{\mu\nu}(x). \tag{5.30}$$

Now we use Equation 5.27 to find,

$$\begin{aligned} F_{\mu\nu} &= -ig\frac{\tau^a}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) - g^2 A_\mu^a A_\nu^b i\varepsilon^{abc}\frac{\tau^c}{2} \\ &= -ig\frac{\tau^a}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc}A_\mu^b A_\nu^c\right), \end{aligned} \tag{5.31}$$

resulting in a decomposition of $F_{\mu\nu}$ in three real components:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc}A_\mu^b A_\nu^c. \tag{5.32}$$

Where ε^{abc} is the Levi-Civita symbol, which emerges from the commutation of the generators $\frac{\tau^a}{2}$ of the Lie algebra. As such, the structure constants of the group $SU(2)$ appear in this equation. Note also that we use the Einstein summation convention for repeating indices. Let us again consider the Lagrangian for the gauge field A_μ and express it in terms of the real components of the strength tensor:

$$\begin{aligned} \mathscr{L}_A &= \frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu} = \frac{1}{2g^2}\mathrm{Tr}(-ig\frac{\tau^a}{2}F_{\mu\nu}^a - ig\frac{\tau^b}{2}F_{\mu\nu}^b) \\ &= \frac{1}{2g^2}(-ig)^2 F_{\mu\nu}^a F_{\mu\nu}^b \mathrm{Tr}(\frac{\tau^a}{2}\frac{\tau^b}{2}) = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a. \end{aligned} \tag{5.33}$$

Here we have used that $\mathrm{Tr}(\tau^a\tau^b) = 0$ for $a \neq b$ and $\mathrm{Tr}(\tau^a\tau^a) = 2$.

To explain the choice of sign and introduction of the factor g we look at small perturbations of the field about the state $A_\mu^a = 0$ for $a = 1, 2, 3$. Using the small field approximation we can neglect all third and fourth order terms in the field. We then find

$$\mathscr{L}_A = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc}A_\mu^b A_\nu^c)^2, \tag{5.34}$$

$$\begin{aligned} \mathscr{L}_A \approx \mathscr{L}_A^{(2)} &= -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 \\ &= -\frac{1}{2}(\partial_\mu A_\nu^a \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \partial_\mu A_\nu^a). \end{aligned} \tag{5.35}$$

This equation displays the sum of three Lagrangian of the form of the Lagrangian of electrodynamics, Equation 4.34. This correspondence explains the sign and the introduction of the factor g.

Lastly, if we look back at the Lagrangian for the scalar field, we find:

$$\mathscr{L}_\varphi = (D_\mu\varphi)^\dagger(D_\mu\varphi) - V(\varphi, \varphi^\dagger), \tag{5.36}$$

where we have

$$D_\mu \varphi = (\partial_\mu + A_\mu)\varphi = (\partial_\mu - ig\frac{\tau^a}{2}A_\mu^a)\varphi. \tag{5.37}$$

We note that in both Lagrangians the factor g appears only in terms of order three or four in the fields. It is thus only present in interaction terms of the Lagrangians, therefore we call the factor g the gauge-coupling constant.

5.2.2 Local $U(1)$ symmetries: The electromagnetic field revisited

Even though $U(1)$ is an abelian group, it is useful to consider the formalism introduced above. This allows us to consider the difference between local and global $U(1)$ symmetries and results in a nice link to electrodynamics.

We consider the Lagrangian

$$\mathscr{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi, \tag{5.38}$$

where φ is a complex scalar field, and demand local $U(1)$ symmetry,

$$\varphi'(x) = e^{i\alpha(x)}\varphi(x) \tag{5.39}$$

Then we obtain a covariant derivative

$$D_\mu(x) = \partial_\mu + A_\mu(x), \tag{5.40}$$

where A_μ takes values in $\mathfrak{u}(1) \simeq \mathbb{R}$ and is a scalar field. This implies that A_μ is precisely the four-vector field that we have in electrodynamics. Indeed, the variational principle on the total Lagrangian with respect to A_μ gives us exactly the non-trivial Maxwell's equations.

5.2.3 Gauge invariance generalised to compact Lie groups

The derivations in the previous section can be generalised to any compact Lie group. First we will generalise it to simple Lie groups. Since compact Lie groups can be written as a products of simple Lie groups and $U(1)$ terms, see Theorem 2.2.30, we can then generalise to arbitrary compact Lie groups.

Gauge theory for simple Lie groups

For a gauge field A_μ corresponding to a simple Lie group G, we find that that A_μ takes values in the Lie algebra of G. We write

$$A_\mu(x) = gt^a A^a_\mu(x), \tag{5.41}$$

where t^a are generators of $\mathrm{Lie}(G) = \mathfrak{g}$, $g \in G$ and A^a_μ are real vector fields. Note that a takes on $\dim G = \dim \mathfrak{g}$ values.

We define the strength tensor similarly to the strength tensor for $SU(N)$ symmetries, see Equation 5.27. $F_{\mu\nu}$ also takes values in $\mathfrak{g}$. Using this we find the same transformation relations as before:

$$A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} + \omega \partial_\mu(\omega^{-1}) \tag{5.42}$$

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu}(x) \omega^{-1}(x). \tag{5.43}$$

By Theorem 2.2.32, the only quadratic invariant is then $\mathrm{Tr} F_{\mu\nu} F_{\mu\nu}$, yielding the following Lagrangian:

$$\mathscr{L}_A = \frac{1}{2g^2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu}. \tag{5.44}$$

This is also were the need for a compact Lie group comes into play. If the Lie group were not compact, by Theorem 2.2.24 there might not exist a positive bilinear form on the Lie algebra. The problem with this is that the absence of a lower bound of the kinetic term Lagrangian would also imply an absence of a lower bound on the energy.

With a similar derivation as above we can express $F_{\mu\nu}$ in terms of real coefficients:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C_{abc} A^b_\mu A^c_\nu. \tag{5.45}$$

In this general form, the structure constants C_{abc} of the Lie group G emerge from the commutator. We can write $F_{\mu\nu}$ as in Equation 5.30.

Combining the Lagrangian of the original matter field φ and the Gauge field, we find the total Lagrangian of our system,

$$\begin{aligned} \mathscr{L} = \mathscr{L}_\varphi + \mathscr{L}_A &= (D_\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi) + \frac{1}{2g^2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \\ &= (D_\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi) - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}. \end{aligned} \tag{5.46}$$

Gauge theory for compact Lie groups

Now that we have found the gauge-field for arbitrary simple Lie groups, we can take a look at compact Lie groups. A compact Lie group can be written as a product of simple Lie groups and $U(1)$ factors, Theorem 2.2.30. Using the framework of simple Lie groups, we find the gauge field for each of these components, each with it's own coupling constant. The combination of these gauge-fields results a single Lagrangian for the original field. Let us look at an example.

Suppose we have a set of $m \cdot n$ complex scalar fields $\varphi_{i\alpha}$, where $i = 1, \ldots, m$ and $\alpha = 1, \ldots, n$. We want $SU(m) \times SU(n)$ symmetry, that is, for $(\omega(x), \Omega(x)) \in SU(m) \times SU(n)$ acting on $\varphi_{i\alpha}$ we have the following transformation,

$$\varphi'_{i\alpha}(x) = \omega_{ij}(x)\Omega_{\alpha\beta}(x)\varphi_{j\beta}(x). \tag{5.47}$$

This means that $\omega(x) \in SU(m)$ acts on the first coordinate of $\varphi_{i\alpha}$ in the usual sense and $\Omega(x) \in SU(n)$ on the second. For global $SU(m) \times SU(n)$ invariance we have the invariant Lagrangian

$$\mathscr{L} = \partial_\mu \varphi^*_{i\alpha} \partial_\mu \varphi_{i\alpha} - \tilde{m}^2 \varphi^*_{i\alpha}\varphi_{i\alpha} - \lambda(\varphi^*_{i\alpha}\varphi_{i\alpha})^2, \tag{5.48}$$

where $\tilde{m}$ indicates the mass of the field. We first introduce the gauge field for the group $SU(m)$, which we express in terms of the real fields A^a_μ,

$$A_\mu(x) = -igt^a A^a_\mu(x). \tag{5.49}$$

Here t^a are the (Hermitian) generators of $\mathfrak{su}(m)$. They act only on the first coordinate of $\varphi_{i\alpha}$, which we can indicate by writing $(t^a)_{ij}$. The constant g is the coupling constant for the gauge field of group $SU(m)$.

Similarly for $SU(n)$ we find:

$$B_\mu(x) = -ihs^p B^b_\mu(x) \tag{5.50}$$

where s^p or $(s^p)_{\alpha\beta}$ are the generators of $\mathfrak{su}(n)$, acting only on the second coordinate of $\varphi_{i\alpha}$. We have an independent coupling constant h for this group. To find the covariant derivative for this Lagrangian, let us first write down the complete expression for the partial derivative:

$$\partial_\mu \varphi_{i\alpha} = \delta^j_i \delta^\beta_\alpha \partial_\mu \varphi_{j\beta}. \tag{5.51}$$

The covariant derivative combines the influence of the gauge-fields A_μ and B_μ,

$$(D_\mu\varphi)_{i\alpha} = (\delta^j_i\delta^\beta_\alpha\partial_\mu - ig\delta^\beta_\alpha t^a_{ij}A^a_\mu - ig\delta^j_i s^p_{\alpha\beta}B^p_\mu)\varphi_{j\beta} \tag{5.52}$$

$$= \partial_\mu\varphi_{i\alpha} + (A_\mu)_{ij}\varphi_{j\alpha} + (B_\mu)_{\alpha\beta}\varphi_{i\beta}. \tag{5.53}$$

We now find the strength tensor for both fields,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC_{abc}A^b_\mu A^c_\nu, \tag{5.54}$$

$$G^p_{\mu\nu} = \partial_\mu B^p_\nu - \partial_\nu B^p_\mu + hD_{pgr}B^q_\mu B^r_\nu. \tag{5.55}$$

In this equation, C_{abc} are the structure constants of the group $SU(m)$ and D_{pqr} are the structure constants of $SU(n)$. With the strength tensors, we can find the gauge-invariant Lagrangian of the model. As expected, we find

$$\begin{aligned} \mathscr{L} =& (D_\mu\varphi)^*_{i\alpha}(D_\mu\varphi)_{i\alpha} - \tilde{m}^2\varphi^*_{i\alpha}\varphi_{i\alpha} - \lambda(\varphi^*_{i\alpha}\varphi_{i\alpha})^2 \\ & - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{4}G^p_{\mu\nu}G^p_{\mu\nu}. \end{aligned} \tag{5.56}$$

This rounds up our discussion of the example of gauge symmetry with the compact Lie group $SU(m) \times SU(n)$. Any product of simple Lie groups and $U(1)$ terms gives a similar derivation with a number of gauge-groups with each their own coupling constant and structure constants. An important observation, is that in some physical systems we do not need to generalise all global symmetries of a simple Lie group to a global symmetry. If this is not necessary we can save ourselves a lot of work by using the easier global symmetry.

5.3 Implications of Yang-Mills theory

Yang-Mills theory is a very influential theory, with implications mainly for the standard model of particle physics. It has generalised the concept of the electromagnetic force, where a local symmetry is associated to a force in nature. One profound consequence was the addition of elementary particles to the standard model [18]. With Yang-Mills theory the three bosons carrying the weak nuclear force can be explained as generators of the Lie algebra $\mathfrak{su}(2)$. Similarly, the Yang-Mills theory of gauge group $SU(3)$, responsible for the strong nuclear force, explains why we have 8 gluons; they are results of the 8 generators of $\mathfrak{su}(3)$, see Part 3, Section 69 in [19].

Additionally, symmetry breaking is used to explain the breaking of the electroweak force in the early universe to the electromagnetic force and the weak nuclear force. In this symmetry breaking phase transition the three bosons mediating the weak force, $W^\pm$ and Z, acquired mass while the photon remained massless. There are theories assuming the unification of the electroweak and the strong force in even earlier stages of the expanding and cooling universe [20]. We will further explore these so called grand unified theories in section 6.3.

Chapter 6

Topological defects in symmetry breaking phase transitions

By now we have accumulated enough ground work in homotopy theory and field theory to delve into the subjects of symmetry breaking and topological defects. The symmetries in the Lagrangian can be spontaneously broken when ground states are not symmetric. The breaking of a symmetry proves a phase transition. In this chapter we will study phase transitions caused by spontaneous symmetry breaking which may result in the formation of topological defects. We will conclude with an example of a grand unified theory, where we will use homotopy groups of the vacuum manifold to determine the types of topological defects that may form.

6.1 Symmetry breaking

Symmetries of a system are said to be spontaneously broken if the ground states of the system are not invariant under the gauge transformations. In this chapter we will describe the mechanism of spontaneous symmetry breaking, using the symmetries of the previous section. This section is based on chapter 5 of [6].

6.1.1 Spontaneous symmetry breaking in the global $U(1)$ model

We again take a look at the Lagrangian

$$\mathscr{L} = \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2 - c. \tag{6.1}$$

We added the constant c, to be able to choose the minimal energy equal to 0. Again φ is a scalar complex field, and we can write $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$, with corresponding Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi_i\partial_\mu\varphi_i - \frac{m^2}{2}\varphi_i\varphi_i - \frac{\lambda}{4}(\varphi_i\varphi_i)^2 - c. \tag{6.2}$$

We know that this Lagrangian has global $U(1)$ invariance,

$$\varphi' = e^{i\alpha}\varphi, \tag{6.3}$$

where $\alpha \in \mathbb{R}$ is fixed. We can also write this invariance in terms of φ_i:

$$\varphi_1' = \cos(\alpha)\varphi_1 - \sin(\alpha)\varphi_2 \tag{6.4}$$

$$\varphi_2' = \sin(\alpha)\varphi_1 + \cos(\alpha)\varphi_2 \tag{6.5}$$

We want to find the ground state of our field. For this we consider the energy functional

$$E = \int \partial_\mu\varphi^*\partial_\mu\varphi + V(\varphi)\, d^3x, \tag{6.6}$$

$$V(\varphi) = m^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 + c, \tag{6.7}$$

where we integrate over real space, $\mu = 1, 2, 3$.

For minimal energy we clearly need φ to be constant, that is $\varphi(x) = \varphi_0$. For the value of φ_0 we need to minimize the potential. If $m^2 \geq 0$ we find that $\varphi_0 = 0$. The field $\varphi \equiv 0$ is clearly invariant under transformations of $U(1)$. In this case the symmetry is not broken.

If $m^2 < 0$, we get a so called 'Mexican hat shaped' potential, see Figure 6.1 . The minima are located on a circle around the central axis, at fixed values of $|\varphi|$. This means all ground states are of the form

$$\varphi(x) = e^{i\alpha}\frac{\varphi_0}{\sqrt{2}}, \tag{6.8}$$

where the factor $\sqrt{2}$ is added for convenience later. These ground states are not invariant under actions of $U(1)$, generally $e^{i\alpha}\frac{\varphi_0}{\sqrt{2}} \neq e^{i\beta}e^{i\alpha}\frac{\varphi_0}{\sqrt{2}}$, and so they are said to break the symmetry.

We want to find φ_0, using the minima of the potential, which by use of rotational symmetry we conveniently express in terms of $|\varphi|$:

$$V(|\varphi|) = m^2|\varphi|^2 + \lambda|\varphi|^4 + c \tag{6.9}$$

$$\frac{\partial V(|\varphi|)}{\partial|\varphi|} = 2m^2|\varphi| + 4\lambda|\varphi|^3 = 0 \tag{6.10}$$

$$|\varphi| = 0 \;\vee |\varphi| = \sqrt{\frac{-2m^2}{4\lambda}}. \tag{6.11}$$

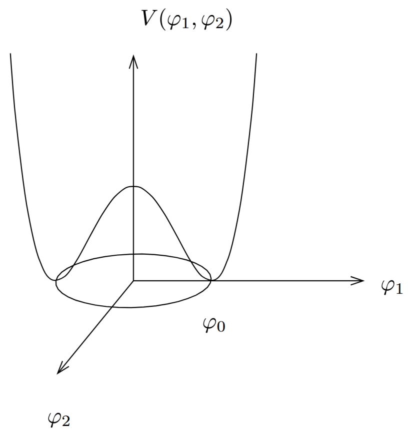


Figure 6.1: *For $m^2 < 0$, the potential of Equation 6.7 has the shape of a Mexican hat. The minima are positioned on a circle around the origin with radius φ_0 and we see a local maximum at $\varphi_i = 0, i = 1, 2$. [6]*

It is clear that $|\varphi| = 0$ is a maximum of the potential, so the second option will be our minimum. If we define $\mu^2 = -m^2 > 0$, $\mu > 0$, we find φ_0 with Equation 6.8,

$$|\varphi| = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{1}{\sqrt{2}} \frac{\mu}{\sqrt{\lambda}} \tag{6.12}$$

$$\varphi_0 = \frac{\mu}{\sqrt{\lambda}}. \tag{6.13}$$

Our next goal will be to find the Lagrangian of perturbations around around a chosen ground state. Perturbations are small deviations to the ground state. We choose a ground state $\vec{\varphi}^{(0)} = (\varphi_1, \varphi_2) = (\varphi_0, 0)$, similarly $\varphi(x) = \frac{\varphi_0}{\sqrt{2}}$. Perturbations around this ground state can be given using

$$\varphi_1(x) = \varphi_0 + \chi(x) \tag{6.14}$$

$$\varphi_2(x) = \theta(x). \tag{6.15}$$

This we can fill in in the Lagrangian, Equation 6.2, to obtain

$$\begin{aligned} \mathscr{L}_{\chi,\theta} =& \frac{1}{2} \left[\partial_\mu \chi(x)\right]^2 + \frac{1}{2} \left[\partial_\mu \theta(x)\right]^2 \\ &+ \frac{\mu^2}{2} \left[(\frac{\mu}{\sqrt{\lambda}} + \chi(x))^2 + \theta(x)^2 \right] \\ &- \frac{\lambda}{4} \left[(\frac{\mu}{\sqrt{\lambda}} + \chi(x))^2 + \theta(x)^2 \right]^2 + \frac{\mu^4}{4\lambda}, \end{aligned}$$

where we have chosen $c = \frac{\mu^4}{4\lambda}$, such that the energy of the ground state is equal to zero. We again restrict this Lagrangian to second order in the perturbations χ and θ:

$$\mathscr{L}^{(2)}_{\chi,\theta} = \frac{1}{2}\left[\partial_\mu \chi(x)\right]^2 + \frac{1}{2}\left[\partial_\mu \theta(x)\right]^2 - \mu^2\chi^2. \tag{6.16}$$

In this approximation we note that the only quadratic term in the fields is $\mu^2\chi^2$, which means that χ is a massive field with mass $m_\chi = \sqrt{2}\mu$ and θ is a massless field. This massless field is known as the Nambu-Goldstone field and it corresponds to the Nambu-Golstone Boson.

There have been particles found of which the Nambu-Golstone bosons are very accurate descriptions, namely the $\pi^\pm$ and π^0 mesons. However, these particles have a non-zero mass, due to more (small) terms in the Lagrangian that were not considered in this example.

6.1.2 Partial breaking of global $SO(3)$ symmetry

At this stage it is time to look at a more extensive example. We will discuss the model of a Lagrangian with $SO(3)$ symmetry. This symmetry will not be completely broken in the ground state, but instead there is a remaining symmetry of $SO(2)$. This will present us with a nice link back to homogeneous manifolds.

Let us consider the symmetries of the following Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi^a \partial_\mu \varphi^a - V(\varphi), \tag{6.17}$$

where φ^a for $a = 1, 2, 3$ are three real fields and

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^a\varphi^a + \frac{\lambda}{4}(\varphi^a\varphi^a)^2 + \frac{\mu^2}{4\lambda}. \tag{6.18}$$

We have again chosen the constant factor such that the energy of the ground states will be equal to zero and we assume $\mu^2 > 0$. If we write φ for the vector of the three components φ^a, we find that this Lagrangian has a global $SO(3)$ symmetry. For $\omega \in SO(3)$ we have:

$$(\varphi^\top \varphi)' = \varphi^\top \omega^\top \omega \varphi = \varphi^\top \varphi. \tag{6.19}$$

As in the previous section we will try to find the ground state of this Lagrangian. The kinetic term shows that minimal energy is obtained, when the field is homogeneous, so the ground state will be a field of the form

$$\varphi^a(x)\varphi^a(x) = |\varphi(x)|^2 = \varphi_0^2. \tag{6.20}$$

A similar derivation as seen in the previous section to minimise the potential yields

$$\varphi_0 = \frac{\mu}{\sqrt{\lambda}}. \tag{6.21}$$

Thus, we find that the set of ground states is a sphere of radius φ_0. The set of ground states is called the classical vacuum, or the vacuum manifold $\mathcal{M}$. For the Lagrangian in Equation 6.17 we obtain $\mathcal{M} \cong S^2$.

We choose the point $\vec{\varphi}^{(0)} = (0, 0, \varphi_0)$ on this sphere as our fixed ground state for further calculation. We will show that, contrary to the previous example, in this case the symmetry $SO(3)$ is not completely broken. For $\tilde{\omega} \in SO(2)$ we can write

$$\tilde{\omega} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & +\cos(\alpha) \end{pmatrix}. \tag{6.22}$$

Now we set

$$\omega = \begin{pmatrix} \tilde{\omega} & 0 \\ 0 & 1 \end{pmatrix} \in SO(3)$$

and find

$$\varphi' = \omega\varphi = \begin{pmatrix} \tilde{\omega} & 0 \\ 0 & 1 \end{pmatrix} \varphi = \begin{pmatrix} \cos(\alpha)\varphi^1 - \sin(\alpha)\varphi^2 \\ \sin(\alpha)\varphi^1 + \cos(\alpha)\varphi^2 \\ \varphi^3 \end{pmatrix}. \tag{6.23}$$

So indeed $\vec{\varphi}^{(0)} = (0, 0, \varphi_0)$ is invariant under the (non-trivial) transformations of $SO(2)$.

To construct the Lagrangian of perturbations around the ground state, we introduce three fields of perturbations, θ^1, θ^2 and χ, such that

$$\varphi^1(x) = \theta^1(x), \tag{6.24}$$

$$\varphi^2(x) = \theta^2(x), \tag{6.25}$$

$$\varphi^3(x) = \varphi_0 + \chi(x). \tag{6.26}$$

We find the potential of the model,

$$\begin{aligned} V = &-\frac{\mu^2}{2}\left[(\theta^1)^2 + (\theta^2)^2\right] - \frac{\mu^2}{4}(\varphi_0 + \chi)^2 \\ &+ \frac{\lambda}{4}\left[(\theta^1)^2 + (\theta^2)^2 + (\varphi_0 + \chi)^2\right] + \frac{\mu^4}{4\lambda} \end{aligned} \tag{6.27}$$

and the kinetic term of the Lagrangian,

$$\mathscr{L}_{kin} = \frac{1}{2}(\partial_\mu \theta^1)^2 + \frac{1}{2}(\partial_\mu \theta^2)^2 + \frac{1}{2}(\partial_\mu \chi)^2. \tag{6.28}$$

It is now clear that the Lagrangian of perturbations also has the $SO(2)$ symmetry in the terms θ^1 and θ^2, but not the full $SO(3)$ symmetry due to the $(\varphi_0 + \chi)^2$ term.

Taking the quadratic approximation we find:

$$\mathscr{L}^{(2)} = \frac{1}{2}(\partial_\mu \theta^1)^2 + \frac{1}{2}(\partial_\mu \theta^2)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2\chi^2. \tag{6.29}$$

Once more, we find that the field χ is massive with mass $\sqrt{2}\mu$, while the fields θ^1 and θ^2 are massless, because there is no second order contribution of them in the Lagrangian. In other words, θ^1 and θ^2 correspond to massless Nambu-Goldstone bosons. In general, "Whenever a continuous global symmetry is spontaneously broken, massless Goldstone bosons appear. Their number is equal to the dimension of the vacuum manifold" [1]. Indeed, in this example we have a vacuum manifold, $\mathcal{M} \cong S^2$, of dimension 2 and we found 2 Nambu-Goldstone bosons.

Now, as you may have noticed, we have seen three differentiable manifolds in this example, namely the Lie group $SO(3)$, the closed subgroup $SO(2)$ and the vacuum manifold, the sphere S^2. If we take the quotient of the symmetry group of the Lagrangian, with the symmetries that are conserved in the ground state of our model, we find the homogeneous manifold of possible ground states, the vacuum manifold. As in Theorem 2.3.9 we have:

$$SO(3)/SO(2) \cong S^2. \tag{6.30}$$

As a rule, the vacuum manifold is a homogeneous manifold and can be described as the quotient of the original symmetries, the Lie group G, and the remaining symmetries, the Lie subgroup $H \subseteq G$. In the next section we will present topological defects and see that they are characterised by the homotopy groups of their (homogeneous) vacuum manifold.

6.2 Topological defects

At last, let us dive into the definition of a topological defect. A topological defect is an excitation of a field that is preserved by the topological structure of the field. In other words, it is a state of matter that cannot continuously be transformed into the ground state, due to topological obstructions.

Topological defects emerge during symmetry breaking phase transitions. We will clarify this process based on [20] and Sections 2.1 and 2.2 of [1]. Consequently, we will discuss three examples of topological defects: domain walls, strings and monopoles based on [1]. The differences between these topological defects can be characterised using the homotopy groups of the vacuum manifold.

6.2.1 Phase transitions

A phase transition is a transition from one state or phase of a model to another. The simplest examples are transitions between a solid, liquid and gaseous state of for example water. During a phase transition the Gibbs free energy, G, changes significantly and abruptly, that is, there is a derivative of the Gibbs function that is discontinuous [20]. As a consequence, parameters that are derived from the Gibbs free energy experience a sudden change as well.

In the phase transition from liquid water to water vapour, 'gaseous water', the significant change of G results in a jump in the density of the water, while the temperature only changes marginally. In some metals there is a phase transition between a superconducting state and a regular state, in this case we observe a big difference in the conductivity. Another example would be the phase transition from a ferromagnet such as iron, to a paramagnet, which happens at the critical temperature known as the Curie temperature T_C.

Phase transitions are generally classified by their order. A phase transition is called a first-order phase transition if the first differential of the Gibbs function is discontinuous. In thermodynamics the step size of a discontinuous first derivative is called the latent heat. More generally, the first order differential of the Gibbs function that is discontinuous is known as the order of the phase transition. If a phase transition has a continuous first derivative, and is thus a phase transition of higher order, it is known as a continuous phase transition.

The water to gas example is a first-order phase transition, because the density, which is a first derivative of the Gibbs free energy, changes discontinuously. The phase transition of a ferromagnet that becomes paramagnetic is a continuous phase transition. The magnetization, which is a first differential of the Gibbs function changes fast, but continuously.

Another classification is that of symmetry breaking phase transitions.

When a symmetry breaks, the model equivalently transitions to a different state. The phase transition of (liquid) water to ice is an example of a symmetry breaking phase transition. Liquid water has continuous translational symmetry, where as the crystal structure of ice only has discrete translational symmetry, only translations by a lattice vector are invariant transformations. The breaking of the continuous symmetry corresponds to the transition from liquid to solid.

The phase transitions between ferromagnetic and paramagnetic comprises a breaking of the rotational symmetry of the magnetic moments in a paramagnetic material, that breaks so that the moments align in a ferro magnetic material. Lastly, in Equation 2.7 and remark on page 113 of [1] we see that the transition between superconducting and regular states of metal is described by the breaking of $U(1)$ symmetry.

It is important to note, that in many cases symmetry is restored above a certain temperature, for example in the ice to water phase transition and in the ferromagnetic state to paramagnetic state transition. Above a critical temperature the kinetic energy becomes significant enough to overrule the broken symmetry caused by potential terms. Below the critical temperature, the potential term become important and the symmetry is broken.

When spontaneous symmetry breaking occurs simultaneously at different places, this might result in fields that are not minimal. For example, if a model has a rotational symmetry that breaks, the field has to pick a direction at each point in space, but when this happens at various points in space simultaneously, the field will not be homogeneous, and thus not minimal. If for example, the field makes a full rotation along a given curve, it costs a lot of energy to break the "circle" and therefore the field, although not of minimal energy, will be stable. We will explain this example in more detail in section 6.2.2.

In short, during a symmetry breaking phase transition non-trivial configurations may form. Usually these are unstable, as they are not constant and thus do not correspond to minimal energy. However, in some cases these non-trivial configurations can survive due to topological obstructions. Such non-trivial configurations are what we call topological defects.

6.2.2 Examples of topological defects

We will take a look at domain walls, strings and monopoles, following section 2.4 of [1]. We will see that they are characterised by the topology of the vacuum manifold, specifically by the zeroth, first, second homotopy groups.

Domain walls

Let us take a look at system with a real scalar field and Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi - V(\varphi), \tag{6.31}$$

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - \eta^2)^2. \tag{6.32}$$

For φ to be in the ground state we need φ to be constant. Minimising the potential we conclude that the vacuum manifold is $\mathcal{M} = \{\pm\eta\}$. At high temperatures, $T \gg \eta$, we need to add corrections to the Lagrangian of the form $T^2\varphi^2$, in which case the vacuum manifold would simply consist of the trivial state, $\mathcal{M}_{T\gg\eta} = \{0\}$. When the temperature drops, we find two possible ground states.

Two points in space $x, y \in \mathbb{R}^3$ that are far apart, may fall into another ground state. Therefore the values $\varphi(x) = -\eta$ and $\varphi(y) = \eta$, for $|x - y| \gg 1$ are independent. To prevent divergence of the $\partial_i\varphi$ terms of the energy, the field φ needs to be continuous. But this means that there is a point z such that $\varphi(z) = 0$, which is a local maximum of the energy. This transition from $-\eta$ to η is a domain wall.

As an example we will explicitly construct a domain wall of a static planar solution, that is φ only depends on the x coordinate. The Euler-Lagrange equation, Equation 4.16, yields

$$\Box\varphi - \frac{\partial V}{\partial \varphi} = 0. \tag{6.33}$$

For a static planar domain wall this can be simplified to

$$\frac{\partial^2 \varphi}{\partial x^2} = \lambda\varphi(\varphi^2 - \eta^2). \tag{6.34}$$

In section 2.4 of [1], we find that this equation is solved by

$$\begin{aligned} \varphi(x) &= \eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta x\right) \\ &= \eta \frac{e^{\sqrt{\frac{\lambda}{2}}\eta x} - e^{-\sqrt{\frac{\lambda}{2}}\eta x}}{e^{\sqrt{\frac{\lambda}{2}}\eta x} + e^{-\sqrt{\frac{\lambda}{2}}}}, \end{aligned} \tag{6.35}$$

which you can check by direct calculation. Hence we have found the solution of a static and planar domain wall in our model. The field φ decays exponentially fast and we can approximate the width of the domain wall by η^{-2}.

To find more information about the domain wall we construct the stress-energy tensor,

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial \mathscr{L}}{\partial \eta_{\rho,\nu}} g_{\rho,\mu} - \eta_{\mu\nu}\mathscr{L} \\ &= \frac{\partial \mathscr{L}}{\partial(\partial_\nu \varphi)}\partial_\mu\varphi - g_{\mu\nu}\mathscr{L} \\ &= \partial_\nu\varphi\partial_\mu\varphi - g_{\mu\nu}\mathscr{L}, \end{aligned} \tag{6.36}$$

where g is the metric, which in our case is the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$, not to be confused with the field variables η_ρ.

In section A.1 we derive the stress-energy tensor of our model,

$$T_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{\lambda}{2}\eta^4 \text{sech}(\eta\sqrt{\frac{\lambda}{2}}x). \tag{6.37}$$

We conclude with the total energy of the domain wall. For this we need to integrate the energy density, T_{00}. The calculations can be found in subsection A.1.1 and yield

$$\sigma = \int_{-\infty}^{\infty} T_{00}dx = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3. \tag{6.38}$$

T_{00} approaches 0 exponentially fast, thus for scales $|x| > \frac{1}{\eta}$, we can approximate the energy in the domain wall as a delta function with height σ, the total energy of the wall,

$$T_{\mu\nu} \approx \sigma\delta(x) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.39}$$

We have seen that a vacuum manifold $\mathcal{M} = \{\pm\eta\}$ can result in a structure that forces the field to take on the value zero at some point in space. This is an energy maximum which results in a domain wall, a topological defect. More generally, when the vacuum manifold is disconnected we find a domain wall between different connected components with similar arguments. For a static planar solution we have found the φ and the energy contained in the wall.

Strings

In this example we will be using some more homotopy. In [21] you can find a physicist explanation of homotopy. We also use some of the explanations there to clarify parts of this section.

We examine a similar model as above, only this time we will consider a complex scalar field and thus the Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi - V(\varphi). \tag{6.40}$$

This Lagrangian has $U(1)$ symmetry. We find constant ground states for $|\varphi| = \eta$, and thus $\mathcal{M} = S^1_\eta \cong S^1$. With S^n_R we denote the n-sphere of radius R.

The total $U(1)$ symmetry is broken, because for $\alpha \in \mathbb{R}$ $\varphi = e^{i\alpha}\varphi$ if and only if $\alpha \in 2\pi\mathbb{Z}$. At high temperatures $T \gg \eta$ symmetry is restored due to correctional terms of the form $T^2|\varphi|^2$, which gives us symmetrical ground states, $\mathcal{M}_{T\gg\eta} = \{0\}$. When the temperature drops, φ randomly chooses values $\eta e^{i\alpha}$ with $\alpha \in \mathbb{R}$. At points in space separated by large distances, the values to which it drops are unrelated.

Let us consider a closed curve in space $\gamma : [0,1] \to \mathbb{R}^3$, $\gamma(0) = \gamma(1)$. We can compose the curve with our field φ, to obtain a curve

$$\Gamma = \varphi \circ \gamma : [0,1] \to S^1_\eta. \tag{6.41}$$

As φ falls in to the vacuum manifold, it may happen that Γ winds around the circle S^1_η a number of times corresponding to a winding number $n \neq 0$. In this case we can write $\Gamma(t) = \eta e^{i\alpha(t)}$, where $\alpha(1) = \alpha(0) + 2\pi n$.

Remember that we started with a loop γ in space. If we continuously shrink γ to a point, the winding number of Γ cannot change, since we are working with continuous maps and the winding number is discrete and it only changes stepwise. If we completely shrink the curve, we find a constant curve, for all $t \in [0,1]$ we have $\Gamma(t) = \tilde{\varphi}$. A constant curve has a winding number $n = 0$.

Continuously shrinking our curve, corresponds to a homotopy between the original curve and the constant curve. The change in winding number means that the two curves are not homotopic. This means that our homotopy, the shrinking of the original curve, must be ill-defined at some point in the interior in space of our original curve. We conclude that there must be a singularity in the field. A singularity can be explained by a point in space $x \in \mathbb{R}^3$ for which $\varphi(x) = 0$.

Yet again, we find that a topological structure may cause the field φ to leave the vacuum manifold and take on the value 0 which is a local maximum for the energy. For this defect to deteriorate, it would have to untwist, which means the whole interior of the loop would have to move through zero to obtain a zero winding number.

In three dimensional space, shrinking the loop can be done along various surfaces. Along all these surfaces the argumentation of above holds, and thus in all these surfaces φ takes on a the value 0 somewhere. These points will line up to minimise the energy needed, creating a string. We can move the surface inside our ring around like a soap bubble, and stretch it infinitely, as long as it stays smooth. This implies that the string cannot end: either the string is a closed loop, or it is infinite.

We again look for an exact solution. We saw already that the derivations for the domain wall were quite cumbersome and for the string they we will even more so. Therefore, we restrict ourselves to a static, infinite and straight string along the z-direction. Still, not all calculations will allow algebraic solutions. For an infinite straight string along the z-direction, we have rotational symmetry in the xy-plane, so we use cylindrical coordinates $\rho = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$. We look for an ansatz for $\varphi(\rho, \theta)$. We will decompose $\varphi(\rho, \theta)$, an element of $\mathbb{C}$, into a modulus and a phase factor. Because of rotational symmetry, the modulus of $\varphi(\rho, \theta)$ can only be dependent on ρ, similarly the phase of $\varphi(\rho, \theta)$ is directly related to θ. With this in mind, we choose the following ansatz,

$$\varphi(\rho, \theta) = \eta f_s(\rho\eta) e^{in\theta}. \tag{6.42}$$

Plugging this into the field equation Equation 6.33, we obtain:

$$f_s'' + \frac{1}{v} f_s' - \frac{n^2}{v^2} f_s - \frac{\lambda}{2} f_s (f_s^2 - 1) = 0. \tag{6.43}$$

For the full derivation, see section A.2. Numerically it is possible to find a solution to this differential equation with boundary conditions $f_s(0) = 0$ and $f_s(\rho\theta) \to 1$, for $\rho\theta \to \infty$. The most important properties for us are

$$f_s \sim 1 - \mathcal{O}(\frac{1}{v^2}) \qquad \text{for } \sqrt{\lambda}\rho\eta \gg 1, \quad \text{and} \tag{6.44}$$

$$f_s \sim \mathcal{O}(\frac{1}{v^n}) \qquad \text{for } \sqrt{\lambda}\rho\eta \ll 1. \tag{6.45}$$

From Equation 6.36 we can determine the stress-energy tensor for the string. We obtain, with a similar derivation as for the stress-energy tensor of the static planar domain wall,

$$T_{00} = -T_{zz} \tag{6.46}$$

$$= -\frac{\lambda\eta^4}{2} \left[f'^2 - \frac{1}{2}(f^2 - 1) + \frac{n^2}{\lambda\eta^2\rho^2} f^2 \right]. \tag{6.47}$$

And for all other components we have,

$$T_{\mu\nu} = 0. \tag{6.48}$$

Now we can find the energy per unit length of a cross section of the string. Taking a cross section of the string with radius R, yields, as in Equation 2.46 in [1],

$$\mu(R) = 2\pi \int_0^R T_{00}\rho d\rho \sim \pi\eta^2 \ln(\sqrt{\lambda}\eta R). \tag{6.49}$$

Note that the energy diverges for $R \to \infty$. This is because the term $\frac{n^2}{\lambda\eta^2\rho^2}f^2$ decays only in the order $\frac{1}{\rho^2}$. In reality, there is an upper cut-off for this integral caused by either the curvature of the string or by a neighbouring string.

Similar to the domain wall, we found a topological obstruction for the field to stay in the ground state: at some point we have $\varphi = 0$. This is a local maximum of the energy which creates a topological defect. The specific structure causing the defect, creates a line in space where $\varphi = 0$, this is a string.

Local strings We can also take a look at strings formed by a Lagrangian that has a gauge symmetry, or in other words a local symmetry. We will discuss the Abelian Higgs model known as the Nielson-Olesen or Abrikosov vortex. This model uses the Lagrangian density of scalar electrodynamics, where we include the gauge field, and the potential remains the same as above, Equation 6.32,

$$\mathscr{L} = (\partial_\mu + ieA_\mu)\varphi^*(\partial_\mu - ieA_\mu)\varphi - \frac{\lambda}{4}(|\varphi|^2 - \eta^2)^2 - \frac{1}{4}F_{\mu\nu}F_{\mu\nu}. \tag{6.50}$$

Following the same procedure as for the global string it is possible to choose ansatz's for φ as well as for A_μ. We will not be performing the (numerical) derivations, as they are not illuminating for our purposes. In section 2.4.2.2 of [1] you can find them.

An interesting result is that local strings have a finite energy per unit length, and thus the problem of the divergence of energy is solved when considering local strings.

Monopoles

The last type of topological defect that we explore is a monopole. For this we consider a three-component scalar field φ_i, $i = 1, 2, 3$, with a Lagrangian that is symmetric under transformations of the Lie group $O(3)$.

We have

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi_i \partial_\mu \varphi_i - V(\varphi), \tag{6.51}$$

where again

$$V(\varphi) = \frac{\lambda}{4}(|\varphi|^2 - \eta^2)^2, \tag{6.52}$$

with $|\varphi|^2 = \sum_{i=1}^{3}(\varphi_i)^2$.

For the vacuum manifold we find $\mathcal{M} = S^2_\eta$ a sphere with radius η. Note that again at high temperature the corrections of the form $T^2\varphi^2$ will give us a vacuum manifold $\mathcal{M}_{T\gg\eta} = \{0\}$. When temperature drops the field falls into the new vacuum manifold, and the values it takes are unrelated for points far apart.

We will give a similar argument as in section 6.2.2. Suppose that $f : S^2 \to \mathbb{R}^3$ is a maps a sphere into real space. We define

$$F = \varphi \circ f : S^2 \to \mathcal{M} = S^2_\eta. \tag{6.53}$$

It is possible that the values φ takes around this sphere have a non-zero winding number. Where in this case the winding number is the amount of times the sphere is fully covered, counted with orientation. The winding number still has to be an integer value. Note that any set of maps $\varphi \circ f$ with the same winding number is equal to a class in the homotopy group $\pi_2(S_2)$,

$$\{F = \varphi \circ f | F \text{ has winding number } n \in \mathbb{Z}\} \in \pi_2(S_2). \tag{6.54}$$

We can continuously shrink the image of f to a point p, giving a homotopy between f and the continuous map c_p. You can visualise this as a balloon deflating, f is the map that sends S^2 tot he surface of the balloon. In this analogy, F gives the field values at the surface of the balloon. Shrinking f to a point also yields a homotopy between F and the constant map $c_p \circ \varphi$.

The map $\varphi \circ c_p$ has winding number zero, therefore, if we started with a non-zero winding number of map F, there needs to be a singularity in φ to account for the change in winding number. The homotopy we constructed cannot exist, and this can only be explained if φ is ill-defined at some point in space. This singularity is again a point in space $\mathbf{x}$ for which $\varphi(\mathbf{x}) = 0$. The local maximum in the energy given by $\varphi = 0$, is what we call the monopole.

Let us take look at an example of a static and spherically symmetric monopole located at the origin, **x**= 0. This gives us the boundary conditions $\varphi(0) = 0$ and $\lim_{r\to\infty} |\varphi(r)| = \eta$. We start with the ansatz

$$\varphi_i = \eta f_m(r)\frac{x_i}{r}, \tag{6.55}$$

which displays the full rotational symmetry, as it is only dependent on the radius. Further, we want the field to vanish at infinity, hence the factor $\frac{1}{r}$

Using Equation 4.16 we find, see Appendix A,

$$f_m'' + \frac{2}{s}f_m' - \frac{2}{s^2}f_m - f_m(f_m^2 - 1) = 0, \tag{6.56}$$

where $s = \sqrt{\lambda}\eta r$. This equation can again be solved numerically and in the limits we find,

$$f_m \sim s(\frac{1}{v^2}) \qquad \text{for } s \ll 1, \quad \text{and} \tag{6.57}$$

$$|f_m - 1| \sim \frac{1}{s^2} \qquad \text{for } s \gg 1. \tag{6.58}$$

Using the energy stress tensor and integrating over T_{00} we can approximate the total energy in a ball of radius R around the monopole. Following [1], we obtain the relation

$$E(R) = 4\pi \int_0^R T_{00} r^2 dr \propto R \quad \text{for } \sqrt{\lambda}\eta R \gg 1. \tag{6.59}$$

The first thing to notice about this result is that it diverges. As in the example for the strings an upper cut-off is provided the distance to the nearest anti-monopole. An anti-monopole is a monopole with negative magnetic charge [22]. The divergence is also a lot faster than the divergence for strings. This long range behaviour is at the origin of the long range interactions of global monopoles and strings.

Again, the structure of the vacuum manifold, forces the field φ to take on the value zero. The energy this requires results in topological defect, which for this structure is a monopole, as the maximum is only attained in one point in space.

Local monopoles We will consider an example of a local monopole, specifically we will discuss the 't Hooft Polyakov monopole. [23, 24]

We begin with our Lagrangian in Equation 6.51, to which we want to add the gauge field to ensure local invariance with the $SO(3)$ group. Using Equation 5.46 we construct the Lagrangian for this system,

$$\mathscr{L} = (D_\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi) - F_{\mu\nu} F_{\mu\nu}. \tag{6.60}$$

Where we have

$$(D_\mu \varphi)^\dagger (D_\mu \varphi) = (\partial_\mu \varphi^a + ie\varepsilon_{abc} A^b_\mu \varphi^c)(\partial_\mu \varphi^a - ie\varepsilon_{abc} (A^\top)^b_\mu \varphi^c), \tag{6.61}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon_{abc} A^b_\mu a^c_\nu, \tag{6.62}$$

and A^a_μ is defined such that

$$A_\mu = A^a_\mu I_a \tag{6.63}$$

where I_a are the standard generators of $\mathfrak{so}(3)$.

We can again plug-in well chosen ansatz's for φ^a and A^a_i, which can be numerically solved. Using these ansatz's it can also be shown that the energy of the local monopole is finite,

$$E = \int \mathcal{H} d^3x = -\int \mathscr{L} d^3x \sim \frac{4\pi}{e}(1 - \frac{\lambda}{e^2}). \tag{6.64}$$

Further results can be found using the the approximation $rm_V \gg 1$, resulting in the magnetic field on the 't Hooft Polyakov monopole

$$B^a_i = \frac{1}{2}\varepsilon_{ijk} F^a_{ik} \approx \frac{\hat{r}_i \hat{r}_a}{er^2}, \tag{6.65}$$

and the magnetic charge of the monopole

$$e_{mag} = \frac{1}{4\pi} \int_{S^2_R} R^2 d\Omega B^a_i B^a_i = e^{-1}. \tag{6.66}$$

Hence, this monopole represents a charge and the surrounding magnetic field, something physicists have been studying for centuries. Being able to describe electronic charges and magnetic fields as a topological defect illustrates how omnipresent topological defects are. This universality makes topological defects a powerful framework in particle physic and high energy physics, especially since it is not limited to classical field theory but also applies to quantum field theory. [25]

6.2.3 Homotopy groups of topological defects

In the examples above we have seen that there is a clear relation between the homotopy groups of the vacuum manifold and the formation of topological defects. For a disconnected vacuum manifold we found domain walls, for "one-dimensional holes" we found strings, and for "two dimensional holes" we found monopoles. These three categories are exactly those of the first three homotopy groups. If the vacuum manifold is disconnected, it has a non-trivial homotopy group $\pi_0(\mathcal{M})$, the existence of "one-dimensional holes", or loops, correspond to the non-triviality of the fundamental group of $\mathcal{M}$ and "two dimensional holes", or spheres, correspond to a non-trivial second homotopy group $\pi_2(\mathcal{M})$.

We conclude the following

If	$\pi_0(\mathcal{M}) \neq \{0\}$	domain walls emerge.	(6.67)
If	$\pi_1(\mathcal{M}) \neq \{0\}$	strings emerge.	(6.68)
If	$\pi_2(\mathcal{M}) \neq \{0\}$	monopoles emerge.	(6.69)

Naturally the question pops up, what happens for the higher homotopy groups? A reasonable suggestion, is to increase the dimension of the vacuum manifold by including a time coordinate. In the four dimensional space time we might be able to find topological defects, that is field configurations on a vacuum manifold $\mathcal{M}$ with $\pi_3(\mathcal{M}) \neq \{0\}$, with a non-zero winding number. Topological defects of this form have been studied and are called instantons [26].

Remarkably, in the three dimensional space we have studied so far, there is also a configuration we can study, yielding non trivial third homotopy groups. If we have a field variable that is asymptotically constant, we can take a look at $\mathbb{R}^3 \cup \{\infty\} \cong S^3$. The field configuration $\varphi : S^3 \to \mathcal{M}$ may give us non-trivial structures wrapping around the three-sphere. Such a structure is known as a texture. It has been shown that textures can only exists as non static solutions. We will not dive into this, but for those interested we recommend Section 2.4.4 of [1].

There are also theories of spacetime consisting of many more dimensions than the four mentioned here. In these theories even higher orders of homotopy groups can be studied, and provoke topological defects of these higher orders. In section 6.3 we will touch upon this subject.

6.3 Grand unified theory

In 1979 the Nobel Prize in Physics "was awarded jointly to Sheldon Lee Glashow, Abdus Salam and Steven Weinberg "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current" [27]. The unification of the electromagnetic and weak interactions solved problem of the inability to normalise the weak interactions using gauge-theory [28]. Essentially Glashow, Salam and Weinberg predicted that at high enough temperatures the electromagnetic interactions and the weak interactions are the same. This has been experimentally observed [29], expanding the already significant success of the theory.

As the strong interaction is now known as a Gauge theory of $SU(3)$, a reasonable question to ask is whether this unification extends to a unified electronuclear interaction. Various theoretical candidates for these so called Grand Unified theories have been conjectured, but none have been observed in experiments.

We will consider the Georgi-Glashow model. [30] This model suggests the gauge group $SU(5)$ as the initial symmetry group, which breaks down to the symmetry groups of the electromagnetic interaction, $U(1)$, the weak interaction, $SU(2)$, and the strong interaction, $SU(3)$. We will take a look at the symmetry breaking phase transitions and determine the topological defects that may emerge.

If we take a look at the $SU(5)$ gauge group as a candidate we need the product group $S := U(1) \times SU(2) \times SU(3)$, to be isomorphic to a subgroup of $SU(5)$.

However, we actually need to work with the true gauge group $H = S/\mathbb{Z}_6$ of the standard model. The quotient with $\mathbb{Z}_6$ arises from the spin requirements on the elementary particles that this gauge group generates. [31]

Let us define a map from S into $SU(5)$,

$$\begin{aligned} \varphi : U(1) \times SU(2) \times SU(3) &\to SU(2 \times SU(3) \subseteq SU(5) \\ (\alpha, g, h) &\mapsto \begin{pmatrix} \alpha^3 g & \\ & \alpha^{-2} h \end{pmatrix} . \end{aligned} \tag{6.70}$$

We check that this is a well-defined map,

$$\begin{aligned}\det\begin{pmatrix}\alpha^3 g & \\ & \alpha^{-2}h\end{pmatrix} &= \det(\alpha^3 g)\cdot\det(\alpha^{-2}h) \\ &= (\alpha^3)^2\det g\cdot(\alpha^{-2})^3\det h = \det g\cdot\det h = 1,\end{aligned} \tag{6.71}$$

$$\begin{aligned}\begin{pmatrix}\alpha^3 g & \\ & \alpha^{-2}h\end{pmatrix}^\dagger &= \begin{pmatrix}(\alpha^3 g)^\dagger & \\ & (\alpha^{-2}h)^\dagger\end{pmatrix} = \begin{pmatrix}g^\dagger(\alpha^3)^\dagger & \\ & h^\dagger(\alpha^{-2})^\dagger\end{pmatrix} \\ &= \begin{pmatrix}g^{-1}\alpha^{-3} & \\ & h^{-1}\alpha^{2}\end{pmatrix} = \begin{pmatrix}\alpha^3 g & \\ & \alpha^{-2}h\end{pmatrix}^{-1}.\end{aligned} \tag{6.72}$$

So $\varphi((\alpha, g, h)) \in SU(5)$.
We now look for the kernel of φ.

$$\ker(\varphi) = \{(\alpha, g, h) \in S | \begin{pmatrix}\alpha^3 g & \\ & \alpha^{-2}h\end{pmatrix} = I_5\} \tag{6.73}$$

$$= \{(\alpha, g, h) \in S | \alpha^3 g = I_2, \alpha^{-2}h = I_3\} \tag{6.74}$$

We want to rewrite $\alpha^3 g = I_2$ as $g = \alpha^{-3}I_2$, but we still need $g \in SU(2)$, this means $\det g = \det(\alpha^{-3}I_2) = (\alpha^{-3})^2 = 1$, and thus $\alpha^{-6} = 1$. Similarly for $h = \alpha^2 I_3$, we find $\alpha^6 = 1$. This yields,

$$\ker(\varphi) = \{(\alpha, \alpha^{-3}I_2, \alpha^2 I_3) | \alpha \in \mathbb{C}, \alpha^6 = 1\} \tag{6.75}$$

$$\cong \mathbb{Z}_6 \tag{6.76}$$

Using the isomorphism theorem for groups, we find the kernel,

$$H := (U(1) \times SU(2) \times SU(3)) / \mathbb{Z}_6 \cong SU(2 \times SU(3) \subseteq SU(5). \tag{6.77}$$

We will try to explain which topological defects occur when we assume the symmetry breaking of $SU(5)$ to H. For this we determine which of the first three homotopy groups of $\mathcal{M} = SU(5)/H$ are non-trivial.

First we gather the homotopy groups of $U(1), SU(2), SU(3)$ an $SU(5)$. They are described in Appendix A, Section 6 paragraph VI of [32]. We also know that $\mathbb{Z}_6$ is a discrete group with 6 points, i.e. 6 path connected components. We use Theorem 3.1.6 to determine the homotopy groups of P. These homotopy groups are summarised in Table 6.1.

With these homotopy groups, we can construct the long exact sequence of Theorem 3.4.6. We start with a long exact sequence based on the projection map $p : S \to H = S/\mathbb{Z}_6$. The space $\mathbb{Z}_6$ is not path connected, however, we can still use the sequence in Theorem 3.4.6, except for the last step, as can be seen in the proof. Luckily, we will not need the last step.

	$\mathbb{Z}_6$	$U(1)$	$SU(2)$	$SU(3)$	$SU(5)$
π_0	$\mathbb{Z}_6$	0	0	0	0
π_1	0	$\mathbb{Z}$	0	0	0
π_2	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
π_4	0	0	$\mathbb{Z}_2$	0	0

Table 6.1: *These are the homotopy groups that we will be using as found in Appendix A, Section 6 paragraph VI of [32]. The large quantity of zero's will give us easier exact sequences.*

We will leave out the basepoint for all connected spaces. Let us calculate $\pi_n(H), n = 0, 1, 2, 3$. For $n > 0$ we use sequences of the form

$$\pi_n(Z_6, z_0) \to \pi_n(S) \to \pi_n(H) \to \pi_{n-1}(\mathbb{Z}_6, z_0) \to \pi_{n-1}(P), \tag{6.78}$$

$\pi_0(H)$ We know that S is connected as a product of connected spaces. Therefore $H = S/\mathbb{Z}_6$ is also connected and we have

$$\pi_0(H) = 0. \tag{6.79}$$

$\pi_1(H)$ We use the exact sequence and fill in the known homotopy groups,

$$0 \to \mathbb{Z} \to \pi_1(H) \to \mathbb{Z}_6 \to 0. \tag{6.80}$$

Since $\ker(\mathbb{Z} \to \pi_1(H)) = \mathrm{Im}(0 \to \mathbb{Z}) = 0$, we find that $\mathbb{Z}$ maps injectively to $\pi_1(H)$. This means that

$$\pi_1(H) \supseteq \mathbb{Z}. \tag{6.81}$$

Working out the exact homotopy group requires explicit construction of the maps in the sequence and it turns out that we do not need this for our conclusions. Therefore, we will make do with this result.

$\pi_2(H)$ Using the exact sequence we find,

$$0 \to \pi_2(H) \to 0. \tag{6.82}$$

With this we conclude,

$$\pi_2(H) = 0. \tag{6.83}$$

$\pi_3(H)$ The exact sequence is given by

$$0 \to \mathbb{Z}^2 \to \pi_2(H) \to 0. \tag{6.84}$$

With this we find

$$\pi_3(H) \cong \mathbb{Z}^2. \tag{6.85}$$

Now that we have found these, we will continue with the homotopy groups of $SU(5)/H$, using the same long exact sequence. Note that we are allowed to do this as we are looking at a fibre bundle of manifolds, and thus at a fibration. In this case the last step of Theorem 3.4.6 holds because H is path connected. For $n > 0$ we will use,

$$\begin{aligned} \pi_n(H) \to \pi_n(SU(5)) \to \pi_n(SU(5)/H) \to \\ \to \pi_{n-1}(H) \to \pi_{n-1}(SU(5)). \end{aligned} \tag{6.86}$$

$\pi_0(SU(5)/H)$ We know that $SU(5)$ is connected and therefore $SU(5)/H$ is also connected. Hence we find:

$$\pi_0(SU(5)/H) = 0. \tag{6.87}$$

A trivial $\pi_0(SU(5)/H)$ means that we will not find domain walls as a result of this symmetry breaking pattern.

$\pi_1(SU(5)/H)$ With the exact sequence we find,

$$0 \to \pi_1(SU(5)/H) \to 0, \tag{6.88}$$

yielding

$$\pi_1(SU(5)/H) = 0. \tag{6.89}$$

Due to a trivial fundamental group of $SU(5)/H$ we can also exclude the possibility of strings in this model.

$\pi_2(SU(5)/H)$ The exact sequence results in

$$0 \to \pi_2(SU(5)/H) \to \pi_1(H) \to 0. \tag{6.90}$$

With this we find

$$\pi_2(SU(5)/H) \cong \pi_1(H) \neq 0. \tag{6.91}$$

We do not know the exact nature of the homotopy group $\pi_2(SU(5)/H)$, but knowing that it is non-trivial is enough to conclude that monopoles can form.

$\pi_3(SU(5)/H)$ The exact sequence becomes as follows

$$0 \to \pi_4(SU(5)/H) \to \mathbb{Z}^2 \to \mathbb{Z} \to \pi_3(SU(5)/H) \to 0. \qquad (6.92)$$

Since $\mathbb{Z} \to \pi_3(SU(5)/H)$ has to be surjective, we can conclude that $\pi_3(SU(5)) \subseteq \mathbb{Z}$. But this does not tell us anything about being trivial or not. We have to take a look at the explicit maps to find out if structures can form. Specifically we need to determine the image of the map $\mathbb{Z}^2 \to \mathbb{Z}$.

We will not construct the explicit maps between $\pi_3(H) \to \pi_3(SU(5))$. Sadly, this means that we cannot complete our characterisation. We do not know if textures may form in the symmetry breaking pattern of the grand unified theory from $SU(5)$ to H.

Using the Georgi-Glashow model, we have seen how grand unified theories try to combine the symmetries of the electromagnetic- and the nuclear interactions. Having found the homotopy groups, we conclude that neither domain walls nor strings can form in the grand unified theory with symmetry breaking pattern $SU(5) \to U(1) \times SU(2) \times SU(3)$. Monopoles can form and on the formation of textures we remain inconclusive.

The absence of strings causes physicists researching cosmic string production in grand unified theories, to appeal to other gauge groups. Another often discussed model uses the $SO(10)$ group, but many more initial symmetry groups can be put forward [19].

Apart from the electromagnetic and nuclear interactions, there is another fundamental force of nature: gravity. Gravity does not fit in the models of the grand unified theory [28]. A model that entails the electromagnetic and nuclear interactions as well as gravity is yet another quest of many theoretical physicists. A model comprehending both gravity and a grand unified theory is humbly called a theory of everything and remains to be found [33] .

Chapter 7

Conclusion

In this thesis we have explored topological defects with a mathematical perspective. We have learned a lot about Lie groups, Lie algebras, homotopy theory, gauge theory and symmetry breaking. In the final chapter we were able to give an explanation of topological defects and three insightful examples to substantiate the explanation.

One of the main challenges of this thesis was to find a topic that contained enough interesting mathematics, preferably in the direction of differential geometry and algebraic topology, while also bearing physical relevance. The subject of topological defect fitted these requirements accurately.

Despite the amount of theory already discussed in this thesis, there are still a number of topics that I would like to learn more about. The most substantial of all being the mathematical frame work underlying the covariant derivate used in non-abelian gauge theory. The theory needed to describe this is covered in graduate courses such as differentiable manifolds.

Another open question is the third homotopy group of $SU(5)/H$, and thereby the appearance of textures during this hypothesised symmetry breaking phase transition.

Furthermore, compact and simple Lie groups could have been studied in more depth. We have had to accept a lot of theorems, to use in Yang-Mills theory, while not understanding their proofs. This is also theory that can be acquired in a graduate course in Lie groups.

There are also questions, that not only remain unanswered in this thesis, but that to this date have not been solved. For example, the question if there exists a grand unified theory that accurately describes the unification of the electromagnetic and nuclear forces.

Chapter 8

Last words and acknowledgements

I really liked working on this thesis. It consisted of both the mathematics that I am interested in, while also giving a real life application of the abstract subjects. The topics I learned about will hopefully be recurring in my masters of mathematics. Even though I have decided to do a masters in mathematics, this thesis showed me that I do actually really like physics as well. As a result I might try to implement a bit of theoretical physics in my masters.

Looking back, I very much underestimated the fraction of work I would spend on the theoretical background. In final product this is also seen, as the ground work takes up most of the thesis. One consequence of the extensive background I researched is that the thesis developed into a substantial work that was a lot longer then anticipated. Still I have decided not to exclude parts or sections, since the theory discussed is all employed. I found that I prefer to give a more comprehensive explanation that my fellow-students can understand than to skip over details for conciseness.

A turning point of the thesis came after I finished most of the theoretical background when both Yang-Mills theory and topological defects presented themselves as feasible objectives. I chose for topological defects as I had become intrigued by the importance of topological structures in physics. Further, I assumed that I may still come across the theoretical framework of Yang-Mills theory during courses in my master of mathematics, for example in the master maths course Differentiable manifolds.

I would like to express thanks to my supervisors Federica Paqsquotto and Subodh Patil for their help during this thesis. Federica has helped me with my many questions, lots of feedback and an always open door, thank you for all your advise and for reassuring me when I was stressed out. Subodh provided me with the subject and physical background of this thesis which shaped this thesis to a project of my liking, thank you for your calm guiding during this project. Finally, I want to thank my friends and family for the (lunch) breaks that took care of my sanity for the past few months.

Appendix A

Calculations on topological defects

This appendix contains some of the computations of subsection 6.2.2.

A.1 Stress-energy tensor of the domain wall example

We calculate the stress-energy tensor of a static planar domain wall.

The first observation to make is that for this model $\partial_x \varphi$ is the only partial derivative that is unequal to zero, this will cancel out a lot of terms. Further more $\eta_{\mu\nu}$ is either 0 if $\mu \neq \nu$, 1 is $\mu = \nu = 0$ and -1 otherwise. Further more we need to remember that $\partial_i \varphi \partial_i \varphi$ comes with a minus sign in the Minkowski metric. Let us first take a look at T_{00}.

$$T_{00} = -\partial_t \varphi \partial_t \varphi - 1 \cdot \left(-\frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi - V(\varphi) \right) \tag{A.1}$$

$$= 0 + \frac{1}{2} \partial_x \varphi \partial_x \varphi + \frac{\lambda}{4} (\varphi^2 - \eta^2)^2 \tag{A.2}$$

Now filling in $\varphi = \eta \tanh(\sqrt{\frac{\lambda}{2}} \eta x)$ we find,

$$T_{00} = \frac{1}{2} \left(\partial_x \eta \tanh(\sqrt{\frac{\lambda}{2}} \eta x) \right)^2 + \frac{\lambda}{4} \left(\left[\eta \tanh(\sqrt{\frac{\lambda}{2}} \eta x) \right]^2 - \eta^2 \right)^2 \tag{A.3}$$

$$= \frac{1}{2} \eta^2 \left(\partial_x \tanh(\sqrt{\frac{\lambda}{2}} \eta x) \right)^2 + \frac{\lambda}{4} \eta^4 \left(\tanh^2(\sqrt{\frac{\lambda}{2}} \eta x) - 1 \right)^2 . \tag{A.4}$$

Now we use the identity $\tanh^2(x) - 1 = \text{sech}^2(x)$, and take the derivative of the hyperbolic tangent, $\partial_x \tanh(ax) = a\,\text{sech}^2(ax)$. The function sech is the hyperbolic secant.

This gives us

$$T_{00} = \frac{1}{2}\eta^2 \left(\sqrt{\frac{\lambda}{2}}\eta\ \text{sech}^2(\sqrt{\frac{\lambda}{2}}\eta x)\right)^2 + \frac{\lambda}{4}\eta^4 \left(\text{sech}^2(\sqrt{\frac{\lambda}{2}}\eta x)\right)^2 \tag{A.5}$$

$$= \frac{\lambda}{2}\eta^4\ \text{sech}^4(\sqrt{\frac{\lambda}{2}}\eta x). \tag{A.6}$$

For T_{yy} T_{zz} we see that we get a nearly equivalent derivation, only the $\eta_{\mu\nu}$ value changes from 1 to -1, and thus we find,

$$T_{yy} = T_{zz} = -T_{00} \tag{A.7}$$

For T_{xx} we find, that $-\eta_{\mu\nu}\mathscr{L}$ gives us the same value as for T_{yy}. Only for T_{xx} the first term $\partial_\mu\varphi\partial_\nu\varphi$ is nonzero. We find $\partial_x\varphi\partial_x\varphi = \frac{\lambda}{2}\eta^4\text{sech}^4(\sqrt{\frac{\lambda}{2}}\eta x)$ simlier to the first term in Equation A.5. This give us

$$\begin{aligned} T_{xx} &= \partial_x\varphi\partial_x\varphi - \mathscr{L}(\varphi) \\ &= \frac{\lambda}{2}\eta^4\text{sech}^4(\sqrt{\frac{\lambda}{2}}\eta x) - \frac{\lambda}{2}\eta^4\text{sech}^4(\sqrt{\frac{\lambda}{2}}\eta x) = 0. \end{aligned} \tag{A.8}$$

Finally for all non-diagonal terms, $\mu \neq \nu$ we note that $\eta_{\mu\nu} = 0$, and because only $\partial_x\varphi \neq 0$ we also see $\partial_\mu\varphi\partial_\nu\varphi = 0$ for $\mu \neq \nu$. Hence, for $\mu \neq \nu$ we find $T_{\mu\nu} = 0$ and therefore

$$T_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{\lambda}{2}\eta^4\text{sech}(\eta\sqrt{\frac{\lambda}{2}}x). \tag{A.9}$$

A.1.1 The total energy in the domain wall

To calculate the total energy in the domain wall, first note that

$$\frac{d}{dx}\tanh(ax) = a\ \text{sech}^2(ax), \tag{A.10}$$

$$\frac{d}{dx}\tanh^2(ax) = 2\tanh(ax)\cdot a\ \text{sech}^2(ax), \tag{A.11}$$

$$\frac{d}{dx}\tanh(ax)(\tanh^2(ax) - 3) = -3a\ \text{sech}^4(ax). \tag{A.12}$$

For the last line we use that $1 - \tanh^2(ax) = \operatorname{sech}^2(ax)$. We thus find the intergral

$$\int_{-\infty}^{\infty} \operatorname{sech}^4(ax)dx = \frac{-\tanh(ax)(\tanh^2(ax) - 3)}{3a}. \tag{A.13}$$

The integral of T_{00} can now be calculated,

$$\sigma = \int_{-\infty}^{\infty} T_{00}dx = \frac{\lambda}{2}\eta^4 \int_{-\infty}^{\infty} \operatorname{sech}^4(\sqrt{\frac{\lambda}{2}}\eta x)dx \tag{A.14}$$

$$= \frac{\frac{\lambda}{2}\eta^4}{-3\sqrt{\frac{\lambda}{2}}\eta} \left[\tanh(\sqrt{\frac{\lambda}{2}}\eta x)(\tanh^2(\sqrt{\frac{\lambda}{2}}\eta x) - 3)\right]_{-\infty}^{\infty} \tag{A.15}$$

$$= -\sqrt{\frac{\lambda}{2}}\eta^3\frac{1}{3}[-4] = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3, \tag{A.16}$$

is the total energy of the domain wall.

A.2 Ordinary differential equation in terms of f_s

We rewrite the Euler-Lagrange equation of a static infinite straight string along the z-direction, with the ansatz given in Equation 6.42. We use

$$\sum_{\mu=x,y} \partial_\mu\rho\partial_\mu\theta = 0, \tag{A.17}$$

$$\sum_{\mu=x,y} \partial_\mu\partial_\mu\theta = 0, \tag{A.18}$$

$$\sum_{\mu=x,y} \partial_\mu\partial_\mu\rho = \frac{1}{\rho}, \tag{A.19}$$

$$\sum_{\mu=x,y} \partial_\mu\theta\partial_\mu\theta = \frac{1}{\rho^2}, \tag{A.20}$$

$$\sum_{\mu=x,y} \partial_\mu\rho\partial_\mu\rho = 1. \tag{A.21}$$

Then we write

$$\Box\varphi = \Box\left(\eta f_s(\rho\eta)e^{in\theta}\right) \tag{A.22}$$

$$= \eta \sum_{\mu=x,y} \partial_\mu\partial\mu\left(f_s(\rho\eta)e^{in\theta}\right). \tag{A.23}$$

Using the product rule we find

$$\begin{aligned}\Box\varphi = e^{in\theta}\{\ & f_s''(\rho\eta)\cdot[(\partial_\mu\rho\eta)^2]+\\ & f_s'(\rho\eta)\cdot[\partial_\mu\partial_\mu(\rho\eta)+2\partial_\mu(\rho\eta)\partial_\mu(in\theta)]+\\ & f_s(\rho\eta)\cdot[-n^2(\partial_\mu\theta)^2+\partial_\mu\partial_\mu(in\theta)] \qquad \}\end{aligned} \tag{A.24}$$

With the above mentioned identities we can simplify this to

$$\begin{aligned}\Box\varphi = e^{in\theta}\{\ & f_s''(\rho\eta)\cdot[\eta^2]+\\ & f_s'(\rho\eta)\cdot[\frac{\eta}{\rho}+0]+\\ & f_s(\rho\eta)\cdot[-\frac{n^2}{\rho^2}+0\}.\end{aligned} \tag{A.25}$$

For the second part of Equation 6.33 we have,

$$\frac{\partial V}{\partial\varphi} = \frac{\partial}{\partial\varphi}(\frac{\lambda}{4}(|\varphi|^2-\eta^2)^2 \tag{A.26}$$

$$= \frac{\lambda}{2}\varphi(|\varphi|^2-\eta^2). \tag{A.27}$$

Now we fill in our ansatz of φ. We use that the norm of the exponential term is equal to 1 and we gather the η terms in front:

$$\frac{\partial V}{\partial\varphi} = \frac{\lambda}{2}\eta^3 e^{in\theta} f_s(f_s^2-1). \tag{A.28}$$

We also gather the η terms in Equation A.25 and plug these in into Equation 6.33,

$$\eta^3 e^{in\theta}\left(f_s''+\frac{1}{v}f_s'-\frac{n^2}{v^2}f_s-\frac{\lambda}{2}f_s(f_s^2-1)\right)=0. \tag{A.29}$$

Finally we note that $\eta^3 e^{in\theta}\neq 0$ and we find the wanted expression.

Bibliography

[1] R. Durrer, *Topological defects in cosmology*, Elsevier Science (1999).

[2] J. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, Springer, 2003.

[3] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*, Springer Science & Business Media, 1983.

[4] R. van Luijk, *Linear Algebra 1*, 2020.

[5] J. Shotten, *Representation theory*, chapter Linear Lie groups and their Lie algebras, Durham University, website, u.k.

[6] V. Robakov, *Classical theory of Gauge fields*, Princeton University Press, 2002.

[7] H. Goldstein, *Classical Mechanics*, Addision-Wesley publishing company, 2 edition, 1980.

[8] A. W. Knapp, *Lie groups Beyond an Introduction*, Birkhäuser Boston, MA, 1 edition, 1996.

[9] J. Pevtsova, *Math 508 - Lie Algebras (Lecture notes)*, 2023.

[10] T. Bröcker and T. Tom Dieck, *Representations of compact Lie groups*, volume 98, Springer Science & Business Media, 2003.

[11] A. Hatcher, *Algebraic Topology*, Algebraic Topology, Cambridge University Press, 2002.

[12] F. Pasquotto, *Topologie 2024-25*, 2025.

[13] S. v. d. L. R. S. de Jong, *Syllabus behorend bij het vak Inleiding in de AlgebraÃ¯sche Topologie*, 2024.

[14] E. H. Spanier, *Algebraic Topology*, chapter 2, Springer New York, NY, 1 edition, 1994.

[15] S. Patil, *Lorentz tranformations*.

[16] C. N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96**, 191 (1954).

[17] L. W. Tu, *Differential Geometry*, Springer Cham, 2017.

[18] G. t. Hooft and G. 't Hooft, *50 years of Yang-Mills theory*, World Scientific Publishing Co. Pte. Ltd, Singapore, 1 edition, 2005.

[19] M. Srednicki, *Quantum Field Theory*, Cambridge University Press, 2007.

[20] S. J. Blundell and K. M. Blundell, *Concepts in Thermal Physics*, chapter 28, Oxford University Press, 2009.

[21] N. D. Mermin, *The topological theory of defects in ordered media*, Rev. Mod. Phys. **51**, 591 (1979).

[22] A. Rajantie, *Introduction to magnetic monopoles*, Contemporary Physics **53**, 195 (2012).

[23] A. Polyakov, *Spectrum of particles in quantum field theory*, JETP lett **20**, 430 (1974).

[24] G. t Hooft, *Magnetic monopoles in unified theories*, Nucl. Phys. B **79**, 276 (1974).

[25] D. S. Freed, G. W. Moore, and C. Teleman, *Topological symmetry in quantum field theory*, 2024.

[26] M. Shifman, *Instantons in gauge theories*, World Scientific, 1994.

[27] *The Nobel Prize in Physics 1979*, `https://www.nobelprize.org/prizes/physics/1979/summary/`, 24 Jun 2025.

[28] S. Coleman, *The 1979 Nobel Prize in Physics*, Science **206**, 1290 (1979).

[29] G. Altarelli, *The Standard Electroweak Theory and Beyond*, 1998.

[30] H. Georgi and S. L. Glashow, *Unity of all elementary-particle forces*, Physical Review Letters **32**, 438 (1974).

[31] J. Baez, *Can we understand the standard model*, `https://math.ucr.edu/home/baez/standard/standard_web.pdf`.

[32] S. Iyanaga, Y. Kawada, K. O. May, N. Sugakkai., and A. M. Society., *Encyclopedic dictionary of mathematics*, The MIT Press, Cambridge, MA, [etc, 1977.

[33] S. Weinberg, *Dreams of a final theory: The scientist's search for the ultimate laws of nature*, Vintage, 1994.