Lie Groups in Particle Physics
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Abstract

Symmetries can be found everywhere in physics, and are usually very important within research. Many symmetries in physics can be visualized using Lie-groups, most importantly the internal- and spacetime symmetries. In this thesis, the symmetries corresponding to the fundamental forces will be studied, and the particles will be classified using their unitary irreducible representations, to eventually create a description of a particle using these representations.

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Chapter 1

Introduction

Since the introduction of quantum mechanics, we've come to understand forces in physics way better. Depending on who you ask, they can be divided into either three or four forces, with the fourth force being gravitation, which can also be seen as the bending of spacetime itself instead of a fundamental force. The other three fundamental forces are generally very well understood since the introduction of the standard model, and many particles are intrinsically linked by the way these forces work. Each of these forces has their respective symmetry group, those being the symmetry groups for phase, spin and colour. Under inspection, they tell us a lot about how particles interact under these different forces.

In this thesis the symmetry groups related to different forces will be researched by classifying their irreducible representations. To conserve probabilities in the quantum mechanical sense, only unitary representations are of interest. Each such representation will then be used to signify the matter in which a fundamental force works on a particle, and therefore we can classify particles based on how they interact with the fundamental forces. Using the representations of all three symmetry groups at once, this thesis will classify each of the particles of the standard model on their internal structure.

This thesis will also aim to take a look at the symmetry groups that maintain the laws of physics as per Einstein's theory of special relativity, and classify its unitary irreducible representations, to get a perspective on how the remaining properties can be understood in a similar manner to the internal properties of a particle.

Chapter 2

Preliminaries

The mathematics required to understand this thesis is mostly contained in the mandatory courses for the Bachelor studies Mathematics and Physics at Leiden University. However, there are a few preliminary topics outside of these courses, which will be covered in this chapter.

2.1 Lie-Groups

Definition 2.1 *A **Lie-group** G is a group which is also a finite-dimensional real smooth manifold, with the requirement that the maps $f\colon G \times G \to G$ defined by $f(x,y) = xy$ and $g\colon G \to G$ defined by $f(x) = x^{-1}$ are both smooth.*

2.1.1 Matrix Lie-Groups

As $\mathrm{GL}(n,K)$ *is a Lie-group for both the fields $K = \mathbb{R}$ and $K = \mathbb{C}$, and subgroups of Lie-groups must be Lie-groups as well, we can refine the definition of Lie-groups for matrix groups as follows:*

Definition 2.2 *A **matrix Lie-group** is a matrix group which is closed in the general linear group $GL(n,K)$, where $K = \mathbb{R} \vee K = \mathbb{C}$.*

Definition 2.3 *The **orthogonal group**, and respectively the **special orthogonal group** are defined by:*

$$\mathrm{O}(n) = \{g \in GL(n,\mathbb{R})\colon gg^T = \mathbb{1}_n\}, \quad \mathrm{SO}(n) = \{g \in \mathrm{O}(n) : Det(g) = 1\}$$

Definition 2.4 *The **unitary group**, and respectively the **special unitary group** are defined by:*

$$\mathrm{U}(n) = \{g \in GL(n, \mathbb{C}) \colon gg^\dagger = \mathbb{1}_n\}, \quad \mathrm{SU}(n) = \{g \in \mathrm{U}(n) : Det(g) = 1\}$$

*A matrix g is **unitary** if $gg^\dagger = \mathbb{1}_n$, or equivalently $g \in \mathrm{U}(n)$ for any $n \in \mathbb{N}$.*

2.2 Lie Algebra's

*A finite-dimensional **Lie algebra** is a finite-dimensional vector space $\mathfrak{g}$ over K with a bilinear map $[\cdot, \cdot]$ for which the following properties hold:*

- $[X, Y] = -[Y, X]$ *for all* $X, Y \in \mathfrak{g}$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$

*We call $\mathfrak{g}$ **real** or **complex** for $K = \mathbb{R}, \mathbb{C}$ respectively.*

2.2.1 Matrix Lie Algebra

Definition 2.5 *A **matrix Lie algebra** is a matrix vector space $\mathfrak{g}$ with the commutator map $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{g}$.*

Definition 2.6 *For any matrix Lie-group G over $\mathcal{V}$, we define its Lie algebra by:*

$$\mathfrak{g} = \left\{ X \in M(\mathcal{V}) \colon \forall_{t \in \mathbb{R}} \colon e^{tX} \in G \right\}$$

We can confirm that this is a vector space, and thus a Lie algebra, as for $X, Y \in \mathfrak{g}$:

- $\forall_{t, \lambda \in \mathbb{R}} \colon t\lambda \in \mathbb{R} \implies e^{t(\lambda X)} \in G$
- $e^{t(X+Y)} = e^{tX} e^{tY} \in G$

Definition 2.7 *The **complexification** $\mathfrak{g}_\mathbb{C}$ of a real Lie algebra $\mathfrak{g}$ is defined by:*

$$\mathfrak{g}_\mathbb{C} = \{X + iY \colon X, Y \in \mathfrak{g}\}$$

2.3 Representation Theory

2.3.1 Group Representations

Definition 2.8 *Let G be a group, then a **representation** of G is a group homeomorphism $\rho\colon G \to GL(\mathcal{V})$, where the **representation space** $\mathcal{V}$ is a vector space with dimension n. We call n the **dimension** of ρ.*

Definition 2.9 *Let ρ_1, ρ_2 be two representations of G. Then we say ρ_1 and ρ_2 are **equivalent**, or $\rho_1 \propto \rho_2$, if for some matrix $S \in \mathrm{GL}(n, K)$ we have $\rho_2 = S\rho_1 S^{-1}$. An equivalence class is therefore independent of basis.*

Definition 2.10 *We say that a representation ρ of G is **reducible** if there exists a subspace $\mathcal{W} \subsetneq \mathcal{V}$ such that for each $g \in G$ we have $\rho(g)\mathcal{W} \subseteq \mathcal{W}$. A non-reducible representation is called **irreducible**. The representation space of a reducible representation, can be broken up into a set of spaces $S = \{\mathcal{W}\}$ such that $\mathcal{V} = \bigoplus_{\mathcal{W}\in S} \mathcal{W}$, and ρ is reducible when restricted over any $\mathcal{W} \in S$. For short, this thesis denotes irreducible representations by irrep.*

Schur's Lemma's

Lemma 2.1 Schur's Lemma's
Let ρ, ρ_1, ρ_2 be an irrep. of a group G, and S a matrix. Then

$$[S\rho_1(g) = \rho_2(g)S\colon \forall_{g\in G}] \implies [S = 0 \vee \rho_1 \propto \rho_2] \text{ and}$$

$$[\forall_{g\in G}\colon S\rho(g) = \rho(g)S] \implies S = \lambda \mathbb{1}, \lambda \in \mathbb{C}$$

Proof: *To proof the first statement, let $\mathcal{V}_1, \mathcal{V}_2$ be the representation spaces of ρ_1, ρ_2 respectively, it follows that $S\colon \mathcal{V}_1 \to \mathcal{V}_2$. We must thus have $\mathrm{Im}\, S \subseteq V_2$ and $\ker S \subseteq V_1$. However we then find*

$$\rho_2(g)\mathrm{Im}\, S = \mathrm{Im}\,(\rho_2(g)S) = \mathrm{Im}\,(S\rho_1(g)) = S\mathrm{Im}\,\rho_1(g) = S\mathcal{V}_1 = \mathrm{Im}\, S$$

So from irreducibility we find that $\mathrm{Im}\, S = 0 \implies S = 0$, or $\mathrm{Im}\, S = \mathcal{V}_2$. Also:

$$S\rho_1(g)\ker S = \rho_2(g)S\ker S = 0 \implies \rho_1(g)\ker S \subseteq \ker S$$

So we also have $\ker S = \mathcal{V}_1 \implies S = 0$ or $\ker S = 0$. Assuming $S \neq 0$ We must then have $\dim \rho_1 = \dim \rho_2$ with S invertible. We can conclude that $\rho_2(g) = S\rho_1(g)S^{-1}$, and thus $\rho_1(g) \propto \rho_2(g)$.

To proof the second statement, let $\mathcal{V}_\lambda$ be the λ-eigenspace of S. We find that for $v \in \mathcal{V}_\lambda$:

$$\lambda\rho(g)v = \rho(g)Sv = S\rho(g)v \implies \rho(g)v \in V_\lambda$$

When allowing complex eigenvalues, we can always find an eigenspaces of S, and it must then stretch the entire representation space to retain irreducibility, thus $S = \lambda\mathbb{1}$.

Lemma 2.2 *Let ρ be an irrep. of an Abelian group G, then* $\dim\rho = 1$
Proof: *Let $g, g_0 \in G$, we then have:*

$$\rho(g)\rho(g_0) = \rho(gg_0) = \rho(g_0g) = \rho(g_0)\rho(g)$$

Then from 2.1 we find that $\rho(g) \propto \mathbb{I}_n$ for all $g \in G$. We can then define a function $\lambda\colon G \to \mathbb{C}$ such that $\rho(g) = \lambda(g)\mathbb{I}_n$. We can easily deduce that all group homomorphism properties of ρ are transferred to λ. Taking any 1-dimensional subspace $\mathcal{W} \subseteq \mathcal{V}$ gives for $w \in \mathcal{W}$ that $\rho(g)w = \lambda(g)w \in \mathcal{W}$. For ρ to be irreducible, we must have $\mathcal{V} = \mathcal{V}$, so therefore $\dim\rho = \dim\mathcal{V} = 1$.

Product of Representations

Let G be a group with representations ρ_1, ρ_2 with respective representation spaces $\mathcal{V}_1, \mathcal{V}_2$. We can define a representation ρ working on $\mathcal{V}_1 \otimes \mathcal{V}_2$ by

$$\rho(g)(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2$$

Even when ρ_1, ρ_2 are irreducible, ρ often is not. It can however be decomposed into its sum of reducible representations.

Representations of products of groups

Let G_1, G_2 be groups, and let $\rho_1\colon G_1 \to \mathcal{V}_1, \rho_2\colon G_2 \to \mathcal{V}_2$ be representations of these groups. We can define a representation of $G_1 \times G_2$ as:

$$\rho\colon G_1 \times G_2 \to \mathcal{V}_1 \otimes \mathcal{V}_2 \qquad \rho(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$$

Suppose ρ_1 is reducible, so we have $\mathcal{W}_1 \subsetneq \mathcal{V}_1 \colon \rho_1(g_1)\mathcal{W}_1 \subseteq \mathcal{W}_1$ for all $g_1 \in G_1$. We then also have:

$$\rho(g_1, g_2)\mathcal{W}_1 \otimes \mathcal{V}_2 = \rho_1(g_1)\mathcal{W}_1 \otimes \rho_2(g)\mathcal{V} = \mathcal{W}_1 \otimes \mathcal{V}_2 \subsetneq \mathcal{V}_1 \otimes \mathcal{V}_2$$

Thus ρ is reducible too. Similarly, if ρ is reducible, there must be some vector space $\mathcal{W} \subsetneq \mathcal{V}_1 \times \mathcal{V}_2$ for which for all $g_1, g_2 \in G$ we have $\rho(g)\mathcal{W} = \mathcal{W}$. However, $\mathcal{W}$ must then be smaller than $\mathcal{V}_1 \otimes \mathcal{V}_2$, thus it must be the product of two vector spaces of which at least one is smaller than its superspace, making at least one of ρ_1, ρ_2 reducible. So ρ is irreducible exactly when ρ_1, ρ_2 both are too.

2.3.2 Algebra Representations

Definition 2.11 *Let $\mathfrak{g}$ be a Lie algebra, then a **representation** of $\mathfrak{g}$ is a linear transformation $\pi\colon \mathfrak{g} \to \mathrm{GL}(\mathcal{V})$, where the **representation space** $\mathcal{V}$ is a vector space with dimension n, and the following identity holds:*

$$\pi([X,Y]) = [\pi(X), \pi(Y)] \colon \forall_{X,Y\in\mathfrak{g}}$$

*We call n the **dimension** of π.*

Proposition 2.1 *Let $\mathfrak{g}$ be a Lie algebra. Then any representation ρ of $\mathfrak{g}$ constructs a unique representation $\rho_{\mathbb{C}}(X+iY) := \rho(X) + i\rho(Y)$ of its complexification $\mathfrak{g}_{\mathbb{C}}$. We can as per usual restrict the complex representation to its real subgroup, though it need not be unique.*

Proposition 2.2 *For every representation ρ of a Lie-group G, there exists a unique representation π of its Lie algebra $\mathfrak{g}$ such that $\rho(e^X) = e^{\pi(X)}$ for all $X \in \mathfrak{g}$. For compact Lie-groups, we can find the inverse:*

$$\begin{aligned}\pi(X) &= \frac{d}{dt} e^{t\pi(X)}\bigg|_{t=0} \\ &= \frac{d}{dt} e^{\pi(tX)}\bigg|_{t=0} \\ &= \frac{d}{dt}\rho(e^{tX})\bigg|_{t=0}\end{aligned}$$

Proposition 2.3 *Let G be a compact matrix Lie-group with Lie algebra $\mathfrak{g}$, and let ρ be a product representation of representations ρ_1, ρ_2 with representation space $\mathcal{V}_1, \mathcal{V}_2$ respectively, as described in 2.2. Now let π, π_1, π_2 be their respective Algebra representations as described in 2.11. Then we can write*

$$\begin{aligned}\rho(e^{tX})(v_1 \otimes v_2) &= \rho_1(e^{tX})v_1 \otimes \rho_2(e^{tX})v_2 \\ &= e^{\pi_1(tX)}v_1 \otimes e^{\pi_2(tX)}v_2 \\ \implies \pi(X)(v_1 \otimes v_2) &= \frac{d}{dt}\left[e^{t\pi_1(X)}v_1 \otimes e^{t\pi_2(X)}v_2\right]\bigg|_{t=0} \\ &= \pi_1(X)v_1 \otimes v_2 + v_1 \otimes \pi_2(X)v_2\end{aligned}$$

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Chapter 3

Symmetries of Particle Physics

Symmetries can be found everywhere in physics, and that's only for the best, as it allows us to explain the laws of physics in a much better fashion. Without these symmetries, we could never explain these laws in a general fashion, as they would differ vastly depending on the circumstance. However, using these symmetries, we can understand the laws of physics more generally, and expand it to each specific circumstance using the mathematics behind the symmetries we've found. This chapter aims to understand the symmetries relevant to each of the fundamental forces of physics, in order to eventually classify where we can apply these symmetries to the laws of physics. The first section of this chapter will focus on the internal symmetries, related to the three internal fundamental forces which we already understand within the standard model. The second section will focus on the spacetime symmetries, which are relevant within the theory of relativity.

3.1 Internal Symmetries

Internal symmetries are the symmetries related to the states of a particle, and each can intrinsically be tied to one of the three fundamental internal forces. As they are related to the states, and can be used as transformations between quantum probabilities, the unitary requirement quickly follows.

3.1.1 Electromagnetic Force

The electromagnetic force, or more accurately the electromagnetic field, is all about waves and the way they interfere. The symmetry related to the electromagnetic force then of course is the symmetry of phase. When looking at a wave, there is no difference between any point, and a point one wavelength further. This is *usually denoted by the complex exponent $e^{i\theta}$, which has the very useful property that $e^{i(\theta+2\pi)} = e^{i\theta}$. This set happens to be exactly the group $\mathrm{U}(1)$, which we can use as the symmetry group for the electromagnetic force.*

3.1.2 Weak Nuclear Force

The weak nuclear force is the force responsible for the transition between particle states. Most commonly for example the up quark to the down quark. To describe the transformation between particle states, we need a 2-dimensional linear transformation that maintains probability of the particle states, which would have to be an element of $\mathrm{U}(2)$. However, we also require the orientation of the particle to be maintained. To achieve this property we must look within the group $\mathrm{SU}(2)$. Similarly as to how the phase is preserved after adding a full wavelength for $\mathrm{U}(1)$, a combination of such transformations will always be contained in $\mathrm{SU}(2)$, making this our symmetry group for the weak nuclear force.

3.1.3 Strong Nuclear Force

The strong nuclear force is the force that binds particles together, where we use the concept of colour to describe these bound particles. We cannot know the colours of the particles separately, but do know that their colour charge must be 0. The particles themselves can therefore be a set of different states, which are entangled by the condition of neutral colour charge. The symmetry group of the strong nuclear force must then of course maintain the neutral colour charge, while it can have an affect on the separate colours of the particles. As particles can have 3 different colours, in oppose to the two states we worked with for the weak nuclear force, we must have a symmetry group contained in $\mathrm{U}(3)$. Even though we don't have to care about orientation, without the special condition we could simply multiply the states with an element of $\mathrm{U}(1)$, which would leave no change on the colour states themselves, making the determinant redundant. We can therefore use $\mathrm{SU}(3)$ to describe the symmetry group of the strong nuclear force.

3.2 Spacetime symmetries

From the theory of relativity, we know that regardless of position, direction and speed, the laws of physics (including the speed of light) stay in tact. The spacetime symmetries are exactly those symmetries. The set of transformation between inertial systems which keep the laws of physics in tact.

The spacetime symmetries can quite intuitively be tied to relativity once you understand both well. That's because the theory of relativity is all about how the *laws of physics, like the speed of light, are completely independent of the inertial system of the observer. The spacetime symmetries explain exactly that, the relation between such inertial systems, so logically these symmetries sustain the laws of relativity.*

3.2.1 Translations

The translations are the set of relative position in spacetime or simply $\mathbb{R}^4$. We of course know from the theory of relativity that position is a symmetry of space, but the temporal position, or simply the point in time, of course is also irrelevant to the laws of physics (disregarding the obvious difference in the environments in the universe through time). This must therefore by one of the spacetime symmetries.

3.2.2 Rotations

The n-dimensional rotation set is the set of transformations $g \in M(n, \mathbb{R})$ for which $gg^T = \mathbb{1}_n$, or phrased differently, transformations that preserve the inner product $x \cdot y = x^T y$. We see that this is true as:

$$(gx) \cdot (gy) = (gx)^T (gy) = x^T g^T g y = x^T y = g \cdot y$$

Though rotational velocity is something that definitely has in impact on the Newtonian laws of physics as it introduces an acceleration, the direction itself does not matter. The rotations are therefore also a set that conserves the laws of physics, and thus a spacetime symmetry.

3.2.3 Lorentz Group

The Lorentz group $\mathrm{SO}(3,1)$ is the set of rotations in spacetime. We firstly introduce a time element, usually denoted with index 0. As time and space work differently, we must also introduce the ***metric tensor*** *$\eta = \mathrm{diag}(1, -1, -1, -1)$, to then define $\mathrm{SO}(3,1) = \{g \in M(4, \mathbb{R}) \colon g\eta g^T = \eta\}$. We can of course rotate within either only of the dimension sets (rotations in 1 dimension are the trivial set), but we can also apply a sort of rotation between the time and spatial dimensions. As usual, the rotation happens in two dimensions, this time being time and one spatial direction, say the x- direction. Then the Lorentz boost is defined by a scalar $\alpha \in \mathbb{R}^2$ for $\mathbf{e}_1$, or for any $\mathbf{n} \in \mathbb{R}^3$ with $R \in \mathrm{SO}(3)$ such that $\mathbf{n} = R\mathbf{e}_1$, as:*

$$B(\alpha, \mathbf{e}_1) = \begin{pmatrix} \cosh(\frac{\alpha}{2}) & \sinh(\frac{\alpha}{2}) & 0 & 0 \\ \sinh(\frac{\alpha}{2}) & \cosh(\frac{\alpha}{2}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \implies B(\alpha, \mathbf{n}) = RB(a, \mathbf{e}_1)$$

We can understand the use of sinh *and* cosh *by the fact that we're solving the equation* $r_0^2 - r_1^2 = 1$. *The set of Lorentz boosts itself isn't a group, but along with* $\mathrm{SO}(3)$ *it forms the entire Lorentz group. In physics we however have no time-reversal, so we more often speak of the positive Lorentz group:*

$$\mathrm{SO}^+(3,1) = \{\Lambda \in \mathrm{SO}(3,1) \colon \Lambda_{00} > 0\}$$

As we know that a Lorentz boost has no effect on the laws of physics, the Lorentz group, which is a combination of the Lorentz boost and rotations which both preserve Newtonian physics, therefore is another symmetry of space-time.

3.2.4 Poincaré Group

Eventually our aim is of course to combine all these spacetime symmetries into on general Lie-group. Our result is the Lorentz group, which we can see as the combination of a Lorentz transformation and a translation. As the Lorentz transformation has an effect on the translation, we denote this by the semi-direct product $\mathrm{SO}^+(3,1) \ltimes \mathbb{R}^4$, *where we use* $\mathrm{SO}^+(3,1)$ *as time-reversal is not a physically possible Lorentz transformation. As this is the combination of all space-time symmetries, this is also the exact group for which the laws of physics are conserved, making* $\mathrm{SO}^+(3,1) \ltimes \mathbb{R}^4$ *the full spacetime symmetry group.*

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Chapter 4

Classifications of fin. unit. irreps. of the Internal Symmetry Groups

In the following chapters, this thesis will delve into the unitary irreducible representations of the symmetry groups related to particle physics, with the intention to use them to classify particles on the way the fundamental forces act on them. This specific chapter will delve into the internal symmetry groups, and keep the representations finite as they should classify finite sets of particles. In the next chapter, this thesis will delve into the unitary irreducible representations of the spacetime symmetry groups, while forgoing the finiteness of the representation.

4.1 Unitary Groups

4.1.1 U(1)

As $\mathrm{GL}(1, K) = K$*, we can deduce from 2.4 that*

$$\mathrm{U}(1) = \{z \in \mathbb{C} \colon z\bar{z} = |z|^2 = 1\} = \{z \in \mathbb{C} \colon |z| = 1\} = \{e^{i\theta} \colon \theta \in \mathbb{R}\} \subseteq \mathbb{C}$$

As $\mathbb{C}$ *is Abelian,* $\mathrm{U}(1)$ *must clearly be too. We can deduce that each unit. irrep.* ρ *of* $\mathrm{U}(1)$ *is of the form* $\rho \colon \mathrm{U}(1) \to \mathrm{U}(1)$*. We now have:*

$$\rho(e^{i(\theta_1})\rho(e^{i\theta_2}) = \rho(e^{i(\theta_1+\theta_2)})$$

As the result must once again lie on the unit circle, and as θ *is continuous, we must have a linear relation in the exponent. We thus have* $\rho(e^{i\theta}) = e^{in\theta}$ *for some* $n \in \mathbb{R}$*. We finally find that* $\rho(1) = \rho(e^{2\pi i}) = e^{2n\pi i} = 1 \implies n \in \mathbb{Z}$

For each n *we indeed have a fin. unit. irrep. of* $\mathrm{U}(1)$*, and as* $\mathrm{GL}(1, \mathbb{C})$ *is Abelian, none are equivalent to each other, giving us that the classification of* $\mathrm{U}(1)$ *is given by* $\mathbb{Z}$*.*

4.1.2 SU(2)

Lie algebra of SU(2)

As the exponent of elements seems to be telling about the representations, and considering 2.11, it is smart to start looking at Lie algebra's. We must the find the Lie algebra of $\mathrm{SU}(2)$*, or more generally,* $\mathrm{SU}(n)$*:*

Let X *be a matrix such that* $e^{tX} \in \mathrm{SU}(n)$*. By definition, we know that* $\det(e^{tX}) = 1$*, and thus* $\mathrm{tr}(tX) = 0$*. As* e^{tX} *is unitary, we also know that* $(e^{tX})^\dagger e^{tX} = \mathbb{1} \implies (tX)^\dagger + tX = 0$*, thus:*

$$\mathfrak{su}(n) = \{X \in M(n, \mathbb{C}) \colon tr(X) = 0, X^\dagger = -X\}$$

As the second constraint affects the real and imaginary part in opposite ways, we can take the complexification of $\mathfrak{su}(n)$ *to remove this constraint, giving us:*

$$\mathfrak{su}(n)_\mathbb{C} = \{X \in M(n, \mathbb{C}) \colon tr(X) = 0\}$$

Applying this to $\mathrm{SU}(2)$ *gives us:*

$$\mathfrak{su}(2)_\mathbb{C} = \{X \in M(2, \mathbb{C}) \colon \mathrm{tr}(X) = 0\} = \mathrm{span}(e^+, e^-, h)$$

$$\text{With} \qquad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representations of $\mathfrak{su}(2)_\mathbb{C}$

As a representation of a Lie algebra maintains commutator relations, we'll first calculate those of the generators of $\mathfrak{su}(2)_\mathbb{C}$*:*

$$[e^+, e^-] = h \qquad [h, e^\pm] = \pm 2e^\pm$$

Let ρ *be a fin. unit. irrep. of* $\mathfrak{su}(2)_\mathbb{C}$*, and define* $E^\pm = \rho(e^\pm)$ *and* $H = \rho(h)$*. The same commutator relations as before hold.*
Let u *be an eigenvector of* H *with eigenvalue* α*:*

$$[H, E^\pm]u = HE^\pm u - E^\pm Hu = (H - \alpha)E^\pm u = \pm 2E^\pm u$$

$$\implies HE^\pm u = (\alpha \pm 2)u$$

So we see that either $E^\pm u = 0$*, or it is an eigenvector of* H *with eigenvalue* $\alpha \pm 2$*. Repeatedly applying this principle* k *times gives either that* $(E^\pm)^k u = 0$*, or that* $(E^\pm)^k u$ *is an eigenvector of* H *with eigenvector* $\alpha \pm 2k$*. Now as* ρ *is finite-dimensional, there must be some* $N \geq 0$ *for which:*

$$(E^\pm)^N u \neq 0 \wedge (E^\pm)^{N+1} u = 0$$

Now define N as such for E^-, and take $u_0 = (E^-)^N u$ with $\lambda = \alpha - 2N$. Further define $u_{k+1} = E^+ u_k \implies Hu_k = (\lambda + 2k)u_k$. Then by induction we find that $E^- u_k = -k(\lambda + (k-1))u_{k-1}$, as $E^- u_0 = 0$ and:

$$\begin{aligned}
[E^+, E^-]u_k &= E^+E^-u_k - E^-E^+u_k \\
&= -E^+k(\lambda + (k-1))u_{k-1} - E^-u_{k+1} \\
&= -k(\lambda + (k-1))u_k - E^-u_{k+1} \\
&= Hu_k = (\lambda + 2k)u_k \\
E^-u_{k+1} &= [-k(\lambda + (k-1)) - \lambda - 2k]u_k = [-k^2 - k - \lambda k - \lambda]u_k \\
&= -(k+1)(\lambda + k)u_k
\end{aligned}$$

We must also have some K for which $u_K \neq 0 \wedge u_{K+1} = 0$. We find:

$$\begin{aligned}
E^-u_{K+1} &= E^-0 = 0 \\
&= (K+1)(\lambda + K)u_K = 0
\end{aligned}$$

As K is positive, and u_K is by definition not 0, we must have $K = -\lambda$. As we find that $\mathcal{W} = \{u_k \colon 0 \leq k \leq K\}$ is a set for which $\rho(\mathcal{W}) \subseteq \mathcal{W}$, it must be the entire representation space. We find that the H-eigenvalues of these basis vectors range from $-K$ to K with intervals of 2, and the effect of E^+ and E^- on these vectors is as described before. Thus, outside of equivalence, this representation is uniquely defined by this value $K \in \mathbb{Z}_{\geq 0}$, and by 2.11, so is SU(2).

4.1.3 SU(3)

Like for SU(2)*, we'll look at the complexification of the Lie algebra of* SU(3)*:*

$$\mathfrak{su}(3)_{\mathbb{C}} = \{X \in M(3, \mathbb{C}) \colon tr(X) = 0\} = \operatorname{span}(e_i^{\pm}, h_1, h_2) \quad \textit{with}$$

$$e_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_3^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad e_3^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Defining $h_3 = h_1 + h_2$, we see that for $i = 1, 2, 3$ we each have the same commutation relations as before, being $[e_i^+, e_i^-] = h_i$ and $[h_i, e_i^{\pm}] = \pm 2e_i^{\pm}$. Let ρ be a fin.

unit. irrep. of $\mathfrak{su}(3)_{\mathbb{C}}$*, and define* $E_i^{\pm} = \rho(e_i^{\pm})$ *and* $H_i = \rho(h_i)$*. After defining* $H_3 = H_1 + H_2 = \rho(h_1 + h_2)$*, the same commutator relations as before once again hold. As* h_1 *and* h_2 *are diagonal matrices, we find they must commute, and therefore that we can find an eigenvector* u *of both* H_1 *and* H_2 *with eigenvalues* (α_1, α_2)*, and as a result it's also an eigenvector of* H_3 *with eigenvalue* $\alpha_1 + \alpha_2$*. Like stated before, taking each* i *separately, gives us the same commutation relations as with* $\mathfrak{su}(2)_{\mathbb{C}}$*. We can therefore apply the same tactics to* $i = 1, 2$ *to find that both must be integers. We can thus define the coordinates* $(n_1, n_2) \in \mathbb{Z}^2$ *to be the eigenvalues of* H_1, H_2 *for any eigenvector of both.*

$$[H_1, \{E_1^{\pm}, E_2^{\pm}, E_3^{\pm}\}] = \pm\{2E_1^{\pm}, -E_2^{\pm}, E_3^{\pm}\}$$
$$[H_2, \{E_1^{\pm}, E_2^{\pm}, E_3^{\pm}\}] = \pm\{-E_1^{\pm}, 2E_2^{\pm}, E_3^{\pm}\}$$

We can use the same reasoning as before to deduce that an eigenvector of H_1 *and* H_2 *either becomes zero under the effect of* $E_i^{\pm}$*, or it once again becomes an eigenvector of both with an effect on the eigenvalue of* $(\pm 2, \mp 1), (\mp 1, \pm 2), (\pm 1, \pm 1)$ *respectively. We can visualize the effect of these matrices by putting them in a grid of weights. Using axes of* $n_1 - n_2$ *and* $n_1 + n_2$ *we get:*

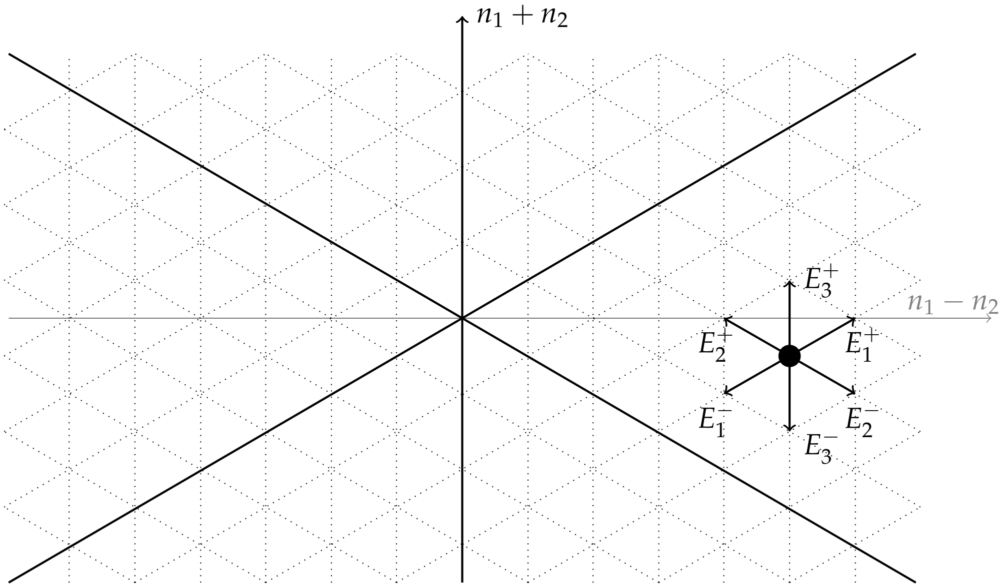

Figure 4.1: The weight diagram to plot representations of $\mathrm{SU}(3)$. The intersections are weights at which eigenvectors of H_1, H_2 can exist, where the eigenvalues are plotted with their difference on the x-axis, and their sum on the y-axis. The effect of the $E_i^{\pm}$ on these eigenvalues is also shown.

Where we also see the effect of $E_i^{\pm}$ on these weights. In the same matter as in the $\mathfrak{su}(2)$ case, for an H_i eigenvector u with eigenvalue n_i for which $E_i^{+}u = 0$, we eventually reach the eigenvector with eigenvalue $-n_i$ but none further, effectively mirroring over the line $n_i = 0$. As ρ is a finite representation, there must be an eigenvector u with weight (N_1, N_2) for which $n_1 + n_2$ is maximized. As E_i^{+} all increase $n_1 + n_2$, we know that $E_i^{+}u = 0$. We can thus reflect all these weights over the before mentioned lines. We can repeat this process until we've got an outline of either a hexagon or a triangle. Lastly, we can again repeat this process for all of the weights on the outline to obtain the center as well, giving us the exact set of weights in our representation.

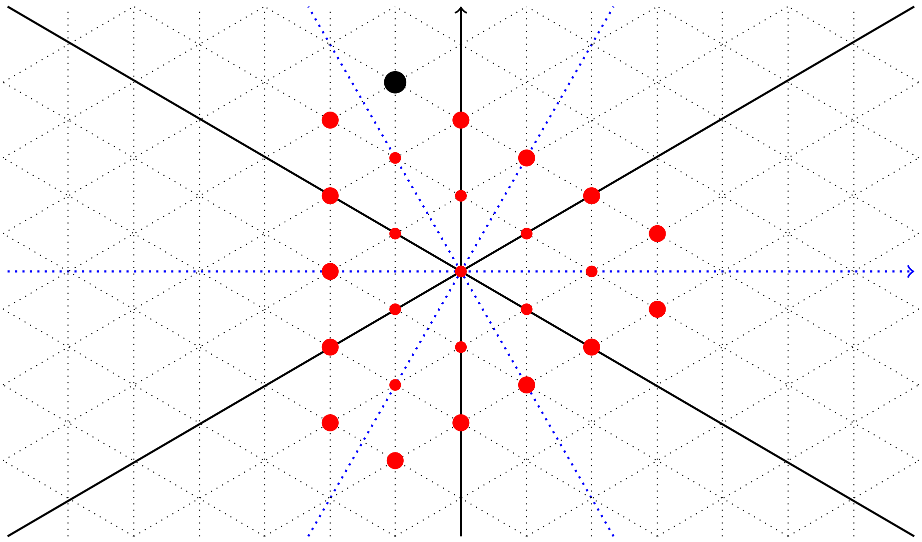

Figure 4.2: *A representation of* SU(3)*, with highest weight vector at (4, 1). The symmetry axes $n_1 = 0, n_2 = 0, n_1 + n_2 = 0$ are shown by the blue dotted lines, and the entire set of weights generated from the highest weight vector is shown in red.*

However, unlike for the $\mathfrak{su}(2)$ case, there may be multiple ways to reach a certain weight (n_1, n_2), meaning there may be weights with higher dimensional eigenspace, whose dimension we call the **multiplicity**. *We find:*

$$[E_1^{\pm}, E_2^{\pm}] = \pm E_3^{\pm} \quad [E_3^{\pm}, E_1^{\mp}] = \pm E_2^{\mp} \quad [E_3^{\mp}, E_2^{\pm}] = \pm E_1^{\pm}$$

While all other commutators are equal to 0. We can again apply combinations of $E_i^{\pm}$ to u. We can already accurately deduce the weight of these eigenvectors, but not necessarily the eigenspaces of these weights. We do know that if we apply $E_3^{\pm}$ to some eigenvector v, it is linearly dependent on $E_2^{\pm}E_1^{\pm}v - E_1^{\pm}E_2^{\pm}v$, so to construct a basis of these eigenspaces, we only need to apply $E_1^{\pm}, E_2^{\pm}$. We also know

that applying E_i^+ before applying E_i^- to u will result in 0, and that $[E_1^\pm, E_2^\mp] = 0$. So suppose we ever apply E_i^+ to u, then we can commute it forward until it inevitably meets its E_i^- counterpart. As we know, $E_i^+ E_i^- v = 0 \vee E_i^+ E_i^- v \propto v$ for each weight vector v, so ever applying E_i^+ gives us no independent vectors.
We can now construct all of these eigenvectors. For any given weight, we know that the total applied E_1^-, E_2^- to u is fixed, the order however isn't. The matrices don't commute, but we can switch the order if we also add a vector where we replace an E_1^- and E_2^- with E_3^-. As E_3^- does commute with these matrices, its placement doesn't matter, giving us a basis of

$$\{(E_1^-)^{i-n}(E_3^-)^n(E_2^-)^{j-n}u \colon 0 \le n \le \min\{i, j\}\}$$

Without loss of generality, we can assume $N_2 \ge N_1$. For $i \le N_1, j \le N_2$, we can see that none of these basis vectors become 0. However, for $j > N_2$ we find that $(E_2^-)^j u = 0$, meaning that the multiplicity can never increase above $N_1 + 1$. Using the same strategy as before, we can mirror multiplicities over the axes $n_1 = 0, n_2 = 0, n_1 + n_2 = 0$. Say that for some weight we have a higher multiplicity than at its mirror. Than we can find a vector u in its eigenspace, for which its mirror $(E_i^\pm)^k u = 0$, which we proofed to not be the case. This only leaves the centre, which with a similar argument must have a weight of at least those of its neighbours, being $N_1 + 1$, which we also proofed to be the maximum. The multiplicity therefore increases by 1 starting at the border with 1, and ending at the centre triangle at $N_1 + 1$.
The fin. unit. irreps. of $\mathfrak{su}_\mathbb{C}(3)$, and thus also $\mathrm{SU}(3)$ can therefore be classified by the highest weight vector $(N_1, N_2) \in \mathbb{Z}^2_{\ge 0}$.

Chapter 5

Classifications of unit. irreps. of Spacetime Symmetry Groups

5.1 Compact Groups

5.1.1 $\mathbb{R}^4$

We know $\mathbb{R}^4$ to be Abelian, so from 2.2 we know that any unit. irreps. of $\mathbb{R}^4$ must be one dimensional. From the unitary property we find that it must also be a complex exponential. As $\mathbb{R}^4$ is a vector space with 4 basis vectors, we can write:

$$\rho(r) = e^{it(r)} = e^{i(p_0r_0+p_1r_1+p_2r_2+p_3r_3)=e^{ip\cdot r}}$$

As for the $\mathrm{U}(1)$ case, because its 1-dimensional no irreps. are equivalent, so the unit. irreps are defined by a vector $p \in \mathbb{R}^4$

5.1.2 SO(3)

Lie algebra of SO(3)

Once again, it is more useful to look at the Lie algebra of $\mathrm{SO}(n)$ instead. Let X be a matrix such that $e^{tX} \in \mathrm{SO}(n)$. We get $(e^{tX})^T e^{tX} = 0 \implies tX^T = -tX$, so X must be anti-symmetric. The trace must still be 0 as for $\mathrm{SU}(n)$, but that is the case for all antisymmetric matrices. We thus find:

$$\mathfrak{so}(3) = \{X \in M(n,\mathbb{R})\colon X^T = -X\} = \mathrm{span}(j_1, j_2, j_3) \quad \textit{with}$$

$$j_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad j_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad j_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Representations of $\mathfrak{so}(3)$

We once again find the commutator relations $[j_i, j_j] = \epsilon_{ijk} j_k$. Let ρ be a fin. unit. irrep. of $\mathfrak{so}(3)$ and define $J_i = \rho(j_i)$, with again $[J_i, J_j] - \epsilon ijk J_k$. We can then define $J_0 = iJ_3$ and $J_\pm = J_1 \pm iJ_2$. We then find:

$$[J_+, J_-] = [J_1, J_1] + i[J_1, -J_2] + i[J_2, J_1] - [J_2, J_2] = -2iJ_3 = -2J_0$$

$$[J_0, J_\pm] = i[J_3, J_1] \mp [J_3, J_2] = iJ_2 \pm J_1 = \pm J_\pm$$

Now let u be an eigenvector of J_0 with eigenvalue α. We find that:

$$\begin{aligned} [J_0, J_\pm]u &= J_0 J_\pm u - J_\pm J_0 u = J_0 J_\pm u - \alpha J_\pm u \\ &= \pm J_\pm \implies J_0 J_\pm u = (\alpha \pm 1) J_\pm u \end{aligned}$$

As we can see, this is very similar to the commutator relations we found for $\mathfrak{su}(2)_\mathbb{C}$, so in a similar manner we can find eigenvectors $\{u_k\}$ of J_0 with $J_- u_0 = 0$ and $u_{k+1} = J_+ u_k$. We can again find that $J_- u_k = k(2\lambda + (k-1)u_{k-1}$ with induction as $J_- u_0 = 0$ and:

$$\begin{aligned} [J_+, J_-]u_k &= J_+ J_- u_k - J_- J_+ u_k \\ &= k(2\lambda + (k-1))J_+ u_{k-1} - J_- u_{k+1} \\ &= k(2\lambda + (k-1))u_k - J_- u_{k+1} \\ &= -2J_0 u_k = -2(\lambda + k)u_k \\ J_- u_{k+1} &= (2k\lambda + k^2 - k + 2\lambda + 2k)u_k \\ &= (k+1)(2\lambda + k)u_k \end{aligned}$$

There must again be a K such that $u_K \neq 0 \wedge u_{K+1} = 0$:

$$u_{K+1} = (K+1)(2\lambda + K)u_K = 0 \implies \lambda = \frac{K}{2}$$

Meaning that each such representation corresponds to a half-integer, which outside of equivalence again uniquely defines ρ. Thus the representations of $\mathfrak{so}(3)$, and therefore $\mathrm{SO}(3)$ correspond to the positive half-integers $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

5.2 Non-Compact Groups

5.2.1 Lorentz Group

Lie algebra of the Lorentz group

To define the Lorentz group, we first must define $\eta = \mathrm{diag}(1, -1, -1, -1)$, the metric tensor, which signifies one time dimension (0), followed by three spatial *dimensions (1, 2, 3). We then define the Lorentz group as:*

$$\mathrm{SO}(3,1) = \{\Lambda \in M(4, \mathbb{R}:)\Lambda^T \eta \Lambda = \eta\}$$

Let X be a matrix such that $e^{tX} \in \mathrm{SO}(3,1)$. *We get* $(e^{tX})^T \eta e^{tX} = \eta$, *of which we can take the derivative and evaluate at* 0 *to get:*

$$\begin{aligned} \frac{d}{dt}\left[(e^{tX})^T \eta e^{tX}\right]\Big|_{t=0} &= X^T (e^{tX})^T \eta e^{tX} + (e^{tX})^T \eta X e^{tX}\Big|_{t=0} \\ &= X^T \eta + \eta X = 0 \end{aligned}$$

Here we find that the diagonal must again be equal to 0, with the spatial dimensions forming a anti-symmetric matrix, and all other elements being symmetric. Thus for the spatial part, we find the same matrices j_i *as for* $\mathrm{SO}(3)$, *and finally we get* $\mathfrak{so}(3,1) = \mathrm{span}(j_1, j_2, j_3, k_1, k_2, k_3)$, *where:*

$$k_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} k_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} k_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For which we can check that all of them do indeed lie in $\mathfrak{so}(3,1)$.

Representations of the Lorentz Group

We first find the commutator relations $[k_i, k_j] = -\epsilon_{ijk} j_k$ *and* $[j_i, k_j] = \epsilon_{ijk} k_k$. *Now let* ρ *be a fin. unit. irrep.* $\mathfrak{so}(3,1)$, *and define* $J_i = \rho(j_i)$ *and* $K_i = \rho(k_i)$. *We can further define* $J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$, *which gives us:*

$$\begin{aligned} [J_i^{\pm}, J_j^{\pm}] &= \frac{1}{4}([J_i, J_j] \pm i[K_i, J_j] \pm i[J_i, K_j] - [K_i, K_j]) \\ &= \frac{1}{4}(\epsilon_{ijk} J_k \pm \epsilon_{ijk} K_k \pm \epsilon_{ijk} K_k + \epsilon_{ijk} J_k) \\ &= \frac{1}{2}\epsilon_{ijk}(J_k \pm iK_k) = \epsilon_{ijk} J_k^{\pm} \\ [J_i^{\pm}, J_j^{\pm}] &= \frac{1}{4}([J_i, J_j] \pm i[K_i, J_j] \mp i[J_i, K_j] + [K_i, K_j]) \\ &= \frac{1}{4}(\epsilon_{ijk} J_k \pm \epsilon_{ijk} K_k \mp \epsilon_{ijk} K_k - \epsilon_{ijk} J_k) = 0 \end{aligned}$$

Both the sets $\{J_i^+\}, \{J_i^-\}$ *give us the same commutator relations as for* $\mathfrak{so}(3)$, *while they commute with each other. We can therefore evaluate them separately, and find that these representations correspond to pairs of half-integers.*

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5.2.2 Poincaré Group

The Poincaré group is defined by the semidirect product $\mathrm{SO}^+(3,1) \ltimes \mathbb{R}^4$

Let $\rho(\Lambda, r)$ *be a unit. irreps. of the Poincaré group over* $\mathcal{V}$*, and define:*

$$\tau(r) = \rho(\mathbb{1}, r) \qquad \sigma(\Lambda) = \rho(\Lambda, 0)$$

We should first point out that for any r, Λ *we have:*

$$\sigma(\Lambda)\tau(r) = \rho(\Lambda, 0)\rho(\mathbb{1}, r) = \rho(\Lambda, \Lambda r) = \tau(\Lambda r)\sigma(\Lambda)$$

From 2.10 and 2.2 we can deduce that we can decompose the representation space into subspaces of $\mathcal{V}$ *for which* $\tau(r)$ *is 1-dimensional. As* τ *must be unitary, for each of these subspaces we can write* $\tau(r) = e^{ip\cdot r}$ *for some* $p \in \mathbb{R}^4$*, so we can label these subspaces* $\mathcal{W}_p$ *such that for each* $w_p \in \mathcal{W}_p$ *we have* $\tau(r)w_p = e^{ip\cdot r}w_p$*, where the dot product is defined by* $r_1 \cdot r_2 = r_1 \eta r_2^T$*, we thus find:*

$$\tau(\Lambda r)\sigma(\Lambda)w_p = \sigma(\Lambda)\tau(r)w_p = e^{ip\cdot r}\sigma(\Lambda)w_p = e^{i(\Lambda p)\cdot(\Lambda r)}\sigma(\Lambda)w_p$$

As $(\Lambda p)\cdot(\Lambda r) = p\Lambda\eta\Lambda^T r^T = p\eta r^T$*, so we find* $\sigma(\Lambda)w_p \in \mathcal{W}_{\Lambda p}$*, and thus:*

$$\left.\begin{array}{r}\sigma(\Lambda)\mathcal{W}_p \subseteq \mathcal{W}_{\Lambda p} \\ \sigma(\Lambda)\mathcal{W}_p \supseteq \sigma(\Lambda)\sigma(\Lambda^{-1})\mathcal{W}_{\Lambda p} = \mathcal{W}_{\Lambda p}\end{array}\right\} \implies \sigma(\Lambda)\mathcal{W}_p = \mathcal{W}_{\Lambda p}$$

We'll now pick any $\bar{p} \in \mathrm{SO}(3,1)$*, and define the* ***little group*** *and* ***orbit****:*

$$G_{\bar{p}} = \{\Lambda \in \mathrm{SO}^+(3,1) \colon \Lambda\bar{p} = \bar{p}\} \qquad \mathcal{O}_{\bar{p}} = \{\Lambda\bar{p} \colon \Lambda \in \mathrm{SO}^+(3,1)\}$$

We can define $\overline{\sigma}(\bar{\Lambda}) = \sigma(\bar{\Lambda})$ *as a representation of* $G_{\bar{p}}$*, and find that* $\overline{\sigma}(\mathcal{W}_{\bar{p}}) = \mathcal{W}_{\bar{p}}$*. Now suppose for some* $\overline{\mathcal{W}}_{\bar{p}} \subsetneq \mathcal{W}_{\bar{p}}$ *we have* $\overline{\sigma}(\overline{\mathcal{W}}_{\bar{p}}) = \overline{\sigma}(\overline{\mathcal{W}}_{\bar{p}})$*, then we could define* $\overline{\mathcal{W}}_p = \sigma(\Lambda)\overline{\mathcal{W}}_{\bar{p}}$*, and find that* $\overline{\mathcal{V}} = \bigoplus_{p\in\mathcal{O}_p}\overline{\mathcal{W}}_p \subsetneq \mathcal{V}$ *with* $\rho\overline{\mathcal{V}} = \overline{\mathcal{V}}$*, thus for* ρ *to be irreducible over* $\mathcal{V}$*,* σ *must be irreducible over* $\mathcal{W}_{\bar{p}}$*. For each* $p \in \mathcal{O}_{\bar{p}}$ *we'll then pick an* $L_p \in \mathrm{SO}^+(3,1)$ *such that* $L_p\bar{p} = p$*. We can check that* $L_pG_{\bar{p}}$ *work as left cosets for* $\mathrm{SO}^+(3,1)$ *from* $G_{\bar{p}}$*, thus we find that for all* $\Lambda \in \mathrm{SO}^+(3,1)$ *and* $p \in \mathcal{O}_{\bar{p}}$ *there is some* $\bar{\Lambda} \in G_{\bar{p}}$ *and* $q \in \mathcal{O}_{\bar{p}}$ *for which* $\Lambda = L_q\bar{\Lambda}L_p^{-1}$*. We can then deduce that* $q = L_q\bar{p} = L_q\bar{\Lambda}\bar{p} = \Lambda L_p\bar{p} = \Lambda p$*, giving us* $\bar{\Lambda} = L_{\Lambda p}^{-1}\Lambda L_p$ *and:*

$$\sigma(\Lambda) = \sigma(L_{\Lambda p})\overline{\sigma}(\bar{\Lambda})\sigma(L_p^{-1})$$

Our results should be independent of our choices for L_p*, so we can write:*

$$\rho(\Lambda, p) = S_{\Lambda p}\rho(\bar{\Lambda}, \bar{p})S_p^{-1} = S_{\Lambda p}\overline{\sigma}(\bar{\Lambda})\tau(\bar{p})S_p^{-1}$$

As each $\mathcal{W}_p$ *is disjoint, we can pick their bases separately, making our choice for each* S_p *insignificant to the classification of these representations. Finding a un. irrep. for* ρ *is therefore equivalent to finding a un. irrep. for* $\bar{\sigma}$*.*

Choice for $\bar{p}$

Suppose we have $p, p_0 \in \mathcal{O}_p$ in the same orbit, then with Λ_0 such that $p_0 = \Lambda_0 p$ we naturally have $G_{p_0} = \Lambda_0 G_p \Lambda_0^{-1}$.

$$\begin{aligned}\Lambda_0 G_p \Lambda_0^{-1} &= \{\Lambda_0 \Lambda \Lambda_0^{-1} \in \mathrm{SO}^+(3,1) \colon \Lambda p = p\} \\ &= \{\Lambda \in \mathrm{SO}^+(3,1) \colon \Lambda_0^{-1} \Lambda \Lambda_0 p = p\} \\ &= \{\Lambda \in \mathrm{SO}^+(3,1) \colon \Lambda p_0 = \Lambda_0 p = p_0 = G_{p_0}\}\end{aligned}$$

*We should first classify each of the orbits, and as the inner product $p \cdot p$ is conserved for each $\Lambda \in \mathrm{SO}^+(3,1)$ we can define the **mass** m of p as $m^2 = p \cdot p$ and conclude that mass is conserved within an orbit.*

For $m^2 > 0$, we find two different kinds of orbit, being those with either negative or positive p_0. We see that either $(\pm m, 0, 0, 0)$ must exist in this orbit, both with little group $\mathrm{SO}(3)$. However, only the positive mass is physically interesting.

For $m^2 < 0$, we have a single orbit, which contains $(0, m, 0, 0)$. Its little group conserves a single spatial direction, so it must be $\mathrm{SO}^+(2,1)$. However, as these aren't physically interesting either, I won't delve further into these representations.

For $m = 0$, we firstly of course have the trivial orbit $\mathcal{O}_0 = \{0\}$, with little group $\mathrm{SO}^+(3,1)$, which isn't all too interesting. However, we also have another pair of orbits, again with negative or positive p_0, which then contain $\pm(1,1,0,0)$. We are again only interested in the positive variant. Its little group can freely rotate in two directions, but it can also freely boost the remaining space direction and time, making its little group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$.

Massive representations

We can pick $\bar{p} = (m, 0, 0, 0)$ with little group $G_{\bar{p}} \equiv \mathrm{SO}(3)$. As we've already covered this group before we can deduce that these representations are classified by their mass $m > 0$ and the spin in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ from the Little group.

Massless representations

We can now pick $\bar{p} = (1, 1, 0, 0)$ as our massless momentum, and find the little group $G_{\bar{p}} = SO(2) \ltimes \mathbb{R}^2$. We can use the same tactic as for the Poincaré group here, however with the much easier to handle group $\mathrm{SO}(2)$ instead of $\mathrm{SO}^+(3,1)$.

Let $\rho(R, r)$ be a unit. irreps. of $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ over $\mathcal{V}$, and define:

$$\tau(r) = \rho(\mathbb{1}, r) \qquad \sigma(R) = \rho(R, 0) \implies \sigma(R)\tau(r) = \tau(rR)\sigma(R)$$

We can again find subspaces $\mathcal{W}_p \ni w_p$ in which $\tau(r)w_p = e^{ip\cdot r}w_p$. And with the same reasoning, we can start with picking a momentum $\bar{p} \in \mathbb{R}^2$ with little group $G_{\bar{p}}$, and orbit $\mathcal{O}_{\bar{p}}$, where $p, \bar{p} \in \mathcal{O}_{\bar{p}}$ again give the same representation. For any momentum $p \neq 0$, we get the little group $G_{\bar{p}} = \{\mathbb{1}_2\}$. This would induce the 1-dimensional representation with label $p \in \mathbb{R}^2 \setminus \{0\}$ here, and the label (0, p) for the Poincaré group itself. As this has no physical relevance, we'll instead pick $\bar{p} = 0$ with little group $G_{\bar{p}} = \mathrm{SO}(2)$ itself. As $\mathrm{SO}(2)$ and $\mathrm{U}(1)$ are isomorphic, it representations are equivalent. We can thus label them by any integer in $h \in \mathbb{Z}$ though we usually instead use the half-integers $\frac{1}{2}\mathbb{Z}$, which are of course isomorphic. The representations of the entire Poincaré group of this form can be thus labelled by the pair $(0, h)$.

Chapter 6

Classifications of Particles within the Fundamental Forces

As we know, the way the fundamental forces work on particles is quite different for different particles. However, if we group these particles properly, we can describe them for each group separately. Using the classifications of fin. unit. irreps. of the symmetry groups of the internal fundamental forces, we can understand them and the way they work on different classes of particles quite well.

6.1 Electromagnetic Force

The electromagnetic force is in its entirety defined by the charge of a particle. In the standard model, the charge always corresponds with a third of an integer. The reason that this is not an integer, however lies more in the human interpretation than in physics itself, as the unit for charge we use comes from the charge of a proton and electron. The proton however, must always consist of three quarks, each of which has a charge of 1/3.

As we've seen, the fin. unit. irreps of $\mathrm{U}(1)$ *can be classified with* $\mathbb{Z}$*. It is therefore a nice fit to classify particles with when it comes to the electromagnetic force. For this, we use the notation* $\mathrm{U}(1)_{EM}$*. As* $\mathrm{U}(1)$ *is related to the phase of a particle, it is logical that we see the representation class* $q \in \mathbb{Z}$ *as the charge of the particle. Though this is a nice way to explain charge on a simpler scale, as we'll see this is a simplified way to look at it.*

6.2 Weak Nuclear Force

The weak nuclear force is responsible for a transformation of a particle. It can for example transform an up quark into a down quark, or a neutrino into an electron. It is therefore useful to view these particles not as separate, but as a set of particles, which we call ***multiplets****, where we assign their proper name (*___singlet___*,* ___doublet___*,* ___triplet___*, etc.) according to the amount of particles in the set. The main difference between particles in such a set, is their charge, which differ by 1 each. For example, the up-, down-quark doublet has charges* $+1/3, -2/3$*, and the* $W^{\pm}, Z_0$ *bosons with charges* $\pm 1, 0$*. This already looks a lot like the representations we've found for* $\mathrm{SU}(2)$*. The dimension of the representation then corresponds with the amount of particles in the multiplet, and the constant difference in charges would come from the constant difference in the eigenvalues.*

The unit. irreps. of $\mathrm{SU}(2)$ *can be classified with* $\mathbb{Z}_{\geq 0}$*, the positive integers, though in this case it's more useful to classify them* $\frac{1}{2}\mathbb{Z}_{\geq 0}$*. A particle belong to a multiplet with n particles, would then have a* ***weak isospin*** *of* $\frac{n-1}{2}$*, with the factor* $\frac{1}{2}$ *of course coming from the adjustment to get half-integers. In this manner, we can directly correlated the difference in charges to the difference in eigenvalues, meaning every particle in the multiplet belongs to one eigenvector. However, with this way of looking at the weak force, there is no use classifying it separately, as it has an influence on the charge.*

Electroweak Property

To solve the problem we found for the weak nuclear force, instead of classifying it on its own using $\mathrm{SU}(2)$*, we instead classify it together with* $\mathrm{U}(1)$*, using the hypercharge.*

Where before we used $\mathrm{U}(1)_{EM}$ *to define the charge of a particle directly, we now use* $\mathrm{U}(1)_Y$ *to define a hypercharge of a multiplet. As we've seen, the charge within such a multiplet changes only based on the eigenvalue related to a specific particle. This means that we can define the entire charge Q as the sum of this eigenvalue and some other value Y, which we then call the* ***hypercharge****. We can then write* $Q = \frac{Y}{2} + J$*, where J is the eigenvalue belonging to that specific particle. As multiplets only contain right-handed variants of particles, we denote this complete set by* $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$*.*

6.3 Strong Nuclear Force

Some particles can never be found alone, they are always accompanied by other particles. This is due to the strong force, a force that bounds particles together.

Though we cannot measure it, we understand what happens within this bundles using the concept of ***colour charge****. It comes in the colours red, green and blue* ($\mathbf{r}$, $\mathbf{g}$, $\mathbf{b}$)*, and their anti-variants* ($\bar{\mathbf{r}}$, $\bar{\mathbf{g}}$, $\bar{\mathbf{b}}$)*, and these bundles of particles can only exist if their total colour charge is neutral. The reason we use this notation is not because it has anything to do with colour, but because of what happens when you combine them. Adding all colours* $\mathbf{r}$, $\mathbf{g}$ *and* $\mathbf{b}$ *gives us colour neutrality, just like with light, and adding a colour with its anti-variant also gives colour neutrality. Other than that, we are just looking at the sum of all colour charges within a bundle of particles. The gluons are the force carriers of the strong nuclear force, and carry a charge of one colour and one anti-colour. Therefore, by absorbing or exerting a gluon, a particle can change one of its positive- or negative colours for another. We can therefore classify sets of particles with the same amount of colour and anti-colours, which can turn into each other by exertion or absorption of gluons. There then are at most 12 such transformations that can be made at any point, depending on which colours a particle contains. For the colour charge, swapping two colours, or the opposite anti-colours, is effectively the same, leaving us with at most 6 changes. The way in which we the colour charge changes, corresponds perfectly with the fin. unit. irreps. of* $\mathrm{SU}(3)$.

We've found that the unit. irreps. of $\mathrm{SU}(3)$ *can be classified by a pair of integers* $(N_1, N_2) \in \mathbb{Z}^2$*, which represent the highest weight of the representations. These values* N_1, N_2 *in physics represent the total of colour- and anti-colour states respectively. The most simple examples (except for the trivial one), are those with highest weights (1, 0) and (0, 1), which in physics represent the quarks and respectively anti-quarks. Their representations are shown in the figures below.*

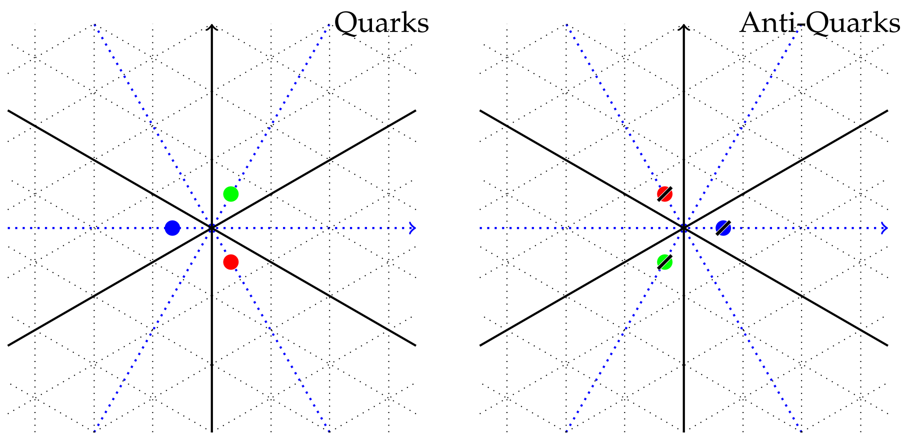

Figure 6.1: *The irreducible representations corresponding to the quark and anti-quark respectively. Both have 3 flavours, naturally for colours and anti-colours respectively.*

We can understand each eigenvector of the representation as a different combination of colours and anti-colours. As adding a colour has the same result as adding the other two anti-colours, the Netto colour of a particle can therefore be represented in a 2D-graph, just as we found for the representations of $\mathrm{SU}(3)$*. We can thus see the weight as a Netto colour of a particle, and the eigenvector itself as a specific orderless combination of colours. This also explains why near the border the multiplicity is low, as there is already a strict requirement by the Netto colour, while near the centre there is barely any requirement. A simple example with higher multiplicity is the meson octet. These particles consists of a quark and anti-quark, so they have one colour and one anti-colour, making their highest weight representation the (1, 1) representation.*

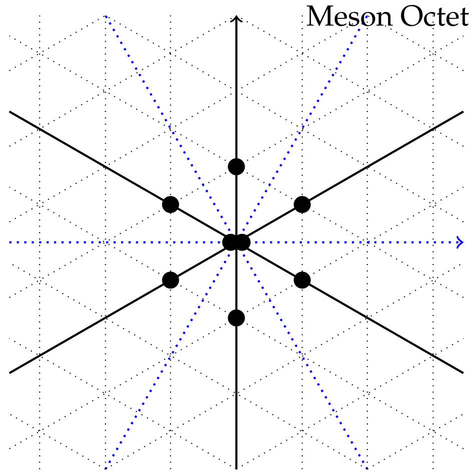

Figure 6.2: *The irreducible representations corresponding to the meson octet. The 6 outer weights represent states with differing colour and anti-colour. The inner two weights are the neutral states with same colour and anti-colour.*

We can thus classify the sets of particles that can transform into each other by gluons by their amount of colours and anti-colours $(N_1, N_2) \in \mathbb{Z}^2_{\geq 0}$ *as a representation of* $\mathrm{SU}(3)_C$*. The eigenvalues and -vectors again have meaning, as the eigenvalues represent the different colour charges, and the eigenvalues a unique combination of colours and anti-colours.*

Chapter 7

Remaining Properties of a Particle

Though we have classified the particles when it comes to the three fundamental forces we know best, there are still properties of a particle outside of these classifications, those being the spin quantum number and the mass. The mass is one of the most important properties of a particle, as on any scale it scales the force applied to the acceleration, is crucial in the theory of general relativity and gravity in its entirety, and of course tells us a lot about the energy and stability of a particle. The spin quantum number however, is a much more complicated property, as it defines the manner in which it interacts with spacetime and quantum fields. For both properties, the unitary irreducible representations of the Poincaré group can give interesting insights.

7.1 Poincaré Group ($SO^+(3,1) \ltimes \mathbb{R}^4$)

For the Poincaré group, the way in which we find the unitary irreducible representations tell us a lot about how they work. Most importantly, the process starts with determining the mass, where in physics we only care about the real non-negative masses. For a positive mass, we find that we do indeed get a half-integer spin as our final property. For massless particles, we get the interesting result that having a negative 'spin' is also allowed. It is actually more appropriate to talk about helicity here instead of spin, as spin is best defined from the rest frame, which massless particles do not have. Helicity however, is dependent on the velocity of a vector, and as a massless vector cannot be boosted it is well defined for these particles. The helicity is the projection of the spin along the direction of motion. This means that for positive helicity, which we call right-handed, the spin and momentum share the same direction. We then naturally call negative helicity left-handed, of course when spin and momentum lie in opposite directions. In op-

pose to massive particles, we cannot boost the reference frame in such a way that the momentum turns around without the spin doing so, meaning that the sign of the helicity for a particle is fixed. We can however still define its spin by taking the absolute value of the helicity.

7.2 The definition of a particle

In this thesis, the internal symmetry groups, and the spacetime symmetry groups, were researched to give us insights into the classifications of particles. The internal symmetry groups were used to classify the internal workings of particles by looking at their finite unitary irreducible representations, and classifying them. It concluded that the entirety of the inner workings of these particles could be classified using representations of the product of these Lie-groups, where the first and special second unitary group were used for the electroweak forces, and the special third unitary group for the strong nuclear force.

The Poincaré group was afterwards used to get an insight into the remaining properties of a particle, the spin quantum number and the mass. The unitary reducible representations of the Poincaré can here be used to define the mass and spin quantum number of a particle.

Chapter 8

Conclusion

8.1 Fundamental Forces and Symmetries

This thesis aimed to classify particles in the standard model by the way the fundamental forces act on them, with the use of their respective symmetry groups. The symmetry groups of these forces were found to be the 1-dimensional unitary group $\mathrm{U}(1)$ *for the electromagnetic force, the 2-dimensional special unitary group* $\mathrm{SU}(2)$ *for the weak nuclear force, and the 3-dimensional special unitary group for the strong nuclear force. This thesis also researched the spacetime symmetry group to get a further insight into the remaining properties of a particle and their relativistic properties, which was found to be the Poincaré group* $\mathrm{SO}(3,1) \ltimes \mathbb{R}^4$.

8.2 Classifications of Particles

This thesis went on to classify the unitary irreducible representations of these symmetry groups, with the additional requirement of finiteness for the unitary groups, to prevent infinite sets of particles. As the irreducible representations of complex Lie algebra's can often be more easily classified than those of Lie-groups, while still having a one to one relation between these representations, this thesis researches the finite unitary irreducible representations of the Lie algebras $\mathfrak{su}(2)_{\mathbb{C}}$ *and* $\mathfrak{su}(3)_{\mathbb{C}}$ *instead.*

For $\mathrm{U}(1)$, *we found that these representations can be classified with an integer* $q \in \mathbb{Z}$, *with which we could indicate the charge of a particle, though we often instead use third integers to indicate charge. For* $\mathrm{SU}(2)$ *we could indicate these representations with a positive integer, where we often instead use half-integers* $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ *to more accurately indicate the weak isospin. The dimension of this representation* $2n+1$ *indicates an amount of particles in a multiplet that can* *transform into each other by the weak nuclear force. By diagonalizing the matrix* $H = \rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$*, we can indicate the different particles in the multiplet by its eigenvectors. As different particles in such a multiplet have different charges, we classify particles by the effect of these forces together. The effect of the particle on the charge is linearly related to the eigenvalue of its respective eigenvector, and each multiplet has a hypercharge to indicate the remaining charge component. We can therefore classify particles by the effect of the electromagnetic- and weak nuclear force together as a finite unitary irreducible representation of:*

$$\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$$

For $\mathrm{SU}(3)$ *we found that its finite unitary irreducible representations can be classified with a pair of positive integers* $(N_1, N_2) \in \mathbb{Z}^2_{\geq 0}$*, with which we could indicate colour/anti-colour combinations of particles. To achieve this, the knowledge of representations of* $\mathrm{SU}(2)$ *was utilized, as they have similar structures. Because of the blindness of the colour properties of a particle, it is appropriate to classify particles by the effect of the strong nuclear force on their colour/anti-colour combinations, as we often have no deeper information of the particles. We therefore classify particles on the effect from the strong nuclear force by:*

$$\mathrm{SU}(3)_S$$

Lastly, the unitary irreducible representations of the Poincaré group were found to best first be classified by their mass, which for physically relevant combinations can be taken to be non-negative. Massive particles can further be classified by a non-negative half-integer, which can be tied to the spin quantum number. For massless particles, the particles can further be classified by any half-integer, which can be tied to the helicity of a particle. As the positive and negative helicity only differ by the left- or right-handedness of a particle, they are usually classified together by their absolute value, which we often also denote with spin to simplify the standard model. The Poincaré group therefore allows us to classify particles by a non-negative mass and non-negative half-integer spin. A particle in the standard model can thus be described by a unitary reducible representation of:

$$\mathrm{SO}(3,1) \ltimes \mathbb{R}^4 \times \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$

Appendix A

Fundamental forces of a composite system

To confirm that we're working with proper classifications of these properties of the particles, it is interesting to see whether these classifications hold up in composite systems. We can do that using the definition from 2.2, and seeing whether the result lines up with our what we would expect in physics.

A.1 Charge

Checking whether the composite charge maintains the properties we would expect is not too difficult, as the product for 1-dimensional representations can be done by simple taking the $\mathbb{R}$*-ring product. As we are working in exponentials, for two charges* q_1, q_2 *we find* $e^{iq_1\theta}e^{iq_2\theta} = e^{i(q_1+q_2)\theta}$*, so we do indeed find the composite charge* $q_1 + q_2$*, as we would expect.*

A.2 Weak isospin

Let ρ_m, ρ_n *be two fin. unit. irreps. of* $\mathrm{SU}(2)$ *of dimension* $m+1, n+1$ *respectively, and let* π_m, π_n *be their corresponding representations of* $\mathfrak{su}(3)_\mathbb{C}$*. We can now define* $H_i = \pi_i(h)$*. Now let* $\{u_{-m}, \dots, u_{m-2}, u_m\}$ *and* $\{v_{-n}, \dots, v_{n-2}, v_n\}$ *be eigenvectors for* H_1, H_2 *respectively, where* u_i, v_j *have eigenvalues* i, j*. We find:*

$$(\pi_m(h) \otimes \pi_n(h))(v_i \otimes v_j) = H_m v_i \otimes v_j + v_i \otimes H_n v_j = (i+j) v_i \otimes v_j$$

Let π *be the Algebra representation corresponding to* $\rho = \rho_1 \otimes \rho_2$*, then the dimension of eigenspaces of* $H = \pi(h)$ *with eigenvalue* k *are given by eigenvalue* *pairs (i,j) for which $i+j=k$. We first see that there is one eigenvector with eigenvalue $\pm(m+n)$, then two with eigenvalue $\pm(m+n-2)$, with eigenspace increasing in dimension until the eigenvalue $\pm|m-n|$, at which point the dimension stays fixed. We can thus conclude there is one subspace of dimension $m+n+1$. From the remaining eigenspaces, we can similarly conclude there is a subspace of dimension $m+n-1$, and so forth. We can thus reduce representations with subspaces $\mathcal{V}_k$ corresponding to dimension $k+1$ as:*

$$\mathcal{V}_m \otimes \mathcal{V}_n = \mathcal{V}_{m+n} \oplus \mathcal{V}_{m+n-2} \oplus \cdots \oplus \mathcal{V}_{|m-n|}$$

After applying the same half-integer adjustment to the weak isospin, this does indeed confirm what we know of physics that two spins:

$$|i\rangle \otimes |j\rangle = \bigoplus_{k=|i-j|}^{i+j} |k\rangle$$

Where we can use this notation as the spins differ by 1 instead of 2.

A.3 Colour

As the combining of colour multiplets can become very complicated for larger representations, it's best to only look at the smaller cases here. The principle however stays the same as for the weak isospin case: For the weights furthest from the centre there is only one way to obtain those colour combinations from combining two eigenvectors. For the weights closest to the centre there are multiple, but only a single of these eigenvectors belongs to its own highest weight representation, where the others belong to representations with higher weights.

An example of this is the product of the quarks and anti-quarks. As we expect, the product contains particles with a colour and an anti-colour, these of course being the meson octet. This almost adds up, except for that we have 3 eigenvectors in the centre. We however know that the combination of $r\bar{r}, g\bar{g}$ and $b\bar{b}$ gives us the neutral singlet. We denote this by writing $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

In general we find that, similarly to the weak force case, we can first find a maximal representation. The weights for which we have eigenvectors left, we can deduce that their respective colour combinations can be formed in multiple ways depending on from which representations you pick a certain colour/anti-colour. These representations must thus have at least a same-colour pair, thus reducing the freedom of colours and anti-colours by at least one. We can thus start with representations with maximal sum of colours and anti-colours, and then find other representations recuresively, which must decrease into the centre.