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The maximum number of lines on a smooth quartic surface

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Abstract. In this thesis, we take a close look at a theorem from Rams and Schütt [20] which states that a smooth quartic surface in 3-dimensional projective space over an algebraically closed field of characteristic not equal to 2 or 3 contains at most 64 lines. In Chapter 1, we cover the well-known theorem that smooth cubic surfaces contain exactly 27 lines. In Chapter 2, we present a basic introduction into elliptic curves and elliptic surfaces. In Chapter 3, we follow Rams' and Schütt's proof to show the aforementioned theorem while providing additional details and presenting computations that they omitted. We improve on one of the intermediate results in [20] and show that if two inflectious lines on a smooth quartic surface intersect, the surface is projectively equivalent to the Schur quartic.

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Contents

Algebraic curves were created by God,
algebraic surfaces by the Devil.

ATTRIBUTED TO MAX NOETHER

0 Introduction

It has been known since the 19^{th} century that a smooth cubic surface in 3-dimensional projective space over an algebraically closed field contains exactly 27 lines, and each of these lines intersects precisely 10 others. According to George Salmon ([22], p.496), this was first proven in 1849 in correspondence between him and Arthur Cayley.

Over the course of the following years, much effort has been put into studying the configurations of these lines, finding symmetries and examples. Notable examples include the Fermat cubic surface, given by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

in $\mathbb{P}^3$, which we will study in Chapter 1, and the Clebsch surface, which can be given in $\mathbb{P}^4$ by the equations

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 + x_4 &= 0 \\ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 &= 0 \end{aligned}$$

and is isomorphic to a cubic surface in $\mathbb{P}^3$ whose 27 lines can all be defined over the number field $\mathbb{Q}(\sqrt{5})$ (see [2, 11, 13]).

A natural question that arises from this result is how far it can be generalised. While general surfaces of degrees higher than 3 do not contain any lines, there are examples of surfaces that do, and in these cases one may still be interested in possible numbers and configurations of lines.

In 1943, Beniamino Segre [26] proved that a smooth surface of degree $d \geq 0$ in $\mathbb{P}^3$ can contain at most $(d-2)(11d-6)$ lines in characteristic 0. However, this bound is far from optimal. For $d = 4$, this gives 80, but as we will soon see the maximum number of lines on a smooth quartic surface is actually 64.

Segre also claimed to have proven this maximum for quartic surfaces in the same paper [26]. However, his proof relied on the claim that any line on such a surface intersects at most 18 other lines. As Sławomir Rams and Matthias Schütt [21] showed in 2014, this claim is false, and in fact there are examples of quartic surfaces which contain a line that intersect as many as 20 other lines.

Segre's end result on quartic surfaces is nonetheless correct, as Rams and Schütt [20] also proved in 2015 in all characteristics except 2 and 3.

In this thesis, we will explore both the cubic and quartic situation.

In the cubic case, we will follow the proof from [5]. We first show that any cubic surface contains a line. We will do this by defining the set Σ which consists of pairs of a cubic surface S and a line L such that $L \subset S$ and showing that the projection from Σ to the first component is surjective with the help of a dimension argument.

We then use the existence of a line to examine its possible intersections with other lines. The key observation is that a given line $L \subset S$ gives rise to a morphism $\pi\colon S \to \mathbb{P}^1$ whose fibres are precisely the conics residual to L in intersections of S with planes containing L. Any line intersecting L must then occur in a fibre of π.

Since fibres of π are plane conics, they are either smooth or consist of two lines. We then go on to show that π has exactly five singular fibres, consisting of ten lines, all of which intersect L. With this observation, we can then show that the surface S must contain exactly 27 lines.

In order to treat the quartic case, we will follow the arguments from [20] which are based upon a similar method. In this case, after assuming that a given smooth quartic surface S contains a line L, the residual curves in the fibres of the corresponding morphism π are no longer conics, but plane cubic curves. This will allow us to make use of the theory of elliptic curves and elliptic surfaces.

An elliptic curve is a smooth curve E of genus 1 which contains a designated point O. Such an elliptic curve admits a group structure with O as its neutral element. Furthermore, the surface S together with the morphism π is a so called elliptic surface.

In Chapter 2, we will explore some general definitions and theorems about these objects that will help us understand lines on quartic surfaces. Our main source on elliptic curves will be [29], and for elliptic surfaces we will draw from [18, 23, 25, 28].

One of the central aspects of elliptic surfaces that play a role in examining lines on quartic surfaces is the theory of fibre types. Almost all fibres of an elliptic fibration are smooth, but some are singular. The singular fibres come in different types which have been classified by Kunihiko Kodaira [14, 15, 16] and André Néron [19]. Similarly to the situation on cubic surfaces, these singular fibres must contain all the lines intersecting L.

In Chapter 3, we will prove that any smooth quartic surface over an algebraically closed field of characteristic not equal to 2 or 3 contains at most 64 lines. We will closely follow the method from [20] while filling in missing details and omitted computations.

We first present the Schur quartic as an example of a smooth quartic that attains the maximum of 64 lines.

In order to prove the theorem, we need to distinguish between regular and inflectious lines, which Segre [26] and Rams and Schütt [20] call lines of the first and second kind, respectively. A line $L \subset S$ is called inflectious if every point of intersection of L with a fibre F of the morphism $\pi: S \to \mathbb{P}^1$ associated to L is a point of inflection of F. A regular line is a line that is not inflectious. A priori, this differs slightly from the definition in [20], but we will see that the definitions are equivalent.

If L is regular, similar logic as in Chapter 1 can be used to prove that it intersects at most 18 other lines on S, where we provide computations that were omitted in [20]. We can then show the main theorem in the case that all lines on S are regular.

If L is inflectious, more work is needed. First we examine which Kodaira types can occur in fibres of π in this case, which depends on the ramification behaviour of the curve morphism $\pi|_L : L \to \mathbb{P}^1$. We then consider the global ramification behaviour of this map, and show that L intersects more than 18 lines on S if and only the map ramifies at exactly two points.

This is the only case that differs significantly from the case where L is regular. From this condition we can derive a simplified equation for S of the form

$$x_2x_0^3 + x_3x_1^3 + x_0x_1q(x_2,x_3) + g(x_2,x_3) = 0$$

where q and g are homogeneous polynomials of degrees 2 and 4, respectively. In Proposition 3.31, we improve on an intermediate result from [20] and show that if two inflectious lines intersect on a surface of this form, it is projectively equivalent to the Schur quartic. We also provide additional computations that were omitted in [20] in Lemma 3.35. We then go on to prove that the maximum of 64 lines still persists in this case.

1 Cubic Surfaces

In this chapter we will study lines on cubic surfaces. Let k be an algebraically closed field not of characteristic 2 or 3. As was mentioned in the introduction, we want to prove the well known theorem that every smooth cubic surface in $\mathbb{P}^3(k)$ contains precisely 27 lines.

Some authors require curves and surfaces to always be irreducible. For the purposes of this thesis, both the term 'curve' as well as the term 'surface' will include reducible instances, unless specifically mentioned otherwise.

Before dealing with general surfaces, it is worth looking into an example.

1.1 Example. The Fermat cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

has 27 lines.

Proof. Let X denote the surface in question. First note that X is smooth because the partial derivatives of $x_0^3 + x_1^3 + x_2^3 + x_3^3$ never simultaneously vanish. Now let $L \subset \mathbb{P}^3$ be a line. Then we can write

$$L = Z(x_0 - (a_2x_2 + a_3x_3), x_1 - (b_2x_2 + b_3x_3))$$

up to some permutation of coordinates. Now $L \subset X$ if and only if for all $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3$, if $x_0 = a_2x_2 + a_3x_3$ and $x_1 = b_2x_2 + b_3x_3$, then $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Substituting the first two equations into the third one gives

$$\begin{aligned} 0 &= (a_2x_2 + a_3x_3)^3 + (b_2x_2 + b_3x_3)^3 + x_2^3 + x_3^3 \\ &= (a_2^3 + b_2^3 + 1)x_2^3 + (3a_2^3a_3 + 3b_2^2b_3)x_2^2x_3 + (3a_2a_3^2 + 3b_2b_3^2)x_2x_3^2 + (a_3^3 + b_3^3 + 1)x_3^3 \end{aligned}$$

which is the case for all x_2, x_3 if and only if all the coefficients of this polynomial expression vanish. So we get equations

$$a_2^3 + b_2^3 = -1 \tag{1.1}$$
$$a_2^2 a_3 = -b_2^2 b_3 \tag{1.2}$$
$$a_2 a_3^2 = -b_2 b_3^2 \tag{1.3}$$
$$a_3^3 + b_3^3 = -1. \tag{1.4}$$

Now we claim that a_2, a_3, b_2 and b_3 can never be simultaneously non-zero: Indeed, if that were the case, then squaring Equation (1.2) and dividing by Equation (1.3) would give $a_2^3 = -b_2^3$, which contradicts Equation (1.1). So we can assume that $a_2 = 0$ (all other cases will be covered by permutations of coordinates). This gives us equations

$$b_2^3 = -1 \tag{1.5}$$
$$b_2^2 b_3 = 0 \tag{1.6}$$
$$b_2 b_3^2 = 0 \tag{1.7}$$
$$a_3^3 + b_3^3 = -1 \tag{1.8}$$

Now from Equation (1.5) we can conclude that $b_2 = \zeta^i$ for a primitive third root of unity ζ and $i \in \{0,1,2\}$. In particular, $b_2 \neq 0$ and so Equation (1.6) gives us $b_3 = 0$, after which Equation (1.8) becomes $a_3^3 = -1$, so $a_3 = \zeta^j$ for some $j \in \{0,1,2\}$. So we obtain nine solutions to these equations, corresponding to the lines

$$L_{ij} = Z(x_0 + \zeta^j x_3, x_1 + \zeta^i x_2), 0 \le i, j \le 2.$$

We can parametrise the line L_{ij} as

$$\left\{(-\zeta^j y : -\zeta^i x : x : y) : (x : y) \in \mathbb{P}^1\right\}.$$

Note that swapping coordinates x_0 and x_3 is the same as replacing j by $3-j$, and swapping x_1 and x_2 is the same as replacing i by $3-i$. Furthermore, swapping the pair (x_0, x_3) with the pair (x_1, x_2) is the same as swapping i and j. Hence, permutations that return a different set of nine lines are precisely the partitions of (x_0, x_1, x_2, x_3) into two groups of two, of which there are three. Therefore, we have precisely $3 \cdot 9 = 27$ lines, whose parametrisations are given by

$$\begin{aligned}
L_{ij} &= \left\{(-\zeta^j y : -\zeta^i x : x : y) : (x:y) \in \mathbb{P}^1\right\} \\
L'_{ij} &= \left\{(-\zeta^j y : -\zeta^i x : y : x) : (x:y) \in \mathbb{P}^1\right\} \\
L''_{ij} &= \left\{(-\zeta^j y : y : -\zeta^i x : x) : (x:y) \in \mathbb{P}^1\right\}
\end{aligned}$$

□

The parametrisations from this proof are very useful in determining the intersections of those lines. First, looking at lines from the same group of nine, we see that the lines L_{ij} and $L_{i'j'}$ intersect if and only if $i = i'$ (at the point $(0 : -\zeta^i : 1 : 0)$) or $j = j'$ (at the point $(-\zeta^j : 0 : 0 : 1)$). So every line of the form L_{ij} intersects precisely four of the other lines of the form L_{ij}. Furthermore, by fixing one i, we can see that the three lines L_{ij} for $j = 0,1,2$ all pass through the same point $(0 : -\zeta^i : 1 : 0)$, and the same holds if we fix a j.

Now if we again fix a line L_{ij}, it will intersect three lines each from the other two groups. To see this, first note that for L_{ij} and $L'_{i'j'}$ to intersect at a point P, all four coordinates must be nonzero. Hence, in the parametrisation for L_{ij}, we can set $y = 1$ to get

$$P = (-\zeta^j : -\zeta^i x : x : 1)$$

for some $x \in k$. Now assuming P also lies on $L'_{i'j'}$, knowing that the fourth coordinate is 1, we can write

$$P = (-\zeta^{j'} y : -\zeta^{i'} : y : 1)$$

for some $y \in k^\times$. Now since $-\zeta^i x = -\zeta^{i'}$, we find that

$$x = \zeta^{i'-i}.$$

Similarly, since $-\zeta^{j'} y = -\zeta^j$, we have

$$y = \zeta^{j-j'}.$$

But since x and y have to be equal, this point of intersection exists if and only if

$$j - j' \equiv i' - i \pmod 3$$

or equivalently

$$i + j \equiv i' + j' \pmod{3}.$$

So for every fixed index pair (i, j), we find three index pairs (i', j') such that L_{ij} and $L'_{i'j'}$ intersect, and the point of intersection is given by

$$(-\zeta^j : -\zeta^{i'} : \zeta^k : 1)$$

where $k \equiv i' - i \equiv j - j' \pmod{3}$. This works analogously if we replace L with L' or L'' or vice versa.

So to summarise, each line on the Fermat cubic surface intersects exactly ten other lines. This is no coincidence: we will soon see that the same thing holds for any cubic surface. However, we also noticed some points where three lines intersect. To be precise, there are six such points among the lines L_{ij}, and six each for L'_{ij} and L''_{ij} as well, so a total of 18. These points are called Eckardt points, and are generally very rare [1.a]. Roughly speaking, the ten lines intersecting a given line L on a smooth surface S come in pairs, each of which forms a triangle with L, and an Eckardt point is a degeneration of such a triangle.

It is also worth noting that while the Fermat cubic is defined over $\mathbb{Q}$, the same can be said for only three out of the 27 lines on it, namely the lines L_{00}, L'_{00} and L''_{00}. So the assumption that k is algebraically closed is in general necessary to find all lines on a cubic surface.

Before we get into the construction of the 27 lines on a cubic surface in general, it is worth noting that while planes and quadratic surfaces are ruled surfaces and can be completely covered with lines, general algebraic surfaces of degree higher than 3 contain no lines at all, as we will see in Remark 1.3 at the end of the next section. Cubic surfaces therefore represent a special case in between. In contrast to the lower degree cases, it is not at all trivial that every cubic surface does indeed contain a line. We will present a proof thereof in the next section.

Both the proof of the existence of a line and the subsequent proof that there exactly 27 lines on a smooth cubic are based primarily on lecture notes 'Algebraic Geometry' by Edixhoven et al[5].

[1.a]In fact, 18 is the highest possible number of Eckardt points on a smooth cubic surface, as we will prove in Proposition 1.14 at the end of this chapter. Furthermore, most cubic surfaces do not have any Eckardt points. Indeed, the set of cubic surfaces with Eckardt points has codimension 1 within the set of all cubic surfaces (see [4] p.440).

1.1 EXISTENCE OF A LINE

In this section, we will prove the following proposition.

1.2 Proposition. *Every cubic surface in $\mathbb{P}^3$ contains a line.*

Note that we do not assume smoothness here. While singular cubics need not contain exactly 27 lines, they do always contain at least one.

In order to prove Proposition 1.2, we first need to introduce Grassmannians. The goal is to equip both the set of cubic surfaces and the set of lines in $\mathbb{P}^3$ with the structure of a projective variety. The former case will be quite straight-forward. As for the lines, we note that lines in $\mathbb{P}^1$ are in canonical bijection with 2-dimensional subspaces of k^4. Now let $\mathrm{Gr}(m,n)$ be the set of m-dimensional subspaces of k^n. An element $V \in \mathrm{Gr}(m,n)$ can be represented by a matrix in $k^{n\times m}$ of rank m, with the columns forming a basis of V. The choice of such matrix is not unique; two matrices A,B define the same subspace if and only if there is an invertible matrix $T \in \mathrm{GL}_n(k)$ such that $A = BT$. Now for every index set $I = \{i_1,\dots,i_m\}$ with $1 \le i_1 < \cdots < i_m \le n$ and matrix $A \in k^{m\times n}$, we define A_I to be the $m\times m$ submatrix of A defined by the rows of indices $i_1,\dots,i_m$. Considering the set of A such that A_I is invertible, we note that this set is invariant under right multiplication with elements of $\mathrm{GL}_m(k)$, so it defines a subset

$$U_I \subset \mathrm{Gr}(m,n).$$

Furthermore, every $m\times n$ matrix of rank m has an invertible $m\times m$ submatrix, so these sets U_I cover $\mathrm{Gr}(m,n)$. Now we can define maps

$$\varphi_I : U_I \to \mathbb{A}^{m(n-m)}, A \mapsto (AA_I^{-1})_{\{1,\dots,n\}\smallsetminus I}.$$

This definition is again invariant under right multiplication by invertible matrices. Indeed, for $T \in \mathrm{GL}(n,k)$ and $A \in k^{m\times n}$ we have $(AT)_I = A_I T$ and then

$$(AT)(AT)_I^{-1} = AT(A_IT)^{-1} = ATT^{-1}A_I^{-1} = AA_I^{-1}.$$

So the φ_I are well-defined maps for every index set I, and it can be shown that the φ_I are bijective, that for index sets I,J the sets $\varphi_I(U_I \cap U_J)$ are open in $\mathbb{A}^{m(n-m)}$, and that the transition functions

$$\varphi_J \circ \varphi_I^{-1} : \varphi_I(U_I \cap U_J) \to \varphi_J(U_I \cap U_J)$$

are rational maps between open subsets of $\mathbb{A}^{m(n-m)}$. This makes $\mathrm{Gr}(m,n)$ a smooth variety of dimension $m(n-m)$. Furthermore, it can be shown that this variety can be embedded into the projective space $\mathbb{P}^N$ where $N = \binom{n}{m} - 1$. So as desired, the lines in $\mathbb{P}^3$ can be given the structure of a smooth projective variety of dimension $2(4-2) = 4$. For a more in depth explanation with proofs of the aforementioned results, see [10], chapter 11.

We can now prove Proposition 1.2.

Proof of Proposition 1.2. We have already equipped the set of lines in $\mathbb{P}^3$ with the structure of a projective variety and will now do the same with the set of cubic surfaces. Note that every cubic surface is defined by a homogeneous polynomial of degree 3 in four variables. Such a polynomial has $\binom{3+3}{3} = 20$ coefficients. Two polynomials define the same surface if and only if one they differ by an element of k^*, or in other words, if their coefficients have the same cross-ratio. We can thus naturally identify the set of cubic surfaces in $\mathbb{P}^3$ with $\mathbb{P}^{19}$. [1.b]

We now consider the subset

$$\Sigma = \{(S,L) : L \subset S\} \subset \mathbb{P}^{19} \times \mathrm{Gr}(2,4).$$

Showing that every cubic surface contains a line is equivalent to showing that the projection $\Sigma \to \mathbb{P}^{19}$ to the first component is surjective. In order to prove this, we want to make use of the projective variety structure on $\mathbb{P}^{19} \times \mathrm{Gr}(2,4)$.

We can see that Σ is closed in $\mathbb{P}^{19} \times \mathrm{Gr}(2,4)$ by considering the standard open cover of $\mathrm{Gr}(2,4)$. For notational convenience, we will only look at U_{12}, but the other opens function in the same way. An element of U_{12} is a plane of the form $\langle (1,0,a,b),(0,1,c,d)\rangle \subset k^4$, where $(a,b,c,d) \in \mathbb{A}^4$, corresponding to the line $L \subset \mathbb{P}^2$ defined by the parametrisation

$$\{(\lambda : \mu : \lambda a + \mu c : \lambda b + \mu d) : (\lambda : \mu) \in \mathbb{P}^1\}.$$

For an element $[f] \in \mathbb{P}^{19}$, where $[f]$ is the equivalence class of a homogeneous polynomial $f \in k[x_0,x_1,x_2,x_3]_3$, the pair $([f],L)$ is in Σ if and only if for all $(\lambda : \mu) \in \mathbb{P}^1$

$$f(\lambda, \mu, \lambda a + \mu c, \lambda b + \mu d) = 0.$$

Looking at this expression as a polynomial in λ and μ, it is homogeneous of degree 3, and thus has four coefficients, all of which have to vanish. Therefore, $\Sigma \cap (\mathbb{P}^{19} \times U_{12})$ is defined by four polynomial equations in a,b,c,d and the coefficients of f, thus closed in $\mathbb{P}^{19} \times U_{12}$.

[1.b] Note that the term 'cubic surface' in this case includes non-reduced surfaces, such as sets of three (not necessarily distinct) planes.

The same argument can be used for the other standard open subsets of $\mathrm{Gr}(2,4)$, so $\Sigma \cap U_{ij}$ is closed in U_{ij} for all pairs of indices $1 \le i < j \le 4$. Since these open sets cover $\mathrm{Gr}(2,4)$, it follows that Σ is closed in $\mathbb{P}^{19} \times \mathrm{Gr}(2,4)$ and of codimension 4, or in other words the dimension of Σ is 19.

Now, noting that π is a morphism between projective varieties and thus closed, we can see that the image of π in $\mathbb{P}^{19}$ must have dimension 19, since otherwise, all nonempty fibres of π would have positive dimension. But that would mean every cubic surface contains either no lines or infinitely many, contradicting Example 1.1. So the image of Σ under π has dimension 19 and because it is closed in $\mathbb{P}^{19}$, the morphism π must be surjective. □

1.3 Remark. The arguments in this proof give rise to another interesting fact. If we consider hypersurfaces of degree d in $\mathbb{P}^n$, we can construct a similar projective variety

$$\Sigma \subset \mathbb{P}^N \times \mathrm{Gr}(2, n+1)$$

like above, where now $N = \binom{n+d}{d} - 1$. The dimension of the Grassmannian $\mathrm{Gr}(2, n+1)$ is now $2n-2$, and computing the codimension of Σ inside $\mathbb{P}^N \times \mathrm{Gr}(2, n+1)$ in the same way gives

$$N + 2n - d - 3.$$

Whenever this codimension exceeds the dimension of $\mathrm{Gr}(2, n+1)$, it follows that Σ has a dimension strictly less than N. As a consequence, the general hypersurface in $\mathbb{P}^n$ of degree d contains no lines at all. This is the case whenever $d > 2n-3$. In particular, by setting $n = 3$, we can see that the general surface in $\mathbb{P}^3$ of degree higher than 3 does not contain any lines.

1.4 Remark. We can also conclude from the proof of Proposition 1.2 that no cubic surface can contain infinitely many lines. Indeed, if there were a cubic surface with infinitely many lines, then there would be a fibre of the morphism $\Sigma \to \mathbb{P}^{19}$ that has positive dimension, which is impossible.

1.2 The 27 lines on a cubic surface

In this section, let S be a smooth algebraic surface of degree $d \ge 3$ containing a line $L \subset S$. In the cubic case, such an L always exists by Proposition 1.2. In general, there may be no lines on S, but under the assumption that there is at least one, we can still draw conclusions on the number and configuration of lines on S.

With L as a starting point, we first want to find out how many other lines on S could possibly intersect L. Since two intersecting lines always lie on a common plane, we can find such lines in the planes through L. The intersection of any plane with S is a plane curve with the same degree as S. Because S is smooth, we can show that this curve is reduced, i.e. it does not contain any components with multiplicity higher than 1.

1.5 Lemma. *Let H be a plane in $\mathbb{P}^3$. Then the intersection $H \cap S$ is a reduced plane curve in $H \cong \mathbb{P}^2$ of degree d.*

Proof. After a change of coordinates, we can assume that H is given by the equation $x_3 = 0$. Now assume there is a plane curve $C \subset S \cap H$ that is contained in the intersection with multiplicity at least 2. Let $g = 0$ be an equation for g in the plane $H \simeq \mathbb{P}^2(x_0, x_1, x_2)$ so that in $\mathbb{P}^3$, C is given by the equations $x_3 = 0$ and $g = 0$. We can then write an equation for S in the form

$$f = x_3 h_1 + g^2 h_2 = 0 \tag{1.9}$$

for some homogeneous polynomials h_1, h_2 of fitting degrees. Now consider the surface defined by the equation $h_1 = 0$. This surface then intersects the curve C in at least one point P. Now P lies on S, because C is contained in S, and P is a singularity of S: indeed, the partial derivatives of f are given by

$$\frac{\partial}{\partial x_i} f = x_3 \frac{\partial}{\partial x_i} h_1 + 2 g h_2 \frac{\partial}{\partial x_i} g + g^2 \frac{\partial}{\partial x_i} h_2 \qquad \text{for } i = 0,1,2 \tag{1.10}$$

$$\frac{\partial}{\partial x_3} f = h_1 + g^2 \frac{\partial}{\partial x_3} h_2 \tag{1.11}$$

and since P is a root of x_3, h_1 and g, it is also a root of all partial derivatives of f and thus a singularity of S. But we assumed S to be smooth, so our initial assumption must have been wrong and the curve C cannot exist. The intersection $S \cap H$ is thus a reduced curve. $\square$

Now note that any line in $\mathbb{P}^3$ skew to L has a unique intersection with any plane through L. A choice of such line L' thus gives us a bijective mapping from L' to the family of planes through L. Instead of looking at this family, we can define a morphism $S \to L'$ whose fibres are exactly the curves residual to L in the intersections of S with these planes.

1.6 Proposition. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree at least 3 containing a line L. Let $L' \subset \mathbb{P}^3$ be a line skew to L and not contained in S. Then there is a unique morphism*

$$\pi_{L,L'} : S \to L'$$

with the property that for every point $P \in S \setminus L$, the image $\pi_{L,L'}(P)$ is given as the unique point of intersection of L' with the plane through P and L.

Proof. Since the set $S \setminus L$ is dense in S and morphisms of varieties are continuous, such a morphism has to be unique if it exists. To prove existence, we can choose coordinates such that

$$L = Z(x_2, x_3)$$
$$L' = Z(x_0, x_1).$$

Since L is contained in S, we can write an equation for S of the form

$$f := x_2 g + x_3 h = 0$$

for some homogeneous polynomials g, h of degree $\deg S - 1$. Given a point $P = (w : x : y : z)$ not on L, the plane through P and L is given as

$$Z(zx_2 - yx_3).$$

Intersecting this plane with the line L' gives the point $(0 : 0 : y : z)$. If P lies on S, we have $yg(P) + zh(P) = 0$, and so we can write this as $(0 : 0 : -h(P) : g(P))$. In particular, we can extend this definition to L as well, since g and h cannot simultaneously vanish on L. Indeed, note that the partial derivatives of f are given by

$$\frac{\partial f}{\partial x_0} = x_2 \frac{\partial g}{\partial x_0} + x_3 \frac{\partial h}{\partial x_0}$$
$$\frac{\partial f}{\partial x_1} = x_2 \frac{\partial g}{\partial x_1} + x_3 \frac{\partial h}{\partial x_1}$$
$$\frac{\partial f}{\partial x_2} = x_2 \frac{\partial g}{\partial x_0} + g + x_3 \frac{\partial h}{\partial x_2}$$
$$\frac{\partial f}{\partial x_3} = x_2 \frac{\partial g}{\partial x_3} + x_3 \frac{\partial h}{\partial x_3} + h.$$

If g and h both vanished at some point $Q \in L$, then Q would be a singularity of S, contradicting the assumption that S is smooth. So indeed we obtain a morphism $S \to L'$ with the desired properties. $\square$

In the following, we will always identify L' with $\mathbb{P}^1$. Note that the specific choice of L' is irrelevant for our purposes, since the set of fibres of $\pi_{L,L'}$ is independent of this choice. We will thus write π_L instead of $\pi_{L,L'}$, and if it is clear from context what L is, we will drop that index as well and simply write π.

This morphism gives us a precise notion of the residual curves that we mentioned earlier. For any $t \in \mathbb{P}^1$, the fibre $\pi^{-1}(t)$ over t is a plane curve of degree $\deg S - 1$ which does not contain L, but together with L forms the complete intersection of some plane H with the surface S.

For the rest of this chapter, S will be a smooth cubic surface, and L will be a line contained in S, whose existence is guaranteed by Proposition 1.2. In this case, the residual curves in the fibres of π will be relatively easy to handle, as they are plane quadrics, which are either smooth or decompose into two lines. The following lemma immediately gives us the exact number of singular quadrics.

1.7 Proposition. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, L a line contained in S, and $\pi = \pi_L : S \to \mathbb{P}^1$ a morphism like in Proposition 1.6. Then there are exactly five distinct elements $t \in \mathbb{P}^1$ such that $\pi^{-1}(t)$ is a singular conic.*

Proof. First take any plane conic of the form $Z(f) \subset \mathbb{P}^2$ for some homogeneous quadratic polynomial f. Then $Z(f)$ is smooth if and only if its Hessian is nonzero. Note that the Hessian is constant, since f has degree 2. If the Hessian is zero, $Z(f)$ consists of two lines. This will allow us to characterise fibres of π which contain lines.

Now going back to $\mathbb{P}^3$, we choose coordinates such that $L = Z(x_2, x_3)$. We can write an equation for S as

$$f = a_{00}x_0^2 + 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{02}x_0 + 2a_{12}x_1 + a_{22}. \tag{1.12}$$

where the a_{ij} are homogeneous in the variables x_2, x_3 of fitting degrees. The a_{ij} are named this way because they will be the entries of the Hessian matrix of f restricted to any plane through L. Note that in order to write f in this way, we require the assumption that k does not have characteristic 2.

Now planes containing L are of the form

$$H_{(\lambda:\mu)} = Z(\lambda x_2 + \mu x_3)$$

for $(\lambda : \mu) \in \mathbb{P}^1$.

Note that there must be at least one value of $(\lambda : \mu)$ such that the corresponding fibre of π is a smooth conic. If all fibres were singular, then S would contain infinitely many lines, which is impossible by Remark 1.4. After a linear change in the coordinates x_2 and x_3, we can assume that the fibre above $(1 : 0)$ is smooth.

We thus only need to consider the planes $H_{(\lambda:1)}$. The intersection with S as a subvariety of $H_{(\lambda:1)} \simeq \mathbb{P}^2$ is then given by

$$f(x_0, x_1, x_2, -\lambda x_2) = 0$$

which we can simplify by using the homogeneity of the a_{ij} to get

$$\begin{aligned} x_2 \big(& a_{00}(1,-\lambda)x_0^2 + 2a_{01}(1,-\lambda)x_0x_1 + a_{11}(1,-\lambda)x_1^2 \\ &+ 2a_{02}(1,-\lambda)x_0x_2 + 2a_{12}(1,-\lambda)x_1x_2 + a_{22}(1,-\lambda)x_2^2\big) \end{aligned}$$

The factor x_2 corresponds to the line L, and the fibre of π above $(\lambda : 1)$ is given by the equation

$$q_\lambda := \tfrac{f(x_0,x_1,x_2,-\lambda x_2)}{x_2} = 0.$$

We claim that the conics defined by equations of this form are singular for precisely 5 different values of λ. Note that we can write

$$q_\lambda = \sum_{i,j} a_{ij} x_i x_j$$

and then the conic is singular if and only if

$$\det \begin{pmatrix} a_{00}(1,-\lambda) & a_{01}(1,-\lambda) & a_{02}(1,-\lambda) \\ a_{01}(1,-\lambda) & a_{11}(1,-\lambda) & a_{12}(1,-\lambda) \\ a_{02}(1,-\lambda) & a_{12}(1,-\lambda) & a_{22}(1,-\lambda) \end{pmatrix} = 0. \tag{1.13}$$

Viewed as polynomials in λ, the coefficients of this matrix have degrees as follows:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

so the determinant defines a degree 5 equation in λ. It should be noted that it is impossible for this equation to degenerate. The leading coefficient of a_{ij} when viewed as a polynomial in λ is precisely the value $a_{ij}(0,1)$. So indeed the leading coefficient of Equation (1.13) were zero, then the Hessian determinant in the fibre above $(0:1)$ would also vanish, contrary to our assumption.

What remains to be shown is that the roots of Equation (1.13) are distinct. Suppose λ_1 is one such root. After a change of coordinates in x_2 and x_3 we can assume that $\lambda_1 = 0$, and the plane $H_{(0:1)}$ is given by the equation $x_3 = 0$. The intersection $S \cap H_{(\lambda_1:1)}$ consists of the line L and two more lines, which we will call L_1 and L_1'. We distinguish between two cases.

If the three lines are concurrent, then we can choose coordinates x_0, x_1 such that the point of intersection is $(1:0:0:0)$. After a further change in the coordinates x_1, x_2, we can assume that the lines are given by $L = Z(x_2, x_3)$, $L_1 = Z(x_1, x_3)$ and $L_1' = Z(b_1x_1 + b_2x_2, x_3)$ for some $(b_1 : b_2) \in \mathbb{P}^1$. We can then write the equation for S as

$$f = x_3F + b_1x_1^2x_2 + b_2x_1x_2^2 = 0$$

for some homogeneous polynomial F of degree 2. In the notation from Eq. (1.12), the monomial x_3 thus divides all a_{ij} with the exception of

$$a_{11} = b_1x_2 + cx_3$$
$$a_{12} = \tfrac{1}{2}b_2x_2^2 + dx_2x_3 + ex_3^2$$

for some constants $c, d, e \in k$. Now the Hessian of f is given by

$$-a_{00}a_{12}^2 + x_3^2G$$

for some polynomial G. Note that x_3 does not divide a_{12}^2 and only divides a_{00} with multiplicity 1, since a_{00} has degree 1. Therefore, λ divides $\det(a_{ij}(1, -\lambda))_{ij}$, but λ^2 does not, and so λ_1 is indeed a simple root of Equation (1.13).

If the lines are not concurrent, we can use similar reasoning. This time, they intersect in a proper triangle, whose corners we can assume are given by the points $(1:0:0:0)$, $(0:1:0:0)$ and $(0:0:1:0)$. The lines are then given by $L = Z(x_2, x_3)$, $L_1 = Z(x_1, x_3)$ and $L_1' = Z(x_0, x_3)$. So now we can write

$$f = x_3F + x_0x_1x_2$$

for some homogeneous polynomial F, which means $a_{01} = \frac{1}{2}x_2 + cx_3$ for some constant $c \in k$ and all other coefficients are multiples of x_3. Similarly to the previous case, the Hessian of f is given by

$$-a_{01}^2a_22 + x_3^2G$$

for some polynomial G. Again we note that x_3 does not divide a_{01}^2, but does divide a_{22}. However, a_{22} cannot be a multiple of x_3^2 if S is smooth.

To see this, note first that a_{00} and a_{11} are multiples of x_3 and have degree 1, so they are constant with respect to x_2. We compute the partial derivatives of f:

$$\begin{aligned}
\frac{\partial}{\partial x_0} f &= 2(a_{00}x_0 + a_{01}x_1 + a_{02}) \\
\frac{\partial}{\partial x_1} f &= 2(a_{01}x_0 + a_{11}x_1 + a_{02}) \\
\frac{\partial}{\partial x_2} f &= x_0x_1 + 2x_0\frac{\partial}{\partial x_2}a_{02} + 2x_1\frac{\partial}{\partial x_2}a_{12} + \frac{\partial}{\partial x_2}a_{22} \\
\frac{\partial}{\partial x_3} f &= x_0^2\frac{\partial}{\partial x_3}a_{00} + 2x_0x_1\frac{\partial}{\partial x_3}a_{01} + x_1^2\frac{\partial}{\partial x_3}a_{11} + 2x_0\frac{\partial}{\partial x_3}a_{02} + 2x_1\frac{\partial}{\partial x_3}a_{12} + \frac{\partial}{\partial x_3}a_{22}
\end{aligned}$$

If x_3^2 divides a_{22}, then x_3 divides $\frac{\partial}{\partial x_3}a_{22}$. But this would imply that every monomial in these partial derivatives is a multiple of x_0, x_1 or x_3, and then $(0:0:1:0)$ would be a singular point of S.

So indeed x_3 only divides a_{22} with multiplicity 1, and λ_1 is once again a simple root of the Hessian of f. Since this applies to all roots of f, there are indeed five distinct roots, corresponding to five distinct fibres of π, each containing two lines intersecting L. □

1.8 Remark. Note that lines from different fibres of π cannot intersect each other. They lie on different planes through L, and their point of intersection would have to lie on L. But because S is smooth, three concurrent lines on S must be coplanar. In particular, we can conclude that there exist two disjoint lines on S.

Now, let L_0 and M_0 be two such disjoint lines. We apply Proposition 1.7 to both of them separately. Applying it to L_0, we obtain five pairs of coplanar lines L_i, N_i for $i = 1, \dots, 5$. Now for every i, the lines L_0, L_i and N_i form the complete intersection of S with some plane. Since M_0 intersects this plane and lies on S, it must intersect exactly one of them. Because L_0 and M_0 are disjoint, it has to be L_i or N_i. Without loss of generality, we can assume that M_0 intersects the lines N_i for every i.

When we apply Proposition 1.7 to M_0, we again obtain five singular fibres of π_{M_0}. Each of these fibres contains one of the lines N_i, and another line M_i. In total, we have now constructed 17 lines on S. Before we can find the final ten, we need to establish some basic properties of mutually disjoint lines in $\mathbb{P}^3$.

1.9 Lemma. *Let $L, M, N \subset \mathbb{P}^3$ be mutually disjoint lines. Then there is exactly one quadric surface Q that contains L, M, N. The surface Q is precisely the union of all lines passing through L, M, N.*

Proof. Since L and M are disjoint, we claim that we can find homogeneous coordinates such that $L = Z(x_0, x_1)$, $M = Z(x_2, x_3)$ and $N = Z(x_0 - x_2, x_1 - x_3)$. This is equivalent to finding a basis (v_1,v_2,v_3,v_4) of k^4 such that the two dimensional linear subspaces $U,V,W \subset k^4$ corresponding to L,M,N are given by

$$U = \langle v_1,v_2\rangle\,, V = \langle v_2,v_3\rangle\,, W = \langle v_1+v_3, v_2+v_4\rangle\,.$$

Since $U \cap V = \{0\}$ (corresponding to the disjointness of L and M) and their dimensions are 2, any two bases of U and V joined together form a basis of k^4. So, taking a basis (w_1,w_2) of W, we can find vectors $v_1,v_2 \in U, v_3,v_4 \in V$ such that $v_1+v_3 = w_1$ and $v_2+v_4 = w_2$. If v_1 and v_2 were linearly dependent, i.e. $\lambda v_1 + \mu v_2 = 0$, then

$$\lambda w_1 + \mu w_2 = \lambda v_3 + \mu v_4 \in V,$$

but because $V \cap W = \{0\}$, that implies that w_1,w_2 are linearly dependent – a contradiction. Thus, both (v_1,v_2) and (v_3,v_4) are linearly independent pairs, hence bases of U,V respectively, and so indeed (v_1,v_2,v_3,v_4) is a basis of k^4.

Given these coordinates, we can now explicitly define the quadric surface Q. Let

$$Q = Z(x_0x_3 - x_1x_2).$$

This Q clearly contains L,M,N. We will postpone the proof of uniqueness for now and first show that Q is the union of all lines intersecting L, M and N.

Let P be a point on Q, with coordinates $(x_0 : x_1 : x_2 : x_3)$ with $x_0x_3 - x_1x_2 = 0$. If P lies on one of the three lines, say $P \in L$, we can consider the plane through P and M and note that this plane intersects N, giving us a line through P, M and N, so P is indeed contained in the union of all lines intersecting L, M, N. From now on assume P lies on neither of the three lines. At least one coordinate of P has to be nonzero, say without loss of generality $\lambda := x_0 \neq 0$. If $x_3 \neq 0$, then all coordinates must be nonzero. Writing $x_2 = r\lambda$ for some $r \in k^\times$ and $x_1 = \mu$, it follows that x_3 must be equal to $r\mu$ and P is given as $(\lambda : \mu : r\lambda : r\mu)$. In this case, the line $Z(\mu x_0 - \lambda x_1, \mu x_2 - \lambda x_3)$ passes through P and intersects the lines L, M and N. On the other hand, if $x_3 = 0$, then either x_1 or x_2 has to be equal to zero. But since we assumed that P does not lie on $M = Z(x_2,x_3)$, it follows that $x_2 \neq 0$ and so we can write P in the form $P = (\lambda : 0 : \mu : 0)$ for some $\lambda, \mu \in k^\times$. Then the line $Z(x_1,x_3)$ contains P and intersects L, M and N.

To show the other inclusion, we will look at any quadric surface Q' containing L, M and N. Consider a line L' that intersects L, M and N. Then the set $L' \cap Q'$ contains all three points of intersection. Using Proposition 7.6 from [9] and the notion of degree introduced in the preceding definition, we can conclude that $\deg L' = 1$, since L' is isomorphic to the projective line and the degree of a variety depends only on the coordinate ring, hence is invariant under isomorphism. From the same proposition it is also clear that $\deg Q' = 2$. It then follows from Theorem 7.7 that L' must be fully contained in Q', since otherwise, their intersection could contain at most two points. So any quadric surface containing L, M and N also contains all lines passing through them.

In particular, this fact can also be used to show that Q is unique. Indeed, if Q' is another quadric surface containing L, M, and N, then Q must be contained in Q'. But since Q is smooth, equality follows. □

This quadric surface is quite useful for examining what can happen if we add a fourth line.

1.10 Lemma. *Suppose $L,M,N,O \subset \mathbb{P}^3$ are mutually disjoint lines. Then*

1. *If O lies on the quadric Q spanned by L,M,N, then there are infinitely many lines intersecting L,M,N,O*
2. *If O is tangent to Q, there is exactly one such line*
3. *If O does not lie on Q and is also not tangent to it, then there are exactly two such lines.*

Proof. We choose the same coordinates as in the proof of Lemma 1.9. Then Q is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ and the lines L,M,N correspond to the lines

$$\{(0:1)\} \times \mathbb{P}^1, \{(1:0\} \times \mathbb{P}^1, \{(1:1)\} \times \mathbb{P}^1.$$

If $O \subset Q$, then it corresponds to $\{P\} \times \mathbb{P}^1$ for some $P \in \mathbb{P}^1$ and all lines $\mathbb{P}^1 \times \{R\}$ intersect all four. If $O \not\subset Q$, then their intersection can only have cardinality 1 or 2, with the former being the case if and only if O is tangent to Q. □

We can now prove the main theorem of this chapter, which was first proven by Cayley and Salmon in 1849 (see [22], p.496), here presented with a proof following [5].

1.11 Theorem (Cayley, Salmon, 1849)**.** *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Then S contains exactly 27 lines.*

Proof. By Proposition 1.2, we know that S contains at least one line. By Remark 1.8, we also know that there must be two disjoint lines on S. We shall fix two such disjoint lines L_0 and M_0.

As was explained before, this allows us to construct 15 more lines, which we shall review.

By Proposition 1.6, the line L_0 is intersected by precisely ten other lines on S, which come in pairs (L_i,N_i) for $i \in \{1,\ldots,5\}$ with the property that for each i, there is a plane $H \subset \mathbb{P}^3$ such that the intersection of S with H consists precisely of the three lines L_0, L_i and N_i.

Given such a plane H, the line M_0 must intersect H, and because M_0 is contained in S, the point of intersection must lie on L_0, L_i or N_i. Since L_0 and M_0 are disjoint by assumption, the line M_0 must intersect exactly one of L_i or N_i.

Without loss of generality, we can assume that M_0 intersects N_i for each i, and is disjoint with all L_i.

After applying Proposition 1.6 to the line M_0 as well, we find that M_0 is intersected by exactly ten other lines on S, five of which are already given in the lines N_i. Therefore, there must be five more lines M_i for $i \in \{1,\ldots,5\}$ such that for each i, the three lines M_0, M_i and N_i are coplanar.

We now have the 17 lines

$$L_0,M_0,L_1,\ldots,L_5,M_1,\ldots,M_5,N_1,\ldots,N_5 \tag{1.14}$$

on S.

Next, we want to prove that S cannot contain more than 27 lines. Let O be any line on S not among the 17 lines (1.14). We claim that O must intersect precisely three of the lines N_i.

Assume first that O meets at least four N_i, and assume without loss of generality that these are given by N_1, N_2, N_3 and N_4. Note that the lines N_i are mutually disjoint. Indeed, two lines N_i and N_j with $i \neq j$ lie in different planes through L_0, so if they intersected, we would have three concurrent lines that are not coplanar, leading to a singularity on S.

Further note that by construction, the four lines $N_1,\ldots, N_4$ are all intersected by three different lines in L_0, M_0, and O. By Lemma 1.10, this implies that N_4 must lie on the quadric surface Q spanned by N_1, N_2 and N_3. By Lemma 1.9, this quadric surface Q also contains the three lines L_0, M_0 and O.

This means that the intersection of the surfaces S and Q has degree at least 7. But by Theorem I.7.7 in [9], this is impossible, since $(\deg Q)(\deg S) = 6$. So O can meet at most three of the lines N_i.

Now suppose that it meets at most two. Note that O does not intersect L_0, since otherwise it would be equal to L_i or N_i for some i. But among each triangle L_0,L_i,N_i, it must intersect exactly one of the three lines, as discussed before. So after renumbering, we find that O must intersect

$$\begin{aligned} &L_1,L_2,L_3,L_4,L_5 \\ \text{or } &N_1,L_2,L_3,L_4,L_5 \\ \text{or } &N_1,N_2,L_3,L_4,L_5 \end{aligned}$$

Since L_0 intersects all five lines in either case, and M_1 intersects N_1 as well as $L_2,\ldots,L_5$ (because M_1 is disjoint with L_0 and N_i for $i \neq 1$), this means that M_1 intersects at least four of the five lines in each case. In particular, there are always four disjoint lines that are all intersected by the three distinct lines L_0, M_1 and O. With the same reasoning as above, Lemmas 1.9 and 1.10 imply that this is impossible.

We conclude that O must intersect precisely three among the five lines N_i. Since there are $\binom{5}{3} = 10$ possibilities to choose three among those five lines, there can only be at most ten additional lines on S, giving us the desired maximum of $17 + 10 = 27$.

It only remains to be shown that this maximum is always reached. Consider the line N_i for a fixed index i. By construction it intersects the lines L_0 and M_0, as well as L_1 and M_1. However, it cannot intersect any of the other lines already constructed, as they all lie in different planes through L_0 or M_0.

By Proposition 1.7, there must be six more lines on S intersecting N_1. As we have shown above, any line O on S distinct from the 17 already constructed ones must intersect precisely three among the lines $N_1,\ldots,N_5$. Therefore, the six lines intersecting N_i must each intersect precisely two other lines N_j and N_k for distinct indices $j,k \neq i$.

Note that if two distinct lines O and O' were to meet the same three lines N_i, N_j and N_k, then they would both lie on the quadric surface spanned by N_i, N_j and N_k, together with L_0 and M_0, which leads to a contradiction like above.

Since for each i there are precisely six tuples of indices (j,k) such that i,j,k are all distinct, there must be precisely one line intersecting the three lines N_i, N_j and N_k for each such choice of j and k.

For $i < j < k$, we shall label this line O_{ijk}. In particular, there are precisely $\binom{5}{3} = 10$ such lines O_{ijk}. Together with the 17 lines from (1.14), that makes 27 lines.

Because we already established that S cannot contain more than 27 lines, it must contain exactly 27. $\square$

While this does finally prove that there are exactly 27 lines on any smooth cubic surface, there are still some open questions. In particular, it is not yet fully clear which ten lines intersect any of L_i, M_i or O_{ijk} in the notation of the proof above. But we already know enough about the intersections to answer this question.

Take any line M_i. It has to intersect any triangle (L, L_j, N_j) exactly once. Since M_i is disjoint with L and intersects N_i, but no other line N_j, it has to intersect all L_j for $j \neq i$.

Furthermore, the line O_{ijk} also intersects each of the triangles (L, L_r, N_r) and (M, M_r, N_r) exactly once. Among the N_r, O_{ijk} meets exactly the three N_i, N_j and N_k, so it must meet the remaining L_r and M_r for $r \neq i, j, k$.

So far, we have found seven intersecting lines for every O_{ijk}, so it must intersect three other lines among this group as well. Take for example O_{123} and note that it forms a triangle with the lines L_4 and M_5. Again, we can conclude that each other line on S must intersect one from this trio. The lines O_{i45} for $i = 1, 2, 3$ all do not intersect L_4 or M_5, so they must intersect L_{123}. Doing this for all indices, we can conclude that the lines O_{ijk} and O_{rst} intersect if and only if $\{i, j, k\} \cup \{r, s, t\} = \{1, 2, 3, 4, 5\}$. This gives precisely three new intersecting lines for each O_{ijk}, meaning we found all ten for every line on S.

Lastly, let us briefly touch on the subject of Eckardt points, which were mentioned at the very beginning of this chapter.

1.12 Definition. A point $P \in S$ is called an Eckardt point if there are three lines on S passing through P.

As we have seen previously, the lines on S come in triangles. Whenever two lines intersect, there is a third line in that same plane also intersecting both. Since each line intersects ten other lines, it is a part of five different triangles. In total, there are thus $\frac{27 \cdot 5}{3} = 45$ triangles of lines on S. An Eckardt point is simply a degeneration of one of these triangles. However, it is not possible for each triangle to be degenerate at the same time. In fact, there can be at most 18 Eckardt points on S. To prove this, we will first show that each line on S can only contain up to two of them.

1.13 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and let L be a line lying on S. Then L contains at most two Eckardt points.*

Proof. Recall the morphism $\pi\colon S \to \mathbb{P}^1$ associated to L as in Proposition 1.6. Restricting it to L, we obtain a morphism of curves

$$\pi\colon L \to \mathbb{P}^1.$$

Note that an Eckardt point occurs precisely when a singular fibre of the morphism $\pi\colon S \to \mathbb{P}^1$ ramified when viewed as a fibre of the morphism $\pi\colon L \to \mathbb{P}^1$. This morphism has degree 2, since fibres of π are intersections of L with plane quadrics. Since both L and $\mathbb{P}^1$

are genus 0 curves, we can use the Riemann-Hurwitz formula (see Corollary IV.2.4 in [9]) to see that

$$2g(\mathbb{P}^1) - 2 = (\deg \pi)(2g(L) - 2) + \sum_{t \in \mathbb{P}^1} e_t - 1,$$

where g is the genus and e_t is the ramification index of π at the fibre above t. Note that in positive characteristic the Riemann-Hurwitz formula is true whenever all ramification is tame. Since the degree of the morphism $L \to \mathbb{P}^1$ is 2, wild ramification can only exist in characteristic 2. Simplifying this equation and substituting $g(L) = g(\mathbb{P}^1) = 0$, we get

$$\sum_{t \in \mathbb{P}^1} e_t - 1 = 2.$$

In other words, there are exactly two ramified fibres, of ramification index 2 each. Whenever one of these two coincides with a singular fibre of $\pi\colon S \to \mathbb{P}^1$, we obtain an Eckardt point. In particular, there can be at most two Eckardt points on L. □

With the help of this lemma, we can prove the aforementioned maximum.

1.14 Proposition. *There can be at most* 18 *Eckardt points on* S.

Proof. By Lemma 1.13, there can be at most two Eckardt points on every line on S. Since there are 27 lines and each Eckardt point is counted thrice, the maximum number of Eckardt points on S is

$$\frac{27 \cdot 2}{3} = 18.$$

□

Note that this upper bound can indeed be obtained, namely by the Fermat cubic from Example 1.1. The proof of Lemma 1.13 perhaps also illustrates why Eckardt points are so rare. For an Eckardt point to occur on the line L, the morphism $L \to \mathbb{P}^1$ has to ramify at one of only finitely many specified fibres; namely the five singular fibres of $S \to \mathbb{P}^1$. Indeed, the set of smooth cubic surfaces with an Eckardt point has codimension 1 in the space of all cubic surfaces (see [4], p.440).

2 | Elliptic Surfaces

Unless otherwise mentioned, let k be an algebraically closed field with char $k \neq 2,3$.

In Chapter 1, we proved that every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines. To be a bit more precise, we actually proved two separate results, with independent methods:

1. Every cubic surface contains a line.

2. Every smooth cubic surface S with a line $L \subset S$ contains exactly 27 lines.

A natural goal would be to generalise this as much as possible. We have already seen in Remark 1.3 that the general surface of degree higher than 3 does not contain any lines, so the first point does not generalise at all. However, for any degree $d > 3$ there is a positive dimensional family of smooth surfaces of degree d which do contain lines, and we will restrict our attention to those.

In Chapter 1, the central observations were Proposition 1.6 and Proposition 1.7. For a smooth cubic S containing a line L we defined a morphism π whose fibres are plane conics, and any lines on S intersecting L are components of such fibres. We then went on to prove that almost all fibres are smooth irreducible conics while exactly five of them decompose into two lines each.

Proposition 1.6 generalises very easily to all smooth surfaces S of higher degrees, as long as we assume that S contains a line L. When it comes to Proposition 1.7, we made heavy use of the fact that when $\deg S = 3$, the residual curves in the fibres of the morphism $\pi_L : S \to \mathbb{P}^1$ are conics. In this case, there are only two different fibre configurations that we need to distinguish: either the fibre is smooth, or it consists of two lines.

We used the Hessian as a tool to distinguish these cases; a conic has a constant Hessian, and is smooth if and only if the Hessian is nonzero. For curves of higher degrees, the Hessian will no longer be constant. Rather than being a measure for the smoothness of the curve, it will instead determine points of inflection.

Denote by $i_P(C,D)$ the intersection multiplicity (see [9], p. 53) of two curves C and D at a point P in which they intersect.

2.1 Definition. Let $C \subset \mathbb{P}^2$ be an algebraic curve, $P \in C$ a smooth point, and let T_P be the tangent line to C at P. We call P an inflection of C if $i_P(T_P, C) \geq 3$.

2.2 Remark. Some authors may require inflections to be smooth points. We will not make this distinction. Note that according to the above definition, singular points are always inflections. Indeed, for any singular point P on a curve C we can find a line L such that $i_P(L,C) \geq 3$. We further note that every point on a line is an inflection since for any line L and $P \in L$ we have $i_P(L,L) = \infty$.

For plane conics, inflection points are not very interesting: when the curve is smooth, there are no inflections, and when it is singular, it decomposes into two lines and all points are inflections. This is reflected in the fact that the Hessian is constant. In fact, the Hessian of a plane curve can generally be used to find its points of inflection.

2.3 Proposition. *Let $C \subset \mathbb{P}^2$ be an algebraic curve and $P \in C$ a point. Then P is an inflection point of C if and only if the Hessian of C vanishes at P.*

A proof of this can be found in [7], p. 67. Curves of degree higher than 2 can contain non-trivial (i.e. not lying on a line component) inflection points, and so while the Hessian necessarily vanishes along every line, it is no longer a sufficient tool to detect lines, unless we pay closer attention to the inflection points in general. Specifically if S is a smooth quartic surface, then the fibres of the morphism π are plane cubic curves. The following proposition gives the number of inflection points on a plane cubic.

2.4 Proposition. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree* 3.

a) If C is smooth, then C has exactly nine points of inflection.

b) If C has a node, then C has exactly four points of inflection, including the node.

c) If C has a cusp, then C has exactly two points of inflection, including the cusp.

Proof. This follows immediately from the Plücker formulas ([7], p.89). One of these formulas states that

$$s^* = 3n(n-2) - 6d - 8s,$$

where C is a curve of degree n, s^* is the number of smooth inflection points, d is the number of nodes, and s is the number of cusps. For $n = 3$, we obtain the desired results. □

In Chapter 3, we will see there is an analogue of Proposition 1.7 for smooth quartic surfaces. However, this will not hold in full generality, as there is a specific type of line that evades this reasoning.

2.5 Definition. Let S be a smooth surface in $\mathbb{P}^3$ containing a line L. Let $\pi\colon S \to \mathbb{P}^1$ be the morphism as defined in Proposition 1.6.

Then L is called inflectious if for every $P \in L$, every point of intersection of L with the fibre $C_P = \pi^{-1}(\{\pi(P)\})$ is a point of inflection of C_P.

Otherwise we call L regular.

2.6 Remark. This is not standard terminology. Segre [26] introduces regular lines as 'lines of the first kind' and inflectious lines as 'lines of the second kind', and Rams and Schütt [20]use the same terms.

2.7 Remark. There are two different definitions for inflectious lines in the literature. The definition used here is similar to the one in [26], where a more general formulation applies to all surfaces in $\mathbb{P}^3$ of degree 3 or higher.

Another definition is used in [20], which requires a weaker property of the line L. There, each fibre F of the morphism π need only intersect L in at least one point which is an inflection of F. We will see later in Remark 3.4 that these two definitions are equivalent. The property from Definition 2.5 is used at several different points in [20] without it being made explicit that it follows from their definition.

On a smooth cubic surface, all lines are regular. When S is a smooth quartic surface however, inflectious lines exist, and a priori they make it very difficult to count how many lines on S could intersect them.

Consider a line $L \subset S$ with associated morphism $\pi\colon S \to L$. If there is fibre F of π that contains a line L', then the point of intersection is automatically an inflection of the curve F by Remark 2.2. Hence such points of inflection in intersections of L with a fibre of π are a necessary, but not sufficient condition for the existence of lines in that fibre. This helps us identify fibres that may potentially contain lines so long as L is regular, and in fact this is good enough to prove a generalised version of Proposition 1.7.

When L is inflectious, every fibre intersects L in points of inflection, and so this condition loses its usefulness in detecting candidate fibres for containing lines. Smooth quartic surfaces with inflectious lines thus require a different approach.

This approach will be the theory of elliptic curves and elliptic surfaces. As we will soon see, smooth plane cubics are examples of elliptic curves, and the morphism π gives S the structure of an elliptic surface.

In the rest of this chapter, we will give an overview over elliptic curves, elliptic surfaces, and some core theorems that will help us analyse lines on quartic surfaces. Our primary source of reference for elliptic curves will be [29], with elliptic surfaces being covered by [18, 23, 25, 28].

2.1 ELLIPTIC CURVES

In this section, let k be any field, not necessarily algebraically closed.

2.8 Definition. An elliptic curve over k is a smooth algebraic curve E over k of genus 1 with a designated point $O \in E(k)$. We will use both (E,O) and just E as notation for the curve E with point O depending on the context.

Elliptic curves are often studied over varying ground fields. The points of E over the field k are denoted by $E(k)$. Since many of these fields are not algebraically closed, the existence of a point on a curve of genus 1is not always guaranteed. However, even while working over algebraically closed fields, the inclusion of O is an essential part of this definition, since O will be the neutral element of a group structure on the points of E. To define this group structure, we consider the Picard group Pic(E), which is the group of divisors on E modulo principal divisors. The elements of degree 0 form a subgroup, which we will denote by $\mathrm{Pic}^0(E)$. We will then induce a group structure on E with the help of the following lemma.

2.9 Lemma. *([29]. Chapter III, Proposition 3.4)*
Let (E,O) be an elliptic curve. Then the map

$$\begin{aligned} \sigma : E(k) &\to \mathrm{Pic}^0(E) \\ P &\mapsto [(P)-(O)] \end{aligned}$$

is a bijection.

Here (P) denotes the divisor corresponding to the point P and by $[(P)-(O)]$ we mean the class of the divisor $(P)-(O)$ modulo principal divisors. This bijection allows us to transfer the group law from $\mathrm{Pic}^0(E)$ to $E(k)$.

2.10 Corollary. *Let (E,O) be an elliptic curve and P,Q two (not necessarily distinct) points on E. We define*

$$P+Q = \sigma^{-1}(\sigma(P)+\sigma(Q))$$

where the addition in the parentheses happens in $\mathrm{Pic}^0(E)$. Then this operation makes $E(k)$ into a group with neutral element O.

Proof. This follows immediately from the fact that σ is bijective and $\sigma(O)=0$. □

This abstract group structure has a very concrete geometric interpretation in the case that E is a plane curve, and we will make use of this interpretation later. Note that by the genus formula (see Example I.7.2 in [9]), smooth plane curves of genus 1 are precisely the smooth cubics in $\mathbb{P}^2$.

2.11 Proposition. *Let (E,O) be an elliptic curve where $E \subset \mathbb{P}^2$ is a smooth plane cubic and P and Q be two (not necessarily distinct) points on E. Then there is a unique line $L \subset \mathbb{P}^2$ such that the intersection of L with E is given by the divisor*

$$L = (P) + (Q) + (R)$$

for some $R \in E$. We define $P * Q$ to be the point R. Then

$$P + Q = (P * Q) * O.$$

Proof. Consider the line L through P and Q, or the tangent line to E at the point P if $P = Q$. Since E is a cubic curve, the line L intersects E in a unique third point R, and indeed the divisor of the intersection is given by $(P) + (Q) + (R)$.

Furthermore, we consider the line L' through R and O (or the tangent at O if $R = O$), which intersects E in a third point R'. The point $(P * Q) * O$ is then precisely this point R'. Since the intersections of E with the lines L and L' are linearly equivalent, we get the equation

$$(P) + (Q) + (R) = (O) + (R) + (R') \tag{2.1}$$

in $\mathrm{Pic}^0(E)$; or equivalently $P + Q = R'$ in the group E, which is what we wanted to show. □

In particular, this leads to a handy statement about collinear points.

2.12 Corollary. *Let (E,O) be an elliptic curve where $E \subset \mathbb{P}^2$ is a smooth plane cubic and O is an inflection of E. Let $L \subset \mathbb{P}^2$ be any line. Then L intersects E in three (not necessarily distinct) points P, Q, R and we have*

$$P + Q + R = O.$$

Proof. We compute

$$P + Q + R = (((P * Q) * O) * R) * O.$$

Since P, Q and R are collinear, we have $P * Q = R$. Furthermore, $(R * O) * R$ is equal to O, so the term simplifies to $O * O$. Since O is a point of inflection, this is equal to O. □

As we have seen at the beginning of this chapter, we are particularly interested in points of inflection of plane cubic curves. Inflections also play a special role in the theory of elliptic curves. In practice, the zero point O of a plane elliptic curve E is often taken to be an inflection of E, and due to Corollary 2.12, we can now see that the inflections of E are precisely its 3-torsion points.

2.13 Corollary. *Let E be a smooth plane cubic. Then E has exactly nine points of inflection. If $O \in E$ is one of them, then the 3-torsion subgroup of the elliptic curve (E,O) consists precisely of these nine inflection points.*

Proof. We have already seen in Proposition 2.4 that a smooth plane cubic has exactly nine inflection points. By Corollary 2.12, for any inflection point P of E, we have $3P = 0$, so P is indeed a 3-torsion point.

Conversely, if P is a 3-torsion point of E, we know that $3P = O$. The tangent L at P has divisor $2(P)+(Q)$ for some point $Q \in E$. By Corollary 2.12, we also have $2P+Q = O$. Taking both equations together, it follows that $P = Q$, and so P must be a point of inflection of E. □

In general, elliptic curves need not be plane cubics. However, every elliptic curve is isomorphic to a plane cubic. More specifically, it can described by a very specific kind of equation.

2.14 Proposition (Weierstrass equation)**.** *Let (E,O) be an elliptic curve. Then there is an isomorphism $\varphi\colon E \to C$ to a smooth projective plane curve C with affine equation of the form*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.2}$$

such that $\varphi(O) = (0:1:0)$. If furthermore the characteristic of k is not equal to 2 or 3, we can assume that $a_1 = a_2 = a_3 = 0$ and so the equation becomes

$$y^2 = x^3 + ax + b. \tag{2.3}$$

An equation of the form Eq. (2.2) is called a Weierstrass equation and Eq. (2.3) is called a short Weierstrass equation.

2.15 Remark. A curve described by a Weierstrass equation always has exactly one point at infinity, the point $(0:1:0)$. When an elliptic curve is given in this form, it is typically assumed that this is the point O. This point is automatically an inflection, as its tangent is the line at infinity.

2.16 Remark. If an elliptic curve E over a field of characteristic other than 2 is given by a short Weierstrass form $y^2 = x^3 + ax + b$, it is symmetric across the x-axis. In particular, for any given point $P = (x, y)$, its inverse $-P$ is given by $(x, -y)$, since the line through these two points is vertical and meets the projective line at infinity in the point $O = (0:1:0)$. In the case that $P = (x, 0)$ lies on the x-axis, its tangent is vertical and $P = -P$. In particular, the nontrivial 2-torsion points of E are precisely the intersection points of E with the x-axis. Over an algebraically closed field, they correspond precisely to the three solutions to the equation $x^3 + ax + b = 0$.

Lastly, there are two important quantities that help explore some properties of curves defined by Weierstrass equations: the discriminant and the j-invariant, both of which are given as rational functions in the coefficients of the equation. For precise formulae, see [29], p.46. Here we only give the formulae for short Weierstrass equations. Note that we only work with fields of characteristic different from 2 or 3, and any elliptic curve over such a field is given by a short Weierstrass equation, up to isomorphism.

2.17 Definition. Let $y^2 = x^3 + ax + b$ be a short Weierstrass equation. Then

$$\Delta = -16(4a^3 + 27b^2)$$

is called the discriminant of this equation and if $\Delta \neq 0$, then

$$j = \frac{1728(4a)^3}{\Delta}$$

is called the j-invariant.

As the name might suggest, the j-invariant does not depend on the choice of equation, but only on the isomorphism class of an elliptic curve. The use for these quantities lies in the following proposition.

2.18 Proposition ([29], Chapter III, Proposition 1.4). *a) Let C be a curve given by a Weierstrass equation. Let Δ be the discriminant of this equation. Then C is smooth if and only if $\Delta \neq 0$.*

b) Two elliptic curves are isomorphic over the algebraic closure of their ground field if and only if their j-invariants are equal.

2.2 ELLIPTIC SURFACES

In this section, let k be an algebraically closed field. As was mentioned at the beginning of this chapter, our goal is to apply the theory of elliptic curves to the morphism $\pi: S \to \mathbb{P}^1$ associated to a smooth quartic surface $S \subset \mathbb{P}^3$ containing a line L.

2.19 Definition. An elliptic surface is a smooth projective surface S with a surjective morphism $\pi\colon S \to C$ to a smooth irreducible projective curve C such that all but finitely many fibres of π are smooth curves of genus 1.

We call π an elliptic fibration and C the base curve of the elliptic surface S.

There are several different ways to think about elliptic surfaces. On the one hand, we have an algebraic surface S which happens to admit an elliptic fibration $\pi\colon S \to C$. On the other hand, we can also view S as the union of its fibres, or a family of curves, almost all of which are smooth and of genus 1. Another way of looking at this family of fibres is by considering the generic fibre, that is the fibre above the generic point of C.

2.20 Definition. Let $f: X \to Y$ be a morphism of schemes, and let $y \in Y$. The scheme theoretic fibre of f above y is the fibre product

$$X \times_Y \operatorname{Spec} \kappa(y)$$

where $\kappa(y)$ is the residue field of the point $y \in Y$. The fibre above the generic point of Y is called the generic fibre.

Note that in our case of a morphism $f: S \to C$ with a surface S and a curve C, the points of C as a scheme are precisely the points of C as a projective variety, which are the closed points, and the generic point $\eta \in C$. The fibre above any point of C is a scheme over the residue field of that point. For a closed point in C, its residue field is k, so the fibre above a closed point is a scheme over k. For the generic point however, we obtain a scheme over the function field $k(C)$.

It can be shown [2.a] that the scheme theoretic fibre over a closed point $P \in C$ is homeomorphic as a topological space to $\pi^{-1}(\{P\})$ as a closed subvariety of S.

2.21 Example. Consider the quartic surface $S \subset \mathbb{P}^3$ defined by the equation

$$x_2\left(x_0^3 - x_2^3\right) - x_3\left(x_1^3 - x_3^3\right) = 0.$$

This surface is called the Schur quartic, and we will later show that it contains exactly 64 lines. From the equation we can immediately see that it contains the line $L = Z(x_2, x_3)$. We can apply the construction from Proposition 1.6 to obtain a morphism

$$\begin{aligned} S &\to \mathbb{P}^1 \\ (x_0 : x_1 : x_2 : x_3) &\mapsto (x_2 : x_3) = \left(x_1^3 - x_3^3 : x_0^3 - x_2^3\right). \end{aligned}$$

[2.a] See Exercise II.3.10 in [9]

Let $c \in k$ and consider the plane $H_c = Z(x_3 - cx_2)\mathbb{P}^3$. The intersection $S \cap H_c$ is the quartic curve in $H_c \simeq \mathbb{P}^2(x_0, x_1, x_2)$ defined by the equation

$$x_2\left(x_0^3 - x_2^3\right) - cx_2\left(x_1^3 - c^3x_2^3\right) = 0.$$

This curve contains L as a component. The fibre above the point $(1:c) \in \mathbb{P}^2$ is the residual cubic curve defined inside H_c by the equation

$$\left(x_0^3 - x_2^3\right) - c\left(x_1^3 - c^3x_2^3\right) = 0. \tag{2.4}$$

This curve is smooth for almost all choices of c: we compute the partial derivatives

$$\begin{aligned} d_0 &= 3x_0^2 \\ d_1 &= 3cx_1^2 \\ d_2 &= 3\left(c^4 - 1\right)x_2^2. \end{aligned}$$

For $c \neq 0$ and $c^4 \neq 1$, these can never simultaneously vanish. For all such c, the fibre above $(1:c)$ is thus smooth. We will examine the singular fibres in more detail later.

Now we want to compute the generic fibre of π. Consider the standard cover of $\mathbb{P}^1$ by the affine open subset $U_0 = \{(c:1) : c \in k\}$ and $U_1 = \{(1:c) : c \in k\}$, and note that the inverse images $\pi^{-1}(U_0), \pi^{-1}(U_1) \subset S$ are both open affine as well; for example $\pi^{-1}(U_1) = D(x_2) \cap S$, where $D(x_2)$ is the open subset $\{x_2 \neq 0\}$ in $\mathbb{P}^3$.

Let $\eta \in \mathbb{P}^1$ be the generic point of $\mathbb{P}^1$, given as a subscheme $\{\eta\} \simeq \operatorname{Spec} k(t)$ of $\mathbb{P}^1$. The generic fibre is then the fibre product of the map $\pi : S \to \mathbb{P}^1$ and the inclusion $\{\eta\} \to \mathbb{P}^1$. This fibre product is covered by the two affine fibre products of the maps $\pi\big|^{U_i} : \pi^{-1}(U_i) \to U_i$ and $\{\eta\} \to U_i$ where $i = 0, 1$. We will explicitly compute this for $i = 1$.

Note that $U_1 = \operatorname{Spec} k[t]$ and $\pi^{-1}(U_1) = \operatorname{Spec} R$ where $R = k[x_0, x_1, x_3]/\left(\left(x_0^3 - 1\right) - x_3\left(x_1^3 - x_3^3\right)\right)$. The restriction of π to a map $\pi^{-1}(U_1) \to U_1$ corresponds to the k-algebra-homomorphism

$$k[t] \to R, t \mapsto x_3.$$

The inclusion $\{\eta\} \to U_1$ corresponds to the inclusion $k[t] \to k(t)$. The fibre product $\pi^{-1}(U_1) \times_{U_1} \{\eta\}$ is thus given as the spectrum of the tensor product

$$R \otimes_{k[t]} k(t).$$

Because t maps to $x_3 \in R$, this k-algebra is isomorphic to

$$k(t)[x_0, x_1]/\left(\left(x_0^3 - 1\right) - t\left(x_1^3 - t^3\right)\right).$$

The generic fibre of π is thus the curve in $\mathbb{P}^2_{k(t)}$ defined by the affine equation

$$\left(x_0^3 - 1\right) - t\left(x_1^3 - t^3\right) = 0$$

or the projective equation

$$\left(x_0^3 - x_2^3\right) - t\left(x_1^3 - t^3 x_2^3\right) = 0. \tag{2.5}$$

This curve is smooth over $k(t)$. By substituting $t = c$ in Eq. (2.5) we obtain Eq. (2.4), the equation of the fibre above $(1:c)$. In this sense, the generic fibre can be seen as a template for the other fibres of π.

We will see later that there is in fact a $1:1$-correspondence between elliptic surfaces over a fixed base curve C and elliptic curves over the function field $k(C)$.

2.2.1 THE GROUP OF SECTIONS

Since almost all fibres of an elliptic fibration $\pi: S \to C$ are smooth, we naturally get a group structure in each individual smooth fibre so long as there is a uniform choice of one specified point per fibre. In order to make such a choice, we will make use of the sections of π. A section of π is a morphism $\sigma: C \to S$ that satisfies $\pi \circ \sigma = \mathrm{id}_C$. In particular, a section gives precisely one point per fibre, and after fixing one section $o: C \to L$ as the zero section, we can define a group structure on the set of sections by viewing a section as a collection of one point on each fibre of π and adding two sections component-wise. In the following proposition we verify that this addition is well-defined.

2.22 Proposition ([28], Chapter III, Proposition 3.10 (a))**.** *Let $\pi: S \to C$ be an elliptic fibration with a section o and let σ_1, σ_2 be two sections of π. Then there is a unique section $\sigma: C \to S$ such that for any $t \in C$ with a smooth fibre above t, we have*

$$\sigma(t) = \sigma_1(t) + \sigma_2(t),$$

where the addition is carried out in the elliptic curve given by the fibre above t with zero point $o(t)$.

2.23 Corollary. *Let $\pi\colon S \to C$ be an elliptic fibration with a section o. Then the set of sections of π obtains a natural group structure with the addition defined as above. We call o the zero section of S.*

Proof. This follows immediately from the fact that the points on any given smooth fibre form a group. □

To formalise the idea from earlier that there is a certain equivalence between elliptic surfaces and their generic fibres, it can be shown that the group structure on the sections of an elliptic surface S over a curve C is isomorphic to the group structure of the generic fibre as an elliptic curve over $k(C)$.

2.24 Proposition. *Let $\pi\colon S \to C$ be an elliptic fibration with zero section o. Then the group of sections as described above is isomorphic to the group of points of the generic fibre as an elliptic curve over $k(C)$.*

Proof. See Chapter III, Proposition 3.10 (c) in [28]. □

2.2.2 KODAIRA'S CLASSIFICATION OF FIBRES

In this section let k be an algebraically closed field.

As we have seen in Example 2.21, not all fibres of an elliptic fibration are smooth. In fact, we are particularly interested in the singular fibres, since the the lines intersecting a fixed line L on a smooth quartic surface $S \subset \mathbb{P}^3$ all lie in singular fibres of the corresponding elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Based on the work of Kunihiko Kodaira [14, 15, 16] and André Néron [19], these singular fibres can be classified into a finite number of different types. We will only be using Kodaira's notation.

2.25 Theorem (Classification of fibre types)**.** *Let $\pi\colon S \to C$ be an elliptic fibration. Then every fibre of π falls under one of the types listed in [28], Chapter IV, Theorem 8.2.*

The same source also contains a table with an overview of the fibre types ([28], p.365), which we will be frequently referring to. This table lists various properties of the fibre types that we will rely on at many points, such as the group structure on the smooth points of each singular fibre which is induced by the group of sections.

In the case of a smooth quartic surface $S \subset \mathbb{P}^3$ and the elliptic fibration $\pi\colon S \to \mathbb{P}^1$ associated to a given line $L \subset S$, only six of these fibre types can occur.

2.26 Example. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a line $L \subset S$. Let $\pi\colon S \to \mathbb{P}^1$ be the morphism associated to L as defined in Proposition 1.6.

Then π is an elliptic fibration and each singular fibre of π is of type I_1, I_2, I_3, *II*, *III*, or *IV*.

Proof. Note that π is an elliptic fibration, since the fibres of π are plane cubic curves, almost all of which are smooth; and smooth plane cubics have genus 1. Furthermore, the fibres are reduced by Lemma 1.5.

An irreducible, reduced plane cubic can either be smooth, have a node, or have a cusp, corresponding to Kodaira types I_0, I_1 and *II* respectively. A reducible, reduced plane cubic can either consist of a line and a smooth conic, or three lines. The first case corresponds to Kodaira types I_2 and *III*, depending on whether the line intersects the conic transversally or tangentially. The second case corresponds to Kodaira types I_3 and *IV*, depending on whether the three lines form a non-degenerate triangle or intersect in a single point. $\square$

Nonetheless, the remaining fibre types will still be relevant when we consider base changes as discussed in the following section.

2.2.3 SMOOTH MINIMAL MODELS AND BASE CHANGE

In this section let k be any field of characteristic not equal to 2, 3, not necessarily algebraically closed.

As was mentioned at the beginning of this chapter, inflection points in the fibres of elliptic fibrations are of particular interest to us. In the case of a smooth quartic surface $S \subset \mathbb{P}^3$ containing a line $L \subset S$ with corresponding elliptic fibration $\pi\colon S \to \mathbb{P}^1$, every line in a fibre of π necessarily intersects L in a point that is an inflection on the fibre. Furthermore, Definition 2.5 also warrants a closer look at inflection points in the fibres of π. We have seen in Corollary 2.13 that inflection points on elliptic curves have an inherent connection to 3-torsion points. In general, the elliptic fibration $\pi\colon S \to \mathbb{P}^1$ does not necessarily have a section at all, let alone a 3-torsion section. However, if L is inflectious we can construct another elliptic surface S' with fibration $\pi'\colon S' \to C$ which does have 3-torsion sections.

For an elliptic curve, we can ensure the existence of 3-torsion points simply by extending the ground field. In fact, over an algebraically closed field k, any elliptic curve has an n-torsion subgroup of order n^2 so long as n is not a multiple of char k ([29], chapter III, Corollary 6.4). In particular, for any elliptic curve E over k, we can always find some algebraic extension ℓ/k such that E has non-trivial 3-torsion over ℓ. For an elliptic surface, we can pursue a similar idea, but instead of extending the ground field, we ‘extend’ the base curve. This is called a base change.

Let S be an elliptic surface with fibration $\pi\colon S \to C$. The analogue of a field extension ℓ/k would be a morphism of curves $\varphi\colon B \to C$ for some other smooth projective curve B. We can then obtain a new surface S' by taking the fibre product

$$\begin{tikzcd} S' = S \times_C B \arrow[r] \arrow[d] & S \arrow[d] \\ B \arrow[r] & C. \end{tikzcd}$$

A priori, the fibre of $S' \to B$ above a point $P \in B$ is the same as the fibre of π above $\varphi(P)$. However, the resulting surface S' might not be smooth. To resolve these possible singularities, we will use blowups.

Let $P \in S$ be a point on an algebraic surface S. The blowup of S at P is a surface $\tilde{S}$ with a morphism $\psi\colon \tilde{S} \to S$ such that ψ induces an isomorphism between $\tilde{S} \setminus \psi^{-1}(P)$ and $S \setminus \{P\}$, and $\psi^{-1}(P)$ is a curve $E \subset \tilde{S}$, which we call the exceptional curve of the blowup. A precise description of a general blowup can be found in [9], pp. 23-30, 163.

It should be noted that if S admits an elliptic fibration $\pi\colon S \to C$ to a curve C, then a blowup of S as an algebraic surface at a point P retains the elliptic fibration by composing it with the blowup map.

2.27 Definition. An elliptic surface S is called minimal if none of its fibres contain (-1)-curves, i.e. smooth irreducible curves with genus 0 and self-intersection -1.

Such curves naturally occur when a smooth point of a surface is blown up. In fact, every (-1)-curve is the exceptional curve of some blowup ([9], chapter V, Theorem 5.7). Thus by reversing that blowup we obtain a 'smaller' surface in the same birational equivalence class. A surface in general is called minimal if it contains no (-1)-curves at all, but for elliptic surfaces we only consider (-1)-curves in the fibres. This is because every (-1)-curve that arises as the result of a blowup of an elliptic surface is automatically contained within the same fibre as the point that was blown up.

2.28 Definition. Let $\tilde{\pi}\colon \tilde{S} \to C$ be an elliptic fibration. We say that S is a model of $\tilde{S}$, if S is an elliptic surface with elliptic fibration $\pi\colon S \to C$ and a birational map $S \to \tilde{S}$ such that the triangle

$$\begin{tikzcd} S \arrow[rr, dashed] \arrow[dr, "\pi"'] & & \tilde{S} \arrow[dl, "\tilde{\pi}"] \\ & C & \end{tikzcd}$$

commutes.

A smooth minimal model of $\tilde{S}$ is a model of $\tilde{S}$ that is smooth as an algebraic surface and minimal as an elliptic surface.

2.29 Proposition (Corollary II.1.3 in [18]). *Every elliptic surface over k has a smooth minimal model, and it is unique up to isomorphism.*

Using the language of minimal models, we can define a unique base change by replacing the fibre product from above by its unique smooth minimal model.

2.30 Definition. Let $\pi\colon S \to C$ be an elliptic fibration and $\varphi\colon B \to C$ a morphism of curves. Consider the cartesian diagram

$$\begin{tikzcd} S \times_C B \arrow[r] \arrow[d] & S \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

Let S' be a smooth minimal model of $S \times_C B$. We call S' the base change of S along $\varphi\colon B \to C$.

Of course during such a base change, blowups can change the types of some of the singular fibres. It is thus important to keep track of what happens to singular fibres under a base change. This is well understood, and depends primarily on the ramification behaviour in the morphism between the base curves.

2.31 Proposition (Table 3 in [23]). *Let $\pi\colon S \to C$ be an elliptic fibration and $\varphi\colon B \to C$ a surjective morphism of smooth irreducible curves. Let S' be the base change of S along φ with elliptic fibration $\pi'\colon S' \to B$. Let $t \in B(k)$ be a point and let e be the ramification index of φ at t. Let $F_{\varphi(t)}$ be the fibre of π above $\varphi(t)$ and let F_t be the fibre of π' above t.*

a) If $F_{\varphi(t)}$ is of type I_n, then F_t is of type I_{en}

b) If $F_{\varphi(t)}$ is of type II, then F_t is of type

- I_0, *if* $e \equiv 0 \pmod 6$
- *II, if* $e \equiv 1 \pmod 6$
- *IV, if* $e \equiv 2 \pmod 6$
- I_0^*, *if* $e \equiv 3 \pmod 6$
- *IV*, if* $e \equiv 4 \pmod 6$
- *II*, if* $e \equiv 5 \pmod 6$

c) If $F_{\varphi(t)}$ is of type III, then F_t is of type

- I_0, *if* $e \equiv 0 \pmod 4$
- *III, if* $e \equiv 1 \pmod 4$
- I_0^*, *if* $e \equiv 2 \pmod 4$

- *III**, *if* $e \equiv 3 \pmod 4$

d) If $F_{\varphi(t)}$ *is of type IV, then* F_t *is of type*

- I_0, *if* $e \equiv 0 \pmod 3$
- *IV, if* $e \equiv 1 \pmod 3$
- *IV**, *if* $e \equiv 2 \pmod 3$

3 | Quartic Surfaces

In this entire chapter, assume that k is an algebraically closed field of characteristic not equal to 2 or 3. We will prove that any smooth quartic surface in $\mathbb{P}^3(k)$ contains at most 64 lines by following the method from [20].

3.1 | Some examples of lines on smooth quartics

While the general smooth quartic contains no lines, there is a family of examples which each contain sixteen designated lines. We will follow the description from [1]. In particular, we will be looking at surfaces of the form

$$S = Z(\varphi - \psi) \subset \mathbb{P}^3$$

where φ is a homogeneous quartic polynomial in x_0, x_1 and ψ is a homogeneous quartic polynomial in x_2, x_3. Now φ cuts out four points on the line $\Phi = Z(x_2, x_3)$, say $P_1, \ldots, P_4$ and ψ cuts out the four points $Q_1, \ldots, Q_4$ on the line $\Psi = Z(x_0, x_1)$. No two of the points P_i or Q_i can coincide, or S would be singular. Indeed, for any i and j in $\{1,2,3,4\}$, the points P_i and Q_j must differ because the lines Φ and Ψ are disjoint; and if $P_i = P_j$ for distinct i and j, then the surface in $\mathbb{P}^3$ defined by φ is singular. Because the polynomial ψ contains no x_0 or x_1, it follows that S is singular as well. On the other hand, if the points P_i and Q_i are all distinct, then S is smooth.

In this case we see that S contains at least 16 lines, namely the lines $\overline{P_i Q_j}$ connecting the points P_i and Q_j, for any $i, j \in \{1, \ldots, 4\}$. Indeed, if

$$P_i = (0 : 0 : x_2 : x_3)$$
$$Q_j = (x_0 : x_1 : 0 : 0),$$

then a point on the line $\overline{P_i Q_j}$ has projective coordinates $(\lambda x_0 : \lambda x_1 : \mu x_2 : \mu x_3)$ for some $(\lambda : \mu) \in \mathbb{P}^1$ and the polynomials φ and ψ both vanish at this point.

Now the intersection of S with the tangent plane T of S at a point P_i consists precisely of the four lines $\overline{P_i Q_1}, \ldots, \overline{P_i Q_4}$, and any other line on S cannot pass through P_i, since more than two lines on S intersecting in one smooth point must be coplanar. Fixing a line $L \subset S$ that is not among the 16 lines $\overline{P_i Q_j}$ thus gives us a homography

$$\rho : \Phi \to \Psi$$

mapping a point $P \in \Phi$ to the point obtained by intersecting the plane through P and L with the line Ψ. This map sends the points P_i to $Q_{\sigma(i)}$ for some permutation $\sigma \in \mathfrak{S}_4$. Indeed, the plane through P_i and Ψ intersects S in four lines, and L must intersect the plane on one of the lines $\overline{P_i Q_j}$, so $\rho(P_i) = Q_j$. After a coordinate change in x_0, x_1, we can assume that ρ is given as $\mathbb{P}^1(x_0, x_1) \to \mathbb{P}^1(x_2, x_3), (\alpha : \beta) \mapsto (\alpha : \beta)$, that the line L is the line $Z(x_0 - x_2, x_1 - x_3)$, and the polynomials φ, ψ satisfy $\varphi(\alpha, \beta) = \psi(\alpha, \beta)$ for all $(\alpha : \beta) \in \mathbb{P}^1$. The line L is now a member of a set of four lines $Z(x_0 - i^k x_2, x_1 - i^k x_3)$ for $k = 1, 2, 3, 4$, which are all contained in S.

Conversely, if we are given a isomorphism $\rho : \Phi \to \Psi$ which maps $\{P_1, \dots, P_4\}$ to $\{Q_1, \dots, Q_4\}$, then we can obtain four lines in the same way. The number of lines on S is thus 16 plus four times the number of such isomorphisms.

Note that an isomorphism between two projective lines is determined uniquely by the images of three different points, so generically there is no isomorphism mapping a given set of four points to another given set of four points. Since there are no constraints on the choice of points P_i and Q_i, the surface S will generically only contain the 16 already determined lines.

If there is an isomorphism $\rho : \Phi \to \Psi$ that maps the set $\{P_1, \dots, P_4\}$ to $\{Q_1, \dots, Q_4\}$, then ρ gives a natural bijection between the set of such isomorphisms and the set of automorphisms of Φ that permutes the points $P_1, \dots, P_4$.

We thus need to determine the number of automorphisms of the projective line that permute four given points. Barth [1] achieves this by considering the double cover, an elliptic curve E with a morphism $E \to \Phi$ that ramifies precisely above the four points. Then one can establish a relation between the automorphisms of $\mathbb{P}^1$ fixing four points, and the elliptic curve automorphisms of the double cover.

However, one can also do this more directly. After a change in coordinates, we can assume that three of the four points are given by $0, 1$, and ∞. The fourth point is then given by some $\lambda \in k \setminus \{0,1\}$. For each permutation σ of the set $\{0,1,\lambda,\infty\}$, we can compute the unique automorphism f of $\mathbb{P}^1$ that satisfies

$$f(0) = \sigma(0)$$
$$f(1) = \sigma(1)$$
$$f(\infty) = \sigma(\infty)$$

and then determine an equation that λ must fulfil so that $f(\lambda) = \sigma(\lambda)$ is also satisfied.

This is done in detail in Proposition A.1 in the appendix. The conclusion is that there are either 4, 8, or 12 automorphisms. Since each automorphism corresponds to four lines on the surface S, and taking into account the possibility that there is no isomorphism $\Phi \to \Psi$ that fulfils our conditions, we find that S contains either 16, 32, 48, or 64 lines.

In the computations in Proposition A.1, it is also shown that the maximum number of lines occurs if and only if in the above notation $\lambda = -\zeta$ or $\lambda = -\zeta^2$, where ζ is a primitive cube root of unity in k. After a change in coordinates [3.a] the four points 0, 1, $-\zeta$, and ∞ can be transformed to the four points 0, ζ, 1, ζ^2, which are precisely the roots of the polynomial $p(x) = x^4 - x$.

In particular, we obtain the following example.

3.1 Example. The quartic surface given by the equation

$$x_0^4 - x_0 x_1^3 = x_2^4 - x_2 x_3^3$$

has precisely 64 lines. We will refer to this surface as the Schur quartic.

The name ‘Schur quartic’ is also used in [26] and [20]; it is named after Friedrich Schur who described it in 1882 [24]. This example will continue to be relevant in this chapter.

3.2 QUARTIC SURFACES WITHOUT INFLECTIOUS LINES

Recall the following definition from Chapter 2.

2.5 Definition. Let S be a smooth surface in $\mathbb{P}^3$ containing a line L. Let $\pi : S \to \mathbb{P}^1$ be the morphism as defined in Proposition 1.6.

Then L is called inflectious if for every $P \in L$, every point of intersection of L with the fibre $C_P = \pi^{-1}(\{\pi(P)\})$ is a point of inflection of C_P.

Otherwise we call L regular.

[3.a] In fact, this coordinate change can be given explicitly by the matrix $\begin{pmatrix} \zeta^2 & 0 \\ 1 & \zeta - 1 \end{pmatrix}$

3.2 Remark. In Proposition 1.6, the morphism π is not unique, and in fact depends on the choice of a line L' skew to L. However, we are primarily interested in the fibres of π, and the set of fibres is independent of this choice. Therefore, we will accept this slight imprecision and speak of 'the' morphism π associated to L.

The goal of this section is to prove that smooth quartic surfaces in $\mathbb{P}^3$ on which all lines are regular contain at most 64 lines. We follow the method from [20], which is based on [26].

For smooth quartic surfaces that do not contain any inflectious lines, we can establish an upper bound on the number of lines with some of the same ideas as in Chapter 1. We begin by proving an analogue of Proposition 1.7 for quartic surfaces with a regular line, where we shall provide additional details to the proof in [20].

3.3 Proposition. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. If $L \subset S$ is a regular line, then it is intersected by at most* 18 *other lines on S.*

Proof. After a coordinate change, we can assume that L is given by

$$x_2 = x_3 = 0.$$

We can then write an equation for S in the form

$$f := \sum_{1 \le i+j \le 4} x_2^i x_3^j \alpha_{i,j}(x_0, x_1) = 0$$

where each $\alpha_{i,j}$ is a homogeneous polynomial of degree $4-(i+j)$ in the variables x_0, x_1. Consider the morphism $\pi\colon S \to \mathbb{P}^1$ from Proposition 1.6. Since π only has finitely many singular fibres, there exists at least one smooth fibre. After a coordinate transformation in x_2, x_3, we can assume this fibre to lie in the plane $x_2 = 0$. Because all lines intersecting L lie in singular fibres of π, we can restrict our attention to planes of the form $H_\lambda = Z(x_3 - \lambda x_2)$ for $\lambda \in k$. In every such plane, the intersection $H_\lambda \subset S$ is given in $H_\lambda \simeq \mathbb{P}^2(x_0, x_1, x_2)$ by the equation

$$\sum_{1 \le i+j \le 4} \lambda^j x_2^{i+j} \alpha_{i,j}(x_0, x_1) = 0.$$

This is a plane quartic curve which contains L as a component, represented by the equation $x_2 = 0$. The residual cubic has equation

$$\sum_{1 \le i+j \le 4} \lambda^j x_2^{i+j-1} \alpha_{i,j}(x_0, x_1) = 0. \tag{3.1}$$

Intersecting this with L, which is given in the plane H_λ by the equation $x_2 = 0$, we obtain three points given by the cubic equation

$$g_\lambda(x_0,x_1) = \alpha_{1,0}(x_0,x_1) + \lambda\alpha_{0,1}(x_0,x_1) = 0$$

on $L \simeq \mathbb{P}^1(x_0,x_1)$. Now we want to investigate when these points are inflections of C_λ. We consider the Hessian of C_λ, obtained from Equation (3.1). After restricting it to L, i.e., setting $x_2 = 0$, this equation takes the form

$$h_\lambda(x_0,x_1) = \det\begin{pmatrix} \frac{\partial^2 g_\lambda}{\partial x_0^2} & \frac{\partial^2 g_\lambda}{\partial x_0 \partial x_1} & \frac{\partial}{\partial x_0}\sum_{i+j=2}\lambda^j\alpha_{i,j} \\ \frac{\partial^2 g_\lambda}{\partial x_0 \partial x_1} & \frac{\partial^2 g_\lambda}{\partial x_1^2} & \frac{\partial}{\partial x_1}\sum_{i+j=2}\lambda^j\alpha_{i,j} \\ \frac{\partial}{\partial x_0}\sum_{i+j=2}\lambda^j\alpha_{i,j} & \frac{\partial}{\partial x_1}\sum_{i+j=2}\lambda^j\alpha_{i,j} & 2\sum_{i+j=3}\lambda^j\alpha_{i,j} \end{pmatrix} = 0.$$

We now claim that because L is a regular line, the polynomials g_λ and h_λ cannot have any common factors in $k[x_0,x_1,\lambda]$. To see this, we first show that g_λ is irreducible. Indeed, since g_λ is linear in λ, it can only be reducible if $\alpha_{1,0}$ and $\alpha_{0,1}$ have a common factor. Because the ground field k is algebraically closed, we can write both as the product of three linear forms. If there is one linear factor that divides both, then after a linear coordinate change, we can assume that this common factor is x_1. Consider the affine open subset $\{x_0 = 1\}$ of $\mathbb{P}^3$. In this subset, the surface S is given by the equation

$$\sum_{1\le i+j\le 4} x_2^i x_3^j \alpha_{i,j}(1,x_1) = 0.$$

Every monomial of this equation is at least quadratic, since $\alpha_{0,1}(1,x_1)$ and $\alpha_{1,0}(1,x_1)$ are both multiples of x_1. In particular, the origin is a singular point, and consequently, the point $(1:0:0:0)$ is a singular point on the projective surface $S \subset \mathbb{P}^3$. Hence, the polynomial g_λ must be irreducible in $k[x_0,x_1,\lambda]$. The only way for g_λ and h_λ to have a common factor is thus if h_λ is a multiple of g_λ. However, if that were the case, then for every $\lambda \in k$, the three points on L where g_λ vanishes, i.e. the three points in the intersection of L with the cubic curve C_λ would also be roots of h_λ, i.e. inflections of C_λ. This would then imply that L is inflectious, contrary to our assumption.

With g_λ and h_λ being coprime, their resultant with respect to x_0 is nonzero. Because g_λ and h_λ are both homogeneous in x_0,x_1, the resultant is of the form

$$r(\lambda)\cdot x_1^k$$

for some polynomial $r \in k[\lambda]$ and $k \in \mathbb{N}$. We can thus compute r as the determinant of the Sylvester matrix of $g_\lambda(x_0,1)$ and $h_\lambda(x_0,1)$ with respect to x_0. Now, noting that $\alpha_{i,j}(x_0,1)$ has degree at most $4-(i+j)$ in x_0, the degrees of g_λ and h_λ are both at most 3. This results in a 6×6 Sylvester matrix, where the coefficients of g_λ have degree at most 1 in λ, and the coefficients of h_λ have degree at most 5 in λ. The degree of $r(\lambda)$ is thus at most $3\cdot 1+3\cdot 5=18$.

As a consequence, at most 18 fibres of π can contain lines. Recall that the number of lines in any fibre is either zero, one, or three. A priori it might be possible that a given fibre contributes only one root to the polynomial $r(\lambda)$, but contains three lines. It thus remains to prove that this cannot be the case.

Suppose there is a fibre of π that consists of three lines. We can assume that this fibre lies in the plane $Z(x_3)$, corresponding to $\lambda=0$. By Equation (3.1), the residual cubic is given by

$$\sum_{i=1}^{4} x_2^{i-1}\alpha_{i,0}(x_0,x_1)=0. \tag{3.2}$$

We will now distinguish two different cases.

First we assume that there are two lines in this fibre whose intersection does not lie on L. In this case, we may assume that within the plane $Z(x_3)\simeq \mathbb{P}^2(x_0,x_1,x_2)$, they are given by the equations $x_0=0$ and $x_1=0$, respectively. Because both lines are contained in the fibre given by Equation (3.2), the polynomial x_0x_1 must divide $\sum_{i=1}^4 x_2^{i-1}\alpha_{i,0}(x_0,x_1)$, and hence each individual $\alpha_{i,0}$. We can thus write

$$\begin{aligned}
\alpha_{1,0} &= bx_0^2x_1+cx_0c_1^2\\
\alpha_{2,0} &= ax_0x_1\\
\alpha_{3,0} &= 0\\
\alpha_{4,0} &= 0.
\end{aligned}$$

A computer-aided computation (see Appendix A.2), which was not provided in [20], then confirms that indeed 0 is a triple root of $r(\lambda)$.

The second case is that two such lines do not exist. In this case, all three lines in the fibre and L must intersect in one single point, and we can assume that the lines are given by equations

$$\begin{aligned} x_1 &= 0 \\ bx_1 + x_2 &= 0 \\ cx_1 + x_2 &= 0. \end{aligned}$$

The fibre is thus given by the equation

$$x_1(bx_1 + x_2)(cx_1 + x_2) = 0.$$

After expanding this product, we can write

$$\begin{aligned} \alpha_{1,0} &= bcx_1^3 \\ \alpha_{2,0} &= (b+c)x_1^2 \\ \alpha_{3,0} &= 0 \\ \alpha_{4,0} &= 0. \end{aligned}$$

Another computer-aided computation (see Appendix A.2), which again was omitted in [20] will reveal that in this case, 0 is in fact a root of degree 5 of $r(\lambda)$.

To conclude, every fibre that contains three lines also contributes at least three roots to $r(\lambda)$, and thus there can only be at most $\deg(r) = 18$ lines on S intersecting L. □

3.4 Remark. This proof also implies that the seemingly weaker definition of an inflectious line from [20], which was mentioned in Remark 2.7, is equivalent to the one from Definition 2.5.

Indeed, we have implicitly shown that a regular line can only intersect at most 18 fibres of its corresponding elliptic fibration in a point of inflection. Therefore, a line that is of the second kind in the sense of [20] must also be inflectious in the sense of Definition 2.5.

For a cubic surface $S \subset \mathbb{P}^3$, recall that lines in fibres of the morphism $\pi\colon S \to \mathbb{P}^1$ always come in pairs, and a trio of coplanar lines is the complete intersection of S with the corresponding plane. Whenever there is a plane such that its intersection with S consists only of lines, every other line on the surface must intersect exactly one of the lines in this plane. In the case of a smooth quartic surface $S \subset \mathbb{P}^3$, a fibre of π can contain either zero, one, or three lines, and only the latter case allows us to use this reasoning. However, this case is common enough to still be useful.

3.5 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. If a line $L \subset S$ is intersected by more than twelve other lines on S, then three of those lines are coplanar.*

Proof. Assume that L is not intersected by three coplanar lines on S. Then all the lines intersecting L occur in fibres of type I_2 or III. Each of those fibres has an Euler-Poincaré characteristic of at least 2. Since the Euler-Poincaré characteristics of all singular fibres must add up to $e(S) = 24$ [3.b], there can be at most twelve such fibres. $\square$

Another tool we will use is the flecnodal divisor. The term flecnode classically refers to a point on a plane curve where an inflection and a node coincide. However, we will use a more general definition. We consider a point P on a curve C a flecnode if there is a line L that intersects C with multiplicity 4 at the point P. Note that a flecnode in the classical sense satisfies this definition, because one can take L to be the tangent line at P of the branch of C that has an inflection at P. The more general definition also includes cases such as tacnodes or smooth points on lines. This motivates the following definition of a flecnodal point on a smooth surface.

3.6 Definition. Let S be a smooth algebraic surface in $\mathbb{P}^3$. A point $P \in S$ is called a flecnodal point of S if there exists a line $L \subset \mathbb{P}^3$ such that

$$i_P(L,S) \geq 4.$$

The definition of a flecnodal point of a surface is closely related to the definition of a flecnode of a plane curve. Let $P \in S$ be a point on S and let H be the tangent plane of S at P. If $L \subset \mathbb{P}^3$ is a line with $i_P(L,S) \geq 4$, it must be contained in H, and we can compute the local intersection number $i_P(L,S) = i_P(L, S \cap H)$ within the plane $H \cong \mathbb{P}^2$. Since $i_P(L,S) \geq 4$, the point P is a flecnode of the plane curve $S \cap H$.

Flecnodal points on S are useful because every point P on a line $L \subset S$ is a flecnodal point. Indeed, the line L itself intersects S with infinite multiplicity at each of its points. We now consider the following result about flecnodal points.

3.7 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth projective surface of degree $d \geq 3$. Then there is a surface $F \subset \mathbb{P}^3$ of degree $(11d - 24)$ such that the intersection $F \cap S$ consists precisely of the flecnodal points of S. Furthermore, if the characteristic of k does not divide $d(d-1)(d-2)$, then S is not a component of F.*

[3.b] See [12], p. 12. The Euler characteristic here is obtained via the second Chern number $c_2(X)$, which is computed on p. 8

Proof. For the existence and degree of F, see [3], p.102f. In [17], Lemma 2.10, it is shown that S is not contained in F in characteristic zero. Lastly, the result for positive characteristic is proven in Theorem 1 in [30]. □

3.8 Remark. For cubic and quartic surfaces, the assumption that the characteristic p of the ground field k does not divide $d(d-1)(d-2)$ is equivalent to the assumption that p is not 2 or 3.

This lemma shows that the locus of the flecnodal points of S is a divisor $\mathcal{F}_S$ on S of degree $d(11d-24)$.

3.9 Definition. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. Then we denote by $\mathcal{F}_S$ the divisor of flecnodal points on S and refer to it as the flecnodal divisor.

3.10 Remark. Note that in the case of quartic surfaces, the divisor $\mathcal{F}_S$ is linearly equivalent to $20 \cdot H$ for any plane section $H \subset S$, because it arises as the intersection of S with a surface of degree $11 \cdot 4 - 24 = 20$. In particular, for any curve $C \subset S$, the intersection number $C \cdot \mathcal{F}_S$ is given by

$$C \cdot \mathcal{F}_S = C \cdot (20H) = 20(C \cdot H) = 20 \deg C$$

where $H \subset S$ is a plane section. We will make use of this fact at multiple points.

Immediately we can see that all lines that lie on a smooth surface must be components of its flecnodal divisor, since all points lying on such lines are flecnodal, as mentioned right before Lemma 3.7. In particular, we obtain an upper bound on the number of lines on any smooth surface of degree at least 3.

3.11 Corollary. *Let $S \subset \mathbb{P}^3$ be a smooth algebraic surface of degree $d \geq 3$. Assume that the characteristic of k does not divide $d(d-1)(d-2)$. Then S contains at most $d(11d-24)$ lines.*

For $d = 3$, this works out to be 27 and the upper bound is in fact optimal. For quartic surfaces however, we have $4(11 \cdot 4 - 24) = 80$, which is strictly larger than the actual maximum of 64. We can still make use of the flecnodal divisor if we can show that it has some components which are either duplicates of some lines, or not lines at all.

We will now prove the main theorem of this section, originally proven by Segre [26], but here presented following the more modern proof from [20].

3.12 Theorem (Segre, 1943). *Let $S \subset \mathbb{P}^3(k)$ be a smooth quartic surface over an algebraically closed field k whose characteristic is not 2 or 3. Assume that all lines on S are regular as per Definition 2.5. Then S contains at most 64 lines.*

Proof. Assume that all lines on S are regular. This proof will be split into several distinct cases.

Let us first assume that S contains four coplanar lines. Then these four lines form the intersection of S with some plane. Consequently, all other lines on S must intersect at least one of these four lines. Since each can only be intersected by at most 15 additional lines by Proposition 3.3, there can be at most $4+4\cdot 15 = 64$ lines on S.

For the rest of the proof, we will always assume that S does not contain four coplanar lines. In particular, whenever two lines on S intersect in a plane H, the intersection $S\cap H$ consists precisely of the two lines and a smooth conic. Furthermore, by Lemma 3.5, every line on S intersects at most twelve other lines.

Now we consider the case where S contains two intersecting lines L_1 and L_2 such that the coplanar conic is not a component of the flecnodal divisor $\mathcal{F}_S$. By Remark 3.10, the conic then has an intersection number of 40 with $\mathcal{F}_S$, and since all lines are components of $\mathcal{F}_S$, the conic can intersect at most 40 lines on S, where L_1 and L_2 are each included twice. Since every additional line on S has to intersect either L_1, L_2 or the conic, there can be at most $2+2\cdot 11+(40-2\cdot 2) = 60$ lines on S.

We can now assume that for any two intersecting lines on S, the corresponding conic is a component of $\mathcal{F}_S$.

If S contains at least eight such pairs, then the eight corresponding conics form components of total degree 16 of $\mathcal{F}_S$. Since $\deg\mathcal{F}_S = 80$, all other components of $\mathcal{F}_S$ have a combined degree of $80-16 = 64$. Since all lines on S are components of $\mathcal{F}_S$, the surface S can contain at most 64 lines in this case.

The last remaining case is that S contains exactly n pairs of incident lines where $0 \le n \le 7$. Any line on S that is not contained in any of these n pairs intersects no other line on S. Consider now the Picard group of S. Note that any set of r disjoint lines on S is $\mathbb{Z}$-linearly independent in $\operatorname{Pic} S$. Indeed, let $L_1,\dots,L_r \subset S$ be mutually disjoint lines, and assume we have a relation

$$a_1L_A + \cdots + a_rL_r = 0$$

with coefficients $a_j \in \mathbb{Z}$. Then we can intersect the divisor on the left hand side with any line L_j for some fixed $j \in \{1,\dots,r\}$. Because $L_j\cdot L_i = 0$ for all $i \neq j$ and $L_j\cdot L_j = -2$, we conclude $-2a_j = 0$ and thus $a_j = 0$. Because this holds for all j, the lines are linearly independent in $\operatorname{Pic} S$.

In particular, such a set of r mutually disjoint lines forms a free sub-$\mathbb{Z}$-module of $\operatorname{Pic} S$ of rank r. Consequently, r is less than or equal to the rank of $\operatorname{Pic} S$. Because $\operatorname{Pic} S$ has rank at most 22 [3.c], there can be at most 22 pairwise skew lines on S. Thus in total,the surface S contains at most $2n+22 \leq 36$ lines in this case. $\square$

3.3 FIBRE TYPES

As mentioned before, inflectious lines evade the reasoning from Proposition 3.3. If a line L on a smooth quartic surface $S \subset \mathbb{P}^3$ is regular, then we saw in Proposition 3.3 that there can be at most 18 fibres of the morphism $\pi_L : S \to \mathbb{P}^1$ from Proposition 1.6 which intersect L in a point that is an inflection of the fibre. These are the only fibres in which lines can occur. On the other hand, if L is inflectious, every fibre of π is a priori a potential candidate for containing lines. In order to establish a sharp upper bound on the lines intersecting an inflectious line L, we thus need to examine more closely which Kodaira types the fibres of the morphism π can have. This depends on the ramification of the curve morphism $\pi_L|_L : L \to \mathbb{P}^1$.

Our goal now is to make use of the theory of elliptic surfaces, but a priori the elliptic fibration π does not necessarily have a section. To deal with this problem, we will use a base change as described in Chapter 2.

3.13 Proposition. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing an inflectious line L. Consider the morphism $\pi = \pi_L : S \to \mathbb{P}^1$ from Proposition 1.6, which makes S into an elliptic surface. Then there exist a smooth curve B and a base change*

$$\begin{tikzcd} S_2 \arrow[r, dashed] \arrow[d, "\pi_2"'] & S \arrow[d, "\pi"] \\ B \arrow[r, "\varphi"'] & \mathbb{P}^1 \end{tikzcd}$$

such that the elliptic fibration $S_2 \to B$ in the diagram above has non-trivial 3-torsion sections.

Proof. Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing an inflectious line L and let $\pi = \pi_L$ be the associated elliptic fibration.

Consider the morphism of curves $L \to \mathbb{P}^1$ induced by π, and the fibre product

$$\begin{tikzcd} S \times_{\mathbb{P}^1} L \arrow[r] \arrow[d] & S \arrow[d] \\ L \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

[3.c]This holds for all K3 surfaces, see [12], Remark 1.3.7. For complex K3 surfaces, this bound can be improved to 20 ([12], p.11).

Consider now the identity $L \to L$ and the inclusion map $L \to S$. The diagram

$$\begin{tikzcd} L \arrow[drr, bend left] \arrow[dr, bend right] & & \\ & S \times_{\mathbb{P}^1} L \arrow[r] \arrow[d] & S \arrow[d] \\ & L \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

commutes and the universal property gives a morphism $L \to S \times_{\mathbb{P}^1} L$ which by construction is a section of the fibration $S \times_{\mathbb{P}^1} L \to L$. By Proposition 2.29, we can find a smooth minimal elliptic surface S_1 over L that is a model of $S \times_{\mathbb{P}^1} L$ with a section.

Now note that the morphism $L \to \mathbb{P}^1$ has degree 3 and corresponds to a field extension $k(L)/k(t)$ of degree 3. Because the characteristic of k is not 3, this extension is separable. Its Galois closure is the function field $k(B)$ of some smooth curve B, and any embedding $k(L) \to k(B)$ corresponds to a curve morphism $B \to L$. Because $[k(L) : k(t)] = 3$, there are three such embeddings. After fixing one, we can do another base change, by considering the cartesian diagram

$$\begin{tikzcd} S_1 \times_L C \arrow[r] \arrow[d] & S_1 \arrow[d] \\ B \arrow[r] & L. \end{tikzcd}$$

For each of the three morphisms $B \to L$, we obtain a map $B \to S_1$ by composing it with the section $L \to S_1$. Similarly to the construction above, by the universal property of the fibre product, these three maps $B \to S_1$, together with the identity map $B \to B$, induce three different sections $B \to S_1 \times_L B$. After replacing the elliptic surface $S_1 \times_L B$ with a smooth minimal model, the sections are retained and we have a commutative base change diagram

$$\begin{tikzcd} S_2 \arrow[r, dashed] \arrow[d] & S_1 \arrow[r, dashed] \arrow[d] & S \arrow[d] \\ B \arrow[r] & L \arrow[r] & \mathbb{P}^1, \end{tikzcd}$$

where the dashed arrows represent rational maps.

We shall fix the three sections and call them o, σ_1, and σ_2. Let o be the zero section. We now claim that σ_1 and σ_2 are 3-torsion sections and each other's inverses. It is enough to show this on the smooth fibres of $S_2 \to B$.

Let F be such a smooth fibre, and let O, Σ_1, Σ_2 be the three points on F intersected by o, σ_1, σ_2 respectively. Consider the base change diagram

$$\begin{tikzcd} S_2 \arrow[r, dashed, "\Phi"] \arrow[d, "\pi_2"'] & S \arrow[d, "\pi"] \\ B \arrow[r, "\varphi"'] & \mathbb{P}^1. \end{tikzcd}$$

Note that this base change preserves all but finitely many smooth fibres of π. More precisely, Let $P \in B$ such that the map φ is unramified at P and let F be the fibre of π_2 above P. Then the rational map Φ induces an isomorphism from F to the fibre F_S of π above $\varphi(P)$. By construction, the three points O, Σ_1, and Σ_2 then correspond to the three points of intersection of F_S and L. In particular, they are collinear. Furthermore, since L is inflectious, all three points are points of inflection of F. Since O is the zero point of the elliptic curve F, we can conclude by Corollary 2.12 that

$$O = O + \Sigma_1 + \Sigma_2 = \Sigma_1 + \Sigma_2.$$

By Corollary 2.13 it follows that they are indeed 3-torsion points of F.

Note that φ can only ramify at finitely many points, and therefore this holds for all but finitely many smooth fibres of π_2. Therefore, indeed σ_1 and σ_2 are 3-torsion sections and each other's inverses. □

3.14 Remark. Because the group of sections of an elliptic surface induces a group structure on the smooth points of each singular fibre by Theorem 5.22 in [25], and because 3-torsion sections are disjoint outside of characteristic 3 by Proposition 6.33 (v) in [25], S_2 can only contain singular fibres which admit non-trivial 3-torsion. By Table 4.1 on page 365 in [28], this leaves only fibres of type I_n, IV, and IV^*.

Note that Proposition 2.31 tells us exactly how singular fibres behave under such a base change $S_2 \dashrightarrow S$. In particular, the important factor is the local ramification behaviour of the morphisms of base curves $L \to \mathbb{P}^1$ and $B \to L$. We can use this to draw conclusions about the possible types of singular fibres on S.

However, before doing so, we can place some inherent restrictions on how an inflectious line can intersect singular fibres. Assume that $S \subset \mathbb{P}^3$ is a smooth quartic surface containg an inflectious line L. Let $I^0 \subset S$ be the set of inflection points on the smooth fibres. Further let L^0 consist of those points of L such that the corresponding fibre of π is smooth. Note that L^0 is dense in L because its complement consists of only finitely many points. Since L^0

is contained in I^0 by definition, because L is inflectious, it follows that the closure of L^0, which is L, must be contained in the closure of I^0.

The following lemma gives some information on how this closure can intersect singular fibres.

3.15 Lemma. *Let I^0 be the set of points $P \in S$ such that P lies on a smooth fibre of π and is an inflection of this fibre. Consider the closure I of I^0. Then the following holds for any singular fibre F of π. Note that by 'smooth point', we mean a point that is smooth on the fibre F.*

a) If F is of type I_1, then $F \cap I$ contains at most three smooth points.

b) If F is of type I_2, then $F \cap I$ contains at most three smooth points, all of which lie on the line component.

c) If F is of type I_3, then $F \cap I$ consists of three smooth points on each line.

d) If F is of type II, then $F \cap I$ contains exactly one smooth point.

e) If F is of type III, then $F \cap I$ contains exactly one smooth point, which lies on the line component.

f) If F is of type IV, then $F \cap I$ contains at most one smooth point on each line.

Proof. We will provide a sketch of the proof, expanding on a brief argument from [20].

By taking a sufficiently general plane section of S, we can find a curve $D \subset S$ such that D is smooth and intersects F transversally. Considering the morphism $\pi\colon S \to \mathbb{P}^1$ and its restriction $\pi|_D\colon D \to \mathbb{P}^1$, we have a base change

$$\begin{tikzcd} S \times_{\mathbb{P}^1} D \arrow[r] \arrow[d, "\rho"'] & S \arrow[d, "\pi"] \\ D \arrow[r, "\pi|_D"'] & \mathbb{P}^1. \end{tikzcd}$$

Similarly to the the proof of Proposition 3.13, the inclusion $D \to S$ together with the identity map on D induces a section $\tau\colon D \to S \times_{\mathbb{P}^1} D$ of ρ.

Like we did before, we can replace the fibre product $S \times_{\mathbb{P}^1} D$ with a smooth minimal model, an elliptic surface S' above D, and refer to this as the base change.

Because D intersects F transversally, the map $D \to \mathbb{P}^1$ is unramified at the transversal point of intersection on F. In particular, the fibre F is isomorphic to the corresponding fibre on the base change.

The section τ does not necessarily intersect each smooth fibre in a point of inflection, so the inflections do not necessarily form the 3-torsion subgroup of any smooth fibre when this section is chosen as the zero section. However, they do form a coset of the 3-torsion subgroup, as we will now show.

For ease of notation, let $s \in D$ be the point above which the fibre F lies. Note that on a smooth fibre, if $\tau(s)$ is a point of inflection, then by Corollary 2.12, any three collinear points add up to zero (i.e. to $\tau(s)$). Because the choice of a different zero point of such a curve corresponds to a translation in its Picard group by the definition in Corollary 2.10, even if $\tau(s)$ is not a point of inflection, any three collinear points add up to the same point, which we will refer to as L. In particular, this implies that for any inflection P on a smooth fibre, it holds that $3P = L$, and the inflection points on a smooth fibre are precisely the translation of the 3-torsion group by L.

This property also carries over to the smooth part of the singular fibres in the sense that the points in $F \cap I$ that are smooth points of F are precisely those points P such that $3P = L$, where L still denotes the sum of any three collinear points. Now consider each type separately, noting that the group structure on the fibre is given by Table 4.1 in [28].

a) A fibre of type I_1 has precisely three 3-torsion points on its smooth part. Any coset of this must also consist of three points on the smooth part of the fibre.

b) A type I_2 fibre has three 3-torsion points, so there are three smooth points in $F \cap I$. We want to show that these smooth points all lie on the line component.

   We claim the point L which is the sum of any three collinear points must lie on the line component. To see this, consider the proof of Corollary 2.12, while dropping the assumption that O is an inflection. It follows that the sum of any three collinear points is $\tau(s) * \tau(s)$, which is obtained by considering the tangent at $\tau(s)$ and then taking the third point of intersection of that tangent with the curve. Because this tangent intersects the conic with multiplicity 2, this point must always lie on the line component.

   Because the component group of F is $\mathbb{Z}/2\mathbb{Z}$, for any smooth point $P \in F$, the point $3P$ lies on the same component as P. In particular, all points that satisfy $3P = L$ must lie on the line component of F.

c) If F has type I_3, note that the 3-torsion subgroup of F has order 9, consisting of three points per component. The same must hold for any coset of this subgroup.

d) If F has type II, then there is no non-trivial 3-torsion, so there can be only one smooth point of F in the intersection $F \cap I$.

e) For type *III*, we can use the same argument as for type I_2, except the 3-torsion subgroup is trivial in this case, so there is only one instead of three smooth points on the line component.

f) Similarly, type *IV* is analogous with type I_3 with the difference being that there is only one 3-torsion point per component rather than three. The coset thus also consists of one smooth point per component.

□

3.16 Remark. The corresponding result in [20] extends also to the singular points of each fibre. It is claimed that the singular points of each fibre F are also contained in the set $F \cap I$, except in the case of type I_3.

Note that I intersects each fibre with multiplicity 9, and since for a fibre F of type I_3 the set $F \cap I$ already contains nine smooth points, it cannot also contain any of the three nodes. This is an important observation that we will use later.

In all other cases, we are content with the observation that the singular points could, but do not necessarily need to, be included in the set $F \cap I$. This will be sufficient for our purposes.

With this lemma in mind, we can now analyse the possible types of singular fibres depending on the ramification behaviour. Recall that a priori, the possible fibre types are I_1, I_2, I_3, *II*, *III*, and *IV* as we saw in Example 2.26.

3.17 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a line L with corresponding elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Let F be a singular fibre of π such that the map $L \to \mathbb{P}^1$ is unramified at F. Then F is of type I_1, I_3, or IV.*

Proof. Since F is an unramified fibre, it meets L in three distinct points. These points must be smooth points of F, since the intersection multiplicity at a singular point would be higher than one, so there could not be three distinct intersection points.

We want to show that this rules out fibres of type *II*, *III*, and I_2.

Indeed, consider first the case that F has type *II*. By Lemma 3.15, there is only one smooth point on F that can be intersected by L.

If F were of type *III* or I_2, then if the fibre were to be unramified, the line L would have to intersect it in two distinct smooth points on the conic. This is impossible by Lemma 3.15. □

Now we consider ramified fibres. These fall into two different categories, with different geometric behaviours. Namely, the degree 3 morphism $L \to \mathbb{P}^1$ can have either zero, one or two preimages at a ramified fibre. In order to make it easier to speak about the different types of ramification, we shall use the following terminology.

3.18 Definition. Let $\varphi\colon C_1 \to C_2$ be a curve morphism of degree 3. We say that φ is totally ramified at a point $P \in C_2$ if its ramification index is 3, i.e. it has one preimage under φ. If P has ramification index 2, i.e. precisely two preimages under φ, then we call it partially ramified.

Note that while ‘total ramification’ is standard terminology, ‘partial ramification’ is not. It simply happens to be useful in the case of degree 3 morphisms due to the limited number of possible ramification indices.

Before examining how local ramification indices affect possible fibre types, it is worth noting that there cannot be any points in $\mathbb{P}^1$ that ramify with index 6 in the Galois closure $B \to \mathbb{P}^1$. This is an important observation which is implicitly used in [20] while dealing with ramified fibres, without ever being explicitly stated. We shall give a proof here.

3.19 Lemma. *Let $\varphi\colon C_1 \to C_2$ be a morphism of smooth curves such that the corresponding extension of function fields $k(C_1)/k(C_2)$ is Galois and its Galois group is the symmetric group S_3.*

Then φ does not admit ramification of index 6.

Proof. Note that the order of the symmetric group S_3 is 6, and thus so is the degree of the morphism φ. In particular, if there is a point $P \in C_2$ such that φ ramifies above P with index 6, it follows that P has only one pre-image, say $Q \in C_1$.

Consider the decomposition group D of Q [3.∂] as defined on page 21 of [27]. Then we can consider the fixed field k_D of this decomposition group in the Galois extension $k(C_1)/k(C_2)$ and by loc. cit., the degree of the extension $k_D/k(C_2)$ is equal to the number of points on C_1 lying above P, which in this case is 1, as only Q lies above P. Therefore, the decomposition group must be equal to the Galois group $\mathrm{Gal}(k(C_1)/k(C_2))$, which is isomorphic to S_3.

Furthermore, the inertia group of Q as defined in loc. cit. is the kernel of a homomorphism ε from D to the Galois group of the extension of the residue fields of the local rings at Q and P. In this case, both residue fields are equal to the ground field k, and so this Galois group is trivial. It follows that the kernel of the group homomorphism ε, which is the inertia group, must be the entire decomposition group. As we have seen, the decomposition group is isomorphic to S_3, and so the same holds for the inertia group.

[3.∂] More precisely it would be the decomposition group of the maximal ideal in the local ring of C_1 associated to Q.

This however leads to a contradiction. To show this, we consider the ramification groups G_i as defined in Proposition 1 in Chapter IV, §1 in [27], where G_0 is the aforementioned inertia group. In the same section of op. cit., it is shown in Corollary 2 that G_0 must be cyclic if the characteristic of the ground field k is zero, which is a contradiction. In positive characteristic, we consider in the same section of op. cit. the Corollaries 1 and 3. Corollary 3 states that G_1, which is a normal subgroup of G_0 is a p-group, where $p = \operatorname{char} k$. Since we assume $\operatorname{char} k \neq 2, 3$, the only possible p-subgroup of S_3 is the trivial subgroup $\{1\}$. It follows that the quotient G_0/G_1 is equal to the group G_0 itself, but by Corollary 1, this quotient must be cyclic. We thus again arrive at a contradiction.

Therefore, ramification of index 6 is impossible. □

Note that if the morphism $L \to \mathbb{P}^1$ is already Galois, then the curve B is equal to L, and the morphism $B \to \mathbb{P}^1$ has degree 3. In this case, it is trivial that ramification of index 6 cannot occur. On the other hand, if $L \to \mathbb{P}^1$ is not Galois, then the field extension $k(B)/k(L)$ must have Galois group S_3, since it arises as the Galois closure of a degree 3 extension. Therefore, we can apply this lemma to the base change in Proposition 3.13.

3.20 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a line L with corresponding elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Let F be a singular fibre of π such that the map $L \to \mathbb{P}^1$ is ramified at F.*

a) If $L \to \mathbb{P}^1$ is partially ramified at the fibre F, then F is of type II.

b) If $L \to \mathbb{P}^1$ is totally ramified at the fibre F, then F is of type I_1, I_2, or IV.

Proof. To prove this, we will go through all fibre types that are possible in the fibration $S \to \mathbb{P}^1$ and determine under which ramification types they can occur. In the following we will always assume that F is a fibre of π such that the map $L \to \mathbb{P}^1$ ramified at F.

We start by considering fibres of type I_3. If F is an I_3 fibre, it consists of three lines, intersecting in three distinct points. If $L \to \mathbb{P}^1$ were to ramify at F, the line L would have to meet one of the three nodes. By Lemma 3.15, this is impossible.

If F has type *IV*, then with similar reasoning the line L it must meet the node of F. It immediately follows that $L \to \mathbb{P}^1$ is totally ramified at F.

If F has type I_2, then note that L can not intersect any smooth points of the conic component of F by Lemma 3.15. Furthermore, if L intersected both nodes, it would be equal to the line component of F, which is impossible. It follows that L must intersect one of the nodes tangentially to the conic, i.e. with multiplicity 3. Therefore, the map $L \to \mathbb{P}^1$ is totally ramified at F.

A similar argument shows that type *III* does not occur at all. Indeed, the line L can only intersect the conic component of a fibre F of type *III* in the tacnode, and would thus have to be tangential to the conic there. But this would make L identical to the line component of F, which is impossible.

Now assume that F has type I_1, and assume for the sake of contradiction that $L \to \mathbb{P}^1$ is partially ramified at F. Let $S_2 \to B$ be the base change from Proposition 3.13. Because the ramification is partial, the extension $k(L)/k(\mathbb{P}^1)$ cannot be Galois, and thus the extension $k(B)/k(L)$ must have degree 2. If $t \in \mathbb{P}^1$ is the point above which F lies, then from the fact that $B \to \mathbb{P}^1$ is Galois, it follows that there must be precisely three points on B lying above t, all with ramification index 2.

By Proposition 2.31 this leads to three fibres of type I_2 on the elliptic surface S_2. Since the component group of a fibre of type I_2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the three 3-torsion sections must then intersect the same component of each I_2 fibre.

However, we can also show that L must meet the node of F. Indeed because we assumed the map $L \to \mathbb{P}^1$ to partially ramify at F, the line L must meet one point of F with multiplicity 1, and another with multiplicity 2. However, if L were to meet a smooth point of F with multiplicity 2, then this point could not be an inflection of F, contradicting the assumption that L is an inflectious line. So L must intersect the node of F transversally to both branches.

Going from the fibre product $S \times_{\mathbb{P}^1} B$ to the smooth minimal model S_2, each fibre of type I_1 gets blown up once, and the two sections of S_2 corresponding to the intersection of L with the node of F intersect each resulting I_2-fibre in the exceptional curve, whereas the third section intersects it in the other component. This directly contradicts the conclusion above that all three sections must meet the same component. Fibres of type I_1 can thus not be partially ramified.

Lastly, we consider the case that F is of type *II*. By Proposition 2.31, after the base change $S_2 \dashrightarrow S$, the fibre gets replaced with fibres of type *IV* or type I_0^*, depending on whether the local ramification index of the map $B \to \mathbb{P}^1$ is 2 or 3 respectively. As discussed above S_2 cannot have fibres of type I_0^*, and thus F must have ramification index 2, i.e. be partially ramified. □

3.21 Remark. Neither here nor in the proof of Lemma 3.17 did we show that the mentioned fibre types can actually occur in practice, so it should be noted that for each fibre type listed in the preceding lemmas, there exists an example of a smooth quartic surface in $\mathbb{P}^3$ containing an inflectious line such that the corresponding elliptic fibration has a fibre of the respective type. We will not prove this, however.

Now that we have examined fibre types dependent on the local ramification behaviour, we want to look at the global consequences. Notably, the degree of the map $L \to \mathbb{P}^1$ and the genera of L and $\mathbb{P}^1$ limit the number of ramified fibres. Because all ramification indices divide 6, and the characteristic of the base field is not 2 or 3, there can be no wild ramification. We can thus use the Riemann-Hurwitz formula (see [9], Corollary IV.2.4), and find that

$$2 \cdot g(L) - 2 = 3 \cdot (2 \cdot g(\mathbb{P}^1) - 2) + \sum_{P \in B} (e_P - 1), \tag{3.3}$$

where e_P is the ramification index of the map $L \to \mathbb{P}^1$ at P and g denotes the genus. Since $g(L) = g(\mathbb{P}^1) = 0$, we find that

$$\sum_{P \in B} (e_P - 1) = 4. \tag{3.4}$$

Since the ramification index at a point P cannot exceed 3, there are the following three different ramification configurations.

3.22 Proposition. *Let L be a projective curve of genus 0. Then a morphism $L \to \mathbb{P}^1$ of degree 3 has*

a) two totally ramified points in $\mathbb{P}^1$,

b) two partially ramified points and one totally ramified point, or

c) four partially ramified points.

where 'totally' and 'partially' ramified are defined like in Definition 3.18.

With this knowledge about the global ramification behaviour of $\pi|_L$ and the local information about which fibre types can occur in which ramified fibres, we will now examine specifically how many lines can lie in the fibres of π when L is inflectious.

3.4 LINES ON S INTERSECTING THE INFLECTIOUS LINE L

In this section let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing an inflectious line L. Regular lines on S intersect at most 18 other lines on S by Proposition 3.3. Segre [26] claimed to have proven that the same maximum also holds for inflectious lines. Example

6.9 in [20] shows that this is false by giving a smooth quartic surface with an inflectious line that is intersected by 20 other lines on the surface. In this section, we will show that 20 is indeed the correct maximum.

Knowing which fibres S can have over points with each ramification type, we can now determine the number of lines in the fibres, i.e. the lines on S intersecting L, in each global ramification configuration. For this, we consider the elliptic surface S_2 from Proposition 3.13. Viewing S_2 as an elliptic curve E over $k(B)$ as explained in Chapter 2, we can identify a 3-torsion subgroup consisting of the three points on the curve corresponding to the sections o, σ_1 and σ_2. This subgroup, call it $T \subset E$, induces an isogeny

$$\pi : E \to E' \tag{3.5}$$

to a different elliptic curve E' over $k(B)$ such that $\ker \pi = T$ [3.c]. On the group level, E' can be thought of as the quotient E/T. But more importantly, E' corresponds to an elliptic surface S_2' and we have a rational map

$$S_2 \dashrightarrow S_2' \tag{3.6}$$

of elliptic surfaces over B. This map becomes very useful when we take into account the following observation, which is mentioned, but not proven in [20].

3.23 Lemma. *The surfaces S_2 and S_2' have the same Euler-Poincaré characteristic.*

Proof. This result can be concluded from Noether's formula, which may be formulated for any algebraic surface S as

$$12\chi(S) = K_S^2 + e(S), \tag{3.7}$$

where $\chi(S) = \chi(\mathcal{O}_S)$ is the characteristic of the structure sheaf on S, which will be defined below; K_S is the canonical divisor on S, with K_S^2 denoting its self-intersection; and lastly $e(S)$ is the topological Euler-Poincaré characteristic. We refer to p. 472 in [8] for this formulation of Noether's formula, where it is proven in Section 6 of Chapter 4.

For a topological space X and a sheaf $\mathcal{F}$ on X, the characteristic $\chi(\mathcal{F})$ is defined to be

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}),$$

[3.c] Proposition III.4.12 in [29]

where $H^i(X,\mathcal{F})$ is the i-th cohomology group (see Exercise III.5.1 in [9] for this definition). By Grothendieck's vanishing theorem (Theorem III.2.7 in [9]) all terms of index i greater than 2 vanish in the case of a surface, and the sum reduces to

$$\chi(\mathcal{O}_S) = h^0 - h^1 + h^2,$$

where h^i denotes $\dim_k H^i(S,\mathcal{O}_S)$.

These numbers h^i for $i = 0,1,2$ can be given explicitly. In Section 6.10 of [23] [3.f] it is stated that $h^0 = 1$, $h^1 = g$, where g is the genus of the base curve, and $h^2 = p_g$, which is the geometric genus of the surface S.

Applying all of this to the surfaces S_2 and S_2' here, we find that the numbers h^0 and h^1 must be equal between the two surfaces. Furthermore, the canonical divisor has self-intersection 0 (see Theorem 6.8 in [23]) on both surfaces. Therefore, it suffices to show that the geometric genera of the surfaces are equal to then conclude from Noether's formula (3.7) that the topological Euler characteristics $e(S_2)$ and $e(S_2')$ must also be equal.

By blowing up S_2 along the locus where the rational map $S_2 \dashrightarrow S_2'$ is undefined, we can obtain another surface $B(S_2)$ with a birational morphism $B(S_2) \to S_2$ such that the composed map $B(S_2) \to S_2 \dashrightarrow S_2'$ is a morphism.

By exercise III.9.3 (a) in [9], this morphism $f : B(S_2) \to S_2'$ is flat.

Generically, f constitutes a 3:1-map, so for any $x \in B(S_2), y = f(x)$, the degree of the extension of residue fields $k(x)/k(y)$ is at most 3. Because k does not have characteristic 2 or 3, this extension must be separable.

Furthermore, there is an open subset $V \subset S_2'$, such that for $U := f^{-1}(V)$, the restriction $f|_U^V : U \to V$ has exactly 3 pre-images for every $v \in V$. Thus $f|_U^V$ is unramified.

By exercise III.10.3 in [9], it follows that $\Omega_{U/V} = 0$, and thus $f|_U^V$ is smooth of relative dimension 1.

For each $u \in U, v = f(u) \in V$, we can conclude, using Proposition III.10.4 in [9], that the induced map on tangent spaces $T_u \to T_v$ is surjective, and thus the map on cotangent spaces

$$f^* : \Omega_{S_2'/k} \otimes k(u) \to \Omega_{B(S_2)/k} \otimes k(u)$$

is injective for every $u \in U$.

It follows that the map on global sections

$$\Gamma(V,\omega_V) \to \Gamma(V,\omega_V)$$

[3.f] Note that h^i in the notation used here corresponds to $h^{i,0}$ in the notation used in [23] is injective as well.

Since V is a dense open subset of S_2', and nonzero global sections on an invertible sheaf cannot vanish on a dense open subset, we can conclude that the map of vector spaces

$$\Gamma(S_2', \omega_{S_2'}) \to \Gamma(U, \omega_U)$$

is injective. Because it factors through

$$f^*: \Gamma(S_2', \omega_{S_2'}) \to \Gamma(B(S_2), \omega_{(B(S_2))}),$$

the latter map is injective as well, and thus we find

$$p_g(B(S_2)) \geq p_g(S_2').$$

Because S_2 and $B(S_2)$ are birationally equivalent, they have the same geometric genus by Theorem II.8.19 in [9], and so we finally conclude that

$$p_g(S_2) \geq p_g(S_2').$$

By Theorem III.6.1 in [29], every isogeny $E \to E'$ of elliptic curves has a corresponding dual isogeny $E' \to E$ with certain properties. Using this dual isogeny, we can apply the same logic to obtain the other inequality, and thus

$$p_g(S_2) = p_g(S_2').$$

□

The Euler characteristic of an elliptic surface is an interesting property because it is equal to the sum of the Euler characteristics [3.g] of all its fibres by Lemma IV.3.3 in [18]. The Euler characteristic of a fibre depends only on its type, and the numbers are given in Table IV.3.1 in [18].

In order to draw conclusions from this, we first need to examine how fibres on S correspond to fibres on the elliptic surfaces S_2 and S_2' from above. We correct a small error in [20] which inadequately dealt with the case where the map $L \to \mathbb{P}^1$ is not Galois, and provide some more detail in the proof.

[3.g] Often referred to as Euler numbers.

3.24 Lemma. *Let $S_2 \to B$ be the base change from Proposition 3.13. Let d be the degree of the field extension $k(B)/k(t)$. The singular fibres on the elliptic surfaces S_2 and S_2' correspond to the singular fibres of S according to the following table.*

	Fibre on S	Fibre on S_2	Fibre on S_2'
Unramified	I_1	$d \times I_1$	$d \times I_3$
	I_3	$d \times I_3$	$d \times I_1$
	IV	$d \times IV$	$d \times IV$
Ramified	II	$3 \times IV$	$3 \times IV$
	I_1	$\frac{d}{3} \times I_3$	$\frac{d}{3} \times I_9$
	I_2	$\frac{d}{3} \times I_6$	$\frac{d}{3} \times I_{18}$
	IV	$\frac{d}{3} \times I_0$	$\frac{d}{3} \times I_0$

Table 3.1.: Singular fibres on S and the corresponding fibres on S_2 and S_2'

Proof. The proof of this lemma consists of two separate parts, the base change $S_2 \dashrightarrow S$, and the isogeny $S_2 \dashrightarrow S_2'$.

For the base change, we use Proposition 2.31. The unramified fibres are replaced by n fibres of the same type. For the ramified fibres, we need to examine the local ramification indices. Note that by Lemma 3.20, type II on S only occurs in partly ramified fibres, and types I_1, I_2 and IV only occur in totally ramified fibres. Since the extension $k(B)/k(\mathbb{P}^1)$ corresponding to the map $B \to L \to \mathbb{P}^1$ is Galois, for any $t \in \mathbb{P}^1$, the local ramification indices of all points of B above t must be identical.

In particular, fibres of type II on S only occur in the non-Galois case, where $B \to L$ is a map of degree 2, and there are three points on B above such a fibre, all with ramification index 2 above $\mathbb{P}^1$. By Proposition 2.31, the fibre of type II on S gets replaced by three fibres of type IV on S_2.

Ramified fibres of the remaining three types have total ramification which could occur in both the Galois or the non-Galois case, and we need to treat these cases separately. If $L \to \mathbb{P}^1$ is Galois, the map $B \to L$ is trivial, and above any totally ramified fibre there is exactly one fibre of S_2 of ramification index 3. By Proposition 2.31, if the fibre on S if of type I_n, the resulting fibre on S_2 will be of type I_{3n}, and if the fibre on S is of type IV, the resulting fibre on S_2 will be smooth. Note that in the Galois case, $n = 3$, and so $\frac{n}{3} = 1$, which is consistent with the table.

Lastly, if the morphism $L \to \mathbb{P}^1$ is not Galois, the Galois closure is a degree 2 morphism $B \to L$, and the composed map $B \to \mathbb{P}^1$ has degree 6. A priori, a totally ramified point in $\mathbb{P}^1$ could split into one point on B of ramification index 6, or two points of ramification index 3 each. However, the first case is not possible, as was discussed ahead of Lemma 3.20. Hence, the ramified fibres of types I_1, I_2 and IV of S each split into $2 = \frac{n}{3}$ fibres on S_2, and because the local ramification indices are all 3, the types are given as in Table 3.1 by Proposition 2.31.

This proves the second column of Table 3.1. To examine how fibres change under the 3-isogeny $S_2 \dashrightarrow S_2'$, we will use section 7.8.1 from [23], which describes the desired behaviour for fibres of type I_n. Notably, a fibre of type I_n on S_2 gets replaced by a fibre of type I_{3n} on S_2' if the 3-torsion section σ_1 meets the zero component of I_n, and by $I_{n/3}$ otherwise. Note that σ_1 meeting a nonzero component of an I_n fibre implies that the component group of the fibre has 3-torsion, and since the component group is $\mathbb{Z}/n\mathbb{Z}$, this is equivalent to n being a multiple of 3, and thus $\frac{n}{3}$ must be an integer in this case. Consequently, we only need to examine how the section σ_1 meets the fibres on S_2, depending on the type of the corresponding fibre on S.

For an unramified fibre F of type I_1, this is obvious, since the fibres above F on S_2 are also of type I_1. All sections must necessarily meet the zero component, since it is the only component available. Therefore, the corresponding fibres on S_2' must be of type I_3.

For an unramified fibre F of type I_3, note that on the surface S, the line L meets all three components of F, since F consists of three lines coplanar to L. On S_2, the three points of intersection of L with F split into the three 3-torsion sections, which all meet different components of the resulting I_3-fibres on S_2. In particular, on S_2', they get replaced by fibres of type I_1.

If F is a ramified fibre of type I_1, then note that by Lemma 3.20, the ramification is total. In particular, the line L intersects a single point on the fibre F with multiplicity 3. Since the corresponding fibres [3.𝔥] of type I_3 on S_2 are obtained via a blowup from the I_1 fibre on the fibre product $S \times_{\mathbb{P}^1} C$, we can conclude that by construction, the three 3-torsion sections must intersect the same component of the I_3 fibres on S_2. Like above, it then follows from section 7.8.1 in [23] that on S_2' we obtain fibres of type I_9.

The same argument also works in the case of ramified fibres of type I_2, since they too are necessarily totally ramified by Lemma 3.20. □

It is interesting to note that in this table, if we compare the Euler numbers of fibres on S_2 and the corresponding fibres on S_2', almost all of them increase or stay the same. The only case where the Euler number decreases is that of an unramified fibre of type I_3 on S. However, the elliptic surfaces S_2 and S_2' have the same Euler-Poincaré characteristic by Lemma 3.23, which is equal to the sum of the Euler numbers of the singular fibres.

[3.𝔥]Note that this could be a single fibre in the Galois case, or two fibres in the non-Galois case.

Hence, the positive change in Euler numbers between S_2 and S_2' that comes from fibres of types I_1 and I_2 on S must be offset by an appropriate number of fibres of type I_3. We make this precise in the following lemma.

3.25 Lemma. *Fibres of type I_n, $n > 0$, on S occur in pairs (I_1, I_3) and triples (I_2, I_3, I_3).*

Proof. Consider Table 3.1 and the Euler numbers of the fibres on S_2 and S_2':

	Fibre on S	Euler nr on S_2	Euler nr on S_2'	Difference
Unramified	I_1	n	$3n$	$+2n$
	I_3	$3n$	n	$-2n$
	IV	$4n$	$4n$	0
Ramified	II	12	12	0
	I_1	n	$3n$	$+2n$
	I_2	$2n$	$6n$	$+4n$
	IV	0	0	0

Table 3.2.: Euler numbers of the singular fibres on S_2 and S_2'

Since the sums of the Euler numbers of singular fibres on S_2 and S_2' must be equal, fibres of type I_3 and fibres of type I_1 and I_2 have to balance out in such a way that the differences add up to zero. Since I_1-fibres always contribute $2n$ to the sum, each I_1 fibre has to be balanced out by precisely one I_3-fibre. Similarly, I_2-fibres contribute $4n$ to the sum and have to be balanced out by two I_3-fibres. □

With this knowledge, there is very little possible variation left in the number of lines intersecting the inflectious line L.

3.26 Proposition (Rams, Schütt, 2015). *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing a line L that is inflectious in the sense of Definition 2.5. Then L is intersected by precisely*

a) 12 *other lines on S if $L \to \mathbb{P}^1$ has no totally ramified points,*

b) 15 *or* 16 *lines if $L \to \mathbb{P}^1$ has one totally ramified point, and*

c) 18, 19 *or* 20 *lines if $L \to \mathbb{P}^1$ has two totally ramified points.*

Proof. Let $\pi : S \to \mathbb{P}^1$ be the elliptic fibration corresponding to L.

Recall Lemmas 3.17 and 3.20, which state that type II fibres of π occur if and only if the fibre is partly ramified. Totally ramified fibres can be of type I_1, I_2 and IV, and unramified fibres are of type I_1, I_3 or IV.

The set of singular fibres is thus made up of three basic 'building blocs'. First, we have pairs of fibres of type II, which have total Euler number 4 and no lines, and make up all partly ramified fibres. Note that partly ramified fibres indeed only occur in pairs, due to Proposition 3.22. Secondly, we have triples (I_2, I_3, I_3), which have Euler number 8, seven lines, and contain one totally ramified fibre. Lastly, there are fibres of type IV, or pairs (I_1, I_3), both of which have Euler number 4 and three lines and can be either unramified or contain exactly one totally ramified fibre. In the following analysis, we will treat them as equal and only mention the pairs (I_1, I_3). It should however not be forgotten that in the following, any pair (I_1, I_3) (ramified or not) could be replaced by a fibre of type IV.

Going through the three cases, we can now determine all possibilities.

a) If $L \to \mathbb{P}^1$ has no totally ramified points, and four partly ramified points, then there are four fibres of type II, which contribute 8 to $e(S) = 24$. The remaining 16 are made up of four unramified pairs (I_1, I_3), which contain exactly 12 lines.

b) If $L \to \mathbb{P}^1$ has one totally ramified point, and two partly ramified points, then there are two fibres of type II, and this time the remaining fibres have a total Euler number of 20. The one totally ramified fibre now leaves us with two possibilites: there could be one triple (I_2, I_3, I_3), which would add seven lines for an Euler number of 8, or two pairs (I_1, I_3), one containing a ramified I_1 and the other unramified, which add only six lines for the same Euler number. In both cases, the remaining 12 comes from three unramified pairs (I_1, I_3), which add another nine lines. Thus in total there can be either $7 + 9 = 16$ or $6 + 9 = 15$.

c) If $L \to \mathbb{P}^1$ has only two totally ramified points, then there are no type II fibres, so the minimal number of lines is 18, corresponding to six pairs (I_1, I_3). There are two independent choices each between one triple (I_2, I_3, I_3) and two pairs (I_1, I_3). Both times, similarly to b), choosing the former adds one line to the minimum of 18. There can thus be $18, 19$ or 20 fibres.

□

3.27 Remark. In cases b) and c), the exact number of lines intersecting L solely depends on the number of fibres of type I_2. In particular, in order for an inflectious line to exceed the maximum of 18 that we established for regular lines, its elliptic fibration needs to contain at least one fibre of type I_2.

3.5 QUARTIC SURFACES WITH INFLECTIOUS LINES

The detailed analysis from the previous section provides us with all the tools that we need to prove the maximum of 64 lines on any smooth quartic.

In this section, we will assume that $S \in \mathbb{P}^3$ is a smooth quartic surface. Due to Proposition 3.26, the only case that we still need to consider is the case that S contains an inflectious line L_0 such that the corresponding morphism $L_0 \to \mathbb{P}^1$ has two totally ramified points. Indeed, because Lemma 3.5 holds for inflectious lines as well, the proof of Theorem 3.12 is valid for any smooth quartic so long as all of its inflectious lines fall under the first two cases of Proposition 3.26, i.e. the morphism $L_0 \to \mathbb{P}^1$ has either zero or one totally ramified points. In this section we are particularly interested in the third case. For the rest of this section, we will assume that $L_0 \subset S$ is an inflectious line with this property, and we will call the associated elliptic fibration π_0. Immediately we can use the existence of the line L_0 to simplify the general equation for S significantly.

3.28 Lemma. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface containing an inflectious line L_0 with corresponding elliptic fibration $\pi_0 \colon S \to \mathbb{P}^1$. Assume that the restricted morphism of curves $\pi_0|_{L_0} : L_0 \to \mathbb{P}^1$ has two totally ramified points.*

Then S is projectively equivalent to a quartic given by an equation of the form

$$x_2x_0^3 + x_3x_1^3 + x_0x_1q(x_2,x_3) + g(x_2,x_3) = 0 \tag{3.8}$$

where q and g are homogeneous polynomials in x_2,x_3 of degree 2 and 4 respectively.

Proof. Like we have done before, we can assume the line L_0 to be $Z(x_2,x_3)$. An equation for S can then be written in the form

$$x_2h_1 + x_3h_2 = 0.$$

Since the morphism $L_0 \to \mathbb{P}^1$ has exactly two ramified points, we can apply a linear transformation to x_2,x_3 and assume that the ramified points are zero and infinity, i.e. the ramified fibres lie in the planes $Z(x_2)$ and $Z(x_3)$. Since the ramification is total in both cases, the residual cubic curves in both planes have a triple intersection with L_0, so the equation can be written as

$$x_2r_0^3 + x_3r_1^3 + x_2^2s_1 + x_2x_3s_2 + x_3^2s_3 = 0, \tag{3.9}$$

where the r_i are homogeneous linear polynomials in x_0,x_1 and the s_i are homogeneous quadratic polynomials in x_0,x_1,x_2,x_3.

We note that r_0 and r_1 are linearly independent. Indeed, if we assume otherwise, then without loss of generality we can write $r_1 = \lambda r_0$ for some $\lambda \in k$. Replacing r_1 with λr_0 in Eq. (3.9) and computing the partial derivatives, the only terms in these partial derivatives that are not divisible by x_2 or x_3 are r_0^3 in the partial derivative by x_2, and $(\lambda r_0)^3$ in the partial derivative by x_3.

Therefore, any point P on the line L_0 that satisfies $r_0(P) = 0$ would be a singularity of S in this case.

After a linear coordinate transformation in x_0, x_1, we can assume that $r_0 = x_0$ and $r_1 = x_1$, giving us an equation of the form

$$x_2x_0^3 + x_3x_1^3 + x_2^2s_1 + x_2x_3s_2 + x_3^2s_3 = 0.$$

Furthermore, we can make sure that all monomials divisible by $x_2x_0^2$ or $x_3x_1^2$ vanish, with the exception of $x_2x_0^3 + x_3x_1^3$. Indeed, we can add terms in x_2 and x_3 to x_0 and x_1 without negating the effects of the earlier coordinate transformations. After replacing x_0 with $x_0 + p(x_2, x_3)$, the term $x_2x_0^3$ becomes

$$x_2x_0^3 + 3x_2x_0^2p(x_2,x_3) + 3x_2x_0p(x_2,x_3)^2 + x_2p(x_2,x_3)^3.$$

Since the characteristic of k is not 3, we can choose p appropriately to eliminate the multiples of $x_2x_0^2$. After doing the same with x_1, the equation of S becomes

$$x_2x_0^3 + x_3x_1^3 + ax_0^2x_3^2 + bx_1^2x_2^2 + x_0p_0(x_2,x_3) + x_1p_1(x_2,x_3) + x_0x_1q(x_2,x_3) + g(x_2,x_3) = 0,$$

where p_i, q and g are all homogeneous polynomials of fitting degrees. It remains to be shown that a, b, p_0 and p_1 are all zero. For this we finally use the assumption that L_0 is inflectious.

Consider at first planes of the form $x_3 = tx_2$ with $t \in k$. The intersection of S with this plane is a quartic curve in $\mathbb{P}^2(x_0, x_1, x_2)$ given by the equation

$$\begin{aligned} x_2x_0^3 + tx_2x_1^3 + at^2x_0^2x_2^2 + bx_1^2x_2^2 + x_0p_0(x_2,tx_2) + x_1p_1(x_2,tx_2)& \\ +x_0x_1q(x_2,tx_2) + g(x_2,tx_2) = 0.& \end{aligned} \tag{3.10}$$

Now note that for any homogeneous polynomial $f \in k[x_1,x_2]$, the polynomial $f(x_2,tx_2)$ is of the form $F(t)x_2^{\deg f}$ for some polynomial $F(t) \in k[t]$. In particular, we can write

$$\begin{aligned} p_i(x_2,tx_2) &= P_i(t)x_2^3 \quad \text{for } i = 1,2 \\ q(x_2,tx_2) &= Q(t)x_2^2 \\ g(x_2,tx_2) &= G(t)x_2^4. \end{aligned}$$

Furthermore note that Polynomial (3.10) is divisible by x_2, with $x_2 = 0$ being the equation of the line L_0 in the plane. After dividing by x_2, we obtain the equation for the residual cubic:

$$x_0^3 + tx_1^3 + at^2x_0^2x_2 + bx_1^2x_2 + P_0(t)x_0x_2^2 + P_1(t)x_1x_2^2 + Q(t)x_0x_1x_2 + G(t)x_2^3 = 0. \tag{3.11}$$

To find the points of inflection, we compute the Hessian matrix of this, which is given by

$$\begin{pmatrix} 6x_0 + 2at^2x_2 & Q(t)x_2 & 2at^2x_0 + 2P_0(t)x_2 + Q(t)x_1 \\ Q(t)x_2 & 6tx_1 + 2bx_2 & 2bx_1 + 2P_1(t)x_2 + Q(t)x_0 \\ 2at^2x_0 + 2P_0(t)x_2 + Q(t)x_1 & 2bx_1 + 2P_1(t)x_2 + Q(t)x_0 & 2P_0(t)x_0 + 2P_1(t)x_1 + 6G(t)x_2 \end{pmatrix}.$$

Restricted to L_0, i.e. substituting $x_2 = 0$, we compute the determinant to be

$$\begin{aligned} &-6Q(t)^2x_0^3 + (72tP_0(t) - 24bQ(t) - 24a^2t^5)x_0^2x_1 \\ &+(72tP_1(t) - 24b^2 - 24at^3Q(t))x_0x_1^2 - 6tQ(t)^2x_1^3 \end{aligned} \tag{3.12}$$

This equation defines three points on the line $L_0 = \mathbb{P}^1(x_0,x_1)$. Since L_0 is inflectious, these must coincide exactly with the three points where the fibre given by Eq. (3.11) meets the line L_0, i.e. the three points given by the equation

$$x_0^3 + tx_1^3 = 0. \tag{3.13}$$

In particular, the Polynomials (3.12) and (3.13) have to be equal up to a constant factor. We obtain the two equations

$$\begin{aligned} 72tP_0(t) - 24bQ(t) - 24a^2t^5 &= 0 \\ 72tP_1(t) - 24b^2 - 24at^3Q(t) &= 0 \end{aligned}$$

Since these need to hold for every $t \in k$, we can substitute $t = 0$ in the second equation to conclude $24b^2 = 0$, which implies that $b = 0$ since char $k \neq 2,3$. Substituting $b = 0$ back into both equations yields

$$P_0(t) = \frac{1}{3}a^2 t^4 \tag{3.14}$$

$$P_1(t) = \frac{1}{3}at^2 Q(t). \tag{3.15}$$

Analogously to how we showed $b = 0$, we can show that $a = 0$ by considering planes of the form $x_2 = tx_3$, and from Eqs. (3.14) and (3.15) it follows that $P_0(t) = P_1(t) = 0$. By definition, this means that $p_0(x_2, tx_2) = p_1(x_2, tx_2) = 0$, and analogously $p_0(tx_3, x_3) = p_1(tx_3, x_3) = 0$ for every value of t, so in particular the polynomials p_0 and p_1 must be identically zero, which concludes the proof. $\square$

3.29 Remark. Note that a quartic surface S given by an equation like (3.8) automatically contains the inflectious line $L_0 = Z(x_2, x_3)$ and admits the elliptic fibration $\pi_0 \colon S \to \mathbb{P}^1$ corresponding to L_0. This morphism π_0 can, similarly to the proof of Proposition 1.6, be given on the open subset $S \setminus L_0$ of S by

$$(x_0, x_1, x_2, x_3) \mapsto (x_2, x_3).$$

Furthermore, the restriction $\pi_0|_{L_0} : L_0 \to \mathbb{P}^1$ ramifies exactly above the points $(1:0)$ and $(0:1)$ in $\mathbb{P}^1$, which correspond to the planes $x_2 = 0$ and $x_3 = 0$ in $\mathbb{P}^3$.

Also note that if S is a surface containing a line L with elliptic fibration π that satisfies the assumptions of Lemma 3.28, then under the projective equivalence, the line L and morphism π correspond to L_0 and π_0 as given here.

We will adopt a notation from [20], albeit slightly differently, by including the line L_0 and the elliptic fibration π_0 in the definition.

3.30 Definition. We define the family $\mathcal{Z}$ to consist of all triples of the form (S, L_0, π_0) where

- S is a smooth quartic surface in $\mathbb{P}^3$ given by an equation like (3.8)
- L_0 is the inflectious line $Z(x_2, x_3)$ contained in S
- π_0 is the elliptic fibration given as in Remark 3.29.

For the rest of this section, we will always assume that (S, L_0, π_0) is a member of the family $\mathcal{Z}$.

We now show that unless S is equal to the Schur quartic from Example 3.1, the line L_0 cannot be intersected by another inflectious line on S. Rams and Schütt [20] prove a slightly weaker version of this lemma which only states that unless S is the Schur quartic, any inflectious line intersecting L_0 can only intersect at most 16 lines on S. Of course this immediately follows from the stronger version.

3.31 Proposition. *Let (S, L_0, π_0) be in the family $\mathcal{Z}$ from Definition 3.30.*

If a fibre of π_0 contains another inflectious line L_1, then S is projectively equivalent to the Schur quartic from Example 3.1.

Proof. Because this proof is rather long, we will only provide a brief overview. The full proof (including code for the computations for which computer assistance was used) can be found in Appendix A.3.

The plane that contains L_1 can be assumed without loss of generality to be given by an equation of the form $x_3 = \lambda x_2$, where λ is 0 or 1, with $\lambda = 0$ representing the case where the fibre is ramified, and $\lambda = 1$ representing the unramified case.

Then the equation of the curve in the intersection of S with such a plane has equation

$$x_2 x_0^3 + \lambda x_2 x_1^3 + c x_0 x_1 x_2 + d x_2^4 = 0,$$

where c and d are constants which can be derived from the coefficients of the polynomials q and g from Polynomial (3.8), depending on the value of λ.

We can then differentiate four subcases of each case, depending on whether c or d vanish. Both times, three of the four subcases can be ruled out immediately because the resulting cubic residual to L_0 is either irreducible or of a Kodaira type that is incompatible with the ramification behaviour by Lemmas 3.17 and 3.20.

For the fourth subcase, we consider the cubic residual to the line L_1, and by computing the intersection points with L_1 and the Hessian of the residual cubic, we can derive equations on the coeffients of q and g which must hold in order for L_1 to be inflectious.

In the unramified case, this leads to an equation $144 = 0$, which cannot happen outside characteristic 2 or 3, thus leading to a contradiction. In the ramified case, we find that $q = 0$ and $g = x_2^4 + x_3^4$, which leads to the Schur quartic. □

In particular, since the Schur quartic contains exactly 64 lines, we immediately get the following corollary.

3.32 Corollary. *If there is an inflectious line L_1 in a fibre of π_0, then S contains at most* 64 *lines.*

3.33 Remark. This corollary also holds with the weaker version of Proposition 3.31 without much extra effort, see [20], p.691.

3.34 Remark. Note that regardless of the existence of such a line L_1, π_0 always has a fibre of type I_3 or *IV* as seen in the proof Proposition 3.26. If all lines intersecting L_0 are regular, then we can apply the same logic as in Theorem 3.12 to obtain a maximum of

$$4+(20-3)+3(18-3)=66$$

lines on S.

For the rest of the section, we can thus assume that any two inflectious lines on S are skew to each other. We can also assume that L_0 intersects at least 19 lines. Indeed, as seen in the proof of Proposition 3.26, π_0 has at least one fibre of type *IV* or I_3. Since all lines in any of these fibres are regular, they intersect at most 18 lines. If L_0 also only intersects 18 lines, then with similar reasoning as in Theorem 3.12, S will contain at most $4+4\cdot 15=64$ lines.

Under the assumption that L_0 intersects 19 or 20 lines, we have seen in Remark 3.27 that π_0 has at least one fibre of type I_2, consisting of a regular line L_1, and a conic Q_1, which we will fix for the rest of this section. It should be noted that by Lemma 3.17, the fibre of type I_2 has to be ramified, and in particular, without loss of generality we can assume that it lies in the plane $Z(x_3)$. A first lemma is directed at the conic in this fibre. In [20] it is only mentioned that one can compute this directly, without providing any details, which we shall do here.

3.35 Lemma. *Let (S,L_0,π_0) be in the family $\mathcal{Z}$ from Definition 3.30. Assume that π_0 has a fibre of type I_2, containing a regular line L_1 and a smooth conic Q_1.*

Then Q_1 is not a component of the flecnodal divisor $\mathcal{F}_S$ of S.

Proof. We will give a brief and informal overview here. The full proof, including code for computer-aided computations can be found in Appendix A.4.

We can make use of the fact that a quartic surface in the family $\mathcal{Z}$ can explicitly be given with an equation like (3.8), and the knowledge that the fibre of I_2 must lie in the plane $x_2=0$ or $x_3=0$.

Up to a change a coordinates, one can then give concrete equations for the line L_1 and the conic Q_1. After parametrising the points on Q_1, one can compute an equation for the tangent plane T to S at such a point P.

Intersecting T with S gives a quartic curve C, and any line intersecting S with multiplicity 4 at P must lie in the plane T and intersect the curve C with multiplicity 4 at P.

It can then be shown that the curve C has a singularity at P, and one can compute the tangent cone, which consists of two lines. Thus it suffices to compute the intersection multiplicity of both of these lines with C at P.

One can then compute a polynomial in the homogeneous parameters s and t for the conic Q_1 whose coefficients are polynomials in the coefficients of the equation of S; and a point on Q_1, given by parameters s and t is flecnodal on S if and only if the polynomial vanishes at these values of s and t.

If Q_1 were to be contained in the flecnodal divisor, then this equation would have to be satisfied for all values of s and t, which implies that all coefficients must be zero. But as the computations in Appendix A.4 will show, there is at least one coefficient that is always non-zero, thereby making this impossible. □

We will now consider the elliptic fibration associated to L_1, and call it π_1. The proof of Lemma 3.20 showed that L_0 meets one of the two intersection points of L_1 and Q_1 tangentially to Q_1. In particular, π_1 has a fibre F_0 of type *III*, consisting of L_0 and Q_1. Because L_1 is a regular line, fibres of π_1 can a priori be of any Kodaira type from Example 2.26. However, we will see that the reducible fibres other than F_0 containing a single line can only occur in triples, and as a consequence, we get the following lemma.

3.36 Lemma. *Let (S,L_0,π_0) be in the family $\mathcal{Z}$, and let L_1 be a regular line in a fibre of type I_2 of π_0. Then the number of lines intersecting L_1 is equal to $3N+1$ for some integer N.*

Proof. By construction, L_1 is intersected by the line L_0. We want to define an automorphism σ of S of order 3 and show that it fixes L_0 and L_1, but does not fix any other line that intersects L_1.

To define σ, let $\zeta \in k$ be a primitive cube root of unity and consider the automorphism of order 3

$$\sigma : \mathbb{P}^3 \to \mathbb{P}^3, (x_0 : x_1 : x_2 : x_3) \mapsto (\zeta x_0 : \zeta^2 x_1 : x_2 : x_3).$$

This map restricts to an automorphism of S, since for any $(x_0 : x_1 : x_2 : x_3)$ satisfying Polynomial (3.8), the image point $(\zeta x_0 : \zeta^2 x_1 : x_2 : x_3)$ also satisfies the equation, and the inverse σ^2 also restricts to a map $S \to S$.

Next note that the line $L_0 = Z(x_2, x_3)$ is fixed by σ as a set, and so are all planes through L_0, which are of the form $ax_2 + bx_3 = 0$. In particular, σ fixes fibres of π_0 as sets, although it does not necessarily fix each component of a reducible fibre. We conclude that L_1 is also fixed by σ as a set, since it occurs in a fibre of π_0 together with the conic Q_1.

What remains to be shown is that σ does not fix any other lines in the fibres of π_1. Let O be such a line, and assume for the sake of contradiction that σ fixes O as a set. Since we have already established that σ fixes planes through L_0, it follows that for any such plane H, the intersection $O\cap H$ would also have to be fixed by σ. However, this intersection consists of a single point, and thus we can conclude that the line O would have to be fixed pointwise by σ.

This is impossible. Indeed, we can directly compute the fixed points of σ; they are given by $(1:0:0:0)$, $(0:1:0:0)$, and every point on the line given by $x_0 = x_1 = 0$. This line cannot be contained in S, because if it were, then in the notation of Polynomial (3.8), that would imply $g = 0$, and then every partial derivative of the equation of S would vanish along that line, contradicting the smoothness of S.

Therefore, the automorphism σ cannot fix any lines on S pointwise and in particular it cannot fix the line O, resulting in the desired contradiction.

Since σ has order 3, it follows that lines intersecting L_1 with the exception of L_0 must come in triples. □

3.37 Corollary. *The line L_1 from Lemma 3.36 meets at most* 16 *other lines on S.*

Proof. Since L_1 is a regular line, it cannot be intersected by more than 18 lines by Proposition 3.3. The result then follows from Lemma 3.36. □

On the other hand, note that every line on S intersects exactly one of L_0, L_1 and Q_1, because the union of these three curves is the intersection of S with some plane. Since Q_1 is not a component of the flecnodal divisor by Lemma 3.35, it can only meet at most $2\cdot 20 = 40$ lines on S, which includes L_0 and L_1 with multiplicity 2. With L_0 intersecting at most 19 lines besides L_1, and Q_1 intersecting at most 36 lines besides L_0, L_1, the only way to reach more than 64 lines on S is if L_1 is intersected by at least 8 lines besides L_0. This means that in the notation of Lemma 3.36, we can restrict our attention to the cases $N = 3$, $N = 4$, and $N = 5$. In the latter two cases, it will now be fairly simple to prove the main theorem with the help of the following lemma.

3.38 Lemma. *Let $(S, L_0, \pi_0) \in \mathcal{Z}$. Let L_1 be a regular line in a fibre of type I_2 of π_0, and let $\pi_1 \colon S \to \mathbb{P}^1$ be the associated elliptic fibration. If π_1 has a fibre of type I_3 or IV, then S contains at most* 64 *lines.*

Proof. Assume that such a fibre exists. Then there is a plane $H \subset \mathbb{P}^3$ such that the intersection $H\cap S$ consists of four lines, one of which being L_1. Each other line on S thus intersects exactly one of these four. By Corollary 3.37, L_1 is met by at most 16, and the other three lines by at most 20 lines on S. If at most one line in this fibre is inflectious, then the number of lines on S is bounded by

$$4+(16-3)+2(18-3)+(20-3)=64.$$

If two or more lines in the fibre are inflectious, note that they intersect each other. By Proposition 3.31, the surface S must then be projectively equivalent to the Schur quartic, which only contains 64 lines. □

3.39 Corollary. *If L_1 is intersected by more than* 10 *lines, then S contains at most* 64 *lines.*

Proof. Note that in this case, L_1 would be intersected by at least 13 lines by Lemma 3.36. Assume for the sake of contradiction that π_1 has no fibres of type I_3 or *IV*. In particular, π_1 has at least 13 fibres of type I_2 or *III*. In both cases, the Euler-Poincaré characteristic of each such fibre is at least 2, and their sum exceeds $e(S)=24$, which is impossible. So, π_1 must have at least one I_3 or *IV* fibre, and the result follows from Lemma 3.38. □

The only case that remains is the case that L_1 is intersected by exactly ten other lines on S. Here, the existence of a fibre of type I_3 or *IV* is not guaranteed. However, we can still prove the main result in this case regardless.

3.40 Lemma. *Let $(S,L_0,\pi_0)\in\mathcal{Z}$ and let L_1 be a regular line in a fibre of π_0 of type I_2. If L_1 is intersected by exactly* 10 *lines, then S contains at most* 64 *lines.*

Proof. Let $\pi_1: S\to\mathbb{P}^1$ be the elliptic fibration associated to L_1 from Proposition 1.6. If π_1 has a fibre of type I_3 or *IV*, the result follows from Lemma 3.38. Assume that this is not the case, i.e. all lines intersecting L_1 occur in fibres of type I_2 or *III*. Every such fibre can be written as

$$F_i=L_i+Q_i, i=2,\dots,10$$

for a line L_i and an irreducible conic Q_i. Note that if all nine conics $Q_2,\dots,Q_{10}$ were contained in the flecnodal divisor, their added degree is 18, which would leave at most $80-18=62$ lines on S. We can thus assume that Q_2 is not a component of $\mathcal{F}_S$. Since the automorphism σ from the proof of Lemma 3.36 maps planes to planes and fixes L_1, it maps each fibre of π_1 to another fibre of π_1, and because we know it cannot fix lines in these fibres, up to renumbering we have

$$\begin{aligned} F_3&=\sigma F_2 & F_4&=\sigma^2F_2\\ F_6&=\sigma F_5 & F_7&=\sigma^2F_5\\ F_9&=\sigma F_8 & F_{10}&=\sigma^2F_8. \end{aligned}\tag{3.16}$$

Since Q_2 is not contained in $\mathcal{F}_S$, the same holds for Q_3 and Q_4. Indeed, any automorphism of S fixes $\mathcal{F}_S$, because it is defined in terms of lines and intersection numbers, which are preserved by σ Since Q_2 meets $\mathcal{F}_S$ with multiplicity $2 \cdot 20 = 40$ and intersects both L_1 and L_2 with multiplicity 2 each, it can intersect at most 36 additional lines.

If L_2 were a regular line, it would meet at most 18 lines by Proposition 3.3, and since every line on S must intersect exactly one of L_1, L_2, Q_2, the total number of lines on S would be bounded by

$$2+(10-1)+(18-1)+36 = 64.$$

For the rest of this proof we will assume, for the sake of contradiction, that S contains more than 64 lines. In particular, L_2 is inflectious and the elliptic fibration corresponding to L_2 has a fibre containing the line L_1 and the irreducible conic Q_2. By Lemmas 3.17 and 3.20, this fibre must be ramified of type I_2. Furthermore, as seen in the proof of Lemma 3.20, L_2 must be tangent to Q_2, which implies that F_2 is a fibre of type *III* of π_1. The same holds for $F_3 = \sigma F_2$ and $F_4 = \sigma^2 F_2$.

While a priori the fibres $F_5,\dots,F_{10}$ could be either of type I_2 or *III*, the latter is impossible. Indeed, since π_1 already has four fibres of type *III*, those being F_2, F_3, F_4 and the fibre containing L_0, whose Euler-Poincaré characteristics add up to 12, the Euler-Poincaré characteristics of the remaining fibres can add up to at most 12 as well. This can only happen if they are all of type I_2.

We conclude that the conics $Q_5,\dots,Q_{10}$ must be contained in $\mathcal{F}_S$. If one of them were not contained in $\mathcal{F}_S$, its fibre F_i would be of type *III* with the same logic that we applied to F_2 above, which is impossible.

We now claim that under the assumption that S contains more than 64 lines, there must be exactly 45 lines skew to L_0. Recall Remark 3.34 and note that we can be at most one line short of the maximum of 66. Similarly to the proof of Corollary 3.39, there must be at least one fibre of type I_3 or *IV*, which consists of three lines M_0, M_1, and M_2. Because σ fixes fibres of π_0, it must permute the three lines M_i. Note that none of the lines M_i can be fixed by σ since otherwise the point of intersection $L_0 \cap M_i$ would also have to be fixed, and as we have seen in the proof of Lemma 3.36, none of the points on L_0 are fixed by σ.

Now the four lines L_0, M_0, M_1, and M_2 together form the intersection of S with some plane, and therefore every line on S that is skew to L_0 must intersect precisely one line M_i. So let $L \subset S$ be a line skew to L_0 and assume it intersects the line M_i for a fixed value of i. Then the line σL intersects σM_i, which is distinct from the line M_i. Therefore, the line σL must also be distinct from L.

It follows that the number of lines on S skew to L_0 must be divisible by 3, since no such line is fixed by σ and σ has order 3.

We can now decompose the flecnodal divisor of S as

$$\mathcal{F}_S = Q_5 + \cdots + Q_{10} + \mathcal{L} + \mathcal{R}$$

where $\mathcal{L}$ is the divisor of degree 65 containing the lines L_0, L_1, the 18 lines besides L_1 in the fibres of π_0, and the 45 lines that are skew to L_0, each with mutiplicity 1. Recall that the flecnodal divisor has degree 80 by Lemma 3.7. We conclude that the degree of the residual divisor $\mathcal{R}$ is 3. Note that a priori it is possible that $\mathcal{R}$ contains one or more of the lines that are included in $\mathcal{L}$.

Next we note that the sum of the conics $Q_5 + \cdots + Q_{10}$ is σ-invariant by Eq. (3.16), and the divisor $\mathcal{L}$ is also σ-invariant, since σ fixes L_0, L_1 and fibres of π_0 and maps sections of π_0 to sections of π_0. Because the flecnodal divisor $\mathcal{F}_S$ as a whole is also σ-invariant, the same must hold for $\mathcal{R}$.

Consider the intersection $L_0 \cdot \mathcal{F}_S$ and note that L_0 does not intersect any of the conics $Q_5, \ldots, Q_{10}$, because fibres of π_1 are pairwise disjoint. Since $L_0^2 = -2$ and L_0 meets exactly 19 of the already counted lines, we find that

$$20 = L_0 \cdot \mathcal{F}_S = L_0 \cdot (Q_5 + \cdots + Q_{10} + \mathcal{L} + \mathcal{R}) = 19 - 2 + L_0 \cdot \mathcal{R}$$

and it follows that $L_0 \cdot \mathcal{R} = 3 = \deg \mathcal{R}$.

We now want to show that every component of $\mathcal{R}$ is contained in some fibre of π_0. Let $R_1, \ldots, R_k$ be the irreducible components of $\mathcal{R}$. First observe that for any plane H containing the line L_0, the intersection number $\mathcal{R} \cdot H$ must be 3 by Theorem 7.7 in Chapter I of [9]. But since $\mathcal{R} \cdot L_0 = 3$, it follows that for any fibre F of π_0, we have $\mathcal{R} \cdot F = 0$.

Further note that for each individual component R_i we have $R_i \cdot L_0 \le R_i \cdot H$ for any plane H containing L_0. Because $\mathcal{R} \cdot L_0 = \mathcal{R} \cdot H$, it follows that for each value of i, we must also have $R_i \cdot L_0 = R_i \cdot H$. But then we also find $R_i \cdot F = 0$ for any fibre F of π_0, and so R_i is either a component of a fibre or disjoint with all fibres of π_0. But because the union of the fibres of π_0 is the entire surface S, the latter case is impossible. Thus $\mathcal{R}$ consists only of components of fibres of π_0, as claimed.

However, $\mathcal{R}$ cannot be a full fibre of π_0. If that were the case, then L_2 would intersect $\mathcal{R}$ with multiplicity 1, since L_2 is a section of π_0. On the other hand, L_2 is also skew to the conics $Q_5,\dots,Q_{10}$ because they occur in different fibres of π_1. Since L_2 intersects at most 20 other lines on S, itself with multiplicity -2, and $\mathcal{R}$ with multiplicity 1, we find

$$20 = L_2 \cdot \mathcal{F}_S = L_2(Q_5 + \cdots + Q_{10} + \mathcal{L} + \mathcal{R}) \leq 20 - 2 + 1 = 19,$$

a contradiction. We conclude that $\mathcal{R}$ contains components of different singular fibres of π_0.

Furthermore, we show that $\mathcal{R}$ cannot contain any irreducible conics. Indeed, the morphism π_0 has no fibres of type *III* by Lemmas 3.17 and 3.20, and the conic in a fibre of type I_2 is not contained in the flecnodal divisor by Lemma 3.35. We can conclude that $\mathcal{R}$ must consist of three lines.

Now recall that $\mathcal{R}$ is σ-invariant, as we noted earlier in this proof. Because σ permutes the components of fibres of π_0 of types I_3 and *IV*, this means that $\mathcal{R}$ cannot contain any lines in such fibres, since otherwise it would have to contain the full fibre.

This only leaves L_1 and possibly a line L_1' in a second fibre of type I_2. However, L_1 cannot be a component of $\mathcal{R}$. To see this, note that $L_1 \cdot \mathcal{F}_S = 20$, and L_1 intersects the six conics $Q_5,\dots,Q_{10}$ each with multiplicity 2, as well as 10 other lines on S, each with multiplicity 1, and lastly itself with multiplicity -2. Thus we find

$$L_1 \cdot \mathcal{R} = L_1 \cdot \mathcal{F}_S - L_1 \cdot (Q_5 + \cdots + Q_{10} + \mathcal{L}) = 20 - (6 \cdot 2 + 10 - 2) = 0.$$

It follows immediately that $\mathcal{R}$ cannot be of the form$3L_1$, since otherwise $L_1 \cdot \mathcal{R}$ would be equal to -6. So π_0 must have a second fibre of type I_2 containing a line L_1'.

Now if we write $\mathcal{R} = aL_1 + bL_1'$ for some $a, b \in \mathbb{Z}$ satisfying $a + b = 3$, then it follows that

$$0 = L_1 \cdot (aL_1 + bL_1') = aL_1^2 + bL_1 \cdot L_1' = -2a + bL_1 \cdot L_1'.$$

Because L_1 and L_1' lie in different fibres of π_0, they must be disjoint, and hence $L_1 \cdot L_1' = 0$. It follows that $-2a = 0$, and thus $a = 0$, and so $\mathcal{R}$ is equal to $3L_1'$.

We conclude that indeed π_0 has a second fibre of type I_2 containing a line L_1' and we have $\mathcal{R} = 3L_1'$. Corollary 3.39 also applies to L_1', and so we can assume that L_1' also intersects at most 10 other lines on S. We compute

$$L_1' \cdot \mathcal{F}_S = L_1' \cdot \mathcal{L} + L_1' \cdot (Q_5 + \cdots + Q_{10}) + L_1' \cdot (3L_1') \leq 10 + 12 - 6 = 16 < 20,$$

which is impossible. We conclude that the assumption that S contains more than 64 lines was false, which completes the proof. □

Taking all the results from this chapter together, we obtain the final theorem of this thesis.

3.41 Theorem (Rams, Schütt, 2015)**.** *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface over an algebraically closed field k of characteristic not equal to* 2 *or* 3*.*

Then S contains at most 64 *lines.*

Proof. Recall Definition 2.5 of regular and inflectious lines. If all lines on S are regular, then the result follows from Theorem 3.12.

Note that this proof remains valid if we drop the assumption that all lines are regular and replace the use of Proposition 3.3 with the added assumption that all lines on S, regular or not, intersect at most 18 other lines on S.

Thus it only remains to cover the case where S contains at least one line L_0 that is inflectious and intersects at least 19 other lines on S. Let $\pi_0 \colon S \to \mathbb{P}^1$ be the elliptic fibration corresponding to L_0 as defined in Proposition 1.6.

By Proposition 3.26, the fact that L_0 intersects at least 19 lines implies that the restriction $\pi_0|_{L_0} : L_0 \to \mathbb{P}^1$ has two totally ramified points. By Lemma 3.28 and Remark 3.29 we can assume that the triple (S, L_0, π_0) is a member of the family $\mathcal{Z}$ from Definition 3.30. Furthermore, as the proof of Proposition 3.26 has shown, if L_0 intersects more than 18 lines, the fibration π_0 has at least one fibre of type I_2, containing a line L_1 and a conic Q_1.

By Corollary 3.32, we can assume that this line L_1 is regular. Then Corollary 3.39 and Lemma 3.40 prove the theorem in the cases where L_1 intersects at least ten other lines on S.

On the other hand, if L_1 intersects fewer than ten other lines on S, then note that by Lemma 3.36, this number can be at most 7. Further note that any line on S distinct from L_0 and L_1 must intersect precisely one of L_0, L_1, and the conic Q_1, as these three curves together form the intersection of S with some plane.

By Proposition 3.26, the line L_0 intersects at most 19 lines besides L_1. By Remark 3.10, the conic Q_1 intersects the flecnodal divisor $\mathcal{F}_S$ from Definition 3.9 with multiplicity 40. Since $\mathcal{F}_S$ contains all lines of S as components, and the conic Q_1 intersects both L_0 and L_1 with multiplicity 2, as they are all coplanar, it follows that Q_1 can intersect at most 36 lines besides L_0 and L_1. Lastly by assumption, the line L_1 intersects at most 6 other lines on S besides L_0.

In this case, the surface S contains at most

$$2 + 19 + 36 + 6 = 63$$

lines, which concludes the proof. □

A | Appendix

A.1 | Computations for Section 3.1

In this section, let k be an algebraically closed field and $\mathbb{P}^1$ the projective line over k. For ease of notation, we will write t for the point $(t:1) \in \mathbb{P}^1$ and ∞ for the point $(1:0)$. We want to prove the following statement:

A.1 Proposition. *Let k be an algebraically closed field and P_1,P_2,P_3,P_4 distinct points on the projective line $\mathbb{P}^1(k)$.*

Then the number of automorphisms f of $\mathbb{P}^1$ with the property that f permutes the points $P_1,\dots,P_4$ is either 4, 8, or 12.

Proof. After a change in coordinates, we can assume that the four points are given by 0, 1, ∞, and λ for some $\lambda \in k \smallsetminus \{0,1\}$.

Note that an automorphism of the projective line is uniquely determined by the images of three points. Thus for any of the 24 permutations of the four points, there can be at most one automorphism of $\mathbb{P}^1$ that respects the permutation.

Given a permutation $\sigma\colon \{0,1,\infty,\lambda\} \to \{0,1,\infty,\lambda\}$, we can compute the unique automorphism f of $\mathbb{P}^1$ satisfying $f(x) = \sigma(x)$ for $x = 0,1,\infty$. We can then compute $f(\lambda)$ and determine an equation that λ must fulfil so that $f(\lambda) = \sigma(\lambda)$.

For example, let σ be the permutation such that

$$\begin{aligned} \sigma(0) &= 0 \\ \sigma(1) &= \infty \\ \sigma(\infty) &= 1 \\ \sigma(\lambda) &= \lambda. \end{aligned}$$

The corresponding automorphism f of $\mathbb{P}^1$ that agrees with σ on 0, 1, and ∞ can be given (uniquely up to scalar multiplication by a unit in k) as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $f((r:s)) = (ar+bs : cr+ds)$.

Now because $f(0) = 0$, it follows that $(b:d) = (0:1)$, i.e.

$$b = 0.$$

Furthermore, since $f(1) = \infty$, we can conclude that $(a+b : c+d) = (1:0)$, and thus

$$c + d = 0.$$

Lastly, since $f(\infty) = 1$, we must have $(a:c) = (1:1)$, i.e.

$$a = c.$$

Taking all three equations together, we find that f is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

We can then compute

$$f(\lambda) = (\lambda : \lambda - 1).$$

In order for $f(\lambda)$ to be equal to λ, we must have

$$\frac{\lambda}{\lambda - 1} = \lambda.$$

Since λ cannot be equal to 1, this is equivalent to

$$\lambda^2 - 2\lambda = 0.$$

We can conclude that for this particular permutation σ an automorphism f respecting σ exists if and only if $\lambda^2 - 2\lambda = 0$, i.e. $\lambda = 2$.

This computation can be done for each of the 24 permutations of the four points. A complete list of all automorphisms and the resulting equations can be found in Tables A.1 and A.2. From these computations, we can conclude the following.

If λ satisfies the equation $\lambda^2 - \lambda + 1 = 0$, i.e. is equal to $-\zeta$ or $-\zeta^2$ for some fixed primitive cubic root of unity in k, then there are exactly 12 automorphisms of the projective line permuting the four points 0, 1, ∞, and λ.

If λ is equal to -1, $\frac{1}{2}$, or 2, then there are exactly 8 such automorphisms.

In all other cases, there are only 4. $\square$

Permutation	Autormorphism f in matrix form	$f(\lambda)$	$\sigma(\lambda)$	Equation
$(0,1,\infty,\lambda)$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	λ	λ	–
$(0,1,\lambda,\infty)$	$\begin{pmatrix}\lambda & 0\\1 & \lambda-1\end{pmatrix}$	$(\lambda^2:2\lambda-1)$	∞	$2\lambda-1=0$
$(0,\infty,1,\lambda)$	$\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}$	$(\lambda:\lambda-1)$	λ	$\lambda^2-2\lambda=0$
$(0,\infty,\lambda,1)$	$\begin{pmatrix}\lambda & 0\\1 & -1\end{pmatrix}$	$(\lambda^2:\lambda-1)$	1	$\lambda^2-\lambda+1=0$
$(0,\lambda,1,\infty)$	$\begin{pmatrix}\lambda & 0\\\lambda & 1-\lambda\end{pmatrix}$	$(\lambda^2:\lambda^2-\lambda+1)$	∞	$\lambda^2-\lambda+1=0$
$(0,\lambda,\infty,1)$	$\begin{pmatrix}\lambda & 0\\0 & 1\end{pmatrix}$	λ^2	1	$\lambda^2-1=0$
$(1,0,\infty,\lambda)$	$\begin{pmatrix}-1 & 1\\0 & 1\end{pmatrix}$	$1-\lambda$	λ	$2\lambda-1=0$
$(1,0,\lambda,\infty)$	$\begin{pmatrix}\lambda & -\lambda\\1 & -\lambda\end{pmatrix}$	∞	∞	–
$(1,\infty,0,\lambda)$	$\begin{pmatrix}0 & -1\\1 & -1\end{pmatrix}$	$(1:1-\lambda)$	λ	$\lambda^2-\lambda+1=0$
$(1,\infty,\lambda,0)$	$\begin{pmatrix}\lambda & -1\\1 & -1\end{pmatrix}$	$(\lambda^2-1:\lambda-1)$	0	$\lambda^2-1=0$
$(1,\lambda,0,\infty)$	$\begin{pmatrix}0 & \lambda\\1-\lambda & \lambda\end{pmatrix}$	$(-1:\lambda-2)$	∞	$\lambda-2=0$
$(1,\lambda,\infty,0)$	$\begin{pmatrix}\lambda-1 & 1\\0 & 1\end{pmatrix}$	$\lambda^2-\lambda+1$	0	$\lambda^2-\lambda+1=0$

Table A.1.: Computations for Proposition A.1, part 1. For a permutation σ of the four points 0, 1, ∞, λ on the projective line, we list the images of these four points in that order; the automorphism f of $\mathbb{P}^1$ that coincides with the given permutations on the points 0, 1, and ∞; the image of λ under that automorphism; the desired value of $f(\lambda)$ (i.e. $\sigma(\lambda)$); and lastly the equation that λ must satisfy in order for $f(\lambda)$ to be equal to $(\sigma(\lambda))$.

Permutation	Autormorphism f in matrix form	$f(\lambda)$	$\sigma(\lambda)$	Equation
$(\infty,0,1,\lambda)$	$\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$	$(\lambda-1:\lambda)$	λ	$\lambda^2-\lambda+1=0$
$(\infty,0,\lambda,1)$	$\begin{pmatrix}\lambda & -\lambda\\ 1 & 0\end{pmatrix}$	$\lambda-1$	1	$\lambda-2=0$
$(\infty,1,0,\lambda)$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$(1:\lambda)$	λ	$\lambda^2-1=0$
$(\infty,1,\lambda,0)$	$\begin{pmatrix}\lambda & 1-\lambda\\ 1 & 0\end{pmatrix}$	$(\lambda^2-\lambda+1:\lambda)$	0	$\lambda^2-\lambda+1=0$
$(\infty,\lambda,0,1)$	$\begin{pmatrix}0 & \lambda\\ 1 & 0\end{pmatrix}$	1	1	–
$(\infty,\lambda,1,0)$	$\begin{pmatrix}1 & \lambda-1\\ 1 & 0\end{pmatrix}$	$(2\lambda-1:\lambda)$	0	$2\lambda-1=0$
$(\lambda,0,1,\infty)$	$\begin{pmatrix}-\lambda & \lambda\\ -\lambda & 1\end{pmatrix}$	$(\lambda^2-\lambda:\lambda^2-1)$	∞	$\lambda^2-1=0$
$(\lambda,0,\infty,1)$	$\begin{pmatrix}-\lambda & \lambda\\ 0 & 1\end{pmatrix}$	$-\lambda^2+\lambda$	1	$\lambda^2-\lambda+1=0$
$(\lambda,1,0,\infty)$	$\begin{pmatrix}0 & \lambda\\ \lambda-1 & 1\end{pmatrix}$	$(\lambda:\lambda^2-\lambda+1)$	∞	$\lambda^2-\lambda+1=0$
$(\lambda,1,\infty,0)$	$\begin{pmatrix}1-\lambda & \lambda\\ 0 & 1\end{pmatrix}$	$(\lambda^2-2\lambda:-1)$	0	$\lambda^2-2\lambda=0$
$(\lambda,\infty,0,1)$	$\begin{pmatrix}0 & \lambda\\ -1 & 1\end{pmatrix}$	$(\lambda:1-\lambda)$	1	$2\lambda-1=0$
$(\lambda,\infty,1,0)$	$\begin{pmatrix}-1 & \lambda\\ -1 & 1\end{pmatrix}$	0	0	–

Table A.2.: Computations for Proposition A.1, part 2. For a permutation σ of the four points 0, 1, ∞, λ on the projective line, we list the images of these four points in that order; the automorphism f of $\mathbb{P}^1$ that coincides with the given permutations on the points 0, 1, and ∞; the image of λ under that automorphism; the desired value of $f(\lambda)$ (i.e. $\sigma(\lambda)$); and lastly the equation that λ must satisfy in order for $f(\lambda)$ to be equal to $(\sigma(\lambda))$.

A.2 Computations for Proposition 3.3

In the proof of Proposition 3.3, we needed to compute the multiplicity of the root 0 of the polynomial r_λ in the case where the fibre in the corresponding plane contains three lines. In Listing A.1, we provide the code for the case where at least one intersection of the lines in the fibre is not on L. The output is provided in Listing A.2. The second case of three concurrent lines whose intersection lies on L is covered in Listings A.3 and A.4.

```
// Setting up variables for the coefficients of the
   polynomials
K<a010,a011,a012,a013,a020,a021,a022,a030,a031,a040,
a100,a101,a110,a111,a112,a120,a121,a130,
a200,a210,a211,a220,a310>
:=FunctionField(Rationals(),23);
R<lambda>:=PolynomialRing(K);
S<x0,x1>:=PolynomialRing(R,2);
T<x>:=PolynomialRing(R);

// Defining the polynomials a_ij
a01:=a010*x0^3+a011*x0^2*x1+a012*x0*x1^2+a013*x1^3;
a02:=a020*x0^2+a021*x0*x1+a022*x1^2;
a03:=a030*x0+a031*x1;
a04:=a040;
a10:=x0*x1*(a100*x0+a101*x1);
a11:=a110*x0^2+a111*x0*x1+a112*x1^2;
a12:=a120*x0+a121*x1;
a13:=a130;
a20:=a200*x0*x1;
a21:=a210*x0+a211*x1;
a22:=a220;
a30:=0;
a31:=a310;
a40:=0;

// Defining the polynomial g_lambda and its second
   derivatives
g:=a10+a01*lambda;
```

```
g0:=Derivative(g,x0);
g1:=Derivative(g,x1);
g00:=Derivative(g0,x0);
g01:=Derivative(g0,x1);
g11:=Derivative(g1,x1);

// Defining the remaining entries of the Hessian matrix of C_lambda
F2:=a20+a11*lambda+a02*lambda^2;
F3:=a30+a21*lambda+a12*lambda^2+a03*lambda^3;

// Defining the matrix and its determinant to get h_lambda
M:=Matrix([
[g00,g01,Derivative(F2,x0)],
[g01,g11,Derivative(F2,x1)],
[Derivative(F2,x0),Derivative(F2,x1),2*F3]
]);

h:=Determinant(M);

// Determining the multiplicity of the root 0 of the
// resultant of g_lambda and h_lambda
res:=Resultant(Evaluate(g,[x,1]), Evaluate(h,[x,1]));

// Printing the degree and roots of r_lambda
Degree(res);
Roots(res);
```

Listing A.1: Magma code to compute the multiplicity of a root of the polynomial $r(\lambda)$ in the proof of Proposition 3.3 in the case that the fibre corresponding to this root contains two lines intersecting in a point not lying on the line L

```
1 18
2 [
3 <0, 3>
4 ]
```

Listing A.2: Output of the code in Listing A.1. Line 1 shows that the degree of the polynomial r_λ is indeed 18. Line 3 shows that 0 is a root of multiplicity 3, as desired.

```
// Setting up variables for the coefficients of the polynomials
K<b,c,
a010,a011,a012,a013,a020,a021,a022,a030,a031,a040,
a110,a111,a112,a120,a121,a130,a210,a211,a220,a310>
:=FunctionField(Rationals(),23);
R<lambda>:=PolynomialRing(K);
S<x0,x1>:=PolynomialRing(R,2);
T<x>:=PolynomialRing(R);


// Defining the polynomials a_ij
a01:=a010*x0^3+a011*x0^2*x1+a012*x0*x1^2+a013*x1^3;
a02:=a020*x0^2+a021*x0*x1+a022*x1^2;
a03:=a030*x0+a031*x1;
a04:=a040;
a10:=b*c*x1^3;
a11:=a110*x0^2+a111*x0*x1+a112*x1^2;
a12:=a120*x0+a121*x1;
a13:=a130;
a20:=(b+c)*x1^2;
a21:=a210*x0+a211*x1;
a22:=a220;
a30:=x1;
a31:=a310;
a40:=0;

```

```
    // Defining the polynomial g_lambda and its second
       derivatives
    g:=a10+a01*lambda;

    g0:=Derivative(g,x0);
    g1:=Derivative(g,x1);
    g00:=Derivative(g0,x0);
    g01:=Derivative(g0,x1);
    g11:=Derivative(g1,x1);

    // Defining the remaining entries of the Hessian matrix
       of C_lambda
    F2:=a20+a11*lambda+a02*lambda^2;
    F3:=a30+a21*lambda+a12*lambda^2+a03*lambda^3;

    // Defining the matrix and its determinant to get
       h_lambda
    M:=Matrix([
    [g00,g01,Derivative(F2,x0)],
    [g01,g11,Derivative(F2,x1)],
    [Derivative(F2,x0),Derivative(F2,x1),2*F3]
    ]);

    h:=Determinant(M);

    // Determining the multiplicity of the root 0 of the
    // resultant of g_lambda and h_lambda
    res:=Resultant(Evaluate(g,[x,1]), Evaluate(h,[x,1]));

    // Printing the degree and roots of r_lambda
    Degree(res);
    Roots(res);
```

Listing A.3: Magma code to compute the multiplicity of a root of the polynomial $r(\lambda)$ in the proof of Proposition 3.3 in the case that the fibre corresponding to this root contains three lines which all intersect in a single point that lies on the line L

```
1    18
2    [
3    <0, 5>
4    ]
```

Listing A.4: Output of the code in Listing A.3. Line 1 shows that the degree of the polynomial r_λ is indeed 18. Line 3 shows that 0 is a root of multiplicity 5, as desired.

A.3 PROOF OF PROPOSITION 3.31

In this section we want to prove Proposition 3.31.

3.31 Proposition. *Let (S, L_0, π_0) be in the family $\mathcal{Z}$ from Definition 3.30.*

If a fibre of π_0 contains another inflectious line L_1, then S is projectively equivalent to the Schur quartic from Example 3.1.

We will fix a notation for the coefficients of the polynomials q and g by defining

$$q(x_2, x_3) = q_0x_3^2 + q_1x_3x_2 + q_2x_2^2$$
$$g(x_2, x_3) = g_0x_3^4 + g_1x_3^3x_2 + g_2x_3^2x_2^2 + g_3x_3x_2^3 + g_4x_2^4$$

Now assume that there is an inflectious line L_1 in a fibre of π_0. There are two important cases to consider. The fibre in which L_1 lies is either ramified or unramified, in the sense that the curve morphism $\pi_0|_{L_0} : L_0 \to \mathbb{P}^1$ does or does not ramify at the point above which the fibre lies. We will be looking at these two cases separately.

A.3.1 UNRAMIFIED FIBRE

In this subsection, we will prove the following.

A.2 Lemma. *The statement of Proposition 3.31 holds under the assumption that the fibre of π_0 in which L_1 lies is unramified.*

Proof. In this case we can assume that the fibre lies in a plane of the form $x_3 = \lambda x_2$ for some nonzero $\lambda \in k$. After a linear change in coordinates, we can assume $\lambda = 1$, i.e. the plane has equation $x_2 = x_3$.

The intersection of S with this plane is given by the equation

$$x_2x_0^3 + x_2x_1^3 + cx_0x_1x_2^2 + dx_2^4 = 0$$

where

$$
\begin{aligned}
c &= q_0 + q_1 + q_2 \\
d &= g_0 + g_1 + g_2 + g_3 + g_4
\end{aligned}
$$

and after dividing by the factor x_2 we obtain the residual cubic

$$x_0^3 + x_1^3 + dx_2^3 + cx_0x_1x_2 = 0.$$

We will now distinguish four subcases depending on whether c or d are zero.

a) $d = 0$ and $c \neq 0$. In this case the residual cubic is irreducible. Indeed, after rescaling x_2, we can assume $c = 1$, and get a cubic curve with equation

$$x_0^3 + x_1^3 + x_0x_1x_2 = 0.$$

We can show that this curve only has one singular point by computing the partial derivatives, which are

$$
\begin{aligned}
3x_0^2 + x_1x_2 \\
3x_1^2 + x_0x_2 \\
x_0x_1
\end{aligned}
$$

In order for x_0x_1 to vanish, one of x_0 or x_1 must vanish, but then by the first two equations, so does the other. This leaves only the point $(0:0:1)$ as a possibility for a singularity, and indeed it does lie on the curve. We conclude that the curve is either irreducible or consists of three lines all meeting in one point. We can rule out the latter case by considering the line $x_0 = x_1$, which intersects the curve not only in the singular point, but also in the point $(1:1:-2)$.

b) $d = 0$ and $c = 0$. In this case, the residual cubic has equation

$$x_0^3 + x_1^3 = 0.$$

This clearly decomposes into the three lines $x_0 + \zeta^i x_1 = 0$ for $i = 0, 1, 2$, where ζ is a primitive cube root of unity. Since all three of these lines meet in the point $(0:0:1)$, which does not lie on L_0, we get a fibre of type *IV*.

If we consider the elliptic fibration π_1 corresponding to the line L_1, then π_1 has a ramified fibre of type I_3 in this plane. But this is impossible by Lemma 3.20, since we assume that L_1 is inflectious.

c) $d \neq 0$ and $c = 0$. After scaling x_2, we can assume $d = 1$, and the equation for the residual cubic becomes

$$x_0^3 + x_1^3 + x_2^3 = 0$$

which gives a smooth curve outside characteristic 3.

d) $d \neq 0$ and $c \neq 0$. Then after scaling x_2 we can assume $d = 1$ and get a residual cubic with equation

$$x_0^3 + x_1^3 + x_2^3 + cx_0x_1x_2 = 0.$$

We will examine singularities in this curve. Consider the three partial derivatives

$$\frac{\partial}{\partial x_0} = 3x_0^2 + cx_1x_2$$
$$\frac{\partial}{\partial x_1} = 3x_1^2 + cx_0x_2$$
$$\frac{\partial}{\partial x_2} = 3x_2^2 + cx_0x_1$$

If $x_0 = 0$, then all three cannot simultaneously vanish, and so we can restrict our attention to the affine patch $x_0 = 1$. We get the three equations

$$3 + cx_1x_2 = 0$$
$$3x_1^2 + cx_2 = 0$$
$$3x_2^2 + cx_1 = 0$$

The first equation implies that both x_1 and x_2 are nonzero. From the second and third equation respectively, we conclude that

$$x_2 = -\frac{3}{c}x_1^2 \tag{A.1}$$
$$x_1 = -\frac{3}{c}x_2^2 \tag{A.2}$$

Substituting x_2 into the second equation here, we get

$$\begin{aligned} x_1 &= -\frac{3}{c}\left(-\frac{3}{c}x_1^2\right)^2 \\ &= -\frac{27}{c^3}x_1^4 \end{aligned}$$

so either $x_1 = 0$, which we ruled out above, or $1 = -\frac{27}{c^3}x_1^3$, i.e.

$$x_1 \in \left\{-\frac{c}{3}, -\frac{\zeta c}{3}, -\frac{\zeta^2 c}{3}\right\}$$

where ζ is a primitive cube root of unity. On the one hand, we can substitute this back into Equation (A.1) to obtain

$$x_2 \in \left\{-\frac{c}{3}, -\frac{\zeta^2 c}{3}, -\frac{\zeta c}{3}\right\}.$$

On the other hand, if we substitute any of the three points

$$\begin{aligned} &(1 : -\frac{c}{3} : -\frac{c}{3}) \\ &(1 : -\frac{\zeta c}{3} : -\frac{\zeta^2 c}{3}) \\ &(1 : -\frac{\zeta^2 c}{3} : -\frac{\zeta c}{3}) \end{aligned}$$

into the equation of the cubic, we get

$$1 - \frac{c^3}{27} - \frac{c^3}{27} + \frac{c^3}{9} = 0$$

from which we can conclude

$$1 = -\frac{c^3}{27}$$

and thus the three points above lie on the cubic if and only if $c^3 = -27$, i.e., if $c = -3$, -3ζ, or $-3\zeta^2$.

In all three cases, we get the same three singular points, and in all other cases the cubic is smooth. Without loss of generality, we shall only consider the case $c = -3$. The cubic now decomposes into the product of three lines

$$(x_0 + x_1 + x_2)(x_0 + \zeta x_1 + \zeta^2 x_2)(x_0 + \zeta^2 x_1 + \zeta x_2) = 0.$$

Without loss of generality we can also assume that the line L_1 is given by the equation $x_0 + x_1 + x_2 = 0$.

We claim that this case is actually impossible if L_1 is inflectious.

Consider the elliptic fibration π_1 associated to L_1. Almost all planes through L_1 have the form

$$x_2 - x_3 = m(x_0 + x_1 + x_2) = 0$$

for some $m \in k$. We compute the intersection of the plane with the surface S by substituting $x_3 = x_2 - m(x_0 + x_1 + x_2)$ in the equation of S, and get

$$\begin{aligned}
0 =& x_2 x_0^3 + (x_2 - m(x_0 + x_1 + x_2))x_1^3 \\
&+ x_0 x_1 \left[q_0(x_2 - m(x_0 + x_1 + x_2))^2 + q_1 x_2 (x_2 - m(x_0 + x_1 + x_2)) + q_2 x_2^2\right] \\
&+ g_0(x_2 - m(x_0 + x_1 + x_2))^4 + g_1 x_2 (x_2 - m(x_0 + x_1 + x_2))^3 \\
&+ g_2 x_2^2 (x_2 - m(x_0 + x_1 + x_2))^2 + g_3 x_2^3 (x_2 - m(x_0 + x_1 + x_2)) + g_4 x_2^4
\end{aligned}$$

We expand the powers of $(x_2 - m(x_0 + x_1 + x_2))$ to obtain

$$\begin{aligned}
0 =& x_2 x_0^3 + x_2 x_1^3 - m(x_0 + x_1 + x_2)x_1^3 \\
&+ q_0 x_0 x_1 \left[x_2^2 - m x_2 (x_0 + x_1 + x_2) + m^2 (x_0 + x_1 + x_2)^2\right] \\
&+ q_1 x_0 x_1 \left[x_2^2 - m x_2 (x_0 + x_1 + x_2)\right] \\
&+ q_2 x_0 x_1 x_2^2 \\
&+ g_0 \left[x_2^4 - 4m x_2^3 (x_0 + x_1 + x_2) + 6m^2 x_2^2 (x_0 + x_1 + x_2)^2 \right. \\
&\qquad \left. -4m^3 x_2 (x_0 + x_1 + x_2)^3 + m^4 (x_0 + x_1 + x_2)^4\right] \\
&+ g_1 \left[x_2^4 - 3m x_2^3 (x_0 + x_1 + x_2) + 3m^2 x_2^2 (x_0 + x_1 + x_2)^2 - m^3 x_2 (x_0 + x_1 + x_2)^3\right] \\
&+ g_2 \left[x_2^4 - 2m x_2^3 (x_0 + x_1 + x_2) + m^2 x_2^2 (x_0 + x_1 + x_2)^2\right] \\
&+ g_3 \left[x_2^4 - x_2^3 (x_0 + x_1 + x_2)\right] \\
&+ g_4 x_2^4
\end{aligned}$$

This equation must be divisible by $(x_0+x_1+x_2)$, since that factor represents the line L_1, which is contained in the fibre. We will thus collect the terms that already contain the factor, and the terms that don't. The equation now takes the form

$$\begin{aligned}
0 =& x_2x_0^3+x_2x_1^3+(q_0+q_1+q_2)x_0x_1x_2^2+(g_0+g_1+g_2+g_3+g_4)x_2^4\\
& m(x_0+x_1+x_2)\big[-x_1^3\\
& -q_0x_0x_1x_2+mq_0(x_0+x_1+x_2)\\
& -4g_0x_2^3+6mg_0x_2^2(x_0+x_1+x_2)-4m^2g_0x_2(x_0+x_1+x_2)^2+m^3g_0(x_0+x_1+x_2)^3\\
& -3g_1x_2^3+3mg_1x_2^2(x_0+x_1+x_2)-m^2g_1x_2(x_0+x_1+x_2)^2\\
& -2g_2x_2^3+mg_2x_2^2(x_0+x_1+x_2)\\
& -g_3x_2^3\big].
\end{aligned}$$

Since we know that $q_0+q_1+q_2=c=-3$ and $g_0+g_1+g_2+g_3+g_4=d=1$, the first line of this equation takes the form

$$x_2(x_0^3+x_1^3+x_2^3-3x_0x_1x_2)$$

of which we already know the factorisation. The residual cubic is thus given by the equation

$$0=x_2(x_0+\zeta x_1+\zeta^2x_2)(x_0+\zeta^2x_1+\zeta x_2)+m[\dots] \tag{A.3}$$

To compute the intersection of this cubic with the line L_1, we set $x_0+x_1+x_2=0$, so a lot of terms within the square brackets already disappear, and we are left with

$$\begin{aligned}
0 =& x_2(x_0+\zeta x_1+\zeta^2x_2)(x_0+\zeta^2x_1+\zeta x_2)\\
&+m\left[-x_1^3-(q_0+q_1)x_0x_1x_2-(4g_0+3g_1+2g_2+g_3)x_2^3\right]
\end{aligned}$$

and after substituting $x_0=-x_1-x_2$, the three points in the intersection are given by the equation

$$0=x_2\left((\zeta-1)x_1+(\zeta^2-1)x_2\right)\left((\zeta^2-1)x_1+(\zeta-1)x_2\right)+m(-x_1^3-\gamma x_2^3)$$

where $\gamma=4g_0+3g_1+2g_2+g_3$. After expanding the parentheses, we can write this as

$$0=-mx_1^3+(m(q_0+q_1)+3)(x_1^2x_2+x_1x_2^2)+(3-m\gamma)x_2^3. \tag{A.4}$$

Since L_1 is assumed to be inflectious, all three of these points must be inflections of the residual cubic. We thus compute the Hessian of Eq. (A.3).

This is rather tedious and so we will use a computer for this. Listing A.5 provides code that computes the Hessian, and then restricts it to the line L_1. The output can be found in Listing A.6.

Now in order for L_1 to be inflectious, the points in the intersection of any fibre with L_1, i.e., the points that satisfy Eq. (A.4), must also be inflections of their respective fibre, i.e. also be roots of the Hessian from Listing A.6.

In particular, the Hessian must be a multiple of Eq. (A.4). If we write this equation as

$$0 = C_0 x_1^3 + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 + C_3 x_2^3$$

and the Hessian as

$$0 = H_0 x_1^3 + H_1 x_1^2 x_2 + H_2 x_1 x_2^2 + H_3 x_2^3$$

where the C_i and H_i are polynomials in the variables m, q_i, and g_i. then this implies that for any $i, j \in \{0,1,2,3\}$, we must have

$$C_i H_j - C_j H_i = 0.$$

With this in mind, we can define the polynomials

$$g_{ij} = C_i H_j - C_j H_i$$

in m, q_i, and g_i. These polynomials are computed in Listing A.7. Because they must vanish in every fibre, i.e., for every value of m, we can interpret them as monovariate polynomials in m, and conclude that each coefficient must vanish. As we can see in Listing A.8, the polynomial p_{01} has 144 as the coefficient of the linear term. This means that because we are not in characteristic 2 or 3, it is impossible for all coefficients to vanish.

We conclude that L_1 cannot be inflectious, contradicting our assumption.

We have thus shown that an inflectious line cannot occur in an unramified fibre of π_0, given that L_0 itself is inflectious. □

```
K<s> := CyclotomicField(3);
R0<m,q0,q1,q2,g0,g1,g2,g3,g4> := PolynomialRing(K,9);
R<x0,x1,x2> := PolynomialRing(R0,3);

//Equation of the residual cubic
c := x2*(x0 + s*x1 + s^2*x2)*(x0 + s^2*x1 + s*x2)
+ m*(-x1^3 - q0*x0*x1*x2 + m*q0*(x0+x1+x2) - 4*g0*x2^3
+ 6*m*g0*x2^2*(x0+x1+x2) - 4*m^2*g0*x2*(x0+x1+x2)^2
+ m^3*g0*(x0+x1+x2)^3 - 3*g1*x2^3
+ 3*m*g1*x2^2*(x0+x1+x2)
- m^2*g1*x2*(x0+x1+x2)^2 - 2*g2*x2^3
+ m*g2*x2^2*(x0+x1+x2) - g3*x2^3);

//First Derivatives
c0 := Derivative(c,x0);
c1 := Derivative(c,x1);
c2 := Derivative(c,x2);

//Second Derivatives
c00 := Derivative(c0,x0);
c01 := Derivative(c0,x1);
c02 := Derivative(c0,x2);
c11 := Derivative(c1,x1);
c12 := Derivative(c1,x2);
c22 := Derivative(c2,x2);

//Determinant of the Hessian matrix
h := c00*c11*c22 + c01*c12*c02 + c02*c01*c12 - c02^2*c11
   - c01^2*c22 - c12^2*c00;

//restrict h to the line L_1
Evaluate(h,[-x1-x2,x1,x2]);
```

Listing A.5: Magma code for computing the Hessian of the residual cubic in case d), i.e. under the assumption that the line L_1 is given by the equation $x_0 + x_1 + x_2 = 0$.

```
(6*m^3*q0^2 + 36*m^2*q0 + 54*m)*x1^3

+ (32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 + 96*m^5*q0*g0 +
24*m^5*q0*g1 + 48*m^4*q0*g0 - 24*m^4*q0*g1 - 24*m^4*q0*g
+ 288*m^4*g0 + 72*m^4*g1 + 2*m^3*q0^3 - 144*m^3*g0
- 144*m^3*g1 - 72*m^3*g2 + 10*m^2*q0^2 + 48*m^2*q0
+ 6*m*q0 + 144*m - 18)*x1^2*x2

+ (32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 - 288*m^5*g0^2
- 288*m^5*g0*g1 - 288*m^5*g0*g2 - 288*m^5*g0*g3
- 72*m^5*g1*g3 + 24*m^5*g2^2 + 192*m^4*q0*g0
+ 48*m^4*q0*g1 + 864*m^4*g0 + 216*m^4*g1 + 2*m^3*q0^3
- 576*m^3*g0 - 360*m^3*g1 - 144*m^3*g2 + 10*m^2*q0^2
+ 288*m^2*g0 + 216*m^2*g1 + 144*m^2*g2 + 72*m^2*g3
+ 6*m*q0 - 18)*x1*x2^2

+ (32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 + 96*m^5*q0*g0^2
+ 96*m^5*q0*g0*g1 + 96*m^5*q0*g0*g2 + 96*m^5*q0*g0*g3
+ 24*m^5*q0*g1*g3 - 8*m^5*q0*g2^2 - 48*m^4*q0^2*g0
- 24*m^4*q0^2*g1 - 8*m^4*q0^2*g2 - 96*m^4*q0*g0
- 24*m^4*q0*g1 + 288*m^4*g0^2 + 288*m^4*g0*g1
+ 288*m^4*g0*g2 + 288*m^4*g0*g3 + 72*m^4*g1*g3
- 24*m^4*g2^2 + 24*m^3*q0^2*g0 + 18*m^3*q0^2*g1
+ 12*m^3*q0^2*g2 + 6*m^3*q0^2*g3 - 576*m^3*g0
- 144*m^3*g1 - 2*m^2*q0^2 + 48*m^2*q0*g0 + 36*m^2*q0*g1
+ 24*m^2*q0*g2 + 12*m^2*q0*g3 + 432*m^2*g0 + 216*m^2*g1
+ 72*m^2*g2 - 12*m*q0 - 72*m*g0 - 54*m*g1 - 36*m*g2
- 18*m*g3 - 18)*x2^3
```

Listing A.6: Output of Listing A.5. Linebreaks have been manually edited for readabiltiy reasons.

```
K<s> := CyclotomicField(3);
S0<q0,q1,q2,g0,g1,g2,g3,g4> := PolynomialRing(K,8);
S<m>:= PolynomialRing(S0);

//Coefficients of the residual cubic restricted to L_1
C0 := -m;
C1 := m*(q0+q1) + 3;
C2 := m*(q0+q1) + 3;
C3 := 3 - m*(4*g0 + 3*g1 + 2*g2 + g3);

//Coefficients of the Hessian restricted to L_1
H0 := 6*m^3*q0^2 + 36*m^2*q0 + 54*m;
H1 := 32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 + 96*m^5*q0*g0
+ 24*m^5*q0*g1 + 48*m^4*q0*g0 - 24*m^4*q0*g1
- 24*m^4*q0*g2 + 288*m^4*g0 + 72*m^4*g1 + 2*m^3*q0^3
- 144*m^3*g0 - 144*m^3*g1 - 72*m^3*g2 + 10*m^2*q0^2
+ 48*m^2*q0 + 6*m*q0 + 144*m - 18;
H2 := 32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 - 288*m^5*g0^2
- 288*m^5*g0*g1 - 288*m^5*g0*g2 - 288*m^5*g0*g3
- 72*m^5*g1*g3 + 24*m^5*g2^2 + 192*m^4*q0*g0
+ 48*m^4*q0*g1 + 864*m^4*g0 + 216*m^4*g1 + 2*m^3*q0^3
- 576*m^3*g0 - 360*m^3*g1 - 144*m^3*g2 + 10*m^2*q0^2
+ 288*m^2*g0 + 216*m^2*g1 + 144*m^2*g2 + 72*m^2*g3
+ 6*m*q0 - 18;
H3 := 32*m^5*q0^2*g0 + 8*m^5*q0^2*g1 + 96*m^5*q0*g0^2
+ 96*m^5*q0*g0*g1 + 96*m^5*q0*g0*g2 + 96*m^5*q0*g0*g3
+ 24*m^5*q0*g1*g3 - 8*m^5*q0*g2^2 - 48*m^4*q0^2*g0
- 24*m^4*q0^2*g1 - 8*m^4*q0^2*g2 - 96*m^4*q0*g0
- 24*m^4*q0*g1 + 288*m^4*g0^2 + 288*m^4*g0*g1
+ 288*m^4*g0*g2 + 288*m^4*g0*g3 + 72*m^4*g1*g3
- 24*m^4*g2^2 + 24*m^3*q0^2*g0 + 18*m^3*q0^2*g1
+ 12*m^3*q0^2*g2 + 6*m^3*q0^2*g3 - 576*m^3*g0
- 144*m^3*g1 - 2*m^2*q0^2 + 48*m^2*q0*g0 + 36*m^2*q0*g1
+ 24*m^2*q0*g2 + 12*m^2*q0*g3 + 432*m^2*g0 + 216*m^2*g1
+ 72*m^2*g2 - 12*m*q0 - 72*m*g0 - 54*m*g1 - 36*m*g2
```

```
    - 18*m*g3 - 18;
    //Only printing p_01 to save space
    H0*C1 - C0*H1;
    //H0*C2 - C0*H2;
    //H0*C3 - C0*H3;
    //H1*C2 - C1*H2;
    //H1*C3 - C1*H3;
    //H2*C3 - C2*H3;
```

Listing A.7: Magma code to compute the polynomials p_{ij} which must vanish in order for the line L_1 to be inflectious. Only p_{01} is printed because it is sufficient to complete the argument.

```
    (32*q0^2*g0 + 8*q0^2*g1 + 96*q0*g0 + 24*q0*g1)*m^6
    + (48*q0*g0 - 24*q0*g1 - 24*q0*g2 + 288*g0 + 72*g1)*m^5
    + (8*q0^3 + 6*q0^2*q1 - 144*g0 - 144*g1 - 72*g2)*m^4
    + (64*q0^2 + 36*q0*q1 + 48*q0)*m^3
    + (168*q0 + 54*q1 + 144)*m^2
    + 144*m
```

Listing A.8: Output of Listing A.7. Linebreaks have been manually edited for readability reasons

A.3.2 RAMIFIED FIBRE

In this subsection we prove the second subcase of Proposition 3.31.

A.3 Lemma. *The statement of Proposition 3.31 holds under the assumption that the fibre of π_0 in which L_1 lies is ramified.*

Proof. Because (S, L_0, π_0) is in the family $\mathcal{Z}$, we know by Remark 3.29 that the ramified fibre in question must lie in the plane $x_2 = 0$ or $x_3 = 0$. Without loss of generality, we can assume it to be $x_3 = 0$. The intersection of S with this plane has equation

$$x_2 x_0^3 + c x_0 x_1 x_2^2 + d x_2^4 = 0$$

where $c = q_2$ and $d = g_4$. The residual cubic is given by

$$x_0^3 + c x_0 x_1 x_2 + d x_2^3.$$

We will distinguish the same four subcases as in the unramified case.

a) $d = 0$ and $c = 0$. This case is impossible, as it leads to the cubic equation $x_0^3 = 0$, which is not a reduced curve.

b) $d = 0$ and $c \neq 0$. In this case, the cubic equation becomes

$$x_0^3 + c x_0 x_1 x_2.$$

Immediately this splits off a linear factor x_0. The conic $x_0^2 + c x_1 x_2$ is smooth, as the partial derivatives $2x_0$, cx_2 and cx_1 cannot simultaneously vanish. Furthermore, the line and the conic intersect in the two distinct points $(0:0:1:0)$ and $(0:1:0:0)$. In particular, this gives a ramified fibre of type I_2. The fibration corresponding to the line L_1 would then have a (ramified) fibre of type *III* in this plane, which is impossible by Lemma 3.20 since L_1 is inflectious.

c) $d \neq 0$ and $c \neq 0$. After scaling x_2, we can assume that $d = 1$. The residual cubic is now given by

$$x_0^3 + x_2^3 + c x_0 x_1 x_2 = 0.$$

Up to renaming coordinates, this is identical to subcase a) in the case $\lambda \neq 0$, where it was shown that this is an irreducible cubic.

d) $d \neq 0$ and $c = 0$. In this case, we can assume $d = 1$ and get

$$x_0^3 + x_2^3 = 0$$

as an equation for the residual cubic. Similarly to subcase b) of the case $\lambda \neq 0$, this gives a fibre of type IV, with the difference being that in this case it is ramified (note that the intersection point here is $(0:1:0:0)$, which lies on L_0). We claim that this implies that S is projectively equivalent to the Schur quartic from Section 3.1 in this case.

We will employ a similar method as in case d) of the previous section. Without loss of generality we shall assume that L_1 is given by $x_0 + x_2 = 0$ and $x_3 = 0$. Almost all planes through L_1 are given by an equation

$$x_3 = m(x_0 + x_2)$$

for some $m \in k$. The intersection of such a plane with S is given by

$$\begin{aligned}
0 =& x_2 x_0^3 + m(x_0 + x_2)x_1^3 \\
&+ x_0 x_1 \left[q_0(m(x_0 + x_2))^2 + q_1 m(x_0 + x_2)x_2 + q_2 x_2^2 \right] \\
&+ g_0(m(x_0 + x_2))^4 + g_1(m(x_0 + x_2))^3 x_2 + g_2(m(x_0 + x_2))^2 x_2^2 \\
&+ g_3(m(x_0 + x_2))x_2^3 + g_4 x_2^4.
\end{aligned}$$

Recall that $q_2 = c = 0$ and $g_4 = d = 1$. After simplifying, this equation takes the form

$$\begin{aligned}
0 =& x_2 x_0^3 + m x_0 x_1^3 + m x_2 x_1^3 + x_2^4 \\
&+ m(x_0 + x_2)[m q_0 x_0 x_1 (x_0 + x_2) + q_1 x_0 x_1 x_2] \\
&+ m(x_0 + x_2) \left[g_0(m(x_0 + x_2))^3 + g_1(m(x_0 + x_2))^2 x_2 \right. \\
&\left. + g_2(m(x_0 + x_2))x_2^2 + g_3 x_2^3 \right]
\end{aligned}$$

Note that the first line splits off a factor of $(x_0 + x_2)$ via

$$(x_0 + x_2)(x_0^2 x_2 - x_0 x_2^2 + m x_1^3 + x_2^3)$$

and so the residual cubic is given by

$$0 = x_0^2 x_2 - x_0 x_2^2 + m x_1^3 + x_2^3 \tag{A.5}$$

$$+ m\left[m q_0 x_0 x_1 (x_0 + x_2) + q_1 x_0 x_1 x_2\right] \tag{A.6}$$

$$+ m\left[g_0(m(x_0+x_2))^3 + g_1(m(x_0+x_2))^2 x_2\right. \tag{A.7}$$

$$\left. + g_2(m(x_0+x_2)) x_2^2 + g_3 x_2^3\right] \tag{A.8}$$

We intersect this with the line L_1 by substituting $x_0 = -x_2$ and get the equation

$$0 = (3 + m g_3) x_2^3 - m q_1 x_1 x_2^2 + m x_1^3. \tag{A.9}$$

Furthermore, we compute the Hessian of Eq. (A.5). Once again, we will use a computer. The code can be found in Listing A.9 and the output is given in Listing A.10. Like we did in the previous section, we will use the notation

$$0 = C_0 x_1^3 + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 + C_3 x_2^3$$
$$0 = H_0 x_1^3 + H_1 x_1^2 x_2 + H_2 x_1 x_2^2 + H_3 x_2^3$$

for the residual cubic restricted to L_1 and the Hessian restricted to L_1, respectively, where the C_i and the H_i are polynomials in the variables m, q_i, g_i. Because these two polynomials must be multiples of each other, and in this case $C_1 = 0$, it follows that H_1 must also be zero as a univariate polynomial in m. Notice that the coefficient of m^3 in the polynomial H_1 is $144 q_0$, and we conclude that q_0 must be zero. Similarly, the coefficient of m^2 is $48 q_1$, which must also vanish, and so $q_1 = 0$.

After substituting these two values back into the polynomials H_i, we get

$$H_0 = 0$$
$$H_1 = 0$$
$$H_2 = (72 g_1 g_3 - 24 g_2^2) m^5 + 216 g_1 m^4 + 144 g_2 m^3 + 72 g_3 m^2$$
$$H_3 = 0.$$

Considering like in the previous subsection the polynomial

$$g_{02} = H_2 C_0 - H_0 C_2,$$

which must be zero as a polynomial in m, note that this is equal to mH_2. It follows that all coefficients of H_2 must vanish, and we conclude that $g_1 = g_2 = g_3 = 0$.

We are left with only g_0 and g_4 being possibly nonzero. The surface S thus has equation

$$x_2x_0^3 + x_3x_1^3 + g_0x_3^4 + g_4x_2^4 = 0,$$

and after a change in coordinates we may assume that $g_0 = g_4 = 1$. Up to flipping signs and renaming coordinates, this is exactly the equation of the Schur quartic as it was defined in Section 3.1.

We conclude that if there is an inflectious line L_1 in a ramified fibre of π_0, then S must be the Schur quartic. □

```
K<s> := CyclotomicField(3);
R0<m,q0,q1,g0,g1,g2,g3> := PolynomialRing(K,7);
R<x0,x1,x2> := PolynomialRing(R0,3);

//Equation of the residual cubic
c := x0^2*x2 - x0*x2^2 + m*x1^3 + x2^3
+ m*(m*q0*x0*x1*(x0+x2) + q1*x0*x1*x2)
+ m*(g0*(m*(x0+x2))^3 + g1*(m*(x0+x2))^2*x2
+ g2*(m*(x0+x2))*x2^2 + g3*x2^3);

//First derivatives
c0 := Derivative(c,x0);
c1 := Derivative(c,x1);
c2 := Derivative(c,x2);

//Second derivatives
c00 := Derivative(c0,x0);
c01 := Derivative(c0,x1);
c02 := Derivative(c0,x2);
c11 := Derivative(c1,x1);
c12 := Derivative(c1,x2);
c22 := Derivative(c2,x2);

//Compute the determinant of the Hessian matrix
h := c00*c11*c22 + c01*c12*c02 + c02*c01*c12 - c02*c11*
   c02
- c01*c01*c22 - c12*c12*c00;

//Restrict to the line L_1
Evaluate(h,[-x2,x1,x2]);
```

Listing A.9: Magma code for computing the Hessian of the residual cubic in the case that L_1 is given by the equation $x_0 + x_2 = 0$

```
(-6*m^5*q0^2 - 12*m^4*q0*q1 - 6*m^3*q1^2)*x1^3
+ (24*m^5*q0*g2 - 24*m^5*q1*g1 +
72*m^4*q0*g3 - 24*m^4*q1*g2 + 144*m^3*q0
+ 48*m^2*q1)*x1^2*x2 +
(-2*m^5*q0^2*q1 + 72*m^5*g1*g3 - 24*m^5*g2^2
- 4*m^4*q0*q1^2 + 216*m^4*g1 -
2*m^3*q1^3 + 144*m^3*g2 + 72*m^2*g3)*x1*x2^2
+ (-6*m^5*q0^2*g3 +
8*m^5*q0*q1*g2 - 8*m^5*q1^2*g1 - 18*m^4*q0^2
+ 12*m^4*q0*q1*g3 -
8*m^4*q1^2*g2 + 12*m^3*q0*q1 - 6*m^3*q1^2*g3
- 2*m^2*q1^2)*x2^3
```

Listing A.10: Output of Listing A.9. Linebreaks have been manually edited for readability reasons.

A.4 COMPUTATIONS FOR LEMMA 3.35

In this section we want to prove Lemma 3.35.

3.35 Lemma. *Let (S, L_0, π_0) be in the family $\mathcal{Z}$ from Definition 3.30. Assume that π_0 has a fibre of type I_2, containing a regular line L_1 and a smooth conic Q_1.*

Then Q_1 is not a component of the flecnodal divisor $\mathcal{F}_S$ of S.

From Lemmas 3.17 and 3.20 we know that the fibre of type I_2 must be ramified, and as we have seen in the proof of Lemma 3.28, the ramified fibres of S lie in the planes $x_2 = 0$ and $x_3 = 0$. Without loss of generality we can assume the fibre of type I_2 lies in the plane $x_3 = 0$.

Using the notation from Lemma 3.28, we write the polynomials q and g as

$$q(x_2, x_3) = q_0 x_3^2 + q_1 x_2 x_3 + q_2 x_2^2$$
$$g(x_2, x_3) = g_0 x_3^4 + g_1 x_2 x_3^3 + g_2 x_2^2 x_3^2 + g_3 x_2^3 x_3 + g_4 x_3^4.$$

As we have seen in Appendix A.3.2, the fibration π_0 has a fibre of type I_2 in the plane $x_3 = 0$ if and only if $g_4 = 0$ and $q_2 \neq 0$, corresponding to case b) in Appendix A.3.2. Therefore, we can assume that the conic is given by the equations$x_3 = 0$ and $x_0^2 + q_2 x_1 x_2 = 0$. We can parametrise this curve; a point P on it has projective coordinates $(q_2 st : -q_2 s^2 : t^2 : 0)$ for some $(s : t) \in \mathbb{P}^1$.

Our goal is to show that it is impossible for all of these points to simultaneously be flecnodal points of S.

Consider such a point P. In order for P to be a flecnode, there has to be a line $L \subset \mathbb{P}^3$ such that L intersects the surface S with multiplicity at least 4 at the point P. This line must necessarily lie in the tangent plane $T = T_P(S)$ of S at P. Within this plane, we can consider the curve $C = S \cap T$, and then P is a flecnode of S if and only if $i_P(C, L) \geq 4$.

The tangent plane T is given by the polynomial in Listing A.12. We can solve that equation for x_0 so that T is given by

$$x_0 = h(x_1, x_2, x_3)$$

where h is a polynomial whose coefficients are rational functions in the coefficients of g and q as well as the parameters s and t. This gives a canonical isomorphism $T = \mathbb{P}^2(x_1, x_2, x_3)$.

After substituting h for x_0 in the equation for the surface S, we obtain an equation $G(x_1, x_2, x_3) = 0$ for the resulting curve $S \cap T$ in the plane $T = \mathbb{P}^2(x_1, x_2, x_3)$.

We will now consider the affine patch $x_2 = 1$ in this plane and translate the coordinates so that P is the origin of this affine plane, which corresponds to the point $(0:0:1:0)$ in $\mathbb{P}^3$. Then we can compute the translated affine equation $H(x_1, x_3) = 0$ for the curve C in this new plane $\mathbb{A}^2(x_1, x_3)$ with P as its origin.

The polynomial H is computed in Listing A.13 and as we can see in the output in Listing A.14, there are no linear coefficients, meaning P is a singular point of the curve $C = S \cap T$. In order to study the local behaviour of C at P, we consider the quadratic part of H, given in the form

$$Ax_1^2 + Bx_1x_3 + Cx_3^2,$$

where A, B and C are rational functions in q_i, g_i, s and t. We compute the discriminant D of this quadratic polynomial. If u is a square root of D, then the tangents to the two branches of C at P are given by the equations

$$\begin{aligned} x_1 &= \frac{-B+u}{2A}x_3 \\ x_1 &= \frac{-B-u}{2A}x_3. \end{aligned}$$

Substituting these two equations back into the equation H, we obtain two univariate polynomial $J(x_3)$ and $\overline{J}(x_3)$, which as we can see in Listing A.16 are of the form

$$\begin{aligned} J &= J_1x_3^3 + J_2x_3^4 \\ \overline{J} &= \overline{J_1}x_3^3 + \overline{J_2}x_3^4 \end{aligned}$$

where J_i and $\overline{J_i}$ are rational functions in s, t and the coefficients of q and g, as well as u; and where $\overline{J_i}$ is the conjugate of J_i with respect to u, i.e. the expression obtained by replacing all instances of u in J_i with $-u$.

Now in order for P to be a flecnode of the surface S, it must be a flecnode of the curve C, and thus one of the two tangents must intersect C with multiplicity 4. This is equivalent to either J_1 or $\overline{J_1}$ vanishing. Because J_1 and $\overline{J_1}$ are each others conjugates, this is the case if and only if the norm of J_1 with respect to u vanishes. We view the norm as a homogeneous polynomial in s,t whose coefficients are polynomials in the coefficients of q and g. If every point on the conic Q_1 is to be a flecnode of S, then this norm must vanish for all values of s and t, which means that all coefficients must vanish.

In Listing A.18 we can see that one of these coefficients is $768q_2^{18}$. Since q_2 is non-zero, this coefficient is also non-zero outside of characteristic 2 or 3. Therefore, the conic Q_1 is not a component of the flecnodal divisor of S.

```
// Define the polynomials q and g, the surface S
// and introduce parameters t and s for the conic

FF:=RationalField();
K<q0,q1,q2,g0,g1,g2,g3>:=FunctionField(FF,7);
L<s,t>:=FunctionField(K,2);
P3<x0,x1,x2,x3>:=ProjectiveSpace(L,3);
g4:=0;
q:=q2*x2^2+q1*x2*x3+q0*x3^2;
g:=g4*x2^4+g3*x2^3*x3+g2*x2^2*x3^2+g1*x2*x3^3+g0*x3^4;
F:=x2*x0^3+x3*x1^3+x0*x1*q+g;
S:=Scheme(P3,F);

// Define the point P on S, compute the tangent plane T

P_coords:=[q2*s*t,-q2*s^2,t^2,0];
P:=S!coords;
T:=TangentSpace(S,P);
DefiningEquation(T); // print the equation for T
```

Listing A.11: Magma code to compute the tangent plane T of the surface S at the point P and an equation $G = 0$ for the curve C in the intersection of T with S.

```
x0 + 1/2*t/s*x1 - 1/2*q2*s/t*x2
   + (-1/2*q2*s^6 - 1/2*q1*s^3*t^3
   + 1/2*g3/q2^2*t^6)/(s^2*t^4)*x3
```

Listing A.12: Output of Listing A.11, giving the polynomial that defines the tangent plane T. Formatting manually edited.

```
// Intersect T with S by substituting x0

X0:=x0-DefiningEquation(T);
G:=Evaluate(F,[X0,x1,x2,x3]);

// Substitute translated variables into G to give an
// affine equation H in the variables x1, x3
// where now P is the origin (0,0)

// Multiply G by 8*q2^6 to simplify denominators

H:=Evaluate(8*q2^6*G,[x0+q2*s*t,x1-q2*s^2,t^2,x3]);
Monomials(H);
```

Listing A.13: Magma code to compute the equation $H = 0$ for the tranlsation of the curve $T \cap S$ in the affine plane $\mathbb{A}^2(x_1, x_3)$ where P corresponds to the origin. Note that this code does not compile on its own, but is specifically a continuation of the code given in Listing A.11.

```
[
x1^3*x3,
x1^2*x3^2,
x1*x3^3,
x3^4,
x1^3,
x1^2*x3,
x1*x3^2,
x3^3,
x1^2,
x1*x3,
x3^2
]
```

Listing A.14: Output of Listing A.13, giving the monomials of the polynomial H which defines the curve C in the translated affine plane. Notably the linear coefficients are absent.

```
A:=MonomialCoefficient(H,x1^2);
B:=MonomialCoefficient(H,x1*x3);
C:=MonomialCoefficient(H,x3^2);
D:=B^2-4*A*C;

// Substitute the two tangent equations back into H

preR<U>:=PolynomialRing(L);
R<u>:=quo<preR|U^2-D>;
X<X0,X1,X2,X3>:=ProjectiveSpace(R,3);
J:=Evaluate(H,[0,(-B+u)/(2*A) * X3,0,X3]);

Monomials(J);
```

Listing A.15: Magma code for computing the quadratic part of the polynomial H and the discriminant D. Note that the code will not compile on its own, but is a continuation of the code in Listings A.11 and A.13.

```
[
X3^4,
X3^3
]
```

Listing A.16: Output of Listing A.15, showing the monomials of the polynomial J, which represents the intersection of the curve C with one of the two tangent lines at the origin. The formatting has been manually edited.

```
NormCoeffX3:=Norm(MonomialCoefficient(J,X3^3));
Coefficients(Numerator(NormCoeffX3));
```

Listing A.17: Magma code for computing the norm with respect to u of J_1. This code will not compile on its own, but is a continuation of Listings A.11, A.13 and A.15.

```
[
    768*q2^18
    [...]
]
```

Listing A.18: Output of Listing A.17, which has been manually truncated.

B | BIBLIOGRAPHY

[1] W. Barth. 'Lectures on K3- and enriques surfaces'. In: *Algebraic Geometry Sitges (Barcelona) 1983*. Ed. by Eduard Casas-Alvero, Gerald Welters and Sebastian Xambó-Descamps. Berlin, Heidelberg: Springer Berlin Heidelberg, 1985, pp. 21–57. ISBN: 978-3-540-39643-7.

[2] A. Clebsch. 'Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5ten Grades und die geometrische Theorie des ebenen Fünfseits'. In: *Mathematische Annalen* 4.2 (1871), pp. 284–345.

[3] A. Clebsch. 'Zur Theorie der algebraischen Flächen'. ger. In: *Journal für die reine und angewandte Mathematik* 58 (1861), pp. 93–108. ISSN: 0075-4102.

[4] I. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge ; New York : Cambridge University Press, 2012.

[5] B. Edixhoven et al. *Algebraic Geometry*. `https://webspace.science.uu.nl~kool0009/AG_notes.pdf`. Last accessed: 2025-07-17. 2021.

[6] D. Eisenbud and J. Harris. *3264 and all that: A second course in algebraic geometry*. Cambridge University Press, 2016.

[7] G. Fischer. *Plane algebraic curves*. Vol. 15. American Mathematical Soc., 2001.

[8] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. eng. 1. Aufl. Wiley Classics Library. Wiley-Interscience, 2011. ISBN: 0471050598.

[9] R. Hartshorne. *Algebraic Geometry*. Springer-Verlag New York Berlin Heidelberg, 1977.

[10] Brendan Hassett. *Introduction to algebraic geometry*. Cambridge University Press, 2007.

[11] F. Hirzebruch. 'Hilbert's modular group of the field and the cubic diagonal surface of Clebsch and Klein'. In: *Russian Mathematical Surveys* 31.5 (1976), p. 96.

[12] D. Huybrechts. *Lectures on K3 surfaces*. Vol. 158. Cambridge University Press, 2016.

[13] F. Klein. 'Ueber flächen dritter ordnung'. In: *Mathematische Annalen* 6.4 (1873), pp. 551–581.

[14] K. Kodaira. 'On Compact Analytic Surfaces, III'. In: *Annals of Mathematics* (1963).

[15] K. Kodaira. 'On Compact Analytic Surfaces: II'. In: *Annals of Mathematics* (1963).

[16] K. Kodaira. 'On Compact Complex Analytic Surfaces, I'. In: *Annals of Mathematics* (1960).

[17] C. McCrory and T. Shifrin. 'Cusps of the projective Gauss map'. eng. In: *Journal of differential geometry* 19.1 (1984), pp. 257–276. ISSN: 0022-040X.

[18] R. Miranda. *The Basic Theory of Elliptic Surfaces*. `https://www.math.colostate.edu/~miranda/BTES-Miranda.pdf`. Last accessed: 2025-07-03. 1989.

[19] A. Néron. 'Modèles minimaux des variétés abéliennes sur les corps locaux et globaux'. In: *Publications Mathématiques de l'IHÉS* 21 (1964), pp. 5–128.

[20] S. Rams and M. Schütt. '64 lines on smooth quartic surfaces'. In: *Mathematische Annalen* 362.1 (2015), pp. 679–698.

[21] S. Rams and M. Schütt. 'On quartics with lines of the second kind'. In: *Advances in Geometry* 14.4 (2014), pp. 735–756.

[22] G. Salmon. *A treatise on the analytic geometry of three dimensions*. Hodges, Smith, and Company, 1865.

[23] M. Schuett and T. Shioda. *Elliptic Surfaces*. 2010. arXiv: 0907.0298 [math.AG].

[24] F. Schur. 'Ueber eine besondre Classe von Flächen vierter Ordnung'. In: *Mathematische Annalen* 20.2 (1882), pp. 254–296.

[25] M. Schütt and T. Shioda. 'Elliptic Surfaces'. In: *Mordell–Weil Lattices*. Singapore: Springer Singapore, 2019, pp. 79–114. ISBN: 978-981-32-9301-4. DOI: 10.1007/978-981-32-9301-4_5. URL: https://doi.org/10.1007/978-981-32-9301-4_5.

[26] B. Segre. 'The maximum number of lines lying on a quartic surface'. In: *The Quarterly Journal of Mathematics* 14.1 (Jan. 1943), pp. 86–96. ISSN: 0033-5606. DOI: 10.1093/qmath/os-14.1.86. eprint: https://academic.oup.com/qjmath/article-pdf/os-14/1/86/4485868/os-14-1-86.pdf. URL: https://doi.org/10.1093/qmath/os-14.1.86.

[27] Jean-Pierre Serre. *Local fields*. Vol. 67. Springer Science & Business Media, 2013.

[28] J.H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*. Springer-Verlag New York, Inc., 1994.

[29] J.H. Silverman. *The Arithmetic of Elliptic Curves*. Springer-Verlag New York, Inc., 1986.

[30] J.F. Voloch. 'Surfaces in $\mathbb{P}^3$ over finite fields'. In: *Topics in Algebraic and Noncommutative Geometry: Proc. in Memory of Ruth Michler (C. Melles et al. eds.), Contemp. Math* 324 (2003), pp. 219–226.