Model Adaptation by Shared Satisfaction—A Meta-Theoretic Strategy for Managing Indeterminacy in Logic and Science

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Chapter 1. Introduction: Model Fragmentation, Theoretical Indeterminacy, and the Possibility of Integration

In 1976, Hilary Putnam opened his Presidential Address to the American Philosophical Association with a striking claim: "Realism is an empirical theory [...explaining...] that scientific theories tend to 'converge' in the sense that earlier theories are, very often, limiting cases of later theories." [1] This introduced the Model-Theoretic Argument (MTA), which challenges the idea that formal systems uniquely determine reference—the relation between terms in a language or theory and what they denote, even in scientific contexts. Putnam linked realism to the unification of disparate models into more general theories.
But nowadays, many philosophers of science would agree with Sandra Mitchell that 'science is disunified, and [...] this disunification [...] brings strength and stability', [2], or that 'integrative pluralism' better reflects practice through 'an expanded epistemology ... embracing both reductive and multilevel, context-dependent approaches'[3] Nancy Cartwright even contends that '[natural] laws form a patchwork, not a pyramid. [...M]uch of nature may follow no law but negotiation between domains. The *dappled world* is what comes naturally; regimented behaviour results from good engineering'.[4] In physics, for instance, Historian-physicist Peter Galison, notices that 'forms of work, modes of demonstration, [and] ontological commitments differ' across its many traditions.[5] Physics practice divides into experiment, theory, and instrumentation 'matching Kuhn's criteria for separate communities'[6]—and 'even specialties within physics cannot be considered homogeneous communities'.[7]

Mitchell also argues that scientific progress often depended on moving beyond the search for 'universal, exceptionless laws, since much of what we now know about complex, contingent, and evolved structures would otherwise be excluded'.[8] Biology illustrates this vividly. In the Enlightenment, natural diversity was modeled by Linnaeus through a universal taxonomy. In the nineteenth century this was refined by Darwinian evolution, Mendelian inheritance, Lamarckian adaptability, and Humboldtian ecology—each capturing different aspects of life's complexity. Rather than unifying these accounts, these separations caused disagreements between competing frameworks—like evolutionists *vs* creationists, or inheritable *vs* adaptable accounts.[9] Today, subfields such as population genetics, developmental biology, ecology, combinations ('evo-devo'[10]) and systems biology remain internally coherent yet structurally divergent. As biologist Eva Neumann-Held argued: 'Biology still needs to perform its integrative descriptive function.'[11] Instead, a mosaic of partially overlapping scientific explanatory models developed—each acting as a 'sub-model' giving a different account of Linnaeus's all-

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[1] Putnam (1977), 483
[2] Galison (1997), 781-2
[3] Mitchell (2009), 2
[4] Cartwright (1999)
[5] Galison (1997), 782
[6] *Ibid.*, 797
[7] *Ibid.*
[8] Mitchell (2009): 2
[9] Huxley (1942, 1974)
[10] Arthur (2002)
[11] Neumann-Held (1998), 107

encompassing taxonomic model: regional variances (Humboldt), adaptive capabilities (Lamarck), evolutionary origins (Darwin), and inheritable traits (Mendel). These 'submodels' act within the Linnaean order of biological diversity,[12] and they are effective within their scope. They are also difficult to reconcile with the others, even though they reference the same biological domain—Linnaeus's *Kingdoms of Nature*.

This fragmentation sharpens a problem identified by Putnam: how can reference remain stable across divergent conceptual frameworks? If models differ in structure and interpretation yet describe the same phenomena, what justifies saying they refer to the same "piece of THE WORLD".[13] Putnam's MTA draws on the Löwenheim–Skolem theorems to show that even our best theories can have multiple, non-isomorphic models satisfying the same axioms, allowing meaning to shift across models.[14] For Putnam, scientists construct "symbolic representations of their environment" shaped by practices, interests, and linguistic conventions.[15] Putnam found a solution in internal realism, which holds that truth and reference are fixed within conceptual frameworks. He opposed the metaphysical realist's unique, theory-independent mapping from language to THE WORLD. Critics, however, disputed this view for blurring truth and justification and inviting relativism. Putnam himself later moved toward a pragmatic natural realism aligned with scientific practice, stating that 'there is no conflict between natural realism and science'.[16,17] However, he never seemed to fully abandoned internal realism.[18,19,20]

The core insight of MTA—the non-uniqueness of models—remains influential. Philosophers continue to ask how best to account for the success of diverse formal models of a single scientific theory, which, despite structural differences, yield accurate or useful descriptions of the same phenomena.[21] Some call for 'a revised and expanded epistemology' to understand complex explanatory structures that, despite disparities, work remarkably well.[22]

Against this background, this thesis revisits Putnam's concerns, seeking to reframe rather than oppose them. It proposes *Model Adaptation by Shared Satisfaction* (*MASS*) as a new formal framework for managing indeterminacy. *MASS* identifies sentences satisfied across structurally divergent submodels—*reference sentences*—that serve as connective elements linking different perspectives without requiring full unification (§3.2.3, §4.1). Fragmentation is treated as multiple vantage points rather than a flaw: unlike pluralists who take diversity to imply disunity,[23,24,25] the world itself remains unified, even if our perspectives do not—and they can mutually enrich one another. As Sandra Mitchell notes, modeling can bridge gaps between pragmatism, complexity,

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[12] Chapter 3 (Insert-1) will define 'submodels' formally.
[13] Putnam (1977), 484
[14] Hale *et al.* (2017) "Chapter 27-Putnam's Model-Theoretic Argument Against Metaphysical Realism,'' 938–970
[15] Putnam (1977), 483
[16] Putnam (1994), "Dewey Lecture I:" 465
[17] Hildebrand (2000), 109-132
[18] Passmore(1985), 104: '[Putnam] still stands by what he calls 'internal realism'.'
[19] Putnam, Hilary (1994), " Dewey Lecture I," 463, note 41: 'Am I then giving up "internal realism?'
[20] Rothmaler (2014)
[21] Hodges (2003), 18–21: "5-Models and Modelling": "We have a confusing halfway situation when a scientist describes a phenomenon in the world by an equation… Is the model the theory consisting of the equation, or are these solutions themselves models of the phenomenon?" This ambiguity also shows in the informal use in this Introduction of terms like 'models', 'sub-models', 'account', 'perspective' or 'viewpoint', as neutral labels for structures satisfying a theory—not for independent theories or informal interpretations. In the rest of this thesis, models and submodels will consistently be understood in a formal model-theoretic sense (See §3.1, Insert 1).
[22] Mitchell (2009), 3
[23] Various standpoints are defended by Cartwright, Galison and Mitchell—and summarized broadly in Galison (1997), 781-797: "Ch.9—The Trading Zone"
[24] Cartwright (1999), 1
[25] Galison (1997)

and theory across scientific disciplines.[26]

MASS offers a model-theoretic alternative to both the metaphysical realist's external reference and the relativist implications of internal realism. Each scientific submodel is treated as a partial realization of a broader, often underspecified theory. When a reference sentence ϕ is satisfied in multiple submodels, it signals a common structural component. This approach can, in principle, link scientific submodels, accepting that indeterminacy is inherent to scientific representation but demonstrating that it is formally manageable. *MASS* provides a meta-theoretical strategy for this, employing model-theoretic tools augmented by graph- and sheaf-theoretic methods to stabilize reference while preserving the specificity of submodels.

In this thesis, the development of (sub)models for biological diversity serves as a recurring case study to illustrate how distinct frameworks can refer to overlapping domains without reduction to a single theory. It will be shown that they can, in principle, be linked by recent advances in molecular biology—although this would require a new 'interactionist molecular paradigm'.[27] The central question of this thesis is whether *MASS* can manage scientific model fragmentation by preserving stable reference, and how shared elements across submodels can connect formal viewpoints that capture specific aspects of a theory while maintaining their distinctiveness. Chapter 2 offers a historical overview of (sub-)models of biological diversity. Chapter 3 introduces model-theoretic concepts more formally, and focuses on the Löwenheim–Skolem theorems and their implications for model indeterminacy. Chapter 4 develops *MASS* in detail, and Chapter 5 how this contributes to unification. Chapter 6 discusses *MASS* in relation to Putnam's MTA, and applies it to (sub-)models of biological diversity—which are presented next.

[26] Mitchell (2009), 85: 'Political scientists, economists, logicians, decision theorists, and others have been modeling for decades'.
[27] Neumann-Held (2006)

Chapter 2. Biological Diversity as a Case Study in Model Fragmentation and Reintegration

Scientific fragmentation is vividly exemplified by theories of biology diversity. What began in the Enlightenment as a systematic classification of nature, evolved into diverse models addressing variation, inheritance, evolution, development, and adaptation from distinct perspectives. These models share the empirical domain of all living organisms but differ in structure, scope, and aims—often sharply enough to generate conflict. This raises the thesis's central challenge: relating divergent models without losing their explanatory values.

In the eighteenth century, Linnaeus's *Systema Naturae* established one of the first formal frameworks for organizing biological knowledge. Amid the age of exploration, Linnaeus classified the increasing numbers of known organisms into nested hierarchies, treating species as fixed natural kinds. Despite its theological roots, Linnaean taxonomy became a durable framework that is still used. By the nineteenth century, ideas about this system fragmented. New models—Humboldt's environmental mappings, Lamarck's organic adaptivity, Darwin's evolution, and Mendel's inheritance—introduced differing assumptions and methods. Though all concerned life's diversity, these models diverged in explanatory structures and concepts.

The twentieth century sought to reconnect these strands, notably via the *Modern Synthesis* merging Darwinian and Mendelian principles——although it did not integrate developmental or ecological perspectives. Recent advances in epigenetics and systems biology have enriched but also multiplied conceptual frameworks. The following sections trace these developments and set the stage for applying *MASS* to submodels of biological diversity in §6.2.

2.1 Linnaean Taxonomy: A Model of Classification

Carolus Linnaeus (1707–1778) considered natural history to reflect a perfect Divine order, and he began classifying plants into nested hierarchies based on shared traits.[28] The species, as the core unit, was a natural kind—fixed, bounded, identified by visible traits and reproductive compatibility. Linnaeus transformed the diverse descriptive practices of his time—based on appearance, use, and tradition—into a coherent system via a binomial nomenclature of genus and species. His final *Systema Naturae* (1735–1758) organized life into Three Kingdoms of Nature: *Regnum Animale*, *Regnum Vegetabile* and *Regnum Lapideum*.[29] Genera and species referred to stable sets under shared morphological predicates, forming the basis for modern botanical and zoological nomenclature. While broadly encompassing all life, the immutability of species was soon challenged by emerging observations in nature.

2.2 Lamarckian 'Life Force': A Model of Developmental Adaptation

Jean-Baptiste de Lamarck (1744–1829) introduced the term *biologie* for the scientific study of life, and moved beyond static morphology to consider development and adaptation. He proposed that a vital organizing force (*le pouvoir de la vie*) drives organisms toward greater complexity, while adaptation to environmental conditions (*l'influence des circonstances*) shapes their specific traits. In *Philosophie Zoologique* (1809), Lamarck formulated two biological laws:[30] organs used frequently develop further while unused ones atrophy; and acquired traits can be passed on to descendants. The giraffe's neck elongation in response to taller trees exemplified this. Lamarck's ideas were influential, but they lacked mechanism and would later be contested by Darwin's evolutionism, which emphasizes group-based selection of random traits rather than individual development of adaptive traits.

2.3 Humboldtian Environmentalism: A Model of Relational Variation

Alexander von Humboldt (1769–1859) world-wide voyages of discovery led him to re-envision

[28] Linnaeus, *Species Plantarum* (1753-1759)
[29] Linnaeus, *Systema Naturae* (1758) the 'Kingdom of Minerals' also included fossils.
[30] Lamarck (1809)

natural history as a dynamic system shaped by environmental conditions. He portrayed biological variation as ecologically modulated continuity rather than fixed taxonomic categories.[31] Organisms formed parts of ecological gradients and geographic patterns—an interconnected *Naturgemälde*.[32] Humboldt stressed connectedness over identity, restructuring Linnaean taxonomy into a relational framework based on altitude, latitude, humidity, and soil. His model partially reorganized life by establishing environmental networks rather than replacing taxonomy, and also lacked mechanisms for adaptation or persistence of ecological variations.

2.4 Darwinian Evolution: A Model of Historical Descent

Charles Darwin (1809–1882) followed Humboldt's idea to understand nature by exploring it, but his *Voyage of the Beagle* (1839) revealed different patterns. *On the Origin of Species (1859)* described species as products of natural selection acting on random inherited variation. The Linnaean tree became genealogical: species were evolutionary entities, not static. Darwin acknowledged Lamarck's and Humboldt's insights that diversity is dynamic rather than static, but he framed 'survival of the fittest' more mechanistically. However, this was still vaguely represented by 'trait transmission' from organs to reproductive cells. The timescales involved and the complexity of transitions between species remained difficult to reconcile.

2.5 Mendelian Genetics: A Model of Generational Hybridization

In 1865, Gregor Mendel (1822–1884), unknown to Darwin, published experiments on pea plants.[33] His systematic results indicated particulate 'factors' (*Anlagen*—later genes) that were predictably inherited as dominant, recessive, or incomplete.[34] Mendel's mechanistic, rigorous model introduced generational hybridization via quantifiable, rule-based interpretations linking 'factors' to binary traits. His controlled experiments abstracted from natural complexity and environment, clashing with Darwin's gradual evolution, Lamarck's individual adaptation, and Humboldt's ecological gradients. Mendel's model did not affect Linnaean taxonomy, which already identified hybrids.

By the early twentieth century, biology remained taxonomically Linnaean but fractured into distinct partial models: Mendelian inheritance focused on discrete factors, Darwinian evolution on chance and selection, Humboldtian ecology on environmental distribution. Lamarck seemed largely forgotten. Though sharing the domain of life, these models differed structurally and conceptually, often incommensurable, forming a disjointed patchwork rather than a pluralistic synthesis.

2.6 The Modern Synthesis

From the 1920s–30s, statistician Ronald Fisher, biologist J.B.S. Haldane, and geneticist Sewall Wright collectively rediscovered Mendel's work and integrated it with Darwinian selection using recent advances in population statistics. In 1942, this became the Modern Neo-Darwinian Synthesis:[35] evolution as changing gene frequencies over time. It preserved the strengths of Darwin's evolutionary and Mendel's generational coherence, and reinterpreted Linnaeus's categories as gene clusters. Humboldt's insights were partially absorbed as ecological pressures. Lamarck's laws were now rejected as incompatible and unsupported.

Molecular explanations of genetic transmission came in sight when Watson and Crick's multidisciplinary team elucidated the structure of DNA (1953). DNA genetics offered powerful

[31] Wulf (2017)
[32] *Ibid.*
[33] Mendel (1866), 3–47
[34] *Ibid.*, 24: 'Soweit die Erfahrung reicht, [werden] constante Nachkommen nur gebildet [...] wenn Keimzellen und befruchtender Pollen gleichartig sind [...und] mit der Anlage ausgerüstet, völlig gleiche Individuen zu beleben.'
[35] Huxley (1942-1974)

tools for species classification, from morphology to gene patterns. Over time, this gave rise to widespread genetic reclassifications.[36,37] This still respects Linnaean hierarchical life model, but does not explain short-term adaptability or resilience.[38] Neo-Darwinism also struggles with the fossil record's lack of expected intermediate forms under gradual mutation and selection.

2.7 Beyond Genetics: Epigenetic and Regulatory Models of Development and Adaptation

Over the past fifty years, biology's focus on molecular life mechanisms deepened, with genetics becoming a biochemistry specialization. Around 2000, it was demonstrated that gene expression can adapt and transmit to offspring without DNA changes.[39] Epigenetic and regulatory mechanisms revealed that inheritance, development, and adaptation are interconnected, extending classical Mendelian genetics. Genes are now seen in regulatory networks where adaptations arise from subtle chemical shifts in cells, rather than isolated mutations.[40,41] These findings enrich debates on processes driving evolutionary adaptations[42] and challenge traditional genetic inheritance.[43]

This historical development illustrates scientific fragmentation, where partial, structurally distinct, and sometimes conflicting submodels represent overlapping features of a general model. Model theory helps understand how models can fragment into submodels, and theoretical possibilities and limitations for reconciliation. The next chapter introduces the essentials of model theory—interpretation, satisfaction, cardinality—which provides the formal structure for understanding model-theoretic indeterminacy and, ultimately, for developing the framework of *Model Adaptation by Shared Satisfaction* (*MASS*) as a strategy for coherent reintegration.

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[36] Wiley & Lieberman(2011)
[37] Hedges & Kumar (2009), 3–18: "Chapter 1 - Discovering the Timetree of Life."
[38] Wulf (2017). Humboldt noted the harm caused by monocultures and the strong adaptability of introduced foreign crops in colonial territories, as well as ecosystems recovering from large-scale agricultural disruptions.
[39] Meaney & Szyf (2005), 103–123
[40] Neumann-Held (2006)
[41] Lee (2014), 4221
[42] Johannes & Becker (2025)
[43] Meaney & Szyf (2005)

Chapter 3 Insert 1—Model Theory[44,45,46,47]
Model theory formalizes the structural relationships between objects in a domain and the terms of a formal **language $\mathcal{L}(\Sigma)$**, where the non-logical symbols of a **signature Σ** are interpreted according to axioms and rules.

Basic Definitions[48]
A model-theoretic structure (a model $\mathcal{M}$ of $\mathcal{L}(\Sigma)$) consists of:

- **Domain *D***: a non-empty set of objects over which variables range
- **Signature Σ**: a set of non-logical symbols, specifying
  - predicate symbols P_i of arity m_i
  - function symbols $\mathcal{F}_j$ of arity n_j
  - constant symbols c_k
- **Language $\mathcal{L}(\Sigma)$**: the set of well-formed formulas generated from Σ using logical connectives and quantifiers
- **Subsignature *L*:** a subset of Σ containing only some of its non-logical symbols: $\Sigma' \subseteq \Sigma$
  The language $\mathcal{L}(\Sigma')$ is then a sublanguage of $\mathcal{L}(\Sigma)$:—a 'rudimentary language'[49] tied to the corresponding submodel $\mathcal{M}'$
- **Interpretation function *I***: assigns meanings to the non-logical symbols
- **Theory *T***: a set of sentences Γ in $\mathcal{L}(\Sigma)$, closed under logical consequence
- **Valid sentences ($\phi, \psi, \ldots$)**: satisfied in all models of $\mathcal{L}(\Sigma)$ (*ie* $\mathcal{M} \vDash \phi$)
- **Model class**: $\mathrm{MOD}(T) = \{\mathcal{M} \mid \mathcal{M} \vDash T\}$.

Semantic assignments:
- $I(P_i) \subseteq D^{m_i}$ assigns a relation to each predicate
- $I(\mathcal{F}_j)$: $D^{n_j} \rightarrow D$ assigns a function to each function symbol
- $I(c_k) \in D$ assigns an element of the domain to each constant.

Embeddings and Submodels[50]
An **embedding** is an injective, structure-preserving map
$h : \mathcal{M} \rightarrow \mathcal{M}$
that preserves the interpretations of relations, functions, and constants.
In model theory, a **partial sub-model** is the special case where h is inclusion ($D' \subseteq D$), so that $\mathcal{M}'$ is fully contained ('nested') in $\mathcal{M}$.
In the *MASS* framework (Insert 2, §4.6), I extend this to a broader concept of *submodels* $\mathcal{M}'$ which include:
- **Partial sub-models**: the signature remains the same (Σ), but the domain is restricted ($D' \subseteq D$).
- **Reducts**: the domain remains the same, but the signature is reduced to a subsignature ($L' \subseteq \Sigma$).
  The submodel $\mathcal{M}'$ then interprets only the non-logical symbols in L'.
- **Hybrid submodels**: combinations of components from different submodels.
  A **partial isomorphism** is a structure-preserving map between overlapping fragments of models—domain elements, predicates, constants, and variables—that preserves satisfaction of sentences restricted to the shared structure.

Interpretation
In this thesis, the general framework assumes a 'universal' model $\mathcal{M}_U$ of a theory T_U. Submodels $\mathcal{M}'$ (*eg* $\mathcal{M}_{A,B,C,..H,H',...S,..\#}$) may be related by partial isomorphisms, ensuring that shared components are consistently interpreted so that a reference sentence ϕ is preserved across them. Such partial isomorphisms must be embeddable in $\mathcal{M}_U$, even when other components differ. This includes reducts, where the domain is identical but the active subsignature is smaller, and hybrid submodels that incorporate elements from models of other theories, provided no logical contradictions arise.

[44] Doets (1996) *Basic Model Theory*
[45] Button (2018), 225-236: "Appendix 1-Model Theory Primer"
[46] Rothmaler (2014), 1-38: "I-Basics"
[47] Lutz, Sebastian, (2015), 563-579: "Partial Model Theory as Model Theory"
[48] Hodges (1997), *Shorter Model Theory*, 2-3
[49] *Ibid.*, 4-5
[50] *Ibid*, *vii*

Chapter 3. Model Theory

3.1. Essentials of Model Theory

This section gives a brief overview of model theory, with technical details in Insert 1.[51]
In first-order logic, a theory $\boldsymbol{T}$ is a set of sentences ($\boldsymbol{\Gamma}$) in a formal language $\mathcal{L}$. Truth is alwaysonly relative to models: a sentence ϕ is true in a model $\mathcal{M}$ (written $\mathcal{M} \vDash \phi$) *iff* $\mathcal{M}$ satisfies ϕ under its interpretation.[52] A model comprises a domain $\boldsymbol{D}$ of objects together with an interpretation function $\boldsymbol{I}$ that assigns: to each constant symbol an element of $\boldsymbol{D}$, to each n-ary function symbol an operation on $\boldsymbol{D}^n \rightarrow \boldsymbol{D}$, and to each n-ary predicate symbol a subset of $\boldsymbol{D}^n$.

A model satisfies $\boldsymbol{T}$ if all sentences in $\boldsymbol{T}$ hold. Even in a fixed language, many distinct models of $\boldsymbol{T}$ arise by varying interpretations, so a sentence from $\boldsymbol{\Gamma}$ may be true in multiple models. This core indeterminacy is closely tied to the model's cardinality $|\mathcal{M}|$, determined by its domain $\boldsymbol{D}$. Domains may be finite, countably infinite (like $\mathbb{N}$ with cardinality $\aleph 0$), or uncountably infinite (like $\mathbb{R}$ or larger—$\aleph 1, \aleph 2$ *etc*). Larger domains yield vastly more structurally distinct models.

These model-theoretic concepts also affect scientific modeling, especially in complex fields like biological diversity. Biological diversity models split into sub-frameworks emphasizing ecology, adaptation, selection, or inheritance. Each can be treated, for present purposes, as a submodel of a broader (essentially Linnaean) 'diversity model': they share a basic commitment to describing and explaining life's diversity within Linnaeus's systematic taxonomy, but differ internally in how their interpretation functions assign meaning to the same non-logical vocabulary. Domains vary from countable discrete traits and parental inheritance, to uncountable continuous ecological gradients and entire populations.

The set of all models satisfying $\boldsymbol{T}$ is $\mathcal{M} \mid \mathcal{M} \vDash \boldsymbol{T}\}$. A sentence ϕ is true in $\mathcal{M}$ if it holds under $\mathcal{M}$'s interpretations. For example, '*x* has a long neck' may be satisfied in Lamarck's model (where the predicate is interpreted as habitual stretching), in Darwin's (as survival advantage), in Mendel's (as inherited trait), or in Humboldt's (as ecological niche). Although $\phi(x)$ refers to the same observable characteristic, its satisfaction conditions differ across models—illustrating the model-relativity of truth.

Philosophers have recognized the importance of model-building in understanding reality. Some philosophers see it as a key method for understanding reality.[53,54] Timothy Williamson, for instance, argues that model-building is an underappreciated method of doing philosophy, especially in formal metaphysics.[55] Others consider modeling as a structured cognitive and epistemic way to theorize about reality.[56, 57] Many philosophers work on mathematical frameworks for metaphysical explanation, proof and grounding.[58, 59] Some even attempt to model the world itself as a metaphysical higher-order graphical structure.[60] Putnam observed a similar "striking connection" between foundational issues in science and mathematics.[61] He also noted that this flexibility challenges the idea of a single true model of THE WORLD: scientific theories may be expressed by many valid but structurally different models, raising questions about unity and reality. These issues are formalized by the Löwenheim–Skolem theorems from which Putnam's Model-Theoretic Argument is derived.

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[51] §3.1 and Insert 1 largely compiled from Doets (1996), Button(2013) and Rothmaler (2014).
[52] Hodges (2018), 448
[53] Mitchell (2012), 85—See footnote 26.
[54] Godfrey-Smith (2006)
[55] Williamson (2022), 372-385
[56] Dym (2004), 3
[57] Weisberg (2018), 7-23, 19-20
[58] Poggiolesi & Genco (2023)
[59] Litland (2023)
[60] Shackel (2011), 10–21
[61] Putnam (1980), 473

3.2. Löwenheim-Skolem Theorems and Indeterminacies of Scientific Fragmentation
Scientific theories often give rise to multiple models that capture certain regularities in the world. Each model selects a different domain and relies on different assumptions about the same underlying reality. Hence, a single unified model is rarely determined. Putnam suggested that realism could be understood as the hypothesis that science converges over time, with earlier theories becoming limiting cases of broader, more accurate ones.[62] However, the Löwenheim–Skolem theorems point in a different direction ny first-order theory with an infinite model has many models of different sizes and internal structure, showing that logical models alone provide no indication that earlier theories become "limiting cases of later ones". Applied to science, this shows that even when different scientific models explain the same phenomena equally well, they may still be structurally incompatible—especially when drawn from different disciplines. Scientific development often results in a collection of specialized models difficult to integrate.

3.2.1 The Löwenheim–Skolem Theorems and Their Structural Implications
As noted, in science it is common to construct general models from limited or sparse data—say, a few measurements in biology or ecology. A simple curve may fit the data well, but infinitely many alternative models—some more complex, or based on different assumptions—could also match the observations. In this sense, the data alone underdetermines the model. The Löwenheim–Skolem theorems reveal a similar phenomenon in logic: even a precisely formulated first-order theory can have many models. These models differ not only in size but also in their internal relational structure and domain composition. The logical form of a theory constrains the relations between its terms, but it does not uniquely determine the domain or structure of any model that satisfies it, because the axioms specify conditions compatible with many different structures. The Downward and Upward Löwenheim–Skolem Theorems formalize this:[63] any first-order theory with an infinite model will have a range of models—some smaller, some larger, and many structurally distinct—each capable of satisfying the same sentences from $\boldsymbol{\Gamma}$, but doing so over different domains.

The Downward Löwenheim–Skolem Theorem states that if a first-order theory is expressed using a countable language—that is, a language with only countably many non-logical symbols, such as predicates, functions, and constants[64]—and this theory has an infinite model, then it also has a model whose domain is countably infinite. This result holds even when a theory is meant to describe an uncountably large domain, such as the real numbers or an entire ecosystem. Smaller models can 'encode' portions of the original domain within a limited structure, yet still satisfies all the theory's axioms.

The Upward Löwenheim–Skolem Theorem shows that any infinite model has models of arbitrarily large cardinality. These larger models can embed the original structure while remaining elementarily equivalent: they satisfy the same first-order sentences, even if instantiated differently. A natural illustration is the extension of Mendelian inheritance into the Modern Synthesis. Mendel's pea experiments form a countable model, studying fifteen to twenty variants from two closely related genera, with three types of transmission across discrete generations for about seven heritable traits. In nature, tens of thousands of potentially hybridizable pea variants and roughly 30,000–40,000 genes give rise to an effectively uncountable range of traits. Mendel's model is thus elementarily embedded within a population-based model (both submodels of biological diversity) that applies the same axioms to a vastly expanded domain, of which Mendel's greenhouse experiments examined only a tiny subset.

[62] Putnam (1977), 483
[63] Rothmaler (2014), 49-50: "5.1-The Löwenheim-Skolem Theorem Upward"; 119: "8.4 Existence of Elementary Substructures and Extensions"
[64] See §3.1 and Insert 1 for clarification of model-theoretic symbols and concepts.

In reality, the Modern Synthesis also included many structural modifications that go beyond what the Löwenheim–Skolem Theorems require—such as multiple populations species, and extinct lineages, additional traits ('fitness'), new interpretation functions ('survival'), and statistical methods for allele frequencies and evolutionary dynamics. §6.2 will discuss how *MASS* can deal with these contributions for 'external theories'. However, if for illustration we restrict population genetics to the original Mendelian rules applied across this larger domain, we obtain a larger, elementarily equivalent model, exemplifying the Upward Löwenheim–Skolem Theorem by showing how a single theory can support models of vastly different cardinality and structure.

3.2.2 Structural Fragmentation and the Local Underdetermination of Scientific Models

The Löwenheim–Skolem theorems show that no first-order theory with an infinite model can single out one canonical model fixing both the identity and structure of its objects. But indeterminacy is not merely about size: models satisfying the same axioms can also diverge in how they represent relations and operations. Hilary Putnam took this structural openness to be central to his argument against metaphysical realism: if a theory is compatible with many models that differ not only in domain size but also in internal relational structure, the idea of a single theory-determined 'external' reference model collapses. Quine similarly warned that cardinality indeterminacy should not be dismissed as a mere technical oddity. [65] For both, the inability to fix a unique 'intended' model undermines the notion that theories straightforwardly correspond to THE WORLD.

Clear counterparts to this form of cardinality indeterminacy are rare in the empirical sciences, but in physics, some theorists recognize direct analogues. Lee Smolin, for example, argued that spacetime itself may not be continuous, but instead could be modeled in radically different yet empirically indistinguishable ways—discrete, continuous, or something in between—without any single one being fixed as the correct structure by current theory or evidence.[66] Scientists usually generate a plurality of domain-specific constructions reflecting different disciplinary emphases, research traditions, methodological constraints, and theoretical inputs. One model may focus on morphology, another on genetics, others on ecology or phylogenetics. Each is internally coherent yet incomplete, and the differences between them are structural, not merely informational: they diverge in how entities, relations, and reference classes are defined and applied. For instance, one model might treat species as discrete units, another as populations with variable traits, and a third might focus on ecological interactions across those populations (see §6.2 for a formal application of *MASS* to submodels of biological diversity).

This fragmentation is not simply a matter of pragmatic limitation. It follows from the model-theoretic fact that a single first-order theory can support multiple, equally legitimate submodels that are not mutually reducible. The underdetermination thus applies not only globally—across the space of all full models MOD($\boldsymbol{T}$)—but also locally, in the construction of submodels adapted to specific research perspectives. Philosophers of science have long recognized this. Patrick Suppes, for example, developed a semantic view of theories that explicitly accommodates a multiplicity of non-equivalent models.[67] Clark Glymour similarly emphasized the context-dependence of modeling practices.[68] Some of these ideas resonate in Kuhn. Although he did not

[65] Putnam (1981), 41: 'W.V.Quine has urged that that is what reference in fact is—indeterminate! [...] If the range of values is infinite, any infinite range can be made to serve; this is the Skolem-Lowenheim theorem. The true sentences stay true under all such changes." Putnam cites Quine (1977), 176-196, 190–191
[66] Smolin (2021)
[67] Suppes (1960), 163–85
[68] Glymour (1980)

directly engage directly with model theory or the fragmented structure of science, [69] Kuhn later described scientific paradigms as sets of 'metaphysical interpretations of basic models' embedded within disciplinary matrices.[70] This notion aligns with the idea that scientific modeling across disciplines involves distinct structural commitments that can diverge even when the overarching theory is shared. Kuhn's concept of incommensurability—where different scientific communities operate with mutually incompatible standards of evidence and meaning—echoes the model-theoretic insight that structurally distinct models may all satisfy the same theory yet resist straightforward translation into one another.

From this perspective, scientific paradigms can be understood as non-isomorphic local models of a broader theoretical space. This allows Kuhn's historical insights to be seen as responses to structural features of formal representation, rather than purely sociological or psychological phenomena. The deeper lesson from model theory is that such fragmentation is not just possible but inevitable under the expressive limitations of first-order logic. The Löwenheim–Skolem theorem shows that any theory represented within this framework admits multiple non-equivalent models, none of which can be singled out as uniquely correct on logical grounds alone. Moreover, there is no guarantee that a given local model is not itself part of a larger model or that it could be straightforwardly extended to one.

In this light, model fragmentation should not be seen as a failure of scientific unity, but rather as a formally grounded expectation. In science, the coherence—if attainable at all—must be achieved not by positing a single privileged model but through processes of coordination, adaptation, and partial integration among multiple models. While model theory offers tools for understanding such relations, it does not supply ready-made theorems that resolve these complex, cross-model interactions. This is the subject of *Model Adaptation by Shared Satisfaction* (*MASS*), which aims to integrate submodels that are part of an encompassing model, as explained in Chapter 4. Their interactions will be examined in Chapter 5, from the perspective of graph-theoretic interconnections between models. Technically, these rely on shared satisfaction of a formal reference sentence by partially isomorphic submodels—a mechanism that also underlies Putnam's model-theoretic argument, which will be revisited in §6.1.

3.2.3 Referential Indeterminacy and the Role of Interpretation

Putnam did not address theory-fragmentation directly, but he rejected the idea of a knowable "God's-eye point of view". We only have situated perspectives shaped by specific purposes[71]—fragmenting any imagined total model into context-specific interpretations. Linnaeus's taxonomy was one such universal view, which was diversified into Darwin's and Humboldt's expeditions, Mendel's experiments, and other approaches. This raises the question: how do individual sentences retain—or lose—reference across diverging interpretations? A single reference sentence ϕ can link otherwise different models. For instance, 'the African Wildcat (*Felis silvestris lybica*) is the progenitor of the domestic cat' may hold in a morphology-based species model, an inheritance model, an evolutionary model, or an ecological model—each with adjustments that interpret the claim in terms of phenotype, descent, population genetics,

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[69] Barnes (1974), 95; cited in Matthews (2022), 26: 'Kuhn's work reveals little sensitivity to the highly differentiated structure of science and the [...] competition [...] between [...] 'schools' or specialities. It leaves us unprepared [...] that a combination of [...] specialties led to the elucidation of the structure of DNA and hence [...] a new basic model for biological investigators.'

[70] Kuhn (1970), 182-187; cited in Matthews (2022), 13: Kuhn replaced 'paradigm' with 'disciplinary matrix,' comprising symbolic generalizations, metaphysical interpretations of models, shared values, and exemplars guiding puzzle-solving.

[71] Putnam (1981), 49-74: "Chapter 3 - Two Philosophical Perspectives," 50. See also 3.2.3 (*re* footnote 74) for a comment on Putnam's consideration of indeterminacies as a problem of persons ('speakers' and 'hearers'), and footnote 74 for my motivation to adopt this terminology.

or hybridization[72]

Putnam's central concern was the deeper indeterminacy of reference itself. He argued that metaphysical realism cannot explain why "cat" refers to cats rather than to some systematically permuted set (*eg* dogs).[73] A formal satisfaction correspondence between words and sets of things may capture regularities in collective speaker behaviour[74]—yet without additional constraints it fails to secure a unique mapping.[75] Even in a unified theory $\boldsymbol{T}_U$, truth is model-relative: its full model $\mathcal{M}_U$ satisfies all its sentences, but submodels may satisfy only a subset. A true sentence in $\mathcal{M}_U$ may be false or irrelevant in one of the submodels, and vice versa when the submodels omit essential fragments of $\mathcal{M}_U$. Satisfaction in one submodel may suggest plausibility but not truth in the theory, unless it holds across all submodels of $\mathcal{M}_U$ (as explained and nuanced by the Compactness Theorem in the next subsection). As Michael Dummett notes, "there may be no such thing as a conclusive verification," so meaning must rest on grounds of assertion that fall short of conclusiveness.[76]

Like indeterminacy by cardinality or model fragmentation, referential instability is no mere technical anomaly—it shapes how theories maintain coherence across diverse modelling contexts. Later chapters develop a method for preserving shared reference despite model fragmentation: *Model Adaptation by Shared Satisfaction* (*MASS*). Chapter 4 (§4.6) formalizes it; Chapter 5 (§5.2) operationalizes it. While most of this thesis addresses the scientific and philosophical aspects of a model-theoretically effective and scientifically meaningful reference sentence (§4.1, Chapter 6), a simple formal example appears in Insert-2.

3.2.4 Compactness and the Constraints on Global Integration

The fragmentation of a universal model into submodels is not necessarily detrimental. As stated earlier, some philosophers and scientists consider pluralism as fundamental to nature[77] and even essential for scientific progress.[78] Many scientific theories—especially those addressing complex empirical domains like biological diversity—are applied through multiple submodels, each emphasizing different structural features (see Chapter 2). However, this fragmentation leads to indeterminacy about how these submodels relate to a unified explanatory theory. The Compactness Theorem offers a partial answer. It states that if every finite subset of a set $\boldsymbol{\Gamma}$ of first-order sentences is satisfiable (*ie* each finite subset has some model that makes all sentences true), then the entire set $\boldsymbol{\Gamma}$ is satisfiable.[79,80,81] In terms of fragmented modeling, if each submodel represents a finite, internally consistent portion of a broader theory $\boldsymbol{T}_U$, then there exists at least one global model $\mathcal{M}_U$ in which all these fragments coexist.[82] However, this guarantee is purely existential and non-constructive. Compactness does not require that this recombined global model either mirrors the original structure or preserves the interpretive integrity of its fragments. For example, consider local theories $\{\boldsymbol{T}_{H1}, \boldsymbol{T}_{H2}, \boldsymbol{T}_{H3}\}$ modeling morphological, genetic and ecological aspects via partial models ($\mathcal{M}_{H1}$, $\mathcal{M}_{H2}$, $\mathcal{M}_{H3}$). Compactness

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[72] This illustrative example of has limited scientific and integrative value. See §§4.1 and 6.2 for more discussion.

[73] Putnam, Hilary, 22-48: "Chapter 2 - A Problem about Reference"

[74] This thesis Is about scientific theory, but I will adopt the 'Tarskian' perspective of a 'speaker' and a 'hearer' who can each have individual 'partial sub-models of THE WORLD. I previously explored this speaker-hearer interaction in an essay, hence the use of $\mathcal{M}_S$ ('speaker') and $\mathcal{M}_H$ ('hearer') for presenting and receiving sub-models. I cannot argue here how interpersonal communication is a case for *MASS*.

[75] Putnam (1977), 483

[76] Dummett (1978), xxxvii

[77] Cartwright (1999), 1: Nature is a jumbled world, mostly governed by negotiation rather than strict laws.'

[78] Galison (1997)

[79] Hodges (1997), 124: "Theorem 5.1.1 (Compactness [...]):
Let $\boldsymbol{T}$ be a first order theory. If every finite subset of [sentences from $\boldsymbol{\Gamma}$] $\boldsymbol{T}$ has a model then $\boldsymbol{T}$ has a model."

[80] Doets (1996), 51-55: "4.1 Compactness"

[81] See Insert 1 for concepts and symbols.

[82] Hodges (1997), 124

ensures there is some model $\mathcal{M}_r$ satisfying all sentences, but this $\mathcal{M}_r$ might combine morphological traits of one species with ecological or genetic features from others—producing an abstract amalgam that is formally consistent but biologically unviable.

This non-constructiveness of Compactness reveals a key limitation: global coherence cannot be assumed from local consistency alone. *MASS* addresses this by *not* demanding one fully unified global model. Instead, it proposes an adaptive process that incrementally adjusts local partial models—through minimal 'push-through' modifications of interpretation functions and other elements (§4.3)—preserving shared reference points at the level of key sentences ϕ. Thus, while Compactness guarantees the theoretical possibility of a global model, *MASS* operationalizes this by ensuring local coherence through progressive, context-sensitive adaptation. Rather than assembling a disjointed collection of fragments in a 'bag of balls', *MASS* uses shared sentences as a kind of 'glue' to shape fragments into a coherent 'sphere' (as visually illustrated in Figure 2, §5.2).[83] Local successes in achieving shared satisfaction form the practical foundation of theoretical coherence despite the multiplicity of models revealed by Löwenheim–Skolem theorems.

Chapters 4 and 5 will develop this further, showing how tools such as the push-through of shared structures, sheaf-theoretic amalgamation, and graph-theoretic mappings serve as technical means to apply this 'glue'. These methods secure points of shared reference and supportive structures across partial models, while allowing each submodel to preserve its essential structure. The next chapter will illustrate this 'toolbox' (schematized in Insert 4 and outlined in §4.6), explaining how a reference sentence can be propagated stepwise through a fragmented network of models with minimal adaptation.

[83] For visual illustration see Figure 2, §5.2.

Chapter 4. Towards an Internal Resolution of Indeterminacy by Model Fragmentation —*Shared Satisfaction of a Reference Sentence with Minimal Model Adaptation*

The preceding chapters have shown that model-theoretic indeterminacy, as revealed by the Löwenheim–Skolem theorems and tempered by Compactness, blocks the straightforward construction of a single, fully unified model representing a complex scientific theory. Compactness guarantees the existence of a global model $\mathcal{M}_U$ consistent with all finite fragments (M_{H1}, M_{H2}, *etc*), but offers no constructive method for achieving coherence across a fragmented network of submodels.

This chapter introduces *Model Adaptation by Shared Satisfaction* (*MASS*), an iterative framework for bridging fragmented models via a carefully chosen reference sentence ϕ. Rather than imposing rigid structural uniformity, *MASS* establishes common ground by ensuring that ϕ is satisfied across submodels through minimal, controlled adaptations. The method preserves diversity in legitimate perspectives while providing a flexible coordination of interpretations. By focusing on shared satisfaction rather than full unification, *MASS* offers a stepwise path to internal theoretical coherence. The chapter begins with the features of a reference sentence that enable effective sharing among partial models (§4.1), before examining how such sentences can bind them together (§4.2).

4.1 Reference Sentences and Their Function in *MASS*

The strategy begins with a reference sentence ϕ: a nontrivial structural claim grounded in the Unified Theoretical Framework ($\boldsymbol{T}_U$). It captures a principle whose truth cannot be confined to any single submodel without undermining the coherence and explanatory power of $\boldsymbol{T}_U$. In this sense, ϕ represents a core hypothesis of $\boldsymbol{T}_U$, designed to unify coherent but not immediately compatible submodels—such as those from distinct disciplines or rival frameworks—that describe different aspects of reality.

As discussed in §3.2.3, the content of ϕ is critical to *MASS*. It must be neither trivial nor too general, nor unduly complex. Full satisfiability across all submodels is neither expected nor required; progress comes through iterative refinement of ϕ and the submodels. ϕ should not be mistaken for a higher-order truth covering all phenomena at once. Because the classical model theory used here is grounded in first-order logic, the expressive scope of model-theoretic sentences remains limited. Attempting to force a universal claim into a fragmented network risks imposing unrealistic structural demands. Nor should ϕ be an axiom of $\boldsymbol{T}_U$ or a self-evident feature of $\mathcal{M}_U$ already shared by every submodel. Models satisfying ϕ must not be mere homomorphic copies of one another, as this would mask meaningful structural differences and defeat the purpose of *MASS* to mediate between genuinely distinct submodels.

This approach accommodates a wide range of complexities in ϕ while remaining adaptable and self-correcting. Later sections (§4.4; Chapter 5) trace how shared satisfaction spreads across submodels and where it encounters limits. To explain *MASS*, ϕ need only be challenging enough to advance both scientific and philosophical insight, while the submodels must be sufficiently distinct to make integration worthwhile. When referring to $\boldsymbol{T}_U$ as the 'Unified Theoretical Framework' and ϕ as a 'meaningful reference sentence,' I aim to keep the same deliberate vagueness—and constructive ambition—that Putnam used for his 'ideal scientific theory $\boldsymbol{T}_1$.' [84] In Chapter 6, however, I return to a richer example from biological diversity than the African Wildcat of §3.2.3.

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[84] Putnam (1980), 473: '[A] possible formalization of present-day total science $\boldsymbol{T}$, and a possible formalization of ideal scientific theory $\boldsymbol{T}_1$. $\boldsymbol{T}_1$ is epistemically 'ideal': [...] when God makes up $\boldsymbol{T}_1$, He constructs [...] a limit of theories [...] rational [for scientists] to accept, as more and more evidence accumulates, [...] relative to which $\boldsymbol{T}$ can be quantitatively compared.'

4.2 Fixation of Indeterminacy by Shared Satisfaction across Fragmented Models

Our reference sentence ϕ is assumed to capture a core aspect of $\boldsymbol{T}_U$ and thius is satisfied in its full model $\mathcal{M}_U$. However, this satisfaction can only be confirmed if ϕ is preserved—*ie* remains true—across all submodels of $\mathcal{M}_U$. When $\boldsymbol{T}_U$ is complex and $\mathcal{M}_U$ is fragmented, direct proof is generally impossible due to indeterminacies from model fragmentation and cardinality.

The strategy is to fix satisfaction through the shared structure of ϕ. If ϕ is true in $\mathcal{M}_U$ and satisfied in one submodel, it should also be satisfiable in others, provided these submodels overlap in the structural core component $\boldsymbol{\pi}$ expressed by ϕ. If ϕ is universally satisfied, it cannot be logically contradictory in any submodel or the full model of $\boldsymbol{T}_U$, and no other true sentence ψ in $\boldsymbol{T}_U$ is refuted by it. This does not mean that every submodel will be able to satisfy ϕ from the outset. For one, Φ must satisfy strict conditions within $\boldsymbol{T}_U$ and interact coherently with overlapping submodels to ensure satisfaction can be maintained or restored. These constraints help mitigate destabilizing consequences of the Löwenheim–Skolem theorems. In this thesis, they are formalized using model-theoretic tools and graph-theoretic descriptions of the network of submodels described in this Chapter and the next.

The remainder of this Chapter demonstrates how the functions and structures of interconnected submodels are internally constrained by imposing a shared isomorphism central to $\mathcal{M}_U$, hence compatible with any submodel—though not necessarily entailed in each. Φ expresses the supportive partial structure $\boldsymbol{\pi}$, enabling shared satisfaction with minimal adaptation. This builds on model theory (§3.1), but employs integrative tools like *push-through construction* (§4.3) and *sheaf-amalgamation* (§4.4). Other advanced approaches are considered (§4.5) but found unsuitable for fragmentation-induced indeterminacy. In §4.6 these tools are employed in $\boldsymbol{A}_{MA\rightarrow SS}$, a model-adaptation axiom schema of *Model Adaptation by Shared Satisfaction*. Insert 2 (above §4.6) shows how their stepwise application leads to progressive adaptation of partial models or refinement of ϕ—or their rejection as unsustainable. The dynamics of these adaptations are analyzed in Chapter 5 via graph theory, showing how increasing satisfaction across submodels generates a coherent network sharing ϕ's content and structure. Chapter 6 discusses how far this reduces indeterminacy from model fragmentation, illustrated with recent molecular (epi)genetic advances integrating distinct models of biological diversity.

4.3 Push-Through Construction of Reference Sentence Structures

Scientific theories often cover various aspects of reality, described by different hypotheses within the same framework. A hypothesis gains support when critically compared with others under the same theory. In model-theoretic terms, this mirrors how sentences in submodels are tested for consistency and satisfaction relative to other submodels.

Here, the reference sentence ϕ plays the role of a unifying hypothesis in our unified theoretical framework $\boldsymbol{T}_U$, even though $\boldsymbol{T}_U$ also accommodates other, more specific hypotheses with their own submodels. If ϕ is true in $\boldsymbol{T}_U$, it should be satisfiable in all submodels of $\mathcal{M}_U$. However, satisfaction is not guaranteed in submodels lacking the structural elements needed for ϕ (see § 3.2.2 on model fragmentation). Thus, ϕ may be provable in some submodels but unprovable in others, without entailing a contradiction.

The push-through construction addresses this by transferring the isomorphic core structure of ϕ—$\boldsymbol{\pi}$—from one submodel to another.[85] In standard model theory, push-through may also include a permutation function giving a one-to-one 'translation' between the mappings of submodels, which can radically alter their referential interpretation. As we will discuss in §6.1, Putnam used this to enforce his model-theoretic argument for indeterminacy of reference. In the present approach, however, the partial isomorphism $\boldsymbol{\pi}$ includes only the elements necessary to

[85] Button & Walsh (2018), 35-37: "2.1 Isomorphism and the Push-Through Construction."

satisfy the reference sentence ϕ. Specifically, push-through constructs the shared elements of two interconnected submodels of $\mathcal{M}_U$—a presenting model $\mathcal{M}_S$ ('speaker') and a receiving model $\mathcal{M}_H$ ('hearer').[86] Satisfaction of ϕ can be achieved via different forms of **π**, and **π** may need to be adapted as ϕ is passed to further receiving models $\mathcal{M}_H$, $\mathcal{M}'_H$, $\mathcal{M}''_H$, $\mathcal{M}'^{\cdots}{}'_H$.

This reveals a key limitation of push-through: in its basic form, it does not modify ϕ or **π**—it merely transfers the structure of **π** into a receiving model, provided no internal conflicts arise. Full satisfaction across all shared submodels is possible only if the structure of $\mathcal{M}_S$ (on which **π** is based) is already compatible with all others. This imposes an unrealistically high demand: ϕ would need to be flawless from the start, with **π** matching every submodel structurally.

A more flexible approach allows ϕ to be progressively refined through its interaction with different submodels. Without such flexibility, only exceptionally well-formed ϕ could be propagated without friction. Model theory offers several constructs to reduce this burden . The rest of this chapter discuss model-theoretic approaches that support structured adaptation of interconnected models—some readily applicable (§ 4.4), others more problematic (§4.5). $\boldsymbol{A}_{MA\rightarrow SS}$ (§4.6) integrates these into a meta-theoretic axiom schema for stepwise model harmonization and refinement of ϕ. Push-through corresponds to Step 3 (Insert 2), where a partial isomorphism **π** between a presenting and a receiving model is established, ensuring that the structural core of ϕ is preserved without disruption. The conditions described here—compatibility of **π**, non-modification of ϕ, and the role of an amalgamated carrier model built on **π** ($\mathcal{M}_A$)—are invoked in that step as discussed next.

4.4 Amalgamation and Other Constructs of Model Integration

4.4.1 Push-Through and Structural Reconciliation

Push-through extends the partial isomorphism **π** from $\mathcal{M}_S$ to $\mathcal{M}_H$. Since **π** is not a model and therefore cannot on its own verify ϕ's satisfaction, it instead scaffolds the construction of an amalgamated model $\mathcal{M}_A$, which consolidates shared structure from $\mathcal{M}_S$, $\mathcal{M}_H$, and other submodels.[87] $\mathcal{M}_A$ integrates information from $\mathcal{M}_S$, $\mathcal{M}_H$, and subsequent 'hearer submodels', preserving their common structure. A ϕ true in both $\mathcal{M}_S$ and $\mathcal{M}_H$ can then be proved in $\mathcal{M}_A$, because $\mathcal{M}_A$ incorporates the elements to **π** necessay to bridge submodels by aligning domains and interpretation functions. Each time $\mathcal{M}_S$ is pushed through to $\mathcal{M}_H$, $\mathcal{M}_A$ helps preserve internal consistency.

$\mathcal{M}_A$ can also operate as a standalone model, integrating elements from receptive submodels. By amalgamating structural elements that consistently accompany ϕ's satisfaction, $\mathcal{M}_A$ could develop into a submodel of $\mathcal{M}_U$ in which ϕ is satisfied along with additional structurally compatible sentences from $\boldsymbol{T}_U$. In this way, $\mathcal{M}_A$ represents a refinement or extension of the original reference sentence ϕ, corresponding to an improved version of ϕ (introduced as a core hypothesis in §4.1), now able to unify a broader set of coherent—but previously incompatible—submodels. The information gathered by $\mathcal{M}_A$ is also useful when revising ϕ and **π** after push-through failure, as discussed in §*4.4.2.2*. This may have several causes. In first-order logic, satisfaction can be flexible, depending on the precision of interpretation functions and the scope of reference. Related but distinct domain elements may be unevenly distributed across submodels, and even when submodels agree on predicates, functions, and constants, their interpretation functions—though derived from $\mathcal{M}_U$—may map to subtly different subdomains. Because **π** encodes only the minimal structure needed to satisfy ϕ, it may eventually encounter an obstructive submodel model $\mathcal{M}_H$—a 'fractured' fragment of $\mathcal{M}_U$ (now $\mathcal{M}^{\#}{}_H$) misaligned with **π** enough to resist push-throughs. For instance, if **π** includes a binary relation $\boldsymbol{R}(x,y)$ necessary for ϕ, but $\mathcal{M}^{\#}{}_H$ contains only unary projections, it may reinterpret $\boldsymbol{R}(x,y)$ as two disjoint unary

[86] See also footnotes 71 and 74 for explanation of S-('speaker') and H–('hearer') suffixes.
[87] Hodges (1997), 134-141: "5.3-Elementary Amalgamation"

predicates $\boldsymbol{R}_1(x)$ and $\boldsymbol{R}_2(y)$, thereby rejecting $\mathcal{M}_U$'s binary structure. Such reinterpretation can be locally favored to preserve internal constraints, making $\boldsymbol{\pi}$ appear as a foreign insertion that threatens established interpretations and defies minimal adaptation.

Thus, satisfaction preservation across submodels is not guaranteed by common origin alone; it also depends on structural compatibility and interpretive flexibility at the point of integration. Resistance of a valid submodel of $\mathcal{M}_U$ to integrating an equally valid isomorphic $\boldsymbol{\pi}$ of $\mathcal{M}_U$ is a key aspect of indeterminacy arising from deep fragmentation. Fragmentation can 'shatter' the structural coherence needed to stabilize ϕ's satisfaction—not through absence of shared origin, but through the difficulty of reconstructing shared structures across divergent interpretations. The Compactness Theorem (§3.2.3) implies that even if $\mathcal{M}_U$ were complete, pushing its full structure through to all submodels would fail: fragmented substructures cannot interpret or sustain the total structure from which they emerge.[88]

One possible response is to formulate a new reference sentence also true in $\mathcal{M}_S$—but this disregards the reasons for $\boldsymbol{\pi}$'s failure. Valuable information can be gained from ϕ's successful transmissions through multiple submodels before encountering a disruptive fragment. While ignoring $\mathcal{M}^{\#}_H$ is possible when ϕ is still tentative, renewed push-through to another $\mathcal{M}_H$ is more likely to succeed if $\boldsymbol{\pi}$'s problematic structural elements are refined. Although $\mathcal{M}_A$ may reveal the cause, it lacks a direct procedure for adjusting conflicting components—and repeated incorporation of seemingly useful elements from receptive submodels risks structural overaccumulation, burdening $\mathcal{M}_A$ with features incidental to ϕ's satisfaction. This may hinder its role in revising $\boldsymbol{\pi}$ when push-through faces genuine obstruction (§*4.4.2.1*). A higher-order integrative mechanism therefore appears necessary—which is addressed in the next subsection which also goes into $\mathcal{M}_A$'s amalgamating function (§*4.4.2.2*).

4.4.2 Sheaf-Theoretic Amalgamation

Some model-theoretic constructs address structural instability across models. Sheaf-theoretic amalgamation is among the most integrative, with numerous applications in AI.[89] It builds on the concrete amalgamated model $\mathcal{M}_A$ by integrating $\boldsymbol{\pi}$'s shared isomorphic elements into a structured local space for each contributing submodel. As clarified in the next subsection (Figure 1), these local spaces correspond to *categorical dimensions* representing structural correspondences—domain types, function arities, and related features—weighted accordingly. The ability to unify evidence from heterogeneous sources and across inquiry levels is increasingly important in both pure and applied sciences.[90] Hodges considers 'the idea of amalgamation… very powerful, and I have used it whenever I can.'[91] The next two subsections provide some technical detail, and foreshadow its relevance for *MASS* (discussed in §4.6).

4.4.2.1 Structural Amalgamation via Sheaf Theory

Sheaf-theoretic amalgamation preserves and coordinates essential model-theoretic features at a higher level of abstraction. As shown in Figure 1, model-theoretic 'sheaves' integrate parallel information from partial isomorphisms into a higher-order structure encoding the elements shared across submodels.

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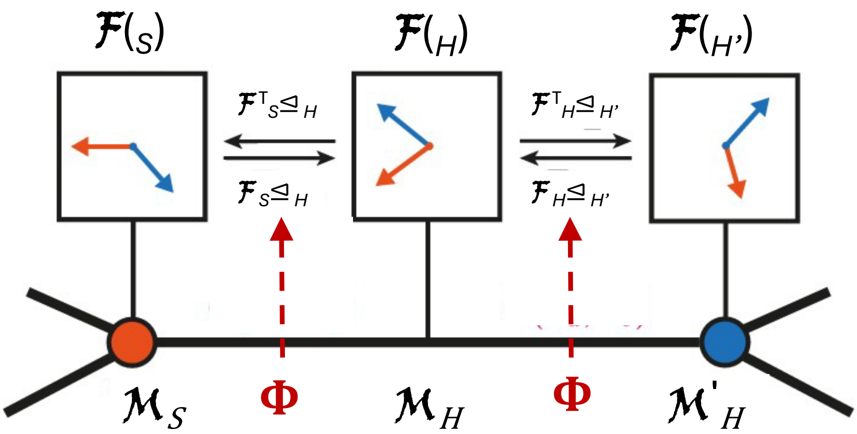
[88] Even with a complete scientific model, the Compactness Theorem (§3.2.3) shows that fragmentation—via disciplinary boundaries, semantic drift, and representational limits—undermines the structural coherence required for universal push-through. Hence, unifying all scientific knowledge into a single fully transmissible model is conceptually incoherent as well as impractical.

[89] Schmid (2025)

[90] Fletcher (2019), 3170-3171

[91] Hodges (1997), 124

Figure 1: [92] **Sheaf composition, illustrated for two push-through attempts with two shared structural categories represented by red and blue arrows.** The three boxes show stalks $\mathcal{F}(S)$, $\mathcal{F}(H)$, $\mathcal{F}(H')$ that correspond to $\mathcal{M}_S$, $\mathcal{M}_H$, $\mathcal{M}'_H$, respectively. Each stalk represents a local structured space populated by the shared isomorphic elements (**π**) of that submodel—the same elements that $\mathcal{M}^+_A$ integrates at the global level. These structured spaces can be described in terms of categorical dimensions (see §4.6), with vector lengths indicating the relative weights assigned to the corresponding categorical features. [93] Functions $\mathcal{F}^{(T)}{}_{S\trianglelefteq H}$ and $\mathcal{F}^{(T)}{}_{H\trianglelefteq H'}$ (so-called 'restriction maps') transfer compatible, weighted information between stalks**.** A parametric function **Φ** governs how features are aligned and 'glued' into the unified fiber bundle χ.



Sheaf theory systematically tracks local data on a topological base space of core model elements (categories) and assembles them into global structures. Basically, the higher-order sheaf function $\mathcal{F}$ integrates the weighted categorical dimensions from the contributing submodels into a unified fiber bundle, preserving relational structure between local contributions. Ideally, a complete model $\mathcal{M}_U$ of a theory $\boldsymbol{T}_U$ could be represented by a single sheaf—a fiber bundle with $\mathcal{F}$ amalgamating all partial isomorphic structures. However, the Compactness Theorem implies that deep fragmentation can make such composition difficult: some 'broken' stalks may lack categorical dimensions or contain incongruent ones that resist incorporation into the global bundle. Sheaf-theoretic amalgamation retains higher-order weighted information from successes and failures, helping diagnose integration breakdowns—as will be discussed in the next sub-section.

4.4.2.2 Push-Through Failure and Structural Modification

If push-through fails because **π** is incompatible with a receiving submodel $\mathcal{M}'_H$, this fractured submodel $\mathcal{M}^\#_H$ blocks further attempts to share our reference sentence ϕ. Push-through can continue with other submodels, or revisit $\mathcal{M}^\#_H$ after modifying **π**' and ϕ'. Sheaf amalgamation can help prevent repeated errors. The sheaf function $\mathcal{F}$ still attempts to glue the structure of $\mathcal{M}^\#_H$'s stalk into the higher-order amalgamated model $\mathcal{M}^+_A$—which is built from all categorical stalks of previously integrated submodels, not just **π**'s essentials. Thus, $\mathcal{M}^+_A$ can reveal incompatible elements of $\mathcal{M}^\#_H$. If possible, $\mathcal{M}'_A$ can be instantiated from $\mathcal{M}^+_A$ in a way that avoids the obstruction. From $\mathcal{M}'_A$, a novel reference sentence ϕ' can be formulated with a refined isomorphism **π**', still matching the earlier receptive models. $\mathcal{M}'_A$ now effectively serves as a renewed presenting model $\mathcal{M}'_S$—not for a naive retry, but based on multiple models already sharing satisfaction of ϕ. Viewed from the broader perspective of our Unified Theoretical Framework, this corresponds to reformulating an improved hypothesis within $\boldsymbol{T}_U$: still consistent

[92] Bodnar *et al.* (2022). This figure is based on neuronal networks, but it generalizes to model theory and highlights relevance to AI and computer science.

[93] Hodges (1997), 137 gives an example elementary amalgamation of two structures with different vector spaces with an overlapping substructure.

with confirmed results but better fitted to avoid incompatibilities with other unchallenged hypotheses.

It is important to realize that push-through failure may also occur because the receiving model is itself inconsistent with $\boldsymbol{T}_U$. In other words, $\mathcal{M}^{\#}{}_H$ may not be a valid—albeit fractured—submodel of $\mathcal{M}_U$, but an 'alien' submodel $\mathcal{M}^*{}_H$ from another theory $\boldsymbol{T^*}$, satisfying sentences that are not satisfiable in $\mathcal{M}_U$. In such cases, it would be better to reject $\mathcal{M}^*{}_H$ altogether, rather than to adapt ϕ to it. The sheaf here acts as a diagnostic tool: by revealing structural misalignment across submodels, it distinguishes between internal fragmentation within $\mathcal{M}_U$ and intrusion from structural categories from $\mathcal{M}^*{}_H$ that are (yet) unknown to amalgamated model $\mathcal{M}'_A$. In both cases, push-through fails because neither $\mathcal{M}^{\#}{}_H$ nor $\mathcal{M}_H$ can identify matching domain elements. Sheaf-theoretic amalgamation can indicate that $\mathcal{M}_H$'s higher-order features make it structurally alien to $\mathcal{M}_U$: if so, $\mathcal{M}_H$'s stalk fails to fit not only $\boldsymbol{\pi}$'s essentials, but also other structural properties of the amalgamated fiber bundle.

The reliability of this indication requires a well-developed sheaf function, built from many successful push-throughs. In early stages, it may be more efficient to bypass resistant models and gather more information, deferring renewed integration until greater coherence exists. Recurrent failures may reveal that some obstructive submodels, $\mathcal{M}^*$, belong to a different theory $\boldsymbol{T^*}$, when they have an incompatible higher-order sheaf structure $\mathcal{X}$. Thus, failed push-through with successful sheaf amalgamation often signals more structural fragmentation of submodels; whereas failure of both indicates more fundamental incompatibilities—possibly belonging to different full models from unrelated theories. In this way, the sheaf helps distinguish between potentially compatible and incompatible submodels.

If a huge proportion of obstructive submodels appear to be alien to $\mathcal{M}_U$, this suggests there is a class of $\mathcal{M}^*{}_H$-submodels that systematically mimic $\mathcal{M}_H$ but remain structurally incompatible. This can occur if $\boldsymbol{T^*}$ is very similar to $\boldsymbol{T}_U$, and perhaps competitive. This may warrant a systematic comparison, which can lead to reconstruction of $\mathcal{M}_U$. Apparently alien $\mathcal{M}^*{}_H$-submodels can also belong to $\mathcal{M}^+{}_U$ with a higher cardinality than $\mathcal{M}_U$.

When successful push-throughs accumulate, the global sheaf structure $\mathcal{X}$ increasingly represents essential features of $\mathcal{M}_U$. This will be revealed by a progressively consistent structure of $\mathcal{M}^+{}_A$. This in turn can supports refinement of $\boldsymbol{\pi}$ and ϕ when they fail in a submodel that can still be sheaf-integrated, by reformulation of ϕ' that avoids the cause of failure. Finally, within first-order model theory, $\mathcal{M}^+{}_A$ serves only as a structural guide for revising ϕ' after ϕ fails. While ϕ' may still not be satisfied in all submodels, it must at least avoid incompatibility with any model contributing to $\mathcal{M}^+{}_A$. $\mathcal{M}^+{}_A$ itself cannot be pushed through, since it contains only higher-order elements—types, categories, predicate arities—without specifying first-order connections in a receiving model. Proof of satisfaction across submodels always requires a first-order reference sentence ϕ. Moreover, while $\mathcal{M}^+{}_A$ may guide the reconstruction of a global model $\mathcal{M}_U$, it does so from a higher-order framework that lacks direct first-order completeness or definability.[94] Nonetheless, sheaf-theoretic amalgamation enriches the framework with information about structures that satisfy ϕ and elements causing conflicts. If push-through fails, $\mathcal{M}^+{}_A$ constrains modification of π' by enforcing shared structural conditions, reducing indeterminacy in revising ϕ'. However, the actual choice of a revised ϕ' requires an additional adaptation principle. This is the purpose of *MASS*, which uses $\mathcal{M}^+{}_A$'s constraints to selects ϕ' that maximizes satisfaction across coherent of $\mathcal{M}_U$ submodels. But before we develop this novel

[94] This reflects a tension between higher-order structures and first-order model theory: while $\mathcal{M}^+{}_A$ organizes relationships across models categorically, these do not necessarily correspond to a first-order model satisfying the original theory. Key properties like compactness and definability may fail or be lost when moving from higher-order frameworks to first-order structures.

approach in §4.6, we might consider other established guiding tools for model optimization from model theory, which is performed in the next subsection.

4.5 Other Model-Theoretic Approaches to Model Expansion and Interaction

Model theory is a vast and intricate field of mathematics. While it is not yet firmly established within philosophy, it is increasingly applied in philosophical contexts.[95] It is useful to briefly consider some alternative model-theoretic strategies that bear on problems of fragmentation and indeterminacy. One relevant concept is conservative extension,[96] which can be viewed as an application of the Upward Löwenheim–Skolem Theorem (§3.2.1). In simple terms, a conservative extension shows that a theory $\boldsymbol{T}$ can be expanded into a richer theory $\boldsymbol{T}^*$ by adding new structural elements—such as additional predicates or constants—without altering any of the original truths of $\boldsymbol{T}$. In the context of model fragmentation, one might apply this to a submodel $\mathcal{M}_S$ by incorporating interpretation functions, predicates, or constants from other submodels like $\mathcal{M}_H$, $\mathcal{M}'_H$,.... However, as noted in the discussion of Compactness (§3.2.4), such extensions cannot i unify the fragments into a coherent total model $\mathcal{M}_U$. The result would be a bag of dissociated submodels—analogous to a textbook on biological diversity with a collection of separate articles on taxonomy, adaptability, ecology, evolution and genetics.

More sophisticated approaches have been developed within advanced model theory to address interaction and coherence among models. Wilfrid Hodges discusses several such frameworks in his *Short History of Model Theory*, included in Button and Walsh's *Philosophy and Model Theory*. For example, geometric model theory uses an integrative approach that resembles sheaf-theoretic amalgamation (§*4.4.2.2*)[97]—with the same problems of trying to reconstruct a first-order model $\mathcal{M}_U$ from a higher-order framework.[98] Hodges also mentions other approaches involving 'atomic', 'compact', and 'saturated' models, which show that local consistency and structural overlap can, under suitable conditions, yield global or quasi-global coherence across fragmented or expanding networks of models.[99] These frameworks offer powerful mathematical tools, but they have not been applied to the philosophical indeterminacies exposed by Putnam's Model-Theoretic Argument. These difficulties are not merely technical, but stem from deep structural features of model theory itself—especially those highlighted by the Löwenheim–Skolem theorems. This essay develops theoretical tools to address its most disruptive consequences while preserving the advantages of diverse perspectives.

§4.6 introduces a meta-theoretic principle: the *Axiom of Model Adaptation by Shared Satisfaction* ($\boldsymbol{A}_{MA\rightarrow SS}$). This axiom schema supports the integration of divergent submodels by requiring the successful 'push-through' of a reference sentence ϕ across submodels that are sufficiently adaptable to satisfy it within a broader unified theory $\boldsymbol{T}_U$, with minimal adaptations when push-through fails. Chapter 5 introduces graph theory as a concrete yet accessible mathematical framework to structure these interactions and visualize the dynamics of model integration.

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[95] Button & Walsh (2018), vi.

[96] Hodges (1997), 59: "Let $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}^*$ be first-order languages with $\boldsymbol{\mathcal{L}} \subseteq \boldsymbol{\mathcal{L}}^*$, and let $\boldsymbol{T}$ and $\boldsymbol{T}^*$ be theories in $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}^*$ respectively. We say that $\boldsymbol{T}^*$ is a conservative extension of $\boldsymbol{T}$ if for every sentence ϕ in $\boldsymbol{\mathcal{L}}$, $\boldsymbol{T}^* \vDash \phi$ *iff* $\boldsymbol{T} \vDash$ ϕ."

[97] Hodges (2018) "18.8 – Geometric Model Theory." in Button & Walsh (2018), 469-472.

Like sheaf-theoretic amalgamation, geometric model theory classifies structures in terms of their combinatorial geometries and the groups and fields that are interpretable in the structures.

[98] See footnote 94

[99] Hodges (2018), "18.5 – Maps between Structures." in Button & Walsh (2018), 455-460

§4.6— Insert 2—Axiom Schema $A_{MA\rightarrow SS}$ (*Model Adaptation by Shared Satisfaction*)[100]

Let $\boldsymbol{L'}$ be a subsignature of theory $\boldsymbol{T}_U$, and let $\mathcal{L}(\boldsymbol{L'})$ be the formal language over $\boldsymbol{L'}$:

- -$\boldsymbol{T}_U$: a theory in $\boldsymbol{L'}$, and MOD($\boldsymbol{T}_U$) the class of all models that satisfy every sentence in $\boldsymbol{T}_U$
- -$\mathcal{M}_U$: a full model of $\boldsymbol{T}_U$ such that $\mathcal{M}_U \vDash \boldsymbol{T}_U$, and let $\mathcal{M}'$ denote submodels of $\mathcal{M}_U$[101] (*ie* $\mathcal{M}_S$, $\mathcal{M}_H$, $\mathcal{M}'_H$..., with $\mathcal{M}_S \in$ MOD($\boldsymbol{T}_U$))
- -$\mathcal{M}_S$ = ($\boldsymbol{D}_S$, $\boldsymbol{L'}_S$, $\boldsymbol{I}_S$) be a presenting ('speaking') submodel and $\mathcal{M}_S \vDash \phi$ (where $\boldsymbol{L'}_S \subseteq \boldsymbol{L'}$)
- -$\mathcal{M}_H$ = ($\boldsymbol{D}_H$, $\boldsymbol{L'}_H$, $\boldsymbol{I}_H$) be a receiving ('hearing') submodel, such that either $\mathcal{M}_H \vDash \phi$ or $\mathcal{M}_H \nvDash \phi$ (and $\boldsymbol{L'}_H \subseteq \boldsymbol{L'}$)
- -$\phi \in \mathcal{L}(\boldsymbol{L'})$: a reference sentence such that $\mathcal{M}_S \vDash \phi$
- -$\psi \in \mathcal{L}(\boldsymbol{L'})$: any sentence satisfied in any submodel $\mathcal{M}'$ such that $\mathcal{M}' \vDash \psi$ for some $\mathcal{M}' \in$ MOD($\boldsymbol{T}_U$)

Let $\mathcal{M}'_H$, $\mathcal{M}''_H$, $\mathcal{M}'^{\cdots\prime}_H$ *etc* denote 'minimally adapted' variants of $\mathcal{M}_H$ such that $\mathcal{M}'_H \vDash \phi$, $\mathcal{M}''_H \vDash \phi$ *etc*

- -$\mathcal{M}^{\#}_H$: a 'fractured' submodel of $\mathcal{M}_U$ such that $\mathcal{M}^{\#}_H \nvDash \phi$, but $\mathcal{M}^{\#}_H \in$ MOD($\boldsymbol{T}_U$)
- -$\mathcal{M}^*_H$: an 'alien' (sub)model such that $\mathcal{M}^*_H \nvDash \phi$ and $\mathcal{M}^*_H \notin$ MOD($\boldsymbol{T}_U$), whereas instead $\mathcal{M}^*_H \in$ MOD($\boldsymbol{T}^*$); $\mathcal{M}^*_H$ satisfies some alien sentence χ ($\chi \in \mathcal{L}(\boldsymbol{L'})$ or $\chi \notin \mathcal{L}(\boldsymbol{L'})$), may satisfy some $\psi \in \mathcal{L}(\boldsymbol{L'})$, but is structurally incompatible with $\mathcal{M}_U$ and cannot be adapted to satisfy ϕ.

1. (Re)Formulation of Reference Sentence $\boldsymbol{\phi}$ (§4.1)

Reference sentence ϕ (or its reformulation ϕ' *etc* following from step 6) is

- -a non-trivial, non-axiomatic sentence from $\boldsymbol{T}_U$ that expresses a core feature of $\boldsymbol{T}_U$
- -satisfied by $\mathcal{M}_U$: $\mathcal{M}_U \vDash \phi$
- -satisfied in presenting submodel $\mathcal{M}_S$: $\mathcal{M}_S \vDash \phi$
- -not in contradiction with any sentence ψ in $\mathcal{M}_S$: $\forall \psi \in \boldsymbol{T}_S$
- -nor with any other submodel $\mathcal{M}'_H$, $\mathcal{M}''_H$, $\mathcal{M}'^{\cdots\prime}_H$ *etc* to which ϕ (or ϕ') has been presented

2. Immediate Shared Satisfaction (Basic Transfer)

If two submodels $\mathcal{M}_S$ and $\mathcal{M}_H$, of $\mathcal{M}_U$ both independently satisfy the same reference sentence ϕ, and there exists a partial isomorphism $\boldsymbol{\pi}$: $\mathcal{M}_S \rightarrow \mathcal{M}_H$ that preserves the structural components relevant to ϕ, such that $\mathcal{M}_S \vDash \phi$ and $\mathcal{M}_H \vDash \phi$:

—then ϕ is immediately and jointly satisfied in both models without need for modification,

—then $\mathcal{M}_H$ alters into $\mathcal{M}'_H$ without introducing additional ambiguity, ensuring that model revision preserves a shared interpretational core within both $\mathcal{M}_H$ and $\mathcal{M}'_H$ that is also contained in $\mathcal{M}_S$,

—then push-through stability ensues in Step 3.

1-ϕ is satisfied in both $\mathcal{M}_S$ and $\mathcal{M}_H$ without modification

2-$\mathcal{M}_H$ is updated to $\mathcal{M}'_H$ preserving shared interpretational structure

3-Push-through stability is recorded (Step 3).

This forms the base case of $\boldsymbol{A_{MA\rightarrow SS}}$: ϕ is stable across distinct submodels just by virtue of shared structure. No adaptation, reinterpretation, or extension is required.

—This immediate agreement serves as the starting point for progressive propagation of ϕ throughout the set of untested submodels $\mathcal{M}_H$ of $\mathcal{M}_U$

—When ϕ cannot be satisfied in some new submodel $\mathcal{M}_H$, minimal model modification occurs in Step 4:

3. Push-Through Stability (Non-Disruptive Adaptation) (§4.3)

If a partial isomorphism $\boldsymbol{\pi}$: $\mathcal{M}_S \rightarrow \mathcal{M}_H$ (or $\mathcal{M}'_H$ *etc*) preserves shared structure such that:

- $\mathcal{M}_S \vDash \phi$ and $\mathcal{M}_H \vDash \phi$ (or $\mathcal{M}'_H \vDash \phi$ *etc*)

—then ϕ is jointly satisfied in both models without any modification,

—otherwise, if $\mathcal{M}_H \nvDash \phi$ due to missing or conflicting components of $\boldsymbol{\pi}$ (*ie* $\mathcal{M}_H$ is a 'fractured' submodel $\mathcal{M}^{\#}_H$), Step 4 is invoked for minimal model modification

— At this stage, the outcomes of successful push-throughs are recorded, preparing the framework for potential adaptations. (The amalgamated model $\mathcal{M}_A$, which will track coherence across all adapted submodels, is formally introduced in Step 4.)

4. Minimal Model Modification (Main Iterative Process)

If there exists a modified model $\mathcal{M}'_H$ = ($\boldsymbol{D}'_H$, $\boldsymbol{L}'_H$, $\boldsymbol{I}'_H$) such that:

- $\mathcal{M}'_H \vDash \phi$ (ensuring shared satisfaction of reference sentences)
- $\forall \psi \in \boldsymbol{T}_H$, $\mathcal{M}'_H \vDash \psi$ (previously satisfied sentences in $\mathcal{M}_H$ remain satisfied)
- $\Delta \boldsymbol{I}'_H \preceq \Delta \boldsymbol{D}'_H \preceq \Delta \boldsymbol{L}'_H$ (specifying the preferred partial order $\preceq$ of adaptations to $\mathcal{M}_H$ to minimize structural impact)
- structural modifications remain in agreement with theory $\boldsymbol{T}_U$ (ensuring compatibility)

Let $\mathcal{M}_A$ be an amalgamated model constructed from all previously successful push-throughs. $\mathcal{M}_A$ ensures that $\mathcal{M}'_H$ is adapted to preserve coherence across the set of previously 'adapted' submodels $\mathcal{M}'_H$, $\mathcal{M}'_H$, $\mathcal{M}'^{\cdots\prime}_H$ *etc*.

[100] See Insert 1 and §3.1 for model-theoretic concepts and symbols

[101] *Ibid.* for *submodels*.

—$\mathcal{M}'_H$ is chosen such that modifications to $\boldsymbol{I}_H$, $\boldsymbol{D}_H$ and $\boldsymbol{L}_H$ are minimal under μ:[102]

$$\mu(\mathcal{M}_H \setminus \mathcal{M}'_H) = \mu\boldsymbol{I}_H+\mu\boldsymbol{D}_H+\mu\boldsymbol{L}_H, \text{ where } \mu\boldsymbol{I}_H \preceq \mu\boldsymbol{D}_H \preceq\mu\boldsymbol{L}_H$$

—where μ is a 'cost function', measuring the extent of structural changes in interpretation, domain, or subsignature, and imposing the partial order of $\mu\boldsymbol{I}_H \preceq \mu\boldsymbol{D}_H \preceq\mu\boldsymbol{L}_H$ to minimize $\mu\boldsymbol{I}_H+\mu\boldsymbol{D}_H+\mu\boldsymbol{L}_H$

Thus, $\mathcal{M}'_H$ satisfies ϕ while minimizing distinctions from the original receiving model $\mathcal{M}_H$ (*ie* μ is minimal), and all sentences ψ previously satisfied in $\mathcal{M}_H$ remain satisfied.

—Step 2 is repeated for previously successfully adapted submodels $\mathcal{M}'^{\cdots\prime}_H$ to ensure stable satisfaction of ϕ.

5. **Sheaf-Theoretic Amalgamation (Higher-Order Stabilization and Modification)** (§4.4.2)

Let $\mathcal{M}^+_A$ be a higher-order amalgamated model, created by combining higher-order structural elements across submodels.

Let $\mathcal{M}^+_A$ contain the structural elements necessary to represent the relationships captured by π.

— then, $\mathcal{M}^+_A$ stabilizes Model Adaptation by Shared Satisfaction, through safeguarding and refinement of partial isomorphism of π that supports reference sentence ϕ across all models, by:

- continuous adaption of $\mathcal{M}^+_A$ to the modifications of consecutive receiving submodels by sharing of ϕ and π from $\mathcal{M}_S$ to $\mathcal{M}_H$, to the next $\mathcal{M}'_H$ (and so on to $\mathcal{M}'^{\cdots\prime}_H$)
- ensuring that each new adaptation in Step 2 or 3 aligns with the shared structural framework of the theory T_U
- in case π fails due to major divergence in structure or cardinality, $\mathcal{M}^+_A$ can provide a reference for extracting a refined partial isomorphism π', incorporating conflicts from the last failing model $\mathcal{M}^\#_H$.
- this refined π' can then be used to reformulate ϕ' and continue the adaptation process.

6. **Reference Sentence Reformulation** (If Minimal Model Modification Fails: Back to Step 1)

If no $\mathcal{M}'_H$ can be constructed that satisfies the above conditions,

—then and only then is ϕ replaced by a modified sentence ϕ' such that:

- ϕ' is based in the refined partial isomorphism π', derived from $\mathcal{M}_A$ and $\mathcal{M}^+_A$ which are constructed by push-through stability and sheaf-theoretic amalgamation.
- **π**' is derived from the coordinated structures of $\mathcal{M}_A$ (first-order interpretations, Step 3, and $\mathcal{M}^+_A$ (higher-order coherence, Step 5), incorporating information from obstructing submodel $\mathcal{M}^\#_H$.
- Formally, π' extends the maximal partial isomorphism of $\mathcal{M}_A$ by incorporating relational constraints and coherence conditions introduced by $\mathcal{M}^+_A$, thereby facilitating the reformulation of ϕ into ϕ'.
- $\mathcal{M}_H \vDash \phi$', ensuring that ϕ' is satisfied in $\mathcal{M}'_H$ while preserving satisfaction in $\mathcal{M}_S$, maintaining shared satisfaction of ϕ' by all submodels previously satisfying ϕ
- ϕ' also maintains the features of a meaningful reference sentence within $\boldsymbol{T}_U$, as described in Step 1
- ϕ' initiates a new cycleof shared satisfaction and minimal adaptation (Step 1), re-entered into $\mathcal{M}_S$, $\mathcal{M}'_H$, $\mathcal{M}'^{\cdots\prime}_H$ *etc*.

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[102] A 'cost function' is not standard in model theory, but μ and the partial order $\Delta\boldsymbol{I}'_H \preceq \Delta\boldsymbol{D}'_H \preceq \Delta\boldsymbol{L}'_H$ formalize minimal adaptations of submodels, inspired by Gärdenfors's 'minimal change principle' that revisions of epistemic states should involve the smallest necessary change (Gärdenfors, 1988, 9–14, 66–68, "3.5—On the Notion of Minimal Change"). Gärdenfors does not quantify 'minimal change', but in graph-theoretic modeling, μ can be seen as a 'distance' controlling the likelihood of successful push-through (*cf* §5.3.1).

Simple formal example illustrating shared satisfaction (without adaptation) by two distinct submodels
Language signature $\boldsymbol{\Sigma}$ ($\boldsymbol{L}_{H\ldots S\ldots}$): one unary predicate P and one constant *c*.

- **Submodel $\mathcal{M}_S$:**
  - Domain $\boldsymbol{D}_S = \{1, 2\}$
  - Interpretation $\boldsymbol{I}_S$:
    - P interpreted as {1}
    - *c* interpreted as 1
- **Submodel $\mathcal{M}_H$:**
  - Domain $\boldsymbol{D}_H = \{a, b\}$
  - Interpretation $\boldsymbol{I}_H$:
    - P interpreted as {a, b}
    - *c* interpreted as b

Reference sentence $\boldsymbol{\phi}$: P(*c*)
Partial isomorphism π: the minimal shared structure needed to satisfy $\boldsymbol{\phi}$ = P(c**)**, consisting of:
- The element interpreting the constant *c* (1 in $\mathcal{M}_S$, b in $\mathcal{M}_H$)
- The predicate P holding for that element ($1 \in$ P in $\mathcal{M}_S$, $b \in$ P in $\mathcal{M}_H$)
- The preservation of this structure under push-through ensures ϕ is satisfied in both submodels.

Check satisfaction:

- In $\mathcal{M}_S$: ϕ = P(*c*) means P(1). Φ is true in $\mathcal{M}_{S,}$ since $1 \in P^{I_S}$
  (*ie* belongs to the set of all elements of $\boldsymbol{D}_S$ that satisfy P under interpretation $\boldsymbol{I}_S$.)
- In $\mathcal{M}_H$: ϕ = P(*c*) means P(b). Since $b \in P^{I_H}$, ϕ is true in $\mathcal{M}_H$.

Interpretation:

- Although $\mathcal{M}_S$ and $\mathcal{M}_H$ differ completely in domain and in extension of P, the same reference sentence ϕ is satisfied in both submodels.
- ϕ thus acts as a *bridging sentence* that both submodels agree on, providing a minimal point of shared satisfaction.

Shared Satisfaction by Minimal Adaptation (*cf* $\boldsymbol{A}_{MA\rightarrow SS}$*, §4.6, Insert 2*)

- **Same as above, but P in $\mathcal{M}_H$ now interpreted as {a}**: $P^{I_H} = \{a\}$
  - Reference sentence ϕ fails in $\mathcal{M}_H$, since P(*c*) with c interpreted in $\boldsymbol{I}_H$ as $b \notin P^{I_H}$:
- **Minimal Adaptation:**
  - Modify $\boldsymbol{I}_H$ minimally to $\boldsymbol{I}'_H$ by expanding the interpretation of P: $P^{I_H} = \{a\} \longrightarrow P^{I'_H} = \{a,b\}$

Now ϕ = P(*c*) is true in $\mathcal{M}'_H$, restoring shared satisfaction.

4.6 $\boldsymbol{A}_{MA\rightarrow SS}$: Axiom of Model Adaptation by Shared Satisfaction[103]

Sections 3.2.1–3.2.3 examined the Löwenheim-Skolem consequences for model cardinalities, submodel variants, and satisfaction across models. A major source of indeterminacy is fragmentation of a full model $\mathcal{M}_U$ of $\boldsymbol{T}_U$, whose subsets of sentences may be satisfied by many distinct submodels. Sections 4.3–5 introduced first- and higher-order push-through techniques to harmonize submodels by ensuring shared satisfaction of a reference sentence ϕ. This formal sentence represents a core structural principle of $\boldsymbol{T}_U$, and preserving its substructure **π** across submodels is central to $\boldsymbol{T}_U$'s coherence. Model theory provides no rules to achieve this. I have therefore formulated a meta-theoretical axiom schema. $\mathcal{M}_U$ contains many subsets of sentences satisfied by different submodels, and a reference sentence ϕ may not hold in all of them. If ϕ is true in $\mathcal{M}_U$, it can be made true in all submodels capable of incorporating its structure. This is done through progressive presentation of ϕ and minimal structural adjustments when satisfaction fails. Each attempt—successful or failed—extends ϕ's reach and/or refines it. If the process succeeds, ϕ unifies $\mathcal{M}_U$'s fragments, reducing fragmentation-induced indeterminacy.
A meaningful reference sentence ϕ may not be satisfied in all submodels—but if ϕ is true in $\mathcal{M}_U$,

[103] See Insert 2.

it can be (made) true in all submodels capable of incorporating the necessary structure. This is achieved through progressive presentation of ϕ to submodels and minimal structural adjustments if satisfaction fails. Each attempt—successful or failed— increasing ϕ's reach or refines it. If the process succeeds, ϕ unifies $\mathcal{M}_U$'s fragments, reducing fragmentation-induced indeterminacy.

Insert 2 formalizes *Model Adaptation by Shared Satisfaction* in a stepwise axiom schema ($A_{MA\to SS}$). $A_{MA\to SS}$ governs the progressive confirmation or refutation of the truth of ϕ, by repeated sharing across submodels of $\mathcal{M}_U$. $A_{MA\to SS}$, along with the graph dynamics in Chapter 5, identifies failures and guides systematic revisions. The goal is to minimally adapt submodels or refine ϕ, until all partial models can satisfy it without internal disruption.

Step 1 entails the formulation on the initial reference ϕ, or its modified version ϕ' after $A_{MA\to SS}$. As considered in §4.1 and §5.3.1, the speed and success of this process depend on strategies for revision and iteration and the integrative force of ϕ. More philosophical and scientific aspects of ϕ will be discussed in Chapter 6.

Step 2 (Basic Transfer) occurs when ϕ's presenting model $\mathcal{M}_S$ shares $\boldsymbol{\pi}$ with a receiving submodel $\mathcal{M}_H$, which satisfies ϕ without needing changes. $\mathcal{M}_H$ can then propagate ϕ to other submodels $\mathcal{M}'_H$.

Step 3 secures agreement between the models by push-through stability, which amalgamates $\mathcal{M}_S$ and $\mathcal{M}_H$ into $\mathcal{M}_A$, preserving $\boldsymbol{\pi}$ and other shared structures. Each successful transfer increases coherence. Push-through may also fail (§*4.4.2.2*). $\mathcal{M}_S \vDash \phi$ but $\mathcal{M}_H \vDash \neg\phi$ if $\mathcal{M}_H$ lacks or contradicts parts of $\boldsymbol{\pi}$ – which makes it a 'fractured' submodel $\mathcal{M}^{\#}_H$.

Step 4 introduces Minimal Model Modification of $\mathcal{M}^{\#}_H$ using structural elements from $\boldsymbol{\pi}$ recorded in Step 3. To minimize structural disruption, modification proceeds in a preferred order, reflecting increasing scope of impact within the model:

1. $\Delta \boldsymbol{I'}_H$: reinterpretation—modifying interpretation functions while preserving domain and subsignature—*ie* reassigning meanings to constants, predicates, or relations.
2. $\Delta \boldsymbol{D'}_H$: domain expansion—adding individuals or types missing from $\mathcal{M}^{\#}_H$.
3. $\Delta \boldsymbol{L'}_H$: signature extension—introducing new symbols or structures needed to express ϕ.

All changes must preserve sentences ψ previously satisfied in $\mathcal{M}^{\#}_H$. The goal is to revise $\mathcal{M}^{\#}_H$ into a submodel $\mathcal{M}'_H$ that satisfies ϕ and retains its original ψ. If some obstructing ψ resists after revision, is $\mathcal{M}^{\#}_H$ skipped.

Step 4 may fail if modifications disrupt prior truths in $\mathcal{M}^{\#}_H$. $\Delta \boldsymbol{I'}_H$, $\Delta \boldsymbol{D'}_H$, or $\Delta \boldsymbol{L'}_H$ can fail due to incompatible assignments, missing types, or incompatible terms. Adaptation failure does not imply ϕ is false, but that $\boldsymbol{\pi}$ and $\mathcal{M}^{\#}_H$ are too distinct for shared satisfaction. This indicates deeper fractures between $\mathcal{M}^{\#}_H$ and MU, assuming ϕ accurately represents $\boldsymbol{T}_U$.

Step 5 becomes active when Step 4 fails. This invokes the amalgamated structure $\mathcal{M}^+_A$ (§*4.4.2.1*). $\mathcal{M}^+_A$ accumulates structural categories from all push-through attempts via Sheaf-Theoretic Amalgamation. When minimal adaptation of $\mathcal{M}^{\#}_H$ fails, $\mathcal{M}^+_A$ extracts a refined $\boldsymbol{\pi}'$ capturing the divergence. This structural information supports ϕ's reformulation into ϕ' and a renewed adaptation attempt. As explained in §*4.4.2.2*, $\mathcal{M}^+_A$'s higher-order function guides repair and coherence. It can distinguish between fractured models ($\mathcal{M}^{\#}_H$) and alien ones ($\mathcal{M}^*$): $\mathcal{M}^{\#}_H$ may be reconstructable; $\mathcal{M}^*$ may be skipped as a probable part of another theory $\boldsymbol{T}^*$—although it could be also revisited after each revision of ϕ.

Step 6 concludes $\boldsymbol{A}_{MA\to SS}$. When minimal adjustments fail, ϕ is refined into ϕ' (or ϕ' into ϕ'' *etc*) using $\mathcal{M}^+_A$ higher-order feedback. If even these refinements cause contradiction in some models—*eg* due to conflicting ψ-sentences—then ϕ is abandoned as unsatisfiable within $\boldsymbol{T}_U$. This initiates an new repeated testing and revision of ϕ'. Each cycle increases shared satisfaction or eventually rules ϕ out as a ubiquitously shared reference sentence. While the

Compactness Theorem (§2.2.2) guarantees that global satisfaction is possible when all local adaptations succeed, it offers no procedure for guiding those adaptations. $\boldsymbol{A}_{MA \to SS}$ fills this gap. It governs how ϕ is repaired or withdrawn, and how $\mathcal{M}_U$'s fragments are revised or bypassed.

This leads to an important conclusion. If (and only if) the collection of submodels fully captured $\mathcal{M}_U$ and $\boldsymbol{A}_{MA \to SS}$ successfully integrated them, ϕ could in principle be formally established as true throughout $\mathcal{M}_U$. Compactness guarantees this theoretical possibility, while $\boldsymbol{A}_{MA \to SS}$ shows how ϕ is progressively satisfied across submodels through minimal adaptation. In practice (as §§3.2.2-3 make clear) submodels cover only fragments of $\mathcal{M}_U$. This limitation motivates viewing $\mathcal{M}_U$ as a dynamic network of partially aligned submodels, progressively integrated through adaptive coherence. Chapter 5 develops this perspective by representing and analyzing the interactions among submodels and the emergent structure of their incremental alignment.

Chapter 5. Graph Theory and the Dynamics of Model Adaptation

5.1 Essentials of Graph Theory [104]

Like model theory, graph theory plays an important role in both science and philosophy. Graph-like systems have long supported logical and conceptual representation—most famously in Frege's *Begriffsschrift* (1879) [105] and Peirce's existential graphs (1885).[106] In contemporary philosophy, graph theory often takes a formal mathematical form, yet as Daniel Parrochia argues in *Graphs, Orders, Infinites and Philosophy*, mathematics itself is a primary source of intelligibility.[107] Graph theory, he suggests, might even serve as a *philosophical system*: a structural account of reality, or a theory that ultimately reveals why such a totalizing account may fail. Others, like Nicholas Shackel, defend metaphysical graphical structuralism, suggesting that the world itself may be modeled as a graph.[108] Graphs can represent not only formal but also metaphysical structures.[109]

In this thesis, I adopt a more focused use of graph theory: to model how submodels of a fragmented theoretical model $\mathcal{M}_U$ interact in response to the shared satisfaction of a scientifically meaningful reference sentence ϕ. Graph structures help trace how ϕ and its core structure $\boldsymbol{\pi}$ propagate through a network of submodels, helping visualize and assess coherence within the broader framework $\boldsymbol{T}_U$. Formally, a graph $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{E})$ is a structure of vertices (or nodes) $\boldsymbol{V}$ connected by edges $\boldsymbol{E}$, which represent relationships or interactions. Graphs are widely used to model relational systems in disciplines ranging from linguistics and computer science to theoretical physics.[110] Here, I treat *submodels* as nodes, and their shared reference sentences or structural overlaps as edges. This lets us treat model adaptation by shared satisfaction in a graph-theoretic way.

Graphs can also be homogeneous or heterogeneous, depending on node type. Model adaptation by shared satisfaction primarily relies on homogeneous isomorphisms: overlaps between models that satisfy the same reference sentence. However, the submodels themselves are heterogeneous, often acting as host graphs that embed isomorphic subgraphs or minors. This higher-order structure is relevant for later discussion of hypergraphs and sheaf-theoretic amalgamation in §5.3.2. Graph theory contains many additional technical tools—such as clustering, centrality, and percolation theory—but I will focus only on those essential to modeling adaptation dynamics in fragmented model systems.

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[104] Extracted from Wilson (2010), and Parrochia (2023), "1-Graphs", 6-9, 11-20
[105] Frege (1879): *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denken*.
[106] Bellucci & Pietarinen 2019): *Pierce, Charles Sanders - General Introduction to Logic of the Future*: *Writings on Existential Graphs*
[107] Parrochia (2023), ix-xv, x
[108] Shackel (2011)
[109] Parrochia (2023), 171, 181: "Assume (again) that the structure of the world is a graph."
[110] *Ibid*.

5.2 Random Graph Evolution and the Erdős–Rényi Model[111-112]

Random network theory, developed by Paul Erdős and Alfréd Rényi[113] (and independently by Gilbert[114]) , offers a formal framework to analyze how connections between nodes emerge and consolidate into large-scale structures. In §5.3, I will draw on this theory to model how the $\boldsymbol{A}_{MA\to SS}$-axiom (explained in §4.6) allows a single reference sentence ϕ—true in our Unified Theoretical Framework $\boldsymbol{T}_U$ and in a presenting 'speaker' model $\mathcal{M}_S$—to progressively establish a coherent network of submodels of $\mathcal{M}_U$ that all satisfy ϕ and share its partial isomorphism $\boldsymbol{\pi}$.

A key property in network evolution is the degree of a node—the number of edges it shares with others. This metric profoundly shapes the global topology of the network, including how components form, expand, and coalesce. The Erdős–Rényi–Gilbert model captures this process in its standard form $\boldsymbol{G}(n, p)$, where n is the number of nodes and p the probability that any given pair is connected. While this model is formally probabilistic, it also approximates the dynamic growth of a network as new links are established. At low p-values, the graph consists of scattered small components. As p increases, these gradually merge into larger structures. Once a critical threshold is crossed, one component outgrows the rest, forming a *giant component* that spans a large part of the network. Figure 2 illustrates this consolidation in a 1000-node graph near the critical edge probability.

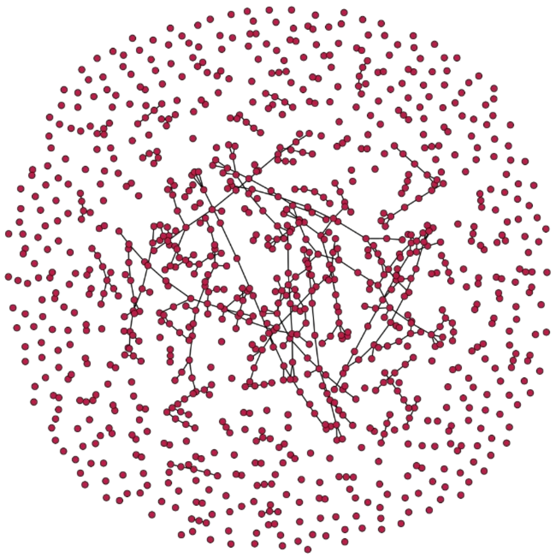

Figure 2: Erdős–Rényi–Gilbert graph with 1000 vertices at the critical edge probability ($\boldsymbol{k}$ = 1) – see explanation in §5.2.[115]

Erdős–Rényi-dynamics describe how these networks evolve. Formally, the total number of possible edges in a graph of n nodes is $n\cdot(n-1)/2$. The parameter p defines the proportion of these edges that are present. That is, p is the probability that any given pair of nodes is connected; this is equivalent to the fraction of all possible edges that are realized.[116] For example, p = 0.1 yields 49,950 edges in a 1000-node graph (out of 499,500 possible). The average degree k of a node is given by $k = p\cdot(n\text{-}1)$. As p (and thus k) increases, disconnected fragments begin to merge into larger components. The graph's global structure is highly sensitive

[111] Barabási (2018), 49-61
[112] Oh & Monge (2016), 9
[113] Erdős & Rényi (1960)
[114] Barabási (2018), 49
[115] <https://en.wikipedia.org/wiki/Erd%C5%91s%E2%80%93R%C3%A9nyi_model#/media/File:Critical_1000-vertex_Erd%C5%91s%E2%80%93R%C3%A9nyi%E2%80%93Gilbert_graph.svg>
[116] The concept of p is slighly confusing. Strictly speaking, 'p' stands for a probability, but it is also treated as a proportion or density parameter reflecting the fraction connected vertices in the graph.

to this process. When k passes 1 (*ie* when $p \approx \ln(n)/n$), the giant component mentioned earlier appears. At this point, the largest connected component suddenly includes a substantial fraction of all nodes. The proportion of nodes in this component, denoted $\boldsymbol{\rho}$ (rho), satisfies the self-referential equation:

$\boldsymbol{\rho}$ = 1 − e^(−k·$\boldsymbol{\rho}$)

This defines the characteristic *S*-shaped transition curve shown in Figure 3, where $\boldsymbol{\rho}$ increases non-linearly as k rises. The graph-model represents this as the 'completion' of the network.[117]

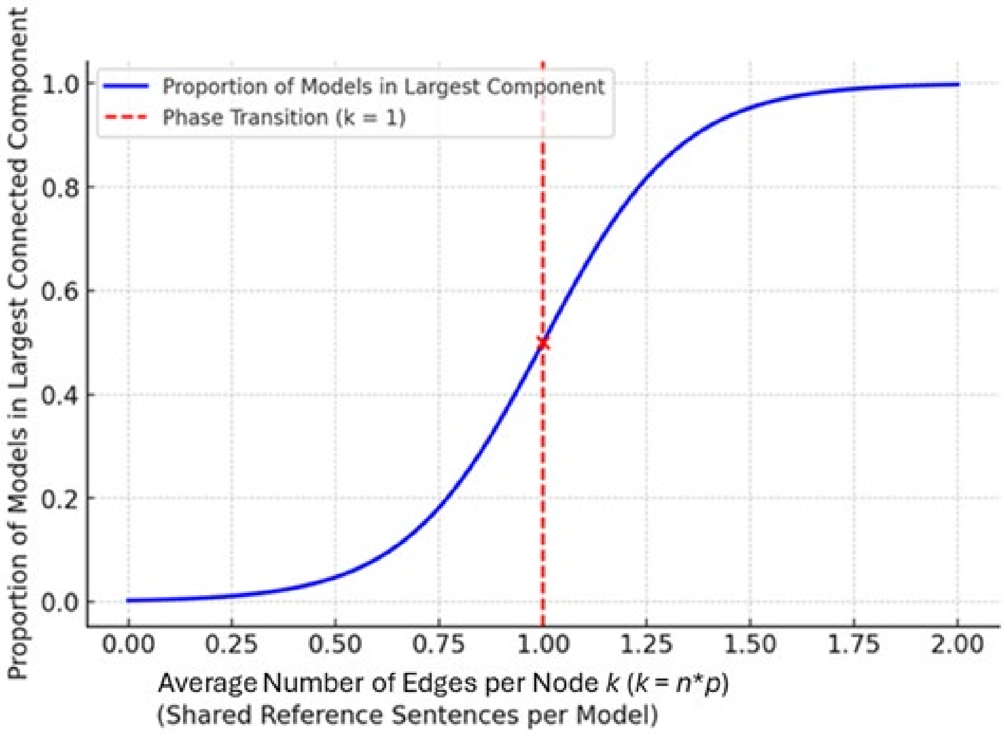

Figure 3: Stabilization of Reference through Model Connectivity, where ***k*** *increases by broader 'sharing' of ϕ across submodels.*[118]

This *connectivity transition*—from fragmentation to integration—has clear implications for the dynamics of *MASS*. The reference sentence ϕ plays a structural role analogous to the probabilistic edge in the Erdős–Rényi model: each instance of ϕ being satisfied across two partial models, effectively creates a link in the interpretive network. As more such links are established, previously disjoint submodel fragments become structurally integrated. This process will be explored further in §5.3.[119]

5.3 Graph Theory Applied to Model Adaptation by Shared Satisfaction

5.3.1 Model Adaptation in Random Graph Models

The progressive dissemination of a reference sentence ϕ across an increasing number of submodels of $\mathcal{M}_U$ can be fruitfully analyzed using graph theory, which offers a mathematical framework for studying the structure and evolution of interconnected systems. Here, the nodes of the graph represent the submodels of the fragmented scientific model $\mathcal{M}_U$, and the edges

[117] Barabási (2018), 58: Image 3.6.a) Evolution of a Random Network.

[118] Adapted from Erdős & Rényi (1960) and *Introduction to Network/Graph Theory*—slide 61 <https://www.cl.cam.ac.uk/teaching/1011/PrincComm/slides/graph_theory_1-11.pdf>

[119] Phase transition in model coherence points to a certain 'persuasive force' of ϕ. This has an interesting analogy with Kuhn's 'paradigm shift'. Further consideration of this is beyond the scope of this thesis.

represent connections formed by the shared satisfaction of ϕ and its supporting partial isomorphic structure $\boldsymbol{\pi}$.

The successful push-through of ϕ from a presenting submodel $\mathcal{M}_S$ to a receiving submodel $\mathcal{M}_H$ establishes a new edge between these nodes, thereby increasing the proportion of submodels $\mathcal{M}'_H$ within $\mathcal{M}_U$ that satisfy ϕ. This process can be represented as an Erdős–Rényi random graph model, where each new push-through event increases the probability parameter p of edge formation as illustrated in Figure 3. If ϕ and $\boldsymbol{\pi}$ can be pushed through completely—without generating conflicts or contradictions in any submodel— ϕ will function as a dominant or 'giant' scientific formula common to all submodels of $\mathcal{M}_U$. The resulting network of interconnected submodels strengthens coherence across the fragmented structure, and widespread satisfaction of ϕ imposes strong constraints on interpretational flexibility arising from model fragmentation.

However, *Model Adaptation by Shared Satisfaction* does not assume that ϕ can always be pushed through without modification. The network's growth will halt at certain submodels where minimal adaptation of $\mathcal{M}_H$ is insufficient to satisfy ϕ. $\boldsymbol{A}_{MA\to SS}$ dictates that these 'obstructing' submodels $\mathcal{M}^{\#}_H$ are temporarily bypassed until the reference sentence ϕ and its partial isomorphism $\boldsymbol{\pi}$ are revised to ϕ' and $\boldsymbol{\pi}'$ (Step 6). Meanwhile, push-through continues with free nodes, connecting additional unexposed submodels and also revisiting established branches of the graph to verify persistent satisfaction. Through this iterative process, the connected subset $\mathcal{C}_{\boldsymbol{\pi}} = \{\mathcal{M}'_H \subseteq \mathcal{M}_U\}$ grows and solidifies,[120] whereas severely fractured submodels $\mathcal{M}^{\#}_H$ and structurally incompatible 'alien' submodels $\mathcal{M}^{*}_H$ remain disconnected.

Figure 3 suggests an idealized scenario where the connected subset $\mathcal{C}_{\boldsymbol{\pi}}$ approaches completeness, with proportion $\boldsymbol{\rho}$ near 100%. In practice, however, the value of $\boldsymbol{\rho}$ depends on several interrelated factors.

First, it hinges on how well ϕ captures the core structure of the Unified Theoretical Framework $\boldsymbol{T}_U$ as represented in $\mathcal{M}_U$. If ϕ expresses a central and ubiquitously shared aspect of $\boldsymbol{T}_U$ preserved across submodels, $\boldsymbol{\rho}$ may approach 1 However, if ϕ is too restrictive or context dependent (on structural elements that are not pushed through), then ubiquitous satisfaction is unattainable and $\boldsymbol{\rho}$ remains low.

Second, the degree of fragmentation in $\mathcal{M}_U$ influences the number of obstructing submodels $\mathcal{M}^{\#}_H$. These submodels initially resist adaptation despite minimal modifications and are bypassed until ϕ' and $\boldsymbol{\pi}'$ are introduced. Each successful revision can increase $\boldsymbol{\rho}$ by 'converting' previous $\mathcal{M}^{\#}_H$-submodels to join $\mathcal{C}_{\boldsymbol{\pi}}$.

Third, some alien submodels $\mathcal{M}^{*}_H$ may be structurally incompatible with any viable revision of ϕ, limiting the maximal achievable $\boldsymbol{\rho}$ to less than 1. As argued in §*4.4.2.2*, this could occur when $\mathcal{M}^{*}_H$ belongs to another theory $\boldsymbol{T}^{*}$. In this case, the impact will be smaller the more $\boldsymbol{T}^{*}$ differs from $\boldsymbol{T}_U$, since this will reduce the chance that their submodels are similar. Large proportions of alien submodels, with a substantial impact on $\boldsymbol{\rho}$, can also point to the existence of $\mathcal{M}^{+}_U$ with a higher cardinality than $\mathcal{M}_U$.

(Finally, the ideal situation shown in Figure 3 requires huge numbers of nodes. This important prerequisite of Erdős–Rényi models will be discussed in the next subsection §5.3.2.)

The process of revising ϕ and $\boldsymbol{\pi}$ in Step 6 of $\boldsymbol{A}_{MA\to SS}$ can vary. Revision may occur after each individual failure or after a batch of push-through attempts, depending on the chosen strategy. This revision is informed by the results of two amalgamation steps: Step 2 (push-through stability) and Step 5 (sheaf-theoretic amalgamation). The first aggregates information from local push-through successes, progressively generalizing $\boldsymbol{\pi}$. The second collects higher-order structural information from both successes and failures. Patterns common to obstructing

[120] Let $\mathcal{C}_{\boldsymbol{\pi}} = \{\mathcal{M}'_H \subseteq \mathcal{M}_U \mid \boldsymbol{\pi}$ is a partial isomorphism between $\mathcal{M}_S$ and $\mathcal{M}'_H$, and ϕ is satisfied in $\mathcal{M}'_H$ via $\boldsymbol{\pi}\}$

submodels $\mathcal{M}^{\#}_{H}$ may reveal systematic causes of failure—for instance, distinguishing between fractured and alien submodels—thus guiding the formulation of a refined ϕ'.

The revision strategy and the modifications of ϕ and $\mathcal{M}_H$ are essentially epistemic, but graph theory offers mathematical tools to model the impact of different scenarios on the growth or size of the subset $\mathcal{M}'_H$ that satisfy ϕ. A key parameter is the Erdős–Rényi probability p,[121] reflecting the likelihood that any two nodes (submodels) are connected by the partial isomorphism $\boldsymbol{\pi}$. Although push-through is not probabilistic in the sense of randomly distributed edges, modeling edge formation as governed by p offers a useful approximation of the evolving connectivity within the ϕ-satisfying subgraph $\mathcal{C}_{\boldsymbol{\pi}}$. Another critical parameter is the cost function μ from $\boldsymbol{A}_{MA\rightarrow SS}$ (Step 4), which ensures minimal structural changes—such as interpretation shifts, domain expansions, or syntactic differences—needed to accommodate ϕ in $\mathcal{M}_H$. In graph-theoretic terms, μ can be interpreted as a measure of 'distance' between $\mathcal{M}_H$ and its minimally adapted variant $\mathcal{M}'_H$, influencing edge weights or thresholds: submodels requiring larger μ are less likely to form successful push-through connections, while smaller μ corresponds to higher likelihood of connectivity in $\mathcal{C}_{\boldsymbol{\pi}}$.[122] For the formal specification of μ and the ordering of adaptations ($\Delta\boldsymbol{I}'_H \preceq \Delta\boldsymbol{D}'_H \preceq \Delta\boldsymbol{L}'_H$), see Step 4 of Insert 2. Translating μ into edge weights or thresholds could model the probability of successful push-through more precisely.

Higher-order amalgamation results may correspond to analyses of more complex graph structures, such as hypergraphs, representing the sheaf-model $\mathcal{M}^{+}_{A}$ (Step 5, §*4.4.2.2*) which shares more structural elements than just $\boldsymbol{\pi}$, or revealing clusters of obstructing submodels. These advanced graph-theoretic approaches go beyond Erdős–Rényi theory, reflecting the nuanced and dynamic nature of model adaptation by shared satisfaction.

In sum, graph theory provides an insightful framework for modeling how a reference sentence propagates through a fragmented scientific model. The approach discussed so far reveals useful aspects of the dynamics of adaptation, resistance, and coherence within $\mathcal{M}_U$—but it also shows that the process of submodel integration is actually quite complicated.

5.3.2 Model Adaptation in Dynamic Networks

While the Erdős–Rényi random graph model offers a useful first approximation, its assumptions of structural symmetry and statistical uniformity—such as Poisson distributions assuming random connectivity—are too restrictive to capture the complex, asymmetric and iterative interactions underlying *Model Adaptation by Shared Satisfaction*. In reality, model adaptation involves heterogeneous connection patterns and feedback-driven adjustments that also affect the nodes (submodels) and edges (ϕ, $\boldsymbol{\pi}$, $\mathcal{M}^{+}_{A}$) themselves, and therefore produce a richer, more coherent network with stronger and weaker subgraphs. In practice, therefore, push-through interactions are unlikely to remain purely bidirectional or uniform. Once the connected subset $\mathcal{C}_{\boldsymbol{\pi}}$ becomes dominant, a small number of highly connected 'presenting' submodels $\mathcal{M}_S$ will serve as hubs that can exert disproportionate influence over numerous receiving $\mathcal{M}_H$ nodes. Such heterogeneous connectivity patterns violate the Erdős–Rényi assumptions, which rely on large numbers of nodes and edges. Only densities of submodels of $\mathcal{M}_U$ like those in Figure 2 will lead to approximation of ubiquitously shared satisfaction as shown in Figure 3. It is not immediately apparent, however, how $\mathcal{M}_U$ could fragment into so many different $\mathcal{M}_S/\mathcal{M}_H$-submodels (although this becomes more realistic if we think of every scientist as a 'speaker' and a 'hearer', exchanging their own submodel versions).[123]

For more limited or realistic scientific theories like biological diversity, alternative network paradigms such as scale-free or small-world networks may better approximate adaptation

[121] See footnote 116
[122] See footnote 120
[123] See footnotes 71 and 74

dynamics. These models account for heterogeneous degree distributions and clustered connectivity, respectively, reflecting the presence of hubs and community structure. These more complex networks typically impose greater constraints on the variability and freedom of individual nodes and edges, resulting in richer dynamical behaviors. For example, if ϕ is widely accepted, repeated sharing across model nodes with multiple connections can lead to the emergence of a power-law degree distribution, where a few nodes have many connections while most have few.[124] This pattern contrasts with Erdős–Rényi's uniform random degree expectations. Persistent hubs connected to dissociated graphs within the set of partial models of $\mathcal{M}_U$ can reveal alien $\mathcal{M}^*_H$ submodels that compete with ϕ. As discussed in the previous section and §*4.4.2.2*, such competition may represent an unrecognized structural regularity within the Unified Theoretical Framework $\boldsymbol{T}_U$, caused by an alternative theory $\boldsymbol{T}^*_U$ or a higher-cardinality model $\mathcal{M}^+_{U.}$

Given the epistemic parallels of $\boldsymbol{A}_{MA\rightarrow SS}$ dynamics, and the increasing prominence of graph and network theory in different scientific areas as mentioned at the start of this Chapter, dynamic network models can provide important tools for understanding the processes that reduce indeterminacy in scientific theorizing. This Chapter used a basic Erdős–Rényi model to show the principles of submodel integration, but graph theory in general supports the broader aim of this thesis: to outline a theoretical framework that systematically reduces the indeterminacies of theories, models, and reference structures by anchoring truth in coherence and relational constraints within model structures, rather than isolated satisfiability or external reference. Graph theory thus contributes not only structural support for model coherence but also a concrete mathematical pathway for mitigating the radical indeterminacy associated with model-theoretic interpretation—central to Putnam's argument. In Chapter 6, I will explore how these model- and graph-theoretic concepts relate to Putnam's own considerations of indeterminacy in science, and with some practical consequences of this.

[124] Oh & Monge (2016), 10: Fig.7

Chapter 6. Discussion and Integration

This final chapter brings together the formal framework of *Model Adaptation by Shared Satisfaction* (*MASS*) with philosophical and scientific issues related to indeterminacy in scientific theorizing. I will begin by situating *MASS* in relation to Putnam's Model-Theoretic Argument, focusing specifically on his challenges of scientific theory and reference. I will then apply the *MASS* framework to biological diversity, illustrating how scientific advances could provide the basis for formulating a realistic reference sentence that can be satisfied by diverse scientific models of related phenomena. The aim is to demonstrate how *MASS* offers a rigorous pragmatic response to longstanding philosophical problems of indeterminacy that are directly relevant for scientific progress.

6.1 Putnam's Model-Theoretic Argument and the Challenge of Model Fragmentation

Hilary Putnam's Model-Theoretic Argument (MTA) remains one of the most influential critiques of metaphysical realism in the philosophy of science. Drawing on the Löwenheim–Skolem theorems, Putnam showed that even rigorously formalized scientific theories admit multiple, inequivalent models that satisfy the same axioms. While any single model can be used effectively for explanations or predictions, the theory itself does not determine a unique model. Putnam opposed the thesis of metaphysical realism that the objects, properties, and relations in the world exist independently of our thoughts or perceptions about them.[125] His argument undermines a central metaphysical realist assumption: that there is a single, theory-independent mapping between language and the external world. Since a first-order theory permits many models, no unique structure in THE WORLD is determined solely by a theory, and reference is therefore not fixed by the theory alone.

Internal realism emerged as a response to this indeterminacy.[126] Rejecting the metaphysical realist's "God's-eye view," Putnam argued that reference is determined not by correspondence with an external reality, but by use within a conceptual, cognitive-linguistic framework.[127] "The world does not pick models or interpret languages. We interpret our languages or nothing does."[128] Objects, signs, and meanings are internal to this scheme of description; reference becomes meaningful only within it.[129] This position has been criticized for blurring the line between truth and justification, and for inviting cognitive subjectivity and relativism, by Lewis,[130] Devitt,[131] Van Fraassen.[132] and others.[133] Putnam's responses evolved over time, prompting some commentators to remark that writing about his philosophy "is like trying to capture the wind with a fishing-net".[134,135]

Ultimately, Putnam adopted *natural realism*,[136] as a form of realism intended to avoid conflict

[125] Khlentzos (2025), 2-8: "1-What is Metaphysical Realism?"
[126] Putnam (1981), 50
[127] *Ibid.* (footnote 25)
[128] Putnam (1980), 482
[129] Putnam (1981), 50
[130] Lewis (1984)
[131] Devitt (1983)
[132] Van Fraassen (1997)
[133] Putnam's Internal Realism and Model-Theoretical Argument (1977-1990), *PhilPapers*, accessed August 10. 2025, <https://philpapers.org/browse/internal-realism>
[134] Passmore (1985), 92
[135] Szubka (2024)
[136] Button (2013), 82-95, 82: '[N]atural realism, [w]as announced most fully in 'Sense, Nonsense, and the Senses' (1994) and *The Threefold Cord* (1999).'

with science.[137, 138] Its exact meaning has been debated: Michael Dummett for instance confessed that "to divine what it is defeats me,"[139] and Tim Button devotes an article and a chapter to it—[140-141]concluding on model-theoretic grounds that natural realism has yet to offer persuasive arguments against skepticism. In broad terms, natural realism is a pragmatic position consistent with how most scientists work: they construct accounts and models of the world that reflect different aims and perspectives, while presupposing that there is a world about which verifiable things can be said. Few claim to make unique, exhaustive mappings of reality, and fewer still see that as problematic.

Yet, as I have argued, the MTA's core insight—the non-uniqueness of models—remains significant, particularly in light of the growing fragmentation of scientific practice. As theories develop, they often diverge into specialised submodels tailored to distinct domains, methods, and explanatory aims. This fragmentation is not merely disciplinary; it reflects deep structural differences in how models represent reality. Even within a single field, such as physics or biology, competing frameworks may coexist without formal integration. The mosaic-like nature of contemporary science raises a pressing question: how can reference remain stable across structurally incompatible models?

Cartwright's image of a 'dappled' world[142] captures how perceived inherent fragmentation can obstruct the transfer of advances from one domain to another. Putnam acknowledged a related tension in *Models and Reality* (1980), where he presents a thought experiment involving an 'ideal scientific theory' T_1—a theoretical endpoint or rational limit of scientific progress[143]—contrasted with our current, partial theory T. Putnam conceived of T_1 as 'a limit of theories' that it would be rational for scientists to accept, even if never fully realised—what Lewis called 'futuristic'[144]—but concrete enough to guide progress as a regulative ideal.[145] By contrast, T reflects our fallible, incomplete knowledge. The distinction parallels scientific fragmentation: T_1 gestures toward a unified framework—illustrated, perhaps naively, by Linnaeus' taxonomy of five kingdoms as a divinely ordered system of life—while T represents a submodel, such as evolution by 'survival of the fittest'.

Model Adaptation by Shared Satisfaction (*MASS*) builds on this distinction, reframing it in model-theoretic terms. *MASS* could extend Putnam's original position of internal realism—for which model theory offers more support than he may have recognized[146]—by clarifying how reference can be coordinated across heterogeneous model spaces. However, rather than adopting Putnam's concept of T_1 as a transcendent endpoint, *MASS* considers a Unified Theoretical Framework ($\boldsymbol{T}_U$) to constitute a pragmatic coordination point for consensus. $\boldsymbol{T}_U$ is not a metaphysical ideal but is specified within *MASS* by reference to sentences jointly satisfied across submodels. The accompanying $\boldsymbol{A}_{MA\rightarrow SS}$ axiom schema functions both as a regulative guide and as a formal mechanism for reconciling divergent models. If the MTA suggests that reference is unstable across models, *MASS* uses this schema to connect them through shared satisfaction of a scientifically relevant reference sentence (bridging hypothesis), enabling partial

[137] Putnam (1994), 465: '[T]here is no conflict between natural realism and science, [nor] between a suitably commonsensical realism about our conceptual powers and science'.
[138] Hildebrand (2000), 109-132
[139] Dummett (2005), in Auxier & Hahn (2007), 168–184: "Reply to Hilary Putnam," 182
[140] Button (2016)
[141] Button (2013), 82-95: "Chapter 10–Natural Realism," 95: 'The natural realist [rightly] aim[s] for a position according to which Cartesian angst does not arise, but she has not given us one.'
[142] Cartwright (1999)
[143] Putnam (1980), 473
[144] Lewis (1984), 230-231
[145] Putnam (1980), 473—see footnote 84
[146] Button & Walsh (2018), 46; 'Internalism about model theory reveals a new understanding of Putnam's internal realism.'

models to adapt, align, and cohere without requiring full unification. In this way, *MASS* preserves the pluralism of scientific practice while offering a framework for conceptual coordination.

In *Philosophy and Model Theory*, Button and Walsh note that Putnam—perhaps implicitly—relies on the push-through construction to challenge metaphysical realism.[147] As discussed in §4.3, model theory supports permutation functions that translate elements across models during push-through. Putnam's argument is that metaphysical (external) realism cannot explain why our terms like "cat" to refer to actual cats, rather than arbitrarily permuted interpretations (for instance, naming them 'dogs', see §3.2.3). [148] Since permutation is a one-to-one bidirectional function, model theory places no internal constraints on it. Therefore, as Button and Walsh observe, indeterminacy 'generalises rapidly' under consecutive permutations.[149] Importantly, however, push-through in *MASS* is structure-preserving, in clear contrast with Putnam's original use of push-through arguments. In $\boldsymbol{A}_{MA\to SS}$, push-through from a submodel $\mathcal{M}_S$ to another $\mathcal{M}_H$ preserves structural alignment, thereby maintaining satisfaction across the network of fragmented models within $\mathcal{M}_U$. Putnam's permutational push-through, however, alters the interpretation function over the domain, disrupting shared satisfaction and leading to indeterminacy. While minimal modifications to the interpretation function, domain, and language—denoted $\Delta \boldsymbol{I}'_H \preceq \Delta \boldsymbol{D}'_H \preceq \Delta \boldsymbol{L}'_H$ in Step 4 of $\boldsymbol{A}_{MA\to SS}$ (Insert 2)—also involve reinterpretation, these are tightly constrained by the ongoing requirement that the reference sentence ϕ remains satisfied after alignment of all submodels. This preservation of satisfaction ensures that reference does not collapse into indeterminacy by permutation but remains coherently regulated across fragmented and adapted models.

Putnam's own framework—though not explicitly mentioning indeterminacy by model fragmentation or potential responses to it—anticipates aspects of this approach. His recognition that "there has to be a determinate relation of reference between terms in ***L*** and pieces (or sets of pieces) of THE WORLD"[6] suggests a model-theoretic awareness of fragmentation. In *MASS* terms, ***L*** corresponds to the vocabulary of a theory T_1 (like $\boldsymbol{T}_U$), 'pieces' represent partial submodels, and 'reference' reflects the dynamic relations among them. While Putnam emphasized epistemic ideals, *MASS* translates these into structural mechanisms for model adaptation. The conceptual overlap between his thought experiment and *MASS* also suggests that *MASS* could address skeptical concerns arising from the MTA's conclusion—that we can never be certain which reference best represents *the* world. If T_1 resembles $\boldsymbol{T}_U$ in its harmonizing role, while T reflects the situated nature of scientific models, *MASS* extends this by formalizing the conditions under which partial models can share satisfaction, thus enabling reference to persist across fragmentation.

As argued in §4.6, compactness shows that in principle *MASS* could procedurally establish ϕ as a true sentence of a general model $\mathcal{M}_1$ of Putnam's T_1, by demonstrating ubiquitous satisfaction in the complete set of fragmented submodels retaining $\mathcal{M}_1$'s structure. Rather than treating indeterminacy as a failure of reference which can basically be ignored in practice, *MASS* regards it as an essential feature of scientific representation that allows both for specialization and adaptive coordination. To illustrate how *MASS* operates in practice, the following section returns to the case of biological diversity presented in Chapter 2, where longstanding fragmentation among models is now gradually yielding to systemic integration.

6.2 Towards Shared Satisfaction of Models of Biological Diversity

MASS addresses the fragmentation of a general model of a Universal Theoretical Framework $\boldsymbol{T}_U$, and I will now apply this to the historical development of biological diversity outlined in Chapter

[147] Button & Walsh (2018), 39-44: "2.3 Putnam's use of Push-Through"
[148] Putnam (1980), 482
[149] Button & Walsh (2018), 40

2. So far, I left $\boldsymbol{T_U}$ deliberately vague to follow Putnam's ideal, divine scientific theory $\boldsymbol{T_1}$.[150] Button characterizes $\boldsymbol{T_1}$ as 'our best all-things-considered theory—an amalgam of physics, semantics, politics, poetry, and everything else—allow[ing] us to represent a physically possible world.'[151] This breadth also explains the inevitable fragmentation of $\boldsymbol{T_1}$'s general model $\mathcal{M}_1$ in science. Putnam presents $\boldsymbol{T_1}$ as a theory that would be rational for scientists to accept, but most act as if 'nature is a jumbled WORLD' and scientific practice a 'trading zone' for 'experiments, theories, and instruments'.[152,153] Model theory suggests that if the WORLD is unified and modellable—as many philosophers and scientists hold[154]—it is possible in principle to formulate true statements about it. Yet on such a scale, a unifying REFERENCE SENTENCE Φ could be satisfied only by the full model $\mathcal{M}_1$, and its truth in $\boldsymbol{T_1}$ would have to be known in advance.

Formulating a flawless Φ would require a God's-eye view if Φ is to be epistemically richer than an axiom or other trivial truth (see §4.1). Without access to $\mathcal{M}_1$, scientists approximate it by assembling submodels and testing candidate Φ for shared satisfaction. This thesis developed a formal strategy for this—*Model Adaptation by Shared Satisfaction*—not to fully construct $\mathcal{M}_1$, but to combine specialized models into a coherent network that incrementally converges toward it. The historical development of models of biological diversity, reviewed in Chapter 2, illustrates how such an approach could work in practice.

Linnaeus developed his divine taxonomic order of nature in the Enlightenment. In line with Putnam, it can be viewed as a bold attempt to formulate a model $\mathcal{M}_1$ for an ideal theory $\boldsymbol{T_1}$ of biological diversity. Were Linnaeus's taxonomy to remain fully accurate, this would suggest that it implicitly already contained, implicitly, the mechanistic structural elements determining its categorical order—elements only revealed by subsequent research. This scenario is unlikely, considering recent genetic revisions of species relationships.[155,156] Nonetheless, a full model need not satisfy every sentence of a theory, allowing $\mathcal{M}_1$ to represent a partial or incomplete realization of $\boldsymbol{T_1}$.

In the eighteenth century, Linnaeus's 'best all-things-considered theory'[157] was increasingly fragmented into specialized submodels of Lamarck's adaptation, Von Humboldt's ecology, Mendel's genes and Darwin's evolution. Each submodel captured essential but distinct aspects of life's diversity, with different emphases but largely the same domains (Linnaeus's *Kingdoms of Life*).[158] Each model had its own followers in science and society, and their interactions ranged from admired inspiration to outright denunciation.[159]

Beginning in the 1920s and 1930s, a group of scientists began to reconcile these fragmented models. Mendel's systemic pea experiments—largely ignored in his lifetime—were rediscovered by scientists like statistician Ronald Fisher, biologist J.B.S. Haldane, and geneticist Sewall Wright. They merged Mendelian genetics with Darwinian selection mainly using newly developed tools of population statistics. In 1942, the result was dubbed the Modern Neo-Darwinian Synthesis:[160] a unified framework in which evolution was understood as changes in gene frequencies over time. This synthesis preserved the strengths of some submodels, while clarifying what each model could explain (or not). Mendelian genes explained how variation

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[150] Putnam (1980), 473—see footnote 84
[151] Button (2013), 131
[152] Putnam (1980), 473—see footnote 84
[153] Galison (1997) 781-797: "Chapter 9: The Trading Zone", 797
[154] Frigg & Nguyen (2020); Morrison (2015)
[155] Wiley & Lieberman(2011)
[156] Hedges & Kumar (2009)
[157] Button (2013), 131
[158] Linnaeus's 'Three Kingdoms'—*Regnum Animale*, *Vegetabile* and *Lapideum*—are expandable with *Regnum Climatis* —see footnote 29
[159] Wulf (2017) for instance describes how biologists responded to Von Humboldt, throughout the 19th and early 20th centuries.
[160] Huxley, Julian S., *Evolution: The Modern Synthesis* (London: Allen & Unwin, 1942-1974)

arose and persisted; Darwinian selection explained how it shaped populations. Linnaean species were now seen as clusters within gene pools. Humboldtian insights were partly absorbed into ecology and evolutionary biology as constraints and pressures acting on populations. Lamarck's laws were largely rejected as being theoretically incompatible and empirically unsupported.

From a model-theoretic perspective, the 'bridging sentences' of the Modern Neo-Darwinian Synthesis largely relied on recent statistical advances—especially Fisher's pioneering work combining biometrical factors (reproduction, fitness, survival) with genetics. These statistical tools act like logical instruments: formal, model-independent frameworks, providing structural rules and scaffolding that supporting integration across diverse biological models.[161] While enabling structural coherence, they do not by themselves supply explanatory content. To achieve shared satisfaction across heterogeneous models of biological diversity, a reference sentence must incorporate not only these statistical frameworks but also appropriate interpretation functions and domain elements within an amalgamated model.

In addition, the Modern Synthesis also introduced several structural modifications: the domain was expanded to include whole populations containing multiple species and extinct lineages ($\Delta\boldsymbol{D}$); interpretation functions were extended to track allele frequencies, genotype distributions, and fitness effects ($\Delta\boldsymbol{I}$); and auxiliary relations and operations were added to model variation, natural selection, gene flow, and genetic drift ($\Delta\boldsymbol{L}$). For these modifications, the Modern Synthesis drew on *external* well-established scientific frameworks, but without disturbing the original Linnaean structure.

In *MASS*, these theories $\boldsymbol{T}^*$ and their (sub)models $\mathcal{M}^*_H$ are classified as 'alien' because they satisfy sentences χ incompatible with $\boldsymbol{T}_U$ (Insert 2). Nevertheless, they may also contain compatible ψ-sentences satisfying $\mathcal{M}_H$, and partial isomorphisms $\boldsymbol{\pi}$ with the supportive structure of Φ. Persistent ψ-sentences can enrich $\mathcal{M}_U$'s submodels structurally—through amalgamation of domain elements, interpretation functions, and auxiliary relations—without introducing new axioms or altering $\boldsymbol{T}_U$. For example, population genetics (Fisher, Haldane, Wright) introduced allele frequency dynamics, genotype distributions, and fitness effects; quantitative genetics contributed statistical methods for trait variation; and phylogenetic theory provided relational scaffolding for species divergence. Each of these modifications preserves the satisfaction of Mendelian axioms while allowing the same reference sentence Φ to hold across heterogeneous submodels—*eg* a classical pea inheritance model, a population genetics account of finch beak variation, or a phylogenetic account of extinct lineages. By combining domain expansion, enriched interpretation, and relational scaffolding, the amalgamated model coherence across submodel. *MASS* formalizes how external ('alien') theories contribute compatible sentences that guide the adaptation of submodels. Through push-through and sheaf-theoretic amalgamation (§4.4.2.1–2), these sentences enrich Φ and $\mathcal{M}_U$'s submodels, coordinating structural integration across diverse accounts of biological diversity while remaining within Linnaeus's overall model.

New interpretations, expanded domains, and statistical innovations facilitated integration of Mendel's inheritable genes into Darwin's evolutionary lineages. However, the hypothetical 'reference sentence' of Modern Synthesis did not yet contain the components needed for satisfaction in submodels of development, adaptability, or ecology. This required new biological elements and mechanisms to interpret their interactions. As described in §2.7, this was provided by new insights into the molecular foundations of life. The discovery of DNA in 1953 (which as Galison pointed out was attributed to Watson and Crick, but actually required a 'coordination of action and belief' among scientist with widely different expertise)[162] expanded

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[161] Frigg (2022), 364-365

[162] Galison (1997) 828-840: "9.8-The Coordination of Action and Belief",

the domain of biological models with new molecular elements—terms like ‘allele,’ ‘gene,’ and ‘sequence’ became constants or predicates within Mendelian and Darwinian submodels but remained disconnected from developmental and ecological mechanisms emphasized in Lamarckian and Humboldtian accounts. This ontological expansion deepened divergence among submodels.

Over subsequent decades, DNA’s role was reconceived as part of a dynamic regulatory system (§2.7) involving gene regulation, RNA expression, and epigenetic modification. These mechanisms revealed that the boundary between inherited and acquired traits is more permeable than previously thought—a phenomenon sometimes called ‘Lamarck’s revenge.’ Epigenetic changes can influence development and traits such as sex, while RNA regulation adds flexibility in gene expression. Together, these mechanisms link inheritance, development, and ecological responsiveness, supporting a scientific bridging hypothesis—a shareable reference sentence Φ —illustrated simply:

> *Distinguishing traits across species develop through regulatory molecular-genetic and epigenetic mechanisms integrating inheritance, individual development and adaptability, and environmental adaptivity.*

Φ encodes a shared structure **π**, although initially only some submodels—at least $\mathcal{M}_S$ and $\mathcal{M}_H$—satisfy it. Other submodels require adaptation under $\boldsymbol{A}_{MA\rightarrow S}$. Broader satisfaction arises only in the amalgamated model $\mathcal{M}_A$ (§4.4.1). $\mathcal{M}_A$ integrates mechanistic components across submodels, so that *Φ* can be progressively accommodated through push-through and adaptation (Steps 3 and 4 of $\boldsymbol{A}_{MA\rightarrow S}$). In this process, molecular-biological constituents are added as domain elements, while cellular and biochemical mechanisms from related theories ***T**** enter as interpretation functions. One striking example is the chemistry of synthetic nucleic acids (‘XNA’), an ‘alien’ theory that nonetheless helps integrate biological submodels: XNA can generate RNA and DNA resistant to biodegradation, enabling new therapeutics (like SARS-Cov19-vaccines) and clarifying which structural features of nucleic acids are essential for heredity. Other chemical, statistical, and mathematical models likewise support integration by providing complementary mechanisms and formal tools.

When push-through fails, sheaf-theoretic amalgamation provides a higher-order structural guide model $\mathcal{M}^+_A$ (§§*4.4.2.1–2*). Rather than altering Φ, $\mathcal{M}^+_A$ generates a unified structural space—via sheaf composition of stalks $\mathcal{F}(_S)$, $\mathcal{F}(_H)$, $\mathcal{F}(_{H'})$,...(Figure 1)—in which overlapping elements (partial isomorphisms **π**) from different submodels are consistently aligned. This alignment restricts the range of admissible reinterpretations of Φ: successive refinements of Φ', Φ'',... must preserve shared structures while integrating distinct structural components (*eg* genetic and ecological factors). Graph-theoretic analysis in *MASS* then tracks how Φ reshapes the evolving network. Minimal modification (Step 4) enforces these constraints in adapting submodels by aligning constants and relations across Mendelian inheritance, Lamarckian adaptation, Darwinian selection, and Humboldtian equilibria. In this way, Φ can evolve into a broadly shareable form that incorporates regulation, inheritance, selection, and environmental interaction, enabling *MASS* to provide a philosophical framework for reconciling fragmented biological models—including a unified perspective on Linnaean taxonomy and diversity.

Undoubtedly, many other scientific areas have also led to—or would allow for—systemic integration. Neuroscience, climate, economy can all provide compelling cases. The work for this needs to be done by scientists willing to collaborate and think outside their own narrow interests. At present, scientists are still more motivated by differentiation than integration. Many philosophers are natural realists because this aligns with science—which also specifically motivated Putnam.[163] But some may also feel the philosophical and theoretical need, with David Lewis, to defuse 'the bomb [Putnam has devised] that threatens to devastate the realist philosophy we know and love'.[164] As long as we need models to understand THE WORLD, we will be faced with the indeterminacies that are inherent to model theory. Scientific models are inevitably partial and intertwined, leaving parts of reality open-ended. *MASS* helps integrate these fragments into coherent networks, offering resilience against the indeterminacy inherent in our understanding of the world.

[163] Putnam (1994) "Dewey Lecture I, 465—see footnote 137
[164] Lewis (1984), 221

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