Invariant MeasuresUsingYoungTowers

Huiszoon, Leon

Leon Huiszoon

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Supervisors: Onno van Gaans, Charlene Kalle

Leiden University
Mathematical Institute

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1 Introduction

Be it population growth [16], particles moving in a space [22] or mixing liquids [19] - many physical processes display a change over time. Since the early 1930s mathematicians have modelled these processes using (discrete) dynamical systems [5] having a measure, to quantify an amount of people or an amount of particles for instance, and an operator to model how the measure changes over time. Natural questions then arise on the long term behaviour of these processes, such as: will our population amount stabilize? Can we predict the size of a volume by following how much time an individual particle spends in the volume on average? Will the liquids mix homogeneously?

Typically, an *invariant measure* is a measure which describes the long term behaviour of our system. A common approach to finding invariant measures is by fixing a *reference measure*, letting the system run for infinitely long and seeing how the reference measure averages over time [23, Chapter 2]. This asymptotically averaged measure is then likely to be an invariant measure.

If we want to be able to answer questions like the ones written earlier however, it is clear that not just *any* reference measure defined on a dynamical system will yield an invariant measure that yields relevant information. Instead, we likely need the reference measure to have some kind of physical meaning, moreover, we then need to be able to describe our invariant measure in terms of this reference measure. Mathematically speaking, we want to find an *absolutely continuous invariant measure* for our reference measure. Or specifically in this thesis, an *absolutely continuous invariant probability measure* also known as *(acip)*.

Turning to a more abstract setting than the physical processes above, in the 1970s it was shown that dynamical systems in which small distances grow uniformly admit *acips* [1]. These systems are called *uniformly expanding*. The uniform expansion allows us to describe the system as a shift over a (possibly infinite) alphabet while still respecting ergodic and measure-theoretic properties such as absolute continuity. For the uninitiated reader, we have added a prototypical example of a uniformly expanding system at the end of this section, seen in Figure 3.1.

However, physical reality is seldom as nice as to assure us of uniformly expanding behaviour in dynamical systems. Hence, Lai-Sang Young in [25] and [24] developed a method stating conditions shared by a wide class of non-uniformly expanding systems under which *acips* exist. Crucially, she showed these non-uniformly expanding systems can be analysed successfully by defining an *induced scheme* (in this thesis referred to as a *tower base*) and a *Young Tower*. Conceptually, the tower base is a small part of the original system where the dynamics are sped up. On a tower base, a Young Tower can then be constructed, which under certain conditions induces an acip for our original (non-uniformly expanding) system.

Since Young's discovery, mathematicians have tried studying *random dynamical* *systems* based on non-uniformly expanding dynamical systems through Young Towers [2],[3],[13], [4]. Random dynamical systems are constructed using a *base dynamic* and a dynamical system we refer to as a *random dynamic*. The base dynamic consists of a family of operators acting on the same set where at every time step an operator is chosen according to the random dynamic. In the literature, there exist two leading paradigms to apply Young Towers to random dynamical systems which are called the *annealed approach* [13], [3] and the *quenched approach* [4], [2]. The annealed approach essentially constructs a Young Tower directly on the random dynamical system allowing us to analyse it with the existing theory on Young Towers. This does force the random dynamic to be subjected to similar conditions as the base dynamic, limiting its applicability to random dynamical systems with a random dynamic that expands. As the annealed approach has already undergone a rigorous treatise in the paper [13] we shall focus ourselves on the quenched approach.

The *quenched approach* was first introduced in [4]. In a nutshell, the quenched approach constructs a Young Tower like structure for almost every element of the random dynamic. This allows for greater flexibility in our choice for a random dynamic, but the construction is much more delicate. To illustrate, the original method as seen in [4] has been reworked several times in papers such as [26], [2] and [7] the latter two being published as recent as 2023. However, each of these approaches is hard to make rigorous, either due to missing measurability of the density of an acip such as in [4], [26], relying on topological conditions not generally satisfied in the context of random Young Towers and/or basing themselves on bounds not applying to typical standard examples such as Young Towers based on the doubling map [1], [7].

The main aim of this text is finding a mathematically rigorous way to use the quenched approach to prove the existence of an absolutely continuous invariant probability measure (acip) for random dynamical systems with a non-uniformly expanding base dynamic and a random dynamic not displaying any form of expandingness.

This thesis is structured as follows. In Section 2, the preliminaries introduce elementary measure-theoretical concepts to the reader necessary to prove the existence of an *acip* using deterministic Young Towers as seen in Section 3. Section 3 is based on papers [25], [24] and the book [1] where we try to make the conditions as imposed by Young as intuitive as possible. Moreover, several proofs on the ergodic properties of the *acip* have been reworked in hopes of providing a clearer exposition than was done in [25]. In Section 3.6 two elementary examples of Young Towers are given, the first one being deterministic and the second being an example of the previously mentioned annealed approach.

Sections 4.1–5.6 are aimed at proving the existence of an acip for quenched random dynamical system. In particular, in Section 4.1 we develop the functional analytic background necessary for our proof from which we in Section 4.1.1 derive an original proof for the deterministic case as well. We do so using a novel measure-theoretical counterpart of the celebrated Arzelà-Ascoli Theorem obtaining L^1 convergence and almost everywhere convergence, without requiring compactness. In Sections 4.2 and 4.3 we shall lay out the measure theoretical foundations necessary to describe Young Towers. Notably, an adapted version of the treatise of Jacobians in [23] is given and subsequently generalised to describe Random Young Towers, or more generally, random dynamical systems. In Sections 5.1-5.3 we shall carefully build up the theory of Random Young Towers to prove the existence of an *acip* for Random Young Towers in Section 5.4. To our knowledge, our approach is novel but does borrow some ideas from papers such as [2] or [7].

Finally, in Section 5.5 we have proven a novel Disintegration Theorem generally applicable to random dynamical systems as it only relies on absolute continuity, avoiding complicated disintegration theorems such as [9, Corollary 6.13] used by [1] that may or may not apply to our setting. We conclude the thesis by presenting an example in Section 5.6 to which the quenched approach applies. The non-expanding random dynamic there will take shape in the form of the irrational rotation.

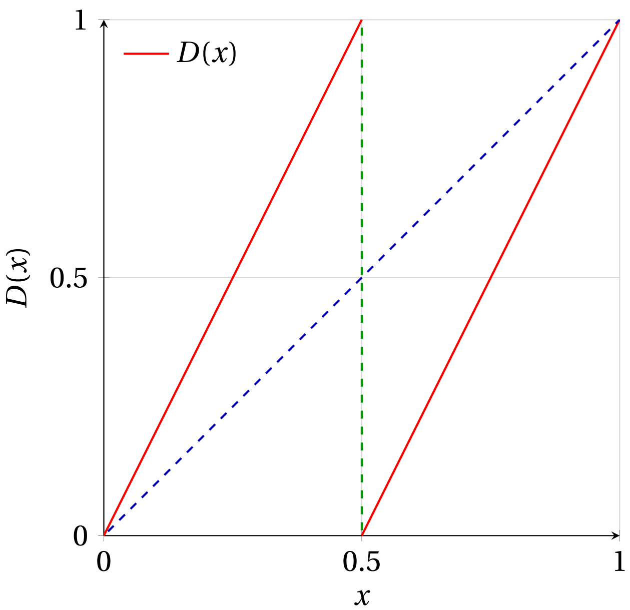


Figure 1: The doubling map

$$D(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}), \\ 2x - 1, & x \in [\frac{1}{2}, 1). \end{cases}$$

2 Preliminaries

We start with some general definitions and notations used in this thesis. The material present here has been written with the help of [6], [23], [1], to which a few results have been added. The material here presented is sufficient for all proofs in Section 3.

2.1 Dynamical Systems

This subsection is structured as follows: In Definition 2.1.1 through Example 2.1.3 first we shall introduce dynamical systems briefly. After that from Definition 2.1.6 through Theorem 2.1.11 we shall then lay out some elementary measure-theoretical concepts, necessary to study dynamical systems. After that we shall formally define an *acip* and for the rest of the section present tools to study these.

We shall generally only consider dynamical systems of positive σ-finite measures with a non-singular operator. Non-singularity we shall require for the Radon-Nikodym Theorem.

Definition 2.1.1. Given a measure space $(X, \mathscr{F}, \mu)$ with μ a σ-finite positive measure and an $\mathscr{F}$-measurable mapping $T: X \to X$, we call T *non-singular* if for all $A \in \mathscr{F}$ with $\mu(A) = 0$ we have $\mu(T^{-1}A) = 0$. If T is non-singular, we refer to the quadruple $(X, \mathscr{F}, \mu, T)$ as a *dynamical system* with T the *operator.*

In ergodic theory we often want to investigate the relation between μ and T by means of the following definitions.

Definition 2.1.2. Given a dynamical system $(X, \mathscr{F}, \mu, T)$ with $\mu(X) \in (0, \infty)$ we say the measure μ is *invariant (for T)* if for all $A \in \mathscr{F}$ we have $\mu(A) = \mu(T^{-1}A)$. If so, we call T *measure preserving.* For measure-preserving T we say, T is:

1. *ergodic,* if for all $A \in \mathscr{F}$ with $T^{-1}A = A$ we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$;
2. *mixing,* if for all $A, B \in \mathscr{F}$ we have
$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B);$$
3. *exact,* if we have
$$\left\{\mu(A) : A \in \bigcap_{n\geq 0} T^{-n}\mathscr{F}\right\} = \{0, \mu(X)\}.$$

We have an order of implications $3 \Rightarrow 2 \Rightarrow 1$, see [10].

Example 2.1.3. The following dynamical systems will be used as examples in Sections 3.6 and 5.6.

Bernoulli shift Given a finite indexed set $\Gamma = \{\gamma\}_{\gamma\in\Gamma}$ with a vector $P = (p_\gamma)_{\gamma\in\Gamma}$ of probabilities, $p_\gamma > 0$, $\sum_{\gamma\in\Gamma} p_\gamma = 1$, we can define $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ where $\mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}$ is the infinite product σ-algebra based on Γ, $\mathbb{P}$ the product measure and σ the left shift. We shall refer to Γ as an *alphabet*.

Rotation Given some $\theta \in \mathbb{R}$, the rotation

$$\sigma_\theta : [0,1) \to [0,1) \qquad \sigma_\theta(x) = x + \theta \mod 1,$$

makes $([0,1), \mathscr{B}[0,1), \lambda, \sigma_\theta)$ into a dynamical system where $\mathscr{B}([0,1))$ is the Borel σ-algebra and λ the Lebesgue measure.

Doubling map Given the Borel space $([0,1), \mathscr{B}[0,1), \lambda)$ equipped with the Lebesgue measure, we can define the doubling map as given by

$$T : [0,1) \to [0,1) \qquad x \mapsto 2x \mod 1.$$

It is well known that *Bernoulli shifts*, the *doubling map* and rotations are measure preserving. On top of that, Bernoulli shifts and the doubling map are exact and rotations σ_θ are not mixing, but σ_θ ergodic if and only if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

As said previously, we present some measure-theoretical tools necessary to study dynamical systems. We incorporated these as they will used extensively and for ease of reference. Functions whose inverse images map a collection of generators into a σ-algebra are measurable, see [21, Lemma 7.2]

Lemma 2.1.4. *Let $(X, \mathscr{F}_X)$ and $(Y, \mathscr{F}_Y)$ be measurable spaces with $T : X \to Y$ some mapping. Suppose we have $\mathscr{W} \subseteq \mathscr{F}_Y$ with $\sigma(\mathscr{W}) = \mathscr{F}_Y$ and $T^{-1}(\mathscr{W}) \subseteq \mathscr{F}_X$. Then T is measurable.*

The following Lemma is also known as the π-λ Theorem. It states that if two measures on a measure space are equal on an intersection stable collection of generators for their σ-algebra, they agree on the entire σ-algebra. See [21, Theorem 5.7].

Lemma 2.1.5. *Let $(X, \mathscr{F})$ be some measurable space and suppose we have finite measures $\mu, \nu \in \mathscr{M}(X)$ such that $\mu(X) = \nu(X)$ and a π-system $\Pi \subseteq \mathscr{F}$ so that $\sigma(\Pi) = \mathscr{F}$, and $\mu(P) = \nu(P)$ for each $P \in \Pi$. Then $\mu = \nu$.*

We shall see many instances of dynamical systems constructed by restricting a measure space to a measurable subset on which on an operator is defined.

Definition 2.1.6. [6, pg 8, 56] Given a measure space $(X, \mathscr{F}, \mu)$ with μ a σ-finite non-negative measure and an $A \in \mathscr{F}$ with $\mu(A) \in [0,\infty)$ we can define the *trace σ-algebra (of A)*

$$\mathscr{F}_A := \{A \cap B \subseteq X : B \in \mathscr{F}\}$$

and the *restricted measure (to A)*

$$\mu_A : \mathscr{F}_A \to [0, \mu(A)] \qquad A \mapsto \mu(A \cap B).$$

Similarly we call $(A, \mathscr{F}_A, \mu_A)$ the *restricted measure space (to A).*

In particular, we shall often use the notion of a restricted measure space when considering product spaces - in fact this is exactly how we will define Young Towers in Section 3.4. The following Proposition will then be useful.

Proposition 2.1.7. *[6, Proposition 3.3.2.] Let $(X, \mathscr{F}_X)$ and $(Y, \mathscr{F}_Y)$ be measurable spaces and construct the measurable space $(X \times Y, \mathscr{F}_{X\times Y})$. Then for each $A \in \mathscr{F}_{X\times Y}$, $x \in X$ and $y \in Y$ we have*

$$A_y := \{x \in X : (x, y) \in A\} \in \mathscr{F}_X \text{ and } A_x := \{y \in X : (x, y) \in A\} \in \mathscr{F}_Y \tag{1}$$

In addition, for every $\mathscr{F}_{X\times Y}$-measurable function f and every $x \in X$ the mapping $y \mapsto f(x, y)$ is $\mathscr{F}_Y$-measurable.

Lastly, for any finite measure ν on $\mathscr{F}_Y$ the mapping $x \mapsto \nu(A_x)$ is $\mathscr{F}_Y$-measurable.

We refer to the sets A_x and A_y as in Equation (1) as *sections.* Proposition 2.1.7 tells us that product measurable sets have measurable sections. The following Theorem will play an important role in Sections 4.2 and 5.5.

Theorem 2.1.8. *[6, Theorem 3.4.4. Fubini] Let μ and ν be σ-finite non-negative measures on the spaces X and Y respectively. Suppose that a function f on $X \times Y$ is integrable with respect to the product measure $\mu \times \nu$. Then, the function $y \mapsto f(x, y)$ is integrable with respect to ν for μ-a.e. x, the function $x \mapsto f(x, y)$ is integrable with respect to μ for ν-a.e. y, the functions*

$$x \mapsto \int_Y f(x, y)\, d\nu(y) \quad \text{and} \quad y \mapsto \int_X f(x, y)\, d\mu(x)$$

are integrable on X and Y respectively, and one has

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \int_X f(x, y)\, d\mu(x)\, \nu(y) = \int_X \int_Y f(x, y)\, \nu(y)\, d\mu(x).$$

To study how measures of dynamical systems change under their operator we define the *pushforward* measure.

Definition 2.1.9. Suppose we have a measure space $(X, \mathscr{F}, \mu)$ with μ a positive measure and a measurable space $(Y, \mathscr{Y})$ with a measurable mapping $T : X \to Y$. We shall refer to the measure $T_\star\mu$ on Y defined by

$$T_\star\mu(B) = \mu(T^{-1}B), \qquad \text{for B} \in \mathscr{Y},$$

as the *pushforward measure* of μ under T.

If $A \in \mathscr{F}$ and $T : A \to Y$ is measurable we will often simply write $T_\star\mu$ instead of $T_\star\mu_A$. Note that measure-preservingness of a $T : X \to X$ with respect to some μ simplifies notationally to $T_\star\mu = \mu$, and non-singularity of T with respect to μ simplifies to requiring $T_\star\mu \ll \mu$.

We now present a technical tool for evaluating pushforward measures.

Lemma 2.1.10 (Change of variables)**.** *Let $(X, \mathscr{F}, \mu)$ be a measure space, and let $(Y, \mathscr{Y})$ be a measurable space. Let $f : X \to Y$ and $g : Y \to \mathbb{R} \cup \{\infty\}$ be measurable mappings. We have $g \in L^1(Y, f_\star\mu)$ if and only if $g \circ f \in L^1(X, \mu)$. In this case, we have*

$$\int_A g \, d(f_\star\mu) = \int_{f^{-1}(A)} g \circ f \, d\mu \qquad \textit{for all } A \in \mathscr{Y}.$$

See [6, Theorem 3.6.1] for a proof.

A useful consequence of assuming absolute continuity of a pushforward measure with respect to reference measure is the ability to define densities using the Radon Nikodym Theorem [6, Theorem 3.2.2.]. The theorem itself, taken from [6, Theorem 3.2.2], is only phrased there for finite measures but the generalisation we phrase here is mentioned in words later in the corresponding section. Note that any measure absolutely continuous with respect to a σ-finite measure is σ-finite.

Theorem 2.1.11 ([6], Theorem 3.2.2)**.** *Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. For any positive measure $\nu : \mathscr{F} \to [0, \infty]$ the following statements are equivalent:*

1. *The measure ν is absolutely continuous with respect to μ.*

2. *There exists a measurable function $f : X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\nu(\cdot) = \int_\cdot f(x) \, d\mu(x)$.*

If such f exists, we refer to it as the Radon-Nikodym derivative of ν with respect to μ and write $f = \frac{d\nu}{d\mu}$. The Radon-Nikodym derivative is unique up to a μ-measure zero set. We have $\frac{d\nu}{d\mu} \in L^1(X, \mu)$ if and only if ν is a finite measure.

Having established the necessary measure-theoretical background, we return to our discussion of dynamical systems. The main goal of this thesis will be finding sufficient conditions under which the following object exists and whether this object is unique in that setting.

Definition 2.1.12. Given a dynamical system $(X, \mathscr{F}, \mu, T)$ we say that ν is an *absolutely continuous invariant probability measure (with respect to μ and T)* if ν is a probability measure satisfying $\nu \ll \mu$ and $T_\star\nu = \nu$. We shall refer to such a measure ν as an *acip* for short.

The following proposition is useful for proving uniqueness of *acips* with the same support. See [1] for a proof.

Proposition 2.1.13. *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system and $A \in \mathscr{F}$ be such that $\mu(A) > 0$. Now let ν be an ergodic acip $\nu \ll \mu$ with $\mu_A \ll \nu_A \ll \mu_A$. Then ν is the unique acip with $\nu \ll \mu$ satisfying $\nu(A) > 0$.*

Combining Theorem 2.1.11 with the non-singularity of dynamical systems, we can study densities of pushforward measures. The following lemma asserts that we can find acip's by these densities. In the proof the L^1 convergence along a subsequence does the heavy lifting.

Lemma 2.1.14. *Suppose we have a dynamical system $(X, \mathscr{F}, \mu, T)$ with $\mu(X) \in (0, \infty)$ and suppose in writing*

$$\phi_n = \frac{1}{n} \sum_{i=0}^{n-1} \frac{d(T^i)_\star \mu}{d\mu}, \qquad \text{for } n \in \mathbb{Z}_{\geq 1},$$

we have a subsequence $(\phi_{n_k})_{k \geq 0}$ converging in $\|\cdot\|_{L^1(X)}$ to some $\phi \in L^1(X)$ as $k \to \infty$. Then there exists an acip *$\nu \ll \mu$ on X. Furthermore, if we have a $C > 1$ such that for all $n \in \mathbb{Z}_{\geq 1}$ we have $\phi_n \in [\frac{1}{C}, C]$ μ-almost surely and $\phi_{n_k} \to \phi$ pointwise μ-almost everywhere as $k \to \infty$, then there exists an acip $\nu \ll \mu$ such that*

$$\frac{1}{C} \leq \frac{d\nu}{d\mu} \leq C \quad \mu\text{-almost everywhere.}$$

Proof. Let $(n_k)_{k \in \mathbb{Z}_{\geq 1}} \subseteq \mathbb{Z}_{\geq 1}$ be a sequence increasing in $k \in \mathbb{Z}_{\geq 1}$ such that for $\phi \in L^1(X)$, $\phi_{n_k} \to \phi$ in $\|\cdot\|_{L^1(X)}$ as $k \to \infty$. In noting $\|\phi_{n_k}\|_1 = 1$ and $\phi_{n_k} \geq 0$ μ-almost surely for all $k \in \mathbb{Z}_{\geq 1}$ we see $\|\phi\|_1 = 1$ and $\phi \geq 0$ μ-almost surely as well, which implies $\nu(\cdot) := \int_\cdot \phi \, d\mu$ is a probability measure. Absolute continuity of ν follows by Theorem 2.1.11. For T-invariance of ν, let $B \in \mathscr{F}$ and note that for all $k \in \mathbb{Z}_{\geq 1}$ we have

$$\begin{aligned}
|\nu(B) - T_\star \nu(B)| &= \left| \int_B \phi \, d\mu - \int_{T^{-1}B} \phi \, d\mu \right| \\
&\leq \left| \int_B \phi \, d\mu - \int_B \phi_{n_k} \, d\mu \right| \\
&\quad + \left| \int_B \phi_{n_k} \, d\mu - \int_{T^{-1}B} \phi_{n_k} \, d\mu \right| + \left| \int_{T^{-1}B} \phi_{n_k} \, d\mu - \int_{T^{-1}B} \phi \, d\mu \right| \\
&\leq \left| \int_B \phi \, d\mu - \int_B \phi_{n_k} \, d\mu \right| \\
&\quad + \frac{1}{n_k} \left| \mu(B) - \mu(T^{n_k - 1} B) \right| + \left| \int_{T^{-1}B} \phi_{n_k} \, d\mu - \int_{T^{-1}B} \phi \, d\mu \right|,
\end{aligned}$$

where in taking the limit of $k \to \infty$ we obtain

$$|\nu(B) - T_\star \nu(B)| = 0.$$

This implies $T_\star \nu = \nu$, proving ν is an *acip*. If $(\phi_{n_k})_{k\geq 1}$ also converges pointwise almost everywhere and $\phi_{n_k} \subseteq [\frac{1}{C}, C]$ for all $k \in \mathbb{Z}_{\geq 1}$, we obtain $\frac{1}{C} \leq \frac{d\nu}{d\mu} \leq C$ μ-almost surely as desired. □

The notion of *ergodic equivalence* below is adopted from[23, Chapter 8].

Definition 2.1.15. Let $(X, \mathscr{F}, \mu, T)$, $(Y, \mathscr{B}, \nu, U)$ be dynamical systems with $\mu(X) = \nu(Y) < \infty$. We say they are *ergodically equivalent* if one can find $A \in \mathscr{F}$, $Y \in \mathscr{B}$ with $\mu(X \setminus A) = 0$, $\nu(Y \setminus B) = 0$, and a mapping $\phi : A \to B$ that is bijective, $\mathscr{F}_A$-$\mathscr{B}_B$ measurable and has a measurable inverse satisfying

$$\phi_\star \mu = \nu \text{ and } \phi \circ T = U \circ \phi.$$

We call the mapping ϕ an *ergodic isomorphism.*

Remark 2.1.16.

1. There is slight abuse of notation in Definition 2.1.15 as copied from [23, Chapter 8], in that $\phi : A \to B$ is not defined on the entirety of X and Y, hence writing

$$\phi_\star \mu = \nu \text{ and } \phi \circ T = U \circ \phi.$$

   is *technically* not well-defined. As $\mu(X \setminus A) = 0$, and $\nu(Y \setminus B) = 0$ and hence $\mu(C \cap A) = \mu(C)$ for all $C \in \mathscr{F}$ and $\nu(D \cap B) = \nu(D)$ for all $D \in \mathscr{B}$ we shall overlook this matter.

2. In [23, Chapter 8] it is also pointed out that the sets A and B in Definition 2.1.15 can be chosen such that $T(A) \subseteq A$ and $U(B) \subseteq B$, allowing us to construct ergodically equivalent dynamical systems

$$(A, \mathscr{F}_A, \mu_A, T) \text{ and } (B, \mathscr{B}_B, \nu_B, U),$$

   where $\phi : A \to B$ serves as an ergodic isomorphism. This will play a role in the proof of Corollary 2.1.19.

The lemma below states that ergodic properties are shared between ergodically equivalent systems. As we shall only need the claim holding for measure-preservingness and exactness we have only included those proofs.

Lemma 2.1.17. *Suppose* $(X, \mathscr{F}, \mu, T)$, $(Y, \mathscr{B}, \nu, U)$ *are dynamical systems satisfying* $\mu(X) = \nu(Y) < \infty$ *and that they are ergodically equivalent. Then* $(X, \mathscr{F}, \mu, T)$ *is measure-preserving/ergodic/mixing/exact if and only if* $(Y, \mathscr{B}, \nu, U)$ *is measure-preserving/ergodic/mixing/exact.*

Proof. Let $A \in \mathscr{F}, B \in \mathscr{B}$ satisfy $\mu(X \setminus A) = 0$, $\nu(Y \setminus B) = 0$ and $\phi : A \to B$ be an ergodic isomorphism. We shall without loss of generality assume our ergodic properties hold on $(X, \mathscr{F}, \mu, T)$ and show these transfer to $(Y, \mathscr{B}, \nu, U)$.

Measure-Preservingness Assume $(X, \mathscr{F}, \mu, T)$ is measure-preserving, and let $K \in \mathscr{B}$. Note we have

$$
\begin{aligned}
(U_\star \nu)(K) &= (U_\star \nu)(K \cap B) \\
&= (U_\star(\phi_\star \mu))(K \cap B) && (2) \\
&= \mu((U \circ \phi)^{-1}(K \cap B) \\
&= \mu((\phi \circ T)^{-1}(K \cap B) && (3) \\
&= \phi_\star \mu(K \cap B) && (4) \\
&= \nu(K),
\end{aligned}
$$

so that we see $U_\star \nu = \nu$. For clarification, in Equation (2) we used $\phi_\star \mu = \nu$; in Equation (3) we used $\phi \circ T = U \circ \phi$ and in Equation (4) we used measure-preservingness of T.

Exactness Suppose $(X, \mathscr{F}, \mu, T)$ is exact and suppose $K \in \bigcap_{n\geq 0} U^{-n}\mathscr{B}$. Then note

$$
\begin{aligned}
\phi^{-1} K &\in \phi^{-1}\left(\cap_{n\geq 0} U^{-n}\mathscr{B}\right) \\
&= \bigcap_{n\geq 0}\left(\phi^{-1} U^{-n}\mathscr{B}\right) \\
&= \bigcap_{n\geq 0}\left(T^{-n}\phi^{-1}\mathscr{B}\right) \quad (\text{using } \phi \circ T = U \circ \phi,\ n \text{ times}) \\
&\subseteq \cap_{n\geq 0}(T^{-n}\mathscr{F}),
\end{aligned}
$$

so that $\nu(K) = \mu(\phi^{-1}K) \in \{0, \infty\}$ proving exactness of ν. □

A canonical (non-trivial) example of an ergodic equivalence between two dynamical systems are the doubling map with the instance of the Bernoulli shift given in Lemma 2.1.18 below. The result is considered 'standard theory'. The author was unable to find a complete proof in existing literature however and hence incorporated a proof in Appendix A.2 or more specifically, below Lemma A.2.4.

Lemma 2.1.18. *Let* $([0,1), \mathscr{B}[0,1), \lambda, D)$ *be the standard Borel measure space equipped with the doubling map and* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma)$ *be the Bernoulli shift with for all* $i \in \mathbb{Z}_{\geq 0}$

$$
\mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 1) = \mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 0) = \frac{1}{2}.
$$

Then

$$
([0,1), \mathscr{B}[0,1), \lambda, D) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)
$$

are ergodically equivalent.

We now prove that acips on dynamical systems induce acips on ergodically equivalent dynamical systems, maintaining additional properties if present, such as the acip having an essentially bounded density or exactness.

Corollary 2.1.19. *Let* $(X, \mathscr{F}, \mu, T)$, $(Y, \mathscr{B}, \nu, U)$ *be dynamical systems with* $\mu(X) = \nu(Y) < \infty$ *that are ergodically equivalent. Suppose we have an acip* $\eta \ll \mu$*. Then there exists an acip* $\zeta \ll \nu$ *on* $(Y, \mathscr{B}, \nu, U)$*. Moreover, if* $\|\frac{d\eta}{d\mu}\|_\infty \le M$ *for some* $M \in \mathbb{R}_{>0}$ *then* $\|\frac{d\zeta}{d\nu}\|_\infty \le M$ *and if* η *is exact then so is* ζ*.*

Proof. For sake of technical ease we shall first assume we have a mapping $\phi : X \to Y$ that is bi-measurable satisfying

$$\phi_\star \mu = \nu \text{ and } \phi \circ T = U \circ \phi.$$

and comment on the more general case, where ϕ is defined on subsets of full measure as in Definition 2.1.15, at the end of the proof.

Using the acip $\eta \ll \mu$, define $\zeta := \phi_\star \eta$. We first want to show $\zeta \ll \nu$. To start, let $N \in \mathscr{B}$ and note

$$\begin{aligned}
\zeta(N) &= (\phi_\star \eta)(N) \\
&= \int_{\phi^{-1}N} \frac{d\eta}{d\mu}(x)\, d\mu(x) \\
&= \int_{\phi^{-1}N} \frac{d\eta}{d\mu}(\phi^{-1} \circ \phi(x))\, d\mu(x).
\end{aligned}$$

In noting $\frac{d\eta}{d\mu} \in L^1(\mu)$, we see by Lemma 2.1.10 that $\frac{d\eta}{d\mu} \circ \phi^{-1} \in L^1(\phi_\star \mu)$. Again by Lemma 2.1.10 we may then write,

$$\begin{aligned}
\int_{\phi^{-1}N} \frac{d\eta}{d\mu}(\phi^{-1} \circ \phi(x))\, d\mu(x) &= \int_N \frac{d\eta}{d\mu}(\phi^{-1}(y))\, d(\phi_\star \mu)(y) \\
&= \int_N \frac{d\eta}{d\mu}(\phi^{-1}(y))\, d\nu(y),
\end{aligned}$$

so that

$$\zeta(N) = \int_N \frac{d\eta}{d\mu}(\phi^{-1}(y))\, d\nu(y).$$

Applying Theorem 2.1.11 we then have $\zeta \ll \nu$ and $\frac{d\zeta}{d\nu} = \frac{d\eta}{d\mu} \circ \phi^{-1}$, ν-almost surely.

We now show $(Y, \mathscr{B}, \zeta, U)$ is a dynamical system to apply Lemma 2.1.17. To do so, we need $U_\star \zeta \ll \zeta$. Let $N \in \mathscr{B}$ and suppose $\zeta(N) = 0$, which is rewritten, $\eta(\phi^{-1}(N)) = 0$. Then note

$$\begin{aligned}
U_\star \zeta(N) &= U_\star \phi_\star \eta(N) \\
&= \eta((U \circ \phi)^{-1}(N)) \\
&= \eta((\phi \circ T)^{-1}(N)) \\
&= T_\star \eta(\phi^{-1}(N)) = 0
\end{aligned}$$

as $T_\star\eta \ll \eta$, showing $U_\star\zeta \ll \zeta$. Having shown non-singularity for ζ we see $(Y,\mathscr{B},\zeta,U)$ is a dynamical system. In noting $\zeta = \phi_\star\eta$, we see ϕ is an ergodic isomorphism between $(Y,\mathscr{B},\zeta,U)$ and $(X,\mathscr{F},\eta,T)$. We may then apply Lemma 2.1.17 so that the invariance of η implies invariance of ζ. Similarly, if η is exact then so is ζ.

Lastly, if $\|\frac{d\eta}{d\mu}\|_\infty \leq M$ for some $M \in \mathbb{R}_{>0}$, we have a set $\dot{X} \in \mathscr{F}$ such that $\mu(X \setminus \dot{X}) = 0$ and $\frac{d\eta}{d\mu}(x) \leq M$ for each $x \in \dot{X}$. Then note $\phi(\dot{X}) = (\phi^{-1})^{-1}(\dot{X}) \in \mathscr{B}$ and

$$\nu(\phi(\dot{X})) = \phi_\star^{-1}\nu(\dot{X}) = \mu(\dot{X}) = \mu(X) = \nu(Y).$$

Consequently, we can see that for $N \in \mathscr{B}$ we have

$$\begin{aligned}\zeta(N) &= \int_N \frac{d\eta}{d\mu}(\phi^{-1}(y))\,d\nu(y)\\ &= \int_{N\cap\phi(\dot{X})} \frac{d\eta}{d\mu}(\phi^{-1}(y))\,d\nu(y)\\ &\leq M\nu(N\cap\phi(\dot{X}))\\ &= M\nu(N),\end{aligned}$$

proving $\|\frac{d\zeta}{d\eta}\|_\infty \leq M$.

Now finally, if ϕ would have only existed between measure-dense subsets of $A \subseteq X$ and $B \subseteq Y$ respectively, we could use Remark 2.1.16 and apply the above proof to $(A,\mathscr{F}_A,\mu_A,T)$ and $(B,\mathscr{B}_B,\nu_B,U)$. The resulting measure ζ (defined on $\mathscr{F}_A$) then induces a measure $\dot{\zeta} : \mathscr{F} \to \mathbb{R}_{\geq 0}$ defined by $\dot{\zeta}(N) = \zeta(N \cap A)$. Using $\mu(X \setminus A) = 0$, we can directly derive that invariance, non-singularity, absolute continuity (with a uniform upper bound on the density) and exactness of $\dot{\zeta}$ are preserved from ζ. □

2.1.1 Partitions and Separation Times

Next, we see the first example (of many) where considering partitions of dynamical systems can be useful. We first phrase a slightly more general claim.

Lemma 2.1.20. *Let $(X,\mathscr{F},\mu)$ and $(Y,\mathscr{B},\nu)$ be σ-finite measure spaces and $f : X \to Y$ a measurable mapping so that $f_\star\mu \ll \nu$. Let $X_0 \in \mathscr{F}$ and $\mathscr{P} \subseteq \mathscr{F}$ be a countable partition of X_0. Then*

$$\frac{d(f|_{X_0})_\star\mu}{d\nu} = \sum_{P\in\mathscr{P}} \frac{d(f|_P)_\star\mu}{d\nu} \quad \textit{holds } \nu\textit{-a.s.} \tag{5}$$

Proof. Let $B \in \mathscr{B}$ be given arbitrarily. We simply note

$$\begin{aligned}\int_B \frac{d(f|_{X_0})_\star\mu}{d\nu}(y)\,d\nu(y) &= \mu(X_0 \cap f^{-1}B)\\ &= \mu(\bigsqcup_{P\in\mathscr{P}} P \cap f^{-1}B)\\ &= \sum_{P\in\mathscr{P}} \mu(P\cap f^{-1}B)\\ &= \sum_{P\in\mathscr{P}}\int_B \frac{d(f|_P)_\star\mu}{d\nu}(y)\,d\nu(y)\\ &= \int_B \sum_{P\in\mathscr{P}} \frac{d(f|_P)_\star\mu}{d\nu}(y)\,d\nu(y), \qquad (6)\end{aligned}$$

where in Equation (6) we used the monotone convergence theorem. As $B \in \mathscr{B}$ was given arbitrarily this implies Equation (5). □

Specified to dynamical systems this reads as follows.

Lemma 2.1.21. *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0,\infty)$. Suppose that $X_0 \in \mathscr{F}$ and $\mathscr{P} \subseteq \mathscr{F}$ is a countable partition of X_0. Then*

$$\frac{d(T|_{X_0})_\star\mu}{d\mu} = \sum_{A\in\mathscr{P}} \frac{d(T|_A)_\star\mu}{d\mu} \quad \textit{holds } \mu\textit{-a.e..} \qquad (7)$$

Also, if there exists an $M > 0$ such that for all $A \in \mathscr{P}$,

$$\left\|\frac{d(T|_A)_\star\mu}{d\mu}\right\|_\infty \le M\,\mu(A) \textit{ for some } M > 0,$$

then $\left\|\frac{d(T|_{X_0})_\star\mu}{d\mu}\right\|_\infty \le \mu(X)\cdot M.$

Proof. Equation (7) is a special case of Lemma 2.1.20. The second statement follows suit by noting that for$B \in \mathscr{F}$,

$$\begin{aligned}\int_B \frac{d(T|_{X_0})_\star\mu}{d\mu}(x)\,d\mu(x) &= \sum_{A\in\mathscr{P}}\int_B \frac{d(T|_A)_\star\mu}{d\mu}(x)\,d\mu(x)\\ &\le \sum_{A\in\mathscr{P}}\int_B M\mu(A)\,d\mu(x)\\ &= \sum_{A\in\mathscr{P}} M\mu(A)\mu(B)\\ &= M\mu(X)\mu(B).\end{aligned}$$

Now in fixing

$$B := \left\{x\in X : \frac{d(T|_{X_0})_\star\mu}{d\mu}(x) > M\right\} \in \mathscr{F},$$

and assuming $\mu(B) > 0$ we have

$$M\mu(B)\mu(X) < \int_B \frac{d(T|_{X_0})_\star\mu}{d\mu}(x)\,d\mu(x) \leq M\mu(X)\mu(B),$$

showing a contradiction, which implies $\mu(B) = 0$, proving the claim. □

In having a partition for some dynamical system, we often want to construct a new partition consisting of smaller sets as these will often be better behaved for higher iterates of an operator. How to do so, we explain below.

Definition 2.1.22. Given a dynamical system $(X, \mathscr{F}, \mu, T)$ and a countable partition $\mathscr{P} \subseteq \mathscr{F}$ of X consisting of sets of positive measure, we say that for $n \in \mathbb{Z}_{\geq 1}$ the partition

$$\mathscr{P}^n := \bigvee_{i=0}^{n-1} T^{-i}\mathscr{P} = \{P_0 \cap T^{-1}P_1 \cap \ldots \cap T^{-n+1}P_{n-1} : P_0, \ldots, P_{n-1} \in \mathscr{P}\} \setminus \{\emptyset\},$$

is the *n'th refinement of* $\mathscr{P}$, and

$$\mathscr{P}^\infty := \left\{ \bigcap_{n=0}^{\infty} T^{-n}P_n \; : (P_n)_{n\geq 0} \subseteq \mathscr{P} \right\},$$

is the *asymptotic refinement*. Furthermore, we say $\mathscr{P}$ is *generating* for $\mathscr{F}$ if the set $\bigcup_{n=1}^{\infty} \mathscr{P}^n$ is generating for $\mathscr{F}$, that is $\sigma(\bigcup_{n=1}^{\infty} \mathscr{P}^n) = \mathscr{F}$. We say $\mathscr{P}$ is *separating* if $\mathscr{P}^\infty$ is the trivial partition of X into singletons. Here we denote for $k \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{F}$,

$$T^{-k}A = \{x \in X : T^k(x) \in A\}.$$

When dealing with partitions and refinements thereof, the following standard set identity is very useful.

Lemma 2.1.23. *Suppose we have a function $f : X \to Y$ with $A \subset X$ and $B \subseteq Y$. Then*

$$f(A \cap f^{-1}(B)) = f(A) \cap B.$$

The Lemma below will be useful to relate the measure of some measurable set to the measure of some element of a generating and separating partition [1, Corollary 2.3] for a proof.

Corollary 2.1.24. *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0, \infty)$ and let $\mathscr{P}$ be a countable generating partition for $\mathscr{F}$. Then for all $\delta > 0$ and $A \in \mathscr{F}$ with $\mu(A) > 0$, there are $n \geq 1$ and $P \in \bigvee_{i=0}^{n-1} T^{-i}\mathscr{P}$ such that $\mu(P \setminus A) < \delta\mu(P)$.*

The following lemma (see [1, Lemma 2.7]) gives a useful criterion for finding whether a partition is generating and separating. We define for metric spaces (X,d) with a countable partition $\mathscr{P} \subseteq X$,

$$\operatorname{diam}(\mathscr{P}) := \sup\{\sup_{x,y\in P} d(x,y) : P \in \mathscr{P}\}.$$

Whenever we refer to a *Borel space* of a metric space, we mean a measure space where the σ-algebra is a Borel σ-algebra induced by the topology of the metric and with its measure a Borel measure.

Lemma 2.1.25. *Given a metric space (X,d), let $(X,\mathscr{F},\mu,T)$ be a dynamical system with $(X,\mathscr{F},\mu)$ a Borel space and let $\mathscr{P} \subseteq \mathscr{F}$ be a countable partition. Then $\mathscr{P}$ is generating and separating if*

$$\operatorname{diam}(\mathscr{P}^n) \to 0 \textit{ as } n \to \infty.$$

The Bernoulli shift naturally admits a generating and separating partition. We shall introduce some terminology to explain this also used for instance in Section 3.6.

Example 2.1.26. Consider a Bernoulli shift $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ as given by Example 2.1.3. Given $n \in \mathbb{Z}_{\geq 1}$ and $a_0,\dots,a_{n-1} \in \Gamma$ we can define a *cylinder of depth n* by

$$[a_0 \cdots a_{n-1}] := \{(\gamma_k)_{k\geq 0} \in \Gamma^{\mathbb{Z}_{\geq 0}} : \gamma_0 = a_0,\dots,\gamma_{n-1} = a_{n-1}\}.$$

Cylinders can be seen as the building blocks of $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}})$ as the set of *cylinders*

$$\mathscr{C} := \{[a_0 \dots a_{n-1}] : n \in \mathbb{Z}_{\geq 1}, i \in \{0,\dots,n-1\}, a_i \in \Gamma,\} \subseteq \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}},$$

generates $\mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}$ by definition. Moreover, for any $n \in \mathbb{Z}_{\geq 1}$, the collection

$$\mathscr{C}^n := \{[a_0 \dots a_{n-1}] : i \in \{0,\dots,n-1\}, a_i \in \Gamma\}\},$$

partitions $\Gamma^{\mathbb{Z}_{\geq 0}}$. We can see $\mathscr{C}^n$ is the n'th refinement of $\mathscr{C}^1$, so that our notation is consistent with Definition 2.1.22. To show $\mathscr{C}^1$ is generating and separating for $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$, we first note the function

$$\begin{aligned} d_{\Gamma^{\mathbb{Z}_{\geq 0}}} &: \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to [0,1] \\ (\gamma,\gamma') &\mapsto \begin{cases} \sup\{2^{-n} : \gamma'_n \neq \gamma_n, n \in \mathbb{Z}_{\geq 0}\} & \text{if } \gamma \neq \gamma' \\ 0 & \text{if } \gamma = \gamma'. \end{cases} \end{aligned} \tag{8}$$

defines a metric on $\Gamma^{\mathbb{Z}_{\geq 0}}$. In fact, $\mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}$ can be seen as the Borel σ-algebra on $\Gamma^{\mathbb{Z}_{\geq 0}}$ induced by $d_{\Gamma^{\mathbb{Z}_{\geq 0}}}$. Then note that for $n \in \mathbb{Z}_{\geq 1}$ we have

$$\operatorname{diam}(\mathscr{C}^n) = \sup\{\sup_{x,y\in C} d_{\Gamma^{\mathbb{Z}_{\geq 0}}}(x,y) : C \in \mathscr{C}^n\} = 2^{-n}.$$

Applying Lemma 2.1.25, then shows that $\mathscr{C}^1$ is generating and separating for $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$.

In this text we shall apply Lemma 2.1.25 to various dynamical systems such as Bernoulli shifts and the doubling map. We postpone formalising this for now. To understand better what it means to have a generating and separating partition for a dynamical system we now show how we can use it to naturally define a metric. For this we require the *separation time* which we will define below.

Definition 2.1.27. Let $(X, \mathscr{F}, \mu, T)$ be some dynamical system with $\mu(X) \in (0, \infty)$ and $\mathscr{P} \subseteq \mathscr{F}$ a countable partition of X. We define the mapping

$$\alpha : X \to \mathscr{P}, \qquad x \mapsto P \quad \text{for the unique } P \ni x,$$

and the *separation time* $s : X \times X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as the mapping

$$s(x, x') = \inf\left\{n \in \mathbb{Z}_{\geq 0} : \alpha\left(T^n(x)\right) \neq \alpha\left(T^n(x')\right)\right\}.$$

We shall now show the separation time induces a metrisable topology on X if the partition it is based on is generating and separating. We start with a preliminary lemma.

Lemma 2.1.28. *Let $(X, \mathscr{F}, \mu, T)$ be some dynamical system with $\mu(X) \in (0, \infty)$ and let $\mathscr{P} \subseteq \mathscr{F}$ be a countable partition of X. Let $n \in \mathbb{Z}_{\geq 1}$ and let $x, y \in X$ be such that there is a $P \in \mathscr{P}^n$ with $x, y \in P$. Then for $i \in \{0, \ldots, n\}$ we have*

$$s\left(T^i(x), T^i(y)\right) = s(x, y) + i. \tag{9}$$

Proof. First, note that if $Q \in \mathscr{P}$ we have for $a, b \in Q$ that $\alpha(a) = \alpha(b)$ so

$$\begin{aligned} s(a, b) &= \inf\{k \in \mathbb{Z}_{\geq 0} : \alpha(T^k(a)) \neq \alpha(T^k(b))\} \\ &= \inf\{k \in \mathbb{Z}_{\geq 1} : \alpha(T^k(a)) \neq \alpha(T^k(b))\} \\ &= \inf\{k \in \mathbb{Z}_{\geq 1} : \alpha(T^{k-1}(T(a))) \neq \alpha(T^{k-1}(T(b)))\} \\ &= \inf\{k \in \mathbb{Z}_{\geq 0} : \alpha(T^k(T(a))) \neq \alpha(T^k(T(b)))\} + 1 \\ &= s(T(a), T(b)) + 1. \end{aligned} \tag{10}$$

Now since $P \in \mathscr{P}^n$, we have some $P_0, \ldots, P_{n-1} \in \mathscr{P}$ such that

$$P = P_0 \cap T^{-1}P_1 \cap \cdots \cap T^{-n+1}P_{n-1}.$$

Note then that for each $i \in \{0, \ldots, n-1\}$ we have $\alpha(T^i(x)) = \alpha(T^i(y))$ as

$$T^i(x), T^i(y) \in T^i[P_0 \cap \cdots \cap T^{-i+1}P_{i-1}] \cap P_i \cap T^i[T^{-i-1}P_{i+1} \cap \cdots \cap T^{-n-1}P_{n+1}] \subseteq P_i,$$

where we applied the set identity Lemma 2.1.23. Knowing this, we can apply Identity (10) i times to obtain

$$s(T^i(x), T^i(y)) = s(x, y) + i.$$

Equation (9) follows. □

Lemma 2.1.29. *Let $(X, \mathscr{F}, \mu, T)$ be some dynamical system with $\mu(X) \in (0, \infty)$ and $\mathscr{P} \subseteq \mathscr{F}$ a generating and separating partition and let $s : X \times X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be the separation time associated with $\mathscr{P}$. For any $C \in \mathbb{R}_{>0}, \beta \in (0,1)$ the mapping*

$$d_{\beta,C} : X \times X \to [0, C]$$
$$(x, y) \mapsto \begin{cases} C\beta^{s(x,y)}, & s(x,y) \neq \infty \\ 0, & \textit{else.} \end{cases}$$

is a metric.

Proof. Fix $\beta \in (0,1)$ and $C \in \mathbb{R}_{>0}$. For notational convenience we write $d := d_{\beta,C}$. To show $d[X \times X] \subseteq [0, C]$ first let $x, y \in X$ and note that if $s(x,y) = \infty$ we have $d(x,y) = 0$. If $s(x,y) \neq \infty$ we have $s(x,y) \in \mathbb{Z}_{\geq 0}$ so $\beta^{s(x,y)} \leq 1$ and thus $d(x,y) = C\beta^{s(x,y)} \leq C$ so that indeed $d[X \times X] \subseteq [0, C]$.

We shall now prove d is a metric and to do so let $x, y, z \in X$ be arbitrary.

Positive Definiteness We already know $d(x,y) \geq 0$. Suppose $d(x,y) = 0$. We know this implies $s(x,y) = \infty$ and so we have for all $n \in \mathbb{Z}_{\geq 0}$, $\alpha(T^n(x)) = \alpha(T^n(y))$. Then write $(P_n)_{n\geq 0} = (\alpha(T^n(y))_{n\geq 0}$ and note by construction this implies $x, y \in \bigcap_{n=0}^{\infty} T^{-n} P_n$. As $\mathscr{P}$ is separating we then have $x = y$.

Symmetry We immediately see $d(x,y) = C\beta^{s(x,y)} = C\beta^{s(y,x)} = d(y,x)$.

Triangle Inequality We discern, without loss of generality, three cases:

$$\alpha(x) = \alpha(y), \quad \alpha(z) \neq \alpha(x) \tag{11}$$
$$\alpha(x) \neq \alpha(y), \quad \alpha(y) \neq \alpha(z), \qquad \alpha(x) \neq \alpha(z) \tag{12}$$
$$\alpha(x) = \alpha(y) = \alpha(z) \tag{13}$$

In Case (11) we can see $\alpha(z) \neq \alpha(y)$ so $\beta^{s(y,z)} = 1$,

$$d(x,y) + d(y,z) = C(\beta^{s(x,y)} + \beta^{s(y,z)}) = C\beta^{s(x,y)} + C \geq C = C\beta^{s(x,z)} = d(x,z).$$

In Case (12) we see $s(x,y) = s(y,z) = s(x,z) = 0$ so

$$d(x,y) + d(y,z) = C(\beta^{s(x,y)} + \beta^{s(y,z)}) = 2C \geq C = C\beta^{s(x,z)} = d(x,z).$$

For Case (13), write $m := \min\{s(x,y), s(y,z), s(x,z)\}$ and note we have $m \geq 1$. As this m represents the highest integer such that there is a $P \in \mathscr{P}^m$ with $x, y, z \in P$, this is then also the highest integer m such that there is

$$P' \in \mathscr{P} \text{ with } T^{m-1}(x), T^{m-1}(y), T^{m-1}(z) \in P',$$

so that $\alpha(T^m(x)), \alpha(T^m(y)), \alpha(T^m(z))$ fall into either one of Cases (11) or (12). We then derive,

$$\begin{aligned} d(x,y) + d(y,z) &= \beta^m \left(d(T^m(x), T^m(y)) + d(T^m(y), T^m(z) \right) && (14) \\ &\geq \beta^m d(T^m(x), T^m(z)) && (15) \\ &= d(x,z) \end{aligned}$$

where in (14) we used Lemma 2.1.28 and in (15) we used either Case (11) or Case (12). We have shown the triangle inequality.

We conclude $d = d_{\beta,C}$ is a metric. □

Remark 2.1.30. In Section 4.1.1 we shall show that for general dynamical systems $(X, \mathscr{F}, \mu, T)$ with a generating and separating partition the topology $\mathscr{T}$ induced by the separation time is complete and separable, making $(X, \mathscr{T})$ a Polish Space, μ a Borel measure and $\mathscr{F}$ a Borel σ-algebra. We have postponed proving this to avoid stretching the preliminaries for too long and as it is not necessary for Section 3.

To the knowledge of the author the statement is seemingly missing from texts such as [25], [24] or [1]. We shall use the Polish space structure to give an alternative proof for Proposition 3.3.2, one of the cornerstones of Section 3.

2.2 Jacobians

The notion of a Jacobian, well-known from analysis, can be adapted to dynamical systems, purely relying on measure theoretical concepts. Relying on measure theory instead of analytical arguments is advantageous as we will frequently encounter mappings that have discontinuities on their domain - or mappings defined on spaces lacking a topology altogether.

Shortly put, in our setting a *Jacobian* of some mapping is a locally integrable function describing how much a small area gets stretched under applying the mapping. Similar to the Radon-Nikodym derivative we shall require some notion of non-singularity, which we call *pbn-singularity*, to be defined in Definition 2.2.3. Additionally, we shall require *forward measurability* of an operator, warranting the need for local invertibility as defined in Definition 2.2.1. These conditions may seem heavy for general dynamical systems but turn out to fit our situation perfectly.

The statements in this section are phrased as a special case of the more general results proven in Section 4.3 where additional results also have been phrased. To elaborate a bit more on this, see Remark 2.2.12 at the end of this section. We included the references to the more general statements in the headers of their respective definitions and claims.

Definition 2.2.1 (4.3.1)**.** Let $(X, \mathscr{F})$ be a measurable space and let $T : X \to X$ be a measurable mapping.

1. If $T : X \to X$ is measurable, bijective and has a measurable inverse, we call T *bi-measurable.*

2. If $A \in \mathscr{F}$ is so that $T(A) \in \mathscr{F}$ and $T|_A : A \to T(A)$ is bi-measurable (onto its image) then we call A an *invertibility domain for T*.

3. If there exists a countable partition $\mathscr{P}$ of X consisting of invertibility domains for T, then we call T *locally invertible.*

Locally invertible mappings map measurable sets to measurable sets and measurable subsets of invertibility domains are again invertibility domains.

Lemma 2.2.2 (4.3.3)**.** *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system. Then,*

1. *if $A \in \mathscr{F}$ is an invertibility domain for T then every $B \in \mathscr{F}$ with $B \subseteq A$ is an invertibility domain for T;*

2. *if $T : X \to X$ is bi-measurable, then for any $A \in \mathscr{F}$, we have that $T|_A : A \to T(A)$ is bi-measurable;*

3. *if $T : X \to X$ is locally invertible, then for any $A \in \mathscr{F}$, we have that $T(A) \in \mathscr{F}$.*

Now we are in the position to define *pullback non-singularity* for locally invertible transformations on finite measure spaces.

Definition 2.2.3 (4.3.4)**.** Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0, \infty)$ and let $T : X \to X$ be a locally invertible mapping. We say T is *pullback non-singular* or *pbn-singular* if for every invertibility domain $A \in \mathscr{F}$, $\mu(A) = 0$ implies $\mu(T(A)) = 0$.

We can verify pullback non-singularity using a single partition consisting of invertibility domains.

Lemma 2.2.4 (4.3.6)**.** *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0, \infty)$ and T be a locally invertible mapping. Then T is pullback non-singular if and only if for some partition $\mathscr{P}$ of invertibility domains for T we have*

$$(T|_P)^{-1}_{\star} \mu_{T(P)} \ll \mu_P \quad \text{for each} \quad P \in \mathscr{P}. \tag{16}$$

We now present an easy example to explain the difference between pbn-singularity and non-singularity.

Example 2.2.5. Define the Borel space $([0,1], \mathscr{B}[0,1], \lambda + \delta_{\{1\}})$ with $\lambda + \delta_{\{1\}}$ being the sum of the Lebesgue measure and the Dirac measure supported on $\{1\}$ respectively. Define

$$G : [0,1] \to [0,1] \qquad\qquad T : [0,1] \to [0,1]$$

$$x \mapsto \begin{cases} x, & x \in (0,1] \\ 1, & x = 0, \end{cases} \qquad\qquad x \mapsto \begin{cases} x, & x \in [0,1) \\ 0, & x = 1. \end{cases}$$

We show G is not pbn-singular but is non-singular, and that T is non-singular but not pbn-singular.

Note both T and G are measurable and $\mathscr{P}_G = \{\{0\},(0,1]\}$ and $\mathscr{P}_T = \{[0,1),\{1\}\}$ partition $[0,1]$ into invertibility domains for G and T respectively.

Note that

$$(\lambda+\delta_{\{1\}})(\{0\}) = 0, \quad (\lambda+\delta_{\{1\}})(G\{0\}) = 1,$$

showing that G is not pbn-singular, however for $A \in \mathscr{B}[0,1]$ we have

$$(\lambda+\delta_{\{1\}})(T(A)) = \lambda(A) \le (\lambda+\delta_{\{1\}})(A),$$

so T *is* pbn-singular. As for non-singularity, note that for $A \in \mathscr{B}[0,1]$

$$G^{-1}(A) = \begin{cases} \{0\} \cup A, & 1 \in A \\ A \setminus \{0\}, & 1 \notin A \end{cases} \qquad T^{-1}(A) = \begin{cases} \{1\} \cup A, & 0 \in A \\ A \setminus \{1\}, & 0 \notin A \end{cases}.$$

We then see

$$(\lambda+\delta_{\{1\}})(G^{-1}(A)) = (\lambda+\delta_{\{1\}})(A) \quad \text{and} \quad (\lambda+\delta_{\{1\}})(T^{-1}(\{0\})) = 1 \text{ while } (\lambda+\delta_{\{1\}})(\{0\}) = 0,$$

showing non-singularity of G and that T is not non-singular.

We are now ready to define the Jacobian. The Jacobian is used to describe how much invertibility domains get 'stretched' under the operator of a dynamical system.

Definition 2.2.6 (4.3.7)**.** Let $(X,\mathscr{F},\mu,T)$ be a dynamical system with $\mu(X) \in (0,\infty)$ and let $T: X \to X$ be a locally invertible, pullback non-singular mapping. A function $JT: X \to [0,\infty)$ such that $JT \cdot \mathbb{1}_P \in L^1(X)$ and

$$\mu(T(P)) = \int_P JT(x)\, d\mu(x), \quad \text{for every invertibility domain } P \in \mathscr{F} \tag{17}$$

is called a *Jacobian* of T.

The following lemma is an existence and uniqueness result for Jacobians in our setting.

Lemma 2.2.7 (4.3.8)**.** *Let $(X,\mathscr{F},\mu,T)$ be a dynamical system with $\mu(X) \in (0,\infty)$ and let $T: X \to X$ be a locally invertible, pullback non-singular mapping. Then a Jacobian $JT: X \to [0,\infty)$ exists and is unique up to a measure zero set. Furthermore, in assuming $\mathscr{P} \subseteq \mathscr{F}$ is a countable partition of X consisting of invertibility domains for T we have for $P \in \mathscr{P}$,*

$$(JT \cdot \mathbb{1}_P)(x) = \begin{cases} \frac{d(T|_P^{-1})_\star \mu}{d\mu}(x), & x \in P \\ 0, & \text{else,} \end{cases} \qquad \mu\text{-a.s.},$$

and

$$JT = \sum_{P \in \mathscr{P}} JT \cdot \mathbb{1}_P, \qquad \mu\text{-a.s.}.$$

We conclude our discussion of Jacobians with two technical characterisations of Jacobians.

Lemma 2.2.8 (4.3.11)**.** *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0,\infty)$ and T a locally invertible, pullback non-singular mapping. Suppose for some invertibility domain $A \in \mathscr{F}$ we have $JT > 0$, μ_A-almost surely. Then*

$$(JT(x))^{-1} = \frac{d(T|_A)_\star \mu}{d\mu}(T(x)), \qquad \text{for } \mu_A\text{-almost every } x \in A.$$

As hinted at before, refinements of partitions consisting of invertibility domains are well-behaved with higher iterates of operators of dynamical systems.

Lemma 2.2.9 (4.3.13)**.** *Let $(X, \mathscr{F}, \mu, T)$ be some dynamical system with $\mu(X) \in (0,\infty)$ and suppose $T : X \to X$ is a locally invertible and pullback non-singular mapping with a partition $\mathscr{P}$ consisting of invertibility domains. Then for each $n \in \mathbb{Z}_{\geq 1}$ and each $k \in \{1, \ldots, n\}$ the n-th refinement $\mathscr{P}^n$ consists of invertibility domains for T^k. Moreover, T^k is pullback non-singular.*

Finally, we phrase the Chain Rule for Jacobians.

Proposition 2.2.10 (4.3.16, Chain Rule For Jacobians)**.** *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system with $\mu(X) \in (0,\infty)$ and let T be locally invertible and pullback non-singular. Then we have for each $n \in \mathbb{Z}_{\geq 1}$ that $J(T^n)$ exists and that*

$$J(T^n) = \prod_{i=0}^{n-1} J(T) \circ T^i \quad \text{holds} \;\; \mu\text{-a.e.}.$$

Below we present a non-trivial example of a Jacobian associated with a locally invertible mapping. We postpone the proof until Lemma 4.3.17 to avoid clouding this subsection with statements we shall not need for any other purpose than this lemma.

Lemma 2.2.11. *Let $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ be some Bernoulli shift with weights $P = (p_\gamma)_{\gamma \in \Gamma}$. Then for each $n \geq 1$ and $k \in \{1, \ldots, n\}$ the collection*

$$\mathscr{C}^n = \left\{ [\gamma_0 \cdots \gamma_{n-1}] \subseteq \Gamma^{\mathbb{Z}_{\geq 0}} : \gamma_0, \ldots, \gamma_{n-1} \in \Gamma \right\}$$

of cylinders of depth n consists of invertibility domains for $\sigma^k : \Gamma^{\mathbb{Z}_{\geq 0}} \to \Gamma^{\mathbb{Z}_{\geq 0}}$. Moreover, $\sigma^k : \Gamma^{\mathbb{Z}_{\geq 0}} \to \Gamma^{\mathbb{Z}_{\geq 0}}$ is locally invertible and pbn-singular with a Jacobian satisfying

$$J\sigma^k \equiv \frac{1}{p_{\gamma_0} \cdots p_{\gamma_{k-1}}}, \quad \mathbb{P}\text{-almost everywhere.}$$

Remark 2.2.12. 1. For a different treatise of the subject, [23, Chapter 9.7] is a good reference, but many proofs are missing there. Moreover, in Section 4.3 we shall need to generalise the notion of a Jacobian in a measure-theoretic setting slightly which is why we have omitted proofs in this section. We point out that our notion of local invertibility is a bit more strict than what is done in [23, Chapter 9.7] where only a countable cover of invertibility domains instead of countable partition is assumed. We believe that our convention leads to an easier understanding of Jacobians however and is sufficient in many cases, in particular for our purpose.

2. In some literature, such as [23, Chapter 9.7], non-singularity is defined as we define pullback non-singularity. We make a distinction between the two to avoid any confusion with the more common definition of non-singularity

3 The Deterministic Case

As said in Section 1 the main goal of this thesis is finding conditions for which random dynamical systems with a non-uniformly expanding base dynamic admit absolutely continuous invariant probability measures. Our method is built upon a framework developed by Lai-Sang Young in the papers [25] and [24] to describe deterministic dynamical systems. We shall start by sketching this method in Section 3.1. After that, we shall formally define a *Tower Base* in Section 3.2 and prove it admits an acip in Section 3.3. Subsequently, in Section 3.4 we shall formalise the notion of a Young Tower and in Section 3.5 prove it admits an acip. We shall close Section 3 with two easy examples to which Young Tower Theory can be applied. Sections 3.2 - 3.5 have been written with the help of [1],[25] and [24].

3.1 Intuition for Young's Theory

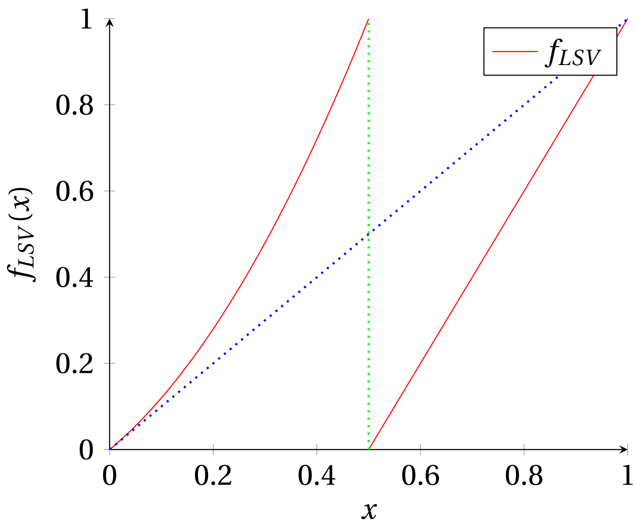
In this (sub)section, we shall do our best to explain what kind of systems Young's theory applies to, why finding acip's can be subtle and why we need Young Towers. To narrow our point of focus we shall mainly consider dynamical systems $([0,1),\mathscr{B}[0,1),\lambda,T)$ where $([0,1),\mathscr{B}[0,1),\lambda)$ is the standard Borel space on $[0,1)$, equipped with the Lebesgue measure and $T:[0,1)\to[0,1)$ is some operator.

We shall start easy. Consider the doubling map from Example 2.1.3. Lemma 2.1.18 states that the doubling map $([0,1),\mathscr{B}[0,1),\lambda,D)$ is ergodically equivalent with a Bernoulli shift on $\{0,1\}$, written as $(\{0,1\}^{\mathbb{Z}_{\geq 0}},\mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}},\mathbb{P},\sigma)$. To sketch the proof, we partition the interval $[0,1)$ in $\mathscr{P}=\{[0,\frac{1}{2}),[\frac{1}{2},1)\}$ up to a set of measure one. Subsequently, we show that refinements of $\mathscr{P}$ can be naturally identified by the cylinders in $\{0,1\}^{\mathbb{Z}_{\geq 0}}$ through an ergodic isomorphism π. The fact that there exists an acip for Bernoulli shifts then implies the existence of an acip for the doubling map (i.e. as follows from Lemma 2.1.17). If we want to find *acip's* for more general systems $([0,1), \mathscr{B}[0,1), \lambda, T)$, the natural question then arises, how big is the class of dynamical systems where $([0,1), \mathscr{B}[0,1), \lambda, T)$ is ergodically isomorphic with some Bernoulli shift? Or, more generally, to what extent does the behaviour of some cleverly chosen partition predict the existence of an acip?

At the very least, we know that if $([0,1), \mathscr{B}[0,1), \lambda, T)$ for some T is ergodically equivalent with some Bernoulli shift $(\Gamma^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Gamma^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ we should be able to encode the dynamics of T symbolically. That is, we should be able to construct a finite partition $\mathscr{P}_T$ on $[0,1)$ having similar properties as a partition of $\Gamma^{\mathbb{Z}_{\geq 0}}$ into cylinder sets. Specifically, we would have

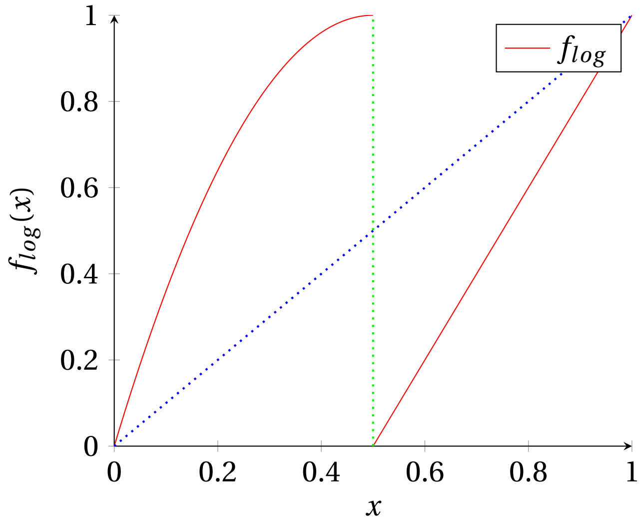
1. For each $P \in \mathscr{P}_T$ the mapping $T|_P : P \to [0,1)$ is bi-measurable.
2. The partition $\mathscr{P}_T$ is generating and separating for $([0,1), \mathscr{B}[0,1), \lambda, T)$.

The next natural question then arises. Are these properties sufficient for obtaining an acip for $([0,1), \mathscr{B}[0,1), \lambda, T)$? As is well known, we can answer this negatively by looking at the instance of the *LSV-map* $([0,1), \mathscr{B}[0,1), \lambda, f_{LSV})$ below.



As shown in [18], the LSV-map does not admit an acip, yet it can be shown the partition $\mathscr{P}_{LSV} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ does satisfy the properties just mentioned. A property the doubling map has, however, which the LSV map lacks, is (uniform) expandingness. When systems are uniformly expanding we expect distances between points that are close to grow uniformly. The LSV-map lacks this property as $f'_{LSV}(x) \downarrow 1$ approaches 1 as $x \downarrow 0$. As a first mention, in [15, Theorem 2.4.6] partitions satisfying the previously mentioned properties are constructed explicitly for expanding maps on the circle. Two other key results linking (a notion of) expandingness and the existence of an acip are [17] and, more generally applicable, [23, Theorem 11.1.2].

Keeping this in mind, one might then assume that being 'expanding' is necessary for the existence of acip for systems of the form $([0,1), \mathscr{B}[0,1), \lambda, T)$. But again, the problem is more subtle: the *logistic map* as graphed below *does* admit an acip with respect to the Lebesgue measure. Despite not being uniformly expanding.



Hence, we need more sophisticated tools to successfully analyse systems such as $([0,1), \mathscr{B}[0,1), \lambda, f_{log})$ and $([0,1), \mathscr{B}[0,1), \lambda, f_{LSV})$.

The crucial remark to make is that even if f_{log} and f_{LSV} are not 'expanding' everywhere, the maps will still separate arbitrarily close points when applied a large amount of times. To use this, Lai-Sang Young defined in [25] and [24] an *induced domain* $\Lambda \subseteq [0,1)$ and a *return time*

$$R : \Lambda \to \mathbb{Z}_{\geq 1} \cup \{\infty\}, \qquad x \mapsto \inf\left\{n \in \mathbb{Z}_{\geq 1} : g^n(x) \in \Lambda\right\}.$$

Assuming the return time indeed takes values in $\mathbb{Z}_{\geq 1}$ we can define an induced dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, T^R)$, which we shall refer to as a tower base. Young then assumes there exists a partition $\mathscr{P}_\Lambda$ with properties similar to the following:

(Constant Return Time) For each $P \in \mathscr{P}_\Lambda$ the mapping $R|_P$ is constant.

(Markov Property) For each $P \in \mathscr{P}_\Lambda$, $g^R|_P : P \to \Lambda$ is bi-measurable and $(g^R|_P^{-1})_\star \mu_\Lambda \ll \mu_\Lambda$.

(Generating and Separating) the partition $\mathscr{P}_\Lambda$ is generating and separating.

Note as we can not expect that $R : \Lambda \to \mathbb{Z}_{\geq 1}$ is bounded the partition $\mathscr{P}_\Lambda$ will likely be countably infinite instead of finite. This will complicate finding an acip but it is something we can overcome.

Finally, this construction allows us to control the distortion of the Jacobian in terms of time it takes points to separate using the *separation time* from Lemma 2.1.27. On our 'naive' partition of $\{[0,\frac{1}{2}),[\frac{1}{2},1)\}$ we could define a separation time, but we can not use this to analyse the distortion of the Jacobian in f_{log} due to the fact that $f'_{log}(x) \to 0$ as $x \uparrow \frac{1}{2}$.

Finding an acip for the *induced systems* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ will be the contents of Sections 3.2 - 3.3. In Sections 3.4-3.5 we will phrase conditions under which this acip implies the existence of an acip on the base system where we shall introduce the concept of a *Young Tower*.

3.2 The Tower Base

For this section we fix a dynamical system $(X, \mathscr{F}, m, g)$ with $m(X) \in (0,\infty]$. Our first goal is to formalise the notion of a Tower Base as mentioned in Section 3.1, and start by defining the *return time* and *induced domain*.

Definition 3.2.1. Suppose we have $\Lambda \in \mathscr{F}$ with $m(\Lambda) \in (0,\infty)$, if the map

$$R : \Lambda \to \mathbb{Z}_{\geq 1} \cup \{\infty\}, \qquad x \mapsto \inf\left\{n \in \mathbb{Z}_{\geq 1} : g^n(x) \in \Lambda\right\},$$

takes values in $\mathbb{Z}_{\geq 1}$ we call R a *return time* and Λ its *induced domain*.

If we have a return time R and induced domain Λ for $(X, \mathscr{F}, m, g)$, we consider the mapping $g^R : \Lambda \to \Lambda$, as given by $g^R(x) = g(x)^{R(x)}$ for $x \in \Lambda$. Then define $\mu = \frac{m}{m(\Lambda)}$. In restricting $(X, \mathscr{F}, \mu)$ to Λ we can then define the tuple $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$, where μ_Λ is a probability measure. The following lemma asserts this tuple is a dynamical system.

Lemma 3.2.2. *Let $(X, \mathscr{F}, \mu, g)$ be some dynamical system and let $\Lambda \in \mathscr{F}$ be such that $\mu(\Lambda) = 1$ and suppose we have an integer valued measurable mapping $R : \Lambda \to \mathbb{Z}_{\geq 1}$ such that $g(x)^{R(x)} \in \Lambda$ for each $x \in \Lambda$. Then $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ is a dynamical system.*

Proof. As Λ is a measurable set of finite measure we can define the finite measure space $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda)$. We need to show g^R is $\mathscr{F}_\Lambda$-measurable. To do so, define

$$\mathscr{P}_R = \{R^{-1}\{n\} : n \in \mathbb{Z}_{\geq 1}\},$$

and note this is a countable collection consisting of $\mathscr{F}_\Lambda$-measurable sets.

We then note that for an arbitrary $A \in \mathscr{F}_\Lambda$ we have

$$(g^R)^{-1}(A) = \bigsqcup_{n \in \mathbb{Z}_{\geq 1}} (g^R|_{R^{-1}\{n\}})^{-1}(A) = \bigsqcup_{n \in \mathbb{Z}_{\geq 1}} R^{-1}\{n\} \cap (g^n)^{-1}(A) \in \mathscr{F}_\Lambda,$$

by measurability of g.

Similarly assuming we have $\mu_\Lambda(A) = 0$ we can see using the non-singularity of g that

$$\mu_\Lambda((g^R)^{-1}A) = \sum_{n\in\mathbb{Z}_{\geq 1}} \mu_\Lambda(R^{-1}\{n\} \cap (g^n)^{-1}(A)) \leq \sum_{n\in\mathbb{Z}_{\geq 1}} \mu((g^n)^{-1}A) = 0,$$

proving $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, T^R)$ is a dynamical system. □

In the sequel until and including Section 3.4 we fix a dynamical system $(X, \mathscr{F}, m, g)$ and assume there exists a return time R with induced domain Λ such that $m(\Lambda) \in (0, \infty)$. We then call $(X, \mathscr{F}, m, g)$ the *base dynamics*. We will focus on the dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ and analyse its properties.

We shall only refer to the underlying *base dynamics* $(X, \mathscr{F}, m, g)$ implicitly until we are readily equipped to analyse it. For now, we focus on the dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$, where a return time and induced domain are implicit. We shall fix $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ until and including Section 3.5.

Definition 3.2.3. Suppose we have a countable partition $\mathscr{P}_\Lambda \subseteq \mathscr{F}_\Lambda$ consisting of sets of positive measure of Λ with the following properties:

(Constant Return Time) For each $P \in \mathscr{P}_\Lambda$ the mapping $R|_P$ is constant.

(Markov Property) For each $P \in \mathscr{P}_\Lambda$, $g^R|_P : P \to \Lambda$ is bi-measurable and $(g^R|_P^{-1})_\star \mu_\Lambda \ll \mu_\Lambda$.

(Generating and Separating) the partition $\mathscr{P}_\Lambda$ is generating and separating.

Then we call $\mathscr{P}_\Lambda$ a *principal partition* for $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$.

Remark 3.2.4. Typically, in the literature, such as in [24] or [1], the Markov property merely assumes the bi-measurability onto a Λ, not also pullback non-singularity. In this text the pullback non-singularity is there to guarantee the existence of the Jacobian. This existence is in the literature usually assumed as part of bounded distortion, seen in Definition 3.2.6, without phrasing conditions for which this object can actually exist. Outside of Young Tower theory the Markov Property is commonly phrased even more leniently, for instance as in [15], for instance not requiring surjectivity. The author is aware of this deviation.

The curious reader can skip ahead to Section 3.6 to see an elementary example of a principal partition. For the rest of this section we shall assume our system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ has a principal partition $\mathscr{P}_\Lambda$. By definition, $\mathscr{P}_\Lambda$ consists of invertibility domains for g^R so g^R is locally invertible and by the Markov Property we can apply Lemma 2.2.4 to see g^R is pbn-singular as well. By Lemma 2.2.7 we then see Jg^R exists and is unique up to a μ_Λ-measure zero set.

The bijective property in the Markov property can be extended inductively to refinements of $\mathscr{P}_\Lambda$.

Lemma 3.2.5. *For each $n \in \mathbb{Z}_{\geq 1}$, and $A \in \mathscr{P}_\Lambda^n$*

1. *the mapping $(g^R)^n|_A : A \to \Lambda$ is bi-measurable, and*
2. *we have $\mu_\Lambda(A) > 0$.*

Proof. 1. By Lemma 2.2.9 we only need to prove surjectivity. In the case $n = 1$ the statement follows directly from the Markov Property. In supposing we have for some $p \in \mathbb{Z}_{\geq 1}$ that for any $A' \in \mathscr{P}_\Lambda^p$ we have $(g^R)^p(A') = \Lambda$that for $A \in \mathscr{P}_\Lambda^{p+1}$ there exist $A_0 \dots, A_p \in \mathscr{P}_\Lambda$ such that

$$A = A_0 \cap \cdots \cap (g^R)^{-p} A_p).$$

Consequently, we have

$$\begin{aligned}(g^R)^{p+1}(A) &= (g^R)^{p+1}(A_0 \cap \cdots \cap (g^R)^{-p+1}(A_{p-1}) \cap (g^R)^{-p} A_p) \\ &= g^R\left((g^R)^p(A_0 \cap \cdots \cap (g^R)^{-p+1}(A_{p-1})) \cap A_p)\right) && (18) \\ &= g^R(A_p) && (19) \\ &= \Lambda, && (20)\end{aligned}$$

where in Equation (18) we used Lemma 2.1.23, in Equation (19) we used the induction hypothesis and in Equation (20) we used the Markov property.

2. Note that for general $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{P}_\Lambda^n$ we know A is an invertibility domain for $(g^R)^n$ by Lemma 2.2.9 and that $(g^R)^n(A) = \Lambda$, so if we would have $\mu(A) = 0$ then $\mu_\Lambda(\Lambda) = \mu_\Lambda((g^R)^n(A)) = 0$ by pullback non-singularity. As we know $\mu_\Lambda(\Lambda) = 1$ we then must have $\mu_\Lambda(A) > 0$. □

As mentioned in Section 3.1 we need to be able to control the distortion of the Jacobian in terms of the separation time metric as defined in Definition 2.1.27. To do so, we shall for the rest of this section fix a $\beta \in (0,1)$ and $C \in \mathbb{R}_{>0}$ and the separation time metric $d_{\beta,C}$ induced by $\mathscr{P}_\Lambda$. Without proof for now, we shall assume the joint measurability of $d_{\beta,C}$, see Remark 2.1.30.

We remind the reader that Jg^R exists and is unique up to a measure zero set.

Definition 3.2.6. If for the dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ for each $A \in \mathscr{P}_\Lambda$ the Jacobian $J(g^R)|_A : A \to [0,\infty)$ is strictly positive μ_A-almost everywhere and satisfies

$$\left|\frac{Jg^R(x)}{Jg^R(y)} - 1\right| \leq d_{\beta,C}(x,y) \qquad \text{for almost every } x, y \in A,$$

we say the dynamical system has *bounded distortion.*

If the dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ has bounded distortion, we refer to it as a *tower base*. From here on out we shall assume $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ is a tower base. We use the non-singularity of g^R to extend bounded distortion to $(g^R)^n$ as follows for general $n \in \mathbb{Z}_{\geq 1}$.

Lemma 3.2.7. *Let $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ be a tower base and let $n \in \mathbb{Z}_{\geq 1}$. For each $A \in \mathscr{P}_\Lambda^n$, we have a set $\dot{A} \in \mathscr{F}_A$ such that $\mu_\Lambda(A \setminus \dot{A}) = 0$ and for each $i \in \{0, \dots, n-1\}$*

$$\left| \frac{J(g^R)((g^R)^i(x))}{J(g^R)((g^R)^i(y))} - 1 \right| \leq d_{\beta,C}((g^R)^i(x), (g^R)^i(y)), \quad \textit{for every } x, y \in \dot{A}. \tag{21}$$

Proof. Let $A \in \mathscr{P}_\Lambda^n$. First we will construct the set $\dot{A}$, show it satisfies Equation (21) and then show why it satisfies $\mu_A(A \setminus \dot{A}) = 0$. Let $A_{(0)}, \dots, A_{(n-1)} \in \mathscr{P}_\Lambda$ be such that

$$A = A_{(0)} \cap \cdots \cap (g^R)^{-(n-1)} A_{(n-1)}.$$

By bounded distortion, we can find for each $i \in \{0, \dots, n-1\}$ an $\dot{A}_{(i)} \in \mathscr{F}_{A_{(i)}}$ such that $\mu_{A_{(i)}}(A_{(i)} \setminus \dot{A}_{(i)}) = 0$ and

$$\left| \frac{Jg^R(x)}{Jg^R(y)} - 1 \right| \leq d_{\beta,C}(g^R(x), g^R(y)) \text{ for all } x, y \in \dot{A}_{(i)}.$$

More so, by non-singularity of g^R we have

$$\mu_\Lambda((g^R)^{-i} A_{(i)} \setminus (g^R)^{-i} \dot{A}_{(i)}) = (g^R)^i_\star \mu_\Lambda(A_{(i)} \setminus \dot{A}_{(i)}) = 0. \tag{22}$$

Then define

$$\dot{A} = \dot{A}_{(0)} \cap \cdots \cap (g^R)^{-(n-1)} \dot{A}_{(n-1)}$$

and note that using Lemma 2.1.23 we have

$$\begin{aligned}(g^R)^i \dot{A} &= (g^R)^i (\dot{A}_{(0)} \cap \cdots \cap (g^R)^{-(i-1)} \dot{A}_{(i-1)}) \cap \dot{A}_{(i)} \cap (g^R)^{-1} \dot{A}_{(i+1)} \cap \dots (g^R)^{-(n-1)+i} \dot{A}_{(n-1)} \\ &\subseteq \dot{A}_{(i)},\end{aligned}$$

so that

$$\left| \frac{Jg^R((g^R)^i(x))}{Jg^R((g^R)^i(y))} - 1 \right| \leq d_{\beta,C}((g^R)^i(x), (g^R)^i(y)) \text{ for all } x, y \in \dot{A}.$$

More so, as $\dot{A}_{(i)} \subseteq A_{(i)}$ for $i \in \{0, \dots, n-1\}$ we can derive using general set identities that we have

$$A \setminus \dot{A} = (A_{(0)} \setminus \dot{A}_{(0)}) \cup \cdots \cup (g^R)^{-(n-1)} (A_{(n-1)} \setminus \dot{A}_{(n-1)}).$$

By Equation (22),$A \setminus \dot{A}$ is the union of a finite amount of measure-zero sets, so that indeed $\mu_\Lambda(A \setminus \dot{A}) = 0$. □

Remark 3.2.8. For tower bases $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ one can show using the chain rule in Proposition 2.2.10 and the positivity phrased in bounded distortion that for $A \in \mathscr{P}_\Lambda$ $J((g^R)^n)$ is strictly positive for $n \in \mathbb{Z}_{\geq 0}$, μ_A-almost everywhere. As the proof is very similar to the proof of Lemmas 3.2.7 and 5.3.6 we shall omit proving this now.

We will see in Proposition 3.3.2 that for each $n \in \mathbb{Z}_{\geq 0}$ the density $\frac{d(g^R)^n_\star \mu_\Lambda}{d\mu_\Lambda}$ displays a remarkable Lipschitz property with respect to the separation time metric. This concept will also return in Section 4.1.1.

Definition 3.2.9. We call a function $f \in L^1(\Lambda)$ *Lipschitz on a set of full measure* if there exists a $\Lambda_f \in \mathscr{F}_\Lambda$ with $\mu(\Lambda \setminus \Lambda_f) = 0$ and a $L \in \mathbb{R}_{>0}$ such that for each $x, y \in \Lambda_f$ we have

$$|f(x) - f(y)| \leq L \cdot d_{\beta,C}(x, y).$$

Remark 3.2.10. As for the well-definedness of Definition 3.2.9 we want that the property 'Lipschitz on a set of full measure' is representation independent. To do so let g, h be two representations of some $f \in L^1(\Lambda)$ such that there exists a $\Lambda_g \in \mathscr{F}_\Lambda$ with $\mu(\Lambda \setminus \Lambda_g) = 0$ and that for each $x, y \in \Lambda_g$

$$|g(x) - g(y)| \leq L \cdot d_{\beta,C}(x, y).$$

As we know g is equal almost everywhere to h, say on some set $X_h \in \mathscr{F}_\Lambda$ we know that $g|_{X_h \cap \Lambda_g} = h|_{X_h \cap \Lambda_g}$, so that

$$|h(x) - h(y)| \leq L \cdot d_{\beta,C}(x, y),$$

for each $x, y \in X_h \cap \Lambda_g$. As we know $\mu_\Lambda(X_h \cap \Lambda_g) = \mu_\Lambda(\Lambda)$ we conclude that being Lipschitz on a set of full measure indeed is a property shared on equivalence classes in $L^1(\Lambda)$.

In the situation of a tower base, we are particularly interested in functions that are Lipschitz on a set of full measure with respect to $d_{\beta,C}$ and that are essentially bounded, as denoted by

$$L_{\beta,C}(\Lambda) := \left\{\phi \in L^\infty(\Lambda) : \phi \text{ is Lipschitz on a set of full measure}\right\}. \tag{23}$$

We can equip this space with

$$|\phi|_\beta = \operatorname{ess\,sup}_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d_{\beta,C}(x, y)} \quad \text{and} \quad ||\phi||_\beta = |\phi|_\beta + ||\phi||_\infty,$$

which are easily verified to be a seminorm and norm, respectively. As Λ is of finite measure, we obtain the inclusions $L_{\beta,C}(\Lambda) \subseteq L^\infty(\Lambda) \subseteq L^1(\Lambda)$.

3.3 The Tower Base Acip

3.3.1 The Existence Of An Acip

For this section, we remain having fixed a tower base $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ with principal partition $\mathscr{P}_\Lambda$ and separation time metric $d_{\beta,C}$ for some $\beta \in (0,1)$, and $C \in \mathbb{R}_{>0}$.

We shall start by proving that a sequence in $L_{\beta,C}(\Lambda)$ consisting of functions that are Lipschitz with the same constant and share a uniform upper bound admits an accumulation point in $L_{\beta,C}(\Lambda)$. The space $L_{\beta,C}(\Lambda)$ will later be shown to contain (convex combinations of) the densities $\frac{d(g^R)^n_\star \mu_\Lambda}{d\mu_\Lambda}$ for $n \in \mathbb{Z}_{\geq 0}$. We will then be able to use Proposition 3.3.2 for proving the existence of an invariant measure in Theorem 3.3.7. The proof of Proposition 3.3.2 below is based on [1] where a diagonalization argument is used. We make this explicit and for ease of reading we state this before phrasing the Lemma.

Lemma 3.3.1 (Diagonalisation Argument [20])**.** *Suppose we have for each $n \in \mathbb{Z}_{\geq 1}$, functions $f_n : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ and consider $(f_n)_{n \in \mathbb{Z}_{\geq 1}}$. Assume there exists a $C \in \mathbb{R}_{>0}$ such that $|f_n(l)| \leq C$ for all $l, n \in \mathbb{Z}_{\geq 1}$. Then there exists a subsequence $(f_{n_k})_{k \geq 0}$ of $(f_n)_{n \geq 0}$ such that $(f_{n_k}(l))_{k \geq 0}$ converges for each $l \in \mathbb{Z}_{\geq 1}$.*

We now prove the promised sequential compactness.

Proposition 3.3.2. *Suppose we have a tower base $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$, and a sequence $(\phi_n)_{n \in \mathbb{Z}_{\geq 1}} \subseteq L_{\beta,C}(\Lambda)$ satisfying for some $M > 1$,*

$$\sup_{n \in \mathbb{Z}_{\geq 1}} ||\phi_n||_\beta \leq M \quad \textit{and} \quad \inf_{n \in \mathbb{Z}_{\geq 1}} \operatorname{ess\,inf}_{x \in \Lambda} \phi_n(x) \geq \frac{1}{M}.$$

Then $(\phi_n)_{n \geq 1}$ converges pointwise almost everywhere and in $L^1(\Lambda)$ to a function $\phi \in L_{\beta,C}(\Lambda)$ with $||\phi||_\beta \leq M$ and $\operatorname{ess\,inf}_{x \in \Lambda} \phi(x) \geq \frac{1}{M}$.

Proof. First, for each $n \in \mathbb{Z}_{\geq 1}$, let $\dot{\Lambda}_n \in \mathscr{F}_\Lambda$ be a set for which $\mu_\Lambda(\dot{\Lambda}_n) = 1$ and that for each $x, y \in \dot{\Lambda}_n$ we have $|\phi_n(x) - \phi_n(y)| \leq M \cdot d_{\beta,C}(x,y)$ and $\phi(x) \leq M$. Note then that on $\dot{\Lambda} := \bigcap_{n=1}^\infty \dot{\Lambda}_n$ we have $|\phi_n(x) - \phi_n(y)| \leq M \cdot d_{\beta,C}(x,y)$ for each $x, y \in \dot{\Lambda}$ and $n \in \mathbb{Z}_{\geq 1}$ and that $\mu_\Lambda(\dot{\Lambda}) = 1$. We keep in mind that any $\mathscr{F}_\Lambda$-measurable set of positive measure has an intersection of positive measure with Λ.

In the proof we shall start by picking a representative of each set in the principal partition and apply Lemma 3.3.1 to find a first statement on convergence. To do so, for each $N \in \mathbb{Z}_{\geq 1}$ we let $\psi_N : \mathbb{Z}_{\geq 1} \to \mathscr{P}^N_\Lambda$ denote an enumeration of $\mathscr{P}^N_\Lambda$. Then define a (countable) choice function

$$\chi : \mathscr{P}^N_\Lambda \to \dot{\Lambda}, \qquad P \mapsto x \quad (s.t. \quad x \in P),$$

and define $(x_{l,N})_{l\in\mathbb{Z}_{\geq 1}} := (\chi \circ \psi_N(l))_{l\in\mathbb{Z}_{\geq 1}}$. Note then for all $N \in \mathbb{Z}_{\geq 1}$ we obtain a sequence of sequences $(\phi_n(x_{\cdot,N}))_{n\geq 1}$, such that

$$|\phi_n(x_{l,N})| \leq \sup_{n\in\mathbb{Z}_{\geq 1}} ||\phi_n||_\beta \leq M \quad \text{for all } n, l \in \mathbb{Z}_{\geq 1}.$$

By Lemma 3.3.1 we then obtain a sequence $(n_k)_{k\geq 0} \subseteq \mathbb{Z}_{\geq 1}$ with $(\phi_{n_k}(x_{l,N})_{k\geq 0}$ converging for all $l \in \mathbb{Z}_{\geq 1}$ (in terms of Lemma 3.3.1 we take $f_n(l) := \phi_n(x_{l,N})$).

Now we use the Lipschitz continuity of ϕ_n on $\dot{\Lambda}$ for each $n \in \mathbb{Z}_{\geq 1}$ to extend this convergence to all elements of $\dot{\Lambda}$. In doing so, let $x \in \dot{\Lambda}$ and $\epsilon \in \mathbb{R}_{>0}$ be arbitrary and $N \in \mathbb{Z}_{\geq 1}$ be such that $M \cdot C\beta^N < \frac{1}{3}\epsilon$. Note we can find exactly one $P \in \mathscr{P}_\Lambda^N$ such that for some $l \in \mathbb{Z}_{\geq 1}$ we have $x, x_{l,N} \in P$. We then have for each $k \in \mathbb{Z}_{\geq 1}$,

$$|\phi_{n_k}(x) - \phi_{n_k}(x_{l,N})| \leq M \cdot d_{\beta,C}(x, x_{l,N}) < \frac{\epsilon}{3}.$$

Now as $(\phi_{n_k}(x_{l,N}))_{k\geq 1}$ is converging in $k \geq 1$ for all $l \in \mathbb{Z}_{\geq 1}$, we may pick $k_0 \in \mathbb{Z}_{\geq 1}$ such that for all $k, m \geq k_0$ we have

$$|\phi_{n_k}(x_{l,N}) - \phi_{n_m}(x_{l,N})| < \frac{\epsilon}{3}.$$

Consequently,

$$\begin{aligned} |\phi_{n_k}(x) - \phi_{n_m}(x)| &\leq |\phi_{n_k}(x) - \phi_{n_k}(x_{l,N})| + |\phi_{n_k}(x_{l,N}) - \phi_{n_m}(x_{l,N})| \\ &\quad + |\phi_{n_m}(x) - \phi_{n_m}(x_{l,N})| \\ &< \epsilon, \end{aligned}$$

which shows $(\phi_{n_k}(x))_{k\geq 0}$ is Cauchy in $\mathbb{R}$ for each $x \in \dot{\Lambda}$. Now define for $x \in \Lambda$,

$$\phi'(x) = \begin{cases} \lim_{k\to\infty} \phi_{n_k}(x), & x \in \dot{\Lambda} \\ 0, & \text{else.} \end{cases},$$

which as a limit of measurable functions is measurable. Now in noting that for $x, y \in \dot{\Lambda}$ we have

$$\frac{|\phi'(x) - \phi'(y)|}{d_{\beta,C}(x,y)} = \lim_{k\to\infty} \frac{|\phi_{n_k}(x) - \phi_{n_k}(y)|}{d_{\beta,C}(x,y)} \leq \sup_{n\in\mathbb{Z}_{\geq 1}} |\phi_n|_\beta \leq M,$$

and

$$|\phi(x)| = \lim_{k\to\infty} |\phi_{n_k}(x)| \leq \sup_{n\in\mathbb{Z}_{\geq 1}} \|\phi_n\|_\infty \leq M,$$

we see $\phi \in L_{\beta,C}(\Lambda)$. The L^1-convergence then follows by the dominated convergence theorem. Lastly, as $(\phi_{n_k}(x)) \geq \frac{1}{M}$, for all $k \in \mathbb{Z}_{\geq 0}$ and $x \in \dot{\Lambda}$, we can easily see

$$\phi(x) = \lim_{k\to\infty} \phi_{n_k}(x) \geq \frac{1}{M},$$

so that indeed $\operatorname{ess\,inf}_{x\in\Lambda} \phi'(x) \geq \frac{1}{M}$. Letting ϕ denote the equivalence class of measurable functions agreeing with ϕ' μ_Λ-almost everywhere proves our claim. □

Remark 3.3.3. 1. It is worth noting that even though the limit point of the sequence is in $L_{\beta,C}(\Lambda)$, we do not have $||\cdot||_\beta$-convergence.

In order to apply Proposition 3.3.2 to the densities $\frac{d(g^{R^n})_\star\mu_\Lambda}{d\mu_\Lambda}$ for $n \in \mathbb{Z}_{\geq 0}$ we need to find a uniform upper bound in $||\cdot||_\beta$-norm. Conceptually, Lemma 3.3.4 below ensures us that the 'symbolic encoding' as mentioned in Section 3.1 can actually lead to an absolutely continuous (invariant probability) measure by showing the bounds in Equation (25)..

A technical difficulty with the proof of Lemma 3.3.4 below is that given $n \in \mathbb{Z}_{\geq 1}$, $A \in \mathscr{P}^n_\Lambda$ we need to find a single set $\dot{A} \in \mathscr{F}_A$ so that we can apply the bounded distortion property 3.2.6, the reciprocal formula in Lemma 2.2.8 and the chain rule in Proposition 2.2.10. For this, we shall need the non-singularity and pullback non-singularity of the mapping g^R. Constructing such a set $\dot{A}$ will be the first task in the lemma.

Lemma 3.3.4. *Suppose we have a tower base $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$. Then for $C' = e^{\frac{C}{1-\beta}}$ and $n \in \mathbb{Z}_{\geq 1}$, and $A \in \mathscr{P}^n_\Lambda$ we have a set $\dot{\Lambda} \in \mathscr{F}_\Lambda$ such that $\mu_\Lambda(\Lambda \setminus \dot{\Lambda}) = 0$ and so that the density $\phi_{n,A} := \frac{d(g^{R^n}|_A)_\star\mu_\Lambda}{d\mu_\Lambda}$ satisfies for all $x, y \in \dot{\Lambda}$*

$$\left|\log \frac{\phi_{n,A}(y)}{\phi_{n,A}(x)}\right| \leq \beta^n \frac{d_{\beta,C}(x,y)}{1-\beta}, \tag{24}$$

and

$$\frac{1}{C'}\mu_\Lambda(A) \leq \phi_{n,A}(x) \leq C'\mu_\Lambda(A) \ \textit{ for all } x \in \dot{\Lambda}. \tag{25}$$

Proof. Let $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{P}^n_\Lambda$. First note that A is an invertibility domain for $(g^R)^n$ by Lemma 3.2.5. By Remark 3.2.8 we may apply Lemma 2.2.8 to claim we have an $A' \in \mathscr{F}_\Lambda$ with $A' \subseteq A$ such that $\mu_A(A \setminus A') = 0$ and

$$\phi_{n,A}((g^R)^n(x')) = (J(g^R)^n(x'))^{-1}, \quad \text{for each } x' \in A'. \tag{26}$$

Furthermore, by Proposition 2.2.10 we can find a set $\Lambda' \in \mathscr{F}_\Lambda$ such that $\mu_\Lambda(\Lambda \setminus \Lambda') = 0$ and

$$J((g^R)^n)(x') = \prod_{i=0}^{n-1} (Jg^R) \circ (g^R)^i(x') \quad \text{for } x' \in \Lambda'. \tag{27}$$

Lastly, by Lemma 3.2.7 we can find a $A'' \in \mathscr{F}_A$ such that $\mu_\Lambda(A \setminus A'') = 0$ and for each $i \in \{0, \ldots, n-1\}$

$$\left|\frac{J(g^R)((g^R)^i(x'))}{J(g^R)((g^R)^i(y'))} - 1\right| \leq d_{\beta,C}((g^R)^i(x'), (g^R)^i(y')), \quad \text{for every } x', y' \in A''. \tag{28}$$

Then define

$$\dot{A} = A' \cap A'' \cap (g^R)^{-n}\Lambda' \in \mathscr{F}_\Lambda.$$

By non-singularity of g^R we then see $\mu_A(A \setminus \dot{A}) = 0$. Moreover, in taking $\dot{\Lambda} = (g^R)^n \dot{A}$, we see by Lemma 2.2.2 $\dot{\Lambda} \in \mathscr{F}_\Lambda$. By Lemma 3.2.5 we have $(g^R)^n(A) = \Lambda$ so

$$\Lambda \setminus \dot{\Lambda} = \Lambda \setminus (g^R)^n(\dot{A}) \subseteq (g^R)^n(A \setminus \dot{A}).$$

Using pbn-singularity of $(g^R)^n$ we then have

$$\mu_\Lambda(\Lambda \setminus \dot{\Lambda}) \le \mu_\Lambda((g^R)^n(A \setminus \dot{A})) = 0.$$

Now we can start with the proof. Proceeding, we find for general $x, y \in \dot{\Lambda}$ unique $x', y' \in \dot{A}$ with $(g^{R^n}|_{\dot{A}})^{-1}(y) = y'$ and $(g^{R^n}|_{\dot{A}})^{-1}(x) = x'$ so that

$$\begin{aligned}
\left|\log\left(\frac{\phi_{n,A}(x)}{\phi_{n,A}(y)}\right)\right| &= \left|\log\frac{(J(g^{R^n})(x'))^{-1}}{(J(g^{R^n})(y'))^{-1}}\right| && (29)\\
&= \left|\log\frac{J(g^{R^n})(x')}{J(g^{R^n})(y')}\right| && (30)\\
&= \left|\log\frac{\prod_{i=0}^{n-1}(Jg^R)((g^R)^i(x'))}{\prod_{i=0}^{n-1}(Jg^R)((g^R)^i(y'))}\right| && (31)\\
&\le \sum_{i=0}^{n-1}\left|\log\frac{(Jg^R)((g^R)^i(x'))}{(Jg^R)((g^R)^i(y'))}\right| \\
&\le \sum_{i=0}^{n-1}\max\left\{\left|1-\frac{(Jg^R)((g^R)^i(y'))}{(Jg^R)((g^R)^i(x'))}\right|,\left|1-\frac{(Jg^R)((g^R)^i(x'))}{(Jg^R)((g^R)^i(y'))}\right|\right\} && (32)\\
&= \sum_{i=0}^{n-1} d_{\beta,C}\left(g^{R^i}(x'), g^{R^i}(y')\right) && (33)\\
&= d_{\beta,C}(x,y)\sum_{i=0}^{n-1}\beta^i && (34)\\
&\le \frac{d_{\beta,C}(x,y)}{1-\beta}, && (35)
\end{aligned}$$

showing Equation (24). For clarification, in Equation (29) we used Equation (26); in Equation (30) we used the identity $\left|\log(z)\right| = \left|(\log(z^{-1}))\right|$ for $z \in (0,\infty)$; in Equation (31) we used Equation (27); in Equation (32) we used $|\log(z)| \le \max\left\{\left|1-\frac{1}{z}\right|, |1-z|\right\}$ for $z \in (0,\infty)$; in Equation (33) we used Equation (28); in Equation (34) we used Lemma 2.1.28; and finally Equation (35) was derived using the expression for a geometric series.

To show Equation (24) we now only need to apply Lemma 2.1.28 and note

$$\left|\log\left(\frac{\phi_{n,A}(x)}{\phi_{n,A}(y)}\right)\right| \le \frac{d_{\beta,C}(x,y)}{1-\beta}.$$

To prove Equation (25) note that as $d_{\beta,C}$ takes values in $[0,C]$, we have

$$\log\left(\frac{\phi_{n,A}(x)}{\phi_{n,A}(y)}\right) \leq \left|\log\left(\frac{\phi_{n,A}(x)}{\phi_{n,A}(y)}\right)\right| \leq C(1-\beta)^{-1}$$

and

$$\log\left(\frac{\phi_{n,A}(y)}{\phi_{n,A}(x)}\right) \leq \left|\log\left(\frac{\phi_{n,A}(x)}{\phi_{n,A}(y)}\right)\right| \leq C(1-\beta)^{-1},$$

so that after exponentiating both equations and rearranging we obtain for all $x, y \in \dot{\Lambda}$

$$(C')^{-1}\phi_{n,A}(y) \leq \phi_{n,A}(x) \leq C'\phi_{n,A}(y), \qquad \text{for } C' = e^{\frac{C}{1-\beta}}.$$

Integrating both sides with respect to y on $\dot{\Lambda}$ then shows

$$(C')^{-1}\mu_\Lambda(A) \leq \phi_{n,A}(x) \leq C'\mu_\Lambda(A).$$

As $x \in \dot{\Lambda}$ was given arbitrarily, we have proven our claim. □

Ancillary, we following result useful for finding bounds in Section 3.3.2.

Corollary 3.3.5. *Let $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ be a tower base, then there exists some $C' > 1$ such that, for all $n \geq 1$, $A \in \mathscr{P}^n_\Lambda$ and almost every $x \in A$ we have*

$$\frac{1}{C'}\mu_\Lambda(A) \leq \frac{1}{J(g^R)^n(x)} \leq C'\mu_\Lambda(A). \tag{36}$$

Proof. Let $C' \in \mathbb{R}_{>1}$ be as in Lemma 3.3.4 and let $n \in \mathbb{Z}_{\geq 1}$, $A \in \mathscr{P}^n_\Lambda$. As seen in the proof of Lemma 3.3.4 we have a $\dot{A} \in \mathscr{F}_\Lambda$ such that $\mu_\Lambda(A \setminus \dot{A}) = 0$,

$$\phi_{n,A}((g^R)^n(x')) = (J(g^R)^n(x))^{-1}, \quad \text{for each } x' \in \dot{A}, \tag{37}$$

and

$$(C')^{-1}\mu_\Lambda(A) \leq \phi_{n,A}(x) \leq C'\mu_\Lambda(A), \quad \text{for each } x \in (g^R)^n(\dot{A}). \tag{38}$$

Combining Equations (37) and (38) then yields Equation (36), proving our claim. □

We are now ready to show the densities $\frac{d(g^{R^n})_\star\mu_\Lambda}{d\mu_\Lambda}$ are in $L_{\beta,L}$ for some $L \in \mathbb{R}_{>1}$. A statement similar to the Lemma below is made in [1, Lemma 3.9] - we provide different bound however.

Conceptually we shall 'sum' the bounds obtained Lemma 3.3.4 over the principal partition.

Lemma 3.3.6. *Suppose $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ is a tower base. We have a $M \in \mathbb{R}_{>1}$ such that for each $n \geq 0$ we have,*

$$\frac{d(g^{R^n})_\star\mu_\Lambda}{d\mu_\Lambda} \in L_{\beta,C}(\Lambda), \quad \left\|\frac{d(g^{R^n})_\star\mu_\Lambda}{d\mu_\Lambda}\right\|_\beta \leq M \quad \textit{and} \quad \operatorname*{ess\,inf}_{x\in\Lambda} \frac{d(g^{R^n})_\star\mu_\Lambda}{d\mu_\Lambda} \geq \frac{1}{M}. \tag{39}$$

Proof. First note for $n = 0$ we have

$$\frac{d(g^{R^n})_\star \mu_\Lambda}{d\mu_\Lambda} = \frac{d\mu_\Lambda}{d\mu_\Lambda} \equiv 1,\ \mu_\Lambda\text{-almost everywhere,}$$

for which we immediately see the claims in Equation (39) hold for any $M \in \mathbb{R}_{>1}$. Now write for $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{P}^n_\Lambda$ the densities

$$\phi_{n,A} := \frac{d(g^{R^n}|_A)_\star \mu_\Lambda}{d\mu_\Lambda}, \qquad \phi_n := \frac{d(g^{R^n})_\star \mu_\Lambda}{d\mu_\Lambda}.$$

We shall start by showing $\inf_{n \in \mathbb{Z}_{\geq 1}} \operatorname{ess\,inf}_{x \in \Lambda} \phi_n(x)$ is positive and that $\sup_{n \in \mathbb{Z}_{\geq 1}} ||\phi_n||_\infty$ is bounded.

Let $C' \in \mathbb{R}_{>1}$ be the constant given by Lemma 3.3.4. For $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{P}^n_\Lambda$ let $\Lambda_{n,A} \subseteq \Lambda$ be the set such that for each $x, y \in \Lambda_{n,A}$,

$$\frac{1}{C'}\mu_\Lambda(A) \leq \phi_{n,A}(x) \leq C'\mu_\Lambda(A), \tag{40}$$

and

$$\left|\log \frac{\phi_{n,A}(y)}{\phi_{n,A}(x)}\right| \leq \frac{d_{\beta,C}(x,y)}{1-\beta}. \tag{41}$$

By Lemma 3.3.4 we know $\Lambda_{n,A} \in \mathscr{F}_\Lambda$ and $\mu_\Lambda(\Lambda \setminus \Lambda_{n,A}) = 0$, so $\mu_\Lambda(\Lambda \setminus (\bigcap_{A \in \mathscr{P}^n_\Lambda} \Lambda_{n,A})) = 0$. More so, we have by Lemma 2.1.21 a $\Lambda' \in \mathscr{F}_\Lambda$ such that $\mu_\Lambda(\Lambda \setminus \Lambda') = 0$ and

$$\sum_{A \in \mathscr{P}^n_\Lambda} \phi_{n,A}(x) = \phi_n(x) \quad \text{for each } x \in \Lambda'. \tag{42}$$

We conclude that for $\dot\Lambda = \bigcap_{A \in \mathscr{P}^n_\Lambda} \Lambda_{n,A} \cap \Lambda'$ we have $\mu_\Lambda(\Lambda \setminus \dot\Lambda) = 0$ and by combining Equation (40) with Equation (42) we see

$$\frac{1}{C'} \leq \phi_n(x) \leq C', \text{ for every } x \in \dot\Lambda. \tag{43}$$

As this holds for arbitrary $n \in \mathbb{Z}_{\geq 1}$ and know $C' > 1$ we have indeed shown $\inf_{n \in \mathbb{Z}_{\geq 1}} \operatorname{ess\,inf}_{x \in \Lambda} \phi_n(x)$ is positive and that $\sup_{n \in \mathbb{Z}_{\geq 1}} ||\phi_n||_\infty$ is bounded. To show there exists an $M \in \mathbb{R}_{>1}$ such that

$$\sup_{n \in \mathbb{Z}_{\geq 1}} \left\| \frac{d(g^{R^n})_\star \mu_\Lambda}{d\mu_\Lambda} \right\|_\beta \leq M$$

let $n \in \mathbb{Z}_{\geq 1}$ be arbitrary and construct $\dot\Lambda$ again as above. Let $x, y \in \dot\Lambda$. Following from Equation (41) we have that

$$\phi_{n,A}(x) \leq \exp\left(\frac{d_{\beta,C}(x,y)}{1-\beta}\right)\phi_{n,A}(y),$$

and

$$\phi_{n,A}(y) \le \exp\left(\frac{d_{\beta,C}(x,y)}{1-\beta}\right)\phi_{n,A}(x).$$

Summing over $\mathscr{P}_\Lambda^n$ as before and rearranging then yields that

$$\left|\log\frac{\phi_n(x)}{\phi_n(y)}\right| \le \frac{d_{\beta,C}(x,y)}{1-\beta}.$$

Furthermore, as by Equation (43) we have that $\frac{\phi_n(x)}{\phi_n(y)} \in \left[\frac{1}{(C')^2}, (C')^2\right]$ and since $|z-1| \le (C')^2|\log(z)|$ for $z \in \left[\frac{1}{(C')^2}, (C')^2\right]$ we can derive

$$|\phi_n(x)-\phi_n(y)| \le C'\left|1-\frac{\phi_n(x)}{\phi_n(y)}\right| \le (C')^3\left|\log\frac{\phi_n(x)}{\phi_n(y)}\right| \le \frac{(C')^3}{1-\beta}d_{\beta,C}(x,y).$$

As $x, y \in \dot{\Lambda}$ and $n \in \mathbb{Z}_{\ge 1}$ were given arbitrarily we have $\sup_{n\ge 1}|\phi_n|_\beta \le \frac{(C')^3}{1-\beta}$. Fixing $M = \frac{(C')^3}{1-\beta} + C'$ we see $M > 1$, which yields our claim. □

Finally, we obtain our *acip* for tower bases.

Theorem 3.3.7. *Suppose $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ is a tower base. Then there exists an* acip ν_Λ *with $\mu_\Lambda \ll \nu_\Lambda \ll \mu_\Lambda$. Furthermore, the density $\frac{d\nu_\Lambda}{d\mu_\Lambda}$ satisfies*

$$\frac{1}{M} \le \frac{d\nu_\Lambda}{d\mu_\Lambda} \le M, \quad \mu_\Lambda\text{-almost surely,} \tag{44}$$

for some $M \in \mathbb{R}_{>1}$.

Proof. Let $n \in \mathbb{Z}_{n\ge 1}$ and define $\phi_i := \frac{d(g^{R^i})_\star\mu_\Lambda}{d\mu_\Lambda}$ for $i \in \{0,\ldots,n-1\}$ write . Then define $\psi_n := \frac{1}{n}\sum_{i=0}^{n-1}\phi_i$. The bounds on ϕ_i, $i \in \{0,...,n-1\}$, from Lemma 3.3.6 are maintained under convex combination of $\{\phi_i\}_{0\le i\le n-1}$ and hence hold for ψ_n as well. Thus, we obtain

$$\psi_n \in L_{\beta,L}(\Lambda), \quad \left|\left|\psi_n\right|\right|_\beta \le M \quad \text{and} \quad \operatorname{ess\,inf}\psi_n \ge \frac{1}{M} \tag{45}$$

for $M, L \in \mathbb{R}_{>1}$ as in Lemma 3.3.6. As Equation (45) holds for all $n \in \mathbb{Z}_{\ge 1}$ we obtain using Proposition 3.3.2 we obtain an accumulation point $\psi \in L^1(\Lambda)$, with $||\psi||_\beta \le M$ and $\psi \ge \frac{1}{M}$ of the sequence $(\psi_n)_{n\ge 1}$. As $||\psi||_\infty \le ||\psi||_\beta \le M$ we obtain by Lemma 2.1.14 an *acip* $\nu_\Lambda \ll \mu_\Lambda$ satisfying Equation (44). Consequently, we see that we have for all $A \in \mathscr{F}_\Lambda$ that $\nu_\Lambda(A) \ge \frac{1}{M}\mu_\Lambda(A)$ so we obtain $\mu_\Lambda \ll \nu_\Lambda$.

□

3.3.2 The uniqueness of the acip

The general strategy for proving uniqueness of the *acip* found in Theorem 3.3.7, is to prove it is ergodic and rely on Lemma 2.1.13. In doing so, we will prove our *acip* is exact. It is a nice ancillary result but we will not revisit similar properties in Sections 4 - 5.6, which can be considered the bulk of this thesis.

Remark 3.3.8. The key point to exactness is 'expanding' behaviour. In constructing a Bernoulli shift $(\Gamma, \mathscr{F}_\Gamma, \mathbb{P}, \sigma)$ based on an alphabet $\Gamma = \{0,1\}$ with weights $\{\frac{1}{2}, \frac{1}{2}\}$ we can see a cylinder can never be an element of $\bigcap_{n\geq 0} \sigma^{-n}\mathscr{F}_\Gamma$. This is most easily illustrated by seeing

$$\sigma[10] = [1] \quad \text{but} \quad \sigma^{-1}[1] = [10] \sqcup [01].$$

That is, the expanding nature of shift forces that there is no set acting as the inverse of $[10]$. In contrast, as rotations on the circle $([0,1), \mathscr{B}[0,1), \lambda, \sigma_\theta)$ are bi-measurable we have $\bigcap_{n\geq 0} \sigma_\theta^{-n}\mathscr{B}[0,1) = \mathscr{B}[0,1)$.

The corollary below is largely inspired by [1, Corollary 3.6] where we have also provided a lower bound we need later on.

Corollary 3.3.9. *Let $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ be a tower base. Then there exists a $C_2 > 1$ such that for all $n \geq 1, A \in \mathscr{P}_\Lambda^n$ and measurable sets $A_1, A_2 \subseteq A$, with $\mu(A_2) > 0$ we have,*

$$\frac{1}{C_2}\frac{\mu_\Lambda(A_1)}{\mu_\Lambda(A_2)} \leq \frac{\mu_\Lambda(g^{R^n}(A_1))}{\mu_\Lambda(g^{R^n}(A_2))} \leq C_2\frac{\mu_\Lambda(A_1)}{\mu_\Lambda(A_2)}. \tag{46}$$

Proof. First let $C' \in \mathbb{R}_{>0}$ be as given in Corollary 3.3.5. Let $n \geq 1, A \in \mathscr{P}_\Lambda^n$, and $A_1, A_2 \in \mathscr{F}_A$, be such that $\mu(A_2) > 0$. By Lemma 2.2.9 we know $\mu_\Lambda(g^{R^n}(A_2)) > 0$.

Rearranging Equation (36) from Corollary 3.3.5 we obtain a $C' > 1$ such that for almost every $x \in A$

$$\frac{1}{C'}\frac{1}{\mu_\Lambda(A)} \leq J(g^R)^n(x) \leq C'\frac{1}{\mu_\Lambda(A)}. \tag{47}$$

Integrating Equation (47) over A_1 yields

$$\frac{1}{C'}\frac{\mu_\Lambda(A_1)}{\mu_\Lambda(A)} \leq \mu_\Lambda(g^{R^n}(A_1)) \leq C'\frac{\mu_\Lambda(A_1)}{\mu_\Lambda(A)}, \tag{48}$$

Integrating Equation (47) over A_2 and taking the reciprocal of both sides yields

$$\frac{1}{C'}\frac{\mu_\Lambda(A)}{\mu_\Lambda(A_2)} \leq \mu_\Lambda(g^{R^n}(A_2))^{-1} \leq C'\frac{\mu_\Lambda(A)}{\mu_\Lambda(A_2)}. \tag{49}$$

Multiplying Equations (48) and (49) then yields Equation (46) for $C_2 = (C')^2$. □

Without relying on invariance, we can show that $\bigcap_{n\geq 0}(g^R)^{-n}\mathscr{F}_\Lambda$ consists of measure-zero and measure-one sets. The proof is largely inspired by [1, Theorem 3.13], where we have included the (necessary) lower bound in Equation (50). Intuitively the proof relies on the fact that any $A \in \mathscr{P}_\Lambda^n$ will be blown up to Λ under n iterations of g^R. We shall make use of the bounds in Corollary 2.1.24 and Corollary 3.3.9 to extend this to more elements of $\mathscr{F}_\Lambda$.

Lemma 3.3.10. *Suppose we have a tower base* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$. *We then have for all* $A \in \bigcap_{n\geq 0}^{\infty}(g^R)^{-n}\mathscr{F}_\Lambda$ *that* $\mu_\Lambda(A) = 0$ *or* $\mu_\Lambda(A) = 1$.

Proof. Fix $A \in \bigcap_{n=0}^{\infty}(g^R)^{-n}\mathscr{F}_\Lambda$ and suppose $\mu(A) > 0$. Fix $C_2 > 1$ as in Corollary 3.3.9. By Corollary 2.1.24 we may for all $\epsilon \in \mathbb{R}_{>0}$ pick an $n \in \mathbb{Z}_{\geq 1}$ and $P \in \mathscr{P}_\Lambda^n$ for which

$$\frac{\mu_\Lambda(P \setminus A)}{\mu_\Lambda(P)} < \frac{\epsilon}{C_2^2}.$$

Define $B = (g^{R^n})^{-1}A$ and note we have by Lemma 2.1.23 that $(g^R)^n(P \setminus A) = \Lambda \setminus B$ so that by Corollary 3.3.9 we obtain

$$\frac{\mu_\Lambda(\Lambda \setminus B)}{\mu_\Lambda(\Lambda)} \leq C_2 \frac{\mu_\Lambda(P \setminus A)}{\mu_\Lambda(P)}.$$

Now note by Lemma 2.1.23 we have for all $\tilde{P} \in \mathscr{P}_\Lambda^n$ that $g^{R^n}(\tilde{P} \setminus A) = \Lambda \setminus B$. In applying Corollary 3.3.9, we then see for each $\tilde{P} \in \mathscr{P}_\Lambda^n$

$$\frac{1}{C_2}\frac{\mu_\Lambda(\tilde{P} \setminus A)}{\mu_\Lambda(\tilde{P})} \leq \frac{\mu_\Lambda(\Lambda \setminus B)}{\mu_\Lambda(\Lambda)}, \tag{50}$$

from which follows

$$\frac{\mu_\Lambda(\tilde{P} \setminus A)}{\mu_\Lambda(\tilde{P})} \leq C_2^2 \frac{\mu_\Lambda(P \setminus A)}{\mu_\Lambda(P)} < \epsilon.$$

Multiplying both sides by $\mu_\Lambda(\tilde{P})$ and summing over $\mathscr{P}_\Lambda^n$ then shows

$$\mu_\Lambda(\Lambda \setminus A) = \sum_{\tilde{P} \in \mathscr{P}_\Lambda^n} \mu_\Lambda(\tilde{P} \setminus A) < \epsilon \sum_{\tilde{P} \in \mathscr{P}_\Lambda^n} \mu_\Lambda(\tilde{P}) = \epsilon.$$

As $\epsilon > 0$ was given arbitrarily we may hence conclude $\mu_\Lambda(\Lambda \setminus A) = 0$ meaning $\mu_\Lambda(A) = 1$, which proves the statement. □

Corollary 3.3.11. *Suppose* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ *is a tower base. The* acip $\nu_\Lambda \ll \mu_\Lambda$ *obtained in Theorem 3.3.7 is exact and is the unique* acip $\nu_\Lambda \ll \mu_\Lambda$.

Proof. Let $\nu_\Lambda \ll \mu_\Lambda$ be an *acip* given by Theorem 3.3.7. We have by Lemma 3.3.10 for any $A \in \bigcap_{n=0}(g^R)^{-n}\mathscr{F}_\Lambda$ that $\mu_\Lambda(A) = 1$ or $\mu_\Lambda(A) = 0$. In the case of the former we have as $\nu_\Lambda \ll \mu_\Lambda$

$$\mu_\Lambda(\Lambda \setminus A) = 0 \Rightarrow \nu_\Lambda(\Lambda \setminus A) = 0 \Rightarrow \nu_\Lambda(A) = 1,$$

and in case of the latter (also using $\nu_\Lambda \ll \mu_\Lambda$)

$$\mu_\Lambda(A) = 0 \Longrightarrow \nu_\Lambda(A) = 0,$$

so that

$$\{\nu_\Lambda(A) : A \in \bigcap_{n=0}(g^R)^{-n}\mathscr{F}_\Lambda\} = \{0,1\}.$$

As we know ν_Λ is invariant under g^R, we can conclude ν_A is exact. As we have $\mu_\Lambda \ll \nu_\Lambda \ll \mu_\Lambda$ as well, it is the unique *acip* $\nu_\Lambda \ll \mu_\Lambda$ by Proposition 2.1.13. □

3.4 The Tower Framework

For this section we remain having fixed the dynamical system $(X, \mathscr{F}, m, g)$ **and tower base** $(\Lambda, \mathscr{F}, \mu_\Lambda, g^R)$**.** As promised in Section 3.1, for the remaining sections 3.4-3.5 we shall complete our exposition on Young's theory by defining a tower and giving conditions under which we can find an acip for our original system $(X, \mathscr{F}, m, g)$ in Corollary 3.5.6. Lemma 3.5.2 through Theorem 3.5.5 cover the ergodic properties of the acip for a tower and are not necessary for understanding the theory in Sections 4 and 5. We have incorporated these results however, as the further ergodic properties of the acip (e.g. the rates of mixing, see for instance [25] or [2]) are central to the theory of Young Towers, and that important bounds necessary to prove this were not found in pieces such as [25] and [24].

As suggested by the notion of a tower base, we shall now define a tower. We encourage the reader to take a brief look at Proposition 2.1.7 for some adopted measure-theoretical conventions.

Conceptually, a tower is a tool to store points of Λ before mapping them back to X. Having found an *acip* for $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ and assuming $R \in L^1(\Lambda)$, showing this induces an acip on the tower is a small step. This acip then induces an acip on our original system $(X, \mathscr{F}, m, g)$.

Definition 3.4.1. We say

1. $\Delta := \{(x,l) \in \Lambda \times \mathbb{Z}_{\geq 0} : 0 \leq l \leq R(x) - 1\}$ is a *tower*;

2. $\Delta_l = \{x \in \Lambda : R(x) > l\}$ for $l \in \mathbb{Z}_{\geq 1}$ is *floor* l *of* Δ;

3. $\Delta_0 = \Lambda$ is the *ground floor of* Δ; and

4. $\mathscr{P}_\Delta := \{P \times \{l\} \subseteq \Lambda \times \mathbb{Z}_{\geq 0} : P \in \mathscr{P}_\Lambda, R[P] > l\}$ is the *principal partition of* Δ.

If we endow $\mathbb{Z}_{\geq 0}$ with the power set $2^{\mathbb{Z}_{\geq 0}}$ as its σ-algebra and the counting measure N we can construct the σ-finite measure space $(\Lambda \times \mathbb{Z}_{\geq 0}, \mathscr{F}_{\Lambda \times \mathbb{Z}_{\geq 0}}, \mu_\Lambda \times N)$. In writing

$$R_{>l} := \{x \in \Lambda : R(x) > l\}$$

we can see that

$$\Delta_l = R_{>l} \quad \text{and} \quad \Delta = \bigsqcup_{l \geq 0} R_{>l} \times \{l\} \in \mathscr{F}_{\Lambda \times \mathbb{Z}_{\geq 0}}, \tag{51}$$

so that $\Delta \in \mathscr{F}_{\Lambda \times \mathbb{Z}_{\geq 0}}$. We can then define the restricted measure space $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$. Using this construction we can see that

$$\mathscr{G}_\Delta := \bigcup_{l \in \mathbb{Z}_{\geq 0}} \{A \times \{l\} \subseteq \mathscr{F}_\Delta : A \subseteq R_{>l}, A \in \mathscr{F}_\Lambda\}, \tag{52}$$

is a set of generators for $\mathscr{F}_\Delta$. We shall fix $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ for the rest of the section, alongside its naturally associated collection $\mathscr{P}_\Delta$. We get the following technicalities sorted.

Lemma 3.4.2. *The following claims hold:*

1. *The collection* $\mathscr{P}_\Delta$ *consists of measurable sets and partitions* Δ.

2. *If* $R \in L^1(\Lambda)$ *we have* $||R||_1 = \mu_\Delta(\Delta)$.

Proof. First we check $\mathscr{P}_\Delta \subseteq \mathscr{F}_\Delta$. For Item 1 note that for any $P \in \mathscr{P}_\Delta$ we have an unique $A \in \mathscr{P}_\Lambda$, $l \in \mathbb{Z}_{\geq 0}$ such that $A \subseteq R_{\geq l}$ so that $A \times \{l\} \subseteq \Delta$ and $A \times \{l\} \in \mathscr{F}_\Lambda \times 2^{\mathbb{Z}_{\geq 0}} \subseteq \mathscr{F}_{\Lambda \times 2^{\mathbb{Z}_{\geq 0}}}$ and so $P = A \times \{l\} \in \mathscr{F}_\Delta$.

To show $\mathscr{P}_\Delta$ coves Δ, recall $\alpha : \Lambda \to \mathscr{P}_\Lambda$ from Definition 2.1.27, we similarly have for any $(x, l) \in \Delta$ that $x \in \alpha(x) \in \mathscr{P}_\Lambda$ and $x \in R_{>l}$ so that $(x, l) \in \alpha(x) \times \{l\} \in \mathscr{P}_\Delta$. Finally, for $P, P' \in \mathscr{P}_\Delta$ we have $A \times \{l\} = P, A' \times \{l'\} = P'$ and so $P \cap P' \neq \emptyset$ implies $l = l'$ and so $A = A'$ as $\mathscr{P}_\Lambda$ is a partition. This implies $P = P'$. We have shown $\mathscr{P}_\Delta$ covers Δ and consists of pairwise disjoint measurable sets, proving our claim.

For Item 2 we note

$$\begin{aligned}
\mu_\Delta(\Delta) &= \sum_{l=0}^{\infty} \mu_\Lambda(R_{>l}) \times N(\{l\}) \\
&= \sum_{l=0}^{\infty} \mu_\Lambda(R_{>l}) \\
&= ||R||_1,
\end{aligned}$$

where we used a standard probabilistic equality (e.g. seen in [14, Lemma 4.4]) in the last line. This proves the claim in Item 2. □

We now the define an operator on $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ called a tower map and show this results in a dynamical system.

Definition 3.4.3. Given a tower Δ and its associated measure space $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ we define the *tower map* $G : \Delta \to \Delta$ as

$$G(x,l) = \begin{cases} (x, l+1), & \text{if } l+1 < R(x) \\ (g^R(x), 0), & \text{otherwise,} \end{cases} \tag{53}$$

Lemma 3.4.4. *The tuple* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *is a dynamical system.*

Proof. We need to show measurability and non-singularity.

Measurability Let $l \in \mathbb{Z}_{\geq 0}$ and define

$$\mathscr{G}_\Delta := \bigcup_{l \in \mathbb{Z}_{\geq 0}} \{A \times \{l\} \subseteq \mathscr{F}_\Delta : A \subseteq R_{>l}, A \in \mathscr{F}_\Lambda\},$$

as in Equation (52). We shall show $G^{-1}(\mathscr{G}_\Delta) \subseteq \mathscr{F}_\Delta$ to use Lemma 2.1.4 and distinguish between the cases $l > 0$ and $l = 0$.

First for $l > 0$, consider an arbitrary $A \times \{l\} \in \mathscr{G}_\Delta$. Note

$$\begin{aligned} G^{-1}(A \times \{l\}) &= \{(x,k) \in \Delta : G(x,k) \in A \times \{l\}, k+1 < R(x)\} \\ &\cup \{(x,k) \in \Delta : G(x,k) \in A \times \{l\}, k+1 = R(x)\} \\ &= \{(x,k) \in \Delta : (x, k+1) \in A \times \{l\}, k+1 < R(x)\} \\ &\cup \{(x,k) \in \Delta : (g^R x, 0) \in A \times \{l\}, k+1 = R(x)\} \\ &= \{(x, l-1) \in \Delta : x \in A \cap R_{>l}\} \cup \emptyset \\ &= (A \cap R_{>l}) \times \{l-1\} && \text{using Equation (51)} \\ &= A \times \{l-1\} && \text{as } A \subseteq R_{>l} \end{aligned}$$

As $A \subseteq R_{>l}$ we see $A \subseteq R_{>l-1}$ and so as $A \in \mathscr{F}_\Lambda$ we see $A \times \{l-1\} \in \mathscr{F}_\Delta$.

For arbitrary $A \times \{0\} \in \mathscr{G}_\Delta$ we see

$$G^{-1}(A \times \{0\}) = \bigsqcup_{P \in \mathscr{P}_\Lambda, R|_P \equiv l} ((g^R)^{-1}(A) \cap P) \times \{l\} \in \mathscr{F}_\Delta.$$

We have shown $G^{-1}(\mathscr{G}_\Delta) \subseteq \mathscr{F}_\Delta$ proving G is measurable by Lemma 2.1.4.

Non-singularity Similarly, for $B \in \mathscr{F}_\Delta$ with $\mu_\Delta(B) = 0$ we see $\mu_\Lambda(B_l) = 0$ for all $l \in \mathbb{Z}_{\geq 0}$.

In writing for $P \in \mathscr{P}_\Lambda$, $r_P \in \mathbb{Z}_{\geq 1}$ for the integer satisfying $R|_P \equiv r_P$ we see

$$\begin{aligned}\mu_\Delta(G^{-1}(B_0 \times \{0\})) &= \sum_{P \in \mathscr{P}_\Lambda} \mu_\Delta((P \cap (g^R)^{-1}(B_0)) \times \{r_P - 1\}) \\ &= \sum_{P \in \mathscr{P}_\Lambda} \mu_\Lambda(P \cap (g^R)^{-1}(B_0)) \\ &\leq \sum_{P \in \mathscr{P}_\Lambda} (g^R)_\star \mu_\Lambda(B_0) \\ &= 0, \qquad \text{by non-singularity of } \mu_\Lambda. \end{aligned} \tag{54}$$

Moreover for $l > 0$,

$$\mu_\Delta(G^{-1}(B_l \times \{l\})) = \mu_\Delta(B_l \times \{l-1\}) = \mu_\Lambda(B_l) = 0. \tag{55}$$

Combining Equation (54) and (55) we see

$$G_\star \mu_\Delta(B) = \sum_{l=0}^{\infty} G_\star \mu_\Delta(B_l \times \{l\}) = 0,$$

proving non-singularity.

Having shown non-singularity and measurability of G we have shown $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is a dynamical system. □

We refer to the dynamical system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ as a *tower*. The following Corollary simplifies questions on measurability greatly.

Corollary 3.4.5. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a tower. Then the following holds.*

1. *The tower map $G : \Delta \to \Delta$ is locally invertible;*
2. *For each $k \in \mathbb{Z}_{\geq 0}$ and each $A \in \mathscr{F}_\Delta$ we have that $G^k(A) \in \mathscr{F}_\Delta$.*

Proof. We prove $\mathscr{P}_\Delta$ consists of invertibility domains for Δ. Note that by definition we have for each $P \in \mathscr{P}_\Delta$ an $A \in \mathscr{P}_\Lambda$ and $l \in \mathbb{Z}_{\geq 0}$ such that $A \times \{l\} = P$ and $A \subseteq R_{>l}$. Note that for $z \in \mathbb{Z}$ the mapping $c_{l,z} : \{l\} \to \{l+z\}, l \mapsto l+z$ is bi-measurable and $\mathrm{Id} : A \to A$ is bi-measurable as well. If $A \subseteq R_{>l+1}$ we have

$$G|_{A \times \{l\}}(x, l) = (x, l+1) = (\mathrm{Id} \times c_{l,1})(x, l),$$

which is bi-measurable by Lemma 4.3.18. If $A \subseteq R^{-1}(l+1)$ we have

$$G|_{A \times \{l\}}(x, l) = (g^R(x), 0) = (g^R|_A \times c_{l,-l})(x, l),$$

which again by Lemma 4.3.18 is bi-measurable. We conclude that $\mathscr{P}_\Delta$ indeed is a partition into invertibility domains and so G is locally invertible. Item 2 is direct consequence of local invertibility. □

In the next section we will prove existence and uniqueness of an *acip* for this system.

3.5 The Tower Acip

We can directly use the acip ν_Λ obtained in Theorem 3.3.7 to obtain an acip ν_Δ for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ - assuming an integrable return time.

Theorem 3.5.1. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be the tower as fixed in Section 3.4 and assume we have* $R \in L^1(\Lambda)$. *Then there exists an* acip $\nu_\Delta \ll \mu_\Delta$ *and an* $M_\Delta \in \mathbb{R}_{>0}$ *such that*

$$\frac{1}{M_\Delta} \leq \frac{d\nu_\Delta}{d\mu_\Delta} \leq M_\Delta, \qquad \text{holds } \mu_\Delta\text{-almost everywhere.} \tag{56}$$

Proof. Let ν_Λ be the acip for $(\Lambda.\mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ obtained in Theorem 3.3.7 and $M \in \mathbb{R}_{>1}$ the constant satisfying Equation (44). Define the measure ν'_Δ on Δ by

$$\nu'_\Delta(B) := \sum_{l=0}^{\infty} \nu_\Lambda(B_l) \qquad \text{for } B \in \mathscr{F}_\Delta. \tag{57}$$

Again, writing for $P \in \mathscr{P}_\Lambda$, $r_P \in \mathbb{Z}_{\geq 1}$ for the integer satisfying $R|_P \equiv r_P$ we have for arbitrary $B \in \mathscr{F}_\Delta$ as in the proof of Lemma 3.4.4,

$$\begin{aligned}
\nu'_\Delta(G^{-1}(B_0 \times \{0\})) &= \sum_{l=0}^{\infty} \nu_\Lambda((G^{-1}(B_0 \times \{0\}))_l) \\
&= \sum_{l=0}^{\infty} \nu_\Lambda\left((\bigsqcup_{P \in \mathscr{P}_\Lambda} (P \cap (g^R)^{-1}(B_0)) \times \{r_P - 1\})_l \right) \\
&= \sum_{l=0}^{\infty} \nu_\Lambda\left(\bigsqcup_{P \in \mathscr{P}_\Lambda, r_P - 1 = l} (P \cap (g^R)^{-1}(B_0)) \right) \\
&= \nu_\Lambda\left(\bigsqcup_{P \in \mathscr{P}_\Lambda} P \cap (g^R)^{-1}(B_0) \right) \\
&= \nu_\Lambda(B_0) \\
&= \nu'_\Delta(B_0 \times \{0\}),
\end{aligned} \tag{58}$$

and for $l' \in \mathbb{Z}_{\geq 1}$ we see

$$\begin{aligned}
\nu'_\Delta(G^{-1}(B_{l'} \times \{l'\})) &= \nu'_\Delta(B_{l'} \times \{l' - 1\}) \\
&= \sum_{l=0}^{\infty} \nu_\Lambda((B_{l'} \times \{l' - 1\})_l) \\
&= \nu_\Lambda(B_{l'}) \\
&= \sum_{l=0}^{\infty} \nu_\Lambda((B_{l'} \times \{l'\})_l) \\
&= \nu'_\Delta(B_{l'} \times \{l'\}).
\end{aligned} \tag{59}$$

Combining Equations (58) and (59) we see

$$\nu_\Delta(G^{-1}B) = \nu_\Delta(G^{-1}(B_0\times\{0\})+\sum_{l=1}^{\infty}\nu_\Delta(G^{-1}[B_l\times\{l\}]) = \nu_\Delta(B_0\times\{0\})+\sum_{l\geq 1}\nu_\Delta(B_l\times\{l\}) = \nu_\Delta(B),$$

proving $G_\star\nu_\Delta = \nu_\Delta$. Furthermore, note that

$$\nu'_\Delta(\Delta) = \sum_{l=0}^{\infty}\nu_\Lambda(\Delta_l) = \sum_{l=0}^{\infty}\nu_\Lambda(R_{>l}) \leq M\sum_{l=0}^{\infty}\mu_\Lambda(R_{>l}) \leq M||R||_1 < \infty,$$

so ν'_Δ is a finite measure. Then define the G-invariant probability measure $\nu_\Delta = (\nu'_\Delta(\Delta))^{-1}\cdot\nu'_\Delta$. Note that as $R(\Lambda)\subseteq\mathbb{Z}_{\geq 1}$ we have $||R||_1\geq 1$.

Lastly, note that for any $B\in\mathscr{F}_\Delta$ we have

$$\nu'_\Delta(B) = \sum_{l=0}^{\infty}\nu_\Lambda(B_l) \leq M\sum_{l=0}^{\infty}\mu_\Lambda(B_l) = M\mu_\Delta(B),$$

and similarly

$$\nu'_\Delta(B) = \sum_{l=0}^{\infty}\nu_\Lambda(B_l) \geq \frac{1}{M}\sum_{l=0}^{\infty}\mu_\Lambda(B_l) = \frac{1}{M}\mu_\Delta(B),$$

which implies $\nu'_\Delta(B)\in\left[\frac{1}{M}\mu_\Delta(B), M\mu_\Delta(B)\right]$, so that we have for

$$M_\Delta := \max\left\{\nu'_\Delta(\Delta)\cdot M, \frac{M}{\nu'_\Delta(\Delta)}\right\}, \quad \text{that} \quad \nu_\Delta(B)\in\left[\frac{1}{M_\Delta}\mu_\Delta(B),\ M_\Delta\mu_\Delta(B)\right].$$

Consequently we have $\mu_\Delta \ll \nu_\Delta \ll \mu_\Delta$, with

$$\frac{1}{M_\Delta} \leq \frac{d\nu_\Delta}{d\mu_\Delta} \leq M_\Delta, \quad \mu_\Delta\text{-almost surely,}$$

proving our claims. □

Similar to the acip for the tower base, we shall now prove uniqueness of the acip obtained in Theorem 3.5.1. We again opt to use Lemma 2.1.13 and for that require ergodicity. In Lemma 3.3.10 we showed this for $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ by proving exactness and did so by showing arbitrary small non-trivial sets can saturate the tower base under enough iterations of g^R. There is one caveat however for applying this method to $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$. If the return times have a greatest common divisor greater than 1, periodic behaviour can occur leading to multiple distinct acips with disjoint support. More precisely in Lemma 3.5.3, we shall rely on the following lemma. The writer was unable to find an explicit reference and has hence proved it - note it is more algebraic in nature than other claims in this thesis.

Lemma 3.5.2. *Suppose we have a countable set $N \subseteq \mathbb{Z}_{\geq 1}$ such that* $\gcd(N) = 1$. *Then there exists $n, d \in \mathbb{Z}_{\geq 1}$ and a finite set $\{a_1, \dots, a_n\} \subseteq N$ such that*

$$\mathbb{Z}_{\geq d} \subseteq \{z \in \mathbb{Z}_{\geq 1} : z = \sum_{i=1}^{n} a_i x_i, \textit{ for } x_i \geq 0\}.$$

Proof. As $\gcd(N) = 1$ we can find finitely many integers $\{a_1, \dots, a_n\} \subseteq N$ for some $n \geq 1$ such that $\sum_{i=1}^{n} a_i x_i = 1$ for some $x_i \in \mathbb{Z}$, $i \in \{1, \dots, n\}$. Write $K = \{i \in \{1, \dots, n\} : x_i < 0\}$ and define

$$C := \{z \in \mathbb{Z}_{\geq 1} : z = \sum_{i=1}^{n} a_i x_i', \text{ for } x_i' \geq 0\}.$$

We seek a $d \in \mathbb{Z}_{\geq 1}$ such that $\mathbb{Z}_{\geq d} \subseteq C$. We claim for

$$d = 1 - \sum_{k \in K} a_k x_k - \sum_{k \in K} (\sum_{i \in K} a_i) a_k x_k$$

that we we have $\mathbb{Z}_{\geq d} \subseteq C$.

First note that for $l \in \{0, \dots, \sum_{k \in K} a_k\}$ we have

$$\begin{aligned}
d + l &= 1 - \sum_{k \in K} a_k x_k - \sum_{k \in K} (\sum_{i \in K} a_i) a_k x_k + l \\
&= \sum_{k \in K^c} a_k x_k - \sum_{k \in K} (\sum_{i \in K} a_i) a_k x_k + l \sum_{i=1}^{n} a_i x_i \\
&= \sum_{k \in K} a_k \left(-l + \sum_{i \in K} a_i \right) (-x_k) + \sum_{k \in K^c} a_k (l+1) x_k
\end{aligned} \tag{60}$$

so that indeed $d + l \in C$, for each $l \in \{0, \dots, \sum_{k \in K} a_k\}$. Moreover, note that $d \in C$ and so $(d + n \sum_{k \in K} a_k)_{n \geq 0} \subseteq C$. Now as C is closed under addition we can see

$$\mathbb{Z}_{\geq d} \subseteq \{d + l + n \sum_{k \in K} a_k\}_{n \geq 0, l \in \{0, \cdots \sum_{k \in K} a_k\}} \subseteq C,$$

proving our claim. □

We now move on to proving our *acip* is unique. A slightly different version of the following Lemma is made as a claim by Young in her [25] paper in the proof of Lemma 5. The claim is made without proof however, which is why we have proven the statement below. In [1] a different approach can be found.

Our proof below goes in two steps: first we derive a bound based on elements directly from our principal partition and thereafter approximate this with a more general set. The statement relies on our ability to saturate Δ with an arbitrarily small set in finite time. In the proof we shall rely on the forward measurability of $G : \Delta \to \Delta$ as seen in Corollary 3.4.5 implicitly.

Lemma 3.5.3. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a tower with return time $R \in L^1(\Lambda)$ and* $\gcd(R(\Lambda)) = 1$. *Then for all $\epsilon \in \mathbb{R}_{>0}$ there exists a $t = t(\epsilon) \in \mathbb{Z}_{\geq 0}$ and $\delta = \delta(t, \epsilon) \in \mathbb{R}_{>0}$ such that all $B \in \mathscr{F}_\Delta$ with $\mu_\Lambda(\Delta_0 \setminus B_0) < \delta$ satisfy $\mu_\Delta(\Delta \setminus G^t(B)) < \epsilon$.*

Proof. Let $\epsilon \in \mathbb{R}_{>0}$ be arbitrary. As $R \in L^1(\Lambda)$ we have $\mu_\Delta(\Delta) = ||R||_1$ by Lemma 3.5.2 and so we may find an $n \in \mathbb{Z}_{\geq 1}$ such that

$$\mu_\Delta\left(\Delta \setminus \bigcup_{i=0}^{n-1} \Delta_i \times \{i\}\right) = ||R||_1 - \sum_{i=0}^{n-1} \mu_\Lambda(R_{>n}) < \frac{1}{2}\epsilon.$$

We shall find a $t = t(\epsilon)$ and $\delta = \delta(t, \epsilon) \in \mathbb{R}_{>0}$ such that for any $B \in \mathscr{F}_\Delta$ with $\mu_\Lambda(\Delta_0 \setminus B_0) < \delta$ we have

$$\mu_\Delta\left(\bigcup_{k=0}^{n-1} \Delta_k \times \{k\} \setminus G^t(B_0 \times \{0\})\right) < \frac{1}{2}\epsilon.$$

(Claim) There exists a $d \in \mathbb{Z}_{\geq 1}$ and $P_i \in \bigvee_{j=0}^{d+i} G^{-j}\mathscr{P}_\Delta$ for $i \in \{0, \ldots, n-1\}$ such that

$$\mu_\Delta\left(\Delta \setminus \bigcup_{i=0}^{n-1} G^{n-1+d}(P_i)\right) < \frac{1}{2}\epsilon.$$

Firstly, as $\gcd(R(\Lambda)) = 1$, we have a finite set $\mathscr{L}$ and a collection $\Delta_0^{\mathscr{L}} := \{P'_l\}_{l \in \mathscr{L}} \in \mathscr{P}_\Lambda$ such that $\gcd\{R(P'_l) : l \in \mathscr{L}\} = 1$. For $l \in \mathscr{L}$ write $r_{P'_l} \in \mathbb{Z}_{\geq 1}$ for the integer satisfying $R|_{P'_l} \equiv r_{P'_l}$. By Lemma 3.5.2 we can then find $d \in \mathbb{Z}_{\geq 1}$ such that for all $i \in \{0, \ldots, n-1\}$ we have *positive* integers $(a_{l,i})_{l \in \mathscr{L}} \subseteq \mathbb{Z}_{\geq 1}$ with $d + i = \sum_{l \in \mathscr{L}} a_{l,i} \cdot R_{P'_l}$.

In particular, this means for each $i \in \{0, \ldots, n-1\}$ there is an $l_i \in \mathscr{L}$ and $P_i \in \bigvee_{j=0}^{d+i} G^{-j}\mathscr{P}_\Delta$, $P_i \subseteq P'_{l_i} \times \{0\}$ such that $G^{d+i}P_i = \Delta_0 \times \{0\}$. Now as, $G^{-l}(\Delta_l \times \{l\}) = \Delta_l \times \{0\} \subseteq \Delta_0 \times \{0\}$ we have

$$\bigcup_{i=0}^{n-1} \Delta_i \times \{i\} \subseteq \bigcup_{i=0}^{n-1} G^{n-1-i}(\Delta_0 \times \{0\}).$$

Thus we can see

$$\bigcup_{i=0}^{n-1} \Delta_i \times \{i\} \subseteq \bigcup_{i=0}^{n-1} G^{n-1-i}(\Delta_0 \times \{0\}) = \bigcup_{i=0}^{n-1} (G^{n-1-i} \circ G^{d+i})(P_i) \subseteq \bigcup_{i=0}^{n-1} G^{n-1+d}(P_i),$$

so that

$$\mu_\Delta\left(\Delta \setminus \bigcup_{i=0}^{n-1} G^{n-1+d}(P_i)\right) \leq \mu_\Delta(\Delta \setminus \bigcup_{i=0}^{n-1} \Delta_i \times \{i\}) < \frac{1}{2}\epsilon,$$

proving our claim.

To proceed with the second part of the proof define $P_{i,0} \in \mathscr{F}_\Lambda$ for the 0-section of P_i, that is $P_i = P_{i,0} \times \{0\}$, we note

$$P_{i,0} \in \bigcup_{j=0}^{d+i} g^{R^{-j}} \mathscr{P}_\Lambda, \quad \text{for} \quad i \in \{0, \ldots, n-1\}. \tag{61}$$

Then define

$$\delta_2 := \min_{i\in\{0,\dots,n-1\}} \mu_\Lambda(P_{i,0}) \quad \text{and} \quad \delta := \frac{\epsilon}{2\cdot C_2\cdot n\cdot\delta_2},$$

where $C_2 \in \mathbb{R}_{>1}$ is as in Lemma 3.3.9.

(Claim) For all $B \in \mathscr{F}_\Delta$ with $\mu_\Lambda(\Delta_0 \setminus B_0) < \delta$ we have

$$\mu_\Lambda(\Delta_0 \setminus g^{d+i}(B_0 \cap P_{i,0})) \le C_2\frac{\delta}{\delta_2}, \quad \textit{for } i \in \{0,\dots,n-1\}.$$

Let $B \in \mathscr{F}_\Delta$. First note

$$\delta > \mu_\Lambda(\Delta_0 \setminus B_0) \ge \mu_\Lambda(\cup_{i=0}^{n-1} P_{i,0} \setminus B_0) \ge \max_{i\in\{0,\dots,n-1\}} \mu_\Lambda(P_{i,0} \setminus B_0). \tag{62}$$

Now by Equation (61) we may for each $i \in \{0,\dots,n-1\}$ find a $n_i \in \mathbb{Z}_{\ge1}$ such that $g^{d+i}[P_{i,0}] = g^{R^{n_i}}[P_{i,0}] = \Delta_0$, consequently, we may apply Lemma 3.3.9 and Lemma 3.2.5 and find

$$\frac{\mu_\Lambda(g^{R^{n_i}}(P_{i,0}\setminus B_0))}{\mu_\Lambda(g^{R^{n_i}}(P_{i,0}))} \le C_2\frac{\mu_\Lambda(P_{i,0}\setminus B_0)}{\mu_\Lambda(P_{i,0})}. \tag{63}$$

As by construction $g^{d+i}(P_{i,0}) = g^{R^{n_i}}(P_{i,0}) = \Delta_0$, and as $g^{d+i}|_{P_{i,0}} : P_{i,0} \to \Delta_0$ is bijective, we can see

$$\Delta_0 \setminus g^{d+i}(B_0 \cap P_{i,0}) = g^{d+i}(P_{i,0} \setminus (B_0 \cap P_{i,0})) = g^{R^{n_i}}(P_{i,0} \setminus B_0 \cap P_{i,0}) = g^{R^{n_i}}(P_{i,0} \setminus B_0),$$

so that $\mu_\Lambda(\Delta_0 \setminus (g^{d+i}(B_0 \cap P_{i,0})) = \mu_\Lambda(g^{R^{n_i}}(P_{i,0} \setminus B_0))$. Combining this with Equation (63) shows

$$\frac{\mu_\Lambda(\Delta_0 \setminus g^{d+i}(B_0\cap P_{i,0}))}{\mu_\Lambda(\Delta_0)} \le C_2\frac{\mu_\Lambda(P_{i,0}\setminus B_0)}{\mu_\Lambda(P_{i,0})},$$

which implies

$$\mu_\Lambda(\Delta_0 \setminus g^{d+i}(B_0 \cap P_{i,0})) \le C_2\frac{\delta}{\delta_2}, \quad \text{for } i \in \{0,\dots,n-1\}, \tag{64}$$

proving our second claim.

Proceeding, we see for $i \in \{0,\dots,n-1\}$ that using the general set identity $f(A) \setminus f(B) \subseteq f(A \setminus B)$ that

$$\begin{aligned}\Delta_{n-i-1} \times \{n-i-1\} \setminus (G^{d+n-1}(B_0 \times \{0\})) &\subseteq G^{n-i-1}(\Delta_{n-i-1} \times \{0\} \setminus G^{d+i}(B_0 \times \{0\})) \\ &\subseteq G^{n-i-1}(\Delta_0 \times \{0\} \setminus G^{d+i}(B_0 \times \{0\})).\end{aligned}$$

Using this bound we then arrive at

$$\begin{aligned}\mu_\Delta\left(\Delta_{n-i-1}\times\{n-i-1\}\setminus G^{d+n-1}(B_0\times\{0\})\right) &\le \mu_\Delta\left(\Delta_0\times\{0\}\setminus G^{d+i}(B_0\times\{0\})\right)\\ &\le \mu_\Delta\left(\Delta_0\times\{0\}\setminus G^{d+i}((B_0\cap P_{i,0})\times\{0\})\right)\\ &= \mu_\Lambda(\Delta_0\setminus g^{d+i}(B_0\cap P_{i,0}))\\ &\le C_2\frac{\delta}{\delta_2}, \qquad (65)\end{aligned}$$

where in Line (65) we used Inequality (64). Then finally (please bear with us), we note

$$\begin{aligned}\mu_\Delta\left(\left(\bigcup_{i=0}^{n-1}\Delta_{n-i-1}\times\{n-i-1\}\right)\setminus G^{d+n-1}(B_0\times\{0\})\right) &= \sum_{i=0}^{n-1}\mu_\Delta\left(\Delta_{n-i-1}\times\{n-i-1\}\setminus G^{d+n-1}(B_0\times\{0\})\right)\\ &\le C_2\frac{n\cdot\delta}{\delta_2}\\ &< \frac{1}{2}\epsilon,\end{aligned}$$

which implies

$$\mu_\Delta(\Delta\setminus G^{d+n-1}(B_0\times\{0\}))\le\mu_\Delta\left(\Delta\setminus\bigcup_{i=1}^{n-1}\Delta_i\times\{i\}\right)+\mu_\Delta\left(\bigcup_{i=1}^{n-1}\Delta_i\times\{i\}\setminus G^{d+n-1}(B_0\times\{0\})\right)<\epsilon.$$

Finally, in noting

$$\mu_\Delta(\Delta\setminus G^{d+n-1}B)\le\mu_\Delta(\Delta\setminus G^{d+n-1}(B_0\times\{0\})),$$

we can see the claim follows (for $t=d+n-1$). □

Finally, we need the following result to prove exactness of G.

Lemma 3.5.4. *Let $G:\Delta\to\Delta$ be a tower map with $R\in L^1(\Lambda)$. If $\gcd(R(\Lambda))=1$, then for all $A\in\mathscr{F}_\Delta$ with $\mu_\Delta(A)>0$ we have for each $\epsilon>0$ an $n\in\mathbb{Z}_{\ge1}$ such that $\mu_\Delta(\Delta\setminus G^n(A))<\epsilon$.*

Proof. First let $A\in\mathscr{F}_\Delta$ with $\mu_\Delta(A)>0$ and let $\epsilon>0$ be given arbitrarily. Now by Lemma 3.5.3 there exists a $\delta\in\mathbb{R}_{>0}$ and $t\in\mathbb{Z}_{\ge0}$ be such that for all $B\in\mathscr{F}_\Delta$ with

$$\mu_\Lambda(\Delta_0\setminus B_0)<\delta \text{ we have } \mu_\Delta(\Delta\setminus G^t(B))<\epsilon.$$

If we can find an $m(A,\delta)\in\mathbb{Z}_{\ge1}$ such that $\mu_\Lambda(\Delta_0\setminus G^m(A)_0)<\delta$ then $\mu_\Delta(\Delta\setminus(G^{m+t}(A)))<\epsilon$, proving our claim for $n=m+t$.

To prove existence of such an $m\in\mathbb{Z}_{\ge1}$, we start by pointing out that as a consequence of pbn-singularity we can pick an $l\in\mathbb{Z}_{\ge0}$ such that $\mu_\Lambda((G^lA)_0)>0$. Fix

$\delta_2 = \frac{\delta}{C_2}$ where C_2 is as given by Corollary 3.3.9. Then by Corollary 2.1.24 there exists an $q \in \mathbb{Z}_{\geq 0}$ and $P \in \mathscr{P}_\Lambda^q$ be such that

$$\frac{\mu_\Lambda(P \setminus (G^l A)_0)}{\mu_\Lambda(P)} < \delta_2.$$

Using Corollary 3.3.9 we then have

$$\frac{\mu_\Lambda(g^{R^q}(P \setminus ((G^l A)_0))}{\mu_\Lambda(g^{R^q}(P))} \leq C_2 \frac{\mu_\Lambda(P \setminus (G^l A)_0)}{\mu_\Lambda(P)}.$$

In noting $g^{R^q}(P) = \Lambda$ we see $\mu_\Lambda(g^{R^q}(P)) = 1$ and

$$\Lambda \setminus g^{R^q}((G^l A)_0) \subseteq g^{R^q}(P \setminus ((G^l A)_0).$$

Combining all the above, we see

$$\begin{aligned}
\mu_\Lambda(\Lambda \setminus g^{R^q}((G^l A)_0)) &\leq \mu_\Lambda(g^{R^q}(P \setminus ((G^l A)_0)) \\
&\leq C_2 \frac{\mu_\Lambda(P \setminus (G^l A)_0)}{\mu_\Lambda(P)} \\
&< C_2 \delta_2 \\
&= \delta.
\end{aligned}$$

Upon fixing $\tilde{R} \in \mathbb{Z}_{\geq 1}$ to be the (smallest) integer satisfying $g^{\tilde{R}}(P) = \Delta_0$ we can see (as $\Lambda = \Delta_0$) that

$$\mu_\Lambda(\Delta_0 \setminus (G^m A)_0) < \delta, \text{ for } m = \tilde{R} + l,$$

so that indeed $\mu_\Delta(\Delta \setminus (G^n(A))) < \epsilon$ for $n = m + t$. □

Theorem 3.5.5. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a tower with* $\gcd(\Lambda) = 1$ *and* $R \in L^1(\Lambda)$*, the* acip $\nu_\Delta \ll \mu_\Delta$ *obtained in Theorem 3.5.1 is exact and unique.*

Proof. We shall first prove that for each $A \in \bigcap_{n=0}^\infty G^{-n}\mathscr{F}_\Delta$, with $\nu_\Delta(A) > 0$ we have $\nu_\Delta(A) = 1$. First, note that by Lemma 3.5.4 we have for all $\epsilon \in \mathbb{R}_{>0}$ an $n \in \mathbb{Z}_{\geq 1}$ such that $\mu_\Delta(\Delta \setminus G^n(A)) < \epsilon$. Continuing, by Theorem 3.5.1 there exists a $M_\Delta \in \mathbb{R}_{>1}$ such that $\mu_\Delta(A) \geq \frac{1}{M_\Delta}\nu_\Delta(A) > 0$ so that we may see

$$\epsilon > \mu_\Delta(\Delta \setminus G^n(A)) \geq \frac{1}{M_\Delta}\nu_\Delta(\Delta \setminus G^n(A)),$$

which implies

$$\nu_\Delta(G^n(A)) > \nu_\Delta(\Delta) - M_\Delta \epsilon.$$

Consequently, as $A \in G^{-n}\mathscr{F}_\Delta$ there exists $A' \in \mathscr{F}_\Delta$ such that $A = G^{-n}A'$. Hence

$$\nu_\Delta(A) = \nu_\Delta(A') = \nu_\Delta(G^n(A)) > \nu_\Delta(\Delta) - M_\Delta\epsilon. \tag{66}$$

As $M_\Delta \in \mathbb{R}_{>0}$ is fixed and $\epsilon > 0$ was chosen arbitrarily, we may conclude $\nu_\Delta(A) = \nu_\Delta(\Delta) = 1$.

Since ν_Δ is invariant under G as well, we conclude that ν_Δ is exact. Using Equation (56) with

$$0 < \frac{1}{M_\Delta}\mu_\Delta(\Delta) < \nu_\Delta(\Delta) < M_\Delta\mu_\Delta(\Delta),$$

we see it is also the unique ergodic *acip* $\nu_\Delta \ll \mu_\Delta$ with $\nu_\Delta(\Delta) > 0$ for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ by Proposition 2.1.13. □

To conclude the section, we end it with a small corollary to show how the results on the tower can be expanded to the rest of the system. Note that relevance of the resulting measure within X depends on the size of Λ. The idea stems from Lemma 4.1 in [13].

Corollary 3.5.6. *Suppose the tower* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *fixed in Section 3.4 satisfies* $R \in L^1(\Lambda)$ *and* $\gcd(R(\Lambda)) = 1$. *Then we can find an exact* acip *for its base dynamics* $(X, \mathscr{F}, m, g)$.

Proof. Write ν_Δ for the exact *acip* found in Theorem 3.5.5. In defining

$$\pi : \Delta \to X \qquad (z, l) \mapsto g^l(z), \tag{67}$$

we note we have for $(z, l) \in \Delta$, $\quad R(z) > l + 1$,

$$(\pi \circ G)(z, l) = \pi(z, l + 1) = g^{l+1}(z) = (g \circ \pi)(z, l)$$

and if $(z, l) \in \Delta, R(z) = l + 1$,

$$(\pi \circ G)(z, l) = \pi(g^{l+1}(z), 0) = g^{l+1}(z) = (g \circ \pi)(z, l)$$

holds, showing $\pi \circ G = g \circ \pi$. In noting that for $A \in \mathscr{F}_X$ we have $\pi^{-1}(A) = \bigsqcup_{l \geq 0}(g^{-l}(A) \cap \Lambda) \times \{l\}$, we can see that π is $\mathscr{F}_\Delta$-measurable. We define $m_g := \pi_\star\nu_\Delta$ and show this is a acip.

To show $m_g \ll m$, we assume $m(A) = 0$ and see

$$\begin{aligned}
m_g(A) = \nu_\Delta(\pi^{-1}(A)) &= \sum_{l \geq 0} \nu_\Delta((g^{-l}(A) \cap \Lambda) \times \{l\}) \\
&\leq M_\Delta \sum_{l \geq 0} \mu_\Delta((g^{-l}(A) \cap \Lambda) \times \{l\}) \\
&= M_\Delta \sum_{l \geq 0} \mu_\Lambda(g^{-l}(A) \cap \Lambda) \\
&= M_\Delta \sum_{l \geq 0} m(\Lambda) \cdot m(g^{-l}(A) \cap \Lambda) \\
&\leq M_\Delta m(\Lambda) \sum_{l \geq 0} (g^l)_\star m(A)
\end{aligned}$$

so that by non-singularity of g we see $m_g \ll m$. Finally, we see

$$g_\star m_g = g_\star(\pi_\star \nu_\Delta) = \pi_\star(G_\star \nu_\Delta) = \pi_\star \nu_\Delta = m_g,$$

and for $K \in \bigcap_{n\geq 0} g^{-n}\mathscr{F}_X$ we see

$$\begin{aligned} \pi^{-1}K \in \pi^{-1}\left[\bigcap_{n\geq 0} g^{-n}\mathscr{F}_X\right] \\ = \bigcap_{n\geq 0} G^{-n}[\pi^{-1}\mathscr{F}_X] \quad (68) \\ \subseteq \bigcap_{n\geq 0} G^{-n}\mathscr{F}_\Delta, \quad (69) \end{aligned}$$

where in Equation (68) we used $\pi\circ G = g\circ\pi$, in Equation (69) we used measurability of π. Using the exactness of ν found in Theorem 3.5.5 we can claim $m_g(K) = \nu_\Delta(\pi^{-1}K) \in \{0,\infty\}$, proving exactness of m_g. □

3.6 Two Examples

In this subsection we give two examples of dynamical systems where Young Towers can be used. The theory thus far has already been used in many different contexts see for instance [13] and [3]. It is for the reasons that

1. Young Tower Theory can seem 'inaccessible' by its many conditions and structures involved as seen in Sections 3.2 and 3.4;

2. Young Tower Theory is still undergoing development in various (stochastic) generalisations such as [2] and [7]; and

3. The literature seems to lack easy paradigm examples to test developed theory against;

that the author feels that the literature may be aided by giving simple paradigm examples already describable by existing theory. Finally, we shall revisit these Examples in a new context in Section 5.6. In this section, after introducing some standard theory and terminology, we shall use Young Towers to construct an exact acip for the doubling map in Section 3.6.1 and in Section 3.6.2 we shall construct an exact acip for the so-called *stalling system* (defined in Equation (74)). The stalling system will be our the occurrence of a random dynamical system. We remind the reader of the adopted conventions on the Bernoulli shift in Examples 2.1.3 and 2.1.26 and of the definition of the doubling map in Examples 2.1.3.

3.6.1 The Doubling Map

The following two elementary lemmas allow us to study the doubling map through the Bernoulli shift. The results are by no means new and are used throughout ergodic theory but the author was unable to find a source with a full proof and believed it to be instructive to provide one. As it is rather lengthy, the proof can be found in Appendix A.2. The results essentially state we can find binary number expansions for every real in $[0,1)$, that these are unique up to a measure zero set and that binary shifting behaves similarly as multiplication by a factor 2.

Lemma 3.6.1 (A.2.2)**.** *Let*

$$X := \left\{(x_n)_{n\geq 0} \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \textit{for all } i \in \mathbb{Z}_{\geq 0} \textit{ there is a } j \geq i \textit{ such that } x_j = 0\right\},$$

$$\begin{aligned} \pi : \qquad X &\to [0,1) \\ (x_n)_{n\geq 0} &\mapsto \sum_{n=0}^{\infty} x_n 2^{-n-1} \end{aligned}$$

and

$$\phi : [0,1) \to X$$
$$x \mapsto \left(n \mapsto \begin{cases} 0, & if 2^n x - \lfloor 2^n x \rfloor < \frac{1}{2} \\ 1, & if 2^n x - \lfloor 2^n x \rfloor \geq \frac{1}{2} \end{cases} \right).$$

Then ϕ and π are well-defined on their respective domains and we have

$$\phi \circ \pi = Id_X \qquad \pi \circ \phi = Id_{[0,1)}.$$

Recall the definition of the Doubling map $([0,1), \mathscr{B}[0,1), \lambda, D)$ from Example 2.1.3.

Lemma 3.6.2 (A.2.4)**.** *Let $([0,1), \mathscr{B}[0,1), \lambda, D)$ be the doubling map and $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ be the Bernoulli shift with*

$$\mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_0 = 1\}) = \mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_0 = 0\}) = \frac{1}{2}.$$

Then the mapping π from Lemma 3.6.1 is an ergodic isomorphism between

$$([0,1), \mathscr{B}[0,1), \lambda, D) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma).$$

We shall now construct a tower on the Bernoulli shift $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$ with $\mathbb{P}$ as seen in Lemma 3.6.2, and fix this system for the rest of this section.

For this system, the pick for an induced domain is not unique: rather *any* cylinder set can be used to construct such a collection. In particular, the easiest choice would be to take the trivial 'cylinder' $\{0,1\}^{\mathbb{Z}_{\geq 0}}$. For sake of future study (and to adopt literary convention e.g. [13] and [3]) our induced domain shall be a subset of the cylinder

$$[10] = \{(\gamma_n)_{n \geq 0} \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \gamma_0 = 1, \gamma_1 = 0\} \in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$$

of full measure. Specifically we restrict this cylinder to the sequences that are not eventually constant to make sure the return time takes finite values[1].

Lemma 3.6.3. *In fixing the set*

$$Y := \left\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \text{ for all } i \in \mathbb{Z}_{\geq 0} \text{ there is a } j \in \mathbb{Z}_{\geq 1} \text{ such that } x_i \neq x_j \right\}, \tag{70}$$

we have $Y \in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$, $\mathbb{P}(Y) = 1$, $\sigma(Y) = Y$ and the return time

$$R : Y \cap [10] \to \mathbb{Z}_{\geq 1}$$
$$x \mapsto \inf\left\{ n \in \mathbb{Z}_{\geq 1} : \sigma^n(x) \in Y \cap [10] \right\},$$

exists on $Y \cap [10]$ and satisfies

$$R(x) = \inf\{n \in \mathbb{Z}_{\geq 2} : x_n = 1, x_{n+1} = 0\}. \tag{71}$$

[1] For a systematic approach in finding suitable induced domains in various non-uniformly expanding systems see [25, Page 29]

Proof. First we show Y is a measurable set. To do so, note that

$$\begin{aligned}
Y &= \bigcap_{i=1}^{\infty} \left\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \text{there exists a } j > i, x_i \neq x_j \right\} \\
&= \bigcap_{i=1}^{\infty} \bigcup_{j=i+1}^{\infty} \left\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i \neq x_j \right\} \\
&= \bigcap_{i=1}^{\infty} \bigcup_{j=i+1}^{\infty} \left\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 1, x_j = 0 \right\} \cup \left\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 0, x_j = 1 \right\} \\
&\in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}.
\end{aligned}$$

Looking at its complement, we see

$$Y^c := \bigcup_{i=0}^{\infty} \left(\{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_j = 1, \text{ for all } j \geq i \} \cup \{ x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_j = 0, \text{ for all } j \geq i \} \right)$$

is the countable union of measure-zero sets, hence $\mathbb{P}(Y) = 1$. Finally note that every sequence is eventually constant if and only if it is eventually constant considered starting the first (or any) index, which implies $\sigma(Y) = Y$.

To show Equation (71) holds, note that for $x \in Y$ we have $x \in [10]$ if and only if $x_0 = 1, x_1 = 0$, so that

$$R(x) \geq \inf\{ n \in \mathbb{Z}_{\geq 2} : x_n = 1, x_{n+1} = 0 \},$$

moreover we see for such $x \in [10] \cap Y$ that $x \in [1]$ and $\sigma(x) \in [0]$, so $\sigma(x) \notin [10]$. Hence

$$R(x) = \inf\{ n \in \mathbb{Z}_{\geq 2} : x_n = 1, x_{n+1} = 0 \}.$$

Lastly, as $x \in Y$ we have $R(x) < \infty$ proving the statement. □

We shall now fix $\Lambda := [10] \cap Y$, and define the measure space $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda)$ as the restriction of $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, 4 \cdot \mathbb{P})$ to Λ. The factor 4 is necessary to ensure $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda)$ is a probability space, as $\mathbb{P}(\Lambda) = \frac{1}{4}$.

Remark 3.6.4. Technically, Λ as a subset of Y does not contain any cylinders of $\{0,1\}^{\mathbb{Z}_{\geq 0}}$ but whenever we write $[a_0 a_1 ... a_{k-1}] \subseteq Y$ for some cylinder $[a_0 a_1 ... a_{k-1}] \subseteq \{0,1\}^{\mathbb{Z}_{\geq 0}}$, we actually mean $[a_0 a_1 ... a_{k-1}] \cap Y$. As Y is closed under σ this slight abuse of notation causes no issues: Lemma 4.3.17 applies to these 'cylinders' as well for instance. We shall do the same for $[a_0 a_1 ... a_{k-1}] \subseteq \Lambda$, that is, if $[a_0 a_1 ... a_{k-1}] \subseteq \Lambda$ then we mean the resctricted cylinder $[a_0 a_1 ... a_{k-1}] \cap \Lambda$.

We remind the reader the return time is measurable and so by Lemma 3.2.2, we can define the dynamical system $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$. We need to fix a principal partition to prove it is a tower base.

Lemma 3.6.5. *Consider the system* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$ *as above. Define for each* $l \in \mathbb{Z}_{\geq 2}$ *the collections* $\mathscr{I}_l \subseteq 2^\Lambda$ *as given by*

$$\mathscr{I}_2 := \{[1010]\}, \quad \mathscr{I}_3 := \{[10110], [10010]\}, \quad \text{and for } l \geq 3$$
$$\mathscr{I}_l := \left\{[10a10] \subseteq \Lambda : \quad a \in \{0,1\}^{l-2},\, a_i a_{i+1} \neq 10,\, i \in \{0, ..., l-3\}\right\}. \tag{72}$$

Then the collection $\mathscr{P}_\Lambda := \bigsqcup_{l \in \mathbb{Z}_{\geq 2}} \mathscr{I}_l$ *is a principal partition in the sense of Definition 3.2.3 for* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$*. Moreover, for* $l \in \mathbb{Z}_{\geq 2}$ *and* $I_l \in \mathscr{I}_l$ *we have* $R|_{I_l} \equiv l$.

Proof. Let $I \in \mathscr{P}_\Lambda$ be given and let $l \in \mathbb{Z}_{\geq 2}$ be such that $I \in \mathscr{I}_l$. Note that I is a cylinder of depth $l+2$ so that by Lemma 4.3.17 we know that $\sigma^l : I_l \to \Lambda$ is bi-measurable. Furthermore by Lemma 3.6.3 we see that $R|_{I_l} \equiv l$. To show $\mathscr{P}_\Lambda$ partitions Λ, note as $x \in \Lambda$ we know $(x_n)_{n\geq 0}$ is not eventually constant and $x \in [10]$. We can then see we have either $x \in [1010]$ or there must exist a smallest $l \geq 3$ and an $a \in \{0,1\}^{l-2}$ with $a_i a_{i+1} \neq 10$ for $i \in \{0, ..., l-3\}$ such that $x \in [10a10]$. We conclude $\mathscr{P}_\Lambda$ covers Λ.

Finally, for $I, J \in \mathscr{P}_\Lambda$ of depth $l, m \in \mathbb{Z}_{\geq 4}$, we have $I \cap J \neq \emptyset$ only if the cylinders consist of sequences that correspond on the first $\min(l, m)$ terms, so that either $I \subseteq J$ or $J \subseteq I$. Assuming $I \subseteq J$ without loss of generality implies that $l \geq m$. To show why $l = m$ must hold, write $I = [10a10]$ and $J = [10b10]$ for $a \in \{0,1\}^{l-4}, b \in \{0,1\}^{m-4}$ so that we have $[10a10] \subseteq [10b10]$. Now if $l > m$ we see this implies that $[a] \subseteq [b10]$ which is a contradiction as $I \in \mathscr{I}_l$. Hence we see that $\mathscr{P}_\Lambda$ indeed consists of disjoint elements.

Looking at the n'th refined partition $\mathscr{P}_\Lambda^n$ as seen in Definition 2.1.22, we can inductively show every $\bar{I}^n \in \mathscr{P}_\Lambda^n$ has some $a_1, ..., a_n \in \bigcup_{k=0}^\infty \{0,1\}^k$ such that

$$\bar{I}^n = [10a_1 10a_2 ... 10a_n 10].$$

Using the metric as defined in Equation (8) we have $\sup_{x,y \in \bar{I}^n} d_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}(x, y) \leq 2^{-n}$ for any $\bar{I}^n \in \mathscr{P}_{\Delta_0}^n$ so by Lemma 2.1.24 we have that $\mathscr{P}_\Lambda$ is generating and separating as in Definition 3.2.3. We conclude $\mathscr{P}_\Lambda$ is indeed a principal partition. □

Lemma 3.6.6. *The dynamical system* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$ *with* $\mathscr{P}_\Lambda$ *as in Lemma 3.6.5 is a tower base. Moreover we have* $R \in L^1(\Lambda)$ *and* $\gcd(R(\Lambda)) = 1$ *and there exists an exact acip* $\nu \ll \mathbb{P}$ *for* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$

Proof. In order for $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$ to be a tower base it remains to prove $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$ satisfies bounded distortion according to Definition 3.2.6. By Lemma 4.3.17 we can see, that given some $P \in \mathscr{P}_\Lambda$ the Jacobian satisfies $J(\sigma^R)|_P \equiv \mu_\Lambda(P)$ so we have

$$\text{for all } x, y \in P, \quad \left|\frac{J(\sigma^R|_P)(x)}{J(\sigma^R|_P)(y)} - 1\right| = 0.$$

So $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, \sigma^R)$ is then a tower base and we may construct a tower $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ as in Definition 3.4.1.

To show the integrability of the return time R, simply note that for any $l \geq 2$, we have $\#\mathscr{I}_l = l-1$ (see Equation (72) for the definition). For each $I_l \in \mathscr{I}_l$ we then have, as they are cylinders of depth $l+2$, that $\mu_\Lambda(I_l) = 4 \cdot 2^{-l-2}$ and $R_{I_l} \equiv l$ by Lemma 3.6.5. Consequently, we see

$$\begin{aligned} ||R||_1 &= \sum_{l=2}^{\infty} \sum_{I_l \in \mathscr{I}_l} l \cdot 4 \cdot \mathbb{P}(I_l) \\ &= 4 \cdot \sum_{l=2}^{\infty} (l-1) \cdot l \cdot 2^{-l-2} \\ &= 4, \end{aligned}$$

so that $R \in L^1(\Lambda)$. Finally we have $[1010], [10110] \in \mathscr{P}_\Lambda$ and $R|_{[1010]} \equiv 2$, $R|_{[10110]} = 3$ so clearly we have $\gcd(R(\Lambda)) = 1$.

By Theorem 3.5.1 we then obtain an exact *acip* ν_Δ for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ with a uniformly bounded density $\frac{d\nu_\Delta}{d\mu_\Delta}$ such that $\frac{1}{C} \leq \frac{d\nu_\Delta}{d\mu_\Delta} \leq C$ for some $C \in \mathbb{R}_{>1}$. By Corollary 3.3.11 we then obtain an exact *acip* for $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P}, \sigma)$. As this system is ergodically equivalent with $([0,1), \mathscr{B}[0,1), \lambda, D)$ we know the latter has an exact acip with a uniformly bounded density as well by Corollary 2.1.19. □

3.6.2 Annealed Stalling System

We shall now showcase how Young Tower Theory can be applied to random dynamical systems. The approach in this example is called *annealed* in the literature, see for instance [3] and [13]. The *annealed approach* is to incorporate the random dynamics into the Young tower. When having a mixing base dynamic this can work rather well - the base dynamic is well-behaved to the extent that is does not interfere with the construction of the Young tower. This method typically fails when considering systems with a non-mixing random dynamic such as the (irrational) rotation. Developing theory to analyse these systems will be done in Sections 4 and 5. The main purpose of this section is to provide a concrete example of the annealed approach to compare with the in Section 5.6 which follows the *quenched approach*.

To keep the example as simple as possible we build upon our previous example in Section 3.6.1. We shall refer to this system as the *(annealed) stalling system*.

Define on the standard Borel space $([0,1), \mathscr{B}[0,1), \lambda)$ with Lebesgue measure two mappings

$$\begin{aligned} f_g : [0,1) &\to [0,1), & f_s : [0,1) &\to [0,1) \\ x &\mapsto 2x \mod 1, & x &\mapsto \begin{cases} x, & x \in [0, \frac{1}{2}) \cup [\frac{3}{4}, 1) \\ 2x-1, & x \in (\frac{1}{2}, \frac{3}{4}) \end{cases}, \end{aligned} \tag{73}$$

where we refer to f_g as a 'go' map and f_s as a 'stall' map. We shall assume non-singularity of f_g and f_s without proof and define the (natural) dynamical systems $([0,1), \mathscr{B}[0,1), \lambda, f_g)$ and $([0,1), \mathscr{B}[0,1), \lambda, f_s)$.

To construct a *random dynamical system*, we consider the Bernoulli shift $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ based on an alphabet $\Sigma = \{s, g\}$ with weights $\{\frac{1}{2}, \frac{1}{2}\}$ respectively, so that $\Omega := \{s, g\}^{\mathbb{Z}_{\geq 0}}$. Now we construct the tuple

$$(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)]}, \mathbb{P} \times \lambda, S), \qquad S(\omega, x) = (\sigma\omega, f_{\omega_0}(x)), \tag{74}$$

where we refer to S as the skew product. To prove the System (74) is a dynamical system we shall make use of the following lemma. In Section 4.2 we shall see a stronger version with a proof.

Lemma 3.6.7. *Let $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ be a dynamical system with $\sigma : \Omega \to \Omega$ bi-measurable and $\mathbb{P}(\Omega) = 1$ and let $(X, \mathscr{F}_X, \mu)$ be a σ-finite measure space. Suppose that we have a measurable mapping*

$$\begin{aligned} f : \Omega \times X &\to X \\ (\omega, x) &\mapsto f_\omega(x) \end{aligned}$$

with $(f_\omega)_\star \mu \ll \mu$ for $\omega \in \Omega$ and then define $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ with

$$\begin{aligned} S : \Omega \times X &\to \Omega \times X \\ (\omega, x) &\mapsto \big(\sigma\omega, f_\omega(x)\big). \end{aligned}$$

Then the mapping S is measurable and non-singular and hence $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ is a dynamical system.

We introduce some terminology for the objects in Lemma 3.6.7. We shall define this in greater generality in Definition 4.2.3.

Definition 3.6.8. We call systems $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ as in Lemma 3.6.7 *random dynamical systems.* More so, we call the mapping $S : \Omega \times X \to \Omega \times X$ a *skew product*, the family $(X, \mathscr{F}, \mu, f_\omega)_{\omega\in\Omega}$ the *base dynamic* and the dynamical system $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ the *random dynamic.*

Applying Lemma 3.6.7, we see the following.

Lemma 3.6.9. *The system $(\Omega \times [0,1), \mathscr{F}_{\Omega\times[0,1)}, \mathbb{P} \times \lambda, S)$ in Equation (74) is a dynamical system.*

Proof. In order to apply Lemma 3.6.7, we first need to show the mapping $f : (\omega, x) \mapsto f_{\omega_0}(x)$ is measurable and non-singular, so let $A \subseteq [0,1)$ be some measurable set. Then we see $f^{-1}(A) = ([g] \times f_g^{-1}(A)) \cup ([s] \times f_s^{-1}(A)) \in \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}$. In assuming $\lambda(A) = 0$ we see similarly,

$$(\mathbb{P} \times \lambda)(f^{-1}A) = (\mathbb{P} \times \lambda)([g] \times f_g^{-1}(A)) + (\mathbb{P} \times \lambda)([s] \times f_s^{-1}(A)) \leq \lambda(f_g^{-1}(A)) + \lambda(f_s^{-1}(A)) = 0.$$

Secondly, the Bernoulli shift is bi-measurable by Lemma 4.3.17 and so by Lemma 3.6.7 we have proven our statement. □

We shall construct a Young Tower on the system (74). We start by transcribing it to a shift system and to do so, use the following Lemma.

Lemma 3.6.10. *Suppose we have dynamical systems $(X, \mathscr{F}_X, \mu, T_i)$, $(Y, \mathscr{F}_Y, \nu, U_i)$ for $i \in \{0,1\}$ and suppose we have sets $U \in \mathscr{F}_X$, $V \in \mathscr{F}_Y$ with $\mu(U) = \nu(V) < \infty$ and an ergodic isomorphism $\alpha : U \to V$ for $(X, \mathscr{F}_X, \mu, T_i)$ to $(Y, \mathscr{F}_Y, \nu, U_i)$ for $i \in \{0,1\}$. Now let $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ be some dynamical system with $\mathbb{P}(\Omega) = 1$ and let $\gamma : \Omega \to \{0,1\}$ be measurable and the mappings*

$$\Omega \times X \to X, \quad (\omega, x) \mapsto T_{\gamma(\omega)}(x), \qquad \Omega \times Y \to Y, \ (\omega, y) \mapsto U_{\gamma(\omega)}(y)$$

be measurable. Lastly suppose that for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have $(T_{\gamma(\omega)})_\star \mu \ll \mu$ and $(U_{\gamma(\omega)})_\star \nu \ll \nu$. Then the random dynamical systems

$$(\Omega \times X, \mathscr{F}_{\Omega \times X}, \mathbb{P} \times \mu, \mathscr{T}) \text{ and } (\Omega \times X, \mathscr{F}_{\Omega \times X}, \mathbb{P} \times \mu, \mathscr{U}), \tag{75}$$

with $\mathscr{T}, \mathscr{U}$ skew products, are ergodically isomorphic.

The proof of Lemma 3.6.10 is straightforward: we can directly verify

$$\text{Id} \times \alpha : \Omega \times U \to \Omega \times V \qquad (\omega, x) \mapsto (\omega, \alpha(x))$$

is an ergodic isomorphism, checking the measurability conditions using Lemma 2.1.4. We have omitted giving a full proof not to interrupt the flow of the text. We now show the system (74) is ergodically equivalent with a shift system.

Lemma 3.6.11. *Define the product measure space $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mu)$ with weights $(p_0, p_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Now define the mappings*

$$\sigma_g : \{0,1\}^{\mathbb{Z}_{\geq 0}} \to \{0,1\}^{\mathbb{Z}_{\geq 0}} \qquad \sigma_s : \{0,1\}^{\mathbb{Z}_{\geq 0}} \to \{0,1\}^{\mathbb{Z}_{\geq 0}}$$

$$(x_n)_{n\geq 0} \mapsto (x_{n+1})_{n\geq 0} \qquad (x_n)_{n\geq 0} \mapsto \begin{cases} (x_{n+1})_{n\geq 0}, & (x_n)_{n\geq 0} \in [10] \\ (x_n)_{n\geq 0}, & (x_n)_{n\geq 0} \in [0] \cup [11], \end{cases}$$

then

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, U) \qquad \textit{and} \qquad (\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, S),$$

with U, S as given by,

$$U(\omega, x) = (\sigma\omega, \sigma_{\omega_0}(x)) \qquad S(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$$

are ergodically equivalent.

Proof. We can see that by Lemma A.2.4 we have

$$([0,1), \mathscr{B}[0,1), \lambda, D) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mu, \sigma_g)$$

are ergodically equivalent and by Lemma A.2.5 the systems

$$([0,1), \mathscr{B}[0,1), \lambda, f_s) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mu, \sigma_s)$$

are ergodically equivalent. We can then use Lemma 3.6.10 to show that the random dynamical systems

$$(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, S) \text{ and } (\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, U)$$

are indeed ergodically equivalent. □

We can equip both Ω and $\{0,1\}^{\mathbb{Z}_{\geq 0}}$ with natural product metrics d_Ω and $d_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$ as in Equation (8) and define the metric

$$\begin{aligned} d_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}} : \quad & (\Omega \times \{0,1\}) \times (\Omega \times \{0,1\}) \to \mathbb{R} \\ & ((\omega, x), (\omega', x')) \qquad \mapsto d_\Omega(\omega, \omega') + d_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}(x, x') \end{aligned}$$

inducing $\mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}$.

For the rest of this section we fix $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, U)$ **as in Lemma 3.6.11**.

In defining an induced domain $\bigwedge \subseteq \Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}$ (not to be confused with Λ) and a principal partition $\mathscr{P}_\bigwedge$ an obvious starting point is the set $[10] \cap Y$ and the collection $\mathscr{P}_\Lambda$ from Lemma 3.6.5. In fact, we shall fix

$$\bigwedge = \dot{\Omega} \times ([10] \cap Y)$$

for some large measurable $\dot{\Omega} \subseteq \Omega$ as defined in Lemma 3.6.12. Further up in Lemma 3.6.14, we shall show we can construct a principal parition $\mathscr{P}_\bigwedge$ consisting of products between elements $I \in \mathscr{P}_\Lambda$ and cylinders $O \in \mathscr{F}_\Omega$ in Ω.

Lemma 3.6.12. *Let $Y \cap [10] = \Lambda \subseteq \{0,1\}^{\mathbb{Z}_{\geq 0}}$ be as in Lemma 3.6.5 and define*

$$\dot{\Omega} := \left\{(\omega_n)_{n \geq 0} \in \Omega : \omega_i = g \text{ for infinitely many } i \in \mathbb{Z}_{\geq 0}\right\}. \tag{76}$$

Then $\bigwedge := \dot{\Omega} \times \Lambda$ is measurable, $(\mathbb{P} \times \mu)(\bigwedge) = \frac{1}{4}$ and $\sigma(\dot{\Omega}) = \dot{\Omega}$.

Proof. First note we may write

$$\dot{\Omega} = \limsup_{n\to\infty} \left\{(\omega_i)_{i\geq 0} \in \Omega : \omega_n = g\right\},$$

so that $\dot{\Omega} \in \mathscr{F}_\Omega$. Furthermore

$$\begin{aligned}\mathbb{P}(\Omega \setminus \dot{\Omega}) &= \mathbb{P}\left(\bigcup_{m=0}^{\infty} \bigcap_{i\geq m} \{\omega \in \Omega : \omega_i = s\}\right) \\ &\leq \sum_{m=0}^{\infty} \mathbb{P}\left(\bigcap_{i\geq m} \{\omega \in \Omega : \omega_i = s\}\right) \\ &= 0,\end{aligned}$$

so that $\mathbb{P}(\dot{\Omega}) = 1$. We then see

$$(\mathbb{P} \times \mu)(\Lambda) = \mathbb{P}(\dot{\Omega})\mu([10] \cap Y) = \frac{1}{4}.$$

Now by Lemma 3.6.3 we have $\Lambda \in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$ so that $\Lambda = \dot{\Omega} \times \Lambda \in \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}$. Lastly, as sequences in Ω have infinitely many terms equal to g starting at index 0 if and only they have infinitely many terms equal to g starting at index 1, we see $\sigma(\dot{\Omega}) = \Omega$. □

We restrict the measure space $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}, 4\cdot\mathbb{P}\times\mu)$ to Λ from Lemma 3.6.12 to obtain the restricted measure space $(\Lambda, \mathscr{F}_\Lambda, (4\cdot\mathbb{P}\times\mu)_\Lambda)$. We shall write $4\mu_\Lambda := (4\cdot\mathbb{P}\times\mu)_\Lambda$ for notational convenience. The factor 4 here once again ensures us we obtain a probability space. In Lemma 3.6.13 below we show there exists a return time on Λ as in Definition 3.2.1 for the dynamical system $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}, 4\cdot\mathbb{P}\times\mu, U)$. In proving this, we also give a convenient expression for $\boldsymbol{R}$.

Lemma 3.6.13. *In letting $R : \Lambda \to \mathbb{Z}_{\geq 2}$ be the return time from Lemma 3.6.3 and defining for $k \in \mathbb{Z}_{\geq 2}$ the measurable mapping*

$$\#_k(\omega) := \#\{i \in \{1,\ldots,k-1\} : \omega_i = g\} + 1, \tag{77}$$

we have a return time $\boldsymbol{R} : \Lambda \to \mathbb{Z}_{\geq 1}$ which is given by

$$\boldsymbol{R}(\omega, x) = \inf\{k \in \mathbb{Z}_{\geq 2} : R(x) = \#_k(\omega)\}. \tag{78}$$

Proof. We recall for $l \in \mathbb{Z}_{\geq 2}$ the definition of $\mathscr{I}_l$ and the relation to $R|_{I_l} \equiv l$ from Lemma 3.6.5. Now let $(\omega, x) \in \Lambda$ and fix $l \in \mathbb{Z}_{\geq 2}$ such that $x \in I_l$ for some $I_l \in \mathscr{I}_l$. First note that by Lemma 3.6.12 we have $\sigma^l(\omega) \in \dot{\Omega}$ for each $l \in \mathbb{Z}_{\geq 0}$ so that $\boldsymbol{R}$ is only dependent on the second component of $U^j(\omega, x)$, for $j \in \mathbb{Z}_{\geq 1}$. It is then clear that $\boldsymbol{R}(\omega, x)$ is the lowest value $k \in \mathbb{Z}_{\geq 2}$ such that there are $l = R(x)$ elements g seen in the first k indices of $\omega = (\omega_0, \omega_1, \ldots) \in \dot{\Omega}$, excluding ω_0 as $\sigma_g|_{[10]} = \sigma_s|_{[10]}$. We conclude that

$$\boldsymbol{R}(\omega, x) = \inf\{k \in \mathbb{Z}_{\geq 2} : R(x) = \#_k(\omega)\},$$

proving our claim. □

Having shown the return time $\boldsymbol{R} : \Lambda \to \mathbb{Z}_{\geq 1}$ exists we can define the dynamical system

$$(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}}) \tag{79}$$

as seen by Lemma 3.2.2. In proving this system is a tower base, we now construct a principal partition. As mentioned earlier, $\mathscr{P}_\Lambda$ is constructed by taking Cartesian products between elements in $\mathscr{P}_\Lambda$ with cylinders in $\dot{\Omega}$.

Lemma 3.6.14. *Let* $(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}})$ *be the dynamical system as in Equation* (79). *Let* $l \in \mathbb{Z}_{\geq 2}$, $\mathscr{I}_l \subseteq \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$ *be as in Lemma 3.6.5. Then for all* $k \geq l$ *define*

$$\mathscr{K}_{k,l} := \left\{[\omega_0 \cdots \omega_{k-1}] \subseteq \#_k^{-1}(l) : \omega_{k-1} = g\right\}.$$

We have for each $k, l \in \mathbb{Z}_{\geq 2}$, $k \geq l$ *that* $\boldsymbol{R}|_{K_{k,l} \times I_l} \equiv k$ *and*

$$JU^{\boldsymbol{R}}(\omega, x) = \frac{1}{\mathbb{P}(K_{k,l})} \cdot \frac{1}{\mu_\Lambda(I_l)} \qquad (\omega, x) \in K_{k,l} \times I_l, \quad 4\mu_{K_{k,l} \times I_l} - \text{ almost surely}. \tag{80}$$

Moreover $\mathscr{P}_\Lambda := \bigsqcup_{l \geq 2} \bigsqcup_{k \geq l} \bigsqcup_{P \in \mathscr{K}_{k,l} \times \mathscr{I}_l} \{P\}$ *is a principal partition for* $(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}})$ *as in Definition 3.2.3.*

Proof. We shall show the conditions as phrased in Definition 3.2.3 and in the process show $\mathscr{P}_\Lambda$ partitions Λ into measurable sets, alongside Identity (80).

(Constant Return Time) Note for general $k \geq 2$ we can write by Lemma 3.6.13

$$\begin{aligned}
\boldsymbol{R}^{-1}(k) &= \left\{(\omega, x) \in \Lambda : k = \inf\{k' \geq 2 : R(x) = \#_{k'}(\omega)\}\right\} \\
&= \bigsqcup_{l \geq 2} \bigsqcup_{I_l \in \mathscr{I}_l} \left\{(\omega, x) \in \dot{\Omega} \times I_l : k = \inf\{k' \geq 2 : R(x) = \#_{k'}(\omega)\}\right\} \\
&= \bigsqcup_{l \geq 2} \bigsqcup_{I_l \in \mathscr{I}_l} \left\{(\omega, x) \in \dot{\Omega} \times I_l : R(x) = \#_k(\omega), \omega_{k-1} = g\right\} \\
&= \bigsqcup_{l \geq 2} \bigsqcup_{I_l \in \mathscr{I}_l} \bigsqcup_{K_{k,l} \in \mathscr{K}_{k,l}} K_{k,l} \times I_l.
\end{aligned}$$

In noting that for $k, l \in \mathbb{Z}_{\geq 2}$ we have $K_{k,l} \neq \emptyset$ if and only if $k \geq l$ we can see as $\Lambda = \boldsymbol{R}^{-1}(\mathbb{Z}_{\geq 2})$,

$$\Lambda = \bigsqcup_{k \geq 2} \boldsymbol{R}^{-1}(k) = \bigsqcup_{l \geq 2} \bigsqcup_{k \geq l} \bigsqcup_{I_l \in \mathscr{I}_l} \bigsqcup_{K_{k,l} \in \mathscr{K}_{k,l}} K_{k,l} \times I_l,$$

so that

$$\mathscr{P}_\Lambda := \bigsqcup_{l \geq 2} \bigsqcup_{k \geq l} \bigsqcup_{I_l \in \mathscr{I}_l} \bigsqcup_{K_{k,l} \in \mathscr{K}_{k,l}} \{K_{k,l} \times I_l\},$$

is a partition. Note it consists of measurable sets as it consists of cylinders (intersected with a measure 1 set closed under $U^{\boldsymbol{R}}$).

(Markov Property) Let $k, l \in \mathbb{Z}_{\geq 2}$, and $K_{k,l} \times I_l \in \mathscr{K}_{k,l} \times \mathscr{I}_l$ be given. First note that as $K_{k,l}$, I_l are cylinders of depth k, l respectively, we have $\sigma^k[K_{k,l}] = \dot{\Omega}$, $\sigma_g^l[I_l] = \Lambda$. Now recalling σ as defined just above Equation (74) we apply Lemma 4.3.17 and see $\sigma^k : K_{k,l} \to \dot{\Omega}$ and $\sigma_g^l : I_l \to \Lambda$ are bi-measurable and pbn-singular. Moreover, we see $J\sigma^k \equiv \frac{1}{\mathbb{P}(K_{k,l})}$, $\mathbb{P}_{K_{k,l}}$-almost surely and $J\sigma_g^l \equiv 4 \cdot \mu(I_l)$, $4 \cdot \mu_{I_l}$ almost surely. By Lemma 4.3.18 we then see $\sigma^k \times \sigma_g^l$ is pbn-singular and that Identity (80) holds.

(Generating and Separating) for $n \in \mathbb{Z}_{\geq 1}$ the sets $A \in \bigvee_{i=0}^{n-1} (U^{\boldsymbol{R}})^{-i} \mathscr{P}_\Lambda$ consist of products of cylinders of depth at least n, meaning that $\operatorname{diam}(\bigvee_{i=0}^{n} (U^{\boldsymbol{R}})^{-i} \mathscr{P}_\Lambda) \to 0$ as $n \to \infty$. Consequently, we see $\mathscr{P}_\Lambda$ generates $\mathscr{F}_\Lambda$ and $\mathscr{P}_\Lambda^\infty$ is the trivial partition into points by Lemma 2.1.25.

We have shown $\mathscr{P}_\Lambda$ is a principal partition. □

We now show $(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}})$ is a tower base.

Lemma 3.6.15. *The dynamical system $(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}})$ with $\mathscr{P}_\Lambda$ as in Lemma 3.6.5 is a tower base. Moreover we have $\boldsymbol{R} \in L^1(\Lambda)$ and $\gcd(\boldsymbol{R}(\Lambda)) = 1$ and there exists a unique mixing acip $\nu \in \mathscr{P}(\Omega \times [0,1)^{\mathbb{Z}_{\geq 0}})$ for $(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, S)$.*

Proof. **Bounded Distortion** We shall first show bounded distortion. To do so, let $P \in \mathscr{P}_\Lambda$. By Equation (80) in Lemma 3.6.14 we then have $l \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{Z}_{\geq l}$ with $P = K_{k,l} \times I_l$ for some $K_{k,l} \in \mathscr{K}_{k,l}$ and $I_l \in \mathscr{P}_\Lambda$, and

$$JU^{\boldsymbol{R}}(\omega, x) = \frac{1}{\mathbb{P}(K_{k,l})} \cdot \frac{1}{\mu_\Lambda(I_l)} \qquad (\omega, x) \in K_{k,l} \times I_l, \quad 4\mu_{K_{k,l} \times I_l} \text{ -almost surely},$$

so that

$$\left| \frac{J(U^{\boldsymbol{R}}|_P)(\omega, x)}{J(U^{\boldsymbol{R}}|_P)(\omega', y)} - 1 \right| = 0$$

holds for $(\omega, x), (\omega', y)$-almost every $4\mu_{K_{k,l} \times I_l}$.

Aperiodic Return Times Note that $[gggg] \times [1010], [ggggg] \times [10110] \in \mathscr{P}_\Lambda$, both have positive measure and $\boldsymbol{R}|_{[gggg] \times [1010]} \equiv 2$, $\boldsymbol{R}|_{[ggggg] \times [10110]} = 3$ so clearly we have $\gcd(\boldsymbol{R}(\Lambda)) = 1$.

Integrability of $\boldsymbol{R}$ We shall now calculate the expectation of the return time. Note for $k \in \{0, 1\}$, $\boldsymbol{R}^{-1}\{k\} = \emptyset$ and for $k \geq 2$ we have by the proof of Lemma 3.6.14,

$$\boldsymbol{R}^{-1}\{k\} = \bigsqcup_{l \geq 2} \bigsqcup_{I_l \in \mathscr{I}_l} \bigsqcup_{K_{k,l} \in \mathscr{K}_{k,l}} K_{k,l} \times I_l. \tag{81}$$

We then see for $l \geq 2$, $k \geq l$ and $K_{k,l} \in \mathscr{K}_{k,l}$, $I_l \in \mathscr{I}_l$ that $K_{k,l}$ is a cylinder of depth k, and I_l is a cylinder of depth $l + 2$ so that $\mathbb{P}(K_{k,l}) = 2^{-k}$ and $\mu(I_l) = 2^{-l-2}$.

Consequently, we have $4\mu_\Lambda(K_{k,l} \times I_l) = 4 \cdot 2^{-k} \cdot 2^{-l-2} = 2^{-k} \cdot 2^{-l}$. Also, we note in writing for $\gamma \in \{g,s\}$ that $\mathscr{K}_{k,l}^{\gamma} := \{[\omega_0 \cdots \omega_{k-1}] \in \mathscr{K}_{k,l} : \omega_0 = \gamma\}$, we see $\#\mathscr{K}_{k,l}^{\gamma} = \binom{k-2}{l-2}$, so that $\#\mathscr{K}_{k,l} = 2 \cdot \binom{k-2}{l-2}$. As seen in the proof of Lemma 3.6.6 we have $\#\mathscr{I}_l = l-1$. Using this, we see

$$
\begin{aligned}
\int_\Lambda \boldsymbol{R}(\omega,x)\, d4\mu_\Lambda(\omega,x) &= \sum_{k=2}^{\infty} \int_{\boldsymbol{R}^{-1}\{k\}} \boldsymbol{R}(\omega,x)\, d4\mu_\Lambda(\omega,x) \\
&= \sum_{k=2}^{\infty} \sum_{l=2}^{k} \sum_{I_l \in \mathscr{I}_l} \sum_{K_{k,l} \in \mathscr{K}_{k,l}} \int_{K_{k,l} \times I_l} \boldsymbol{R}(\omega,x)\, d4\mu_\Lambda(\omega,x) \\
&= \sum_{k=2}^{\infty} \sum_{l=2}^{k} \sum_{I_l \in \mathscr{I}_l} \sum_{K_{k,l} \in \mathscr{K}_{k,l}} k \cdot 4\mu_\Lambda(K_{k,l} \times I_l) \\
&= \sum_{k=2}^{\infty} \sum_{l=2}^{k} \sum_{I_l \in \mathscr{I}_l} \sum_{K_{k,l} \in \mathscr{K}_{k,l}} k \cdot 2^{-k} \cdot 2^{-l} \\
&= \sum_{k=2}^{\infty} \sum_{l=2}^{k} 2 \cdot \binom{k-2}{l-2} k \cdot 2^{-k} \cdot 2^{-l}(l-1) \\
&= \sum_{k=2}^{\infty} 2^{-k+1} \cdot k \sum_{l=2}^{k} \binom{k-2}{l-2} 2^{-l}(l-1). \qquad (82)
\end{aligned}
$$

Focussing on the inner sum, we see

$$
\begin{aligned}
\sum_{l=2}^{k} \binom{k-2}{l-2} 2^{-l}(l-1) &= \sum_{l'=0}^{k-2} \binom{k-2}{l'} 2^{-l-2}(l'+1) \\
&\leq \frac{k-1}{4} \cdot \sum_{l'=0}^{k-2} \binom{k-2}{l'} 2^{-l'} \\
&= \frac{k-1}{4} \left(1 + \frac{1}{2}\right)^{k-2},
\end{aligned}
$$

which we substitute in Equation (82) to see

$$
\begin{aligned}
\sum_{k=2}^{\infty} 2^{-k+1} \cdot k \sum_{l=2}^{k} \binom{k-2}{l-2} 2^{-l}(l-1) &\leq \sum_{k=2}^{\infty} 2^{-k+1} \cdot k \cdot \frac{k-1}{4} \left(1 + \frac{1}{2}\right)^{k-2} \\
&= \sum_{k=2}^{\infty} k \cdot \frac{k-1}{4} \cdot 2 \left(\frac{3}{2}\right)^{-2} \cdot \left(\frac{3}{4}\right)^{k} \\
&\leq \sum_{k=0}^{\infty} k^2 \left(\frac{3}{4}\right)^{k},
\end{aligned}
$$

which we can show converges using the Ratio Test [12, Theorem 2.31]. Hence we see $\boldsymbol{R} \in L^1(\Lambda)$.

As the system $(\Lambda, \mathscr{F}_\Lambda, 4\mu_\Lambda, U^{\boldsymbol{R}})$ satisfies bounded distortion it is a tower base, so that we can construct a tower $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ as in Section 3.4. As we have shown the return times are aperiodic and that return time is integrable we then can apply Theorem 3.5.1 to obtain an exact *acip* ν_Δ for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ such that there exists a $M_\Delta > 1$ with $\frac{1}{M_\Delta} \leq \frac{d\nu_\Delta}{d\mu_\Delta} \leq M_\Delta$. By Corollary 3.3.11 we then obtain an exact *acip* for $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, U)$. As this system is ergodically equivalent with $(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, S)$ we know this has an exact acip as well, by Lemma 2.1.17. □

4 Preliminaries for the Quenched Case

As mentioned in Section 1, *the quenched approach* allows us to study random dynamical systems as defined in Section 4.2 with a non-uniformly expanding base dynamic. To motivate the quenched approach we revisit the *annealed approach* as seen in Section 3.6.2.

In Section 3.6.2, we applied Young Towers as introduced in Sections 3.2 and 3.3 to show the existence of an acip on the *(annealed) stalling system*. As we constructed our Young Tower directly on

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, U),$$

we essentially interpret it as a deterministic dynamical system. As Young Towers are designed to analyse non-uniformly expanding dynamical systems, we can then reasonably expect that the annealed approach only applies to random dynamical systems with a non-uniformly expanding random dynamic, such as a Bernoulli shift. This limits the possible applications for the annealed way of analysing random dynamical systems.

The *quenched approach* tackles this issue, by constructing a Young Tower-like structure Δ_ω for each $\omega \in \Omega$ individually, yielding a *random Young Tower*

$$\Delta = \{(\omega, x) \in \Omega \times X \times \mathbb{Z}_{\geq 0} : x \in \Delta_\omega\}.$$

This way, we can make sure Young's conditions only have to apply to the Ω-sections of Δ, rather than to the entirety of Δ itself. This construction is very delicate, however as technical conditions such as measurability are no longer guaranteed.

The outline of this section is then as follows. In Section 4.1 we will show using functional analytic arguments that a sequence of measures with a uniformly bounded density admits an accumulation point in the topology of setwise convergence. Obtaining this convergence argument is important as we will not be able to generalise the notion of a tower base effectively to the quenched setting. This makes us unable to make use of a potentially generalised version of Proposition 3.3.2. In the adjacent Section 4.1.1 we shall also use this convergence argument to phrase an alternative proof for Proposition 3.3.2. After that, we shall continue with Section 4.2 where we shall define a *random dynamical system* as was already hinted at in Section 3.6.2. Importantly, Lemma 4.2.7 allows us to describe the density associated with skew products through its Ω-sections. As our analysis in Section 5 will mainly happen on the Ω-sections of Δ, this will be vital. Lastly, in Section 4.3 we shall generalise the notion of a Jacobian in order to fit our random dynamical systems framework. As Jacobians are heavily interlinked with Young Towers through bounded distortion, and as in the literature authors commonly assume a Jacobian exists without stating conditions under which they actually do, they are deserving of their own section.

4.1 Uniform Integrability and Weak-* Compactness

Definition 4.1.1. Let $(X, \mathscr{T})$ be a topological vector space over the field $\mathbb{R}$. We call

1. the space X' of bounded linear functionals $\phi : X \to \mathbb{R}$ the *dual (vector space) of* X;
2. the topology $\sigma(X, X') \subseteq \mathscr{T}$ being the weakest topology on X so that all $\phi \in X'$ are continuous, the *weak topology* on X; and
3. the topology $\sigma(X', X)$, being the weakest topology on X' so that for all $x \in X$ the mappings $x' : X' \to \mathbb{R}$ as given by $x'(\phi) = \phi(x)$ are continuous, the *weak-* topology* on X'.

Note that X' with the weak star topology is again a topological vector space, so we may define $X'' = (X')'$. We shall refer to this space as the *bidual* of X.

Remark 4.1.2.
1. Whenever X is a normed vector space, we can equip X' with the norm $||\phi||_{X'} = \sup_{||x|| \leq 1} |\phi(x)|$. Similarly we can endow X'' with a norm based on X'. It is known that whenever X is a normed space, X' and (hence) X'' are Banach spaces.
2. Chapter V of [8] hosts an extensive treatise of these topologies and spaces, but we shall only discuss the material essential for our application.

Example 4.1.3. Let $(X, \mathscr{F}, \mu)$ be a measure space, let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ and define their associated $L^p(X)$ and $L^q(X)$. By Hölders inequality, we know that for $f \in L^p(X), g \in L^q(X)$ we have $||fg||_1 \leq ||f||_p ||g||_q$, and it is then not hard to see that for every $f \in L^p(X)$ we have

$$\begin{aligned} F_f : L^q(X) &\to \mathbb{R} \\ g &\mapsto \int_X fg \, d\mu, \end{aligned}$$

to be a bounded linear functional on $L^q(X)$. The fact that the mapping $f \mapsto F_f$ is actually an isometric isomorphism from $L^p(X)$ to $L^q(X)'$, is the content of Theorem A.3.1.

Building on Example 4.1.3, given again some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we can apply Theorem A.3.1 to $L^q(X, \mu)$. In doing so, we obtain an isomorphism J_p from $L^p(X, \mu)$ to $L^p(X, \mu)''$. such that for $f \in L^p(X)$ and for all $g \in L^q(X, \mu)$ we have

$$J_p(f)(F_g) = F_g(f) = \int_X fg \, d\mu.$$

We generalise this to arbitrary Banach Spaces.

Definition 4.1.4. [8, Definition III.11.2] Let $(X, ||\cdot||)$ be a Banach space over a field $\mathbb{R}$, and let $X'' := (X')'$ be its bidual. If the mapping

$$J_X : X \to \quad X''$$
$$x \mapsto \begin{pmatrix} x' : X' \to \mathbb{R} \\ \qquad \phi \mapsto \phi(x) \end{pmatrix},$$

is an isometric isomorphism from $(X, ||\cdot||)$ onto $(X'', ||\cdot||_{(X')'})$ then we call X *reflexive*.

The main reason we are interested in weak and weak-* topologies is the weak sequential compactness of (norm bounded) balls in reflexive Banach spaces. We will quickly go over the method for proving this and assume X is a reflexive Banach space. Firstly, assuming reflexivity, we can show the mapping $J_X : X \to X''$ is a weak to weak-** continuous linear isomorphism. To prove this, we shall rely on nets, (see [8, Appendix A.2]). Additionally the Banach-Alaoglu Theorem A.3.2 allows to show norm-bounded balls in X'' are weakly compact. Using the Eberlein-Smulian Theorem A.3.3 then implies the weak sequential compactness of these balls. The (just established) properties of J_X can then show the weak sequential compactness of (norm bounded) unit balls in X. Generally, for sequences $(x_n)_{n\geq 0}$ we shall write $x_n \rightsquigarrow x$ to say $(x_{n_k})_{k\geq 0} \to x$ as $k \to \infty$ for some sequence $(n_k)_{k\geq 0}$. That is, *x is an accumulation point* of $(x_n)_{n\geq 0}$.

Proposition 4.1.5. *Let $(X, ||\cdot||)$ be a normed vector space. Then $J_X : X \to J_X[X]$ is a weak to weak-$**$ continuous linear isomorphism.*

Moreover if $(X, ||\cdot||)$ is a reflexive Banach space, we have for each $M \in \mathbb{R}_{>0}$ the unit ball $B_M := \{x \in X : ||x|| \leq M\}$ to be weakly compact.

Proof. For the proof we rely on net convergence. Let $(x_\alpha)_{\alpha\in A} \subseteq X$ be some net convergent to some $x \in X$ in the weak topology on X. Note by definition we then have $\phi(x_a) \to \phi(x)$ for each $\phi \in X'$.

Consequently, we see that for each $\phi \in X'$ we have

$$J_X(x_\alpha)(\phi) = \phi(x_\alpha) \to \phi(x) = J(x)(\phi)$$

proving weak to weak-** continuity of J_X. To prove invertibility of $J_X : X \to J_X[X]$, note that J_X is injective as it is an isometry and so every element $y \in J_X[X]$ is uniquely determined by some $x \in X$ with $y = J_X(x)$. In letting $(y_a)_{a\in A}$ be some net in X'' weak-** convergent to some $y \in X''$ we then have unique $(x_a)_{a\in A} \subseteq X$ and $x \in X$ such that $J_X^{-1}(y_a) = x$ and $J_X^{-1}(y) = x$. We can then show continuity of J_X^{-1} in seeing that for any $\phi \in V$ we have

$$\phi(J_X^{-1}(y_a)) = \phi(x_\alpha) = J(x_a)(\phi) \to J(x)(\phi) = \phi(J_X^{-1}(y)),$$

so $J_X^{-1}(y_a) \to J_X^{-1}(y)$ weakly which implies $(J_X)^{-1} : J_X[X] \to X$ is indeed weak-** to weak continuous. As $J_X : X \to J_X[X]$ is now a continuous linear mapping with continuous inverse we can claim it is a continuous linear isomorphism.

Assuming $(X, ||\cdot||)$ is a reflexive Banach space, we then see $J_X : X \to X''$ is surjective as well, so $J_X : X \to X''$ is a weak to weak-** isomorphism and for each $M \in R_{>0}$ we then have

$$J_X[B_M] = \{x'' \in X'' : ||x''||_{X''} \leq M\},$$

as J_X is an isometry. In knowing that by the Banach-Alaoglu Theorem A.3.2 we have B_M to be weak-** compact, we then may conclude B_M is weakly compact. □

Corollary 4.1.6. *Let $(X, ||\cdot||)$ be a reflexive Banach space over the field $\mathbb{R}$ and let $(x_n)_{n\geq 0}$ be some sequence such that $\sup_{n\geq 0} ||x_n|| \leq M$ for some $M \in \mathbb{R}_{>0}$. Then there exists a $x \in X$ such that $x_n \rightsquigarrow x$ weakly.*

Proof. Immediate in combining Theorem A.3.3 and Proposition 4.1.5 on the ball $B_M := \{x \in X : ||x|| \leq M\}$. □

Remark 4.1.7.

1. Should it be the case that in Corollary 4.1.6 we would have $||x_n|| = M$ for all $n \in \mathbb{Z}_{\geq 1}$, one might be tempted to think the accumulation point $x \in X$ obtained in Corollary 4.1.6 also has a norm $||x|| = M$. It is a well-known fact, however that the spheres in infinite dimensional normed vector spaces are not weak-* compact and that the accumulation point x may even be the zero vector.

2. An important class of examples for reflexive Banach spaces are when given some σ-finite measure space $(X, \mathscr{F}, \mu)$, and $p \in (1, \infty)$ the spaces $L^p(X, \mu)$, see [8, Examples III.1.8 and III.11.2)]

The following Proposition will be a main tool in finding accumulation points of sequences of the densities of the measures as mentioned at the start of this section.

Proposition 4.1.8. *Let $(X, \mathscr{F}, \mu_0)$ be a measure space with $\mu_0(X) \in (0, \infty)$ and suppose we have a sequence of finite (positive) measures $(\mu_n)_{n\geq 1}$ for which there exists $M \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{Z}_{\geq 1}$ we have $\mu_n \ll \mu_0$ and*

$$\left\| \frac{d\mu_n}{d\mu_0} \right\|_\infty \leq M.$$

Then there exists a finite positive measure $\mu \in \mathscr{M}(X)$ such that $\mu_n \rightsquigarrow \mu$ set-wise. Furthermore, we have $\mu \ll \mu_0$ and $\left\| \frac{d\mu}{d\mu_0} \right\|_\infty \leq M$.

Proof. Let $(X, \mathscr{F}, \mu_0)$ be a finite measure space and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Firstly, note that for any $g \in L^\infty(X, \mu_0)$ we have

$$||g||_q \leq ||g||_\infty \mu_0(X)^{\frac{1}{q}}, \tag{83}$$

so $g \in L^q(X,\mu_0)$ and in particular all indicator functions are contained in $L^q(X,\mu_0)$. Using Hölders inequality we can then see for any $f \in L^p(X,\mu_0)$ that

$$||f||_1 = ||f \cdot \mathbb{1}_X||_1 \leq ||f||_p ||\mathbb{1}_X||_q = ||f||_p \cdot \mu_0(X)^{\frac{1}{q}} < \infty \tag{84}$$

and so $L^p(X,\mu_0) \subseteq L^1(X,\mu_0)$ holds. Now let $(\mu_n)_{n\geq 0}$ be a sequence of finite measures with for each $n \in \mathbb{Z}_{\geq 1}$:

$$\mu_0 \ll \mu_n \quad \text{and} \quad \sup_{n\geq 1} \left|\left| \frac{d\mu_n}{d\mu_0} \right|\right|_\infty \leq M \text{ for some } M > 0.$$

Note then, as seen by Equation (83) that we have

$$\left\{ \frac{d\mu_n}{d\mu_0} : n \in \mathbb{Z}_{\geq 1} \right\} \subseteq \left\{ g \in L^p(X,\mu_0) : ||g||_p \leq M\mu_0(X)^{\frac{1}{p}} \right\},$$

so that we can invoke Corollary 4.1.6 to claim there exists a $g \in L^p(X,\mu_0)$ such that $\frac{d\mu_n}{d\mu_0} \rightsquigarrow g$, by reflexivity of $L^p(X,\mu_0)$. As follows from Equation (84) we then have $g \in L^1(X,\mu_0)$ as well and may define the finite measure $\mu(\cdot) = \int_\cdot g \, d\mu_0$.

Note for the $q \in (1,\infty)$ as before we have $L^\infty(X,\mu_0) \subseteq L^q(X,\mu_0)$ so any $A \in \mathscr{F}$ satisfies

$$\mu_n(A) = \int_X \frac{d\mu_n}{d\mu_0} \cdot \mathbb{1}_A d\mu_0 \rightsquigarrow \int_X g \cdot \mathbb{1}_A \, d\mu_0 = \mu(A),$$

showing set-wise convergence and that $\mu(A) \geq 0$ for all $A \in \mathscr{F}$.

Lastly, note that for all $A \in \mathscr{F}, n \geq 1$ we have

$$\mu_n(A) = \int_X \frac{d\mu_n}{d\mu_0}(x) \mathbb{1}_A(x) \, d\mu_0(x) \leq M\mu_0(A),$$

and as $\mu_n \rightsquigarrow \mu$ set-wise we have a sequence $(n_k)_{k\geq 0} \subseteq \mathbb{Z}_{\geq 1}$ such that then for any $A \in \mathscr{F}$ we have

$$\mu(A) = \lim_{k\to\infty} \mu_{n_k}(A) \leq M\mu_0(A)$$

which in turn implies $\mu \ll \mu_0$ and $\left|\left| \frac{d\mu}{d\mu_0} \right|\right|_\infty \leq M$. □

The above proof will be sufficient in proving the existence of an *acip*, but considering densities as objects of L^p spaces for $p \in (1,\infty)$ may seem a bit artificial as densities of finite measures can be just L^1. The issue with expanding this argument to L^1 is, however, that for measure spaces $(X,\mathscr{F},\mu)$ whenever a space $L^1(X,\mu)$ is infinite dimensional we do not have reflexivity. Instead we have $(L^1(X,\mu))' \cong L^\infty(X,\mu)$, but $L^1(X,\mu) \subsetneq (L^\infty(X,\mu))'$. That is, J_X is not surjective as can be seen in [8, Section V.4]. Assuming μ is a finite measure, we can still obtain a beautiful characterisation of the weak closure of subsets of $L^1(X,\mu)$ however.

Definition 4.1.9. [6, Definition 4.5.1] Suppose $(X, \mathscr{F}, \mu)$ is a finite measure space. We call a collection $\mathscr{A} \subseteq L^1(X,\mu)$ *uniformly integrable* if

$$\lim_{C\to\infty} \sup_{f\in\mathscr{A}} \int_{\{|f|>C\}} |f| \, d\mu = 0.$$

The *Dunford-Pettis* Theorem below now gives us the topological characterisation we are looking for. We point out once more that the dual of $L^1(X,\mu)$ is $L^\infty(X,\mu)$.

Theorem 4.1.10. *[6, Theorem 4.7.18] Suppose we have a finite measure space $(X, \mathscr{F}, \mu)$ and let $\mathscr{A} \subseteq L^1(X,\mu)$. Then $\mathscr{A}$ is uniformly integrable if and only if it has compact closure in the weak topology of $L^1(X,\mu)$.*

This way, we obtain an alternative proof of Proposition 4.1.8. Note that we make a much stronger assumption than strictly necessary by assuming the densities are uniformly bounded in $||\cdot||_\infty$ instead of just uniformly integrable.

Alternative proof of 4.1.8. First we show uniform integrability of $\left(\frac{d\mu_n}{d\mu_0}\right)_{n\geq 1}$. Note that we have by assumption

$$\sup_{n\geq 1} \left|\left|\frac{d\mu_n}{d\mu_0}\right|\right|_\infty \leq M,$$

so that $\sup_{n\geq 1}\left|\left|\frac{d\mu_n}{d\mu_0}\right|\right|_1 \leq M\mu_0(X)$ and, in particular, $\frac{d\mu_n}{d\mu_0} \in L^1(X,\mu_0)$ for $n \in \mathbb{Z}_{\geq 1}$. Now we see for any $C > M$ that

$$\sup_{n\geq 1} \int_{\left\{|\frac{d\mu_n}{d\mu_0}|>C\right\}} \left|\frac{d\mu_n}{d\mu_0}\right| d\mu_0 = 0.$$

So the set $\left\{\frac{d\mu_n}{d\mu_0} : n \geq 1\right\}$ is uniformly integrable.

Combining Theorem 4.1.10 with Theorem A.3.3 we then obtain an accumulation point $f \in L^1(X,\mu_0)$ for $\left\{\frac{d\mu_n}{d\mu_0} : n \geq 1\right\}$ in the weak topology on $L^1(X,\mu_0)$. In defining $\mu(\cdot) = \int_\cdot f \, d\mu_0$ we may claim $f = \frac{d\mu}{d\mu_0}$ by (almost everywhere) uniqueness of the Radon-Nikodym derivative, and then have for any $A \in \mathscr{F}$,

$$\mu_n(A) = \int_X \frac{d\mu_n}{d\mu_0} \cdot \mathbb{1}_A d\mu_0 \rightsquigarrow \int_X g \cdot \mathbb{1}_A \, d\mu_0 = \mu(A),$$

showing set-wise convergence as requested. Moreover μ is a positive measure.

The bound $\frac{d\mu}{d\mu_0} \leq M$ is derived perfectly analogously to the last paragraph of Proposition 4.1.8. □

4.1.1 An alternative proof for Proposition 3.3.2

In this section we shall give an alternative proof to Proposition 3.3.2. While the proof of Proposition 3.3.2 is valid, it relies on conditions specific to that of a tower base and it hides some of the more general structure. The alternative proof will also give us more insight into the relationship between weak-* convergence and L^1-convergence and the topological nature of the tower base.

Before starting rigorously, we shall briefly go over the method. Fix some tower base $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ with a metric $d_{\beta,C}$ as in Definition 2.1.29. Firstly, it is clear by Lemma 3.3.6 that for each $n \in \mathbb{Z}_{\geq 0}$ we have $\left\| \frac{d(g^{R^n})_\star \mu_\Lambda}{d\mu_\Lambda} \right\|_\infty \leq M$ for some $M \in \mathbb{R}_{>1}$, independent of n. By Proposition 4.1.8 we then see that $\left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{d(g^{R^k})_\star \mu_\Lambda}{d\mu_\Lambda} \right)_{n \geq 0}$ has got an accumulation point μ so that $\frac{1}{n} \sum_{k=0}^{n-1} \frac{d(g^{R^k})_\star \mu_\Lambda}{d\mu_\Lambda} \rightsquigarrow \mu$ set-wise. To get some intuition on how to strengthen this convergence to obtain L^1-convergence we include an example where this can not be done.

Example 4.1.11. We define for $n \in \mathbb{Z}_{\geq 1}$,

$$
\begin{aligned}
f_n : (0,1) &\to (0,1) \\
x &\mapsto \frac{1}{2} \sin(2n\pi x).
\end{aligned}
$$

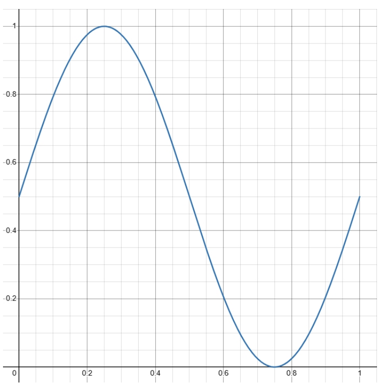


Figure 2: Graph of f_1

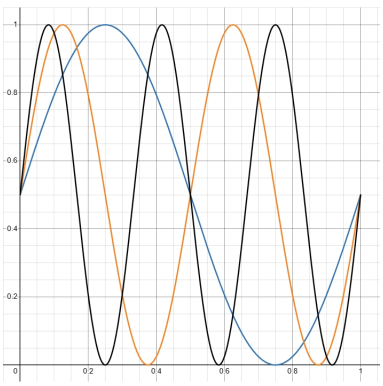


Figure 3: Graph of f_1, f_2, f_3

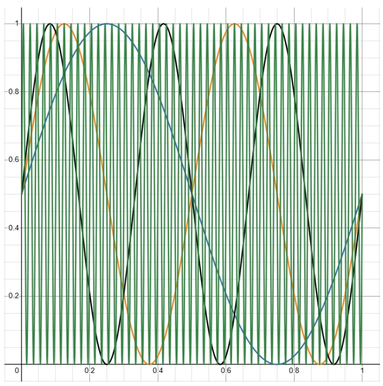


Figure 4: Graph of f_1, f_2, f_3, f_{50}

We can interpret curves $(f_n)_{n \geq 1}$ as densities $\frac{df_n}{d\lambda}$ of measures taken with respect to the Lebesgue measure λ on $(0,1)$. We then know we have an accumulation point μ such that $f_n d\lambda \rightsquigarrow \mu$ set-wise. One can prove that due to the ever increasing speed of oscillations the areas below and above the line $y = \frac{1}{2}$ will even out when integrated over, so that $\mu(\cdot) = \int_\cdot \frac{1}{2} \mathbb{1}_{(0,1)}(x)\, d\lambda(x)$. One can show however that $\int_0^1 \left| f_n - \frac{1}{2} \mathbb{1}_{(0,1)} \right| d\lambda$

will be bounded away from zero uniformly in $n \in \mathbb{Z}_{\geq 0}$, so L^1-convergence can not happen.

The question then remains, is oscillatory behaviour the only way in which weak-* convergence can not lead to L^1-convergence. We shall answer this question through the following concepts, as seen in [6, Page of Theorem 4.7.29.].

Definition 4.1.12. Let $(X,\mathscr{F},\mu)$ be a finite measure space, let $f \in L^1(\mu)$ and let $A \in \mathscr{F}$ be such that $\mu(A) > 0$. The quantity

$$\overline{\mathrm{osc}}(f|_A) := \mu(A)^{-1}\int_A \left| f(x) - \mu(A)^{-1}\int_A f(y)\,d\mu(y)\right| d\mu(x),$$

is called the *average oscillation* of the function f on A.

Now we present the main tool of this section (see [6, Theorem 4.7.29.] for a proof).

Theorem 4.1.13. *Suppose we have a finite measure space $(X,\mathscr{F},\mu)$ with $\mu(X) \in (0,\infty)$ and suppose that a set $F \subseteq L^1(\mu)$ has compact closure in the weak topology. Then, the closure of F is compact in the norm of $L^1(\mu)$ precisely when F satisfies the following condition:*

For every $\epsilon \in \mathbb{R}_{>0}$ and every set $A \in \mathscr{F}$ of positive μ-measure there exists a finite collection of sets $A_1,\dots,A_n \subseteq A$ of positive measure such that every function $f \in F$ has the average oscillation less than ϵ on at least one of the sets A_j, $j \in \{1,\dots,n\}$.

Returning to the setting of a tower base, we also know as a consequence of Lemma 3.3.6 that the convex combinations $\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{d(g^{R^k})_\star\mu_\Lambda}{d\mu_\Lambda}\right)_{n\geq 1}$ are Lipschitz on a set of full measure *with a uniform bound on the Lipschitz constant.* It is then likely we can use Theorem 4.1.13 to prove L^1-convergence. To do so, we do need to be able to relate the measure of measurable sets to distances. A common class of topological spaces where this is possible are Polish Spaces.

Definition 4.1.14. A Polish space $(X,\mathscr{T})$ is a separable topological space for which there exists a metric that is complete.

A strength of Polish spaces is that any finite Borel measure μ on a Borel space $(X,\mathscr{F})$ is *outer regular*, that is for any $A \in \mathscr{F}$ we have

$$\mu(A) = \inf\{\mu(O) : O \supseteq A, \text{with } O \text{ open}\},$$

see [11]. We shall now prove that any dynamical system with a generating and separating partition can be equipped with a natural topology making it Polish. To do so, we recap the separation time from Definition 2.1.27.

Definition 4.1.15. Let $(X, \mathscr{F}, \mu, T)$ be some dynamical system with $\mu(X) \in (0, \infty)$ and $\mathscr{P} \subseteq \mathscr{F}$ a countable finite partition of X. We define the mapping

$$\alpha : X \to \mathscr{P}, \qquad x \mapsto P \quad \text{for the unique } P \ni x,$$

and the *separation time* $s : X \times X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as the mapping

$$s(x, x') = \inf\left\{n \in \mathbb{Z}_{\geq 0} : \alpha\left(T^n(x)\right) \neq \alpha\left(T^n(x')\right)\right\}.$$

Recall that if for some dynamical system $(X, \mathscr{F}, \mu, T)$ with $\mu(X) \in (0, \infty)$ we have a countable generating and separating partition $\mathscr{P} \subseteq \mathscr{F}$ then by Lemma 2.1.29, we know that for any $\beta \in (0,1), C \in \mathbb{R}_{>1}$ that

$$d_{\beta,C}(x, y) := C\beta^{s(x,y)}$$

is a metric on X and we denote the topology it induces by $\mathscr{T}$. We shall characterise the open balls in this topology. We use the notation

$$B_x(\epsilon) := \{y \in X : d_{\beta,C}(x, y) < \epsilon\},$$

for an open ball with radius $\epsilon \in \mathbb{R}_{>0}$ around a point $x \in X$.

Lemma 4.1.16. *Let $(X, \mathscr{F}, \mu, T)$ be a dynamical system and let $\mathscr{P}$ be a generating and separating partition as in Definition 2.1.22. And let $\epsilon \in \mathbb{R}_{>0}$, $C \in \mathbb{R}_{>1}$, $\beta \in (0,1)$ and $x \in X$. If $\epsilon \leq C$ there exists an $n \in \mathbb{Z}_{\geq 1}$ and an $A \in \mathscr{P}^n$ such that*

$$x \in A \quad \text{and } A = \{y \in X : d_{\beta,C}(x, y) < \epsilon\}. \tag{85}$$

If $\epsilon > C$ then we have

$$X = \{y \in X : d_{\beta,C}(x, y) < \epsilon\}. \tag{86}$$

Proof. First note the sequence $(C\beta^n)_{n\geq 0}$ is strictly decreasing to zero and has a maximum C for $n = 0$.

Now suppose $\epsilon \leq C$ and let $n \in \mathbb{Z}_{\geq 1}$ be such that $\epsilon \in (C\beta^n, C\beta^{n-1}]$. Let $\zeta \in (0, C\beta^{n-1} - C\beta^n]$ be so that $\epsilon = C\beta^n + \zeta$. Then let $A \in \mathscr{P}^n$ be such that $x \in A$. Note

$$\begin{aligned} B_x(\epsilon) &:= \{y \in X : d_{\beta,C}(x, y) < \epsilon\} \\ &= \{y \in X : C\beta^{s(x,y)} < C\beta^n + \zeta\} \\ &= \{y \in X : \beta^{s(x,y)} < \beta^{n-1+\log_\beta(\beta+\beta^{-n+1}\zeta/C)}\} \\ &= \{y \in X : s(x, y) > n - 1 + \log_\beta(\beta + \beta^{-n+1}\zeta/C)\}, \end{aligned} \tag{87}$$

where in Equation (87) we used the fact that $\log_\beta$ is strictly decreasing. To show

$$\{y \in X : s(x, y) > n - 1 + \log_\beta(\beta + \beta^{-n+1}\zeta/C)\} = \{y \in X : s(x, y) > n - 1\}, \tag{88}$$

note that $\beta + \beta^{-n+1}\zeta/C \in (\beta, 1]$ so that $\log_\beta(\beta + \beta^{-n+1}\zeta/C) \in [0,1)$ implying

$$\{y \in X : s(x,y) > n-1+\log_\beta(\beta+\beta^{-n+1}\zeta/C)\} = \{y \in X : s(x,y) > n-1\},$$

as s takes values in the (extended) non-negative integers.

Now note that for $y \in X$ we have that

$$s(x,y) > n-1 \text{ if and only if } \alpha(T^k(x)) = \alpha(T^k(y)) \text{ for each } k \in \{0,\dots,n-1\}.$$

As we have $A = A_0 \cap \cdots \cap T^{-n+1}A_{n-1}$ for some $A_0,\dots,A_{n-1} \in \mathscr{P}$ we can see having $\alpha(T^k(x)) = \alpha(T^k(y))$ for each $k \in \{0,\dots,n-1\}$ is equivalent with having $A_k = \alpha(T^k(y))$ for each $k \in \{0,\dots,n-1\}$, which is equivalent with $y \in A$. We conclude $B_x(\epsilon) = \{y \in X : s(x,y) > n-1\} = A$, proving Equation (89).

Now in supposing $\epsilon > C$, we simply note that $d_{\beta,C}$ takes values in $[0,C]$ so that $X = \{y \in X : d_{\beta,C}(x,y) < \epsilon\}$ holds for each $x \in X$. □

From the proof of Lemma 4.1.16 we obtain the following Corollary if we consider the case $\epsilon = C\beta^{n-1}$ for $n \in \mathbb{Z}_{\geq 1}$.

Corollary 4.1.17. *Let $(X,\mathscr{F},\mu,T)$ be a dynamical system and let $\mathscr{P}$ be a generating and separating partition as in Definition 2.1.22. And let $n \in \mathbb{Z}_{\geq 1}$, $C \in \mathbb{R}_{>1}$, $\beta \in (0,1)$ and $x \in X$. Then for $\epsilon = C\beta^{n-1}$ there exists an $A \in \mathscr{P}^n$ such that we have*

$$x \in A \quad \textit{and } A = \{y \in X : d_{\beta,C}(x,y) < \epsilon\}. \tag{89}$$

We now apply Lemma 4.1.16 to prove we can equip dynamical systems with a generating and separating partition with a Polish topology.

Proposition 4.1.18. *Let $(X,\mathscr{F},\mu,T)$ be a dynamical system and let $\mathscr{P}$ be a generating and separating partition as in Definition 2.1.22. Then for any $\beta \in (0,1)$ and $C > 1$, $(X, d_{\beta,C})$ is a complete separable metric space. Moreover, $\mathscr{F}$ is a Borel σ-algebra for this topology and μ is outer regular.*

Proof. Let $\beta \in (0,1), C \in \mathbb{R}_{>1}$ be arbitrary. We show $(X, d_{\beta,C})$ is a complete, separable metric space. To do so, let $(x_n)_{n\geq 0}$ be a Cauchy sequence in X. We shall use the separating property of $\mathscr{P}$ to show $(x_n)_{n\geq 0}$ converges.

As $(x_n)_{n\geq 0}$ is Cauchy, we have for all $\epsilon \in \mathbb{R}_{>0}$ an $N \in \mathbb{Z}_{\geq 0}$ such that for all $n, m \geq N(l)$ we have $d_{\beta,C}(x_n,x_m) < \epsilon$. Now, in letting $(\epsilon_l)_{l\in\mathbb{Z}_{\geq 1}}$ be such that $\epsilon_l = C\beta^{l-1}$ for $l \in \mathbb{Z}_{\geq 1}$, we see that we have for each $l \in \mathbb{Z}_{\geq 1}$ an $N(l) \in \mathbb{Z}_{\geq 0}$ such that for all $n, m \geq N$ we have $d_{\beta,C}(x_n,x_m) < \epsilon_l$. In particular, by Corollary 4.1.17 we have for each $l \in \mathbb{Z}_{\geq 1}$ and $x_{N(l)} \in X$ an $A_l \in \mathscr{P}^l$ such that

$$A_l = \{y \in X : d_{\beta,C}(x_{N(l)}, y) < \epsilon_l\} \tag{90}$$

and $(x_n)_{n\geq N(l)} \in A_l$ for each $n \geq N(l)$. We now show $A_l \supseteq A_{l+1}$ for any $l \in \mathbb{Z}_{\geq 1}$, with A_l and A_{l+1} as constructed in Equation (90).

To do so, let $l \in \mathbb{Z}_{\geq 1}$ and note we have $d_{\beta,C}(x_{N(l)}, x_{N(l+1)}) < \epsilon_l$ so that $x_{N(l+1)} \in A_l$. As $\mathscr{P}^l$ is a partition for X, we then know this A_l satisfies $x_{N(l+1)} \in A_l$ uniquely for $\mathscr{P}^l$, so then, again by Corollary 4.1.17, we know that

$$A_l = \{y \in X : d_{\beta,C}(x_{N(l+1)}, y) < \epsilon_l\}.$$

As we have

$$A_{l+1} = \{y \in X : d_{\beta,C}(x_{N(l+1)}, y) < \epsilon_{l+1}\},$$

we then see as $\epsilon_l > \epsilon_{l+1}$ that $A_l \supseteq A_{l+1}$. As this holds for general $l \in \mathbb{Z}_{\geq 1}$ and as $A_l \in \mathscr{P}^l$ we can see $\bigcap_{l=0}^{\infty} A_l \in \mathscr{P}^{\infty}$. As $\mathscr{P}$ is separating, we then know $\bigcap_{l=0}^{\infty} A_l = \{x\}$ for some $x \in X$. The limit $\lim_{n\to\infty} x_n = x$ follows directly. As $(x_n)_{n\geq 0}$ was a given arbitrarily, we can conclude every Cauchy sequence converges in this metric space. Hence the space is complete.

As for separability note that by Lemma (4.1.16), the collection $\bigcup_{l=0}^{\infty} \mathscr{P}^l$ consists of the open balls in $\mathscr{T}$, so that $\mathscr{T}$ is clearly second countable. As metric spaces are separable if and only if they are second countable, the separability of $\mathscr{T}$ follows.

Having shown $d_{\beta,C}$ endows X with a topology that is separable and completely metrisable (as $d_{\beta,C}$ is complete), we conclude X is a Polish space.

Finally, by definition the collection $\bigcup_{l=0}^{\infty} \mathscr{P}^l$ generates $\mathscr{F}$, that is, $\mathscr{F}$ is generated by the open balls in $\mathscr{T}$ and as such is the Borel σ-algebra. As any finite signed Borel measure on a Polish space is outer regular, μ is outer regular. □

We now provide two auxillary results to apply Theorem 4.1.13.

Lemma 4.1.19. *Let (X, d) be a separable metric space with topology $\mathscr{T}$. Then for every $\delta \in \mathbb{R}_{>0}$ there exists a countable collection of open balls $B_\delta \subseteq X$ of radius δ covering X.*

Proof. Let $\delta \in \mathbb{R}_{>0}$ be given, and let $(x_n)_{n\geq 0}$ be a dense subset of X. We claim $B_\delta(x_n) \subseteq \mathscr{T}$ is an open cover for X. Note that for any $y \in X$ we have $x_n \in B_\delta(y)$ for some $n \in \mathbb{Z}_{\geq 0}$ as $(x_n)_{n\geq 0}$ is a dense in X. Consequently, we see $y \in B_\delta(x_n)$ proving our claim. □

Remark 4.1.20. The concept of an equicontinuous collection of continuous functions can be interpreted as 'continuous in the same way', meaning that rather than imposing a stronger notion of continuity (such as Lipschitz continuity) it is about the relation between the continuous functions themselves. Under the euclidian metric on $(0,1)$, functions as $x \to x^\alpha$ for $\alpha \in (-1,1)$, and $x \mapsto \sqrt{x}$ for $x \in (0,1)$ can hence be part of equicontinuous collections but neither of them are Lipschitz.

Lemma 4.1.21. *Let X be some Polish space and let $(X, \mathscr{F}, \mu)$ be a finite measure space with $\mathscr{F}$ the Borel σ-algebra. Then for any set $A \in \mathscr{F}$ with $\mu(A) > 0$ and any $\delta \in \mathbb{R}_{>0}$ there exists an open ball $B_\delta \subseteq \mathscr{F}$ such that $\mu(B_\delta \cap A) > 0$.*

Proof. Suppose we have $A \in \mathscr{F}$ with $\mu(A) > 0$ and suppose there exists a $\delta > 0$ such that $\mu(A \cap B_\delta) = 0$ for every open ball $B_\delta \subseteq X$.

Then, using Lemma 4.1.19, there is a countable cover $(B_{\delta,n})_{n\geq 0}$ of X consisting of open balls of radius δ and

$$0 = \sum_{n\geq 0} \mu(B_{\delta,n} \cap A) \geq \mu\left(A \cap \bigcup_{n\geq 0} B_{\delta,n}\right) = \mu(A) > 0,$$

which leads to a contradiction meaning that $\mu(A \cap B_\delta) > 0$ for at least one open ball of radius $\delta > 0$. □

To make our convergence result as generally applicable as possible, we use the concept of equicontinuity. Readers acquainted with Arzela-Ascoli's Theorem (see for example [8, Theorem IV.3.8]), a Theorem ubiquitous throughout functional analysis, will be familiar with it. Note, as opposed to Arzela-Ascoli's Theorem, we do not require compactness of our topological space.

Definition 4.1.22. Let X be some Polish space and let $(X, \mathscr{F}, \mu)$ be its standard Borel space. We say a collection $F \subseteq L^0(X)$ is *equicontinuous on a set of full measure* if there exists a complete separable metric d on X, an $X_e \in \mathscr{F}$, $\mu(X \setminus X_e) = 0$ such that for each $\epsilon \in \mathbb{R}_{>0}$ and $x \in X_e$ there exists a $\delta \in \mathbb{R}_{>0}$ such that for each $y \in X_e$, with $d(x, y) < \delta$ we have for each $f \in F$, $|f(x) - f(y)| < \epsilon$.

We quickly prove that every countable set of functions that are Lipschitz on a set of full measure with the same Lipschitz constant, is equicontinuous on a set of full measure.

Lemma 4.1.23. *Let X be some Polish space, let $(X, \mathscr{F}, \mu)$ be its standard Borel space and let d be some complete separable metric on X. Fix $L \in \mathbb{R}_{>1}$. Any countable set $C \subseteq L^1(X)$ consisting of Lipschitz functions on a set of full measure with Lipschitz constant at most L is equicontinuous on a set of full measure.*

Proof. For every $f \in C$, write $X_f \subseteq X$ for a set $\mu(X \setminus X_f) = 0$ upon which we have $|f(x) - f(y)| < Ld(x, y)$ for every $x, y \in X_f$. Then, note $\dot{X} = \bigcap_{f\in C} X_f$ is a set with $\mu(X \setminus \dot{X}) = 0$ for which for each $f \in C$ we have

$$|f(x) - f(y)| < Ld(x, y) \text{ for every } x, y \in \dot{X}.$$

Then note for arbitrary $\epsilon \in \mathbb{R}_{>0}$ we have for $\delta = \epsilon/L$ that for each $x, y \in \dot{X}$, $d(x, y) < \delta$ that

$$|f(x) - f(y)| < Ld(x, y) < \epsilon,$$

proving our claim. □

Finally we arrive at our 'measure-theoretical version' of the Arzela-Ascoli Theorem, not requiring compactness.

Theorem 4.1.24. *Let X be a Polish space and let $(X, \mathscr{F}, \mu)$ be some standard Borel space, with $\mu(X) \in (0,\infty)$. Now let $\{\phi_n\}_{n\geq 0} \subseteq L^1(\mu)$ be a uniformly integrable, equicontinuous on a set of full measure, sequence. Then there exists a subsequence $(\phi_{n_k})_{k\geq 0}$ and a $\phi \in L^1(\mu)$ such that $\lim_{k\to\infty} \phi_{n_k} = \phi$ in L^1 and μ-almost everywhere.*

Proof. By Theorem 4.1.10, we know that $(\phi_n)_{n\geq 0}$ has compact closure in the weak topology of $L^1(X,\mu)$. Now let $\epsilon \in \mathbb{R}_{>0}$ and let $A \in \mathscr{F}$ with $\mu(A) > 0$. Now fix a metric d for X, inducing the topology on X. As $(\phi_n)_{n\geq 0}$ is equicontinuous on a set of full measure we have a set $\dot{X} \in \mathscr{F}$ with $\mu(\dot{X}) = 1$ and a $\delta_\epsilon \in \mathbb{R}_{>0}$ such that

$$\text{for each } x, y \in \dot{X} \text{ with } d(x,y) < \delta_\epsilon \text{ we have } |\phi_n(x) - \phi_n(y)| < \epsilon.$$

Next, as X is Polish and $\mu(A) > 0$ we can by Lemma 4.1.21 find an open ball $B_\delta \subseteq X$ such that $\mu(A \cap B_\delta) > 0$. In writing $A_\delta = A \cap B_\delta \cap \dot{X}$, we can directly verify that the probability measure $\mathbb{P}_{A_\delta} = \frac{\mu_{A_\delta}}{\mu(A_\delta)}$ satisfies $\mathbb{P}_{A_\delta} \ll \mu_{A_\delta}$ and $\frac{d\mathbb{P}_{A_\delta}}{d\mu_{A_\delta}} \equiv \frac{1}{\mu_{A_\delta}(A_\delta)}$, μ_{A_δ}-almost surely. Having verified this, we see that for each $n \in \mathbb{Z}_{\geq 0}$ we obtain

$$\begin{aligned}
\overline{\text{osc}}(\phi_n|_{A_\delta}) &= \mu(A_\delta)^{-1} \int_{A_\delta} \left| \phi_n(x) - \mu(A_\delta)^{-1} \int_{A_\delta} \phi_n(y)\, d\mu(y) \right| d\mu(x) \\
&= \mu(A_\delta)^{-1} \int_{A_\delta} \left| \phi_n(x) - \mu(A_\delta)^{-1} \int_{A_\delta} \phi_n(y)\, d\mu_{A_\delta}(y) \right| d\mu(x) \\
&= \mu(A_\delta)^{-1} \int_{A_\delta} \left| \phi_n(x) - \int_{A_\delta} \phi_n(y)\, d\mathbb{P}_{A_\delta}(y) \right| d\mu(x) \\
&\leq \mu(A_\delta)^{-1} \int_{A_\delta} \int_{A_\delta} \left| \phi_n(x) - \phi_n(y) \right| d\mathbb{P}_{A_\delta}(y)\, d\mu(x) \qquad (91) \\
&< \mu(A_\delta)^{-1} \int_{A_\delta} \int_{A_\delta} \epsilon\, d\mathbb{P}_{A_\delta}(y)\, d\mu(x) \\
&= \mu(A_\delta)^{-1} \cdot \epsilon \cdot \mu(A_\delta) \cdot \mathbb{P}_{A_\delta}(A_\delta) \\
&= \epsilon.
\end{aligned}$$

In Equation (91) we used Jensens Inequality [6, Theorem 2.12.19.] using the convex function

$$\begin{aligned}
\Psi_y : \mathbb{R} &\to \mathbb{R} \\
x &\to |x - y|,
\end{aligned}$$

for any $y \in \mathbb{R}$. Using Theorem 4.1.13 we then see $(\phi_n)_{n\geq 0}$ has compact closure in $L^1(\mu)$. As $L^1(\mu)$ is a metric space, we then obtain a subsequence $(\phi_{n_l})_{l\geq 0}$ converging to some $\phi \in L^1(\mu)$ as $l \to \infty$ in L^1-norm. As this implies that $\phi_{n_l} \to \phi$ in measure (see e.g. [6, Definition 2.2.3]), we can use [6, Theorem 2.2.5] to obtain a subsequence $(\phi_{n_{l_k}})_{k\geq 0}$ of $(\phi_{n_l})_{l\geq 0}$ which converges to ϕ pointwise almost everywhere and in $L^1(\mu)$. We have proven our claim. □

Finally we rephrase Proposition 3.3.2.

Proposition 4.1.25. *Suppose we have a tower base* $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$, *and a sequence* $(\phi_n)_{n\in\mathbb{Z}_{\geq 1}} \subseteq L_{\beta,C}(\Lambda)$ *satisfying for some* $M > 1$,

$$\sup_{n\in\mathbb{Z}_{\geq 1}} ||\phi_n||_\beta \leq M \quad \text{and} \quad \inf_{n\in\mathbb{Z}_{\geq 1}} \operatorname{ess\,inf}_{x\in\Lambda} \phi_n(x) \geq \frac{1}{M}.$$

Then $(\phi_n)_{n\geq 1}$ *converges pointwise almost everywhere and in* $L^1(\Lambda)$ *to a function* $\phi \in L_{\beta,C}(\Lambda)$ *with* $||\phi||_\beta \leq M$ *and* $\operatorname{ess\,inf}_{x\in\Lambda} \phi \geq \frac{1}{M}$.

Proof. (Sketch) By Lemma 4.1.18 we see that for $(\Lambda, \mathscr{F}_\Lambda, \mu_\Lambda, g^R)$ there exists a natural topology $\mathscr{T} \subseteq \mathscr{F}_\Lambda$ such that $(X, \mathscr{T})$ is Polish space, $\mathscr{F}_\Lambda$ is the Borel σ-algebra, and μ_Λ is a Borel measure.

By Lemma 3.3.6 we can then as in the (alternative) proof of Proposition 4.1.8 at the end of Section 4.1 obtain uniform integrability of $(\phi_n)_{n\in\mathbb{Z}_{\geq 1}}$ with respect to μ_Λ. Then, as $(\phi_n)_{n\in\mathbb{Z}_{\geq 1}}$ are Lipschitz on a set of full measure with the same Lipschitz constant, we can apply Theorem 4.1.24 to obtain a $\phi \in L^1$ and a subsequence $(\phi_{n_k})_{k\geq 0}$ such that $(\phi_{n_k})_{k\geq 0} \to \phi$ in L^1 and almost everywhere as $k \to \infty$. Having obtained L^1-convergence and convergence a.e., the other claims in the statement can then be verified as in the proof of Proposition 3.3.2. □

4.2 Random Dynamical Systems

In this section we define a notion of a random dynamical system (RDS) and prove all measure-theoretical properties to fit our *Random Young Towers* framework later. The author has put special care into defining the RDS in such a way that its conditions suffice for our application, are easy to check and apply as general as reasonable.

The proofs in this section are rather dry and technical, but have been incorporated for sake of completeness and for a lack of convenient overview elsewhere. Conceptually it is important to note that

1. bi-measurability of the *random dynamic* plays a central role in the construction of RDS's; and

2. joint integrability and measurability are stronger than integrability and measurability over the sections, but we can use sections to derive statements in the joint setting.

We start by constructing a *Random Dynamical System* in Definition 4.2.3 and to do so we verify basic properties on measurability and non-singularity in Lemmas 4.2.1 and 4.2.2. As mentioned before, for functions $f : X \times Y \to Z$ and elements $x \in X$

$$f_x : Y \to Z : \quad y \mapsto f(x, y).$$

Lemma 4.2.1. *Let $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ be a dynamical system with $\sigma : \Omega \to \Omega$ bi-measurable and $\mathbb{P}(\Omega) = 1$ and let $(X, \mathscr{F}_X, \mu)$ be a σ-finite measure space. Suppose we have a set $\Delta \in \mathscr{F}_{\Omega \times X}$ such that $0 < (\mathbb{P} \times \mu)(\Delta) < \infty$ and a measurable function $f : \Delta \to X$ so that for each $\omega \in \Omega$ we have $f_\omega[\Delta_\omega] \subseteq \Delta_{\sigma\omega}$, then for every $\omega \in \Omega$ the mapping*

$$f_\omega : \Delta_\omega \to \Delta_{\sigma\omega}, \qquad x \mapsto f(\omega, x).$$

is $\mathscr{F}_{\Delta_\omega}$-$\mathscr{F}_{\Delta_{\sigma\omega}}$-measurable.

Proof. First define

$$\begin{aligned} \tilde{f} : \Omega \times X &\to X \\ (\omega, x) &\mapsto \begin{cases} f_\omega(x), & x \in \Delta_\omega \\ x, & x \notin \Delta_\omega. \end{cases} \end{aligned}$$

Then, note that for arbitrary $A \in \mathscr{F}_X$ we have $\tilde{f}^{-1}(A) = f^{-1}(A) \sqcup (\Delta^c \cap (\Omega \times A)) \in \mathscr{F}_{\Omega \times X}$, as $f^{-1}(A) \in \mathscr{F}_\Delta \subseteq \mathscr{F}_{\Omega \times X}$ and $\Delta^c \in \mathscr{F}_{\Omega \times X}$. By Proposition 2.1.7 we then have for each $\omega \in \Omega$ that $x \mapsto \tilde{f}_\omega(x)$ is $\mathscr{F}_X$-measurable. Finally, note that for $B \subseteq \mathscr{F}_{\Delta_{\sigma\omega}}$ we have

$$f_\omega^{-1}(B) = \tilde{f}_\omega^{-1}(B) \setminus (B \cap \Delta_\omega^c) \in \mathscr{F}_{\Delta_\omega},$$

from which we may conclude $\mathscr{F}_{\Delta_\omega}$-$\mathscr{F}_{\Delta_{\sigma\omega}}$ measurability of f_ω. □

We remind the reader of our notation on restricted measures as defined in Definition 2.1.6.

Lemma 4.2.2. *Let $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ be a dynamical system with $\sigma : \Omega \to \Omega$ bi-measurable and $\mathbb{P}(\Omega) = 1$ and let $(X, \mathscr{F}_X, \mu)$ be a σ-finite measure space. Suppose we have a set $\Delta \in \mathscr{F}_{\Omega \times X}$ such that $0 < (\mathbb{P} \times \mu)(\Delta) < \infty$ and a measurable function $f : \Delta \to X$ so that for each $\omega \in \Omega$ we have $f_\omega[\Delta_\omega] \subseteq \Delta_{\sigma\omega}$ and $(f_\omega)_\star \mu_{\Delta_\omega} \ll \mu_{\Delta_{\sigma\omega}}$ for almost every $\omega \in \Omega$. Define $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ as the finite measure space obtained by restricting $\Omega \times X$ to Δ and define*

$$\begin{aligned} S : \qquad \Delta &\to \Delta \\ (\omega, x) &\mapsto \big(\sigma\omega, f_\omega(x)\big). \end{aligned}$$

Then the mapping S is measurable and non-singular and hence $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$ is a dynamical system.

Proof. We start by proving $\mathscr{F}_\Delta$-measurability of S. Note we may write $S : \Delta \to \Omega \times X$ as $\Delta \subseteq \Omega \times X$. For general $O \in \mathscr{F}_\Omega$, $B \in \mathscr{F}_X$ we have

$$\begin{aligned} S^{-1}(O \times B) &= \{(\omega, x) \in \Delta : (\sigma\omega, f_\omega(x)) \in O \times B\} \\ &= (\sigma^{-1} O \times X) \cap f^{-1}[B] \\ &\in \mathscr{F}_{\Omega \times X} \end{aligned} \tag{92}$$

so $S^{-1}(\mathscr{F}_\Omega \times \mathscr{F}_X) \subseteq \mathscr{F}_{\Omega\times X}$. As we know $S^{-1}[\Omega \times X] \subseteq \Delta$ we can then see $S^{-1}(\mathscr{F}_X \times \mathscr{F}_\Omega) \subseteq \mathscr{F}_\Delta$ so that by Lemma 2.1.4 the mapping $S : \Delta \to \Omega \times X$ is $\mathscr{F}_\Delta$-$\mathscr{F}_{\Omega\times X}$ measurable. As $S[\Delta] \subseteq \Delta$ and $\mathscr{F}_\Delta \subseteq \mathscr{F}_{\Omega\times X}$, the $\mathscr{F}_\Delta$-measurability of S follows.

We now note by Lemma 2.1.7 for each $\omega \in \Omega$ the mapping $x \mapsto f(\omega, x)$ is $\mathscr{F}_{\Delta_\omega}$-$\mathscr{F}_{\Delta_{\sigma\omega}}$ measurable.

We write for fixed $\omega \in \Omega$ $x \mapsto f_\omega(x)$ for $x \mapsto f(\omega, x)$. To prove non-singularity of S note that for $A \in \mathscr{F}_{\Omega\times X}$, $\omega \in \Omega$ we have

$$\begin{aligned} (S^{-1}(A))_\omega &= \{(\dot\omega, x) \in \Delta : S(\dot\omega, x) \in A\}_\omega \\ &= \{(\dot\omega, x) \in \Delta : f_{\dot\omega}(x) \in A_{\sigma\dot\omega}\}_\omega \\ &= f_\omega^{-1}(A_{\sigma\omega}). \end{aligned} \tag{93}$$

Then note that by Fubini's Theorem 2.1.8, if $(\mathbb{P} \times \mu)(A) = 0$ we have $\mu(A_\omega) = 0$, $\mathbb{P}$-almost surely. Assuming $(\mathbb{P} \times \mu)(A) = 0$, we then see

$$\begin{aligned} S_\star(\mathbb{P} \times \mu)(A) &= (\mathbb{P} \times \mu)(S^{-1} A) \\ &= \int_\Omega \mu((S^{-1}A)_\omega)\, d\mathbb{P}(\omega) \\ &= \int_\Omega (f_\omega)_\star \mu(A_{\sigma\omega})\, d\mathbb{P}(\omega) \\ &= \int_\Omega (f_{\sigma^{-1}\omega})_\star \mu(A_\omega)\, d\mathbb{P}(\omega) && (94) \\ &= 0, && (95) \end{aligned}$$

where in Equation (94) we used Proposition 2.1.10 and the invariance of σ. As Equation (95) holds for general $A \in \mathscr{F}_{\Omega\times X}$ with $(\mathbb{P} \times \mu)(A) = 0$ we may conclude $S_\star\mu_\Delta = S_\star(\mathbb{P} \times \mu) \ll \mathbb{P} \times \mu$. As μ_Δ is the restriction of $\mathbb{P} \times \mu$ to Δ, we may conclude $S_\star\mu_\Delta \ll \mu_\Delta$. □

Definition 4.2.3. We shall call a dynamical system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$ as constructed in Lemma 4.2.2 a *random dynamical system* or *RDS* for short. The mapping $S : \Delta \to \Delta$ is called a *skew product*, the family $(\Delta_\omega, \mathscr{F}_{\Delta_\omega}, \mu_{\Delta_\omega}, f_\omega)_{\omega\in\Omega}$ the *base dynamic* and the dynamical system $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ the *random dynamic*.

Remark 4.2.4. Whenever we define an RDS $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$ and omit mentioning the spaces $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$, $(X, \mathscr{F}_X, \mu)$ or the function $f : \Delta \to X$ used to construct it as in Lemma 4.2.2, we shall assume these exist implicitly. For this f, we shall often write for $(\omega, x) \in \Delta$ and $n \geq 1$ that

$$f_\omega^n(x) = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_\omega(x),$$

which is well-defined as we assume that $f_\omega(\Delta_\omega) \subseteq \Delta_{\sigma\omega}$. Furthermore, we define $f_\omega^0(x) = \mathrm{Id}|_{\Delta_\omega}$. In similar vein, we shall often want to define ω-sections of partitions $\mathscr{P}$ of Δ. That is, given a partition $\mathscr{P} \subseteq \mathscr{F}_\Delta$, we write

$$\mathscr{P}_\omega := \{P_\omega \in \mathscr{F}_X : P_\omega \neq \emptyset, P \in \mathscr{P}\}. \tag{96}$$

In Lemma 4.2.5 we prove $\mathscr{P}_\omega$ partitions Δ_ω. A final notational comment is that for $\omega \in \Omega$ we shall often write $\mu_{\Delta_\omega} := (\mu)_{\Delta_\omega}$. That is, μ_{Δ_ω} is the restriction of the measure μ to Δ_ω.

Lemma 4.2.5. *Let $(\Omega, \mathscr{F}_\Omega)$ and $(X, \mathscr{F}_X)$ be measurable spaces and suppose for $\Delta \in \mathscr{F}_{\Omega \times X}$ we have a countable partition $\mathscr{P} \subseteq \mathscr{F}_\Delta$ for Δ. Then for each $\omega \in \Omega$ with $\Delta_\omega \neq \emptyset$ the collection $\mathscr{P}_\omega$ as defined in Equation (96) is a countable partition for Δ_ω consisting of $\mathscr{F}_{\Delta_\omega}$-measurable sets.*

Proof. Let $\omega \in \Omega$ and with $\Delta_\omega \neq \emptyset$. Note then for each $x \in \Delta_\omega$ we have $(\omega, x) \in \Delta$ so that there is a unique $P \in \mathscr{P}$ with $(\omega, x) \in P$ so that $x \in P_\omega$. We show $\mathscr{P}_\omega$ consists of disjoint sets. To do so assume that $P_\omega \cap Q_\omega \neq \emptyset$ for some $Q' \in \mathscr{P}_{\Delta_\omega}$ and note by definition of $\mathscr{P}_\omega$ we have a $Q \in \mathscr{P}$ such that $Q_\omega = Q'$. Note we have $\emptyset \neq P_\omega \cap Q_\omega = (P \cap Q)_\omega$ so that $P \cap Q \neq \emptyset$ and so $P = Q$, so that $P_\omega = Q_\omega$. Finally, as every element of $\mathscr{P}$ gives rise to at most one element of $\mathscr{P}_\omega$, we see $\mathscr{P}_\omega$ is countable as well. □

We extend the map σ of an RDS to a mapping $\sigma_\Omega : \Omega \times X \to \Omega \times X$. We will see this mapping σ_Ω many times in Section 5.

Lemma 4.2.6. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$ be an RDS and define*

$$\begin{aligned} \sigma_\Omega : \Omega \times X &\to \Omega \times X \\ (\omega, x) &\mapsto (\sigma\omega, x). \end{aligned}$$

This mapping σ_Ω is bi-measurable, $(\mathbb{P} \times \mu)$-invariant and has a $(\mathbb{P} \times \mu)$-invariant inverse.

Proof. Note that $\sigma : \Omega \to \Omega$ is by definition invariant, bi-measurable with an invariant inverse (which is then also bi-measurable). Similarly, the identity on X is bi-measurable, μ-invariant and has an invariant inverse. We can then apply Lemma 4.3.18 to both $\sigma_\Omega = \sigma \times \mathrm{Id}$ and $\sigma_\Omega^{-1} = \sigma^{-1} \times \mathrm{Id}$ to find both are bi-measurable and $\mathbb{P} \times \mu$-invariant. □

Our analysis of Deterministic Young Towers relied heavily on densities associated with dynamical systems. The following lemma relates the density of the operator of a RDS to 'section-wise' densities. The proof relies on (corollaries of) Fubini's Theorem 2.1.8 and once again explicitly on the bi-measurability of σ. In Section 5.5 we will develop a tool to making these derivations easier, which we, for didactical purposes, have postponed.

Lemma 4.2.7. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$ be a random dynamical system, let $\Delta_0 \in \mathscr{F}_\Delta$ be so that $\mu_\Delta(\Delta_0) > 0$ and let $n \in \mathbb{Z}_{\geq 1}$. For every $\omega \in \Omega$, write $\Delta_{0,\omega} := (\Delta_0)_\omega$. Then we have for almost every $\omega \in \Omega$ and μ_{Δ_ω}-almost every $x \in \Delta_\omega$*

$$\frac{dS^n_\star \mu_{\Delta_0}}{d\mu_\Delta}(\omega, x) = \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x) \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega). \tag{97}$$

Moreover if σ is measure-preserving we obtain for almost every $\omega \in \Omega$ and μ_{Δ_ω}-almost every $x \in \Delta_\omega$,

$$\frac{dS^n_\star \mu_{\Delta_0}}{d\mu_\Delta}(\omega, x) = \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x). \tag{98}$$

Proof. Recall that $\Delta \subseteq \Omega \times X$ and hence we may write $S^n : \Delta \to \Omega \times X$ by embedding its codomain (so that $(S^n)^{-1}[\Omega \times X \setminus \Delta] = \emptyset$). In doing so, note that by non-singularity of S we have

$$S^n_\star \mu_{\Delta_0} \ll S^n_\star \mu_\Delta \ll \mu_\Delta \ll \mathbb{P} \times \mu.$$

Moreover, as $S^n_\star \mu_{\Delta_0}$ is a finite measure we know $\frac{dS^n_\star \mu_{\Delta_0}}{d\mathbb{P} \times \mu} \in L^1(\mathbb{P} \times \mu)$. We shall prove Equation (97). To do so, note that for $O \in \mathscr{F}_\Omega$, $A \in \mathscr{F}_X$ we have

$$\begin{aligned} S^{-n}(O \times A) &= \{(\omega, x) \in \Delta : S^n(\omega, x) \in O \times A\} \\ &= \{(\omega, x) \in \Delta : (\sigma^n(\omega), f^n_\omega(x)) \in O \times A\} \\ &= \{(\omega, x) \in \Delta : f^n_\omega(x) \in A\} \cap \{(\omega, x) \in \Delta : \sigma^n \omega \in O\}. \end{aligned}$$

Now define for $\omega \in \Omega$ the mapping

$$\begin{aligned} F^n_{\omega, A} &: X \to \mathbb{R} \\ x &\mapsto \begin{cases} \mathbb{1}_A(f^n_\omega(x)), & x \in \Delta_\omega \\ 0, & x \notin \Delta_\omega. \end{cases} \end{aligned}$$

and note that $\mathbb{1}_{S^{-n}[O\times A]}(\omega,x) = \mathbb{1}_O(\sigma^n(\omega))\cdot F^n_{\omega,A}(x)$. Knowing this we can see

$$\begin{aligned}\int_{O\times A}\frac{dS^n_\star\mu_{\Delta_0}}{d\mathbb{P}\times\mu}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x) &= S^n_\star\mu_{\Delta_0}(O\times A)\\ &= \int_{\Omega\times X}\frac{d\mu_{\Delta_0}}{d\mathbb{P}\times\mu}(\omega,x)\cdot\mathbb{1}_{S^{-n}[O\times A]}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x)\\ &= \int_\Omega\int_X \mathbb{1}_{\Delta_0}(\omega,x)\cdot F^n_{\omega,A}(x)\cdot\mathbb{1}_O(\sigma^n(\omega))\,d\mu(x)\,d\mathbb{P}(\omega) \qquad (99)\\ &= \int_\Omega \mathbb{1}_O(\sigma^n(\omega))\int_X F^n_{\omega,A}(x)\,d\mu_{\Delta_{0,\omega}}(x)\,d\mathbb{P}(\omega) \qquad (100)\\ &= \int_\Omega \mathbb{1}_O(\sigma^n(\omega))d(f^n_\omega)_\star\mu_{\Delta_{0,\omega}}(A)\,d\mathbb{P}(\omega) \qquad (101)\end{aligned}$$

where in Equation (99) we applied Theorem 2.1.8. As a consequence of Theorem 2.1.8 we also know that $\omega\mapsto(f^n_\omega)_\star\mu_{\Delta_{0,\omega}}(A)$ is in $L^1(\Omega)$. So we may apply Lemma 2.1.10 and continue writing Equation (101) by

$$\begin{aligned}&= \int_\Omega \mathbb{1}_O(\omega)\cdot(f^n_{\sigma^{-n}\omega})_\star\mu_{\Delta_{0,\sigma^{-n}\omega}}(A)\,d\sigma^n_\star\mathbb{P}(\omega)\\ &= \int_O\int_A \frac{d(f^n_{\sigma^{-n}\omega})_\star\mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu}(x)\frac{d\sigma^n_\star\mathbb{P}}{d\mathbb{P}}(\omega)\,d\mu(x)\,d\mathbb{P}(\omega).\end{aligned}$$

To conclude, we can see for every $O\in\mathscr{F}_\Omega$, $A\in\mathscr{F}_X$

$$\begin{aligned}&\int_{O\times A}\frac{dS^n_\star\mu_{\Delta_0}}{d\mathbb{P}\times\mu}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x)\\ &\qquad = \int_O\int_A \frac{d(f^n_{\sigma^{-n}\omega})_\star\mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu}(x)\frac{d\sigma^n_\star\mathbb{P}}{d\mathbb{P}}(\omega)\,d\mu(x)\,d\mathbb{P}(\omega). \qquad (102)\end{aligned}$$

Define

$$\tilde{S}(\omega,x) = \begin{cases}\frac{dS^n_\star\mu_{\Delta_0}}{d\mu_\Delta}(\omega,x), & \text{if } (\omega,x)\in\Delta\\ 0, & \text{else,}\end{cases}$$

and

$$\tilde{f}(\omega,x) = \begin{cases}\frac{d(f^n_{\sigma^{-n}\omega})_\star\mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x)\frac{d\sigma^n_\star\mathbb{P}}{d\mathbb{P}}(\omega), & x\in\Delta_\omega\\ 0, & \text{else.}\end{cases}$$

Furthermore we note that for any $O\in\mathscr{F}_\Omega$, $A\in\mathscr{F}_X$ we have following from Theorem 2.1.8 that

$$\int_O\int_A\tilde{S}(\omega,x)\,d\mu(x)\,d\mathbb{P}(\omega) = S^n_\star\mu_{\Delta_0}(O\times A) = \int_O\int_A\frac{dS^n_\star\mu_{\Delta_0}}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\,d\mathbb{P}(\omega), \qquad (103)$$

and similarly we see that

$$\begin{aligned}
&\int_O \int_A \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu}(x) \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega)\, d\mu(x)\, d\mathbb{P}(\omega) \\
&\quad = \int_O \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega) (f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}(A)\, d\mathbb{P}(\omega) \\
&\quad = \int_O \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega) (f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}(A \cap \Delta_\omega)\, d\mathbb{P}(\omega) \\
&\quad + \int_O \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega) (f^n_{\sigma^{-n}})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}(A \cap \Delta^c_\omega)\, d\mathbb{P}(\omega) \qquad (104) \\
&\quad = \int_O \int_{A\cap\Delta_\omega} \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x) \frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}}(\omega)\, d\mu_{\Delta_\omega}(x)\, d\mathbb{P}(\omega) \\
&\quad = \int_O \int_A \tilde{f}(\omega, x)\, d\mu(x)\, d\mathbb{P}(\omega), \qquad (105)
\end{aligned}$$

where the term in Equation (104) equals zero. Combining Equations (102), (103) and (105) we see that we have

$$\int_O \int_A \tilde{S}(\omega, x)\, d\mu(x)\, d\mathbb{P}(\omega) = \int_O \int_A \tilde{f}(\omega, x)\, d\mu(x)\, d\mathbb{P}(\omega). \qquad (106)$$

As $A \in \mathscr{F}_X, O \in \mathscr{F}_\Omega$ we given arbitrarily we see $\tilde{S}(\omega, x) = \tilde{f}(\omega, x)$ for almost every $\omega \in \Omega$ and almost every $x \in X$ and in particular for almost every $\omega \in \Omega$ and almost every $x \in \Delta_\omega$, which yields Equation (97).

As for Equation (98), if $\mathbb{P}$ is invariant under σ we have $\frac{d\sigma^n_\star \mathbb{P}}{d\mathbb{P}} \equiv 1$ and Equation (98) follows. □

Remark 4.2.8. It is tempting to claim Equation (98) hold μ_Δ-almost everywhere but there is no guarantee the right-hand side of the equation is jointly $\mathscr{F}_\Delta$-measurable implying the set

$$B = \left\{ (\omega, x) \in \Delta : \frac{dS^n_\star \mu_{\Delta_0}}{d\mu_\Delta}(\omega, x) = \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x) \right\}$$

may not be measurable. Contrarily, in fixing $\omega \in \Omega$, the mapping $x \mapsto \frac{d(f^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{0,\sigma^{-n}\omega}}}{d\mathbb{P}\times\mu}(x)$ is $\mathscr{F}_X$-measurable (and integrable) as a consequence of the Radon-Nikodyn theorem. Phrased differently, Lemma 4.2.7 asserts that for almost every $\omega \in \Omega$ we have $\mu_{\Delta_\omega}(B^c_\omega) = 0$.

The following rather straightforward lemma is an elementary example of a disintegration. In Proposition 4.1.8 we require a uniform upper bound on densities to find an acip. In the RDS setting, this means the uniform upper bound needs be found on the density associated with the skew product - just obtaining the bound section-wise is not enough. The lemma below together with Lemma 4.2.7 above are exactly what is necessary to make this translation happen.

Lemma 4.2.9. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a RDS and let $\Delta_0 \in \mathscr{F}_\Delta$ be some set of (finite) positive measure. Then for any $n \in \mathbb{Z}_{\geq 1}$, $A \in \mathscr{F}_\Delta$ we have*

$$\int_A \frac{d(G^n)_\star \mu_\Delta}{d\mu_\Delta}(\omega, x)\, d\mu_\Delta(\omega, x) = \int_\Omega \int_{A_\omega} \frac{d(G^n)_\star \mu_\Delta}{d\mu_\Delta}(\omega, x)\, d\mu_{\Delta_\omega}(x)\, d\mathbb{P}(\omega).$$

Proof. Let $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathscr{F}_\Delta$, then define

$$\begin{aligned} \tilde{G} : \Omega \times X &\to [0, \infty) \\ (\omega, x) &\mapsto \begin{cases} \frac{d(G^n)_\star \mu_\Delta}{d\mu_\Delta}(\omega, x) & \text{if } (\omega, x) \in \Delta \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Now note we have

$$\begin{aligned} \int_A \frac{d(G^n)_\star \mu_{\Delta_0}}{d\mu_\Delta}(\omega, x)\, d\mu_\Delta(\omega, x) &= \int_A \tilde{G}(\omega, x)\, d(\mathbb{P} \times \mu)(\omega, x) \\ &= \int_\Omega \int_{A_\omega} \tilde{G}(\omega, x)\, d\mu(x)\, d\mathbb{P}(\omega) \qquad (107) \\ &= \int_\Omega \int_{A_\omega} \tilde{G}(\omega, x) \cdot \mathbb{1}_{\Delta_\omega}(x)\, d\mu(x)\, d\mathbb{P}(\omega) \qquad (108) \\ &= \int_\Omega \int_{A_\omega} \frac{d(G^n)_\star \mu_{\Delta_0}}{d\mu_\Delta}(\omega, x)\, d\mu_{\Delta_\omega}(x)\, d\mathbb{P}(\omega), \end{aligned}$$

proving the lemma. In Equation (107) we used Theorem 2.1.8 and in Equation (108) we simply note that for $x \in A_\omega$, we have $\mathbb{1}_{\Delta_\omega}(x) = 1$. □

We conclude the section with the following Lemma, allowing us to integrate efficiently over subsets of RDS's.

Lemma 4.2.10. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a RDS and let $\Delta_0 \in \mathscr{F}_\Delta$ be such that $\mu(\Delta_0) < \infty$. Then we have*

$$\mu_\Delta(\Delta_0) = \int_\Omega \mu_{\Delta_\omega}(\Delta_{0,\omega})\, d\mathbb{P}(\omega).$$

Proof. First note that $\mathbb{1}_{\Delta_0} \in L^1(\Delta)$ so that $1_{\Delta_0} \in L^1(\Omega \times X)$. In noting that

$$\mu_{\Delta_\omega}(\Delta_{0,\omega}) = \int_{\Delta_\omega} \mathbb{1}_{\Delta_{0,\omega}}(x)\, d\mu_{\Delta_\omega}(x) = \int_{\Delta_\omega} \mathbb{1}_{\Delta_0}(\omega, x)\, d\mu_{\Delta_\omega}(x) = \int_X \mathbb{1}_{\Delta_0}(\omega, x)\, d\mu(x),$$

we can apply Fubini's Theorem 2.1.8 to see $\omega \mapsto \mu_{\Delta_\omega}(\Delta_{0,\omega})$ is in $L^1(\Omega)$. More so, we see

$$\int_\Omega \int_X \mathbb{1}_{\Delta_0}(\omega, x)\, d\mu(x) = (\mathbb{P} \times \mu)(\Delta_0) = \mu_\Delta(\Delta_0),$$

proving the statement. □

4.3 Jacobians With Distinct Domains

In Definition 2.2.6 we defined the Jacobian as a locally integrable function JT of a locally invertible, pbn-singular measurable transformation $T : X \to X$ on a (σ-finite) measure space $(X, \mathscr{F}, \mu)$. In the *quenched approach* however, we are more interested in the densities associated with the sections $S_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ so we need to define a Jacobian for suitable measurable mappings not necessarily having an identical domain and codomain.

Finally it is worth noting that local invertibility of the section of a mapping S does necessarily translate to the local invertibility of S in its entirety. Luckily, the material developed in Section 4.2 makes the potential local invertibility of S irrelevant for our theory.

We start with some preliminary definitions.

Definition 4.3.1. Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be measurable spaces and let $T : X \to Y$ be a measurable mapping.

1. If $T : X \to Y$ is measurable, bijective and has a measurable inverse, we call T *bi-measurable*;

2. If $A \in \mathscr{F}$ is so that $T(A) \in \mathscr{G}$ and $T|_A : A \to T(A)$ is bi-measurable (onto its image) then A is called an *invertibility domain* for T;

3. If there exists a countable partition $\mathscr{P}$ of X consisting of invertibility domains for T, then we call T *locally invertible*.

Remark 4.3.2. In definition 4.3.1 we assume that X admits a partition of invertibility domains. This is slightly more restrictive than what is done in Section 9.7.3 of [23] but is more easily understood, shortens our exposition on Jacobians significantly, and is sufficient for our theory. It seems our approach is novel and we provide more details than given in [23].

We shall now prove three basic statements on invertibility domains.

Lemma 4.3.3. *Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be measurable spaces, and let $T : X \to Y$ be a measurable mapping. Then,*

1. *if $A \in \mathscr{F}$ is an invertibility domain for T then every measurable $B \in \mathscr{F}$, $B \subseteq A$ is an invertibility domain for T;*

2. *if $T : X \to Y$ is bi-measurable, then for any $A \in \mathscr{F}$, we have that $T|_A : A \to T(A)$ is bi-measurable;*

3. *if $T : X \to Y$ is locally invertible, then for any $A \in \mathscr{F}$, we have that $T(A) \in \mathscr{G}$.*

Proof. 1. Suppose $A \in \mathscr{F}$ is an invertibility domain for T. Then note for any measurable $B \subseteq A$, we have $T|_B : B \to T(B)$ to be bijective and $(T|_A)^{-1}|_{T(B)}$ to be an inverse for $T|_B$. We only need show that $T(B) \in \mathscr{G}$. To do so, note that $T|_A^{-1}$ is measurable and $T(B) \subseteq T(A)$ so we can see $T(B) = (T|_A^{-1})^{-1}(B) \in \mathscr{G}$.

2. If $T : X \to Y$ is bi-measurable, item 1 shows that any $A \in \mathscr{F}$ is an invertibility domain for T, making $T|_A : A \to T(A)$ bi-measurable.

3. If $T : X \to Y$ is locally invertible, let $\mathscr{P} \subseteq \mathscr{F}$ be a partition into invertibility domains, and note that for $A \in \mathscr{F}$ we have

$$T(A) = T(\sqcup_{P\in\mathscr{P}} A \cap P) = \cup_{P\in\mathscr{P}} T(A \cap P) \in \mathscr{G},$$

as by Item 1 we have $T(A \cap P) \in \mathscr{G}$ for each $P \in \mathscr{P}$.

We have shown our claim. □

Now we are in the position to define pbn-singularity for locally invertible transformations on finite measure spaces. Be aware that this notion of pbn-singularity is different from Definition 2.2.3.

Definition 4.3.4. Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible mapping. We say T is *pullback non-singular* or *pbn-singular* if for every invertibility domain $A \in \mathscr{F}$ we have $\mu(A) = 0$ to imply $\nu(T(A)) = 0$.

Remark 4.3.5. In the context of Definition 4.3.4 we can by Lemma 4.3.3 rewrite pbn-singularity as the property $(T|_P^{-1})_\star \nu \ll \mu|_P$ for any invertibility domain $P \in \mathscr{F}$.

Lemma 4.3.6. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible mapping. Then we have T to be pbn-singular if and only if for some partition $\mathscr{P}$ of invertibility domains for T we have*

$$(T|_P^{-1})_\star \nu \ll \mu_P \quad \textit{for each} \quad P \in \mathscr{P}. \tag{109}$$

Proof. The implication assuming T is pbn-singular is trivial. So assume we have a partition $\mathscr{P}$ of invertibility domains for T for which Equation (109) holds. Then let $A \in \mathscr{F}$ be some invertibility domain such that $\mu(A) = 0$. Then note,

$$\begin{aligned} \mu(A) = 0 &\implies \sum_{P\in\mathscr{P}} \mu(A \cap P) = 0 \\ &\implies \sum_{P\in\mathscr{P}} \nu(T(A \cap P)) = 0 && (110) \\ &\implies \nu(T(A)) = 0, && (111) \end{aligned}$$

where we used Equation (109) in Line (110) and the injectivity of $T|_A$ in line (111). This proves our claim. □

We now define the Jacobian.

Definition 4.3.7. Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible, pbn-singular mapping. A function $JT : X \to [0, \infty)$ such that $JT \cdot \mathbb{1}_P \in L^1(\mu)$ and

$$\nu(T(P)) = \int_P JT(x)\, d\mu(x), \quad \text{for every invertibility domain } P \in \mathscr{F} \tag{112}$$

is called a *Jacobian* of T.

Our Definition 4.3.1 of local invertibility allows us to give a concise characterisation of the Jacobian. Included within the proof is an existence and uniqueness condition up to a measure zero set.

Lemma 4.3.8. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible, pbn-singular mapping. Then a Jacobian $JT : X \to [0, \infty)$ exists and is unique up to a measure zero set. Furthermore, in assuming $\mathscr{P} \subseteq \mathscr{F}$ is a countable partition of X consisting of invertibility domains for T we have for every $P \in \mathscr{P}$,*

$$(JT \cdot \mathbb{1}_P)(x) = \begin{cases} \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x), & x \in P \\ 0, & \text{else}, \end{cases} \quad \text{for } \mu\text{-a.e. } x \in X, \tag{113}$$

and

$$JT = \sum_{P \in \mathscr{P}} JT \cdot \mathbb{1}_P, \qquad \mu\text{-a.e.}. \tag{114}$$

Proof. Let $\mathscr{P} \subseteq \mathscr{F}$ be a countable partition of X consisting of invertibility domains for T. We shall first give a characterisation of a Jacobian for T.

In doing so, fix $P \in \mathscr{P}$ and let $B \in \mathscr{F}$ with $B \subseteq P$. Note we have by Lemma 4.3.3, B to be an invertibility domain so $T(B) \in \mathscr{G}$ and note we may write $\nu(T(B)) = (T|_P^{-1})_\star \nu(B)$. Using pbn-singularity as seen in Remark 4.3.5 yields

$$(T|_P^{-1})_\star \nu(B) = \int_B \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x)\, d\mu(x),$$

from which we conclude

$$\nu(T(B)) = \int_B \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x)\, d\mu(x), \quad \text{for arbitrary measurable } B \subseteq P. \tag{115}$$

As the measure ν is finite, we see $(T|_P^{-1})_\star \nu$ is a finite measure as well and so $\mathbb{1}_P \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu} \in L^1(\mu)$ and $\mathbb{1}_P \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu} < \infty$ μ-almost everywhere. As the countably many elements of $\mathscr{P}$ are disjoint, we may then assume $\sum_{P\in\mathscr{P}} \mathbb{1}_P \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu}$ takes finite values μ-almost everywhere.

Now for an arbitrary invertibility domain $B \in \mathscr{F}$ we can see

$$\nu(T(B)) = \sum_{P\in\mathscr{P}} \nu(T(P\cap B)) \tag{116}$$

$$= \sum_{P\in\mathscr{P}} \int_{P\cap B} \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x)\, d\mu(x) \tag{117}$$

$$= \sum_{P\in\mathscr{P}} \int_{B} \mathbb{1}_P(x) \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x)\, d\mu(x)$$

$$= \int_B \sum_{P\in\mathscr{P}} \mathbb{1}_P(x) \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu}(x)\, d\mu(x), \tag{118}$$

where in Equation (116) we used injectivity of T on B and σ-additivity of measures over disjoint measurable sets; in Equation (117) we applied Equation (115); and Equation (118) follows from the monotone convergence theorem, summing over non-negative elements.

From Equation (118) then follows

$$\tilde{J}T := \sum_{P\in\mathscr{P}} \mathbb{1}_P \cdot \frac{d(T|_P^{-1})_\star \nu}{d\mu}$$

is a Jacobian for T, from which Equations (113) and (114) follow immediately

Having found this, we show Equation (112) defines Jacobians up to a measure-zero set uniquely. To do so, note that any measurable function $f : X \to [0,\infty)$ satisfying

$$\nu(T(P)) = \int_P f(x)\, d\mu(x), \quad \text{for every invertibility domain } P \in \mathscr{F},$$

satisfies for arbitrary $B \in \mathscr{F}$,

$$\int_B \tilde{J}T(x)\, d\mu(x) = \sum_{P\in\mathscr{P}} \int_{B\cap P} \tilde{J}T(x)\, d\mu(x) \tag{119}$$

$$= \sum_{P\in\mathscr{P}} \nu(T(B\cap P)) \tag{120}$$

$$= \sum_{P\in\mathscr{P}} \int_{B\cap P} f(x)\, d\mu(x) \tag{121}$$

$$= \int_B f\, d\mu(x), \tag{122}$$

so that f and $\tilde{J}T$ differ up to a measure zero set. Here in Equations (119) and (122) we used the Monotone Convergence Theorem and in Equations (120) and (121) we used Lemma 4.3.3, item 1. □

Remark 4.3.9. Using the uniqueness property proven in Lemma 4.3.8 we can show Equations (113) and (114) hold for any Jacobian JT of T.

The Corollary below shows that Equation (114) is independent of a choice for a partition of invertibility domains, in the sense that J only depends on the points x on which it is evaluated.

Corollary 4.3.10. *Let $(X,\mathscr{F},\mu)$ and $(Y,\mathscr{G},\nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible, pbn-singular mapping with Jacobian JT. Let $A \in \mathscr{F}$ be some invertibility domain for T. We then have*

$$JT(x) = \frac{d(T|_A^{-1})_\star \nu}{d\mu}(x), \quad \textit{for almost every } x \in A. \tag{123}$$

Proof. By Lemma 4.3.8 we have the Jacobian JT to exist. Now fix an invertibility domain $A \in \mathscr{F}$, and suppose $B \in \mathscr{F}, B \subseteq A$. The Jacobian then satisfies

$$\int_B JT(x)\,d\mu(x) = \nu(T(B)) = \nu(T(A\cap B)) = (T|_A^{-1})_\star \nu(B) = \int_B \frac{d(T|_A^{-1})_\star \nu}{d\mu}(x)\,d\mu(x).$$

That is, $JT - \frac{d(T|_A^{-1})_\star \nu}{d\mu}$ integrates to zero on any measurable subset of A and hence Equation (123) holds. □

We conclude our discussion of Jacobians with three technical characterisations of Jacobians. The first in Lemma 4.3.11 characterises the Jacobian in terms of a Radon-Nikodym Derivative. After we show the Chain Rule generalises to Jacobians in Proposition 4.3.16 and that - under the right conditions - the Jacobian factors on product spaces in Lemma 4.3.18.

Lemma 4.3.11. *Let $(X,\mathscr{F},\mu)$ and $(Y,\mathscr{G},\nu)$ be finite measure spaces and let $T : X \to Y$ be a measurable, locally invertible, pbn-singular mapping with Jacobian JT. Suppose for some invertibility domain $A \in \mathscr{F}$ for T we have $JT > 0$, μ_A-almost surely. Then*

$$(JT(x))^{-1} = \frac{d(T|_A)_\star \mu}{d\nu}(T(x)), \qquad \textit{for } \mu\textit{-a.e. } x \in A. \tag{124}$$

Proof. We shall show that on any $B \in \mathscr{F}$, $B \subseteq A$ we have

$$\mu(B) = \int_B \frac{d(T|_A)_\star \mu}{d\nu}(T(x)) \frac{d(T|_A^{-1})_\star \nu}{d\mu}(x)\,d\mu,$$

so that

$$\frac{d(T|_A)_\star \mu}{d\nu} \circ T \cdot \frac{d(T|_A^{-1})_\star \nu}{d\mu} \equiv 1 \quad \mu-\text{a.e. on } A, \tag{125}$$

by Corollary 4.3.10. To do so, note

$$\begin{aligned}
\int_B \frac{d(T|_A)_\star\mu}{d\nu}(T(x))\frac{d(T|_A^{-1})_\star\nu}{d\mu}(x)\,d\mu(x) &= \int_B \frac{d(T|_A)_\star\mu}{d\nu}(T(x))\,d(T|_A^{-1})_\star\nu(x)\\
&= \int_B \frac{d(T|_A)_\star\mu}{d\nu}((T|_A^{-1})^{-1}(x))\,d(T|_A^{-1})_\star\nu(x)\\
&= \int_{T(B)} \frac{d(T|_A)_\star\mu}{d\nu}(y)\,d\nu(y) \qquad (126)\\
&= ((T|_A)_\star\mu)(T(B))\\
&= \mu(B),
\end{aligned}$$

where in Equation (126) we relied on Lemma 2.1.10, pulling back $(T|_A^{-1})$ which we may do as $\frac{d(T|_A)_\star\mu}{d\nu} \in L^1(\nu)$ following from μ being a finite measure. As $B \subseteq A$, $B \in \mathscr{F}$ was taken arbitrarily we may conclude Equation (125), as $JT = \frac{d(T|_A^{-1})_\star\nu}{d\mu}$ by Corollary 4.3.10. □

The following lemma shows that local invertibility and pbn-singularity as defined in Definitions 4.3.1 and 4.3.4 are preserved under composition. This is central to proving the Chain Rule in Proposition 4.3.16

Lemma 4.3.12. *Let $(X,\mathscr{F},\mu)$, $(Y,\mathscr{G},\nu)$, and $(Z,\mathscr{H},\eta)$ be finite measure spaces with $T : X \to Y$ and $U : Y \to Z$ locally invertible mappings. In letting $\mathscr{P}_X, \mathscr{P}_Y$ be partitions of X and Y into invertibility domains, respectively, then the collection*

$$\mathscr{P}_{U\circ T} := \left\{A \cap T^{-1}B : A \in \mathscr{P}_X, B \in \mathscr{P}_Y\right\} \setminus \{\emptyset\} \qquad (127)$$

partitions X into invertibility domains for $U \circ T$, so that $U \circ T$ is a locally invertible measurable mapping. Additionally, if U and T are pbn-singular then $U \circ T$ is pbn-singular.

Proof. Let $(X,\mathscr{F},\mu)$, $(Y,\mathscr{G},\nu)$, and $(Z,\mathscr{H},\eta)$ be measurable spaces with $T : X \to Y$ and $U : Y \to Z$ locally invertible, pbn-singular mappings. The measurability of $U \circ T$ is imminent as composition of measurable mappings.

Now let $\mathscr{P}_X, \mathscr{P}_Y$ be partitions of X and Y consisting of invertibility domains for T and U respectively. Define the collection $\mathscr{P}_{U\circ T}$ as in Equation (127) which partitions X and consists of $\mathscr{F}$-measurable sets. Furthermore, we have for arbitrary $A \in \mathscr{P}_X, B \in \mathscr{P}_Y$ with $A \cap T^{-1}B \in \mathscr{P}_{U\circ T}$, that $A \cap T^{-1}B \subseteq A$ and $T(A \cap T^{-1}B) = T(A) \cap B \subseteq B$ so that by Lemma 4.3.3, item (1) we have $A \cap T^{-1}B$ to be an invertibility domain for T and $T(A \cap T^{-1}B)$ to be an invertibility domain for U. This implies

$$(T|_{A\cap T^{-1}B})^{-1} : T(A) \cap B \to A \cap T^{-1}B$$

and

$$(U|_{T(A)\cap B})^{-1} : U(T(A)\cap B) \to T(A)\cap B$$

are well-defined measurable bijections and it is then easily seen that

$$((U\circ T)|_{A\cap T^{-1}B})^{-1} = (T|_{A\cap T^{-1}B})^{-1} \circ (U|_{T(A\cap T^{-1}B)})^{-1},$$

holds. Hence $((U\circ T)|_{A\cap T^{-1}B})^{-1}$ is a measurable bijection onto its image as a composition of measurable bijections. Lastly as $(U\circ T)(A\cap T^{-1}B) = U(T(A)\cap B) \in \mathscr{H}$ the partition $\mathscr{P}_{U\circ T}$ indeed consists of invertibility domains for $U\circ T$. We conclude $U\circ T$ is locally invertible. We now only need to show pbn-singularity of $U\circ T$.

To do so, let $P\in\mathscr{F}$ be an arbitrary invertibility domain for $U\circ T$ such that $\mu(P)=0$. Then note we can immediately derive

$$\begin{aligned}
\eta(U\circ T(P)) &= \eta(U\circ T(\bigsqcup_{A\in\mathscr{P}_{U\circ T}} P\cap A)) \\
&= \eta(\bigsqcup_{A\in\mathscr{P}_{U\circ T}} U\circ T(P\cap A) \qquad (128)\\
&= \sum_{A\in\mathscr{P}_{U\circ T}} \eta(U\circ T(P\cap A)) \\
&= 0, \qquad (129)
\end{aligned}$$

where in Equation (128) we used the injectivity of $U\circ T$ on P. Equation (129) is seen by noting

$$\mu(P)=0 \implies \mu(A\cap P)=0 \implies \nu(T(A\cap P)=0 \implies \eta(U\circ T(A\cap P))=0,$$

using the pbn-singularity of T and U consecutively. □

We swiftly extend the above inductively.

Corollary 4.3.13. *Let $n\in\mathbb{Z}_{\geq 1}$, $(X_i,\mathscr{F}_i,\mu_i)_{0\leq i\leq n}$ be finite measure spaces, let $T_i : X_i \to X_{i+1}$ for $i\in\{0,\dots,n-1\}$ be locally invertible and let for $i\in\{0,\dots,n-1\}$, $\mathscr{P}_i$ be partitions of X_i consisting of invertibility domains. Then we have for each $k\in\{1,\dots,n\}$,*

$$\mathscr{P}_{T^n} := \left\{A_0\cap T^{-1}A_1\cap\cdots\cap T^{-n+1}A_{n-1} : A_i\in\mathscr{P}_i,\ i\in\{0,\dots,n-1\}\right\}\setminus\{\emptyset\} \qquad (130)$$

to consist of invertibility domains for $T^k := T_{k-1}\circ\cdots\circ T_0$, so that $T^k : X_0\to X_k$ is a locally invertible measurable mapping. Additionally, if for each $i\in\{0,\dots,n-1\}$, T_i is pbn-singular then T^k is pbn-singular as well.

Proof. First we prove that $\mathscr{P}_{T^n}$ consists of invertibility domains for T^n and that T^n is pbn-singular. We will use this to show that $\mathscr{P}_{T^n}$ also consists of invertibility domains for T^k for each $k\in\{1,\dots,n\}$ proving local invertibility and pbn-singularity of T^k. We proceed inductively. The case $n = 1$ is trivial. Furthermore, note the case $n = 2$ is proven in Lemma 4.3.12.

In case $n > 2$, we assume $\mathscr{P}_{T^{n-1}}$ consists of invertibility domains for T^{n-1} and that T^{n-1} is pbn-singular. We can then conclude again by Lemma 4.3.12, $T^n := T_{n-1} \circ \cdots \circ T_0 = T_{n-1} \circ T^{n-1}$, that T^n is locally invertible, pbn-singular as a composition of locally invertible, pbn-singular mappings. Similarly, by Lemma 4.3.12, we see $\mathscr{P}_{T^n} = \mathscr{P}_{T^{n-1}} \vee \mathscr{P}_{n-1}$ is indeed a partition of invertibility domains for T^n. Having shown the induction step and the initial case $n = 2$ we have shown consists of invertibility domains for T^n and that T^n is pbn-singular for general $n \in \mathbb{Z}_{\geq 1}$

In particular for $1 \leq k < n$, note that, applying the above on T^k, we now know T^k is locally invertible and pbn-singular with $\mathscr{P}_{T^k}$ consisting of invertibility domains. As for every $P \in \mathscr{P}_{T^n}$ we have a $Q \in \mathscr{P}_{T^k}$ such that $P \subseteq Q$ we can see by Lemma 4.3.3 $\mathscr{P}_{T^n}$ partitions X into invertibility for T^k as well, as $\mathscr{P}_{T^n}$ partitions X. □

The following Corollary is a direct consequence of Corollary 4.3.13 combined with Lemma 4.3.3.

Corollary 4.3.14. *Let $n \in \mathbb{Z}_{\geq 1}$, $(X_i, \mathscr{F}_i, \mu_i)_{0 \leq i \leq n}$ be finite measure spaces, let $T_i : X_i \to X_{i+1}$ for $i \in \{0, \ldots, n-1\}$ be locally invertible. Then for each $A \in \mathscr{F}_0$ and each $i \in \{1, \ldots, n\}$ we have $T^i(A) \in \mathscr{F}_i$.*

The following Corollary is of no relevance to Jacobians per se, but is useful for future reference.

Corollary 4.3.15. *Let for $n \in \mathbb{Z}_{\geq 1}$, $(X_i, \mathscr{F}_i, \mu_i)_{0 \leq i \leq n}$ be finite measure spaces, $T_i : X_i \to X_{i+1}$ be locally invertible mappings, $\mathscr{P}_i$ be partitions of X_i consisting of invertibility domains for $i \in \{0, \ldots n-1\}$ and $\mathscr{P}_{T^n}$ as defined in Equation* (130)*. For every $k \in \{0, \ldots, n-1\}$ and for each $B_n \in \mathscr{P}_{T^n}$ we have exactly one $A_k \in \mathscr{P}_k$ so that $T^k B_n \subseteq A_k$.*

Proof. In case $n = 1$ simply pick for arbitrary $B_0 \in \mathscr{P}_0$, $A_0 = B_0$ so that $T^0 B_0 \subseteq A_0$ for $A_0 \in \mathscr{P}_0$.

Let $k \in \{1, \ldots, n-1\}$ and $B_n \in \mathscr{P}_{T^n}$. By construction we have

$$B_n = A_0 \cap \cdots \cap T^{-n+1} A_{n-1}, \quad \text{with } A_i \in \mathscr{P}_i, \text{ for } i \in \{0, \ldots, n-1\}.$$

Consequently, we see that

$$\begin{aligned}
T^k[B_n] &= T^k[A_0 \cap \cdots \cap T^{-n+1} A_{n-1}] \\
&= T^k[A_0 \cap \cdots \cap T^{-k+1} A_{k-1} \cap T^{-k}(A_k \cap \cdots \cap T_k^{-n+k+1} A_{n-1})] \\
&= T^k[A_0 \cap \cdots \cap T^{-k+1} A_{k-1}] \cap A_k \cap T_k^{-1} A_{k+1} \cap \cdots \cap T_k^{-n+k+1} A_{n-1} && (131) \\
&\subseteq A_k \in \mathscr{P}_k, && (132)
\end{aligned}$$

where in Equation (131) we used the general set identity $f(A \cap f^{-1}[B]) = f(A) \cap B$ which holds for all functions f and subsets A, B of the domain and codomain of f respectively. □

We shall now show the chain rule for Jacobians holds.

Proposition 4.3.16 (Chain Rule For Jacobians)**.** *Let $n \in \mathbb{Z}_{\geq 2}$, $(X_i, \mathscr{F}_i, \mu_i)_{0 \leq i \leq n}$ be finite measure spaces and let $T_i : X_i \to X_{i+1}$ for $i \in \{0, \dots, n-1\}$ be locally invertible, pbn-singular mappings. Then $J(T_{n-1} \circ \cdots \circ T_0)$ exists and*

$$J(T_{n-1} \circ \cdots \circ T_0) = \prod_{i=0}^{n-1} J(T_i) \circ (T_{i-1} \circ \cdots \circ T_0) \quad \text{holds } \mu\text{-a.e.}, \tag{133}$$

where we interpret $T_{i-1} \circ \cdots \circ T_0 = Id$ for $i = 0$.

Proof. First we will prove the statement for the case $n = 2$. Extending it to a general finite amount comes down to applying the case $n = 2$ and Lemma 4.3.12 $n - 1$ times.

So let $(X, \mathscr{F}, \mu)$, $(Y, \mathscr{G}, \nu)$, and $(Z, \mathscr{H}, \eta)$ be finite measure spaces with $T : X \to Y$ and $U : Y \to Z$ locally invertible, pbn-singular mappings. Then by Lemma 4.3.12 we know $U \circ T$ to be locally invertible and pbn-singular so we may apply Lemma 4.3.8 as well to claim the Jacobian $J(U \circ T)$ exists, as the measure spaces involved are finite.

Now let $A \in \mathscr{F}$ be such that A is an invertibility domain for T and $T(A)$ is an invertibility domain for U. Then we see

$$\begin{aligned} \int_A J(U \circ T)(x)\, d\mu(x) &= \eta(U \circ T(A)) \\ &= \int_{T(A)} JU(y)\, d\nu(y) \qquad (134) \\ &= \int_A JU(T(x))\, d(T|_A^{-1})_\star \nu(x) \qquad (135) \\ &= \int_A JU(T(x)) \frac{d(T|_A^{-1})_\star \nu}{d\mu}(x)\, d\mu(x) \qquad (136) \\ &= \int_A JU(T(x)) JT(x)\, d\mu(x), \qquad (137) \end{aligned}$$

where Equation (134) is by definition of the Jacobian; Equation (135) follows from Lemma 4.3.11; Equation (136) holds by pbn-singularity of T; and (137) follows from Corollary 4.3.10.

Now in fixing $\mathscr{P}_{U \circ T}$ as in Lemma 4.3.12, we see that the conditions under which Equation (137) holds, applies by Lemma 4.3.3 to all measurable sets $A \in \mathscr{F}$ such that there is a $P \in \mathscr{P}_{U \circ T}$ with $A \subseteq P$. Using this, we can see that for an arbitrary $B \in \mathscr{F}$ we have

$$\begin{aligned}
\int_B J(U\circ T)(x)\,d\mu(x) &= \int_X \sum_{P\in\mathscr{P}_{U\circ T}} \mathbb{1}_{B\cap P}(x) J(U\circ T)(x)\,d\mu(x) \\
&= \sum_{P\in\mathscr{P}_{U\circ T}} \int_{B\cap P} J(U\circ T)(x)\,d\mu(x) && (138)\\
&= \sum_{P\in\mathscr{P}_{U\circ T}} \int_{B\cap P} JU(T(x))JT(x)\,d\mu(x) && (139)\\
&= \int_X \sum_{P\in\mathscr{P}_{U\circ T}} \mathbb{1}_{B\cap P}(x) JU(T(x))JT(x)\,d\mu(x) && (140)\\
&= \int_B JU(T(x))JT(x)\,d\mu(x),
\end{aligned}$$

to hold, where in Equations (138), (140) we have used the Monotone Convergence Theorem, and in Equation (139) we applied Equation (137) to the sets $B\cap P$. As $B\in\mathscr{F}$ was given arbitrarily, we may conclude

$$J(U\circ T) = (JU\circ T)\cdot JT, \quad \text{to hold} \quad \mu\text{-a.e..} \tag{141}$$

Now for general $n>2$ let $(X_i,\mathscr{F}_i,\mu_i)_{0\le i\le n}$ be finite measure spaces and let $T_i : X_i \to X_{i+1}$ for $i\in\{0,\dots,n-1\}$ be locally invertible, pbn-singular mappings. We then have by Corollary 4.3.13 and Lemma 4.3.8 the Jacobian $J(T_{n-1}\circ\dots\circ T_0)$ to exist. Assuming Equation (133) holds for the case $n-1$, we can see

$$\begin{aligned}
J(T_{n-1}\circ\dots\circ T_0) &= J(T_{n-1}\circ(T_{n-2}\circ\dots T_0)) \\
&= (JT_{n-1}\circ T_{n-2}\circ\dots\circ T_0)\cdot J(T_{n-2}\circ\dots\circ T_0) && (142)\\
&= (JT_{n-1}\circ T_{n-2}\circ\dots\circ T_0)\cdot \prod_{i=0}^{n-2} JT_i\circ(T_{i-1}\circ\dots\circ T_0) \\
&= \prod_{i=0}^{n-1} JT_i\circ(T_{i-1}\circ\dots\circ T_0) && (143)
\end{aligned}$$

holds μ-*a.e.*., where in Equation (142) we used the case $n=2$ and in Equation (143) we used our induction hypothesis. Note that - like in the claim of the Proposition - we say $T_{i-1}\circ\dots\circ T_0 = \mathrm{Id}$ for $i=0$. We conclude the statement. □

The following lemma showcases nicely the efficiency that can be achieved in calculating Jacobians when standard results have been established. For notation on cylinders see Section 3.6.

Lemma 4.3.17. *Let* $(\Gamma^{\mathbb{Z}_{\ge 0}},\mathscr{F}_\Gamma,\mathbb{P},\sigma)$ *be some Bernoulli shift with weights* $P=(p_\gamma)_{\gamma\in\Gamma}$. *Then for each* $n\ge 1$ *and* $k\in\{1,\dots,n\}$ *we have the collection*

$$\mathscr{C}^n := \left\{[\gamma_0\cdots\gamma_{n-1}]\subseteq\Gamma^{\mathbb{Z}_{\ge 0}} : \gamma_0,\dots,\gamma_{n-1}\in\Gamma\right\}$$

of cylinders of depth n to consist of invertibility domains for $\sigma^k : \Gamma^{\mathbb{Z}_{\geq 0}} \to \Gamma^{\mathbb{Z}_{\geq 0}}$. *Moreover,* $\sigma^k : \Gamma^{\mathbb{Z}_{\geq 0}} \to \Gamma^{\mathbb{Z}_{\geq 0}}$ *is locally invertible and pbn-singular with a Jacobian satisfying*

$$J\sigma^k(x) = \frac{1}{p_{\gamma_0} \cdots p_{\gamma_{k-1}}}, \quad \textit{for almost every } x \in \Gamma^{\mathbb{Z}_{\geq 0}}. \tag{144}$$

Proof. First fix the cylinders of depth 1, $\mathscr{C}^1 := \{[\gamma] \subseteq \Gamma^{\mathbb{Z}_{\geq 0}} : \gamma \in \Gamma\}$, and note for any $\gamma_0 \in \Gamma$ we have

$$\begin{aligned} \sigma_{\gamma_0} : \qquad & \Gamma^{\mathbb{Z}_{\geq 0}} \to [\gamma_0] \\ & (\xi_n)_{n\geq 0} \mapsto \left(n \mapsto \begin{cases} \gamma_0, & n = 0 \\ \xi_{n-1}, & n \geq 1 \end{cases} \right) \end{aligned}$$

to satisfy $(\sigma_{\gamma_0} \circ \sigma)|_{[\gamma_0]} = \mathrm{Id}|_{[\gamma_0]}$, $\sigma \circ \sigma_{\gamma_0} = \mathrm{Id}$. We shall show the measurability of σ_{γ_0}. In doing so, note that for any $n' \geq 1$ and $(\gamma'_i)_{0\leq i \leq n'-1} \subseteq \Gamma^{n'}$ such that $\gamma'_0 = \gamma_0$ we have

$$\sigma_{\gamma_0}^{-1}[\gamma'_0 \cdots \gamma'_{n'-1}] = [\gamma'_1 \cdots \gamma'_{n'-1}] \in \mathscr{F}_\Lambda,$$

proving σ_{γ_0} is measurable by Lemma 2.1.4 and so $\sigma|_{[\gamma_0]}$ is bi-measurable. As $\mathscr{C}^1$ partitions $\Gamma^{\mathbb{Z}_{\geq 0}}$ we have σ to be locally invertible. To prove pbn-singularity, we derive the Jacobian directly. That is, note that for a $\gamma_0 \in \Gamma$ and a cylinder $[\gamma_0\gamma'_1 \cdots \gamma'_{n'-1}] \subseteq [\gamma_0]$ we have

$$\begin{aligned} (\sigma|_{[\gamma_0]}^{-1})_\star \mathbb{P}|_{[\gamma_0]} \left([\gamma_0\gamma'_1 \cdots \gamma'_{n-1}]\right) &= \mathbb{P}(\sigma[\gamma_0\gamma'_1 \cdots \gamma'_{n-1}]) \\ &= \mathbb{P}([\gamma'_1 \cdots \gamma'_{n-1}]) \\ &= \frac{1}{p_{\gamma_0}} \mathbb{P}(\left([\gamma_0\gamma'_1 \cdots \gamma'_{n-1}]\right). \end{aligned} \tag{145}$$

The cylinders starting with γ_0 are a generating collection for the restricted σ-algebra $\mathscr{F}_{[\gamma_0]}$, so we can apply Lemma 2.1.5 to show $(\sigma|_{[\gamma_0]}^{-1})_\star \mathbb{P}_{[\gamma_0]} = \frac{1}{p_{\gamma_0}} \mathbb{P}$. Using Equation (145) we have for $A \in \mathscr{F}_{[\gamma_0]}$ that

$$\mathbb{P}(\sigma(A)) = (\sigma|_{[\gamma_0]}^{-1})_\star \mathbb{P}_{[\gamma_0]}(A) = \frac{1}{p_{\gamma_0}} \mathbb{P}_{[\gamma_0]}(A) = \int_A \frac{1}{p_{\gamma_0}} \, d\mathbb{P}(\omega),$$

so that as γ_0 was given arbitrary we see by pbn-singularity and $J\sigma|_{[\gamma_0]} \equiv \frac{1}{p_{\gamma_0}}$, $\mathbb{P}$-almost surely. Now for $n \geq 1$, $k \in \{1, \ldots, n\}$ the collection $\mathscr{C}^n$ is exactly the n'th refined partition of $\mathscr{C}^1$ and so by Lemma 4.3.15 $\mathscr{C}^n$ consists of invertibility domains of σ^k and hence we may apply Proposition 4.3.16 to see for $[\gamma_0 \cdots \gamma_{n-1}] \in \mathscr{C}$ that

$$J\sigma^k(x) = \prod_{i=0}^{k-1} (J\sigma)(\sigma^i(x)) = \frac{1}{p_{\gamma_0} \cdots p_{\gamma_k}} \qquad \text{for almost every } x \in [\gamma_0 \ldots \gamma_{n-1}].$$

□

The lemma below is one of the few instances in which measure theoretic properties of sections are maintained on their product.

Lemma 4.3.18. *For each $i \in \{0,1\}$ let $(X_i, \mathscr{F}_i, \mu_i)$, $(Y_i, \mathscr{F}_i, \nu_i)$ be finite measure spaces and $f_i : X_i \to Y_i$ mappings. Define*

$$\begin{aligned} H : X_0 \times X_1 &\to Y_0 \times Y_1 \\ (x, y) &\mapsto (f_0(x), f_1(y)). \end{aligned}$$

Then we have

1. *H is $\mathscr{F}_{X_0 \times X_1}$-measurable if f_0 and f_1 are measurable;*

2. *H is bi-measurable if f_0 and f_1 are bi-measurable;*

3. *H is measure-preserving if f_0 and f_1 are measure-preserving and measurable.*

Moreover, if f_0 and f_1 are pbn-singular and bi-measurable the mapping H is pbn-singular and bi-measurable, so that JH exists, and we have $JH(x, y) = Jf_0(x)Jf_1(y)$, for $\mu_0 \times \mu_1$ almost every $(x, y) \in X_0 \times X_1$.

Proof. The first claim follows from [6, Lemma 2.12.5] phrased for general measurable spaces.

Now assuming bi-measurability of f_0 and f_1, we can see H is bijective with $H^{-1} = f_0^{-1} \times f_1^{-1}$, which then as a product of measurable mappings is measurable by [6, Lemma 2.12.5]. We then conclude H is bi-measurable.

If f_0 and f_1 are measure-preserving (and measurable), we can show

$$\mathscr{D} := \{C \in \mathscr{F}_{X_0 \times X_1} : H_\star(\mu_0 \times \mu_1)(C) = (\mu_0 \times \mu_1)(C)\}$$

contains the π-system

$$\mathscr{I} := \{A_0 \times A_1 \in \mathscr{F}_0 \times \mathscr{F}_1 : A_0 \in \mathscr{F}_0, A_1 \in \mathscr{F}_1\}.$$

As $\mathscr{D}$ is also a Dynkin system, we can apply Lemma 2.1.5 to show $\mathscr{D} = \mathscr{F}_{X_0 \times X_1}$, proving $H_\star(\mu_0 \times \mu_1) = \mu_0 \times \mu_1$.

For the remaining statements, assuming f_0 and f_1 are bi-measurable and pbn-singular, we see the Jacobians Jf_0, Jf_1 to exist by Lemma 4.3.8 and be integrable. We can directly verify joint measurability of $(x, y) \mapsto Jf_0(x) \cdot Jf_1(y)$ and the mapping $(x, y) \mapsto Jf_0(x) \cdot Jf_1(y)$ satisfies $Jf_0 \cdot Jf_1 \in L^1(X_0 \times X_1)$ follows by Theorem 2.1.8. Consequently, we see

$$\begin{aligned} \nu : \mathscr{F}_{X_0 \times X_1} &\to [0, \infty) \\ A &\mapsto \int_A Jf_0(x)Jf_1(y)\, d(\mu_0 \times \mu_1)(x, y) \end{aligned}$$

to be a measure by the Radon Nikodym Theorem 2.1.11. Moreover, by bi-measurability of H, we can see that

$$(\nu_0 \times \nu_1)(H(\cdot)) := (H^{-1})_\star(\mu_0 \times \mu_1)$$

is a measure as well.

Finally, on the π-system

$$\mathscr{I} := \{A_0 \times A_1 \in \mathscr{F}_0 \times \mathscr{F}_1 : A_0 \in \mathscr{F}_0, A_1 \in \mathscr{F}_1\},$$

we see that for any $A_0 \times A_1 \in \mathscr{I}$ we have

$$\begin{aligned}(\nu_0 \times \nu_1)(H(A_0 \times A_1)) &= (\nu_0 \times \nu_1)(f_0[A_0] \times f_1[A_1]) \\ &= \nu_0(f_0[A_0]) \cdot \nu_1(f_1[A_1]) \\ &= \int_{A_0} Jf_0(x)\, d\mu_0(x) \cdot \int_{A_1} Jf_1(y)\, d\mu_1(y) \\ &= \int_{A_0 \times A_1} Jf_0(x) Jf_1(y)\, d(\mu_0 \times \mu_1)(x, y),\end{aligned}$$

by Theorem 2.1.8. Applying Lemma 2.1.5 then shows

$$(\nu_0 \times \nu_1)(H(\cdot)) = \int_\cdot Jf_0(x) Jf_1(y)\, d(\mu_0 \times \mu_1)(x, y),$$

from which we can directly derive pbn-singularity of H. To conclude, JH indeed exists and $JH(x, y) = Jf_0(x)Jf_1(y)$, for $\mu_0 \times \mu_1$-almost every $(x, y) \in X_0 \times X_1$. □

A (trivial) application of Lemma 4.3.18 is Lemma 4.3.19 below. We have put it here to avoid repeating the same claim later on.

Lemma 4.3.19. *Let $(X, \mathscr{F}, \mu)$ be some finite measure space and let $(\mathbb{Z}, 2^{\mathbb{Z}}, c)$ be the integers with the counting measure. Let $k, l \in \mathbb{Z}$ and define the restricted measure spaces $(\{k\}, \{\emptyset, \{k\}\}, c_{\{k\}})$ and $(\{l\}, \{\emptyset, \{l\}\}, c_{\{l\}})$. Then the mapping*

$$\begin{aligned} Id \times t_k : X \times \{k\} &\to X \times \{l\} \\ (x, k) &\mapsto (x, l), \end{aligned}$$

is $\mathscr{F}_{X \times \{k\}}$-$\mathscr{F}_{X \times \{l\}}$ measurable, bi-measurable and satisfies $(Id \times t_k)_\star(\mu \times c_{\{k\}}) = \mu \times c_{\{l\}}$. Moreover, $J(Id \times t_k)$ exists and we have $J(Id \times t_k) \equiv 1$, $\mu \times c_{\{k\}}$-almost surely.

5 Quenched Random Young Towers

In this section we finally start our study of Random Young Towers in a quenched setting. At the start of Section 4 we have motivated the necessity for the *quenched approach* for describing certain random dynamical systems and now focus on its method. To the knowledge of the author, the quenched setting was first introduced in [4] and later discussed again in papers such as [26], [2] and [7].

As mentioned in Section 4, in the *quenched approach* the *random tower* Δ is defined on a random dynamical system $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ in such a way that its Ω-sections Δ_ω are Young Tower-like structures. In doing so, we shall also need to define a *(random) induced domain* Λ, a *(random) principal partition* $\mathscr{P}_\Delta$, a *(random) tower map* and a *(random) return time* R. In order to find an acip for the resulting *(random) tower system*

$$(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G),$$

we will then subject the Ω-sections of Λ, $\mathscr{P}_\Delta$, R and Δ to adapted versions of Young's conditions. As mentioned in Section 4, this allows for more flexibility than in the *annealed approach*.

The price we pay for this is that we lack typical properties on $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ such as local invertibility as we are no longer sure if $\mathscr{P}_\Delta$ partitions Δ into invertibility domains. More so, as we can not assume a constant return time on the elements of $\mathscr{P}_\Delta$, finding a useful expression of

$$\frac{d(G^R)_\star \mu_\Lambda}{d\mu_\Lambda}$$

is hard, if not impossible. This forces us to abandon the concept of a quenched tower base and forces us to conduct our analysis on the random tower directly.

Effectively, this will result in us being able to show the acip ν_Δ we obtain for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ in Section 5.4 has a density with a uniform lower bound. This also makes us unable extend the ergodic properties found in Section 3.5 to ν_Δ nor are we able to find ν_Δ as a limit point in total variation norm. Even though a stronger result is commonly accepted in the literature, its proofs in [4], [26], [2] and [7] are significantly lacking in mathematical rigour. We used intuitive concepts presented [2], formalised them to precise definitions and give rigorous proofs for according results. The argument in Section 5.4 is to our knowledge novel. Section 5.5 proves and identifies a disintegration theorem perfectly fitted to Random Towers as it essentially only on absolute continuity instead of topological arguments.

To make sure the reader is not overwhelmed by the technicalities and conditions we shall introduce the (random) Markov property and (random) bounded distortion as late as possible in the text.

5.1 The Core Definitions

In this section we start building our theory necessary for treating random towers. Definition 5.1.1 through Proposition 5.1.9 show the construction of a *random tower system* and prove this is a random dynamical system in the sense of Definition 4.2.3. After doing this, we will define the *random principal partition* in Definition 5.1.10 and characterise its sections in Corollary 5.1.12. We close the section with two technical results showing we can embed higher floors of random towers into lower floors. In the section thereafter we shall impose extra conditions on random tower systems to guarantee the existence of Jacobians for the sections of the tower map.

Suppose we have a measure-preserving dynamical system $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma)$ with $\mathbb{P}(\Omega) = 1$, $\sigma : \Omega \to \Omega$ bi-measurable and $\mathbb{P}$-invariant with bi-measurable $\mathbb{P}$-invariant inverse, a finite measure space $(X, \mathscr{F}_X, \mu)$ and a measurable set $\Lambda \in \mathscr{F}_{\Omega\times X}$, such that $\mu(\Lambda_\omega) = 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Furthermore, suppose we have a measurable mapping

$$\begin{aligned} g: \quad & \Omega \times X \to X \\ & (\omega, x) \mapsto g_\omega(x) \end{aligned}$$

such that for $\mathbb{P}$-almost every $\omega \in \Omega$ we have $(g_\omega)_\star\mu \ll \mu$ so that we may construct a skew product S and a random dynamical system

$$(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S), \tag{146}$$

as seen by Lemma 4.2.2 and Definition 4.2.3. We shall refer to (146) as the *Base RDS*.

Furthermore suppose we have an integrable return time

$$\begin{aligned} R: \quad & \Lambda \to \mathbb{Z}_{\geq 1} \\ & (\omega, x) \mapsto \inf\left\{n \in \mathbb{Z}_{\geq 1} : S^n_\omega(x) \in \Lambda_{\sigma^n\omega}\right\}, \end{aligned}$$

and suppose we have a countable partition $\mathscr{P}_\Lambda \subseteq \mathscr{F}_{\Omega\times X}$ of Λ such that for every $\omega \in \Omega$ and every $P \in \mathscr{P}_\Lambda$ such that $P_\omega \neq \emptyset$ we have $R_\omega|_{P_\omega} \equiv c_{P_\omega}$ for some $c_{P_\omega} \in \mathbb{Z}_{\geq 0}$. That is, for every $P \in \mathscr{P}_\Lambda$ the return time R is constant on the ω-sections of P. We then call Λ the *(random) induced domain*, $\mathscr{P}_\Lambda$ the *(random) principal partition (of* Λ*)* and the family $(g_\omega)_{\omega\in\Omega}$ the *random map*. We shall fix all objects thus far defined until Section 5.6. For sake of notational brevity we shall omit the adjective 'random' whenever it is clear from the context we are not dealing with an exclusively deterministic system. To define a *random tower* in Definition 5.1.1 concisely and correctly we need the objects,

$$\begin{aligned} \sigma_\Omega : \Omega \times X &\to \Omega \times X \\ (\omega, x) &\to (\sigma\omega, x), \end{aligned} \qquad R_{>l} := \{(\omega, x) \in \Lambda : R(\omega, x) > l\}, \text{ for } l \in \mathbb{Z}_{\geq 0}. \tag{147}$$

Recall Lemma 4.2.6 for the properties of σ_Ω and note that for any $l \in \mathbb{Z}_{\geq 0}$ we have

$$R_{>l} = \mathbb{Z}_{\geq 1} \setminus R^{-1}\{1, \ldots, l\} \in \mathscr{F}_\Lambda \text{ by measurability of } R.$$

Below we define a *random tower*. The definition may seem rather contrived but turns out to be useful when interpreted in the context of a random dynamical system.

Definition 5.1.1. Define

$$\Delta := \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \sigma_\Omega^l(R_{>l}) \times \{l\}. \tag{148}$$

We call Δ a *(random) tower*, for $l \in \mathbb{Z}_{\geq 0}$, Δ_l a *(random) floor* of Δ and $\Delta_0 = \Lambda$, the *(random) ground floor of* Δ.

Remark 5.1.2.

1. Note that for each $l \in \mathbb{Z}_{\geq 0}$ we have $\Delta_l = \sigma_\Omega^l(R_{>l})$.
2. For the rest of this section, when mentioning towers, we shall refer to towers as defined in Definition 5.1.1, with its associated objects $\mathscr{P}_\Lambda, R, (g_\omega)_{\omega \in \Omega}$.

We endow Δ with a σ-algebra and a finite measure.

Lemma 5.1.3. *Let* $(\mathbb{Z}_{\geq 0}, \mathscr{F}_{\mathbb{Z}_{\geq 0}}, c)$ *be the integers equipped with the* σ*-algebra* $\mathscr{F}_{\mathbb{Z}_{\geq 0}} := 2^{\mathbb{Z}_{\geq 0}}$ *and the counting measure* c*. Let* $\Delta \subseteq \Omega \times X \times \mathbb{Z}_{\geq 0}$ *be a tower as in Definition 5.1.1. Then* Δ *is* $\mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}}$*-measurable with* $(\mathbb{P} \times \mu \times c)(\Delta) \in [1, \infty)$ *and hence we may define* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ *as a (restricted) finite measure space.*

Proof. Note that for $l \in \mathbb{Z}_{\geq 0}$ we have $R_{>l} \in \mathscr{F}_\Lambda$. As $\Lambda \subseteq \Omega \times X$ we then see $R_{>l} \in \mathscr{F}_{\Omega \times X}$ and so by bi-measurability of σ_Ω we have for each $l \in \mathbb{Z}_{\geq 0}$ that $\Delta_l = \sigma_\Omega^l(R_{>l}) \in \mathscr{F}_{\Omega \times X}$. We then see $\Delta = \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \Delta_l \times \{l\}$ is a countable union of $\mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}}$-measurable sets and hence it is $\mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}}$-measurable. We conclude we may construct $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ as the restricted measure space of

$$(\Omega \times X \times \mathbb{Z}_{\geq 0}, \mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}}, \mathbb{P} \times \mu \times c) \text{ to } \Delta.$$

What remains to be proven is that $\mu_\Delta(\Delta) = [1, \infty)$. Note

$$\mu_\Delta(\Delta) = \sum_{l \in \mathbb{Z}_{\geq 0}} (\mathbb{P} \times \mu)(\Delta_l) = \sum_{l \in \mathbb{Z}_{\geq 0}} (\mathbb{P} \times \mu)(\sigma_\Omega^l[R_{>l}]) = \sum_{l \in \mathbb{Z}_{\geq 0}} (\mathbb{P} \times \mu)(R_{>l}),$$

by invariance of the mapping σ_Ω^{-1}. By using a standard probabilistic equality (e.g. seen in [14, Lemma 4.4]) we see

$$\sum_{l \in \mathbb{Z}_{\geq 0}} (\mathbb{P} \times \mu)(R_{>l}) = \|R\|_1 < \infty,$$

As by construction we have $\Lambda \times \{0\} \subseteq \Delta$, $\Lambda \in \mathscr{F}_{\Omega \times X}$ and $\mu(\Lambda_\omega) = 1$, $\mathbb{P}$-almost surely, we can use Proposition 2.1.7 to show

$$\mu_\Delta(\Delta) \geq (\mathbb{P} \times \mu)(\Lambda) = \int_\Omega \mu(\Lambda_\omega)\, d\mathbb{P}(\omega) = 1,$$

so that indeed $\mu_\Delta(\Delta) \in [1, \infty)$. □

We call the measure space $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ as seen in Lemma 5.1.3 a *random tower (space)*. As mentioned, most of the analysis on random tower spaces takes place their sections. We make a quick notational comment.

Remark 5.1.4. When taking multiple sections on the same set, we adopt the following notation. For $\dot{\omega} \in \Omega$ and $\dot{l} \in \mathbb{Z}_{\geq 0}$ we write

$$\Delta_{\dot{\omega}} := \{(x, l) \in X \times \mathbb{Z}_{\geq 0} : (\dot{\omega}, x, l) \in \Delta\} \text{ and } \Delta_{\dot{\omega}, \dot{l}} := \{x \in X : (\dot{\omega}, x, \dot{l}) \in \Delta\}.$$

Similarly, we write for $A \in \mathscr{F}_\Delta$, $\dot{\omega} \in \Omega$ and $\dot{l} \in \mathbb{Z}_{\geq 0}$ that,

$$A_{\dot{\omega}} = \{(x, l) \in \Delta_{\dot{\omega}} : (\dot{\omega}, x, l) \in A\}, \quad \text{and} \quad A_{\dot{\omega}, \dot{l}} = \{x \in \Delta_{\dot{\omega}, \dot{l}} : (\dot{\omega}, x, \dot{l}) \in A\}.$$

By Lemma 2.1.7 we see that $A_{\dot{\omega}} \in \mathscr{F}_{\Delta_{\dot{\omega}}}$, and $A_{\dot{\omega}, \dot{l}} \in \mathscr{F}_{\Delta_{\dot{\omega}.\dot{l}}}$. We can similarly define for $\dot{l} \in \mathbb{Z}_{\geq 0}$ and $\dot{\omega} \in \Omega$ $A_{\dot{l}, \dot{\omega}}$ first taking the $\dot{l}$-section and then the $\dot{\omega}$-section. As we then see $A_{\dot{l}, \dot{\omega}} = A_{\dot{\omega}, \dot{l}}$ we are free to use both notations interchangeably.

Note we only have required $\mu(\Lambda_\omega) = 1$ and $(g_\omega)_\star \mu \ll \mu$, to hold $\mathbb{P}$-almost everywhere. This has been done to accommodate random dynamical systems where the random dynamic behaves abnormally for particular $\omega \in \Omega$. An example of such an $\omega \in \Omega$ is for instance in the annealed example in Section 3.6.2 a sequence $\omega = (\omega_n)_{n \geq 0} \in \{s, g\}^{\mathbb{Z}_{\geq 0}}$ where ω is eventually constant s. This disrupts the dynamics as points are no longer sure to return to their induced domain.

Like in the annealed approach, this is not a problem in the quenched approach if these 'abnormal' ω only add up to a measure zero set. We shall formalise this by defining a set $\dot{\Omega} \subseteq \Omega$ of full measure, consisting of well-behaved points in Ω.

Remark 5.1.5. Given a random tower space $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ we have by construction for almost every $\omega \in \Omega$, $\mu_{\Delta_\omega}(\Delta_\omega) \geq \mu(\Lambda_\omega) = 1$. Moreover, as a consequence of Lemma 5.1.3 we have $\mu_{\Delta_\omega}(\Delta_\omega) < \infty$ for almost every $\omega \in \Omega$. Lastly we now by construction $(g_\omega)_\star \mu \ll \mu$ for almost every $\omega \in \Omega$. Combined, we can see

$$\tilde{\Omega} := \{\omega \in \Omega : \mu_{\Delta_\omega}(\Delta_\omega) \in [1, \infty), (g_\omega)_\star \mu \ll \mu\} \in \mathscr{F}_\Omega,$$

and $\mathbb{P}(\Omega \setminus \tilde{\Omega}) = 0$. Finally we close $\tilde{\Omega}$ under σ by writing

$$\dot{\Omega} := \bigcap_{n \in \mathbb{Z}}^{\infty} \sigma^n(\Omega).$$

Note again we have $\dot{\Omega} \in \mathscr{F}_\Omega$ and $\mathbb{P}(\Omega \setminus \dot{\Omega}) = 0$. As we now have $(g_\omega)_\star \mu \ll \mu$ for each $\omega \in \dot{\Omega}$, and $\dot{\Omega}$ is closed under σ, we can see that for any $\omega \in \dot{\Omega}$ and any $m \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$ we have $(g^m_{\sigma^l \omega})_\star \mu \ll \mu$ as $g^m_{\sigma^l \omega}$ is a composition of measurable, non-singular mappings. The set $\dot{\Omega}$ will be useful in almost every statement onwards and will save us many technical complications. As there are no drawbacks for us to use $\dot{\Omega}$ instead of Ω, we shall do so consistently.

Remark 5.1.6. Below we have provided a schematic representation of (a part of) a random tower $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$. On the last row of the figure we have specified which Ω-section of the tower is being shown in the column above. As can be seen each row represents a floor of the tower.

$$
\begin{array}{ccccc}
 & \cdots & \cdots & \cdots & \\
\cdots & \Delta_{\sigma^{-1}\omega,3} \times \{3\} & \Delta_{\omega,3} \times \{3\} & \Delta_{\sigma\omega,3} \times \{3\} & \cdots \\
\cdots & \Delta_{\sigma^{-1}\omega,2} \times \{2\} & \Delta_{\omega,2} \times \{2\} & \Delta_{\sigma\omega,2} \times \{2\} & \cdots \\
\cdots & \Delta_{\sigma^{-1}\omega,1} \times \{1\} & \Delta_{\omega,1} \times \{1\} & \Delta_{\sigma\omega,1} \times \{1\} & \cdots \\
\cdots & \Delta_{\sigma^{-1}\omega,0} \times \{0\} & \Delta_{\omega,0} \times \{0\} & \Delta_{\sigma\omega,0} \times \{0\} & \cdots \\
\hline
 & \Delta_{\sigma^{-1}\omega} & \Delta_{\omega} & \Delta_{\sigma\omega} &
\end{array}
$$

Figure 5: A schematic representation of a part of a random tower

To make some more comments on how the sections of a tower in the figure relate, it can be useful to rewrite the above figure in term of the return time. That is, for $\omega \in \dot{\Omega}$ and $l \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}$ we can directly derive from the definition of Δ that

$$\Delta_{\sigma^k\omega,l} \times \{l\} = (R_{>l})_{\sigma^{k-l}\omega} \times \{l\} \tag{149}$$

yielding the figure

$$
\begin{array}{ccccc}
 & \dots & \dots & \dots & \\
\dots & (R_{>3})_{\sigma^{-4}\omega} \times \{3\} & (R_{>3})_{\sigma^{-3}\omega} \times \{3\} & (R_{>3})_{\sigma^{-2}\omega} \times \{3\} & \dots \\
\dots & (R_{>2})_{\sigma^{-3}\omega} \times \{2\} & (R_{>2})_{\sigma^{-2}\omega} \times \{2\} & (R_{>2})_{\sigma^{-1}\omega} \times \{2\} & \dots \\
\dots & (R_{>1})_{\sigma^{-2}\omega} \times \{1\} & (R_{>1})_{\sigma^{-1}\omega} \times \{1\} & (R_{>1})_{\omega} \times \{1\} & \dots \\
\dots & (R_{>0})_{\sigma^{-1}\omega} \times \{0\} & (R_{>0})_{\omega} \times \{0\} & (R_{>0})_{\sigma\omega} \times \{0\} & \dots \\
 & \overline{} & \overline{} & \overline{} & \\
 & \Delta_{\sigma^{-1}\omega} & \Delta_{\omega} & \Delta_{\sigma\omega} &
\end{array}.
$$

Figure 6: Figure 5 reimagined using Identity (149)

Looking at this figure intent-fully we can make two important remarks

1. The individual floors of an Ω-section of a tower do not have to be related. Formalising this, suppose $\omega \in \dot{\Omega}$ and $l, l' \in \mathbb{Z}_{\geq 0}$ and note that the sets $\Delta_{\omega,l}$ and $\Delta_{\omega,l'}$ are floors in the same Ω-section Δ_ω however, we have

$$\Delta_{\omega,l} = (R_{>l})_{\sigma^{-l}\omega} \subseteq \Lambda_{\sigma^{-l}\omega}, \quad \Delta_{\omega,l'} = (R_{>l'})_{\sigma^{-l'}\omega} \subseteq \Lambda_{\sigma^{-l'}\omega}.$$

   Conceptually, the sets $\Lambda_{\sigma^{-l'}\omega}$ and $\Lambda_{\sigma^{-l}\omega}$ may be unrelated other than both being subsets of X. For instance in restricting the skew product S to Λ we can see that

$$S|_\Lambda(\sigma^{-l}\omega, \cdot) = S_{\sigma^{-l}\omega}|_{\Lambda_{\sigma^{-l}\omega}}(\cdot) \text{ and } S|_\Lambda(\sigma^{-l'}\omega, \cdot) = S_{\sigma^{-l'}\omega}|_{\Lambda_{\sigma^{-l'}\omega}}(\cdot),$$

   which can be very different functions.

2. Rather that the floors of an individual Ω-section of Δ, there is a more direct connection along the diagonals in the figures above. Indeed, we clearly have $(R_{>l})_\omega \supseteq (R_{>k+l})_\omega$ for each $k, l \in \mathbb{Z}_{\geq 0}$ so

$$\Delta_{\omega,0} = (R_{>0})_\omega \supseteq (R_{>1})_\omega \supseteq (R_{>2})_\omega \supseteq \dots.$$

   By Equation (149) we have $\Delta_{\sigma^i\omega,i} = (R_{>i})_\omega$, so that

$$\Lambda_\omega = \Delta_{\omega,0} \supseteq \Delta_{\sigma\omega,1} \supseteq \Delta_{\sigma^2\omega,2} \supseteq \dots$$

Embedding floors $\Delta_{\sigma^k\omega,l}$ into $\Delta_{\sigma^{k-l}\omega,0}$ will be central to our to our theory and will be formalised in Lemma 5.1.13 and Corollary 5.1.14.

Now we will equip a random tower with a mapping making it into a dynamical system.

Definition 5.1.7. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ be a (random) tower. Then define the *(random) tower map* by

$$\begin{aligned} G: \quad & \Delta \to \Delta \\ (\omega, x, l) & \mapsto (\sigma\omega, G_\omega(x,l)) \end{aligned}$$

where for $\omega \in \Omega$, with $\Delta_\omega \neq \emptyset$, $G_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ is such that for $(x,l) \in \Delta_\omega$ we have

$$G_\omega(x,l) = \begin{cases} (x, l+1), & \text{if } R_{\sigma^{-l}\omega}(x) > l+1, \\ (g^{l+1}_{\sigma^{-l}\omega}(x), 0), & \text{if } R_{\sigma^{-l}\omega}(x) = l+1. \end{cases}$$

Remark 5.1.8. Before proving the measure-theoretical properties of a random tower map on a technical level, we explain the way the tower map acts on the tower using the Figures 7 and 8. Both figures are based on Figure 6 where we use for $l, k \in \mathbb{Z}_{\geq 0}$ the identity

$$(R_{>l})_{\sigma^k\omega} \times \{l\} = (R_{>l+1})_{\sigma^k\omega} \times \{l\} \sqcup (R^{-1}(l+1))_{\sigma^k\omega} \times \{l\}.$$

As we have $\Delta_{\sigma^{k+l}\omega} \times \{l\} = (R_{>l})_{\sigma^k\omega} \times \{l\}$ by Equation (149) we can then write

$$\begin{aligned} G_{\sigma^{k+l}\omega}|_{\Delta_{\sigma^{k+l}\omega,l} \times \{l\}} : \Delta_{\sigma^{k+l}\omega,l} \times \{l\} & \to \Delta_{\sigma^{k+l+1}\omega} \\ (x,l) & \mapsto \begin{cases} (x, l+1), & x \in (R_{>l+1})_{\sigma^k\omega} \\ (g^{l+1}_{\sigma^k\omega}(x), 0), & x \in (R^{-1}(l+1))_{\sigma^k\omega} \end{cases} \end{aligned}$$

In the Figures 7 and 8 the arrows represent the action of

$$G_{\sigma^i\omega} : \Delta_{\sigma^i\omega} \to \Delta_{\sigma^{i+1}\omega}, i \in \{-1, 0, 1\}$$

restricted to the sets as specified in the figures.

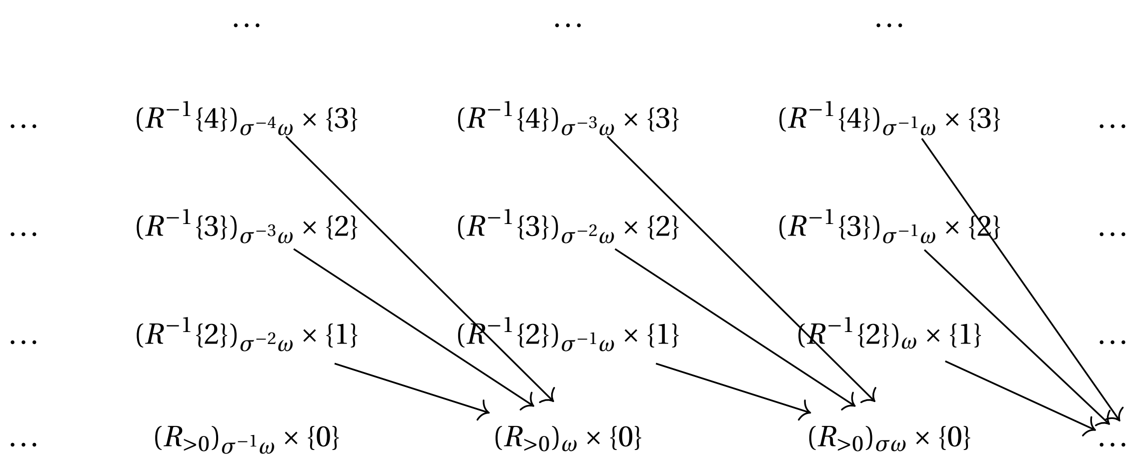


Figure 8: Subsets of floors mapped to the ground floor under the tower map

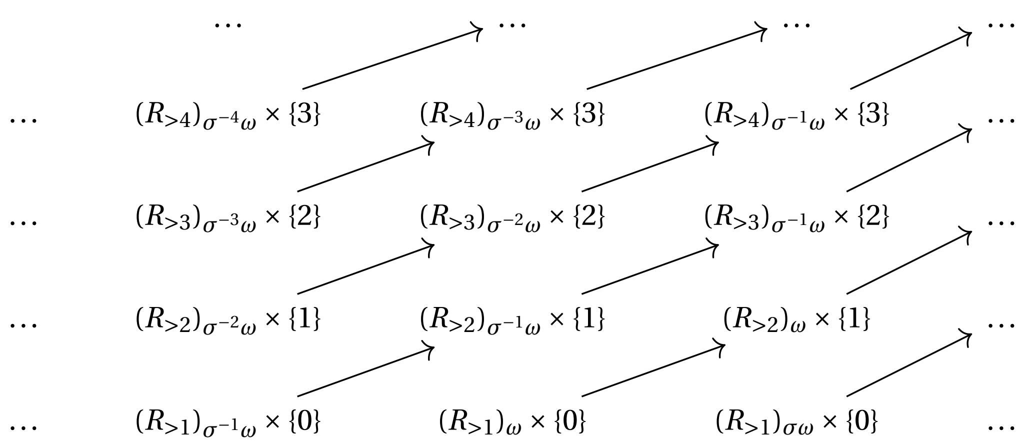


Figure 7: Subsets of floors mapped to a higher floor under the tower map

The following lemma affirms the quadruple $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is a random dynamical system. We refer to it as a *random tower system.*

Proposition 5.1.9. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ *be a tower, with* $G : \Delta \to \Delta$ *a tower map as in Definition* 5.1.7. *Then the mapping* G *is measurable and non-singular so that* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *is a random dynamical system as seen in Definition 4.2.3. Moreover, we have* $(G_\omega)_\star \mu_{\Delta_\omega} \ll \mu_{\Delta_{\sigma\omega}}$*, for each* $\omega \in \dot{\Omega}$.

Proof. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ be a random tower and let G be its random tower map. We first prove measurability of G. So let $A \in \mathscr{F}_\Delta$ and write $A = (A_0 \times \{0\}) \sqcup \bigsqcup_{l \geq 1} A_l \times \{l\}$. First we calculate $G^{-1}(A_0 \times \{0\})$. Note that (keeping Figure 8 in mind)

$$\begin{aligned}
G^{-1}(A_0 \times \{0\}) &= \{(\omega, x, l) \in \Delta : (\sigma\omega, G_\omega(x, l)) \in A_0 \times \{0\}\} \\
&= \{(\omega, x, l) \in \Delta : (\sigma\omega, g^{l+1}_{\sigma^{-l}\omega}(x)) \in A_0,\ R_{\sigma^{-l}\omega}(x) = l+1\} \\
&= \{(\omega, x, l) \in \Delta : S^{l+1}(\sigma^{-l}\omega, x) \in A_0,\ x \in (R^{-1}(l+1))_{\sigma^{-l}\omega}\} \\
&= \{(\omega, x, l) \in \Delta : S^{l+1}(\sigma_\Omega^{-l}(\omega, x)) \in A_0,\ \sigma_\Omega^{-l}(\omega, x) \in R^{-1}(l+1)\} \\
&= \{(\omega, x, l) \in \Delta : (\omega, x) \in \sigma_\Omega^{l}(S^{-(l+1)}(A_0) \cap R^{-1}(l+1))\} \\
&= \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \sigma_\Omega^{l}(S^{-(l+1)}(A_0) \cap R^{-1}(l+1)) \times \{l\} \in \mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}},
\end{aligned}$$

as a countable union of measurable sets. Note we have

$$\sigma_\Omega^{l}(S^{-(l+1)}(A_0) \cap R^{-1}(l+1)) \times \{l\} \subseteq \Delta_l \times \{l\} \text{ for } l \in \mathbb{Z}_{\geq 0}$$

as well, so that

$$G^{-1}(A_0 \times \{0\}) = \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \sigma_\Omega^{l}(S^{-(l+1)}(A_0) \cap R^{-1}(l+1)) \times \{l\} \in \mathscr{F}_\Delta, \tag{150}$$

and for $l \in \mathbb{Z}_{\geq 1}$ we see (keeping Figure 7 in mind)

$$G^{-1}(A_l \times \{l\}) = \sigma_\Omega^{-1}(A_l) \times \{l-1\} \in \mathscr{F}_\Delta. \tag{151}$$

Combining Equations (150) and (151) we obtain

$$G^{-1}A = G^{-1}(A_0 \times \{0\}) \cup G^{-1}\left(\bigsqcup_{l \geq 1} A_l \times \{l\}\right) \in \mathscr{F}_\Delta. \tag{152}$$

To show non-singularity of G we calculate the inverse images of the tower map section wise. Let $\omega \in \dot{\Omega}$ and suppose we have a $B \in \mathscr{F}_{\Delta_{\sigma\omega}}$. We can derive

$$G_\omega^{-1}(B_0 \times \{0\}) = \bigsqcup_{l' \in \mathbb{Z}_{\geq 0}} \left((g^{l'+1}_{\sigma^{-l'}\omega})^{-1}(B_0) \cap (R_{\sigma^{-l'}\omega})^{-1}\{l'+1\}\right) \times \{l'\}, \tag{153}$$

and for $l \in \mathbb{Z}_{\geq 1}$

$$G_\omega^{-1}(B_l \times \{l\}) = B_l \times \{l-1\}.$$

Now assuming $\mu_{\Delta_{\sigma\omega}}(B) = 0$ we see $\mu_{\Delta_{\sigma\omega}}(B_l \times \{l\}) = 0$ for each $l \in \mathbb{Z}_{\geq 0}$.

We then see using Equation (153) that

$$\begin{aligned}
\mu_{\Delta_\omega}(G_\omega^{-1}(B_0 \times \{0\})) &\leq \sum_{l' \in \mathbb{Z}_{\geq 0}} \mu_{\Delta_\omega}\left((g^{l'+1}_{\sigma^{-l'}\omega})^{-1}(B_0) \cap (R_{\sigma^{-l'}\omega})^{-1}\{l'+1\} \times \{l'\}\right) \\
&\leq \sum_{l' \in \mathbb{Z}_{\geq 0}} \mu\left((g^{l'+1}_{\sigma^{-l'}\omega})^{-1}(B_0)\right) = 0,
\end{aligned} \tag{154}$$

as $(g^{l'+1}_{\sigma^{-l'}\omega})_\star\mu \ll \mu$ and $\mu(B_0) = \mu_{\Delta_{\sigma\omega}}(B_0 \times \{0\}) = 0$. Similarly we can obtain using c to denote the counting measure

$$\begin{aligned}\mu_{\Delta_\omega}(G_\omega^{-1}(B_l \times \{l\})) &= \mu_{\Delta_\omega}(B_l \times \{l-1\}) \\ &= \mu(B_l)\cdot c(\{l-1\}) \\ &= \mu(B_l)\cdot c(\{l\}) \\ &= \mu_{\Delta_{\sigma\omega}}(B_l \times \{l\}) \\ &= 0. \end{aligned} \tag{155}$$

Combining Equations (154) and 155 we then see

$$(G_\omega)_\star\mu_{\Delta_\omega}(B) = \sum_{l\in\mathbb{Z}_{\geq 0}} (G_\omega)_\star\mu_{\Delta_\omega}(B_l \times \{l\}) = 0,$$

proving $(G_\omega)_\star\mu_{\Delta_\omega} \ll \mu_{\Delta_{\sigma\omega}}$ for each $\omega \in \dot{\Omega}$. Applying Proposition 4.2.2 then shows that $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is a random dynamical system. □

As mentioned in the introduction of this section, we now turn to defining a *(random) principal partition* for $\mathscr{P}_\Delta$. We do so using the collection $\mathscr{P}_\Lambda$ at the start of this section, similar to the deterministic case. In Lemma 5.1.11 and Corollary 5.1.12 the namesake of the objects defined in Definition 5.1.10 below will be justified.

Definition 5.1.10. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. We say the collection

$$\mathscr{P}_\Delta := \{\sigma^l_\Omega[P \cap R_{>l}] \times \{l\} : P \in \mathscr{P}_\Lambda, l \in \mathbb{Z}_{\geq 0}\} \setminus \{\emptyset\} \tag{156}$$

is the *(random) principal partition* for Δ and for $\omega \in \dot{\Omega}$ we say its ω-section $\mathscr{P}_{\Delta_\omega}$ as defined in (96) is the *(random) principal partition* for Δ_ω.

We now prove the random principal partition $\mathscr{P}_\Delta$ for a random tower system Δ is a countable partition for Δ consisting of measurable sets.

Lemma 5.1.11. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ *be a random tower. Then the collection* $\mathscr{P}_\Delta$ *as defined in Definition 5.1.10 partitions* Δ*, is countable and consists of* $\mathscr{F}_\Delta$*-measurable sets.*

Proof. To start, the countability of $\mathscr{P}_\Delta$ follows from countability of $\mathscr{P}_\Lambda$ and $\mathbb{Z}_{\geq 0}$. We proceed with the proof.

(1: $\mathscr{P}_\Delta$ covers Δ) Let $(\omega, x, l) \in \Delta$. This implies $(\sigma^{-l}\omega, x) \in \Lambda$ and $R(\sigma^{-l}\omega, x) > l$ so that we may fix the unique $P \in \mathscr{P}_\Lambda$ with $(\sigma^{-l}\omega, x) \in P$. We note $(\sigma^{-l}\omega, x) \in P \cap R_{>l}$ so $(\omega, x) \in \sigma^l_\Omega(P \cap R_{>l})$ and hence $(\omega, x, l) \in \sigma^l_\Omega(P \cap R_{>l}) \times \{l\}$. We then have by definition $\sigma^l_\Omega(P \cap R_{>l}) \times \{l\} \in \mathscr{P}_\Delta$.

(2: $\mathscr{P}_\Delta$ consists of disjoint sets) Suppose we have $Q, Q' \in \mathscr{P}_\Delta$, $Q \cap Q' \neq \emptyset$ then there must exist $P, P' \in \mathscr{P}_\Lambda$ and $l, l' \in \mathbb{Z}_{\geq 0}$ such that

$$Q = \sigma_\Omega^l [P \cap R_{>l}] \times \{l\}, \qquad Q' = \sigma_\Omega^{l'} [P' \cap R_{>l'}] \times \{l'\}.$$

Clearly $Q \cap Q \neq \emptyset$ implies $l = l'$ and as σ_Ω^l is a bijection on $\Omega \times X$ we can see $P \cap P' \neq \emptyset$ so $P = P'$ as $P, P' \in \mathscr{P}_\Lambda$. We conclude $Q = Q'$, and so $\mathscr{P}_\Delta$ consists of disjoint elements of Δ

(3: $\mathscr{P}_\Delta \subseteq \mathscr{F}_\Delta$) Finally, in noting that by construction $\mathscr{P}_\Lambda \subseteq \mathscr{F}_\Lambda$ and that for any $l \in \mathbb{Z}_{\geq 0}$ we have $R_{>l} \in \mathscr{F}_\Lambda$ by measurability of R, we can see $P \cap R_{>l} \in \mathscr{F}_\Lambda$. In particular $P \cap R_{>l} \in \mathscr{F}_{\Omega \times X}$ and $\sigma_\Omega^l (P \cap R_{>l}) \in \mathscr{F}_{\Omega \times X}$, by bi-measurabilty of σ_Ω, Lemma 4.2.6. We finally see $\sigma_\Omega^l (P \cap R_{>l}) \times \{l\} \in \mathscr{F}_\Delta$ as $\sigma_\Omega^l (P \cap R_{>l}) \times \{l\} \subseteq \Delta_l \times \{l\} \subseteq \Delta$ and $\sigma_\Omega^l (P \cap R_{>l}) \times \{l\} \in \mathscr{F}_{\Omega \times X \times \mathbb{Z}_{\geq 0}}$.

We have shown the claims in the Lemma. □

To conclude our discussion on principal partitions we show that for $\omega \in \dot{\Omega}$ the collection $\mathscr{P}_{\Delta_\omega}$ is a partition of Δ_ω and give an explicit characterisation of its elements.

Corollary 5.1.12. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta)$ be a random tower. For each $\omega \in \dot{\Omega}$ the collection $\mathscr{P}_{\Delta_\omega}$ is a countable partition of $\mathscr{P}_{\Delta_\omega}$ consisting of $\mathscr{F}_{\Delta_\omega}$-measurable. Moreover, we have*

$$\mathscr{P}_{\Delta_\omega} = \left\{ P_{\sigma^{-l}\omega} \times \{l\} \in \mathscr{F}_{\Delta_\omega} : R_{\sigma^{-l}\omega}|_{P_{\sigma^{-l}\omega}} > l, P \in \mathscr{P}_\Lambda \right\}. \tag{157}$$

Proof. Let $\omega \in \dot{\Omega}$. Lemma 5.1.11 and Lemma 4.2.5 together imply that $\mathscr{P}_{\Delta_\omega} \subseteq \mathscr{F}_{\Delta_\omega}$, that $\mathscr{P}_{\Delta_\omega}$ partitions Δ_ω and that it consist of countably many sets.

To prove Equation (157), we first derive that for any $P \in \mathscr{P}_\Lambda$ and $l \in \mathbb{Z}_{\geq 0}$ we have that $Q = \sigma_\Omega^l (P \cap R_{>l}) \times \{l\}$ satisfies

$$Q_\omega = P_{\sigma^{-l}\omega} \times \{l\} \quad \text{with} \quad R_{\sigma^{-l}\omega}|_{P_{\sigma^{-l}\omega}} > l \quad \text{if } Q_\omega \neq \emptyset. \tag{158}$$

To do so, let $P \in \mathscr{P}_\Lambda$, and $l \in \mathbb{Z}_{\geq 0}$ and note that for $Q = \sigma_\Omega^l (P \cap R_{>l}) \times \{l\}$ we have

$$\begin{aligned} Q &= \{(\sigma_\Omega^l(\rho, x), l) \in \Omega \times X \times \{l\} : (\rho, x) \in P \cap R_{>l}\} \\ &= \{(\rho, x, l) \in \Omega \times X \times \{l\} : (\sigma^{-l}\rho, x) \in P \cap R_{>l}\} \\ &= \{(\rho, x, l) \in \Omega \times X \times \{l\} : x \in P_{\sigma^{-l}\rho} \cap (R_{>l})_{\sigma^{-l}\rho}\}, \end{aligned}$$

so that we see $Q_\omega = P_{\sigma^{-l}\omega} \cap (R_{>l})_{\sigma^{-l}\omega} \times \{l\}$. Note that as $R_{\sigma^{-l}\omega}|_{P_{\sigma^{-l}\omega}}$ is constant, we have either

$$Q_\omega = P_{\sigma^{-l}\omega} \cap (R_{>l})_{\sigma^{-l}\omega} = \emptyset \quad \text{or} \quad P_{\sigma^{-l}\omega} \subseteq (R_{>l})_{\sigma^{-l}\omega}.$$

So if $Q_\omega \neq \emptyset$, we see $P_{\sigma^{-l}\omega} \cap (R_{>l})_{\sigma^{-l}\omega} \times \{l\} = P_{\sigma^{-l}\omega} \times \{l\}$ proving Equation (158).

Now to show Equation (157), note for $Q \in \mathscr{P}_\Delta$ we have a $P \in \mathscr{P}_\Lambda$ and $l \in \mathbb{Z}_{\geq 0}$ such that $Q = \sigma^l_\Omega(P \cap R_{>l}) \times \{l\}$. We then see that

$$\begin{aligned} \mathscr{P}_{\Delta_\omega} &= \{Q_\omega \in \mathscr{F}_{X \times \mathbb{Z}_{\geq 0}} : Q_\omega \neq \emptyset, Q \in \mathscr{P}_\Delta\} && \text{by Lemma 4.2.5} \\ &= \{P_{\sigma^{-l}\omega} \times \{l\} \in \mathscr{F}_{\Delta_\omega} : R_{\sigma^{-l}\omega}|_{P_{\sigma^{-l}\omega}} > l, P \in \mathscr{P}_\Lambda\}, && \text{by Equation (158)} \end{aligned}$$

proving our claim. □

We postpone any measure theoretical properties of the random principal partition until the next section and focus on its 'mechanical' nature first. We point out once more that in Remark 4.2.4 we have defined

$$G^n_\omega = G_{\sigma^{n-1}\omega} \circ \cdots \circ G_\omega, \quad \text{for } n \in \mathbb{Z}_{\geq 1}.$$

Adding to this we shall fix $G^0_\omega := \mathrm{Id}|_{\Delta_\omega}$.

We now formalise the embedding that was hinted at in Remark 5.1.6. Its use will become clear in the sections ahead and is also a good way to get acquainted further with 'section' notation.

Lemma 5.1.13. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system and let $\omega \in \dot{\Omega}$, $l \in \mathbb{Z}_{\geq 0}$ such that $\Delta_{\omega,l} \neq \emptyset$. Then we have for each $l' \in \{0,...,l\}$ that $\Delta_{\omega,l} \times \{l'\} \subseteq \Delta_{\sigma^{-(l-l')}\omega,l'} \times \{l'\}$ and the mapping*

$$\begin{aligned} G^{l-l'}_{\sigma^{-(l-l')}\omega}|_{\Delta_{\omega,l}\times\{l'\}} : \Delta_{\omega,l} \times \{l'\} &\to \Delta_{\omega,l} \times \{l\} \\ (x,l') &\mapsto (x,l), \end{aligned} \tag{159}$$

is bijective. Lastly, $(G^{l-l'}_{\sigma^{-(l-l')}\omega})^{-1}(\Delta_{\omega,l} \times \{l\}) = \Delta_{\omega,l} \times \{l'\}$.

Proof. The case $l' = l$ is trivial so assume $l > 0$ and $l' \in \{0,\dots,l-1\}$. Note that

$$\begin{aligned} \Delta_{\omega,l} \times \{l'\} &= (R_{>l})_{\sigma^{-l}\omega} \times \{l'\} \\ &= (R_{>l})_{\sigma^{-l'}(\sigma^{-l-l'}\omega)} \times \{l'\} \\ &\subseteq (R_{>l'})_{\sigma^{-l'}(\sigma^{-l-l'}\omega)} \times \{l'\} \\ &= \Delta_{\sigma^{-l+l'}\omega,l'} \times \{l'\}, \end{aligned}$$

proving our first claim. Having shown this, we can indeed write

$$G^{l-l'}_{\sigma^{-(l-l')}\omega}|_{\Delta_{\omega,l}\times\{l'\}} : \Delta_{\omega,l} \times \{l'\} \to \Delta_\omega.$$

We have to show injectivity on $\Delta_{\omega,l} \times \{l'\}$. To do so, note that for any $l'' \in \{l',\dots,l\}$ we have

$$R_{\sigma^{-(l-l'')}(\sigma^{-l''}\omega)}(x) = R_{\sigma^{-l}\omega}(x) > l \quad \text{for any} \quad x \in \Delta_{\omega,l},$$

so that $G_{\sigma^{-(l-l'')}\omega}(x,l'') = (x,l''+1)$ for any $x \in \Delta_{\omega,l}$. Having shown this we can conclude

$$G^{l-l'}_{\sigma^{-(l-l')}\omega}(x,l') = G_{\sigma^{l-1}\omega} \circ \cdots \circ G_{\sigma^{-(l-l')}\omega}(x,l') = (x,l),$$

so $G^{l-l'}_{\sigma^{-(l-l')}\omega}|_{\Delta_{\omega,l}\times\{l'\}}$ is injective. Moreover, we see $G^{l-l'}_{\sigma^{-(l-l')}\omega}(\Delta_{\omega,l}\times\{l'\}) = \Delta_{\omega,l}\times\{l\}$.

To show $(G^{l-l'}_{\sigma^{-(l-l')}\omega})^{-1}(\Delta_{\omega,l}\times\{l\}) = \Delta_{\omega,l}\times\{l'\}$, we derive

$$\begin{aligned}(G^{l-l'}_{\sigma^{-(l-l')}\omega})^{-1}(\Delta_{\omega,l}\times\{l\}) &= \{(x,l'') \in \Delta_{\sigma^{l'-l}\omega} : G^{l-l'}_{\sigma^{l'-l}\omega}(x,l'') \in \Delta_{\omega,l}\times\{l\}\}\\ &= \{(x,l') \in \Delta_{\sigma^{l'-l}\omega,l'}\times\{l'\} : G^{l-l'}_{\sigma^{l'-l}\omega}(x,l') \in \Delta_{\omega,l}\times\{l\}\}\\ &= \Delta_{\omega,l}\times\{l'\}.\end{aligned}$$

□

In the following sections we shall be most interested in the special case of Lemma 5.1.13 where we have an embedding between a (general) floor and a subset of the ground floor which we phrase in Corollary 5.1.14 below. We also show this embedding has nice measure theoretical-properties.

Corollary 5.1.14. *Let $(\Delta,\mathscr{F}_\Delta,\mu_\Delta,G)$ be a random tower system and let $\omega \in \dot\Omega$, $l \in \mathbb{Z}_{\geq 0}$ such that $\Delta_{\omega,l} \neq \emptyset$. Then the mapping*

$$\begin{aligned}G^l_{\sigma^{-l}\omega}|_{\Delta_{\omega,l}\times\{0\}} : \Delta_{\omega,l}\times\{0\} &\to \Delta_{\omega,l}\times\{l\} \\ (x,0) &\mapsto (x,l),\end{aligned} \tag{160}$$

satisfies $(G^l_{\sigma^{-l}\omega}|_{\Delta_{\omega,l}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-l}\omega}} = \mu_{\Delta_{\omega,l}\times\{l\}}$ and is bi-measurable with invariant inverse.

Proof. In noting $G^l_{\sigma^{-l}\omega}|_{\Delta_{\omega,l}\times\{0\}} = \mathrm{Id}|_{\Delta_{\omega,l}}\times t_l$ by Lemma 5.1.13 and that $\mu_{\Delta_{\omega,l}}(\Delta_{\omega,l}) < \infty$ our claim is immediate by Lemma 4.3.18. □

5.2 Measure-Regularity of random towers

As said in Section 5, a difficulty with analysing random towers Δ is that we do not know if for elements $A \in \mathscr{P}_\Delta$ the mapping $G|_A : A \to G(A)$ has a measurable inverse, making analysing $\frac{dG_\star\mu_\Delta}{d\mu_\Delta}$ through Jacobians as in the deterministic case harder. Additionally, mimicking the deterministic case and defining some sort of *random tower base* $(\Delta_0,\mathscr{F}_{\Delta_0},\mu_{\Delta_0},G^R)$ appears to be fruitless as well as there does not seem to be a clear identification of the densities associated with $(G^R)_\star\mu_{\Delta_0}$. For this reason, we, like the papers [26], [2], [4], [7] before us, will continue the work on the sections of the random tower. That is, we fix an $\omega \in \dot\Omega$ (with $\dot\Omega$ as in Remark 5.1.5) and use the local behaviour of G_ω to study the global behaviour of G. As said in Section 5.1, we do need to assume some extra regularity on the random tower System in order to apply the theory from Section 4.3. This section is aimed at showing the sections of random tower maps are pbn-singular and locally invertible allowing us to analyse its measure-theoretical properties through refined principal partitions and Jacobians. Moreover, this will mean the (sections) of the random tower map are forward measurable - a property fundamental to almost all of our proofs.

Definition 5.2.1. A random tower system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is *measure-regular* if we have for all $\omega \in \dot{\Omega}$ and all $A \in \mathscr{P}_\Delta$ with $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$ that

$$G_\omega(A_\omega) \in \mathscr{F}_{\Delta_{\sigma\omega}}, \text{ that } G_\omega|_{A_\omega} : A_\omega \to G_\omega(A_\omega) \text{ is bi-measurable,}$$

and we have

$$(G_\omega|_{A_\omega}^{-1})_\star \mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega}.$$

From hereon out we shall assume every random tower system is measure-regular.

We now show the existence of the Jacobian for sections of the random tower map. We remind the reader that we have already obtained non-singularity on the Ω-sections of random tower systems almost surely in Proposition 5.1.9.

Lemma 5.2.2. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. Then for every $\omega \in \dot{\Omega}$ the map $G_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ is locally invertible with $\mathscr{P}_{\Delta_\omega}$ partitioning Δ_ω into invertibility domains for G_ω. Moreover, for each $\omega \in \dot{\Omega}$ the mapping G_ω is pbn-singular so that there exists a Jacobian $JG_\omega : \Delta_\omega \to [0,\infty)$.*

Proof. Let $A_\omega \in \mathscr{P}_{\Delta_\omega}$. We prove A_ω is an invertibility domain and do so by distinguishing two cases. If $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$ we have the bi-measurability of $G_\omega|_{A_\omega} : A_\omega \to G_\omega(A_\omega)$ by measure-regularity, so A_ω is an invertibility domain (see Definition 4.3.1). Moreover, again by measure-regularity, we have $(G_\omega|_{A_\omega}^{-1})_\star \mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega}$. Ancillary, we have as $A_\omega \in \mathscr{P}_{\Delta_\omega}$, with $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$ that $G_\omega(A_\omega) \subseteq \Delta_{\sigma\omega,0} \times \{0\}$.

Now suppose $A_\omega \in \mathscr{P}_{\Delta_\omega}$ satisfies $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) = \emptyset$. By Corollary 5.1.12 we then know there exists $l \in \mathbb{Z}_{\geq 0}$ and $P \in \mathscr{P}_\Lambda$ such that $A_\omega = P_{\sigma^{-l}\omega} \times \{l\} \subseteq R_{>l,\sigma^{-l}\omega} \times \{l\}$. As $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) = \emptyset$ it follows from the definition of the tower map that $A_{\omega,l} = P_{\sigma^{-l}\omega} \subseteq R_{>l+1,\sigma^{-l}\omega}$. In particular, we have

$$\begin{aligned} G_\omega|_{A_\omega} : A_{\omega,l} \times \{l\} &\to A_{\omega,l} \times \{l+1\} \\ (x,l) &\mapsto (x, l+1), \end{aligned} \tag{161}$$

which is bi-measurable and satisfies $(G_\omega|_{A_\omega}^{-1})_\star \mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega}$ as well. Finally,

$$\begin{aligned} G_\omega(A_\omega) &= P_{\sigma^{-l}\omega} \times \{l+1\} \\ &= P_{\sigma^{-l-1}(\sigma\omega)} \times \{l+1\} \\ &\in \mathscr{P}_{\Delta_{\sigma\omega}}, \end{aligned}$$

and $G_\omega(A_\omega) \in \mathscr{F}_{\Delta_{\sigma\omega}}$ by Corollary 5.1.12. We conclude the partition $\mathscr{P}_{\Delta_\omega}$ consists of invertibility domains making G_ω locally invertible. Moreover, we have shown

$$(G_\omega|_{A_\omega}^{-1})_\star \mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega} \quad \text{for each} \quad A_\omega \in \mathscr{P}_{\Delta_\omega}.$$

This together with $\mu_{\Delta_\omega}(\Delta_\omega) \in [1,\infty)$ as $\omega \in \dot{\Omega}$ implies pbn-singularity of G_ω by Lemma 4.3.6. As $G_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ for $\omega \in \dot{\Omega}$ is measurable, locally invertible and pbn-singular between two finite measure spaces, we have by Lemma 4.3.8 that the Jacobian JG_ω exists. □

In the corollary below we have phrased ancillary results following directly from the proof of Lemma 5.2.2. We note the mapping in Equation (161) has a Jacobian that is constant by Lemma 4.3.18.

Corollary 5.2.3. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. Let $\omega \in \dot{\Omega}$, $A_\omega \in \mathscr{P}_{\Delta_\omega}$. We have either*

$$G_\omega(A_\omega) \subseteq \Delta_{\sigma\omega,0} \times \{0\}$$

or

$$G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) = \emptyset \text{ and } G_\omega(A_\omega) \in \mathscr{P}_{\Delta_{\sigma\omega}}.$$

In case of the latter, we have additionally

$$JG_\omega \equiv 1, \qquad \mu_{A_\omega}\text{-almost surely.} \tag{162}$$

The rest of this section is dedicated to deriving the chain rule of the Ω-sections of a random tower map. To this end, it is natural to refine our principal partition. We shall write for $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$,

$$\mathscr{P}_\Delta^n := \bigvee_{i=0}^{n-1} G^{-i} \mathscr{P}_\Delta \quad \text{and} \quad \mathscr{P}_{\Delta_\omega}^n = \bigvee_{i=0}^{n-1} G_\omega^{-i} \mathscr{P}_{\Delta_{\sigma^i\omega}}. \tag{163}$$

We derive the existence of the Jacobian for iterated the tower maps.

Lemma 5.2.4. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system and $n \in \mathbb{Z}_{\geq 1}$. Then for every $\omega \in \dot{\Omega}$ the mapping G_ω^n is pbn-singular and $\mathscr{P}_{\Delta_\omega}^n$ partitions Δ_ω into invertibility domains for $G_\omega^n : \Delta_\omega \to \Delta_{\sigma^n\omega}$. Moreover, the Jacobian JG_ω^n exists and is given by*

$$J(G_\omega^n) = \prod_{i=0}^{n-1} (JG_{\sigma^i\omega}) \circ G_\omega^i \quad \text{for } \mu_{\Delta_\omega}\text{-almost every } x \in \Delta_\omega. \tag{164}$$

Proof. In Lemma 5.2.2 we showed that for $\omega \in \dot{\Omega}$ the partition $\mathscr{P}_{\Delta_\omega}$ consists of invertibility domains for G_ω and that G_ω is pbn-singular. We can see that for $\omega \in \dot{\Omega}$, we can apply Corollary 4.3.13 to see that $G_\omega^n : \Delta_\omega \to \Delta_{\sigma^n\omega}$ is pbn-singular and that $\mathscr{P}_{\Delta_\omega}^n$ partitions Δ_ω into invertibility domains for G_ω^n. The existence of JG_ω^n and its characterisation in Equation (164) then follow from Proposition 4.3.16. □

The following Corollary is obvious by Lemma 5.2.4 and Lemma 4.3.3. We have included it separately to make sure the reader is left without questions on measurability.

Corollary 5.2.5. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system and* $n \in \mathbb{Z}_{\geq 1}$. *Then for every* $\omega \in \dot{\Omega}$ *and every* $A_\omega \in \mathscr{F}_{\Delta_\omega}$ *we have* $G_\omega^n(A_\omega) \in \mathscr{F}_{\Delta_{\sigma^n\omega}}$.

In the next section we will use Equation (164) to prove that $J(G_\omega^n)$ is positive μ_{Δ_ω}-almost surely for $\omega \in \dot{\Omega}$, assuming an extra condition.

5.3 The Markov Property and Bounded Distortion for random towers

In this section we shall continue building the theory of Random Young Towers and to do so we define random equivalent of the Markov Property (3.2.3), Separation Time (5.3.9) and Bounded Distortion (5.3.12), cumulating in the notion of *(acip) admissibility* at the end of this section. Our Definitions, 2.1.27 and 3.2.6 are consistent with [2] with the exception that we have placed the positivity of the Jacobian under the Markov Property as seen in Definition 5.3.1. This way we can derive the reciprocal identity in Lemma 5.3.7 before needing to introduce the bounded distortion for random towers.

We briefly highlight the n'th-Markov Collections as defined in Definition 5.3.2. This collection contains the sets that for a given $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$ map onto a ground floor under G_ω^n, as can be seen in Corollary 5.3.4. The Markov property will ensure this happens bi-measurably with a positive Jacobian. These sets will form the backbone of the arguments in Section 5.4. We still assume measure-regularity on random tower systems throughout this section.

Definition 5.3.1. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. If for each $\omega \in \dot{\Omega}$ and $A \in \mathscr{P}_\Delta$ with $G_\omega A_\omega \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$, the mapping $G_\omega|_{A_\omega}$ is bi-measurable onto $\Delta_{\sigma\omega,0} \times \{0\}$ with $JG_\omega|_{A_\omega} > 0$, μ_{A_ω}-almost surely, we say $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ satisfies the *(Random) Markov Property.*

Additionally to measure-regularity, for the rest of this section we shall assume all random tower Systems satisfy the Markov Property.

Definition 5.3.2. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system, $n \in \mathbb{Z}_{\geq 0}$ and $\omega \in \dot{\Omega}$. We then say

$$\mathscr{R}_\omega^n := \left\{ A_\omega \in \mathscr{P}_{\Delta_\omega}^n : A_\omega \subseteq \Delta_{\omega,0} \times \{0\}, \, G_\omega^n(A_\omega) \cap (\Delta_{\sigma^n\omega,0} \times \{0\}) \neq \emptyset \right\}$$

is the *n'th Markov collection with respect to* ω. We call elements of $\mathscr{R}_\omega^n$ *Markov sets.*

We present two consequences of the Markov Property.

Lemma 5.3.3. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system,* $n \in \mathbb{Z}_{\geq 1}, \omega \in \dot{\Omega}$. *Then for each* $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$ *we have either*

1. $G^n_\omega(A_\omega) \in \mathscr{P}_{\Delta_{\sigma^n\omega}}$ *with* $G^n_\omega(A_\omega) \cap (\Delta_{\sigma^n\omega,0} \times \{0\}) = \emptyset$ *or*

2. $G^n_\omega(A_\omega) = \Delta_{\sigma^n\omega,0} \times \{0\}$.

Proof. We prove by induction. Note for $n = 1$ our claims follow from Corollary 5.2.3 combined with the Markov Property. Now suppose that for some $p \in \mathbb{Z}_{\geq 1}$ we have for each $B_\omega \in \mathscr{P}^p_{\Delta_\omega}$, that either

1. $G^p_\omega[B_\omega] \in \mathscr{P}_{\Delta_{\sigma^p\omega}}$ with $G^p_\omega[B_\omega] \cap \Delta_{\sigma^p\omega,0} \times \{0\} = \emptyset$ or

2. $G^p_\omega[B_\omega] = \Delta_{\sigma^p\omega,0} \times \{0\}$.

Then note that for any $A_\omega \in \mathscr{P}^{p+1}_{\Delta_\omega}$ we have a $C_{\sigma^p\omega} \in \mathscr{P}_{\Delta_{\sigma^p\omega}}$ and a $B_\omega \in \mathscr{P}^p_{\Delta_\omega}$ such that $A_\omega = B_\omega \cap G^{-p}_\omega(C_{\sigma^p\omega})$. Using the induction hypothesis we then see

$$\begin{aligned}
G^p_\omega(A_\omega) &= G^p_\omega(B_\omega \cap G^{-p}_\omega C_{\sigma^p\omega}) \\
&= G^p_\omega(B_\omega) \cap C_{\sigma^p\omega} && \text{using Lemma 2.1.23} \\
&= \begin{cases} C_{\sigma^p\omega} & \text{if } G^p_\omega(B_\omega) \in \mathscr{P}_{\Delta_{\sigma^p\omega}}, \quad \text{(as overlapping elements from the same partition)} \\ C_{\sigma^p\omega} & \text{if } G^p_\omega(B_\omega) = \Delta_{\sigma^p\omega,0} \times \{0\} \quad \text{(as then necessarily } C_{\sigma^p\omega} \subseteq \Delta_{\sigma^p\omega,0} \times \{0\}) \end{cases} \\
&= C_{\sigma^p\omega}.
\end{aligned}$$

Hence $G^{p+1}_\omega(A_\omega) = G_{\sigma^p\omega}(C_{\sigma^p\omega})$. As $C_{\sigma^p\omega} \in \mathscr{P}_{\Delta_{\sigma^p\omega}}$ the claim follows from Corollary 5.2.3 applied with the Markov Property/ □

Directly from Lemma 5.3.3 we obtain a nice characterisation of the Markov Collections.

Corollary 5.3.4. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system,* $n \in \mathbb{Z}_{\geq 1}$ *and* $\omega \in \dot{\Omega}$. *We then have*

$$\mathscr{R}^n_\omega = \left\{ A_\omega \in \mathscr{P}^n_{\Delta_\omega} : A_\omega \subseteq \Delta_{\omega,0} \times \{0\},\, G^n_\omega|_{A_\omega} : A_\omega \to \Delta_{\sigma^n\omega,0} \times \{0\} \text{ is bi-measurable} \right\}, \tag{165}$$

moreover, in writing $R^n_\omega := \bigsqcup_{R \in \mathscr{R}^n_\omega} R$ *we have*

$$(G^n_\omega|_{\Delta_{\omega,0}\times\{0\}})^{-1}(\Delta_{\sigma^n\omega,0} \times \{0\}) = R^n_\omega. \tag{166}$$

Proof. If $\mathscr{R}^n_\omega = \emptyset$, there is nothing to prove. If $\mathscr{R}^n_\omega \neq \emptyset$, let $A_\omega \in \mathscr{R}^n_\omega$ so that $G^n_\omega(A_\omega) \cap (\Delta_{\sigma^n\omega,0} \times \{0\}) \neq \emptyset$. Lemma 5.3.3 then shows $G^n_\omega(A_\omega) = \Delta_{\sigma^n\omega,0} \times \{0\}$ and bi-measurability of $G^n_\omega|_{A_\omega}$ follows directly from Lemma 5.2.4.

Now to prove Equation (166), note that $(G^n_\omega|_{\Delta_{\omega,0}\times\{0\}})^{-1}(\Delta_{\sigma^n\omega,0} \times \{0\}) \supseteq R^n_\omega$ follows by Characterisation (165). Conversely, note that

$$\begin{aligned}(G^n_\omega|_{\Delta_{\omega,0}\times\{0\}})^{-1}(\Delta_{\sigma^n\omega,0} \times \{0\}) &= \{(x,0) \in \Delta_{\omega,0} \times \{0\} : G^n_\omega(x,0) \in \Delta_{\sigma^n\omega,0} \times \{0\}\} \\ &= \bigsqcup_{A_\omega \in \mathscr{P}^n_{\Delta_\omega}, A_\omega \subseteq \Delta_{\omega,0}\times\{0\}} (G^n_\omega|_{A_\omega})^{-1}(\Delta_{\sigma^n\omega,0} \times \{0\}) \\ &\subseteq R^n_\omega,\end{aligned}$$

so that we may claim Equation (166). □

Remark 5.3.5. In Figure 9 on the next page, we have, as a visual aid for the theory, pictured how an element of a refined principal partition can behave under the application of the random tower map. No new concepts will be explained and thus this remark can be skipped. Suppose, given a random tower system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ and $\omega \in \dot{\Omega}$ we have a $A_{\sigma^{-5}\omega} \in \mathscr{R}^6_{\sigma^{-5}\omega}$, such that $R_{\sigma^{-5}\omega}|_{A_{\sigma^{-5}\omega,0}} \equiv 2$ and $R_{\sigma^{-3}\omega}|_{(G^2_{\sigma^{-5}\omega}(A_{\sigma^{-5}\omega}))_0} \equiv 4$. Due to the Markov Property we then have $g^6_{\sigma^{-5}\omega}(A_{\sigma^{-5}\omega}) \times \{0\} = \Delta_{\sigma\omega,0} \times \{0\}$ and by Lemma 5.2.4 we know

$$G^6_{\sigma^{-5}\omega} : A_{\sigma^{-5}\omega} \to \Delta_{\sigma\omega,0} \times \{0\},$$

is bi-measurable. In Figure 9 the straight arrows display applications of $G_{\sigma^i\omega} : \Delta_{\sigma^i\omega} \to \Delta_{\sigma^{i+1}\omega}$ to $G^{5-i}_{\sigma^i\omega}(A_{\sigma^{-5}\omega})$ for $i \in \{-5, \dots, 0\}$ which we have rewritten using the definition of G_ω. The curved arrows below the figure display the (again bi-measurable) mappings

$$G^2_{\sigma^{-5}\omega} : A_{\sigma^{-5}\omega} \to G^2_{\sigma^{-5}\omega}(A_{\sigma^{-5}\omega}), \quad G^6_{\sigma^{-5}\omega} : A_{\sigma^{-5}\omega} \to \Delta_{\sigma\omega,0} \times \{0\},$$

and

$$G^4_{\sigma^{-5}\omega} : G^2_{\sigma^{-5}\omega}(A_{\sigma^{-5}\omega}) \to \Delta_{\sigma\omega,0} \times \{0\},$$

from left to right as the labels appear in the figure. These we have also rewritten using the definition of G_ω.

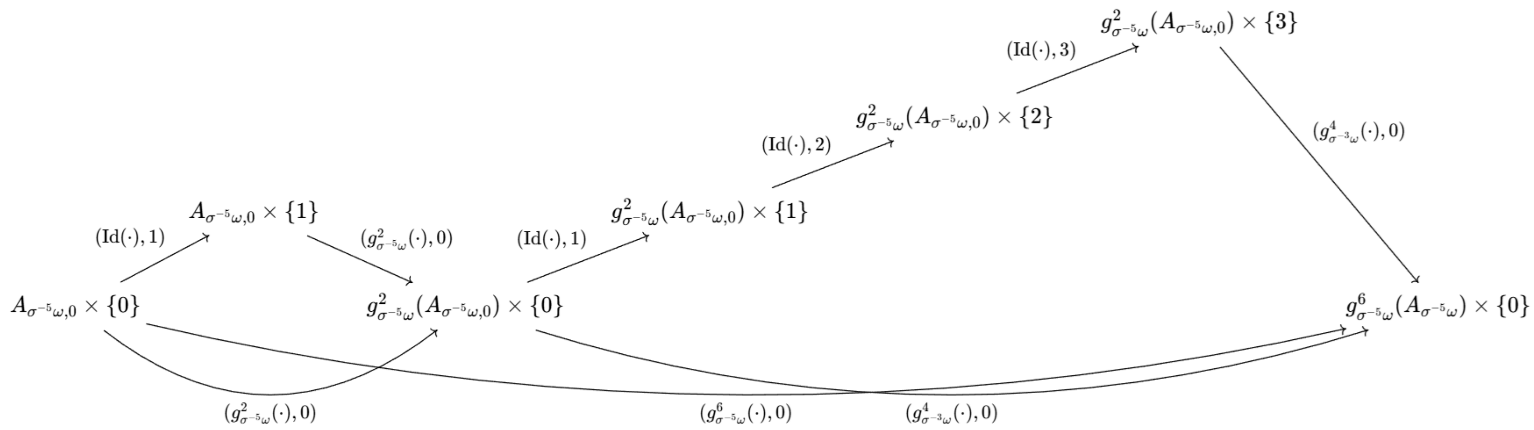


Figure 9: The tower map applied to a markov set

We now start analysing densities associated with random tower maps. First we prove the positivity (almost everywhere) of the iterated Jacobian in Lemma 5.3.6 below, before proving a reciprocal identity in Lemma 5.3.7. Eyeing the Markov Property, Corollary 5.2.3 and the chain rule in Lemma 5.2.4, the result in Lemma 5.3.6 should come to no surprise.

Lemma 5.3.6. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. Then for $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$, we have that $JG_\omega^n > 0$, μ_{Δ_ω}-almost surely.*

Proof. Let $\omega \in \dot{\Omega}$ and $n \in \mathbb{Z}_{\geq 1}$. By Lemma 5.2.4 we may assume the Jacobians JG_ω^n and $J(G_{\sigma^i\omega})$ exist. Note that by Lemma 5.2.4 we can find a set $\Delta'_\omega \subseteq \Delta_\omega$ such that $\mu_{\Delta_\omega}(\Delta_\omega \setminus \Delta'_\omega) = 0$ for which we have

$$JG_\omega^n(x) = \prod_{i=0}^{n-1} (JG_{\sigma^i\omega}) \circ G_\omega^i(x) \text{ for each } x \in \Delta'_\omega.$$

Claim: for each $i \in \{0, \dots, n-1\}$ there exists a set $\dot{\Delta}_{\sigma^i\omega} \in \mathscr{F}_{\Delta_{\sigma^i\omega}}$ for which we have

$$\mu_{\Delta_{\sigma^i\omega}}(\Delta_{\sigma^i\omega} \setminus \dot{\Delta}_{\sigma^i\omega}) = 0 \textit{ and } JG|_{\dot{\Delta}_{\sigma^i\omega}} > 0. \tag{167}$$

Let $i \in \{0, \dots n-1\}$ be arbitrary. By Lemma 5.3.3 we have for any $A_{\sigma^i\omega} \in \mathscr{P}_{\Delta_{\sigma^i\omega}}$ either

$$G_{\sigma^i\omega}(A_{\sigma^i\omega}) \in \mathscr{P}_{\Delta_{\sigma^{i+1}\omega}} \text{ and } G_{\sigma^i\omega}(A_{\sigma^i\omega}) \cap (\Delta_{\sigma^{i+1}\omega,0} \times \{0\}) = \emptyset,$$

or

$$G_{\sigma^i\omega}(A_{\sigma^i\omega}) = (\Delta_{\sigma^{i+1}\omega,0} \times \{0\}).$$

In both cases, using Corollary 5.2.3 or the Markov Property respectively, we find for each $A_{\sigma^i\omega} \in \mathscr{P}_{\Delta_{\sigma^i\omega}}$ a set $\dot{A}_{\sigma^i\omega} \in \mathscr{F}_{\Delta_{\sigma^i\omega}}$ such that

$$\mu_{A_{\sigma^i\omega}}(A_{\sigma^i\omega} \setminus \dot{A}_{\sigma^i\omega}) = 0 \text{ and } JG_{\sigma^i\omega}|_{\dot{A}_{\sigma^i\omega}} > 0.$$

Write

$$\dot{\Delta}_{\sigma^i\omega} = \bigsqcup_{A_{\sigma^i\omega} \in \mathscr{P}_{\Delta_{\sigma^i\omega}}} \dot{A}_{\sigma^i\omega},$$

and note $\mu_{\Delta_{\sigma^i\omega}}(\Delta_{\sigma^i\omega} \setminus \dot{\Delta}_{\sigma^i\omega}) = 0$ and $JG_{\sigma^i\omega}|_{\dot{\Delta}_{\sigma^i\omega}} > 0$, showing our claim.

Now using our claim, construct for each $i \in \{0, \dots, n-1\}$ a set $\dot{\Delta}_{\sigma^i\omega}$ satisfying Equation (167). Note that we have

$$\mu_{\Delta_\omega}(\Delta_\omega \setminus (G_\omega^i)^{-1}\dot{\Delta}_{\sigma^i\omega}) = (G_\omega^i)_\star \mu_{\Delta_\omega}(\Delta_{\sigma^i\omega} \setminus \dot{\Delta}_{\sigma^i\omega}) = 0 \text{ as } (G_\omega^i)_\star \mu_{\Delta_\omega} \ll \mu_{\Delta_{\sigma^i\omega}}.$$

Then note that

$$\ddot{\Delta}_\omega := \dot{\Delta}_\omega \cap \dots \cap (G_\omega^{n-1})^{-1}\dot{\Delta}_{\sigma^{n-1}\omega},$$

satisfies $\mu_{\Delta_\omega}(\Delta_\omega \setminus \ddot{\Delta}_\omega) = 0$ and that for each $i \in \{0,\dots,n-1\}$ we have $G^i_\omega(\ddot{\Delta}_\omega) \subseteq \dot{\Delta}_{\sigma^i\omega}$ by Lemma 2.1.23 and that $G^i_\omega(\ddot{\Delta}_\omega) \in \mathscr{F}_{\Delta_{\sigma^i\omega}}$ by Corollary 5.2.5. Consequently, for each $x \in \ddot{\Delta}_\omega \cap \Delta'_\omega$ we have

$$JG^n_\omega(x) = \prod_{i=0}^{n-1}(JG_{\sigma^i\omega}) \circ G^i_\omega(x) > 0,$$

and $\mu_{\Delta_\omega}(\ddot{\Delta}_\omega \cap \Delta'_\omega) = \mu_{\Delta_\omega}(\Delta_\omega)$, proving our statement. □

We now use the positivity found in Lemma 5.3.6, to express the Radon-Nikodym derivative in terms of the Jacobian with the following reciprocal identity. Using this instead of directly using Lemma 4.3.11 will save us some tedious calculations at the end of Lemma 5.4.2, which might be the most tedious proof in this thesis.

Lemma 5.3.7. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. Then we have for $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$ and any $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$ that,*

$$JG^n_\omega((G^n_\omega|_{A_\omega})^{-1}(\cdot))^{-1} = \frac{d(G^n_\omega|_{A_\omega})_\star\mu_{\Delta_\omega}}{d\mu_{\Delta_{\sigma^n\omega}}}(\cdot), \text{ holds } \mu_{G^n_\omega(A_\omega)}\text{-almost surely.}$$

Proof. Let $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$ and $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$. Note that Lemma 5.3.6 implies that $JG^n_\omega > 0$, μ_{Δ_ω}-almost surely, so in particular $JG^n_\omega > 0$ μ_{A_ω}-almost surely. Lemma 5.2.4 then allow us to apply Lemma 4.3.11 and obtain a set $\dot{A}_\omega \in \mathscr{F}_{\Delta_\omega}$ such that $\mu_{\Delta_\omega}(A_\omega \setminus \dot{A}_\omega) = 0$ and

$$JG^n_\omega(x')^{-1} = \frac{d(G^n_\omega|_{A_\omega})_\star\mu_{\Delta_\omega}}{d\mu_{\Delta_{\sigma^n\omega}}}(G^n_\omega(x')), \text{ for all } x' \in \dot{A}_\omega. \tag{168}$$

Note by Corollary 5.2.5 we have $G^n_\omega(A_\omega), G^n_\omega(\dot{A}_\omega) \in \mathscr{F}_{\Delta_{\sigma^n\omega}}$ and by pbn-singularity of G^n_ω (see Lemma 5.2.4) we then see

$$\mu_{\Delta_{\sigma^n\omega}}(G^n_\omega(A_\omega) \setminus G^n_\omega(\dot{A}_\omega)) \leq \mu_{\Delta_{\sigma^n\omega}}(G^n_\omega(A_\omega \setminus \dot{A}_\omega)) = 0.$$

By bi-measurabilty of $G^n_\omega|_{A_\omega} : A_\omega \to G^n_\omega(A_\omega)$ we can then substitute $x' = (G^n_\omega|_{A_\omega})^{-1}(x)$ in Equation (168), which yields

$$JG^n_\omega((G^n_\omega|_{A_\omega})^{-1}(x))^{-1} = \frac{d(G^n_\omega|_{A_\omega})_\star\mu_{\Delta_\omega}}{d\mu_{\Delta_{\sigma^n\omega}}}(x), \text{ for every } x \in G^n_\omega(\dot{A}_\omega),$$

so that our claim follows. □

For ease of reference, we have rephrased Lemma 5.3.7 for Markov sets using Corollary 5.3.4.

Corollary 5.3.8. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. Then we have for $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$ and any $A_\omega \in \mathscr{R}^n_\omega$ that,*

$$JG^n_\omega((G^n_\omega|_{A_\omega})^{-1}(\cdot))^{-1} = \frac{d(G^n_\omega|_{A_\omega})_\star\mu_{\Delta_\omega}}{d\mu_{\Delta_{\sigma^n\omega}}}(\cdot), \text{ holds } \mu_{\Delta_{\omega,0}\times\{0\}}\text{-almost surely.}$$

Finally, to define the random equivalent to bounded distortion we define the random separation time. Note that under this notion of the separation time, we assume points belonging to different ω-sections of Δ are always separated. Note that in Definition 5.3.9, Lemma 5.3.10 and Definition 5.3.12 we denote for $\omega \in \dot{\Omega}$ elements of $\Delta_\omega \subseteq X \times \mathbb{Z}_{\geq 0}$ as a single symbol x or y.

Definition 5.3.9. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. For $\omega \in \dot{\Omega}$ we define the mapping

$$\alpha_\omega : \Delta_\omega \to \mathscr{P}_{\Delta_\omega}, \qquad x \mapsto A_\omega \quad \text{for the unique} \quad A_\omega \ni x.$$

We then define the *(random) separation time on* Δ as the mapping

$$s : \Delta \times \Delta \to \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

$$\big((\omega, x), ((\omega', y)\big) \mapsto \begin{cases} s_\omega(x, y), & \text{in case } \omega = \omega' \in \dot{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$$

where we define for $\omega \in \dot{\Omega}$,

$$s_\omega : \Delta_\omega \times \Delta_\omega \to \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

$$(x, y) \mapsto \inf\left\{n \in \mathbb{Z}_{\geq 0} : \alpha_{\sigma^n \omega}(G^n_\omega(x)) \neq \alpha_{\sigma^n \omega}(G^n_\omega(y))\right\},$$

Similar to the deterministic case, the random separation time can give rise to a metric, making Δ into a topological space. For the proofs in Section 5.4 this is not necessary however. Instead, we merely state that the tower map is expanding section-wise in Lemma 5.3.10. As the proof is very similar to Lemma 2.1.28 we shall refrain from giving a full proof here.

Lemma 5.3.10 (Expandingness)**.** *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system, let* $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$, $C \in \mathbb{R}_{>0}$ *and* $\beta \in (0, 1)$. *Then for each* $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$ *with* $x', y' \in A_\omega$ *the mapping*

$$\begin{aligned} d_{\beta,C,\omega} : \Delta_\omega \times \Delta_\omega &\to [0, C] \\ (x, y) &\mapsto C\beta^{s_\omega(x,y)}, \end{aligned}$$

satisfies for each $i \in \{1, \ldots, n\}$

$$d_{\beta,C,\sigma^i\omega}(G^i_\omega x', G^i_\omega y') = \beta^{-i} d_{\beta,C,\omega}(x', y')$$

As said, the mapping $d_{\beta,C,\omega}$ in Lemma 5.3.10 above is not a metric unless we assume extra conditions. As such, we shall simply refer to it as *the mapping* $d_{\beta,C,\omega}$. We shall phrase Lemma 5.3.10 conveniently as follows.

Corollary 5.3.11. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system, let* $n \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$, $C \in \mathbb{R}_{>0}$ *and* $\beta \in (0,1)$. *Then for each* $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$ *and* $x', y' \in A_\omega$ *we have*

$$d_{\beta,C,\sigma^n\omega}(G^n_\omega(x'), G^n_\omega(y')) = \beta^{-n+i} d_{\beta,C,\sigma^i\omega}(G^i_\omega(x'), G^i_\omega(y')) \text{ for } i \in \{0,\ldots,n\}. \tag{169}$$

Proof. Let $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$ with $x', y' \in A_\omega$. Then note

$$d_{\beta,C,\sigma^n\omega}(G^n_\omega(x'), G^n_\omega(y')) = \beta^{-n} d_{\beta,C,\omega}(x', y'), \tag{170}$$

by Lemma 5.3.10 and $i \in \{0,\ldots,n\}$ again by Lemma 5.3.10

$$d_{\beta,C,\omega}(x', y') = \beta^{i} d_{\beta,C,\sigma^i\omega}(G^i_\omega(x'), G^i_\omega(y')). \tag{171}$$

Equation (169) then follows from substituting Equation (171) in Equation (170). □

Finally, we phrase the random counterpart of bounded distortion.

Definition 5.3.12. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. We say it satisfies *bounded distortion* if there are constants $C > 1$, $0 < \beta < 1$ such that for $\omega \in \dot{\Omega}$ and each $A_\omega \in \mathscr{P}_{\Delta_\omega}$, we have a set $\dot{A}_\omega \in \mathscr{F}_{A_\omega}$ such that $\mu_{A_\omega}(A_\omega \setminus \dot{A}_\omega) = 0$ and

$$\left| \frac{J(G_\omega)(x)}{J(G_\omega)(y)} - 1 \right| \leq d_{\beta,C,\omega}(x,y), \quad \text{for every } x, y \in \dot{A}_\omega. \tag{172}$$

with $d_{\beta,C,\omega}$ as in Lemma 5.3.10.

We can extend Equation (172) inductively. As the proof relies on methods already shown in Lemma 5.3.6 and Lemma 3.2.7 in particular we have incorporated it without proof.

Lemma 5.3.13. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a random tower system satisfying bounded distortion and let* $n \in \mathbb{Z}_{\geq 1}$. *For each* $\omega \in \dot{\Omega}$ *and each* $A_\omega \in \mathscr{P}^n_{\Delta_\omega}$, *we have a set* $\dot{A}_\omega \in \mathscr{F}_{A_\omega}$ *such that* $\mu_{A_\omega}(A_\omega \setminus \dot{A}_\omega) = 0$ *and for each* $i \in \{0,\ldots,n-1\}$

$$\left| \frac{J(G_{\sigma^i\omega})(G^i_\omega(x))}{J(G_{\sigma^i\omega})(G^i_\omega(y))} - 1 \right| \leq d_{\beta,C,\sigma^i\omega}(G^i_\omega(x), G^i_\omega(y)), \quad \text{for every } x, y \in \dot{A}_\omega. \tag{173}$$

with $d_{\beta,C,\omega}$ *as in Lemma 5.3.10.*

For the readers' convenience we make the following definition, summarising all our conditions on random towers so far.

Definition 5.3.14. Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system. We say it is *(acip) admissible* if the following conditions hold.

Measure-Regularity For all $\omega \in \dot{\Omega}$ and all $A \in \mathscr{P}_\Delta$ with $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$ we have that

$$G_\omega(A_\omega) \in \mathscr{F}_{\Delta_{\sigma\omega}}, \text{ and that } G_\omega|_{A_\omega} : A_\omega \to G_\omega(A_\omega) \text{ is bi-measurable,}$$

and we have

$$(G_\omega|_{A_\omega}^{-1})_\star \mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega}.$$

Markov Property For each $\omega \in \dot{\Omega}$ and $A \in \mathscr{P}_\Delta$ with $G_\omega A_\omega \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$, the mapping $G_\omega|_{A_\omega}$ is bi-measurable onto $\Delta_{\sigma\omega,0} \times \{0\}$ with $JG_\omega|_{A_\omega} > 0$, μ_{A_ω}-almost surely.

Bounded Distortion There exist constants $C > 1$, $0 < \beta < 1$ such that for $\omega \in \dot{\Omega}$ and each $A_\omega \in \mathscr{P}_{\Delta_\omega}$, we have a set $\dot{A}_\omega \in \mathscr{F}_{A_\omega}$ such that $\mu_{A_\omega}(A_\omega \setminus \dot{A}_\omega) = 0$ and

$$\left| \frac{J(G_\omega)(x)}{J(G_\omega)(y)} - 1 \right| \leq d_{\beta,C,\omega}(x,y), \quad \text{for every } x, y \in \dot{A}_\omega. \tag{174}$$

with $d_{\beta,C,\omega}$ as in Lemma 5.3.10.

We say the random dynamical system $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ is *(acip) admissible* if we can use it to construct an admissible random tower system as done in Sections 5.1–5.3

5.4 An Acip In The Quenched Case

In this section we shall assume every random tower system is acip admissible

The main result of this section (and of this thesis) is the Theorem phrased below.

Theorem 5.4.1. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system, then there exists an $M \in \mathbb{R}_{>1}$ and a probability measure ν on Δ such that $\nu \ll \mu_\Delta$, $\frac{d\nu}{d\mu_\Delta} \leq M$, μ_Δ-a.e., and $G_\star \nu = \nu$. That is, there exists an* acip *ν for $(\Delta, \mathscr{F}, \mu_\Delta, G)$.*

As we have already developed all other measure-theoretical and functional analytic machinery, the main goal of the proofs of this section are to show that for a random tower system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$, we have an $M \in \mathbb{R}_{>1}$ such that for all $\omega \in \dot{\Omega}$ and $n \in \mathbb{Z}_{\geq 0}$ we have

$$\frac{d(G^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\sigma^n\omega}}} \leq M, \quad \mu_{\Delta_{\sigma^n\omega}}\text{-almost surely,}$$

which we shall do in Proposition 5.4.6. In evaluating $\frac{d(G^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\sigma^n\omega}}}$, we shall partition its domain $\Delta_{\sigma^n\omega}$ and evaluate $\frac{d(G^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\sigma^n\omega}}}$ on each floor individually .

For the set $\Delta_{\omega,0} \times \{0\}$ we prove an adaptation of Lemma 3.3.4 using Markov sets making us obtain the bound (175) in Lemma 5.4.2. We shall relate this bound to higher floors using Lemma 5.4.5, and combine our knowledge to obtain the promised uniform upper bound in Proposition 5.4.6.

Lemma 5.4.2. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system, let $n \in \mathbb{Z}_{\geq 0}$, $A \in \mathscr{P}^n_\Delta$ and $\omega \in \dot{\Omega}$ such that we have $A_{\sigma^{-n}\omega} \in \mathscr{R}^n_{\sigma^{-n}\omega}$. For μ_{Δ_ω}-almost all $x \in \Delta_{\omega,0} \times \{0\}$, the density*

$$\phi_{A_{\sigma^{-n}\omega}} := \frac{d(G^n_{\sigma^{-n}\omega}|_{A_{\sigma^{-n}\omega}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}} \qquad \textit{satisfies}$$

$$\frac{1}{M}\mu_{\Delta_{\sigma^{-n}\omega}}(A_{\sigma^{-n}\omega}) \leq \phi_{A_{\sigma^{-n}\omega}}(x) \leq M\mu_{\Delta_{\sigma^{-n}\omega}}(A_{\sigma^{-n}\omega}), \tag{175}$$

where $M \in \mathbb{R}_{>1}$ is independent of ω, n, A and x, and

$$\phi_{A_{\sigma^{-n}\omega}}(x) = 0 \quad \textit{for } \mu_{\Delta_\omega}\textit{-almost every } x \in \Delta_\omega \setminus \Delta_{\omega,0} \times \{0\}. \tag{176}$$

Proof. For notational convenience we shall write $\alpha := \sigma^{-n}\omega$. Keeping Lemma 3.3.4 in mind, we start with constructing a set $\dot{A}_\alpha \in \mathscr{F}_{A_\alpha}$ such that $\mu_{A_\alpha}(A_\alpha \setminus \dot{A}_\alpha) = 0$ upon which we can apply the random equivalent of the chain rule in Lemma 5.2.4, bounded distortion as in Lemma 5.3.13 and reciprocal identity of Corollary 5.3.8.

As $A_\alpha \in \mathscr{R}^n_\alpha$ we have by Corollary 5.3.4 that

$$G^n_\alpha|_{A_\alpha} : A_\alpha \to \Delta_{\omega,0} \times \{0\}$$

is bi-measurable. Furthermore, as we know $(G_\alpha)_\star \mu_{\Delta_\alpha} \ll \mu_{\Delta_\omega}$ by Proposition 5.1.9, we can apply Lemma A.1.3 to show $(G^n_\alpha|_{A_\alpha})_\star \mu_{\Delta_\alpha} \ll \mu_{\Delta_\omega}$, and allowing us to make the definition

$$\phi_{A_\alpha} := \frac{d(G^n_\alpha|_{A_\alpha})_\star \mu_{\Delta_\alpha}}{d\mu_{\Delta_\omega}}.$$

Now, using Corollary 5.3.8 we then obtain a set $(\Delta_{\omega,0} \times \{0\})' \in \mathscr{F}_{\Delta_{\omega,0} \times \{0\}}$, so that $\mu_{\Delta_\omega}(\Delta_\omega \times \{0\} \setminus (\Delta_{\omega,0} \times \{0\})') = 0$ and

$$\phi_{A_\alpha}(z) = JG^n_\alpha((G^n_\alpha|_{A_\alpha})^{-1}(z))^{-1}, \tag{177}$$

for each $z \in (\Delta_{\omega,0} \times \{0\})'$. Furthermore, by Lemma 5.2.4 we can find a set $\Delta'_\alpha \in \mathscr{F}_{\Delta_\alpha}$, so that $\mu_{\Delta_\alpha}(\Delta_\alpha \setminus \Delta'_\alpha) = 0$ and

$$J(G^n_\alpha)(x) = \prod_{i=0}^{n-1} (JG_{\sigma^i\alpha}) \circ G^i_\alpha(x) \quad \text{for } x \in \Delta'_\alpha. \tag{178}$$

Lastly, by Lemma 5.3.13 we find an $A'_\alpha \in \mathscr{F}_{A_\alpha}$ with $\mu_{A_\alpha}(A_\alpha \setminus A'_\alpha) = 0$ such that for each $i \in \{0, \ldots, n-1\}$

$$\left| \frac{J(G_{\sigma^i\alpha})(G^i_\alpha(x))}{J(G_{\sigma^i\alpha})(G^i_\alpha(y))} - 1 \right| \leq d_{\beta,C,\sigma^i\alpha}(G^i_\alpha(x), G^i_\alpha(y)), \quad \text{for every } x, y \in A'_\alpha. \tag{179}$$

In writing $\dot{A}_\alpha := (G_\alpha^n|_{A_\alpha})^{-1}((\Delta_{\omega,0}\times\{0\})')\cap\Delta'_\alpha\cap A'_\alpha$, we then have that

$$\mu_{A_\alpha}(A_\alpha\setminus\dot{A}_\alpha)=\mu_{A_\alpha}(A_\alpha\setminus(G_\alpha^n|_{A_\alpha})^{-1}((\Delta_{\omega,0}\times\{0\})')),$$

as $\mu_{A_\alpha}(\Delta'_\alpha\cap A'_\alpha)=\mu_{A_\alpha}(A_\alpha)$ and more so

$$\begin{aligned}\mu_{A_\alpha}(A_\alpha\setminus\dot{A}_\alpha)&=\mu_{A_\alpha}((G_\alpha^n|_{A_\alpha})^{-1}(\Delta_{\omega,0}\times\{0\})\setminus(G_\alpha^n|_{A_\alpha})^{-1}(\Delta_{\omega,0}\times\{0\})')\\&\leq(G_\alpha^n|_{A_\alpha})_\star\mu_{A_\alpha}((\Delta_{\omega,0}\times\{0\})\setminus(\Delta_{\omega,0}\times\{0\})')\\&=0.\end{aligned}$$

As $\dot{A}_\alpha\in\mathscr{F}_{A_\alpha}$ the mapping $G_\alpha^n|_{\dot{A}_\alpha}:\dot{A}_\alpha\to G_\alpha^n(\dot{A}_\alpha)$ is bi-measurable by Lemma 4.3.3 item 2. Moreover, by pbn-singularity of G_α^n we have as $\Delta_{\omega,0}\times\{0\}\setminus G_\alpha^n(\dot{A}_\alpha)\subseteq G_\alpha^n(A_\alpha\setminus\dot{A}_\alpha)$ that

$$\mu_{\Delta_\omega}(\Delta_{\omega,0}\times\{0\}\setminus G_\alpha^n(\dot{A}_\alpha))\leq(G_\alpha^{-n})_\star\mu_{\Delta_\alpha}(A_\alpha\setminus\dot{A}_\alpha)=0.$$

For any $x,y\in G_\alpha^n(\dot{A}_\alpha)$ we then have unique elements

$$x',y'\in\dot{A}_\alpha\ \text{ such that }\ x'=(G_\alpha^n|_{\dot{A}_\alpha})^{-1}(x)\text{ and }\ y'=(G_\alpha^n|_{\dot{A}_\alpha})^{-1}(y).$$

By Equation (177) we then have $\phi_{A_\alpha}(x)=(J(G_\alpha^n)(x'))^{-1}$ and $\phi_{A_\alpha}(y)=(J(G_\alpha^n)(y'))^{-1}$. Finally as $\dot{A}_\alpha\subseteq A_\alpha$ we obtain

$$d_{\beta,C,\omega}(x,y)=\beta^{-n+i}d_{\beta,C,\sigma^i\alpha}(G_\alpha^i x',G_\alpha^i y')\text{ for }i\in\{0,\ldots,n\},\tag{180}$$

by Corollary 5.3.11. Combining our efforts, we see

$$\begin{aligned}\left|\log\left(\frac{\phi_{A_\alpha}(x)}{\phi_{A_\alpha}(y)}\right)\right|&=\left|\log\left(\frac{J(G_\alpha^n)(y')}{J(G_\alpha^n)(x')}\right)\right|\\&=\left|\log\left(\frac{\prod_{i=0}^{n-1}(JG_{\sigma^i\alpha})(G_\alpha^i(y'))}{\prod_{i=0}^{n-1}(JG_{\sigma^i\alpha})(G_\alpha^i(x'))}\right)\right|&&(181)\\&\leq\sum_{i=0}^{n-1}\left|1-\frac{(JG_{\sigma^i\alpha})(G_\alpha^i(y'))}{(JG_{\sigma^i\alpha})(G_\alpha^i(x'))}\right|&&(182)\\&\leq\sum_{i=0}^{n-1}d_{\beta,C,\sigma^i\alpha}(G_\alpha^i(x'),G_\alpha^i(y'))&&(183)\\&\leq d_{\beta,C,\omega}(x,y)\sum_{i=0}^{n-1}\beta^i&&(184)\\&\leq\frac{d_{\beta,C,\omega}(x,y)}{1-\beta}.&&(185)\end{aligned}$$

In Equation (181) we used Equation (178); in Equation (182) we used $|\log(z)|\leq\max(|1-z|,|1-\frac{1}{z}|)$ and $|\log(z)|=|\log(\frac{1}{z})|$ for $z>0$, and in Equation (183) we used Equation

(179); and in Equation (184) we used Equation (180). As Equation (185) holds for general $x, y \in G^n_\omega(\dot{A}_\omega)$ we can see that for all $x, y \in G^n_\omega(\dot{A}_\omega)$ we have

$$\log\left(\frac{\phi_{A_\alpha}(x)}{\phi_{A_\alpha}(y)}\right) \le \frac{d_{\beta,C,\omega}(x,y)}{1-\beta} \quad \text{and} \quad \log\left(\frac{\phi_{A_\alpha}(y)}{\phi_{A_\alpha}(x)}\right) \le \frac{d_{\beta,C,\omega}(x,y)}{1-\beta}. \tag{186}$$

Exponentiating both equations in line (186) yields

$$\frac{1}{M}\phi_{A_\alpha}(y) \le \phi_{A_\alpha}(x) \le M\phi_{A_\alpha}(y), \tag{187}$$

for $M := e^{\frac{C}{1-\beta}}$, so that $M > 1$. Integrating (187) with respect to y on $G^n_\alpha(\dot{A}_\alpha)$ shows

$$\frac{1}{M}\mu_{\Delta_{\alpha,0}\times\{0\}}(A_\alpha) \le \phi_{A_\alpha}(x) \le M\mu_{\Delta_{\alpha,0}\times\{0\}}(A_\alpha),$$

as $\mu_{\Delta_{\alpha,0}\times\{0\}}(\dot{A}_\alpha) = \mu_{\Delta_{\alpha,0}\times\{0\}}(A_\alpha)$ and $\mu_{\Delta_{\omega,0}\times\{0\}}(G^n_\alpha(\dot{A}_\alpha)) = \mu_{\Delta_{\omega,0}\times\{0\}}(\Delta_{\omega,0}\times\{0\})$. Substituting $\sigma^{-n}\omega = \alpha$ back, we obtain the statement in Equation (175). As for Equation (176), note that $(\Delta_\omega \setminus \Delta_{\omega,0}\times\{0\}) \cap G^n_\alpha(A_\alpha) = \emptyset$ as $A_\alpha \in \mathscr{R}^n_\alpha$, using Corollary 5.3.4. As $G^n_\alpha(A_\alpha) \in \mathscr{F}_{\Delta_\omega}$ by Corollary 5.2.5 we can apply Lemma A.1.1 and derive Equation (175). □

Similar to the deterministic case, in order to find an acip, we now need to obtain an uniform upper bound for a density associated with the tower map, here

$$\frac{d(G^n)_\star\mu_{\Delta_0\times\{0\}}}{d\mu_\Delta}, \text{for any } n \in \mathbb{Z}_{\ge 1}.$$

As we are working directly on the tower our analysis starts deviating significantly from the deterministic case however.

First, we need to wrest thorough control of densities associated with the tower map. To do so, we shall in Lemma 5.4.3 verify some statements on absolute continuity to make sure we can define these densities to begin with. Here, we shall also introduce the set $\Delta^{(l)}$ for $l \in \mathbb{Z}_{\ge 0}$. Technically, this set is dependent on a specified $\omega \in \dot{\Omega}$ and $n \in \mathbb{Z}_{\ge 0}$ but we have omitted this for notational brevity. This set contains all information of floor l of Δ for which for a given $\omega \in \dot{\Omega}$ we expect the density

$$\frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}$$

to have a non-zero value. In Lemma 5.4.3 we show some elementary properties of this set. In Lemma 5.4.4 we shall then calculate an upper bound for

$$\frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}.$$

Importantly, as we offset $\Delta^{(l)}$ against $G^{n-l}_{\sigma^{-n}}$, we shall for this only require Markov sets for which the bounds obtained in Lemma 5.4.2 suffice. Lemma 5.4.5 then shows how this relates to the density

$$\frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}.$$

After combining these statements into a more general upper bound in Proposition 5.4.6 we are ready to prove the main Theorem of this text.

The key concept in Lemma 5.4.3 below is that for $l \in \mathbb{Z}_{\geq 1}$, $\omega \in \dot{\Omega}$ and $(x,0) \in \Delta_{\omega,0} \times \{0\}$ we can only have $G^l_\omega(x,0) \in \Delta_{\sigma^l\omega,l} \times \{l\}$ if for all $i \in \{1,\dots,l\}$ we have $G^i_\omega(x,0) \notin \Delta_{\sigma^i\omega,0} \times \{0\}$.

Lemma 5.4.3. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system and let $\omega \in \dot{\Omega}$, $n \in \mathbb{Z}_{\geq 1}$, $l \in \{1,\dots,n\}$. Define*

$$\Delta^{(l)} := (G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})^{-1}(\Delta_{\omega,l} \times \{l\}).$$

Then we have

$$\Delta^{(l)} = (\Delta_{\sigma^{-n}\omega,0} \times \{0\}) \cap G^{-n+l}_{\sigma^{-n}\omega}(\Delta_{\omega,l} \times \{0\}) \in \mathscr{F}_{\Delta_{\sigma^{-n}\omega}}, \quad G^{n-l}_{\sigma^{-n}\omega}(\Delta^{(l)}) \in \mathscr{F}_{\Delta_{\omega,l}\times\{0\}}, \tag{188}$$

and

$$(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}} \ll \mu_{\Delta_{\omega,l}\times\{0\}} \text{ and } (G^n_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}} \ll \mu_{\Delta_{\omega,l}\times\{l\}}. \tag{189}$$

Proof. First note that by Lemma 5.1.13 we can derive (using $l' = 0$) that $G^{-l}_{\sigma^{-l}\omega}(\Delta_{\omega,l} \times \{l\}) = \Delta_{\omega,l} \times \{0\}$. Using this we see

$$\begin{aligned}
\Delta^{(l)} &= (G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})^{-1}(\Delta_{\omega,l} \times \{l\}) \\
&= (G^{n-l}_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})^{-1}(\Delta_{\omega,l} \times \{0\}) \\
&= (\Delta_{\sigma^{-n}\omega,0} \times \{0\}) \cap G^{-n+l}_{\sigma^{-n}\omega}(\Delta_{\omega,l} \times \{0\}) \in \mathscr{F}_{\Delta_{\sigma^{-n}\omega}}.
\end{aligned}$$

Moreover, we note that by Lemma 2.1.23 we have

$$G^{n-l}_{\sigma^{-n}\omega}(\Delta^{(l)}) = G^{n-l}_{\sigma^{-n}\omega}(\Delta_{\sigma^{-n}\omega,0} \times \{0\}) \cap (\Delta_{\omega,l} \times \{0\}) \subseteq \Delta_{\sigma^{-l}\omega,0} \times \{0\}.$$

To show Equation (189) we note

$$\Delta^{(l)} \subseteq G^{-n}_{\sigma^{-n}\omega}(\Delta_{\omega,l} \times \{l\}) \text{ and } \Delta^{(l)} \subseteq G^{-n+l}_{\sigma^{-n}\omega}(\Delta_{\omega,l} \times \{0\}).$$

Equation (189) then follows from Lemma A.1.3. □

The following Lemma is the first step towards obtaining an almost surely uniform upper bound for densities of Ω-sections of random tower maps.

Lemma 5.4.4. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system, let $M \in \mathbb{R}_{>1}$ be as in Lemma 5.4.2 and let $\omega \in \dot{\Omega}$, $n \in \mathbb{Z}_{\geq 1}$ and $l \in \{0, \dots, n\}$. Let $\Delta^{(l)}$ be as in Lemma 5.4.3. Then we have for each $A \in \mathscr{F}_{\Delta_{\sigma^{-l}\omega}}$*

$$\int_A \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}(x, l')\, d\mu_{\Delta_{\sigma^{-l}\omega}}(x, l') \leq M \mu_{\Delta_{\sigma^{-l}\omega,0} \times \{0\}}(A). \tag{190}$$

Proof. First note that for the case $n = l$, Equation (190) can be rewritten as

$$\int_A \mathbb{1}_{\Delta^{(n)}}(x, l')\, d\mu_{\Delta_{\sigma^{-n}\omega}}(x, l') \leq M \mu_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}}(A),$$

for each $A \in \mathscr{F}_{\Delta_{\sigma^{-n}\omega}}$. As we know $M \in \mathbb{R}_{>1}$ and $\Delta^{(n)} \subseteq \Delta_{\sigma^{-n}\omega,0} \times \{0\}$ this is immediate.

Now supposing $0 \leq l < n$, we shall derive our bound through the bound obtained for Markov sets in Lemma 5.4.2 using Equation (166) in Corollary 5.3.4. To start, define

$$R^{n-l}_{\sigma^{-n}\omega} := \bigsqcup_{K \in \mathscr{R}^{n-l}_{\sigma^{-n}\omega}} K$$

and note that we have

$$\begin{aligned}
\Delta^{(l)} &= (\Delta_{\sigma^{-n}\omega,0} \times \{0\}) \cap G^{-n+l}_{\sigma^{-n}\omega}(\Delta_{\omega,l} \times \{0\}) && \text{Using Lemma 5.4.3}\\
&\subseteq (\Delta_{\sigma^{-n}\omega,0} \times \{0\}) \cap G^{-n+l}_{\sigma^{-n}\omega}(\Delta_{\sigma^{-l}\omega,0} \times \{0\}) && \text{Using Lemma 5.1.13}\\
&= R^{n-l}_{\sigma^{-n}\omega}, && \text{Using Corollary 5.3.4.}
\end{aligned}$$

Now applying Lemma 5.4.3 we have

$$(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}} \ll \mu_{\Delta_{\omega,l} \times \{0\}},$$

and so by Lemma A.1.2 we then have

$$\frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}} \leq \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{R^{n-l}_{\sigma^{-n}\omega}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}, \mu_{\Delta_{\sigma^{-l}\omega}}\text{-almost surely.} \tag{191}$$

Now note that for arbitrary $A \in \mathscr{F}_{\Delta_{\sigma^{-l}\omega}}$ we have

$$\begin{aligned}
&\int_A \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}(x,l')\, d\mu_{\Delta_{\sigma^{-l}\omega}}(x,l') \\
&\quad \le \int_A \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{R^{n-l}_{\sigma^{-n}\omega}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}(x,l')\, d\mu_{\Delta_{\sigma^{-l}\omega}}(x,l') \\
&\quad = \int_{A_0\times\{0\}} \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{R^{n-l}_{\sigma^{-n}\omega}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}(x,l')\, d\mu_{\Delta_{\sigma^{-l}\omega}}(x,l') && (192) \\
&\quad = \sum_{K\in\mathscr{R}^{n-l}_{\sigma^{-n}\omega}} \int_{A_0\times\{0\}} \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_K)_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l}\omega}}}(x,l')\, d\mu_{\Delta_{\sigma^{-l}\omega}}(x,l') && (193) \\
&\quad = \sum_{K\in\mathscr{R}^{n-l}_{\sigma^{-n}\omega}} M\cdot \mu_{\Delta_{\sigma^{-l}\omega}}(A_0\times\{0\})\mu_{\Delta_{\sigma^{-n}\omega}}(K) && (194) \\
&\quad \le M\cdot \mu_{\Delta_{\sigma^{-l}\omega,0}\times\{0\}}(A), && (195)
\end{aligned}$$

where in Equation (192) we used Lemma A.1.1 and $G^{n-l}_{\sigma^{-n}\omega}(\Delta^{(l)}) \in \mathscr{F}_{\Delta_{\sigma^{-l}\omega,0}\times\{0\}}$ as seen in Lemma 5.4.3; in Equation (193) we used Lemma 2.1.20 and the Monotone Convergence Theorem; in Equation (194) we used Lemma 5.4.2; in Equation (195) we used $R^{n-l}_{\sigma^{-n}\omega} \subseteq \Delta_{\sigma^{-n}\omega,0}\times\{0\}$ so that

$$\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}(R^{n-l}_{\sigma^{-n}\omega}) \le \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}(\Delta_{\sigma^{-n}\omega,0}\times\{0\}) = 1,$$

proving our claim. □

Conceptually the rest of this section will rely on Corollary 5.1.14.

Lemma 5.4.5. *Let $(\Delta,\mathscr{F}_\Delta,\mu_\Delta,G)$ be a random tower system and let $\omega\in\dot{\Omega}$, $n\in\mathbb{Z}_{\ge1}$, $l\in\{1,\dots,n\}$. Let $\Delta^{(l)}$ be as in Lemma 5.4.3. Then we have for each $A\times\{l\}\in\mathscr{F}_{\Delta_{\omega,l}\times\{l\}}$ that*

$$\begin{aligned}
&\int_{A\times\{l\}} \frac{d(G^{n}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{l\}}}(x,l')\, d\mu_{\Delta_{\omega,l}\times\{l\}}(x,l') \\
&\quad = \int_{A\times\{0\}} \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{0\}}}(x,l')\, d\mu_{\Delta_{\omega,l}\times\{0\}}(x,l'). && (196)
\end{aligned}$$

Proof. We shall use Lemma A.1.6 and quickly verify its conditions. By Lemma 5.4.3 we can write

$$G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}} : \Delta^{(l)} \to \Delta_{\omega,l}\times\{0\},$$

and have $(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-l}\omega}} \ll \mu_{\Delta_{\omega,l}\times\{0\}}$. By Corollary 5.1.14 we know

$$G^l_{\sigma^{-l}\omega}|_{\Delta_{\omega,l}\times\{0\}} : \Delta_{\omega,l}\times\{0\} \to \Delta_{\omega,l}\times\{l\}$$

is bi-measurable and satisfies

$$(G^l_{\sigma^{-l}\omega}|_{\Delta_{\omega,l}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-l}\omega,0}\times\{0\}} = (G^l_{\sigma^{-l}\omega})_\star\mu_{\Delta_{\omega,l}\times\{0\}} = \mu_{\Delta_{\omega,l}\times\{l\}}.$$

Combining the above we can see that on the composition $G^n_{\sigma^{-n}\omega}|_{\Delta^{(l)}} = G^l_{\sigma^{-l}\omega}\circ G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}}$ we can apply Lemma A.1.6 and obtain for $\mu_{\Delta_{\omega,l}\times\{l\}}$-almost every $(x,l)\in\Delta_{\omega,l}\times\{l\}$,

$$\frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{l\}}}(x,l) = \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{0\}}}(G^{-l}_{\sigma^{-l}\omega}(x,l)).$$

Note this implies that for each $A\times\{l\}\in\mathscr{F}_{\Delta_{\omega,l}\times\{l\}}$ we have

$$\begin{aligned}
&\int_{A\times\{l\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{l\}}}(x,l)\,d\mu_{\Delta_{\omega,l}\times\{l\}}(x,l')\\
&\quad= \int_{A\times\{l\}} \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{0\}}}(G^{-l}_{\sigma^{-l}\omega}(x,l'))\,d\mu_{\Delta_{\omega,l}\times\{l\}}(x,l')\\
&\quad= \int_{A\times\{0\}} \frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{0\}}}(x,l')\,d\mu_{\Delta_{\omega,l}\times\{0\}}(x,l'),
\end{aligned}$$

where we used Lemma 2.1.10 (applicable as $\|\frac{d(G^{n-l}_{\sigma^{-n}\omega}|_{\Delta^{(l)}})_\star\mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_{\omega,l}\times\{0\}}}\|_1 \le 1$) and Corollary 5.1.14 in the last step, proving Equation (196). □

Proposition 5.4.6. *Let $(\Delta,\mathscr{F}_\Delta,\mu_\Delta,G)$ be a random tower system, let $\omega\in\dot{\Omega}$, $n\in\mathbb{Z}_{\ge 0}$ and let $M\in\mathbb{R}_{>1}$ be as in Lemma 5.4.4. Then we have for each $A\in\mathscr{F}_{\Delta_\omega}$*

$$\int_A \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}} \le M\mu_{\Delta_\omega}(A).$$

Proof. Let $A\in\mathscr{F}_{\Delta_\omega}$ be arbitrary. First note that we have

$$G^n_{\sigma^{-n}\omega}(\Delta_{\sigma^{-n}\omega,0}\times\{0\}) \subseteq \bigcup_{l\in\{0,\dots,n\}} \Delta_{\omega,l}\times\{l\},$$

so that in writing

$$A_{>n} := A\cap\left(\bigcup_{l>n}\Delta_{\omega,l}\times\{l\}\right),$$

we have $G^n_{\sigma^{-n}\omega}(\Delta_{\sigma^{-n}\omega,0}\times\{0\})\cap A_{>n}=\emptyset$. Hence, by Lemma A.1.1

$$\int_{A_{>n}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x,l')\,d\mu_{\Delta_\omega}(x,l')=0.$$

Proceeding, we see

$$\begin{aligned}
&\int_{A} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x,l')\,d\mu_{\Delta_\omega}(x,l')\\
&=\sum_{l''=0}^{n}\int_{A_{l''}\times\{l''\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x,l')\,d\mu_{\Delta_\omega}(x,l')\\
&=\sum_{l''=0}^{n}\int_{A_{l''}\times\{l''\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{l''\}}(x,l').
\end{aligned}$$

Recapping the definition $\Delta^{(l'')}:=(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})^{-1}(\Delta_{\omega,l''}\times\{l''\})$ from Lemma 5.4.3 we then see using Lemma A.1.1 that

$$\begin{aligned}
&\sum_{l''=0}^{n}\int_{A_{l''}\times\{l''\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{l''\}}(x,l')\\
&\qquad=\sum_{l''=0}^{n}\int_{A_{l''}\times\{l''\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta^{(l'')}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\omega,l''}\times\{l''\}}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{l''\}}(x,l').
\end{aligned}$$

Finishing our proof we then see that,

$$\begin{aligned}
&\sum_{l''=0}^{n}\int_{A_{l''}\times\{l''\}} \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta^{(l'')}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\omega,l''}\times\{l''\}}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{l''\}}(x,l')\\
&=\sum_{l''=0}^{n}\int_{A_{l''}\times\{0\}} \frac{d(G^{n-l''}_{\sigma^{-n}\omega}|_{\Delta^{(l'')}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\omega,l''}\times\{0\}}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{0\}}(x,l') &(197)\\
&=\sum_{l''=0}^{n}\int_{A_{l''}\times\{0\}} \frac{d(G^{n-l''}_{\sigma^{-n}\omega}|_{\Delta^{(l'')}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l''}\omega}}}(x,l')\,d\mu_{\Delta_{\omega,l''}\times\{0\}}(x,l') &(198)\\
&=\sum_{l''=0}^{n}\int_{A_{l''}\times\{0\}} \frac{d(G^{n-l''}_{\sigma^{-n}\omega}|_{\Delta^{(l'')}})_\star\mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_{\sigma^{-l''}\omega}}}(x,l')\,d\mu_{\Delta_{\sigma^{-l''}\omega}}(x,l') &(199)\\
&\le\sum_{l''=0}^{n} M\mu_{\Delta_{\sigma^{-l''}\omega}}(A_{l''}\times\{0\}) &(200)\\
&=\sum_{l''=0}^{n} M(G^{l''}_{\sigma^{-l''}\omega}|_{\Delta_{\omega,l''}\times\{0\}})_\star\mu_{\Delta_{\sigma^{-l''}\omega}}(A_{l''}\times\{l''\}) &(201)\\
&=\sum_{l''=0}^{n} M\mu_{\Delta_{\omega,l''}\times\{l''\}}(A_{l''}\times\{l''\}) &(202)\\
&\le M\mu_{\Delta_\omega}(A),
\end{aligned}$$

where in Equation (197) we used Lemma 5.4.5; in Equation (198) we used Lemma A.1.4, as $\Delta_{\omega,l''} \times \{0\} \subseteq \Delta_{\sigma^{-l''}\omega}$ by Lemma 5.1.13; in Equation (199) we used Lemma A.1.4; in Equation (200) we used Lemma 5.4.4; in Equation (201) we used Lemma 5.1.13 and in Equation (202) we used Corollary 5.1.14. This proves our claim. □

We can now combine all our work in the following theorem. That is:

1. Proposition 5.4.6 provided us with a uniform upper bound for the density of the sections of the tower map.

2. Lemma 4.2.9 will then allow us to claim the bound holds for the density of associated with the tower map itself as well.

3. Proposition 4.1.8 will then supply us with the necessary convergence result to obtain our acip for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$.

Theorem 5.4.7. *Let $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ be a random tower system and $M > 1$ as in Proposition 5.4.6, then there exists a probability measure ν on Δ such that $\nu \ll \mu_\Delta$, $\frac{d\nu}{d\mu_\Delta} \le M$, μ_Δ-a.e., and $G_\star \nu = \nu$. That is, there exists an* acip *ν for $(\Delta, \mathscr{F}, \mu_\Delta, G)$.*

Proof. First note in Lemma 5.1.9 we proved that $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is a random dynamical system. Moreover $\mu_\Delta(\Delta_0 \times \{0\}) = (\mathbb{P} \times \mu)(\Lambda) = 1$. First let $A \in \mathscr{F}_\Delta$ and $n \in \mathbb{Z}_{\ge 0}$ be arbitrary. Note that by Lemma 4.2.10 we have

$$\int_\Omega \mu_{\Delta_\omega}(A_\omega)\, d\mathbb{P}(\omega) = \mu_\Delta(A),$$

and by Lemma 4.2.7

$$\int_A \frac{d(G^n)_\star \mu_{\Delta_0 \times \{0\}}}{d\mu_\Delta}(\omega, x)\, d\mu_\Delta(\omega, x) = \int_\Omega \int_{A_\omega} \frac{d(G^n)_\star \mu_{\Delta_0 \times \{0\}}}{d\mu_\Delta}(\omega, x)\, d\mu_{\Delta_\omega}(x)\, d\mathbb{P}(\omega).$$

Note that according to Lemma 4.2.9 we have for almost every $\omega \in \Omega$ that for each $B \in \mathscr{F}_{\Delta_\omega}$ that

$$\int_B \frac{d(G^n)_\star \mu_{\Delta_0 \times \{0\}}}{d\mu_\Delta}(\omega, x)\, d\mu_{\Delta_\omega}(x) = \int_B \frac{d\left(G^n_{\sigma^{-n}\omega}\right)_\star \mu_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}}}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x).$$

Furthermore, as $(G^n_{\sigma^{-n}\omega})_\star \mu_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}} = (G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}})_\star \mu_{\Delta_\omega}$ we can write

$$\int_B \frac{d\left(G^n_{\sigma^{-n}\omega}\right)_\star \mu_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}}}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x) = \int_B \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x).$$

By Proposition 5.4.6 we then have for each $\omega \in \dot{\Omega}$ that

$$\int_B \frac{d(G^n_{\sigma^{-n}\omega}|_{\Delta_{\sigma^{-n}\omega,0} \times \{0\}})_\star \mu_{\Delta_{\sigma^{-n}\omega}}}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x) \le M \mu_{\Delta_\omega}(B).$$

To conclude we see

$$\begin{aligned}\int_A \frac{d(G^n)_\star \mu_{\Delta_0\times\{0\}}}{d\mu_\Delta}(\omega,x)\,d\mu_\Delta(\omega,x) &= \int_\Omega\int_{A_\omega} \frac{d(G^n)_\star \mu_{\Delta_0\times\{0\}}}{d\mu_\Delta}(\omega,x)\,d\mu_{\Delta_\omega}(x)\,d\mathbb{P}(\omega)\\ &= \int_\Omega\int_{A_\omega} \frac{d\left(G^n_{\sigma^{-n}\omega}\right)_\star \mu_{\Delta_{\sigma^{-n}\omega,0}\times\{0\}}}{d\mu_{\Delta_\omega}}(x)\,d\mu_{\Delta_\omega}(x)\,d\mathbb{P}(\omega)\\ &\le M\int_{\dot\Omega} \mu_{\Delta_\omega}(A_\omega)\,d\mathbb{P}(\omega)\\ &= M\mu_\Delta(A).\end{aligned}$$

As $A \in \mathscr{F}_\Delta$ was given arbitrarily we have $\frac{d(G^n)_\star\mu_{\Delta_0\times\{0\}}}{d\mu_\Delta} \le M$, μ_Δ-a.e.. for each $n \in \mathbb{Z}_{\ge 0}$. Consequently, for $n \in \mathbb{Z}_{\ge 0}$ we have μ_Δ-almost surely,

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{d(G^n)_\star\mu_{\Delta_0\times\{0\}}}{d\mu_\Delta} \le M.$$

We can then apply Proposition 4.1.8 to infer

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1} G^i_\star \mu_{\Delta_0\times\{0\}} \to \nu \text{ set-wise as } k\to\infty,$$

for some strictly increasing sequence $(n_k)_{k\ge 0}$ and a finite positive measure ν on Δ, with $\nu \ll \mu_\Delta$, and $\frac{d\nu}{d\mu_\Delta} \le M$. We can then see by set-wise convergence

$$1 = \frac{1}{n_k}\sum_{i=0}^{n_k-1} G^i_\star \mu_{\Delta_0\times\{0\}}(\Delta) \to \nu(\Delta) \text{ as } k\to\infty,$$

so that ν is a probability measure. Finally note that for general $A \in \mathscr{F}_\Delta$ we have

$$\begin{aligned}\nu(A) - G_\star\nu(A) &= \lim_{k\to\infty}\frac{1}{n_k}\left(\sum_{i=0}^{n_k-1} G^i_\star\mu_{\Delta_0}(A) - \sum_{i=0}^{n_k-1} G^{i+1}_\star\mu_{\Delta_0}(A)\right)\\ &= \lim_{k\to\infty}\frac{1}{n_k}(\mu_{\Delta_0}(A) - \mu_{\Delta_0}(G^{-n_k}(A)))\\ &= 0,\end{aligned}$$

proving the invariance of ν. We have shown our statement. □

5.5 Shattered Measures

When dealing with measures associated with RDS's it is usually convenient to *disintegrate* measures over the random dynamic and the base dynamic. Within existing theory *random measures* are usually used to fulfil this purpose. These objects are well-studied but have two shortcomings: firstly as existence of the disintegration usually relies on the topological assumptions on the RDS and secondly as the *fibers* of their disintegration must be probability measures. Both these conditions can make it hard to use these objects in our situation.

Alternatively, if we have a measure on an RDS that is absolutely continuous with respect to the overlying product measure, we can 'disintegrate' said measure in such a way that we are not faced with either of the previously mentioned constraints. As this method is different than what is commonly understood as a *disintegration* we refer to it as a *shattering*. We remind the reader that for a measurable space $(X, \mathscr{F}_X)$ the spaces $\mathscr{M}^+_\infty(X), \mathscr{M}^+(X)$ denote the positive measures on X and positive finite measures on X respectively.

Proposition 5.5.1. *Suppose $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ is a probability space, $(X, \mathscr{F}_X, \mu)$ is a σ-finite measure space and let $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu)$ be their product measure space. Suppose we have a measure $\nu \in \mathscr{M}^+(\Omega \times X)$, with $\nu \ll \mathbb{P} \times \mu$ and let $\frac{d\nu}{d\mathbb{P}\times\mu}$ denote a postive version of the Radon-Nikodym derivative of ν with respect to $\mathbb{P} \times \mu$. Then the mapping*

$$\begin{aligned} \nu : \Omega \to \ & \mathscr{M}^+_\infty(X) \\ \omega \mapsto & \begin{pmatrix} \nu_\omega : \mathscr{F}_X \to [0,\infty] \\ \qquad A \mapsto \int_X \mathbb{1}_A(x) \dfrac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x) \end{pmatrix} \end{aligned} \tag{203}$$

is so that for each $A \in \mathscr{F}_X$ the mapping $\omega \mapsto \nu_\omega(A)$ is $\mathscr{F}_\Omega$-measurable and $\nu_\cdot(A) \in L^1(\Omega)$. Moreover we hav $\mathbb{P}$-almost every $\omega \in \Omega$, $\nu_\omega \ll \mu$ with

$$\frac{d\nu_\omega}{d\mu}(\cdot) = \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, \cdot)$$

and $\nu_\omega \in \mathscr{M}^+(X)$.

Proof. Note that by Proposition 2.1.7 for each $x \in X$ the mapping

$$\omega \mapsto \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x),$$

is $\mathscr{F}_\Omega$-measurable. By the Radon-Nikodym Theorem 2.1.11 we then see that $\nu_\omega \in \mathscr{M}^+_\infty(X)$ and $\nu_\omega \ll \mu$, with $\frac{d\nu_\omega}{d\mu}(\cdot) = \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,\cdot)$. Moreover for arbitrary $A \in \mathscr{F}_X$ we have $\mathbb{1}_{\Omega\times A} \in L^1(\nu)$ so that by Lemma A.1.7 Item (3) we have $\mathbb{1}_{\Omega\times A} \cdot \frac{d\nu}{d(\mathbb{P}\times\mu)} \in L^1(\mathbb{P}\times\mu)$. By Fubini's Theorem 2.1.8 we then see that

$$\begin{aligned} \nu_\cdot(A) : \Omega \to & \ \mathbb{R} \cup \{\infty\} \\ \omega \mapsto & \int_X \mathbb{1}_{\Omega\times A}(\omega, x) \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x) \end{aligned}$$

is $\mathscr{F}_\Omega$-integrable. In particular, note that this implies $\omega \mapsto \nu_\omega(A)$ is integrable and hence also measurable. As, again by Fubini's Theorem 2.1.8, the mapping

$$x \mapsto \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)$$

is in $L^1(X)$, $\mathbb{P}$-almost surely, we see that (again) by the Radon-Nikodym Theorem 2.1.11 we have $\nu_\omega \in \mathscr{M}^+(X)$, $\mathbb{P}$-almost surely.

□

Theorem 5.5.2 (Disintegration Theorem - Existence and Uniqueness)**.** *Suppose* $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ *is a probability space,* $(X, \mathscr{F}_X, \mu)$ *is a* σ*-finite measure space and let* $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P}\times\mu)$ *be their product measure space. Suppose we have a finite measure* $\nu \in \mathscr{M}^+(\Omega \times X)$*, with* $\nu \ll \mathbb{P}\times\mu$*. Then a mapping* $\nu_\cdot : \Omega \to \mathscr{M}^+_\infty(X)$ *as in Proposition 5.5.1 satisfies for each* $f \in L^1(\nu)$

$$\int_{\Omega\times X} f(\omega, x)\, d\nu(\omega, x) = \int_\Omega \int_X f(\omega, x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega). \tag{204}$$

Finally, the equation

$$\int_\Omega \nu_\omega(A_\omega)\, d\mathbb{P}(\omega) = \nu(A) \quad \textit{for all } A \in \mathscr{F}_{\Omega\times X}, \tag{205}$$

defines the mapping $\omega \mapsto \nu_\omega$ *uniquely* $\mathbb{P}$*-a.s..*

Proof. Our proof relies on the standard machinery. First, to erase any doubts on measurability, let $B \in \mathscr{F}_{\Omega\times X}$ and we prove that we have $\omega \mapsto \nu_\omega(B_\omega) \in L^1(\Omega)$. To do so, we note

$$\begin{aligned}\nu_\omega(B_\omega) &= \int_X \mathbb{1}_{B_\omega}(x) \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x) \\ &= \int_X \mathbb{1}_B(\omega, x) \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x).\end{aligned} \tag{206}$$

Now as $\mathbb{1}_B \in L^1(\Omega\times X) \cap L^\infty(\Omega\times X)$ and $\frac{d\nu}{d\mathbb{P}\times\mu} \in L^1(\Omega\times X)$, we see $\mathbb{1}_B \cdot \frac{d\nu}{d\mathbb{P}\times\mu} \in L^1(\Omega\times X)$. Integrability with respect to $\mathbb{P}$ of

$$\omega \mapsto \nu_\omega(B_\omega) = \int_X \mathbb{1}_B(\omega, x) \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x),$$

then follows from Fubini's Theorem 2.1.8. Now note we have

$$\begin{aligned}\int_\Omega \int_X \mathbb{1}_B(\omega, x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) &= \int_\Omega \nu_\omega(B_\omega)\, d\mathbb{P}(\omega) \\ &= \int_\Omega \int_X \mathbb{1}_B(\omega, x) \frac{d\nu}{d\mathbb{P}\times\mu}(\omega, x)\, d\mu(x)\, d\mathbb{P}(\omega) \\ &= \nu(B) \\ &= \int_{\Omega\times X} \mathbb{1}_B(\omega, x)\, d\nu(\omega, x),\end{aligned}$$

using Fubini's Theorem 2.1.8.

For an arbitrary simple function $s = \sum_{i=0}^{n-1} \alpha_i \mathbb{1}_{B_i}$, with $B_i \in \mathscr{F}_{\Omega\times X}$, $\alpha_i \in \mathbb{R}$, $0 \leq i \leq n-1$, we then see as $s \in L^1(\nu)$:

$$\begin{aligned}
\int_{\Omega\times X} s(\omega,x)\, d\nu(\omega,x) &= \sum_{i=0}^{n-1} \alpha_i \nu(B_i) \\
&= \sum_{i=0}^{n-1} \alpha_i \int_\Omega \nu_\omega((B_i)_\omega)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{i=0}^{n-1} \alpha_i \int_X \mathbb{1}_{(B_i)_\omega}(x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \sum_{i=0}^{n-1} \alpha_i \int_X \mathbb{1}_{B_i}(\omega,x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) \\
&= \int_\Omega \int_X s(\omega,x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) \qquad (207)
\end{aligned}$$

so that Equation (204) also holds for simple functions.

General integrable functions $f \in L^1(\nu)$ can be split into a positive and negative part $f^+, f^- \in L^1(\nu)$ respectively, both of which we can approximated by simple functions, that is, there exists sequences of simple functions $(s_n)_{n\geq 0}, (r_n)_{n\geq 0} \subseteq L^1(\nu)$ such that $s_n \uparrow f^+$, $r_n \uparrow f^-$ pointwise as $n \to \infty$. Knowing this, it is easily seen that

$$\begin{aligned}
\int_{\Omega\times X} f(\omega,x)\, d\nu(\omega,x) &= \sup_{n\geq 0} \int_{\Omega\times X} s_n(\omega,x)\, d\nu(\omega,x) - \sup_{n\geq 0} \int_{\Omega\times X} r_n(\omega,x)\, d\nu(\omega,x) \qquad (208) \\
&= \sup_{n\geq 0} \int_\Omega \int_X s_n(\omega,x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) \\
&\quad - \sup_{n\geq 0} \int_\Omega \int_X r_n(\omega,x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega) \qquad (209) \\
&= \int_\Omega \int_X f(\omega,x)\, d\nu_\omega(x)\, d\mathbb{P}(\omega), \qquad (210)
\end{aligned}$$

holds, from which we conclude that Equation (204) indeed holds. In Equations (208) and (210) use was made of the monotone convergence theorem whereas (209) relies on Equation (207).

Finally, as for uniqueness suppose we have two mappings $\omega \mapsto \nu_\omega$, $\omega \mapsto \nu'_\omega$ both satisfying Equation (205). We can see that for any $A \in \mathscr{F}_X$ and for any $O \in \mathscr{F}_\Omega$ we have

$$\int_O \nu_\omega(A)\, d\mathbb{P}(\omega) = \int_\Omega \nu_\omega((O\times A)_\omega)\, d\mathbb{P}(\omega) = \int_\Omega \nu'_\omega((O\times A)_\omega)\, d\mathbb{P}(\omega) = \int_O \nu'_\omega(A)\, d\mathbb{P}(\omega),$$

so $\nu_\omega(A) = \nu'_\omega(A)$, $\mathbb{P}$-a.s. for any $A \in \mathscr{F}_X$. So $\omega \mapsto \nu_\omega$ is indeed defined uniquely $\mathbb{P}$-a.s. by Equation (205). □

Definition 5.5.3. Suppose $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ is a probability space, $(X, \mathscr{F}_X, \mu)$ is a σ-finite measure space and let $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P}\times\mu)$ be their product measure space. Suppose we have a measure $\nu \in \mathscr{M}^+(\Omega\times X)$, with $\nu \ll \mathbb{P}\times\mu$. We call a mapping $\nu_\cdot$ as in Equation (203) a *shattered measure* and for each $\omega \in \Omega$ we call ν_ω a *shard*.

Remark 5.5.4. As in Theorem 5.5.2, any version of the Radon-Nikodym derivative $\frac{d\nu}{d\mathbb{P}\times\mu}$ satisfies Equation (205), we can see that any two versions $\frac{d\nu}{d\mathbb{P}\times\mu}'$, $\frac{d\nu}{d\mathbb{P}\times\mu}''$ give rise to shattered measures $\nu'_\cdot, \nu''_\cdot$ respectively for which we have $\nu'_\omega = \nu''_\omega$, $\mathbb{P}$-almost surely. This means that shattered measures constructed for some ν are unique up to a $\mathbb{P}$-measure zero set. As all claims on shattered measures we make hold up to a $\mathbb{P}$-measure zero set we shall speak of *the* shattered measure $\nu_\cdot$ when we mean its naturally induced equivalence class.

Remark 5.5.5. Strictly speaking we do not require $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ to be a probability space - even a σ-finite space will yield the same result as long as ν is a positive finite measure. We shall, however, not encounter this and hence we impose $\mathbb{P}(\Omega) = 1$ in Definition 5.5.3.

We will typically want to think of shattered measures in the context of Random Dynamical Systems and we shall give an elementary example. We remind the reader of Remark 4.2.4.

Example 5.5.6. Given a random dynamical system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, S)$, we naturally have $\mu_\Delta \ll \mathbb{P}\times\mu$ so that by Theorem 5.5.2 we obtain a shattered measure $\mu_{\Delta,\cdot}$. Note then by Proposition 2.1.7 we have for each $A \in \mathscr{F}_\Delta$ that $\omega \mapsto \mu((\Delta\cap A)_\omega)$ is $\mathscr{F}_\Omega$-measurable and as $\mu((\Delta\cap A)_\omega) \le \mu(\Delta_\omega)$ we have $\|\mu((\Delta\cap A)_\cdot)\|_{L^1(\Omega)} \le (\mathbb{P}\times\mu)(\Delta)$. Consequently, we may write

$$\int_\Omega \mu_{\Delta,\omega}(A_\omega)\, d\mathbb{P}(\omega) = \mu_\Delta(A) = \int_\Omega \mu((\Delta\cap A)_\omega)\, d\mathbb{P}(\omega) = \int_\Omega \mu_{\Delta_\omega}(A_\omega)\, d\mathbb{P}(\omega),$$

so that by Theorem 5.5.2 we can see $\mu_{\Delta,\omega} = \mu_{\Delta_\omega}$ holds $\mathbb{P}$-a.s.. Using this, we can for instance see that Lemma 4.2.9 immediately holds, again using the disintegration in Theorem 5.5.2. Moreover, *any* statement in Section 4.2 on densities can be phrased in terms of shattered measures.

Lemma 5.5.7. *Let $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ be a probability space, $(X, \mathscr{F}_X, \mu)$ be a σ-finite measure space and let $(\Omega\times X, \mathscr{F}_{\Omega\times X}, \mathbb{P}\times\mu)$ be their product space. Now suppose we have $\nu, \eta \in \mathscr{M}^+(\Omega\times X)$ such that $\nu \ll \eta \ll \mathbb{P}\times\mu$. Then we have for $\mathbb{P}$-almost every $\omega\in\Omega$ that $\nu_\omega \ll \eta_\omega$ and*

$$\frac{d\nu_\omega}{d\eta_\omega}(\cdot) = \frac{d\nu}{d\eta}(\omega,\cdot), \quad \eta_\omega\text{-almost everywhere.}$$

Proof. First as $\eta \ll \mathbb{P}\times\mu$ we have a set $\dot{\Omega}\in\mathscr{F}_\Omega$ with $\mathbb{P}(\Omega\setminus\dot{\Omega})=0$ and for each $\omega\in\dot{\Omega}$ $\frac{d\eta_\omega}{d\mu}(\cdot)=\frac{d\eta}{d\mathbb{P}\times\mu}(\omega,\cdot)$, μ-almost surely, by Proposition 5.5.1. Similarly, due to Proposition 5.5.1 we can find a set $\ddot{\Omega}\in\mathscr{F}_\Omega$ with $\mathbb{P}(\Omega\setminus\ddot{\Omega})=0$ such that $\eta_\omega,\nu_\omega\in\mathscr{M}^+(X)$ for each $\omega\in\ddot{\Omega}$. Then note that for $\omega\in\dot{\Omega}\cap\ddot{\Omega}$ we have for arbitrary $A\in\mathscr{F}_X$,

$$\begin{aligned}
\nu_\omega(A) &= \int_A \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\\
&= \int_A \frac{d\nu}{d\eta}(\omega,x)\cdot\frac{d\eta}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\\
&= \int_A \frac{d\nu}{d\eta}(\omega,x)\cdot\frac{d\eta_\omega}{d\mu}(x)\,d\mu(x) && \text{as } \omega\in\dot{\Omega}\\
&= \int_A \frac{d\nu}{d\eta}(\omega,x)\,d\eta_\omega(x).
\end{aligned}$$

As $A\in\mathscr{F}_X$ was given arbitrarily, we see in particular that for $\omega\in\dot{\Omega}\cap\ddot{\Omega}$ we have

$$\int_X \left|\frac{d\nu}{d\eta}(\omega,x)\right| d\eta_\omega(x)=\nu_\omega(X)<\infty,$$

implying $\frac{d\nu}{d\eta}(\omega,\cdot)\in L^1(X,\eta_\omega)$ and so by Theorem 2.1.11 we have $\nu_\omega\ll\eta_\omega$ and $\frac{d\nu}{d\eta}(\omega,\cdot)=\frac{d\nu_\omega}{d\eta_\omega}(\cdot)$, η_ω-almost everywhere, for each $\omega\in\ddot{\Omega}\cap\dot{\Omega}$. □

To close our discussion on shattered measures, we show how invariance of shattered measures and bounds on densities of shattered measures carry over to their shards.

Corollary 5.5.8. *Let $(\Delta,\mathscr{F}_\Delta,\mu_\Delta,S)$ be an RDS with $\sigma:\Omega\to\Omega$ measure preserving. Furthermore, suppose we have an acip $\nu\ll\mu_\Delta$ on Δ. Then for $\mathbb{P}$-almost every $\omega\in\Omega$ we have $\nu_\omega\ll\mu_{\Delta_\omega}$ and*

$$\frac{d\nu}{d\mu_\Delta}(\omega,x)=\frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x),\quad \text{for } \mu_{\Delta_\omega}\text{-almost every } x\in\Delta_\omega. \tag{211}$$

If additionally we have an $M\in\mathbb{R}_{>0}$ such that $\frac{d\nu}{d\mu_\Delta}\le M$, μ_Δ-almost surely then

$$\frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x)\le M,\quad \text{for } \mu_{\Delta_\omega}\text{-almost every } x\in\Delta_\omega, \tag{212}$$

and if $S_\star\nu=\nu$, then

$$(f_\omega)_\star\nu_\omega=\nu_{\sigma\omega},\mathbb{P}\text{-almost surely.} \tag{213}$$

Proof. As seen in Example 5.5.6 we have $\mu_\Delta \ll \mathbb{P} \times \mu$. As $\nu \ll \mu_\Delta$, Equation (211) then follows directly from Lemma 5.5.7.

Assuming we have $M \in \mathbb{R}_{>0}$ such that $\frac{d\nu}{d\mu_\Delta}$, μ_Δ-almost surely, the ($\mathscr{F}_\Delta$-measurable) set

$$Y := \left\{(\omega, x) \in \Delta : \frac{d\nu}{d\mu_\Delta}(\omega, x) > M\right\},$$

has $\mu_\Delta(Y) = 0$. Using Theorem 5.5.2 we then have $\mu_{\Delta_\omega}(Y_\omega) = 0$, on some set $\Omega_Y \in \mathscr{F}_\Omega$ with $\mathbb{P}(\Omega \setminus \Omega_Y) = 0$. Furthermore, by Lemma 5.5.7 we have a $\ddot{\Omega} \in \mathscr{F}_\Omega$ such that $\mathbb{P}(\ddot{\Omega}) = \mathbb{P}(\Omega)$ and that for each $\omega \in \ddot{\Omega}$ we have $\frac{d\nu}{d\mu_\Delta}(\omega, x) = \frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x)$ on some set $x \in \dot{\Delta}_\omega$ with $\mu_{\Delta_\omega}(\Delta_\omega \setminus \dot{\Delta}_\omega) = 0$. We then note $\mathbb{P}(\ddot{\Omega} \cap \Omega_Y) = \mathbb{P}(\Omega)$ and that for each $\omega \in \ddot{\Omega} \cap \Omega_Y$ and $A \in \mathscr{F}_{\Delta_\omega}$ we have

$$\begin{aligned}
\int_A \frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x) &= \int_{A \cap \dot{\Delta}_\omega \cap (\Delta_\omega \setminus Y_\omega)} \frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x)\, d\mu_{\Delta_\omega}(x) \\
&= \int_{A \cap \dot{\Delta}_\omega \cap (\Delta_\omega \setminus Y_\omega)} \frac{d\nu}{d\mu_\Delta}(\omega, x)\, d\mu_{\Delta_\omega}(x) \qquad (214) \\
&\le M \mu_{\Delta_\omega}(A \cap \dot{\Delta}_\omega \cap (\Delta_\omega \setminus Y_\omega)) \\
&= M \mu_{\Delta_\omega}(A),
\end{aligned}$$

which shows $\frac{d\nu_\omega}{d\mu_{\Delta_\omega}} \le M$, μ_{Δ_ω}-almost surely on the set $\ddot{\Omega} \cap \Omega_Y$ which has measure $\mathbb{P}(\Omega) = \mathbb{P}(\ddot{\Omega} \cap \Omega_Y)$, proving Equation (212).

Finally assuming $S_\star \nu = \nu$ we shall exploit the uniqueness of shards to prove Equation (213). To do so, let $A \in \mathscr{F}_{\Omega \times X}$ be given arbitrarily and note

$$\begin{aligned}
\int_\Omega \nu_\omega(A_\omega)\, d\mathbb{P}(\omega) &= \nu(A) \qquad (215) \\
&= (S_\star \nu)(A) \\
&= \int_\Omega \nu_\omega((S^{-1}A)_\omega)\, d\mathbb{P}(\omega)) \qquad (216) \\
&= \int_\Omega (f_\omega)_\star \nu_\omega(A_{\sigma\omega})\, d\mathbb{P}(\omega) \\
&= \int_\Omega (f_{\sigma^{-1}\omega})_\star \nu_{\sigma^{-1}\omega}(A_\omega)\, d\mathbb{P}(\omega), \qquad (217)
\end{aligned}$$

where in Equation (215) we used Theorem 5.5.2; in Equation (216) we used Theorem 5.5.2 and in Equation (217) we used Lemma 2.1.10 with the invariance of σ^{-1}, showing Equation (213) by the uniqueness condition phrased in Theorem 5.5.2. □

As an application to shattered measures we state the following consequence of Theorem 5.4.7 together with Proposition 5.5.8.

Corollary 5.5.9. *Let* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *be a acip admissible random tower system and* $M \in \mathbb{R}_{>1}$ *as in Theorem 5.4.7, and let* $\nu : \mathscr{F}_\Delta \to [0,1]$ *be an acip for* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$. *Then for* $\mathbb{P}$*-almost every* $\omega \in \Omega$ *we have* $\nu_\omega \ll \mu_{\Delta_\omega}$,

$$\frac{d\nu}{d\mu_\Delta}(\omega, x) = \frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x), \quad \text{for } \mu_{\Delta_\omega}\text{-almost every } x \in \Delta_\omega,$$

and

$$\frac{d\nu_\omega}{d\mu_{\Delta_\omega}}(x) \leq M, \quad \text{for } \mu_{\Delta_\omega}\text{-almost every } x \in \Delta_\omega,$$

and

$$(G_\omega)_\star \nu_\omega = \nu_{\sigma\omega}.$$

Returning to the original problem we indeed obtain an acip for our skew product S as in Section 4. We remind ourself that an *acip admissible* random dynamical system is a random dynamical system for which we can construct a random tower, that is measure-regular and satisfies both the Markov Property and bounded distortion. In particular, we can apply Theorem 5.4.7 to this random tower. We shall now show how we can use this to find an acip for our underlying system.

Corollary 5.5.10. *Let* $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ *be a random dynamical system that is acip admissible. Then there exists an acip* η *on* $\Omega \times X$ *and a shattered measure* $\eta_\cdot : \Omega \to \mathscr{M}^+_\infty(X)$ *such that for almost every* $\omega \in \Omega$ *the shard* η_ω *is a probability measure with* $(g_\omega)_\star \eta_\omega = \eta_{\sigma\omega}$.

Proof. As $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ is acip admissible we can construct a random tower system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ on $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ for which, by Theorem 5.4.7, there exists an acip $\nu \ll \mu_\Delta$ so that $\frac{d\nu}{d\mu_\Delta} \leq M$ μ_Δ-almost surely, for some $M \in \mathbb{R}_{>1}$. We show this induces an acip on $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$ using the mapping

$$\begin{aligned} \pi : \quad & \Delta \to \Omega \times X \\ & (\omega, x, l) \mapsto (\omega, g^l_{\sigma^{-l}\omega}(x)). \end{aligned}$$

We claim $\eta := \pi_\star \mu_\Delta$ is an acip for $(\Omega \times X, \mathscr{F}_{\Omega\times X}, \mathbb{P} \times \mu, S)$. First we prove π is measurable, and let $A \in \mathscr{F}_{\Omega\times X}$. Note that by definition of π we have $\pi^{-1}(A) \subseteq \Delta$. Proceeding, we see,

$$\begin{aligned}
\pi^{-1}(A) &= \{(\omega,x,l)\in\Delta : \pi(\omega,x,l)\in A\}\\
&= \{(\omega,x,l)\in\Delta : g^{l}_{\sigma^{-l}\omega}(x)\in A_\omega\}\\
&= \bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \{(\omega,x)\in\Omega\times X : g^{l}_{\sigma^{-l}\omega}(x)\in A_\omega, x\in\Lambda_{\sigma^{-l}\omega}\}\times\{l\}\\
&= \bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \{(\omega,x)\in\Omega\times X : S^{l}(\sigma^{-l}\omega,x)\in A, (\sigma^{-l}\omega,x)\in\Lambda\}\times\{l\}\\
&= \bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \{(\omega,x)\in\Omega\times X : (\omega,x)\in\sigma^{l}_{\Omega}(S^{-l}(A)\cap\Lambda)\}\times\{l\}\\
&= \bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \sigma^{l}_{\Omega}(S^{-l}(A)\cap\Lambda)\times\{l\}\in\mathscr{F}_{\Omega\times X},
\end{aligned}$$

so that $\pi^{-1}(A)\in\mathscr{F}_\Delta$.

As for absolute continuity of η with respect to $\mathbb{P}\times\mu$, assuming $\mathbb{P}\times\mu(A)=0$ we see

$$\begin{aligned}
\mu_\Delta(\pi^{-1}(A)) &= \mu_\Delta\left(\bigsqcup_{l\in\mathbb{Z}_{\geq 0}} \sigma^{l}_{\Omega}(S^{-l}(A)\cap\Lambda)\times\{l\}\right)\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}} \mu_\Delta(\sigma^{l}_{\Omega}(S^{-l}(A)\cap\Lambda\times\{l\}))\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}} (\mathbb{P}\times\mu)(\sigma^{l}_{\Omega}(S^{-l}(A)\cap\Lambda))\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}} (\mathbb{P}\times\mu)(S^{-l}(A)\cap\Lambda) \qquad (218)\\
&\leq \sum_{l\in\mathbb{Z}_{\geq 0}} (\mathbb{P}\times\mu)(S^{-l}A)\\
&= \sum_{l\in\mathbb{Z}_{\geq 0}} S^{l}_{\star}(\mathbb{P}\times\mu)(A)\\
&= 0, \qquad (219)
\end{aligned}$$

where in Equation (218) we used the invariance and bi-measurability of σ_Ω as seen in Lemma 4.2.6 and in Equation (219) we used the non-singularity of S as proven in Lemma 4.2.2.

Finally, we show $\pi\circ G = S\circ\pi$. Namely, note for $(\omega,x,l)\in\Delta$ we have

$$\begin{aligned}
(\pi\circ G)(\omega,x,l) &= \begin{cases}\pi(\sigma\omega,x,l+1), & R_{\sigma^{-l}\omega}(x)>l+1\\ \pi(\sigma\omega,g^{l+1}_{\sigma^{-l}\omega}(x),0), & R_{\sigma^{-l}\omega}(x)=l+1\end{cases}\\
&= (\sigma\omega, g^{l+1}_{\sigma^{-l}\omega}(x))\\
&= (S\circ\pi)(\omega,x,l),
\end{aligned}$$

so that $\pi \circ G = S \circ \pi$ holds. For general $A \in \mathscr{F}_{\Omega \times X}$ we then see

$$\begin{aligned} S_\star(\pi_\star \mu_\Delta)(A) &= \mu_\Delta(\pi^{-1}(S^{-1}A)) \\ &= \mu_\Delta((\pi \circ G)^{-1}A) \\ &= \pi_\star \mu_\Delta(A), \end{aligned}$$

proving invariance of $\eta = \pi_\star \mu_\Delta$ under S.

Having proven that $\eta \ll \mathbb{P} \times \mu$ and $S_\star \eta = \eta$ we see there exists a shattered measure $\eta_\cdot$ for η by Definition 5.5.3 (and Proposition 5.5.1 to be precise). Corollary 5.5.8 then yields our result. □

5.6 An Example

We shall now showcase the quenched approach by applying the theory of Section 5.4 to a random dynamical system with a non-mixing ergodic driver, namely the irrational rotation from Example 2.1.3. For our definition of the irrational rotation of the circle $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma_\theta)$ over some $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we have based ourselves on the conventions as seen on Page 16 of [23]. We give some further explanation in subsection below.

5.6.1 Conventions made for the (irrational) rotation

We define an equivalence relation $\sim$ on $\mathbb{R}$ such that for $x, y \in \mathbb{R}$ we have

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}.$$

Define $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ as its quotient mapping. We then define $\Omega := \mathbb{R}/\sim$ and endow this with the quotient topology $\mathscr{T}_\sim$. We then naturally have a quotient metric. More specifically, if we let $d_\mathbb{R}$ denote the Euclidian metric on $\mathbb{R}$ we can define the quotient metric as

$$d([x],[y]) = \inf\left\{d_\mathbb{R}(u,v) : u \in [x], v \in [y]\}, \text{for } [x],[y] \in \Omega\right\}.$$

Equipping Ω with $\mathscr{T}_\sim$ yields a compact topological space. More so, for each $x \in \mathbb{R}$ we can represent its equivalence class $[x]$ by $x - \lfloor x \rfloor \in [0,1)$, so that we can write $\Omega = \mathbb{R}/\mathbb{Z} \cong [0,1)$.

As for the σ-algebra $\mathscr{F}_\Omega$ on Ω, we can define

$$\mathscr{F}_\Omega := \{A \subseteq \Omega : \pi^{-1}(A) \in \mathscr{B}(\mathbb{R})\},$$

as the push-forward σ-algebraof the Borel σ-algebra$\mathscr{B}(\mathbb{R})$ on $\mathbb{R}$. In denoting m for the Lebesgue measure on the real line we can define the Lebesgue measure on the circle as

$$\mathbb{P}(E) = m(\pi^{-1}(E) \cap [0,1)).$$

The measure $\mathbb{P}$ is a probability measure which is invariant under the operation $\sigma_\theta : [0,1) \to [0,1)$, $x \mapsto x+\theta \mod 1$. It is well known that $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma_\theta)$ is a uniquely ergodic dynamical system (e.g. as seen in [10, Example 7.2.1], for a definition of unique ergodicity, see Definition 5.6.1). Lastly, we define, given $a, b \in [0,1)$ with $a \neq b$ the open interval $(a,b) \subseteq [0,1)$ as

$$(a,b) = \begin{cases} (a,b), & a <_{\mathbb{R}} b \\ (a,1) \cup (0,b), & a >_{\mathbb{R}} b, \end{cases},$$

where with $<_{\mathbb{R}}$ and $>_{\mathbb{R}}$ we mean the natural ordering on $\mathbb{R}$. The open intervals in $[0,1)$ are open (contained in $\mathscr{T}_\sim$) and measurable. We shall now very briefly explain the ergodic property to prove Lemma 5.6.3 below. Namely, for irrational $\theta \in \mathbb{R}$ the rotation $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma_\Omega)$ is *uniquely ergodic* as defined below.

Definition 5.6.1. [10, Section 7.2] Let $(X, \mathscr{B}, \lambda, T)$ be a dynamical system with X a compact metric space, $(X, \mathscr{B}, \lambda)$ the standard Borel space with the Lebesgue measure, and $T : X \to X$ a continuous transformation. If there exists only one T-invariant probability measure we call T *uniquely ergodic.*

In our case, we shall use unique ergodicity in that is strengthens typical theorems such as the *Pointwise Ergodic Theorem* in obtaining a claim that holds *everywhere* instead of just *almost everywhere* claims in this thesis. In particular, we shall use the following Theorem.

Theorem 5.6.2 ([10] Theorem 7.2.1)**.** *Let $(X, \mathscr{B}, \mu, T)$ be a dynamical system with X a compact metric space, $(X, \mathscr{B}, \mu)$ a standard Borel space and $T : X \to X$ a continuous transformation. Then the following are equivalent:*

1. *There exists an T-invariant probability measure μ on $(X, \mathscr{B})$ such that for all $f \in C(X)$ and all $x \in X$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_X f \, d\mu. \tag{220}$$

2. *T is uniquely ergodic.*

A proof that irrational rotations are uniquely ergodic can be found in Example [13, Example 7.2.1].

Before going into random towers we now prove an observation on circle dynamics that will be vital to proving integrability of return times in Lemma 5.6.7.

Lemma 5.6.3. *Let $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma_\theta)$ be the irrational rotation. Let $P, Q \in \mathscr{F}_\Omega$ be such that $P \bigsqcup Q = \Omega$ and suppose $P^\circ \neq \emptyset$. Then there exists an $M \in \mathbb{R}_{>0}$ such that for every $x \in \Omega$ we have* $\inf\{n \in \mathbb{Z}_{\geq 1} : \sigma_\theta^n x \in P\} \leq M$.

Proof. First note, $\sigma_\theta : \Omega \to \Omega$ is a continuous mapping, so that $\{\sigma_\theta^{-n} P^\circ : n \in \mathbb{Z}_{\geq 1}\}$ is a collection of non-empty open sets. As P° is open there exists a $y \in P^\circ$, $\delta \in \mathbb{R}_{>0}$ such that $(y-\delta, y+\delta) \subseteq P^\circ$. Now note that the mapping

$$f : \Omega \to \Omega, \qquad x \mapsto \max\left(1 - \frac{d(x,y)}{\delta}, 0\right),$$

is continuous and that we have

$$\int_\Omega f(x)\, d\mathbb{P}(x) = \delta > 0.$$

By unique ergodicity of σ_θ we may apply Theorem 5.6.2 and derive that for all $x \in [0,1)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n(x)) = \delta,$$

so $\sigma^n x \in (y-\delta, y+\delta)$ for infinitely many $n \in \mathbb{Z}_{\geq 0}$, and in particular, at least once.

Note this implies that for any $x \in \Omega$ we have an $n \in \mathbb{Z}_{\geq 1}$ such that $\sigma^n x \in (y-\delta, y+\delta)$. Consequently, $\{\sigma^{-n}(y-\delta, y+\delta) : n \in \mathbb{Z}_{\geq 1}\}$ is an open cover for Ω and by compactness of Ω, there exists an $M \in \mathbb{Z}_{\geq 1}$ and a finite sub-cover

$$\mathscr{C} = \{\sigma^{-n}(y-\delta, y+\delta) : n \in \{1,\ldots,M\}\}$$

for Ω. For an arbitrary $x \in \Omega$ we then have a $C \in \mathscr{C}$ and a $k \in \{1,\ldots,M\}$ such that $x \in C = \sigma^{-k}(y-\delta, y+\delta)$ so that $\sigma^k(x) = (y-\delta, y+\delta) \subseteq P^\circ$. We then see

$$\sup_{x\in\Omega} \inf\{n \in \mathbb{Z}_{\geq 1} : \sigma^n x \in P\} \leq M$$

as desired. □

5.6.2 The Quenched Stalling System

We define the base RDS (see (146)) of the random tower constructed later in this section. The ergodic (non-mixing) driver we will equip the base dynamics with, is the irrational rotation. To do so, similar as in Section 3.6.2 we define two mappings

$$f_g : [0,1) \to [0,1) \qquad\qquad f_s : [0,1) \to [0,1)$$

$$x \mapsto 2x \mod 1 \qquad\qquad x \mapsto \begin{cases} x, & x \in (0,\frac{1}{2}) \cup (\frac{3}{4}, 1) \\ 2x-1, & x \in (\frac{1}{2}, \frac{3}{4}), \end{cases}$$

where refer to f_g as a 'go' and f_s as a 'stop'. We now use $([0,1), \mathscr{F}_{[0,1)}, \lambda)$ as the standard Borel space on $[0,1)$. Now, fix $(\Omega, \mathscr{F}_\Omega, \mathbb{P}, \sigma_\theta)$ as the irrational rotation on the circle. Partition Ω into two non-empty half-open intervals $I_g, I_s \in \mathscr{F}_\Omega$, define a mapping

$$\alpha : \Omega \to \{g, s\}$$
$$\omega \mapsto \gamma, \quad \text{such that} \quad \omega \in I_\gamma, \tag{221}$$

and construct the tuple

$$(\Omega \times [0,1), \mathscr{F}_{\Omega\times[0,1)}, \mathbb{P} \times 4\cdot\lambda, U), \qquad U(\omega, x) = (\sigma_\theta \omega, f_{\alpha(\omega)}(x)). \tag{222}$$

We shall now show it is a random dynamical system.

Lemma 5.6.4. *The system* $(\Omega \times [0,1), \mathscr{F}_{\Omega\times[0,1)}, \mathbb{P} \times 4\cdot\lambda, U)$ *as in Equation* (222) *is a (random) dynamical system.*

Proof. We shall use Lemma 3.6.7 and to do so, we need to prove measurability of $(\omega, x) \mapsto f_{\alpha(\omega)}(x)$. To do so, note that for an arbitrary $C \in \mathscr{F}_{[0,1)}$ and $\omega \in \Omega$ we have

$$f^{-1}_{\alpha(\omega)}(C) = I_s \times f_s^{-1}(C) \sqcup I_g \times f_g^{-1}(C) \in \mathscr{F}_{\Omega\times[0,1)},$$

proving measurability of $f_{\alpha(\cdot)}$. Similarly, in having $C \in \mathscr{F}_{[0,1)}$, with $4\cdot\lambda(C) = 0$, we can see that for $\omega \in \Omega$ we have

$$\begin{aligned}\lambda(f^{-1}_{\alpha(\omega)}(C)) &\le \max\left\{\lambda(f_g^{-1}(C)), \lambda(f_s^{-1}(C))\right\}\\ &= 0, \text{by non-singularity of } f_s \text{ and } f_g.\end{aligned}$$

Applying Lemma 3.6.7 yields our result. □

Similar to Lemma 3.6.11 we derive an ergodic equivalence of the system in (222) with an RDS that nicer to work with. To do so, we recall the following definitions from Section 3.6. Define the product measure space $(\{0,1\}^{\mathbb{Z}_{\ge 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\ge 0}}}, \mu)$ with weights $(p_0, p_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Now define the mappings

$$\begin{aligned}\sigma_g &: \{0,1\}^{\mathbb{Z}_{\ge 0}} \to \{0,1\}^{\mathbb{Z}_{\ge 0}} & \sigma_s &: \{0,1\}^{\mathbb{Z}_{\ge 0}} \to \{0,1\}^{\mathbb{Z}_{\ge 0}}\\ &(x_n)_{n\ge 0} \mapsto (x_{n+1})_{n\ge 0} & &(x_n)_{n\ge 0} \mapsto \begin{cases}(x_{n+1})_{n\ge 0}, & (x_n)_{n\ge 0} \in [10]\\ (x_n)_{n\ge 0}, & (x_n)_{n\ge 0} \in [0] \cup [11],\end{cases}\end{aligned}$$

and construct the tuple

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\ge 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\ge 0}}}, \mathbb{P} \times 4\cdot\mu, S), \qquad S(\omega, x) = (\sigma_\theta \omega, \sigma_{\alpha(\omega)}(x)). \tag{223}$$

This system we also show is an RDS.

Lemma 5.6.5. *The system* $(\Omega \times \{0,1\}^{\mathbb{Z}_{\ge 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\ge 0}}}, \mathbb{P} \times 4\cdot\mu, S)$ *in Equation* (223) *is a (random) dynamical system.*

Proof. We shall use Lemma 3.6.7 and to do so, need to prove measurability of $(\omega, x) \mapsto \sigma_{\alpha(\omega)}(x)$. To do so, note that for an arbitrary $C \in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\ge 0}}}$ and $\omega \in \Omega$ we have

$$\sigma^{-1}_{\alpha(\omega)}(C) = I_s \times \sigma_s^{-1}(C) \sqcup I_g \times \sigma_g^{-1}(C) \in \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\ge 0}}},$$

proving measurability of $\sigma_{\alpha(\cdot)}$. Similarly, in having $C \in \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}$, with $4 \cdot \mu(C) = 0$, we can see that for $\omega \in \Omega$ we have

$$\begin{aligned}\mu(\sigma_{\alpha(\omega)}^{-1}(C)) &\leq \max\left\{\mu(\sigma_g^{-1}(C)), \mu(\sigma_s^{-1}(C))\right\} \\ &= 0, \text{by non-singularity of } \sigma_s \text{ and } \sigma_g.\end{aligned}$$

Applying Lemma 3.6.7 yields our result. □

Concluding our first task, we prove the ergodic equivalence between the systems (222) and (223).

Lemma 5.6.6. *The systems*

$$(\Omega \times [0,1), \mathscr{F}_{\Omega\times[0,1)}, \mathbb{P} \times 4 \cdot \lambda, U)$$

and

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$$

as defined in (222) *and* (223) *respectively are ergodically isomorphic.*

Proof. By Lemma A.2.4 we know the systems

$$([0,1), \mathscr{F}_{[0,1)}, \lambda, f_g) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, 4 \cdot \mu, \sigma_g)$$

are ergodically isomorphic. More so, by Corollary A.2.5the dynamical systems

$$([0,1), \mathscr{F}_{[0,1)}, 4 \cdot \lambda, f_s) \text{ and } (\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}^{\mathbb{Z}_{\geq 0}}}, 4 \cdot \mu, \sigma_s)$$

are ergodically isomorphic. Our claim is then immediate by Lemma 3.6.10. □

For the rest of this Section we shall fix $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$ **as in Equation** (223).

We shall now construct a random tower on $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega\times\{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$, and to do so we reuse objects from Section 3.6. We again define

$$Y := \left\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \text{ for all } i \in \mathbb{Z}_{\geq 0} \text{ there is a } j \in \mathbb{Z}_{\geq 1} \text{ such that } x_i \neq x_j\right\}, \tag{224}$$

and fix $\Lambda = Y \cap [10]$. For the reader's convenience we restate the following notation from Lemma 3.6.5,

$$\begin{aligned}&\mathscr{I}_2 := \{[1010]\}, \quad \mathscr{I}_3 := \{[10110], [10010]\}, \quad \text{and for } l \geq 3, \\ &\mathscr{I}_l := \left\{[10a10] \subseteq \Lambda : \quad a \in \{0,1\}^{l-2}, \, a_i a_{i+1} \neq 10, \, i \in \{0, ..., l-3\}\right\},\end{aligned} \tag{225}$$

and $\mathscr{P}_\Lambda := \{I : I \in \mathscr{I}_l, l \in \mathbb{Z}_{\geq 2}\}$. We then define $\Lambda' := \Omega \times \Lambda$ and let

$$\mathscr{P}_{\Lambda'} := \{\Omega \times I : I \in \mathscr{I}_l, l \in \mathbb{Z}_{\geq 2}\}. \tag{226}$$

In noting $\Lambda' = \bigsqcup_{I \in \mathscr{P}_\Lambda} \Omega \times I$ we can apply Lemma 3.6.5 and see $\mathscr{P}_{\Lambda'}$ is indeed a partition of Λ' consisting of countably many sets. It is important to note that the sections of Λ' are constant over Ω, that is $\Lambda'_\omega = \Lambda = [10] \cap Y$ for each $\omega \in \Omega$.

We now need to prove Λ' and $\mathscr{P}_{\Lambda'}$ are a (random) induced domain and a (random) principal partition for $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$. As a step up to this, we first obtain a convenient expression for

$$\begin{aligned} R' : \qquad \Lambda' &\to \mathbb{Z}_{\geq 1} \cup \{\infty\} \\ (\omega, x) &\mapsto \inf\{n \in \mathbb{Z}_{\geq 1} : S^n_\omega(x) \in \Lambda'\}, \end{aligned} \tag{227}$$

in Lemma 5.6.7 showing it indeed takes values in $\mathbb{Z}_{\geq 1}$ and is integrable on Λ'. Note the similarity of the claim with Lemma 3.6.13, and recall the definition of α from Equation (221).

Lemma 5.6.7. *In letting $R : \Lambda \to \mathbb{Z}_{\geq 1}$ be the return time from Lemma 3.6.3 and defining for $k \in \mathbb{Z}_{\geq 2}$ the measurable mapping*

$$\#_k(\omega) := \#\{i \in \{1, \dots, k-1\} : \alpha(\sigma^i_\theta(\omega)) = g\} + 1, \tag{228}$$

we have a return time $R' : \Lambda' \to \mathbb{Z}_{\geq 2}$ which is given by

$$R'(\omega, x) = \inf\{k \in \mathbb{Z}_{\geq 2} : R(x) = \#_k(\omega)\}. \tag{229}$$

Moreover, we have $R' \in L^1(\Lambda')$, making $\mathscr{P}_{\Lambda'}$ into a random principal partition with random induced domain Λ' for $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$.

Proof. First, note that in denoting $\sigma : Y \to Y$ for the shift $\sigma((x_n)_{n \geq 0}) = (x_{n+1})_{n \geq 0}$ we have

$$\sigma_g|_\Lambda = \sigma_s|_\Lambda = \sigma|_\Lambda \quad \text{and} \quad \sigma_s|_{Y \setminus \Lambda} = \mathrm{Id}|_{Y \setminus \Lambda}, \, \sigma_g|_{Y \setminus \Lambda} = \sigma|_{Y \setminus \Lambda}.$$

Recall we have $\Lambda'_\omega = \Lambda = [10] \cap Y$ for each $\omega \in \Omega$. Then note that for each $\omega \in \Omega$ we have $S_\omega|_\Lambda = \sigma|_\Lambda$ and for $x \in Y \setminus \Lambda$ that

$$S_\omega|_{Y \setminus \Lambda}(x) = \begin{cases} \sigma(x), & \alpha(\omega) = g \\ x, & \alpha(\omega) = s. \end{cases}$$

Now let R' be as given by (227). First, we note that for any $(\omega, x) \in \Lambda'$ we have as $x \in [10]$ that

$$\begin{aligned} R'(\omega, x) &= \inf\left\{n \in \mathbb{Z}_{\geq 1} : S^n_\omega(x) \in \Lambda'_{\sigma^n \omega}\right\} \\ &\geq \inf\left\{n \in \mathbb{Z}_{\geq 1} : \sigma^n_g(x) \in [10]\right\} \\ &\geq 2. \end{aligned}$$

We can then see that for any $\omega \in \Omega$ and $x \in \Lambda'_\omega$ we have that for $n \in \mathbb{Z}_{\geq 2}$, $2 \leq n \leq R'(\omega, x)$ we see that

$$S^n_\omega(x) = S_{\sigma_\theta^{n-1}(\omega)} \circ \cdots \circ S_{\sigma_\theta(\omega)}(x) = \sigma^{\#_n(\omega)}(x).$$

We then derive that for any $(\omega, x) \in \Lambda'$ we have as $R'(\omega, x) \geq 2$

$$\begin{aligned} R'(\omega, x) &= \inf\left\{n \in \mathbb{Z}_{\geq 2} : S^n_\omega(x) \in \Lambda'_{\sigma^n \omega}\right\} \\ &= \inf\left\{n \in \mathbb{Z}_{\geq 2} : \sigma^{\#_n(\omega)}(x) \in \Lambda'_{\sigma^n \omega}\right\} \\ &= \inf\left\{n \in \mathbb{Z}_{\geq 2} : \sigma^{\#_n(\omega)}(x) \in [10] \cap Y\right\}. \end{aligned}$$

Now to prove R' is constant over the elements of the partition $\mathscr{P}_{\Lambda'}$, we note that for any $\omega \in \Omega$ and any $P_\omega \in \mathscr{P}_{\Lambda'_\omega}$ we have an $l \in \mathbb{Z}_{\geq 2}$ and $I_l \in \mathscr{I}_l$ such that $P_\omega = I_l$. Recalling the property $R|_{I_l} \equiv l$ from Lemma 3.6.5, and that

$$R(x) = \inf\{n \in \mathbb{Z}_{\geq 2} : \sigma^n(x) \in [10] \cap Y\},$$

with R as in Lemma 3.6.3 we note that for any $x \in I_l$ we have

$$R'(\omega, x) = \inf\left\{n \in \mathbb{Z}_{\geq 2} : \sigma^{\#_n(\omega)}(x) \in [10] \cap Y\right\} = \inf\{n \in \mathbb{Z}_{\geq 2} : \#_n(\omega) = l\}, \tag{230}$$

proving that $R'_\omega(\cdot) := R'(\omega, \cdot)$ is constant on elements $P_\omega \in \mathscr{P}_{\Lambda'_\omega}$. More so, this implies for general $x \in \Lambda_\omega$ that

$$R'(\omega, x) = \inf\left\{n \in \mathbb{Z}_{\geq 2} : \sigma^{\#_n(\omega)}(x) \in [10] \cap Y\right\} = \inf\{n \in \mathbb{Z}_{\geq 2} : \#_n(\omega) = R(x)\},$$

proving Equation (229). To show R'_ω takes values in $\mathbb{Z}_{\geq 1}$ note that by Lemma 5.6.3 we have an $M \in \mathbb{Z}_{\geq 1}$ so that

$$\#\{i \in \{1, \ldots, M\} : \alpha(\sigma_\theta^i \omega) = g\} \geq 1.$$

This implies that for any $\omega \in \Omega$ we have

$$\#_n(\omega) \geq \left\lfloor \frac{n}{M} \right\rfloor. \tag{231}$$

Combining this with Equation (229) then shows that for $\omega \in \Omega$ and $x \in I_l$ we have

$$R'(\omega, x) = \inf\{k \in \mathbb{Z}_{\geq 2} : R(x) = \#_k(\omega)\} \leq M(l-1) \tag{232}$$

proving the return time indeed takes finite values (as for any $\omega \in \Omega$ and any $P_\omega \in \mathscr{P}_{\Lambda'_\omega}$ we have $P_\omega = I_l$ for some $l \in \mathbb{Z}_{\geq 2}$ and $I_l \in \mathscr{I}_l$). Finally, to show $R' \in L^1(\Lambda')$, we note that for all $\omega \in \Omega$ we have

$$4 \cdot \mu(\Lambda'_\omega) = \mu([10] \cap X) = \frac{1}{4}, \quad \text{so that} \quad 4 \cdot (\mathbb{P} \times \mu)(\Lambda') = 1.$$

Moreover, as seen in the proof of Lemma 3.6.6 we have for each $l \in \mathbb{Z}_{\geq 2}$ and $I_l \in \mathscr{I}_l$ that $\mu_\Lambda(I_l) = 4 \cdot 2^{-l-2}$ and $\#\mathscr{I}_l = l-1$. As we know $\mu_\Lambda = (4 \cdot \mu)_\Lambda$, we then calculate

$$
\begin{aligned}
\int_{\Lambda'} R(\omega, x)\, d(4 \cdot \mathbb{P} \times \mu)(\omega, x) &= 4 \int_\Omega \int_{\Lambda'_\omega} R_\omega(x)\, d\mu(x)\, d\mathbb{P}(\omega) \\
&= 4 \cdot \sum_{l \geq 2} \sum_{I_l \in \mathscr{I}_l} \int_\Omega \int_{I_l} R_\omega(x)\, d\mu(x)\, d\mathbb{P}(\omega) && (233) \\
&\leq 4 \cdot \sum_{l \geq 2} \sum_{I_l \in \mathscr{I}_l} \int_\Omega \int_{I_l} M(l-1)\, d\mu(x)\, d\mathbb{P}(\omega) && (234) \\
&\leq 4 \cdot \sum_{l \geq 2} \int_\Omega M(l-1) \cdot 2^{-l-2}(l-1)\, d\mathbb{P}(\omega) \\
&= \sum_{l \geq 2} (l-1)^2 \cdot M \cdot 2^{-l} < \infty, && (235)
\end{aligned}
$$

where in Equation (233) we used the Monotone Convergence Theorem; in Equation (234) we used (232); and in Equation (235) we used the Ratio Test [12, Theorem 2.31]. We conclude $R \in L^1(\Lambda')$ and that Λ' is a random principal partition with random induced domain Λ'. □

As in Definition 5.1.7 we can then construct a random tower system $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ on Λ' and $\mathscr{P}_{\Lambda'}$ on

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S), \quad S(\omega, x) = (\sigma_\theta \omega, \sigma_{\alpha(\omega)}(x)).$$

We shall now prove $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is acip admissible as seen in Definition 5.3.14. That is, it is measure-regular (see Definition 5.2.1) and satisfies the Markov Property (see Definition 5.3.1) and has bounded distortion (see Definition 5.3.12). As in Remark 5.1.5 we shall fix $\dot{\Omega}$ as well (but note that in this particular instance we can likely take $\dot{\Omega} = \Omega$).

Proposition 5.6.8. *The system* $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ *is acip-admissable. In particular there exists an acip* $\nu \ll \mu_\Delta$.

Proof. We need to verify measure-regularity, the markov property and bounded distortion for $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$. Fix $\omega \in \dot{\Omega}$.

First suppose we have an $A \in \mathscr{P}_{\Lambda'}$ such that $G_\omega(A_\omega) \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$. Then according to Corollary 5.1.12 we have a $P \in \mathscr{P}_{\Lambda'_\omega}$, $k \in \mathbb{Z}_{\geq 0}$ such that $A_\omega = P_{\sigma^{-k}\omega} \times \{k\}$. If $G_\omega[A_\omega] \cap (\Delta_{\sigma\omega,0} \times \{0\}) \neq \emptyset$ we then have by Definition 5.1.7 for each $x \in A_\omega$ that $G_\omega|_{A_\omega}(x,k) = (S^{k+1}_{\sigma^{-k}\omega}(x), 0)$, where $R_{\sigma^{-k}\omega}|_{A_{\omega,k}} \equiv k+1$. By definition of $\mathscr{P}_{\Lambda'}$ we then know there exists a $l \in \mathbb{Z}_{\geq 2}$ and an $I_l \in \mathscr{I}_l$ with $P_{\sigma^{-k}\omega} = I_l$, so that for any $x \in P_{\sigma^{-k}\omega}$ we have

$$
\begin{aligned}
S^{k+1}_{\sigma^{-k}\omega}(x) &= S^{k}_{\sigma^{-k+1}\omega}(\sigma_g(x)) \\
&= \sigma_b^{k-(l-1)}\sigma_g^{l-1}(\sigma_g(x)) \\
&= \sigma_g^l(x),
\end{aligned}
$$

so that we have $G_\omega|_{A_\omega}(x,k) = (\sigma_g^l(x), 0)$. By Lemma 3.6.5 we know $\sigma_g^l|_{I_l} : I_l \to \Lambda$ is bi-measurable satisfying $((\sigma_g^l)^{-1})_\star\mu \ll \mu$. Finally, we know

$$
\left|\frac{J\sigma_g^l(x)}{J\sigma_g^l(y)} - 1\right| = \left|\frac{(1/2^{-1})^l}{(1/2^{-1})^l} - 1\right| = 0, \tag{236}
$$

holds for almost every $x, y \in I_l$.

Proceeding, in denoting $c_{\{k\}}, c_{\{0\}}$ for the counting measures restricted to $\{k\}$ and $\{0\}$ respectively, we see the mapping $t_k : \{k\} \to \{0\}, k \mapsto 0$ is trivially bi-measurable and pbn-singular with respect to the measure spaces $(\{k\}, \{\emptyset, \{k\}, c_{\{k\}})$ and $(\{0\}, \{\emptyset, \{0\}, c_{\{0\}})$. More so, we see $J(c_k)(k) = 1$. In noting that $G_\omega|_{A_\omega} = \sigma_g^l|_{I_l} \times t_k$, we can apply Lemma 4.3.18 to show $G_\omega|_{A_\omega}$ is bi-measurable and satisfies $(G_\omega|^{-1}_{A_\omega})_\star\mu_{\Delta_{\sigma\omega}} \ll \mu_{A_\omega}$, showing measure-regularity. Moreover, as again by Lemma 4.3.18 we have

$$
J(G_\omega)|_{A_\omega} = J(\sigma_g^l|_{I_l})J(t_k) = (1/2^{-1})^l \cdot 1 \quad \mu_{A_\omega}\text{-almost surely,}
$$

so that $J(G_\omega|_{A_\omega})$ is constant and positive almost surely so that the Markov property and bounded distortion follow as well.

As $(\Delta, \mathscr{F}_\Delta, \mu_\Delta, G)$ is acip admissible, we obtain an acip $\nu \ll \mu_\Delta$ with $\frac{d\nu}{d\mu_\Delta} \leq M$ for some $M \in \mathbb{R}_{>1}$ by Theorem 5.4.7. □

From Proposition 5.6.8 we can directly apply Corollary 5.5.9 to obtain an acip ν' for $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S)$. The same measure ν' will then be an acip for $(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, S)$. We shall use the ergodic equivalence of this system with the system $(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \mu, U)$ as seen in Equation (222) to show there exists an acip ν for $(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \mu, U)$.

Corollary 5.6.9. *The system $(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \mu, U)$ admits an acip ν and a shattered measure $\nu.$ such that for almost every $\omega \in \Omega$ the shard ν_ω is a probability measure and satisfies $(U_\omega)_\star\eta_\omega = \eta_{\sigma\omega}$.*

Proof. By Lemma 5.6.6 we know that

$$
(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, U) \text{ and } (\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times \mu, S)
$$

are ergodically equivalent. By Proposition 5.6.8 we can then construct an acip-admissible tower on

$$(\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\Omega \times \{0,1\}^{\mathbb{Z}_{\geq 0}}}, \mathbb{P} \times 4 \cdot \mu, S),$$

so that we may find an acip $\nu' \ll \mathbb{P} \times 4 \cdot \mu$. Note that we have $\nu' \ll \mathbb{P} \times \mu$ and $S_\star \nu' = \nu'$ as well. By Corollary 2.1.19 we obtain an acip $\eta \ll \mathbb{P} \times \lambda$ for

$$(\Omega \times [0,1), \mathscr{F}_{\Omega \times [0,1)}, \mathbb{P} \times \lambda, U).$$

The existence of a shattered measure $\eta_\cdot$ of η satisfying $(U_\omega)_\star(\eta_\omega) = \eta_{\sigma\omega}$ then follows from Corollary 5.5.8 (note this Corollary applies to *general* random dynamical systems so in particular to this case). □

A Appendix

A.1 Some Radon-Nikodym Derivative Identities

The following Lemma relates the image of the push-forward mapping with the support of the Radon-Nikodym derivative.

Lemma A.1.1. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces with $f : X \to Y$ a measurable mapping with $f_\star \mu \ll \nu$. Now let $A \in \mathscr{F}$ with $f(A) \in \mathscr{B}$. We then have*

$$\frac{d(f|_A)_\star \mu}{d\nu}(y) = 0 \quad \textit{for almost every } y \notin f(A). \tag{237}$$

Proof. Let $B \in \mathscr{B}$, with $B \subseteq Y \setminus f(A)$ be given arbitrarily. Note this implies $B \cap f(A) = \emptyset$ and we (generally) have $f(A) \cap B = f[A \cap f^{-1}(B)]$, so that

$$A \cap f^{-1}B = \emptyset \Leftrightarrow f[A \cap f^{-1}(B)] = \emptyset \Leftrightarrow f(A) \cap B = \emptyset.$$

Consequently, we have

$$\int_B \frac{d(f|_A)_\star \mu}{d\nu}(y)\, d\nu(y) = (f|_A)_\star \mu(B) = \mu((f|_A)^{-1}(B)) = \mu(A \cap f^{-1}(B)) = 0,$$

which shows Equation (237) as $B \in \mathscr{B}, B \subseteq Y \setminus f(A)$, was given arbitrarily. □

Lemma A.1.2. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and $f : X \to Y$ a measurable mapping so that $f_\star \mu \ll \nu$. Then for $A, B \in \mathscr{F}$ with $A \subseteq B$ we have*

$$\frac{d(f|_A)_\star \mu}{d\nu} \leq \frac{d(f|_B)_\star \mu}{d\nu}, \textit{ holds } \nu\textit{-a.s.}.$$

Proof. Let $C \in \mathscr{F}_Y$ be arbitrary and note

$$\begin{aligned}\int_C \frac{d(f|_A)_\star\mu}{d\nu}(y)\,d\nu(y) &= \mu(A \cap f^{-1}(C)) \\ &\leq \mu(B \cap f^{-1}(C)) \\ &= \int_C \frac{d(f|_B)_\star\mu}{d\nu}(y)\,d\nu(y).\end{aligned}$$

As $C \in \mathscr{F}_Y$ was given arbitrarily, our claim follows. □

Lemma A.1.3. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be σ-finite measure spaces and $f : X \to Y$ a measurable mapping so that $f_\star\mu \ll \nu$. Then suppose $A \in \mathscr{F}$, $B \in \mathscr{B}$ so that $A \subseteq f^{-1}(B)$, then we have*

$$(f|_A)_\star\mu \ll \nu_B.$$

Proof. Note that for general $C \in \mathscr{B}$ with $\nu_B(C) = 0$ that $f_\star\mu(B \cap C) = 0$ as $f_\star\mu \ll \nu$. We see

$$(f|_A)_\star\mu(C) = \mu(A \cap f^{-1}(C)) \leq f_\star\mu(B \cap C) = 0.$$

□

Lemma A.1.4. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be finite measure spaces and $f : X \to Y$ a measurable mapping so that $f_\star\mu \ll \nu$. Then suppose $A \in \mathscr{F}$, $B \in \mathscr{B}$ so that $A \subseteq f^{-1}(B)$. We then have*

$$\frac{d(f|_A)_\star\mu}{d\nu}|_B = \frac{d(f|_A)_\star\mu}{d\nu_B}, \quad \nu_B\text{-almost surely.}$$

Proof. First note that by Lemma A.1.3 and Theorem 2.1.11 we have $\frac{d(f|_A)_\star\mu}{d\nu_B} \in L^1(\nu_B)$. Moreover, note that

$$(f|_A)_\star \ll \nu_B \ll \nu$$

so that we may write

$$\frac{d(f|_A)_\star\mu}{d\nu} = \frac{d(f|_A)_\star\mu}{d\nu_B}\frac{d\nu_B}{d\nu}, \nu\text{-almost surely,}$$

so that in particular

$$\frac{d(f|_A)_\star\mu}{d\nu}|_B = \frac{d(f|_A)_\star\mu}{d\nu_B}|_B \cdot \frac{d\nu_B}{d\nu}|_B,\ \nu_B\text{-almost surely.}$$

In noting that $\frac{d\nu_B}{d\nu}|_B \equiv 1$, ν_B-almost surely, we then see that for arbitrary $C \in \mathscr{B}_B$

$$\begin{aligned}\int_C \frac{d(f|_A)_\star\mu}{d\nu}|_B(y)\,d\nu_B(y) &= \int_C \frac{d(f|_A)_\star\mu}{d\nu}(y)\,d\nu_B(y) \\ &= \int_C \frac{d(f|_A)_\star\mu}{d\nu_B}(y)\,d\nu_B(y),\end{aligned}$$

proving the statement. □

Lemma A.1.5. *Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be finite measure spaces and $f : X \to Y$ a measurable mapping so that $f_\star \mu \ll \nu$. Let $A \in \mathscr{F}$ then we have*

$$\frac{d(f|_A)_\star \mu}{d\nu} = \frac{d f_\star \mu_A}{d\nu}.$$

Proof. Note that for $C \in \mathscr{B}$ we have

$$(f|_A)_\star \mu(C) = \mu(A \cap f^{-1}(C)) = f_\star \mu_A(C),$$

and so

$$f_\star \mu_A(C) \leq f_\star \mu(C)$$

so $f_\star \mu_A \ll \mu$. The claim follows. □

Lemma A.1.6. *Let $(X, \mathscr{F}, \mu)$, $(Y, \mathscr{G}, \nu)$ and $(Z, \mathscr{H}, \eta)$ be finite measure spaces and let $T : X \to Y$, and $U : Y \to Z$ be measurable mappings such that $T_\star \mu \ll \nu$ and $U_\star \nu = \eta$ with U bi-measurable. Then we have*

$$\frac{d(U \circ T)_\star \mu}{d\eta}(z) = \frac{d T_\star \mu}{d\nu}(U^{-1} z), \quad \eta\text{-almost surely.}$$

Proof. Let $C \in \mathscr{H}$ be arbitrary. Note then

$$\begin{aligned}
\int_C \frac{d(U \circ T)_\star \mu}{d\eta}(z)\, d\eta(z) &= (T_\star \mu)(U^{-1} C) \\
&= \int_{U^{-1}C} \frac{d(T_\star \mu)}{d\nu}(y)\, d\nu(y) \\
&= \int_{U^{-1}C} \frac{d(T_\star \mu)}{d\nu}(U^{-1} \circ U(y))\, d\nu(y) \\
&= \int_C \frac{d(T_\star \mu)}{d\nu}(U^{-1}(y))\, dU_\star \nu(y) \qquad (238) \\
&= \int_C \frac{d(T_\star \mu)}{d\nu}(U^{-1}(z))\, d\eta(z), \qquad (239)
\end{aligned}$$

where in Equation (238) we used Lemma 2.1.10 with the integrability of the Radon-Nikodym derivative. □

The following lemma is here for easy reference in Proposition 5.5.1.

Lemma A.1.7. *Suppose $(\Omega, \mathscr{F}_\Omega, \mathbb{P})$ is a probability space and $(X, \mathscr{F}_X, \mu)$ is a σ-finite measure space. Construct $(\Omega \times X, \mathscr{F}_{\Omega \times X}, \mathbb{P} \times \mu)$. Suppose we have a finite positive measure ν on $\Omega \times X$, with $\nu \ll \mathbb{P} \times \mu$. Then we have:*

(1) *The Radon-Nikodym derivative* $\frac{d\nu}{d\mathbb{P}\times\mu}$ *is non-negative* $\mathbb{P}\times\mu$*-almost everywhere;*

(2) *For almost every* $\omega\in\Omega$ *and for each* $A\in\mathscr{F}_X$ *we have*

$$\int_A \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\geq 0;$$

(3) *For every* $f\in L^1(\nu)$ *we have* $f\cdot\frac{d\nu}{d\mathbb{P}\times\mu}\in L^1(\Omega\times X)$.

Proof. Firstly note that for general $A\in\mathscr{F}_{\Omega\times X}$ we have

$$\int_A \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x)=\nu(A)\geq 0,$$

showing Claim (1) of the lemma. Furthermore, for general $A\in\mathscr{F}_X$ and $O\in\mathscr{F}_\Omega$ we have using Theorem 2.1.8

$$\begin{aligned}\int_O\int_A \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\,d\mathbb{P}(\omega) &= \int_{O\times A}\frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x)\\ &=\nu(O\times A)\\ &\geq 0,\end{aligned}$$

so that indeed for almost every $\omega\in\Omega$ and for each $A\in\mathscr{F}_X$

$$\int_A \frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d\mu(x)\geq 0,$$

proving Claim (2). Lastly, note that for $f\in L^1(\nu)$ we have

$$\int_{\Omega\times X}\left|f(\omega,x)\right|\frac{d\nu}{d\mathbb{P}\times\mu}(\omega,x)\,d(\mathbb{P}\times\mu)(\omega,x)=\int_{\Omega\times X}\left|f(\omega,x)\right|\,d\nu(\omega,x)<\infty,$$

so $f\cdot\frac{d\nu}{d\mathbb{P}\times\mu}\in L^1(\Omega\times X)$. □

A.2 Ergodic Equivalence of Bernoulli shift with Doubling map

The rest of this Section is dedicated to proving the equivalence of the doubling maps and the binary shift. A technical complication in the lemma below is that binary expansions need not be unique. Luckily, there are at most countably many real values having a non-unique expansion.

Lemma A.2.1. *Let*

$$\begin{aligned}\bar{\pi}:\quad \{0,1\}^{\mathbb{Z}_{\geq 0}}\setminus\{(1_n)_{n\geq 0}\} &\to [0,1) \\ (x_n)_{n\geq 0} &\mapsto \sum_{n=0}^{\infty} x_n 2^{-n-1}\end{aligned}\tag{240}$$

and

$$\begin{aligned} \phi : [0,1) &\to \{0,1\}^{\mathbb{Z}_{\geq 0}} \\ x &\mapsto \left(n \mapsto \begin{cases} 0, & if 2^n x - \lfloor 2^n x \rfloor < \frac{1}{2} \\ 1, & if 2^n x - \lfloor 2^n x \rfloor \geq \frac{1}{2} \end{cases} \right), \end{aligned} \tag{241}$$

then we have $\bar{\pi} \circ \phi(x) = x$ *for each* $x \in [0,1)$.

Proof. **Well-definedness of** $\bar{\pi}$ Note that for every $x \in \{0,1\}^{\mathbb{Z}_{\geq 0}}$ we have $x_n \geq 0$ for all $n \in \mathbb{Z}_{\geq 0}$ and $\bar{\pi}(x) \leq \sum_{n=0}^{\infty} 2^{-n-1} = 1$, so $\bar{\pi}(x)$ is absolutely convergent. Note that $\bar{\pi}(x) = 1$ only holds for $x \in \{(1_n)_{n \geq 0}\}$, so $\bar{\pi}$ indeed takes values in $[0,1)$.

ϕ **is a right-inverse for** $\bar{\pi}$**:** We claim that

$$\bar{\pi}(\phi(x)) := \sum_{n=0}^{\infty} 2^{-n-1} (\phi(x))_n = x \text{ for each } x \in [0,1).$$

For a proof, suppose $N \geq 0$ and write $(x_n)_{n \geq 0} := \phi(x)$. We will show

$$x - \sum_{n=0}^{N} 2^{-n-1} x_n \in [0, 2^{-N-1}) \tag{242}$$

by induction. First for the case $N = 0$, note that

$$x - \sum_{n=0}^{0} 2^{-n-1} x_n = \begin{cases} x - 0 \cdot \frac{1}{2} \text{ if } x < \frac{1}{2} \\ x - 1 \cdot \frac{1}{2} \text{ if } x \geq \frac{1}{2} \end{cases} \in \left[0, \frac{1}{2}\right) = [0, 2^{-N-1}).$$

Then suppose for some fixed $N \geq 0$ Equation (242) to holds. Then we have

$$x - \sum_{n=0}^{N+1} 2^{-n-1} x_n \in [0, 2^{-N-1} - 2^{-N-2}) = [0, 2^{-N-2}),$$

so that by induction Equation (242) holds for all $N \geq 0$. Consequently, we note

$$x - \bar{\pi}((x_n)_{n \geq 0}) = \lim_{N \to \infty} x - \sum_{n=0}^{N} (2^{-n-1} x_n) = 0,$$

so that indeed

$$\bar{\pi}(\phi(x)) = \bar{\pi}((x_n)_{n \geq 0}) = x \tag{243}$$

□

We shall now show we obtain a bijection when we restrict the domain of $\bar{\pi}$ to a suitable subset.

Lemma A.2.2. *Let*

$$X := \{(x_n)_{n\geq 0} \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : \textit{for all } i \in \mathbb{Z}_{\geq 0} \textit{ there is a } j \geq i \textit{ such that } x_j = 0\}, \tag{244}$$

$$\begin{aligned} \pi : \qquad X &\to [0,1) \\ (x_n)_{n\geq 0} &\mapsto \sum_{n=0}^{\infty} x_n 2^{-n-1} \end{aligned} \tag{245}$$

and

$$\begin{aligned} \phi : [0,1) &\to X \\ x &\mapsto \left(n \mapsto \begin{cases} 0, \ if 2^n x - \lfloor 2^n x \rfloor < \frac{1}{2} \\ 1, \ if 2^n x - \lfloor 2^n x \rfloor \geq \frac{1}{2} \end{cases} \right), \end{aligned} \tag{246}$$

then ϕ *and* π *are well-defined on their respective domains and we have*

$$\phi \circ \pi = Id_X, \qquad \pi \circ \phi = Id_{[0,1)}. \tag{247}$$

Proof. ϕ **takes values in** X**:** Suppose we have an $x \in [0,1)$, such that $(x_n)_{n\geq 0} := \phi(x) \notin X$. Then there exists a lowest $N \in \mathbb{Z}_{\geq 0}$ such that $(n \mapsto x_n)|_{n\geq N} \equiv 1$.

If $N = 0$ then $\sum_{n=0}^{\infty} 2^{-n-1} x_n = 1$, which contradicts Lemma A.2.1 so suppose $N \geq 1$. We then necessarily have $x_{N-1} = 0$ and define

$$(y_n)_{n\geq 0} := \begin{cases} x_n, & n < N-1 \\ 1, & n = N-1 \\ 0, & n > N-1 \end{cases}.$$

Note $(y_n)_{n\geq 0} \in X$ and $y := \sum_{i=0}^{N-1} 2^{-i-1} y_i \in [0,1)$. We then see

$$\begin{aligned} \bar{\pi}((x_n)_{n\geq 0}) - \bar{\pi}((y_n)_{n\geq 0}) &= 2^{-n-1} - \sum_{i=n}^{\infty} 2^{-i-2} \\ &= 2^{-n-1} - \sum_{i=0}^{\infty} 2^{-n-i-2} \\ &= 2^{-n-1}(1 - \sum_{i=0}^{\infty} 2^{-1-i}) \\ &= 0, \end{aligned}$$

so by Lemma A.2.1 we see $x = \bar{\pi}(\phi(x)) = \bar{\pi}(\phi(y))$, so that $x = y$. Finally, note then that

$$2^{N-1}x - \lfloor 2^{N-1}x \rfloor = 2^{N-1}y - \lfloor 2^{N-1}y \rfloor = 1,$$

contradicting $x_{N-1} = 0$, so $x \in X$.

ϕ is a left-inverse for π: Suppose $x \in X$. We shall show

$$x_m = (\phi \circ \pi)(x)_m, \quad \text{for all } m \in \mathbb{Z}_{\geq 0}. \tag{248}$$

To do so let $m \in \mathbb{Z}_{\geq 0}$, and to evaluate $(\phi \circ \pi)(x)_m$ we note

$$\begin{aligned} 2^m \sum_{n=0}^{\infty} x_n 2^{-n-1} - \lfloor 2^m \sum_{n=0}^{\infty} x_n 2^{-n-1} \rfloor &= 2^m \sum_{n=m}^{\infty} x_n 2^{-n-1} \\ &= \sum_{n=0}^{\infty} x_{n+m} 2^{-n-1} \\ &\geq \frac{1}{2} \text{ if and only if } x_m = 1 \end{aligned}$$

where the last inequality holds as $x \in X$. Thus,

$$(\phi \circ \pi)(x)_m = 1 \Leftrightarrow \sum_{n=0}^{\infty} x_{n+m} 2^{-n-1} \geq \frac{1}{2} \Leftrightarrow x_m = 1.$$

Similarly, we can derive

$$(\phi \circ \pi)(x)_m = 0 \Leftrightarrow \sum_{n=0}^{\infty} x_{n+m} 2^{-n-1} < \frac{1}{2} \Leftrightarrow x_m = 0,$$

as $x \in X$. We conclude Equation (248) for all $m \in \mathbb{Z}_{\geq 0}$, so that $\phi \circ \pi = \mathrm{Id}|_X$.

In addition to the above, Lemma A.2.1 yields $\pi \circ \phi = \mathrm{Id}_{[0,1)}$ from which Identity (247) follows. □

The following corollary is useful when applying measure theory to the set X of Equation (244) as in Lemma 4.3.17. For the notation on cylinders, see Section 3.6.

Corollary A.2.3. *Let, $n \in \mathbb{Z}_{\geq 1}$, and $\gamma_i \in \{0,1\}$ for $i \in \{0,\ldots,n-1\}$. For the cylinder $[\gamma_0 \cdots \gamma_{n-1}] \subseteq X$ we have that*

$$\left[\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \quad \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n}\right) = \pi[\gamma_0 \cdots \gamma_{n-1}]. \tag{249}$$

Proof. First we prove

$$\left[\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \quad \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n}\right) \supseteq \pi[\gamma_0 \cdots \gamma_{n-1}]. \tag{250}$$

To do so, let $(x_k)_{k\geq 0} \in [\gamma_0 \cdots \gamma_{n-1}]$ and note

$$\pi((x_k)_{k\geq 0}) = \sum_{k=0}^{\infty} 2^{-k-1} x_k = \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + \sum_{k=n}^{\infty} 2^{-k-1} x_k.$$

Now as $\sum_{k=n}^{\infty} 2^{-k-1} x_k < 2^{-n}$, we have

$$\pi((x_k)_{k\geq 0}) \in \left[\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \quad \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n} \right).$$

To show

$$\left[\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \quad \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n} \right) \subseteq \pi[\gamma_0 \cdots \gamma_{n-1}], \tag{251}$$

let $x \in [\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \quad \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n})$. Note that for $l \in \{0,\ldots,n-1\}$ we have

$$2^l x - \lfloor 2^l x \rfloor = \sum_{k=l}^{n-1} 2^{-k-1} \cdot \gamma_k \cdot 2^l,$$

so that

$$2^l x - \lfloor 2^l x \rfloor \geq \frac{1}{2} \text{ if } \gamma_l = 1, \text{ and } 2^l x - \lfloor 2^l x \rfloor < \frac{1}{2} \text{ if } \gamma_l = 0,$$

meaning that $\phi(x)_l = \gamma_l$ for $l \in \{0,\ldots,n-1\}$. As $\phi = \pi^{-1}$ this implies Equation (251).

Equations (250) and (251) imply Equation (249), proving the corollary. □

Lemma A.2.4. *Let* $([0,1), \mathscr{B}[0,1), \lambda, D)$ *be the standard Borel measure space equipped with the doubling map and* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma)$ *be the Bernoulli shift with for all* $i \in \mathbb{Z}_{\geq 0}$

$$\mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 1) = \mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 0) = \frac{1}{2}.$$

Then the mapping π *from Lemma A.2.2 is an ergodic isomorphism between* $([0,1), \mathscr{B}[0,1), \lambda, D)$ *and* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma)$.

Proof. The proof comes down to verifying the properties of Definition 2.1.15. We fix X as in Equation 244.

Measurability of X Taking for all $n \in \mathbb{Z}_{\geq 0}$, the cylinders

$$C_{n,0} := \{(x_n)_{n\geq 0} \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_n = 0\}$$

we can write

$$X = \limsup_{n\to\infty} C_{n,0} := \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} C_{m,0} \right) \in \mathscr{F}_{\{0,1\}}.$$

The set X is of full measure First note that for each $l \in \mathbb{Z}_{\geq 0}$ we have

$$\mathbb{P}(\bigcup_{m=n}^{n+l} C_{m,0}) = 1 - 2^{-l} \text{ so that } \mathbb{P}(\bigcup_{m=n}^{\infty} C_{m,0}) = 1.$$

The set X is, as the countable intersection of sets of full measure, has full measure, that is, $\mathbb{P}(X) = 1$.

π and ϕ are measurable We shall make use of Lemma 2.1.5. First write $\mathscr{I} = \pi^{-1}\mathscr{C}$, where $\mathscr{C}$ denote the cylinders in X. For clarity sake, note that by Corollary A.2.3 we have

$$\mathscr{I} := \left\{ \left[\sum_{k=0}^{n-1} 2^{-k-1}\gamma_k, \sum_{k=0}^{n-1} 2^{-k-1}\gamma_k + 2^{-n} \right) \subseteq [0,1) : \gamma_k \in \{0,1\}, k \in \{0,\dots,n-1\}, n \geq 1 \right\}.$$

Using this, we note for any open interval $[a,b) \subseteq [0,1)$ we have by Lemma A.2.2

$$[a,b) = \bigcap_{n=0}^{\infty} \left[\sum_{i=0}^{n} 2^{-i-1}(\phi(a))_i, 1 \right) \quad \cap \quad \bigcup_{n=0}^{\infty} \left[0, \sum_{i=0}^{n} 2^{-i-1}(\phi(b))_i \right),$$

so that $\sigma(\pi^{-1}\mathscr{C}) = \mathscr{B}[0,1)$. Now as $\sigma(\mathscr{C}) = \mathscr{F}_X$, we see by Lemma 2.1.4 that $\pi : X \to [0,1)$ is measurable. Conversely $\phi^{-1}(\mathscr{I}) = \mathscr{C} \subseteq \mathscr{F}_X$ and $\sigma(\mathscr{I}) = \mathscr{B}[0,1)$ so $\phi : [0,1) \to X$ is also measurable by Lemma 2.1.4.

The property $\pi \circ S = D \circ \pi$ holds Let $x \in X$. Note by Corollary A.2.3 we have $\pi(x) \in [0, \frac{1}{2})$ if and only if $x_0 = 0$ and $\pi(x) = [\frac{1}{2}, 1)$ if and only if $x_0 = 1$, so that $x_0 = \lfloor 2 \cdot \sum_{n=0}^{\infty} 2^{-n-1} x_n \rfloor$. Note then

$$\begin{aligned}
\pi \circ S(x) &= \pi((x_{n+1})_{n\geq 0}) \\
&= \sum_{n=0}^{\infty} 2^{-n-1} x_{n+1} \\
&= \sum_{n=0}^{\infty} 2^{-n} x_n - x_0 \\
&= \sum_{n=0}^{\infty} D(2^{-n-1} x_n) - \left\lfloor \sum_{n=0}^{\infty} D 2^{-n-1} x_n \right\rfloor \\
&= D \sum_{n=0}^{\infty} (2^{-n-1} x_n) - \left\lfloor D(\sum_{n=0}^{\infty} 2^{-n-1} x_n) \right\rfloor \\
&= D \circ \pi(x).
\end{aligned}$$

The property $\mathbb{P} = \phi_\star \lambda$ holds We shall use Lemma 2.1.5. We denote the set of cylinders in X with the empty set as given by

$$\mathscr{C} = \left\{ [\gamma_0 \cdots \gamma_{n-1}] \subseteq X : n \in \mathbb{Z}_{\geq 1}, \gamma_i \in \{0,1\}, i \in \{0,\dots,n-1\} \right\} \cup \{\emptyset\}.$$

Then $\mathscr{C}$ contains the empty set, generates $\mathscr{F}_X$, and is closed under finite intersection. Moreover, by Lemma A.2.3 we see $\mathbb{P}[\gamma_0 \cdots \gamma_{n-1}] = (\frac{1}{2})^n$, and $\phi_\star \lambda[\gamma_0 \cdots \gamma_{n-1}] = (\frac{1}{2})^n$ so that by Lemma 2.1.5 we have $\phi_\star \lambda = \mathbb{P}$.

We have shown the desired equivalence. $\square$

Corollary A.2.5. *Let* $([0,1), \mathscr{B}[0,1), \lambda, D_s)$ *be the standard Borel measure space equipped with the mapping*

$$D_s : [0,1) \to [0,1)$$
$$x \to \begin{cases} D(x), & x \in \left(\frac{1}{2}, \frac{3}{4}\right) \\ x, & \textit{else}, \end{cases}$$

where D *is the doubling map. Also let* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma_s)$ *be with* $\mathbb{P}$ *defined by*

$$\mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 1) = \mathbb{P}(\{x \in \{0,1\}^{\mathbb{Z}_{\geq 0}} : x_i = 0) = \frac{1}{2},$$

for all $i \in \mathbb{Z}_{\geq 0}$ *and let* $\sigma_s : \{0,1\}^{\mathbb{Z}_{\geq 0}} \to \{0,1\}^{\mathbb{Z}_{\geq 0}}$ *be as given by*

$$\sigma_s((x_n)_{n\geq 0}) = \begin{cases} \sigma((x_n)_{n\geq 0}), & x_0 = 1, x_1 = 0 \\ (x_n)_{n\geq 0}, & \textit{else.} \end{cases} \tag{252}$$

Then the mapping π *from Lemma A.2.2 is an ergodic isomorphism between* $([0,1), \mathscr{B}[0,1), \lambda, D_s)$ *and* $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma_s)$.

Proof. By Lemma A.2.4, the only thing we need to prove is for every $(x_n)_{\geq 0} \in X$ with X as in Equation (244) we have $\pi \circ \sigma_s = D_s \circ \pi$. To do so, let $x \in X$ and note that by Corollary A.2.3 we have $\pi(x) \in [\frac{1}{2}, \frac{3}{4})$ if and only if $x_0 = 1$ and $x_1 = 0$. Then note

$$\begin{aligned} D_s \circ \pi(x) &= \begin{cases} D \circ \pi(x), & \pi(x) \in (\frac{1}{2}, \frac{3}{4}) \\ \pi(x), & \text{else} \end{cases} \\ &= \begin{cases} D \circ \pi(x), & x_0 = 1, x_1 = 0 \\ \pi(x), & \text{else} \end{cases} \\ &= \begin{cases} (\pi \circ \sigma)(x), & x_0 = 1, x_1 = 0 \\ \pi(x), & \text{else} \end{cases} \\ &= (\sigma_s \circ \pi)(x) \end{aligned}$$

proving $\pi \circ \sigma_s = D_s \circ \pi$ on X. Again, by the claims proven in Lemmas A.2.4 and A.2.2 we then have the ergodic equivalence of $([0,1), \mathscr{B}[0,1), \lambda, D_s)$ and $(\{0,1\}^{\mathbb{Z}_{\geq 0}}, \mathscr{F}_{\{0,1\}}, \mathbb{P}, \sigma_s)$. □

A.3 Functional Analysis

In this Appendix we have compiled some functional analytic results needed in Section 4.1.

The following Theorem characterises duals of L^p spaces.

Theorem A.3.1 ([8] Appendix B). *Let* $(X, \mathscr{F}, \mu)$ *be a measure space, let* $1 \leq p, q < \infty$ *and* $1/p + 1/q = 1$*. If* $g \in L^q(X)$*, define* $F_g : L^p(X) \to \mathbb{F}$ *by*

$$F_g(f) = \int f g \, d\mu.$$

If $1 < p < \infty$ *the map* $g \mapsto F_g$ *defines an isometric isomorphism of* $L^q(X)$ *onto* $L^p(X)'$*. If* $p = 1$ *and* $(X, \mathscr{F}, \mu)$ *is* σ*-finite,* $g \mapsto F_g$ *is an isometric isomorphism of* $L^\infty(X)$ *onto* $L^1(\mu)'$*.*

The following Theorem is known as the Banach-Alaoglu Theorem. It is perhaps the main result on compactness within functional analysis.

Theorem A.3.2. *[8, Theorem V.3.1] Let* $(X, ||\cdot||)$ *be a normed vector space. Then the closed balls in the dual space* X' *are compact in the weak* topology.*

The following Theorem is known as the Eberlein-Smulian Theorem. It shows that weak compactness and weak sequential compactness are equivalent in Banach spaces. Note we do not require a separability condition on our space.

Theorem A.3.3. *[8, Theorem V.13.1] If* X *is a Banach space and* $A \subseteq X$*, then the following statements are equivalent.*

1. *Each sequence of elements of A has a subsequence that is weakly convergent.*

2. *The weak closure of A is weakly compact.*

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