Elliptic Fibrations on the Fermat Quartic

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MASTER THESIS

Elliptic Fibrations on the Fermat Quartic

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Introduction

In this thesis, we study elliptic fibrations on K3 surfaces. An elliptic fibration allows us to view a surface as a family of curves over a base curve, and many geometric properties can be understood by analysing the behaviour of its fibres. Given an elliptic fibration, one can associate to it a Jacobian fibration, which has isomorphic generic fibre and comes equipped with a section. Understanding this construction requires first studying the Jacobian of a genus one curve, which is an elliptic curve. If an elliptic fibration does not have a section, it is often useful to pass to the Jacobian fibration, since this still allows us to study many of the properties of the original fibration. Among algebraic surfaces, K3 surfaces form a particularly important and well-studied class. Our main example is the Fermat quartic surface, for which we consider two elliptic fibrations. One of them has a section and is obtained by projecting from a line. The other one is obtained by the intersection of two quadrics and does not admit a section. For both elliptic fibrations, we compute the Weierstrass equation of the generic fibre and classify the singular fibres.

To analyse the fibration given by the intersection of two quadrics in $\mathbb{P}^3$, we first recall that the intersection defines a genus one curve X in $\mathbb{P}^3$. In his PhD thesis [Rei72], Miles Reid describes the relationship between X and the double cover of $\mathbb{P}^1$ branched over the singular members of the pencil generated by the two quadrics. In this thesis, we give details of this analysis and show that this double cover is the Jacobian of X. This correspondence allows an explicit computation of the Jacobian which will be useful in the analysis of the fibration without a section on the Fermat quartic.

The thesis is organized as follows. In Chapter 1, we review the necessary background from algebraic geometry, including line bundles, divisors, and the adjunction formula, as well as basic results on elliptic curves and lattices. In Chapter 2, we study the construction of the Jacobian of a genus one curve and the associated torsors. We also introduce the notions of period and index and illustrate them with an explicit example. Chapter 3 is dedicated to elliptic fibrations: we introduce sections, multisections, and the classification of singular fibres. Finally, in Chapter 4, we focus on the Fermat quartic surface. We compute its Picard group by exhibiting a basis of lines, and we describe two explicit elliptic fibrations, realizing the only possible multisection indices 1 and 2 that occur for elliptic fibrations on the Fermat quartic.

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Chapter 1

Preliminaries

Let X be a smooth algebraic variety over an algebraically closed field k.

1.1 Algebraic geometry prerequisites

Recall that an $\mathcal{O}_X$-module $\mathcal{L}$ is called *locally free of rank one* if there exists an open covering $\{U_i\}$ of X such that

$$\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$$

for all i. Such a sheaf is also called an *invertible sheaf*, since its dual

$$\mathcal{L}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$$

satisfies

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X.$$

There is a natural bijection between isomorphism classes of invertible sheaves on X and isomorphism classes of line bundles on X (rank-1 vector bundles over X). For details on the bijection see [Har77, Exercise II.5.18]. In this thesis, we will use the terms invertible sheaf and line bundle interchangeably.

Definition 1.1.1. The *Picard group* of X, denoted $\mathrm{Pic}(X)$, is the group of isomorphism classes of line bundles on X with group operation given by the tensor product.

There is a natural identification of the Picard group with the first cohomology group of the sheaf of invertible regular functions:

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$$

For further details of this identification see [Har77, Exercise III.4.5].

Divisors and line bundles are closely related.

Definition 1.1.2. A *prime divisor* on X is a closed irreducible subvariety $Z \subset X$ of codimension 1.

Definition 1.1.3. A *divisor* on X is an element of the free abelian group generated by the prime divisors of X. We denote this group by $\mathrm{Div}(X)$.

For example, if X is a smooth surface, a divisor D can be expressed as

$$D = \sum_i n_i C_i$$

where the $C_i \subset X$ form a finite collection of irreducible, reduced curves and the n_i are integers.

Definition 1.1.4. A *principal divisor* on X is a divisor of the form

$$\mathrm{div}(f) = \sum_Z \mathrm{ord}_Z(f) \cdot Z$$

where the sum runs over all prime divisors $Z \subset X$, $f \in K(X)^*$ is some nonzero rational function and ord_Z is the valuation of the discrete valuation ring $\mathcal{O}_{X,Z}$.

Definition 1.1.5. The *line bundle associated to* D, denoted $\mathcal{O}_X(D)$, is the subsheaf of the sheaf of rational functions $K(X)$ defined by

$$\mathcal{O}_X(D)(U) := \{f \in K(X)^* \mid \mathrm{div}(f)|_U + D|_U \geq 0\} \cup \{0\}$$

for every open set $U \subset X$.

Definition 1.1.6. Two divisors D and D' are said to be *linearly equivalent*, denoted $D \sim D'$, if their difference is a principal divisor i.e.

$$D - D' = \mathrm{div}(f)$$

for some non-zero rational function f on X.

Theorem 1.1.7. [Har77, Corollaries II.6.14, II.6.16] Let X be a smooth variety. The map $D \mapsto \mathcal{O}_X(D)$ defines a surjective group homomorphism

$$\varphi : \mathrm{Div}(X) \to \mathrm{Pic}(X)$$

The kernel of φ is precisely the subgroup of principal divisors. Consequently, φ induces an isomorphism of groups

$$\mathrm{Div}(X)/\sim \;\cong\; \mathrm{Pic}(X).$$

Definition 1.1.8. [GH94, Section 0.3 p.37] Let X be a smooth variety over $k = \mathbb{C}$. The *exponential sheaf sequence* is the following short exact sequence of sheaves on X:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

where the first map sends n to $2\pi i n$ and the second map is the exponential map sending f to e^f.

As a consequence, we have a long exact sequence in cohomology:

$$\cdots \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{\exp} H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow \cdots$$

The map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ assigns to each line bundle its *first Chern class*. The kernel of the first Chern class map corresponds to line bundles that are *algebraically equivalent* to the trivial bundle, which we define now formally for divisors.

Definition 1.1.9. Let X be a smooth projective variety over an algebraically closed field. Two divisors D_1 and D_2 on X are said to be *algebraically equivalent*, written $D_1 \approx D_2$, if there exists:

- a smooth connected curve T,
- a divisor $D \subset X \times T$,

such that D is flat over T, and the fibers of D over two closed points $t_1, t_2 \in T$ are precisely D_1 and D_2, i.e.,

$$D_{t_1} = D_1 \quad \text{and} \quad D_{t_2} = D_2.$$

This defines an equivalence relation on the group of divisors as it is reflexive, symmetric and transitive, since the class of divisors algebraically equivalent to zero form a subgroup $\operatorname{Div}(X)$ [Har77, Exercise V.1.7].

Definition 1.1.10. Two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(X)$ are algebraically equivalent if their associated divisors are algebraically equivalent.

The set of isomorphism classes of divisors algebraically equivalent to 0 forms a subgroup of the Picard group:

$$\operatorname{Pic}^0(X) := \{D \in \operatorname{Pic}(X) \mid D \approx 0\} \subseteq \operatorname{Pic}(X).$$

Definition 1.1.11. The *Néron-Severi group* of X is the quotient

$$\operatorname{NS}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^0(X).$$

Equivalently,

$$\operatorname{NS}(X) = \operatorname{Div}(X)/\approx.$$

Proposition 1.1.12. The Néron-Severi group $\operatorname{NS}(X)$ of X is a finitely generated abelian group.

Proof. See [Har77, Exercise V.1.7-8]. □

Definition 1.1.13. The *Picard number* of X is the rank of the Néron-Severi group

$$\rho(X) := \operatorname{rk} \operatorname{NS}(X) = \dim_{\mathbb{Q}} \operatorname{NS}(X) \otimes \mathbb{Q}$$

Definition 1.1.14. Two divisors D and D' on X are said to be *numerically equivalent*, denoted $D \equiv D'$, if for all irreducible curves $C \subset X$ we have

$$(D.C) = (D'.C)$$

where $(D.C)$ denotes the intersecion number of D and C on X. The subgroup of $\operatorname{Pic}(X)$ of all divisors classes numerically equivalent to 0 is denoted by $\operatorname{Pic}^{\tau}(X)$ and we define

$$\operatorname{Num}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^{\tau}(X).$$

Proposition 1.1.15. For any two divisors D_1, D_2 on X we have the following implications:

$$D_1 \sim D_2 \implies D_1 \approx D_2 \implies D_1 \equiv D_2.$$

This induces surjections $\operatorname{Pic}(X) \twoheadrightarrow \operatorname{NS}(X) \twoheadrightarrow \operatorname{Num}(X)$.

Proof. See [Har77, Exercise V.1.7 (b),(d)]. □

1.2 Adjunction formula

Definition 1.2.1. Let $\mathcal{F}$ be a locally free sheaf of rank r. The determinant of $\mathcal{F}$ is

$$\det(\mathcal{F}) := \wedge^r \mathcal{F}.$$

Definition 1.2.2. Let X be a smooth variety. The *canonical bundle* of X, denoted ω_X, is defined as determinant of the sheaf Ω^1_X of Kähler differentials:

$$\omega_X := \det(\Omega^1_X).$$

The canonical bundle ω_X is a locally free $\mathcal{O}_X$-module of rank one, that is, an invertible sheaf.

Definition 1.2.3. A *canonical divisor* K_X is a divisor on X such that the associated line bundle is isomorphic to the canonical sheaf ω_X:

$$\mathcal{O}_X(K_X) \cong \omega_X.$$

Definition 1.2.4. The tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ on $\mathbb{P}^n$ is defined by

$$\mathcal{O}_{\mathbb{P}^n}(-1) := \{(l, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} | \, z \in l\}.$$

It comes with a projection map

$$\pi : \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n, \quad \pi(l, z) = l.$$

The projection π makes $\mathcal{O}_{\mathbb{P}^n}(-1)$ a line bundle, whose fibre over each point $l = [z_0 : \cdots : z_n] \in \mathbb{P}^n$ is

$$\pi^{-1}(l) = \{(l, z) \,|\, z \in l\} \cong l \subset \mathbb{C}^{n+1}$$

i.e. a 1-dimensional complex subspace of $\mathbb{C}^{n+1}$.

Its dual line bundle is $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}_{\mathbb{P}^n}(-1)^\vee$. For $k > 0$, we let $\mathcal{O}_{\mathbb{P}^n}(k) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes k}$ and, analogously, for $k < 0$ we define $\mathcal{O}_{\mathbb{P}^n}(k) := \mathcal{O}_{\mathbb{P}^n}(-k)^\vee$.

Definition 1.2.5 (Euler sequence)**.** [Har77, II.8.13] The *Euler sequence* of $\mathbb{P}^n$ is the following short exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0,$$

In particular, taking determinants gives

$$\det(\Omega^1_{\mathbb{P}^n}) \otimes \det(\mathcal{O}_{\mathbb{P}^n}) \cong \det(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}).$$

Since

- $\det(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$
- $\det(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}$
- $\det(\Omega^1_{\mathbb{P}^n}) = \omega_{\mathbb{P}^n}$

we obtain

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

To relate the canonical bundle of a variety to the canonical bundle of a divisor inside that variety we use the adjunction formula.

Theorem 1.2.6. (Adjunction formula) [Har77, Proposition II.8.20] Let X be a smooth variety, and let $Y \subset X$ be a smooth and effective divisor. Then the canonical bundle of Y, denoted ω_Y, is given by:

$$\omega_Y \cong (\omega_X \otimes \mathcal{O}_X(Y))|_Y.$$

Equivalently, in terms of canonical divisors, if K_X is a canonical divisor on X, then the canonical divisor K_Y on Y satisfies:

$$K_Y \sim (K_X + Y)|_Y.$$

We can use the adjunction formula to compute the genus of a smooth plane curve.

Example 1.2.7 (Genus-degree formula)**.** Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Denote by H the class of a line in $\mathbb{P}^2$. Then taking determinants in the Euler sequence for $\mathbb{P}^2$ gives

$$\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3) \quad \text{i.e.} \quad K_{\mathbb{P}^2} \sim -3H.$$

Apply the adjunction formula

$$K_C \sim (K_{\mathbb{P}^2} + C)|_C \sim (-3H + dH)|_C.$$

Now take degrees, using the fact that the degree of a canonical divisor of a curve satisfies $\deg(K_C) = 2g - 2$, where g denotes the genus of C. This gives

$$2g - 2 = (d-3)H \cdot C.$$

In $\mathbb{P}^2$, by Bézout's theorem a line H intersects a degree d curve C in d points (counted with multiplicities) which gives

$$g = \frac{(d-1)(d-2)}{2}.$$

We now state some results that we will need in the next chapters.

Theorem 1.2.8 (Zariski Main Theorem)**.** [Har77, III.Corollary 11.4] Let $f : X \to Y$ be a birational morphism of varieties, with Y normal. Then all fibres of f are connected.

Lemma 1.2.9. [Sta24, Stacks Project Tag 0AB1] A finite birational morphism $f : X \to Y$ of varieties with Y normal is an isomorphism.

Remark 1.2.10. The above results have the following consequences:

- **Curves:** If $f : C \to D$ is a birational morphism of smooth projective curves, then f is an isomorphism. Indeed, f is automatically finite and birational, and Lemma 1.2.9 applies.
- **Surfaces:** If $f : X \to X'$ is a birational morphism between smooth projective surfaces and both X and X' are minimal (as in Definition 3.1.11), then f is an isomorphism. Here, Theorem 1.2.8 ensures that all fibres are connected, and minimality prevents contraction of divisors.

1.3 Elliptic curve preliminaries

In this section we recall some basic facts about elliptic curves over a field k. Throughout, we assume that $\mathrm{char}(k) \neq 2, 3$.

Definition 1.3.1. An *elliptic curve* over k is a smooth, projective curve E of genus one defined over k, together with a distinguished k-rational point $O \in E(k)$, called the *origin*.

We will denote the curve with its distinguished point by (E, O), or just E if the rational point O is clear. We will also write E/k to indicate that the elliptic curve E is defined over k.

Definition 1.3.2. The set of k-rational points of E, denoted $E(k)$, forms an abelian group with O as the identity, which is called the *Mordell-Weil group* of E.

Definition 1.3.3. Let (E_1, O_1) and (E_2, O_2) be elliptic curves over k. A *morphism of elliptic curves* is a morphism of curves

$$\varphi\colon E_1 \longrightarrow E_2$$

defined over k such that $\varphi(O_1) = O_2$.

Definition 1.3.4. A *Weierstrass equation* over k is an equation of the form

$$E : Y^2Z = X^3 + a_4XZ^2 + a_6Z^3, \qquad a_4, a_6 \in k,$$

which defines a projective plane cubic curve E. Usually E is called *Weierstrass curve*. The point

$$O = (0 : 1 : 0) \in \mathbb{P}^2(k)$$

lies on E and is called the *point at infinity*.

In affine coordinates $x = X/Z$ and $y = Y/Z$, the Weierstrass equation takes the form

$$E : y^2 = x^3 + a_4x + a_6.$$

The pair (E, O) is an elliptic curve over k.

Theorem 1.3.5. [Sil09, Proposition 3.1] Let (C, P) be an elliptic curve over k. Then there exists a smooth Weierstrass equation E over k and an isomorphism of curves defined over k

$$\phi\colon C \longrightarrow E$$

such that $\phi(P) = (0 : 1 : 0)$.

Associated to a Weierstrass equation we define two important invariants:

$$\Delta(E) = -16\,(4a_4^3 + 27a_6^2), \qquad j(E) = 1728\,\frac{4a_4^3}{4a_4^3 + 27a_6^2}.$$

The discriminant $\Delta(E)$ determines whether the curve is smooth.

Proposition 1.3.6. [Sil09, Proposition 1.4(a)(i)] The curve E is smooth if and only if $\Delta(E) \neq 0$.

The j-invariant remains constant under affine transformations of the form

$$(x, y) \mapsto (u^2x, u^3y) \qquad \text{for } u \in \bar{k}^*.$$

Proposition 1.3.7. Let $E : y^2 = x^3 + a_4x + a_6$ and $E' : y^2 = x^3 + a_4'x + a_6'$ be elliptic curves over k with $j(E) = j(E')$. Then there exists $u \in \bar{k}^*$ such that

$$a_4 = u^4a_4' \qquad a_6 = u^6a_6' \qquad \Delta = u^{12}\Delta'.$$

Proof. See the proof of [Sil09, Proposition 1.4(b)]. □

The j-invariant classifies elliptic curves over $\bar{k}$ up to isomorphism.

Proposition 1.3.8. [Sil09, Proposition 1.4(b)] Two elliptic curves E, E' are isomorphic over $\bar{k}$ if and only if they have the same j-invariant.

Proposition 1.3.9. [Sil09, Proposition 1.4(c)] Let $j_0 \in \bar{k}$. There exists an elliptic curve defined over $k(j_0)$ whose j-invariant is equal to j_0.

If two elliptic curves E/k and E'/k are isomorphic over k, then they necessarily have the same j-invariant. However, the converse does not always hold: two elliptic curves can have the same j-invariant without being isomorphic over k as we shall see with the following example.

Example 1.3.10. For any $d \in k^*$, consider the following elliptic curves for $a_4, a_6 \in k^*$

$$E : y^2 = x^3 + a_4x + a_6$$

and

$$E_d : y^2 = x^3 + d^2a_4x + d^3a_6$$

Both curves are defined over k and have the same j-invariant $j(E) = j(E_d)$. By Proposition 1.3.8, there is an $u \in \bar{k}^*$ such that $a_4 = u^4d^2a_4$ and $a_6 = u^6d^3a_6$. We find only the solutions $u = \pm\frac{1}{\sqrt{d}}$. Therefore, the only isomorphisms between E and E_d are

$$\phi_+ : (x, y) \mapsto (dx, d^{3/2}y) \qquad \text{and} \qquad \phi_- : (x, y) \mapsto (dx, -d^{3/2}y).$$

When d is not a square in k, these isomorphisms are only defined over field extensions of k containing $\sqrt{d}$. In particular, E and E_d are elliptic curves defined over k with the same j-invariant, but they are not isomorphic over k.

1.4 Lattices

In this section we fix some terminology of lattices that will be relevant in later chapters.

Definition 1.4.1. A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a symmetric non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Q}.$$

Definition 1.4.2. A lattice is *integral* if the bilinear pairing $\langle \cdot, \cdot \rangle$ is $\mathbb{Z}$-valued.

Definition 1.4.3. An integral lattice is called *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for every $x \in L$, otherwise L is called *odd*.

Suppose that $\{x_1, ..., x_r\}$ is a $\mathbb{Z}$-basis of L where r is the rank of L. The matrix

$$A = (\langle x_i, x_j \rangle)_{1 \leq i,j \leq r}$$

is the *Gram matrix of* L with respect to the basis $\{x_i\}$.

Definition 1.4.4. The *discriminant* of a lattice L is defined as $\mathrm{disc}(L) := \det(A)$, where A is its Gram matrix with respect to any $\mathbb{Z}$-basis of L.

Definition 1.4.5. A lattice L is *unimodular* if $\mathrm{disc}(L) = \pm 1$.

Definition 1.4.6. A lattice L is called a *positive-definite* lattice if

$$\langle x, x \rangle > 0 \quad \text{for every } x \in L, x \neq 0.$$

This is the case if and only if the Gram matrix, i.e. the matrix of inner products of a basis A is positive-definite. Similarly, L is called *negative-definite* if

$$\langle x, x \rangle < 0 \quad \text{for every } x \in L, x \neq 0.$$

Definition 1.4.7. The *signature* of L is defined as the signature of any Gram matrix representing L, after viewing its coefficients as real numbers. For a lattice L of signature (n_+, n_-), its *index* is

$$\tau(L) := n_+ - n_-.$$

Definition 1.4.8. A *sublattice* L' of L is a submodule of L such that the restriction of $\langle \cdot, \cdot \rangle$ to $L' \times L'$ is non-degenerate. A sublattice L' of L is called *primitive* if L/L' is torsion-free. If $L' \subseteq L$ is a sublattice of the same rank then the *index* of L' in L is

$$[L : L'] := \#(L/L').$$

Definition 1.4.9. The *orthogonal complement* of a sublattice L' of L, denoted $L'^{\perp}$ is defined by

$$L'^{\perp} = \{x \in L \mid \langle x, y \rangle = 0 \text{ for all } y \in L'\}$$

The orthogonal complement $L'^{\perp}$ is a primitive sublattice of L of rank $\mathrm{rk}\, L - \mathrm{rk}\, L'$

Lemma 1.4.10. [Shi90, Section 6, (6.7)] If L' is a sublattice of finite index in L, then

$$\mathrm{disc}\, L' = [L : L']^2 \,\mathrm{disc}\, L$$

Definition 1.4.11. The *dual lattice* L^* of an integral lattice L is

$$L^* = \{x \in L \otimes \mathbb{Q} | \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L\}$$

Lemma 1.4.12. [Shi90, Section 6, (6.4)] Let L be an integral lattice. Then

$$\mathrm{disc}\, L^* = 1/\,\mathrm{disc}\, L \quad \text{ and } \quad |\,\mathrm{disc}\, L| = [L : L^*]$$

Chapter 2

The Jacobian

Now we consider genus one curves, not necessarily equipped with a distinguished rational point. Let C/k be a genus one curve. If $C(k) = \varnothing$, then C is not an elliptic curve over k. However, over $\bar{k}$ we can choose a point $P \in C(\bar{k})$ such that the pair $(C_{\bar{k}}, P)$ is an elliptic curve over $\bar{k}$. Here $C_{\bar{k}}$ denotes the base change of C to the algebraic closure $\bar{k}$. Such curves arise as torsors under their Jacobian elliptic curve, and while they may not have k-rational points, they become isomorphic to their Jacobian over $\bar{k}$. In this chapter, we follow closely the lectures notes by Sutherland [Sut13, Chapter 26].

2.1 Action of the Galois group

In this section, we consider the action of the Galois group $\mathrm{Gal}(\bar{k}/k)$. For each $\sigma \in \mathrm{Gal}(\bar{k}/k)$, we describe how σ acts on points, curves, and morphisms over $\bar{k}$. By definition, an element $\sigma \in \mathrm{Gal}(\bar{k}/k)$ is a field automorphism

$$\sigma : \bar{k} \longrightarrow \bar{k},$$

fixing the points of k.

Definition 2.1.1. For a point $P \in \mathbb{P}^n_{\bar{k}}$, σ acts on P by acting on its coordinates. We denote the resulting point by P^σ or $\sigma(P)$.

Definition 2.1.2. For a curve $\bar{C}$ over $\bar{k}$ defined by polynomial equations with coefficients in $\bar{k}$, we denote by $\bar{C}^\sigma$ the curve obtained by applying σ to all coefficients of the defining equations.

Therefore, σ in $\mathrm{Gal}(\bar{k}/k)$ defines a map

$$\sigma : \bar{C}(\bar{k}) \longrightarrow \bar{C}^\sigma(\bar{k}), \quad P \longmapsto P^\sigma,$$

Definition 2.1.3. Let $\bar{C}$ and $\bar{E}$ be curves defined over $\bar{k}$ and

$$\phi : \bar{C} \longrightarrow \bar{E}$$

be a morphism of curves over $\bar{k}$. We define

$$\phi^\sigma : \bar{C}^\sigma \longrightarrow \bar{E}^\sigma$$

as the morphism obtained by applying σ to the coefficients of the rational functions defining ϕ. Equivalently, for any point $P \in \bar{C}^{\sigma}(\bar{k})$, we have

$$\phi^{\sigma}(P) = \sigma\Big(\phi\big(\sigma^{-1}(P)\big)\Big).$$

That is, to compute $\phi^{\sigma}(P)$, we first pull P back to $\bar{C}_{\bar{k}}$ via σ^{-1}, then apply ϕ, and finally apply σ to get a point on $\bar{E}^{\sigma}_{\bar{k}}$.

2.2 Genus one curves

For this section, C denotes a smooth, projective curve of genus one defined over k. Let $Q_0 \in C(\bar{k})$ and consider the elliptic curve $(C_{\bar{k}}, Q_0)$ over $\bar{k}$.

Definition 2.2.1. For any point $P \in C(\bar{k})$ the *translation-by-P* morphism is

$$\tau_P : (C_{\bar{k}}, Q_0) \to (C_{\bar{k}}, Q_0), \qquad \tau_P(Q) := Q + P \quad \text{for } Q \in C(\bar{k}).$$

Proposition 2.2.2. Let $Q_0, Q_1 \in C(\bar{k})$ be two $\bar{k}$-rational points. Then the translation-by-Q_1 morphism induces a group isomorphism over $\bar{k}$ between the elliptic curves $(C_{\bar{k}}, Q_0)$ and $(C_{\bar{k}}, Q_1)$. In particular, the two elliptic curves $(C_{\bar{k}}, Q_0)$ and $(C_{\bar{k}}, Q_1)$ over $\bar{k}$ have the same j-invariant.

Therefore, the j-invariant of an elliptic curve is independent of the choice of origin. This allows us to define the j-invariant of a genus one curve.

Definition 2.2.3. The j-invariant $j(C)$ of C is defined to be the j-invariant of the elliptic curve $(C_{\bar{k}}, Q_0)$ over $\bar{k}$ for any $Q_0 \in C(\bar{k})$.

By Theorem 1.3.5, the elliptic curve $(C_{\bar{k}}, Q_0)$ admits a Weierstrass equation over $\bar{k}$: there exists a Weierstrass curve $E_{\bar{k}}$ over $\bar{k}$ and an isomorphism

$$\phi : C_{\bar{k}} \to E_{\bar{k}}$$

over $\bar{k}$ with $\phi(Q_0) = O$. Since C is defined over k, the coefficients of any defining equation of $C_{\bar{k}}$ are fixed by every $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Hence,

$$C^{\sigma}_{\bar{k}} = C_{\bar{k}}.$$

It follows that, for each σ, the map

$$\phi^{\sigma} : C_{\bar{k}} \longrightarrow E^{\sigma}_{\bar{k}}$$

is an isomorphism from $C_{\bar{k}}$ to $E^{\sigma}_{\bar{k}}$.

The j-invariant of an elliptic curve defined over k lies in k. For a genus one curve C defined over k, although we define its j-invariant as the j-invariant of an elliptic curve over $\bar{k}$, one can show that $j(C)$ actually lies in k.

Proposition 2.2.4. Let C/k be a curve of genus one. Then $j(C)$ lies in k.

Proof. Recall that $j(C) := j(E_{\bar{k}})$. With the above construction, we define

$$\varphi_\sigma := \phi^\sigma \circ \phi^{-1} : E_{\bar{k}} \to E_{\bar{k}}^\sigma$$

which is an isomorphism over $\bar{k}$. Thus,

$$j(E_{\bar{k}}^\sigma) = j(E_{\bar{k}}).$$

Let the short Weierstrass equations for $E_{\bar{k}}$ and $E_{\bar{k}}^\sigma$ be

$$E_{\bar{k}} : y^2 = x^3 + a_4x + a_6, \qquad E_{\bar{k}}^\sigma : y^2 = x^3 + \sigma(a_4)x + \sigma(a_6) \qquad \text{for } a_4, a_6 \in \bar{k}$$

. Then for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$, we have

$$\sigma(j(E_{\bar{k}})) = \sigma\left(1728\frac{4a_4^3}{4a_4^3 + 27a_6^2}\right) = 1728\frac{4(\sigma(a_4))^3}{4(\sigma(a_4))^3 + 27(\sigma(a_6))^2} = j(E_{\bar{k}}^\sigma).$$

Therefore, $\sigma(j(E_{\bar{k}})) = j(E_{\bar{k}}^\sigma) = j(E_{\bar{k}})$ for every $\sigma \in \mathrm{Gal}(\bar{k}/k)$, so the element $j(E_{\bar{k}})$ is fixed by $\mathrm{Gal}(\bar{k}/k)$ and therefore lies in k. Thus, we have $j(C) = j(E_{\bar{k}}) \in k$. ☐

Since $j(E_{\bar{k}})$ lies in k, by Proposition 2.2.4 we can choose a Weierstrass equation

$$E : y^2 = x^3 + a_4x + a_6$$

with coefficients $a_4, a_6 \in k$ such that $j(E) = j(E_{\bar{k}})$. Therefore, without loss of generality we can assume that $E_{\bar{k}}$ is given by a Weierstrass equation with coefficients in k.

Definition 2.2.5. Two varieties over a field k are called *twists* of each other if they are isomorphic over the algebraic closure $\bar{k}$.

Corollary 2.2.6. Every genus one curve C/k is a twist of an elliptic curve E/k.

However, Corollary 2.2.6 does not determine the elliptic curve E/k uniquely, not even up to k-isomorphism. For example, recall the two elliptic curves defined over k from Example 1.3.10:

$$E : y^2 = x^3 + a_4x + a_6$$
$$E_d : y^2 = x^3 + d^2a_4x + d^3a_6.$$

for $d \in k^*$. Then E is clearly a twist of itself, but E is also a twist of E_d since there is an isomorphism over $\bar{k}$. However, E and E_d are not isomorphic over k as shown in Example 1.3.10.

Our goal now is to distinguish, up to k-isomorphism, a particular elliptic curve E/k that is a twist of a given genus one curve C/k. We have shown that to a genus one curve C/k corresponds an elliptic curve E/k and an isomorphism

$$\phi : C_{\bar{k}} \to E_{\bar{k}}$$

over $\bar{k}$. We have also constructed maps

$$\phi^\sigma : C_{\bar{k}} \to E_{\bar{k}} \qquad \text{and} \qquad \varphi_\sigma := \phi^\sigma \circ \phi^{-1} : E_{\bar{k}} \to E_{\bar{k}}$$

for each $\sigma \in \mathrm{Gal}(\bar{k}/k)$. By construction, φ_σ is an automorphism of $E_{\bar{k}}$ over $\bar{k}$. In particular, φ_σ can be decomposed as a translation followed by an automorphism fixing the identity. Concretely, let

$$P_\sigma := \varphi_\sigma(O) \in E(\bar{k}),$$

then we have

$$\varphi_\sigma = \tau_{P_\sigma} \circ \tau_{-P_\sigma} \circ \varphi_\sigma = \tau_{P_\sigma} \circ (\tau_{-P_\sigma} \circ \varphi_\sigma)$$

where the first map is a translation by P_σ and the second map is an automorphism of $E_{\bar{k}}$, since it fixes the identity, i.e.

$$(\tau_{-P_\sigma} \circ \varphi_\sigma)(O) = \tau_{-P_\sigma} \circ \varphi_\sigma(O) = \tau_{-P_\sigma}(P_\sigma) = O.$$

Thus, we can write the map φ_σ as

$$\varphi_\sigma = \tau_{P_\sigma} \circ \varepsilon_\sigma$$

where $\varepsilon_\sigma \in \mathrm{Aut}(E_{\bar{k}})$. We want to choose E/k in such a way that the automorphisms $\varepsilon_\sigma \in \mathrm{Aut}(E_{\bar{k}})$ are the identity automorphism for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, making $\varphi_\sigma = \tau_{P_\sigma}$ a translation. This motivates the following theorem and definition of the *Jacobian* of a curve.

Theorem 2.2.7. Let C/k be a genus one curve. There exists a unique, up to k-isomorphism, elliptic curve E/k and an isomorphism $\phi : C_{\bar{k}} \to E_{\bar{k}}$ defined over $\bar{k}$ such that for each $\sigma \in \mathrm{Gal}(\bar{k}/k)$, the map

$$\varphi_\sigma := \phi^\sigma \circ \phi^{-1} : E_{\bar{k}} \to E_{\bar{k}}$$

is a translation by $P_\sigma = \varphi_\sigma(O) \in E(\bar{k})$.

Proof. For the proof see [Sut13, Theorem 26.12]. □

Definition 2.2.8. The unique, up to k-isomorphism, elliptic curve given by Theorem 2.2.7 is called the *Jacobian* of C.

Now we give a different characterization of the Jacobian.

Proposition 2.2.9. Let C be a curve of genus one over k, and fix a point $Q_0 \in C(\bar{k})$. Then the map

$$\phi_{Q_0} : C \longrightarrow \mathrm{Pic}^0(C), \qquad Q \longmapsto [Q - Q_0]$$

is an isomorphism of varieties over $\bar{k}$. Moreover, the Picard variety $\mathrm{Pic}^0(C)$ is canonically isomorphic to the Jacobian $\mathrm{Jac}(C)$.

Proof. That ϕ_{Q_0} is an isomorphism over $\bar{k}$ is shown in [Sil09, Chap. III, Prop. 3.4]. To identify $\mathrm{Pic}^0(C)$ with the Jacobian of C over k, we have to show that for each $\sigma \in \mathrm{Gal}(\bar{k}/k)$ the map

$$\varphi_\sigma = \phi_{Q_0}^\sigma \circ \phi_{Q_0}^{-1} \colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$$

is a translation. Let $[D] \in \operatorname{Pic}^0(C)$, where $D = \sum n_i P_i$ with $\sum n_i = 0$ and $P_i \in C(\bar{k})$. Let $R = \phi_{Q_0}^{-1}([D])$, so that $[D] = [R - Q_0]$. For convenience, write $\phi = \phi_{Q_0}$. Then

$$\begin{aligned}
\varphi_\sigma([D]) &= \phi^\sigma \circ \phi^{-1}([D]) \\
&= \phi^\sigma(R) \\
&= \phi(R^{\sigma^{-1}})^\sigma \\
&= [R^{\sigma^{-1}} - Q_0]^\sigma \\
&= [R - Q_0^\sigma] \\
&= [R - Q_0 + Q_0 - Q_0^\sigma] \\
&= [R - Q_0] + [Q_0 - Q_0^\sigma] \\
&= [D] + [Q_0 - Q_0^\sigma].
\end{aligned}$$

Thus, φ_σ is a translation by $[Q_0 - Q_0^\sigma]$ and therefore, $\operatorname{Pic}^0(C)$ is the Jacobian of the curve C. □

2.3 The Weil–Châtelet group

2.3.1 Torsors

For a group G and a set S we recall the definition of an *action* of G on S.

Definition 2.3.1. An *action* of a group G on a set S is a map $S \times G \to S$ such that the identity acts trivially and $s(gh) = (sg)h$ for all $g, h \in G$ and $s \in S$.

Definition 2.3.2. A group action of a group G on a set S is:

- *Free* if no nontrivial element of G fixes any point of S:

$$sg = s \implies g = e.$$

- *Transitive* if for all $s, t \in S$, there exists $g \in G$ such that $sg = t$.
- *Regular* if the action is both free and transitive. Equivalently, for all $s, t \in S$, there exists a unique $g \in G$ such that $sg = t$.

Definition 2.3.3. Let G be an abelian group. A *principal homogenous space of* G, or G*-torsor*, is a non-empty set S equipped with a regular group action by G.

Example 2.3.4. Let $G = E(\bar{k})$ be the group of points of an elliptic curve E/k, and let $S = C(\bar{k})$ be the set of $\bar{k}$-points on a genus one curve C/k. Once we fix a point $Q_0 \in C(\bar{k})$, we obtain an isomorphism $\phi : (C_{\bar{k}}, Q_0) \to (E_{\bar{k}}, O)$ over $\bar{k}$. Using the group law on $E(\bar{k})$, we define an action of $E(\bar{k})$ on $C(\bar{k})$ by

$$\mu : C(\bar{k}) \times E(\bar{k}) \longrightarrow C(\bar{k}), \qquad (Q, P) \longmapsto \phi^{-1}\big(\phi(Q) + P\big),$$

and we write $\mu(Q, P) =: Q + P$. With this action, $C(\bar{k})$ becomes an $E(\bar{k})$-torsor. By the regularity of the action, for any $Q, Q' \in C(\bar{k})$ there exists a unique $P \in E(\bar{k})$ such that $Q' = Q + P$. We denote this element by

$$P := Q' - Q.$$

Definition 2.3.5. Let E/k be an elliptic curve. A *principal homogeneous space* for E, or E*-torsor*, is a genus one curve C/k such that:

(i) The set $C(\bar{k})$ is an $E(\bar{k})$-torsor.

(ii) The action $\mu : C(\bar{k}) \times E(\bar{k}) \longrightarrow C(\bar{k})$ is a morphism of varieties defined over k.

Therefore, given a genus one curve C/k, a point $Q_0 \in C(\bar{k})$, an elliptic curve $(E_{\bar{k}}, O)$ and an isomorphism

$$\phi : (C_{\bar{k}}, Q_0) \to (E_{\bar{k}}, O),$$

by Example 2.3.4 we have that $C(\bar{k})$ is an $E(\bar{k})$-torsor. However, in order for C to be an E-torsor, we need the action to be defined over k. The following theorem from [Sut13, Theorem 26.16] characterizes precisely when this condition is satisfied.

Theorem 2.3.6. *Let C/k be a genus one curve and let E/k be an elliptic curve. Then C is an E-torsor if and only if E is the Jacobian of C.*

Proof. Assume first that C is an E-torsor. Denote by O the identity element of E and choose a point $Q_0 \in C(\bar{k})$. Using the torsor structure, we obtain an isomorphism

$$\phi : C_{\bar{k}} \to E_{\bar{k}}, \quad Q \mapsto Q - Q_0.$$

sending Q_0 to O. The inverse morphism ϕ^{-1} sends O back to Q_0 and is given by

$$\phi^{-1} : E_{\bar{k}} \to C_{\bar{k}}, \quad P \mapsto Q_0 + P.$$

For each $\sigma \in \mathrm{Gal}(\bar{k}/k)$, consider the automorphism

$$\varphi_\sigma = \phi^\sigma \circ \phi^{-1}$$

of $E_{\bar{k}}$. For E to be the Jacobian of C we have to check that the map φ_σ is a translation for every $\sigma \in \mathrm{Gal}(\bar{k}/k)$. For a point $P \in E_{\bar{k}}$, we have that

$$\begin{aligned}
\varphi_\sigma(P) &= \phi^\sigma \circ \phi^{-1}(P) \\
&= \phi^\sigma(Q_0 + P) \\
&= \Big(\phi((Q_0 + P)^{\sigma^{-1}})\Big)^\sigma \\
&= \Big((Q_0 + P)^{\sigma^{-1}} - Q_0\Big)^\sigma \\
&= P + Q_0 - Q_0^\sigma.
\end{aligned}$$

which is a translation by the point $P_\sigma = Q_0 - Q_0^\sigma$. This shows that the E is the Jacobian of C. For the converse, we assume that E is the Jacobian of C. Fix a point $Q_0 \in C(\bar{k})$. Since E is the Jacobian of C, we may choose an isomorphism

$$\phi : C_{\bar{k}} \to E_{\bar{k}}.$$

sending Q_0 to the identity element O of E and such that φ_σ is a translation by P_σ. Using this identification, we introduce an action of E on C over $\bar{k}$ as in Example 2.3.4 by setting

$$\mu : C(\bar{k}) \times E(\bar{k}) \to C(\bar{k}), \qquad (Q, P) \mapsto \phi^{-1}(\phi(Q) + P).$$

To verify that this action is defined over k, it suffices to check that it is invariant under the action of the Galois group. Thus, for each $\sigma \in \mathrm{Gal}(\bar{k}/k)$, we compare μ^σ with μ. Let $Q \in C(\bar{k})$ and $P \in E(\bar{k})$. Then

$$\begin{aligned}
\mu^\sigma(Q,P) &= \Big(\mu(Q^{\sigma^{-1}}, P^{\sigma^{-1}})\Big)^\sigma \\
&= \Big(\phi^{-1}(\phi(Q^{\sigma^{-1}}) + P^{\sigma^{-1}})\Big)^\sigma \\
&= (\phi^\sigma)^{-1}\Big(\phi(Q^{\sigma^{-1}})^\sigma + P\Big) \\
&= (\phi^\sigma)^{-1}\left(\phi^\sigma(Q) + P\right) \\
&= (\phi^\sigma)^{-1}\left(\varphi_\sigma \circ \phi(Q) + P\right) \\
&= (\phi^\sigma)^{-1}\left(Q - Q_0 + P_\sigma + P\right) \\
&= \phi^{-1} \circ \varphi_\sigma^{-1}(Q - Q_0 + P_\sigma + P) \\
&= \phi^{-1}(Q - Q_0 + P) \\
&= \phi^{-1}(\phi(Q) + P) \\
&= \mu(Q,P).
\end{aligned}$$

where we use $(\phi^{-1}(P))^\sigma = (\phi^\sigma)^{-1}(P^\sigma)$ and $(\phi(Q))^\sigma = \phi^\sigma(Q^\sigma)$, $\varphi_\sigma = \phi^\sigma \circ \phi^{-1}$ to derive $\phi^\sigma = \varphi_\sigma \circ \phi$ and the relations $\varphi_\sigma(Q) = Q + P_\sigma$ and $\varphi_\sigma^{-1}(P) = P - P_\sigma$. This shows that $\mu^\sigma = \mu$, and hence the action is defined over k.

□

Definition 2.3.7. Let E/k be an elliptic curve. Two E-torsors C/k and C'/k are *equivalent* if there is an isomorphism $\theta : C \to C'$ defined over k such that

$$\theta(Q + P) = \theta(Q) + P$$

holds for all $Q \in C(\bar{k})$ and $P \in E(\bar{k})$.

The set of E-torsors under this equivalence relation has a group structure, as we will see in the next section.

Definition 2.3.8. The set of E-torsors modulo equivalence relation is called the *Weil–Châtelet group* $\mathrm{WC}(E/k)$.

The identity element of $\mathrm{WC}(E/k)$ is the equivalence class of the trivial torsor E, which is precisely the set of elliptic curves that are k-isomorphic to E. To define the group law on $\mathrm{WC}(E/k)$, we will introduce the concept of a *crossed homomorphism*, which we do in the next section.

2.3.2 Crossed homomorphisms

Let E/k be an elliptic curve and let C/k be an E-torsor. Fix a point $Q_0 \in C(\bar{k})$ and for each $\sigma \in \mathrm{Gal}(\bar{k}/k)$, write P_σ for the unique point in $E(\bar{k})$ such that

$$P_\sigma = Q_0^\sigma - Q_0.$$

This defines a map

$$\alpha : \mathrm{Gal}(\bar{k}/k) \longrightarrow E(\bar{k}), \qquad \sigma \mapsto P_\sigma.$$

For any $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$ we have

$$\alpha(\sigma)^\tau = (Q_0^\sigma - Q_0)^\tau = Q_0^{\tau\sigma} - Q_o^\tau = (Q_0^{\tau\sigma} - Q_0) - (Q_0^\tau - Q_0) = \alpha(\tau\sigma) - \alpha(\tau)$$

Definition 2.3.9. A *crossed homomorphism* is a map

$$\alpha : \mathrm{Gal}(\bar{k}/k) \to E(\bar{k})$$

such that $\alpha(\tau\sigma) = \alpha(\sigma)^\tau + \alpha(\tau)$ for any $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$.

Proposition 2.3.10. The set of all crossed homomorphisms from $\mathrm{Gal}(\bar{k}/k)$ to $E(\bar{k})$ forms an abelian group under pointwise addition. The identity is the zero map 0 that sends every σ to O.

Proof. Let α and β be two crossed homomorphism and define the map $(\alpha+\beta)(\sigma) := \alpha(\sigma)+\beta(\sigma)$. Then for any $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$ we have

$$(\alpha+\beta)(\tau\sigma) = \alpha(\tau\sigma)+\beta(\tau\sigma) = \alpha(\tau)+\alpha(\sigma)^\tau+\beta(\tau)+\beta(\sigma)^\tau = (\alpha+\beta)(\tau)+(\alpha+\beta)(\sigma)^\tau$$

so $(\alpha+\beta)$ is a crossed homomorphism. The zero map $0 : \sigma \mapsto O$ is a crossed homomorphism since $O^\tau = O$ for any $\tau \in \mathrm{Gal}(\bar{k}/k)$. The inverse $(-\alpha)(\sigma) := -\alpha(\sigma)$ is a crossed homomorphism since

$$(-\alpha)(\tau\sigma) = -\alpha(\tau\sigma) = -\alpha(\sigma)^\tau - \alpha(\tau) = (-\alpha)(\sigma)^\tau + (-\alpha)(\tau)$$

and it satisfies $\alpha + (-\alpha) = 0$. Associativity and commutativity follows because addition in $E(\bar{k})$ is associative and commutative. Thus, the set of crossed homomorphisms is an abelian group with identity the zero map. □

Definition 2.3.11. Crossed homomorphisms of the form $\sigma \mapsto Q_0^\sigma - Q_0$ that come from an E-torsor C/k with $Q_0 \in C(\bar{k})$ are said to be *continuous*.

Definition 2.3.12. The subgroup of crossed homomorphisms of the form $\sigma \mapsto P^\sigma - P$ with $P \in E(\bar{k})$ are called *principal*.

Definition 2.3.13. Let E/k be an elliptic curve. The group of continuous crossed homomorphisms of E/k modulo its subgroup of principal crossed homomorphisms is the *first Galois-cohomology group* of $E(\bar{k})$, denoted by $H^1(\mathrm{Gal}(\bar{k}/k), E(\bar{k}))$ or shortly by $H^1(k, E)$.

The cohomology class in $H^1(k, E)$ associated to an E-torsor C/k is independent of the choice of $Q_0 \in C(\bar{k})$. Let $Q_0, Q_1 \in C(\bar{k})$ and let

$$\alpha_0(\sigma) := Q_0^\sigma - Q_0, \qquad \alpha_1(\sigma) := Q_1^\sigma - Q_1$$

be the corresponding crossed homomorphisms. Since the action of $E(\bar{k})$ on $C(\bar{k})$ is regular, there exists a unique point $P := Q_1 - Q_0 \in E(\bar{k})$ such that $Q_1 = Q_0 + P$. Then for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have

$$\alpha_1(\sigma) - \alpha_0(\sigma) = (Q_1^\sigma - Q_1) - (Q_0^\sigma - Q_0) = P^\sigma - P,$$

which is a principal crossed homomorphism. Hence α_0 and α_1 represent the same class in $H^1(k, E)$.

Theorem 2.3.14 ([Sut13, Theorem 26.24])**.** Let E be an elliptic curve over a field k. There is a bijection

$$\mathrm{WC}(E/k) \xrightarrow{\sim} H^1(k, E)$$

where for an E-torsor C over k, we choose a point $Q_0 \in C(\bar{k})$ and map

$$[C] \longmapsto \big[\sigma \longmapsto Q_0^\sigma - Q_0\big].$$

This bijection allows us to define the structure of an abelian group on $\mathrm{WC}(E/k)$.

2.4 Period and index

We will now define the *period* and the *index* of a genus one curve. These invariants measure in different ways how far a genus one curve is from possessing a k-rational point. Let C/k be a genus one curve and $E = \mathrm{Jac}(C)$ its Jacobian. Then C determines a class $[C] \in \mathrm{WC}(E/k)$. Fixing a point $Q_0 \in C(\bar{k})$, this class may be represented by the crossed homomorphism $\alpha : \sigma \mapsto P_\sigma$ in $H^1(k, E)$.

Definition 2.4.1. The *period* of C/k is defined to be the order of the corresponding crossed homomorphism $\alpha : \sigma \mapsto P_\sigma$ in $H^1(k, E)$.

Note that the order of the crossed homomorphism α is the minimal positive integer n such that $n[\alpha] = [0]$ where $[0]$ denotes the class of the principal crossed homomorphisms.

For a genus one curve C/k, we can sometimes bound the period by examining the orders of the points $P_\sigma = Q^\sigma - Q$ for a chosen point $Q \in C(\bar{k})$. For example, if for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have

$$mP_\sigma = O \in E(\bar{k}),$$

then $m\alpha$ is principal and therefore, the period of C divides m. Moreover, the period of C/k is the smallest positive integer m with this property [Sil09, Exercise X.10.11(a)].

Definition 2.4.2. The *index* of C/k is the smallest degree of an extension L/k such that $C(L) \neq \varnothing$.

The period and index are closely related by the following divisibility relations:

Proposition 2.4.3. Let C/k be a genus one curve with Jacobian E/k. Then:

- The period divides the index:

$$\mathrm{per}(C) \mid \mathrm{ind}(C).$$

- The index divides the square of the period:

$$\mathrm{ind}(C) \mid \mathrm{per}(C)^2.$$

Proof. See [Lic69, Theorem 8]. □

2.5 Example of Jacobian and period-index calculation

In this section, we compute explicitly the Jacobian of a genus one curve C and we calculate the period and index of C. We work over a field $k = \mathbb{Q}(\rho)$ where ρ is a primitive third root of unity.

Proposition 2.5.1. Let $a, b, c \in k^*$ and $[k(a^{1/3}, b^{1/3}) : k] = 9$. Consider the smooth projective cubic

$$C : \ aU^3 + bV^3 + cW^3 = 0,$$

and put $A := abc$. Then the Jacobian of C is the elliptic curve

$$E : \quad y^2 = x^3 - 432A^2.$$

Proof. To show that E is the Jacobian of C, we have to construct an isomorphism $\phi : C \to E$ and show that the map $\varphi_\sigma := \phi^\sigma \circ \phi^{-1}$ is translation by $P_\sigma = \varphi_\sigma(O)$. Consider the field extension $L := k(a^{1/3}, b^{1/3}) = \mathbb{Q}(\rho, a^{1/3}, b^{1/3})$ and the point $Q_0 = (-a^{-1/3} : b^{-1/3} : 0) \in C(L)$. To compute the corresponding map $\phi : C \to E$, we do some intermediate calculations

$$\begin{tikzcd} C \arrow[r, "\psi_1"] & C_1 \arrow[r, "\psi_2"] & C_2 \arrow[r, "\psi_3"] & E \end{tikzcd}$$

where the intermediate curves are

$$\begin{aligned} &C : aU^3 + bV^3 + cW^3 = 0, \quad C_1 : X^3 + Y^3 + abcZ^3 = 0 \\ &C_2 : 2U^3 + 6UV^2 + AZ^3 = 0, \quad E : y^2 z = x^3 - 432A^2 z^3 \end{aligned}$$

and the isomorphisms are defined in the table below.

Map	**Dom**	**Codom**	**Map in coordinates**
ψ_1	C	C_1	$(U,V,W) \mapsto (a^{1/3}U,\ b^{1/3}V,\ a^{-1/3}b^{-1/3}W)$
ψ_1^{-1}	C_1	C	$(X,Y,Z) \mapsto (a^{-1/3}X,\ b^{-1/3}Y,\ a^{1/3}b^{1/3}Z)$
ψ_2	C_1	C_2	$(X,Y,Z) \mapsto \left(\frac{X+Y}{2},\ \frac{Y-X}{2},\ Z\right)$
ψ_2^{-1}	C_2	C_1	$(U,V,Z) \mapsto (U-V,\ U+V,\ Z)$
ψ_3	C_2	E	$(U,V,Z) \mapsto \frac{1}{216A^2}(-6AZ,\ 36AV,\ U)$
ψ_3^{-1}	E	C_2	$(x,y,z) \mapsto 216A^2\left(z,\ \frac{y}{36A},\ \frac{-x}{6A}\right)$
ϕ	C	E	$(U,V,W) \mapsto \left(\frac{1}{216A^2}(-6A\,a^{-1/3}\,b^{-1/3}\,W,\ 18A(b^{1/3}V - a^{1/3}U),\ \frac{a^{1/3}U+b^{1/3}V}{2}\right)$
ϕ^{-1}	E	C	$(x,y,z) \mapsto 216A^2\left(a^{-1/3}(z - \frac{y}{36A}),\ b^{-1/3}(z + \frac{y}{36A}),\ -a^{1/3}b^{1/3}\frac{x}{6A}\right)$

Table 2.1: Explicit description of the morphisms between the curves.

The map $\phi : C \to E$ is an isomorphism over L. The point $Q_0 = (a^{-1/3}, b^{1/3}, 0) \in C(L)$ is sent under ϕ to $O = (0,1,0)$, the origin of the elliptic curve E. Consider the Galois group of $L/k = \mathbb{Q}(\rho, a^{1/3}, b^{1/3})/\mathbb{Q}(\rho)$. Every $\sigma \in \mathrm{Gal}(L/k)$ is uniquely determined by its action on $a^{1/3}$ and $b^{1/3}$, and is given by

$$\sigma_{(i,j)} : \begin{cases} a^{1/3} \longmapsto \rho^i a^{1/3}, \\ b^{1/3} \longmapsto \rho^j b^{1/3}, \end{cases} \qquad i, j \in \{0,1,2\}.$$

The Galois conjugate $\sigma \circ \phi \circ \sigma^{-1}$ of ϕ is given by applying σ to the coefficients of the map ϕ, that is,

$$\begin{aligned} \phi^\sigma : (U,V,W) &\mapsto \left(\frac{1}{216A^2}(-6A\,\sigma(a^{-1/3})\,\sigma(b^{-1/3})\,W,\ 18A(\sigma(b^{1/3})V - \sigma(a^{1/3})U),\ \tfrac{\sigma(a^{1/3})U+\sigma(b^{1/3})V}{2}\right) \\ &\mapsto \left(\frac{1}{216A^2}(-6A\,\rho^{3-i}a^{-1/3}\,\rho^{3-j}b^{-1/3})\,W,\ 18A(\rho^j b^{1/3})V - \rho^i a^{1/3}U),\ \tfrac{\rho^i a^{1/3}U+\rho j b^{1/3}V}{2}\right) \end{aligned}$$

for $\sigma = \sigma_{(i,j)}$. The composition $\varphi_\sigma = \phi^\sigma \circ \phi^{-1} : E \to E$ is then given by

$$\varphi_\sigma : (x, y, z) \mapsto \left(\rho^{3-i}\rho^{3-j}x, 18A\left(\rho^j(z + \frac{y}{36A}) - \rho^i(z - \frac{y}{36A})\right), \frac{\rho^i(z - \frac{y}{36A}) + \rho^j(z + \frac{y}{36A})}{2}\right).$$

The point of translation P_σ is then given by

$$P_\sigma := \varphi_\sigma(O) = (0, \frac{\rho^j + \rho^j}{2}, \frac{\rho^j - \rho^i}{72A})$$

The code provided in the Appendix A.1 verifies that, for a general point $P \in E(L)$, one has

$$\varphi_\sigma(P) = P + P_\sigma.$$

In particular, φ_σ is a translation on E, and hence E is the Jacobian of C. □

We now calculate the period and the index of C as an E-torsor.

Proposition 2.5.2. Let C/k be the genus one curve

$$C : aU^3 + bV^3 + cW^3 = 0$$

for $a, b, c \in k^*$. Then the period and index of C/k equals 3.

Proof. The Jacobian of C/k is the elliptic curve E/k

$$E : y^2 = x^3 - 432A^2$$

for $A := abc$. In the proposition above, we showed that the isomorphism $\varphi_\sigma : E \to E$ is a translation by $P_\sigma = (0, \frac{\rho^j+\rho^j}{2}, \frac{\rho^j-\rho^i}{72A})$. Thus, the period of C/k corresponds to the minimum positive integer m such that $mP_\sigma = O$ for all σ. Using explicit computations implemented in Python (see Appendix A.1), we obtain the following results for the points

$P = P_\sigma$ corrsponding to the various values of i, j:

$$\begin{aligned}
\text{For } i,j = 0,0: \quad & P = (0:1:0)\\
\text{For } i,j = 0,1: \quad & P = (0:(-24\rho - 12)A:1)\\
& 2P = (0:(24\rho + 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 0,2: \quad & P = (0:(24\rho + 12)A:1)\\
& 2P = (0:(-24\rho - 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 1,0: \quad & P = (0:(24\rho + 12)A:1)\\
& 2P = (0:(-24\rho - 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 1,1: \quad & P = (0:1:0)\\
\text{For } i,j = 1,2: \quad & P = (0:(-24\rho - 12)A:1)\\
& 2P = (0:(24\rho + 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 2,0: \quad & P = (0:(-24\rho - 12)A:1)\\
& 2P = (0:(24\rho + 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 2,1: \quad & P = (0:(24\rho + 12)A:1)\\
& 2P = (0:(-24\rho - 12)A:1)\\
& 3P = (0:1:0)\\
\text{For } i,j = 2,2: \quad & P = (0:1:0)
\end{aligned}$$

Hence, the period of C/k is 3. For the index of C, we first obtain an upper bound. We found an L-rational point

$$Q_0 := (-a^{-1/3} : b^{1/3} : 0) \in C(L),$$

where

$$L = k(a^{1/3}, b^{1/3}) = \mathbb{Q}(\rho, a^{1/3}, b^{1/3}),$$

so that

$$[L : k] = [\mathbb{Q}(\rho, a^{1/3}, b^{1/3}) : \mathbb{Q}(\rho)] = 9.$$

By definition of the index, this shows that $\mathrm{ind}(C)$ is at most 9. Since the period of C divides its index, the only remaining possibilities for the index are

$$\mathrm{ind}(C) \in \{3, 6, 9\}.$$

On the other hand, consider the point

$$Q := \big((b/a)^{1/3} : -1 : 0\big).$$

This point is defined over the field

$$L_0 := k\big((b/a)^{1/3}\big),$$

which satisfies $[L_0 : k] = 3$. Hence C admits a rational point over L_0 which is a degree 3 extension of k, and therefore

$$\mathrm{ind}(C) \mid 3.$$

so we conclude that the index of C is 3.

$\square$

Chapter 3

Elliptic K3 surfaces

In this chapter we introduce K3 surfaces and their elliptic fibrations. We first recall the definition and basic properties of K3 surfaces, and then focus on elliptic fibrations and the classification of singular fibres, which will be used in the next chapter for the study of the Fermat quartic.

3.1 K3 surfaces

Let k be an algebraically closed field of characteristic 0.

Definition 3.1.1. A *K3 surface* over k is a smooth, complete surface X such that

$$\Omega_X^2 \cong \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

Remark 3.1.2. Any smooth complete surface is projective, so K3 surfaces are always projective [Băd01, Theorem 1.28].

We now see some standard examples of K3 surfaces.

Example 3.1.3. A smooth quartic $X \subset \mathbb{P}^3$ is a $K3$ surface. Indeed, by the adjunction formula, we have

$$K_X = (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X \cong \mathcal{O}_X$$

and hence, the canonical bundle of X is trivial. Now, consider the standard short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Taking the associated long exact sequence in cohomology gives:

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \longrightarrow \cdots$$

From [Har77, Theorem III.5.1(b)], one deduces

$$H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0, \qquad H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0.$$

Hence, it follows from exactness that

$$H^1(X, \mathcal{O}_X) = 0.$$

Therefore, X is a K3 surface.

Example 3.1.4. In the following example from [VA17, Example 1.2] and [Huy16, Example 1.3 (ii)], we describe smooth complete intersections in projective space that give rise to K3 surfaces. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection cut out by $r = n-2$ hypersurfaces of degrees $d_1, \dots, d_r$ respectively. By construction, X is a surface. By the adjunction formula for complete intersections (see [Har77, Exercise II.8.4(d)]) the canonical bundle of X is

$$K_X \cong \mathcal{O}_{\mathbb{P}^n}\Big(-n-1+\sum_{i=1}^{r} d_i\Big)\Big|_X.$$

Thus X has trivial canonical bundle if and only if

$$\sum_{i=1}^{r} d_i = n+1.$$

We may assume $d_i \geq 2$ for all i, since a hypersurface of degree 1 is a hyperplane and intersecting with it realizes the same surface as a complete intersection in a lower dimensional projective space. Under this constraints, the only possibilities that can occur are: a smooth quartic in $\mathbb{P}^3$ (this is Example 3.1.3), a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ and a complete intersection of three quadrics in $\mathbb{P}^5$. Each of the these three types of complete intersections, by a similar argument as in Example 3.1.3, satisfies $H^1(X, \mathcal{O}_X) = 0$ so all of them are K3 surfaces.

Example 3.1.5. As a further example, we consider a smooth complete intersection in $\mathbb{P}^1 \times \mathbb{P}^3$. Let $\mathbb{P}^1 \times \mathbb{P}^3$ have homogenous coordinates $[t_0 : t_1]$ on the first factor and $[x_0 : x_1 : x_2 : x_3]$ on the second. Consider two different bidegree $(1,2)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$, that is, two equations of the form:

$$\begin{aligned} f_1 t_0 + f_2 t_1 &= 0 \\ g_1 t_0 + g_2 t_1 &= 0 \end{aligned} \tag{3.1}$$

where f_i and g_i for $i = 1, 2$ are polynomials on $\mathbb{P}^3$ of degree 2. Let X be the common zero locus of these two equations. Then X is a K3 surface. Indeed, the variety X is the complete intersection of two different hypersurfaces of bidegree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. Therefore, its dimension is

$$\dim(X) = \dim(\mathbb{P}^1 \times \mathbb{P}^3) - 2 = 2,$$

so X is a surface.

Now we compute the canonical bundle using the adjunction formula. Recall that the canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^3$ is

$$K_{\mathbb{P}^1 \times \mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(-2, -4),$$

and for a hypersurface of bidegree $(1,2)$, the normal bundle is $\mathcal{O}(1,2)$. Then by the adjunction formula for a complete intersection of two hypersurfaces of bidegree $(1,2)$,

$$K_X = (K_{\mathbb{P}^1 \times \mathbb{P}^3} + \mathcal{O}(1,2) + \mathcal{O}(1,2))|_X = (\mathcal{O}(-2,-4) + \mathcal{O}(2,4))|_X = \mathcal{O}_X$$

so X has trivial canonical bundle.

Finally, since X is a smooth complete intersection in a projective space, one can show that it is connected and has $H^1(X, \mathcal{O}_X) = 0$. Hence, X is a K3 surface.

For the remainder of this section, we work over $k = \mathbb{C}$.

For a K3 surface X over $\mathbb{C}$, Serre duality gives $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$, so X has holomorphic Euler characteristic $\chi(\mathcal{O}_X) = 2$. On the other hand, Noether's formula ([Huy16, Section 2.4]) gives the topological Euler characteristic

$$e(X) = 12\chi(\mathcal{O}_X) - K_X^2 = 24.$$

In fact, we can write the the topological Euler characteristic of X as an alternating sum of the Betti numbers $b_i = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$,

$$e(X) = \sum_{i=0}^{4} (-1)^i b_i.$$

Since $b_1(X) = b_3(X) = 0$ and $b_0(X) = b_4(X) = 1$, we get $b_2(X) = 22$. The Hodge decomposition of the cohomology group $H^2(X, \mathbb{C})$ is fundamental in the study of K3 surfaces. Since K_X is trivial, Serre duality and Hodge symmetry imply that the non-zero Hodge numbers are

$$h^{0,0} = h^{2,2} = 1, \quad h^{2,0} = h^{0,2} = 1, \quad h^{1,1} = 20.$$

Thus the Hodge diamond of a K3 surface [Huy16] is:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad (*)$$

The cup product endows the second cohomology group $H^2(X, \mathbb{Z})$ with the structure of an even unimodular lattice of signature $(3, 19)$, which is uniquely determined up to isometry and is referred to as the *K3 lattice* [VA17, Proposition 1.12, Theorem 1.14]. We denote by

$$B \colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

the corresponding symmetric bilinear form induced by the cup product.

Recall that the Picard group of X can be identified as

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*).$$

Since $H^1(X, \mathcal{O}_X) = 0$, we have $\mathrm{Pic}^0(X) = 0$, and hence the Picard group coincides with the Néron–Severi group,

$$\mathrm{Pic}(X) = \mathrm{NS}(X).$$

The first Chern class map induces an injective homomorphism

$$c_1 \colon \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z}),$$

allowing us to view $\mathrm{Pic}(X)$ as a subgroup of $H^2(X, \mathbb{Z})$.

Proposition 3.1.6. Let X be a K3 surface. Then the natural surjections

$$\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X)$$

are isomorphisms.

Proof. Since $h^1(X, \mathcal{O}_X) = 0$, we have $\mathrm{Pic}^0(X) = 0$, so the first map $\mathrm{Pic}(X) \to \mathrm{NS}(X)$ is an isomorphism. For the second map, let $L \in \mathrm{Pic}(X)$ be numerically trivial. Assume $L \not\cong \mathcal{O}_X$ and choose an ample line bundle L_0 on X. Then $L \cdot L_0 = 0$, which implies $H^0(X, L) = 0$. By Serre duality, we have $H^2(X, L) \cong H^0(X, L^{-1})^\vee = 0$. Therefore

$$\chi(X, L) = h^0(X, L) - h^1(X, L) + h^2(X, L) \leq 0.$$

On the other hand, Riemann-Roch for surfaces (see [Har77, Theorem V.1.6]) applied to K3 surfaces gives $\chi(X, L) = \frac{1}{2}L^2 + 2$. Comparing, we find $L^2 < 0$, contradicting that L is numerically trivial. Hence $L \cong \mathcal{O}_X$ and the map $\mathrm{NS}(X) \to \mathrm{Num}(X)$ is injective. □

The bilinear form B on $H^2(X, \mathbb{Z})$ restricts via the first Chern class map to a bilinear form on $\mathrm{Pic}(X)$. For divisor classes $D, D' \in \mathrm{Pic}(X)$, this restriction agrees with the intersection pairing on divisors [GH94, Section 4.1]:

$$B(c_1(D), c_1(D')) = (D \cdot D'),$$

where $(D \cdot D')$ denotes the intersection number of D and D' on X. In this way, the Picard group (and hence the Néron–Severi group) obtains the structure of an integral lattice.

Definition 3.1.7. The rank of the Néron–Severi group,

$$\rho(X) := \mathrm{rank}\,\mathrm{NS}(X),$$

is called the *Picard number* of X.

For a K3 surface, one has

$$1 \leq \rho(X) \leq 20$$

which follows from the Hodge diamond $(*)$.

Proposition 3.1.8. For a K3 surface X, the intersection pairing on $\mathrm{Pic}(X)$ is even and has signature $(1, \rho(X) - 1)$.

Proof. The Riemann–Roch theorem for K3 surfaces yields

$$\chi(X, D) = \frac{1}{2}(D \cdot D) + 2$$

for any divisor $D \in \mathrm{Pic}(X)$, which implies that $(D \cdot D) \in 2\mathbb{Z}$. The signature statement follows from the fact that the intersection form is negative definite on the orthogonal complement of an ample divisor class. □

3.1.1 Elliptic K3 surfaces

For the rest of the chapter we let k be an algebraically closed field of characteristic 0.

Definition 3.1.9. An *elliptic fibration* on a surface X is a surjective morphism

$$f : X \to \mathbb{P}^1$$

such that all but finitely many fibres of f are smooth curves of genus one. The fibres that are not smooth are called *singular fibres*.

Definition 3.1.10. An *elliptic surface* X is a surface X equipped with an elliptic fibration. In particular, if X is $K3$, we call X an *elliptic* $K3$ *surface.*

Elliptic fibrations allow us to view elliptic $K3$ surfaces as a family of genus one curves, parametrized by the projective line $\mathbb{P}^1$, where all but finitely many are smooth.

Definition 3.1.11. An ellipic surface X is *minimal* if no fibre of f contains a smooth rational curve $C \subset X$ with self-intersection $C^2 = -1$.

Any elliptic $K3$ surface X is always minimal: since $K_X = 0$, the adjunction formula gives

$$2g(C) - 2 = C^2 + K_X \cdot C = C^2.$$

If $C \subset X$ is a smooth rational curve (i.e. if $g(C) = 0$), then $C^2 = -2$. Therefore, X is minimal.

We note that for schemes X and S, a morphism of schemes $f\colon X \to S$, and a point $s \in S$, the *(scheme-theoretic) fibre of* f *over* s, denoted by X_s, is defined as the fibre product

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & \mathrm{Spec}(k(s)) \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

That is,

$$X_s = X \times_S \mathrm{Spec}(k(s)).$$

Definition 3.1.12. The *generic fibre* X_η of an elliptic fibration $f : X \to \mathbb{P}^1$ is defined as the fibre of f over the generic point η of $\mathbb{P}^1$.

We use t for the inhomogeneous coordinate of $\mathbb{P}^1$. The generic fibre X_η is a genus one curve defined over the function field $k(\mathbb{P}^1) = k(t)$. For a closed point t_0 of $\mathbb{P}^1$, the underlying topological space of X_{t_0} is homeomorphic to $f^{-1}(t_0)$.

Recall that an elliptic curve over k is a smooth projective curve of genus one equipped with a distinguished k-rational point that serves as the origin for its group law. In an elliptic fibration, most fibres are smooth curves of genus one over k, so if we can specify a k-rational point on each smooth fibre in a compatible way, we obtain a family of elliptic curves over k. Choosing such a point consistently as $t \in \mathbb{P}^1$ varies corresponds to the existence of a *section* of the fibration.

Definition 3.1.13. A *section* of an elliptic fibration $f : X \to \mathbb{P}^1$ is a morphism

$$\sigma : \mathbb{P}^1 \to X$$

such that $f \circ \sigma = \mathrm{id}_{\mathbb{P}^1}$.

Intuitively, a section determines a point on each fibre of the fibration, providing a compatible choice of origin for the group law for the smooth fibres. Fixing a section

$$\sigma_0 : \mathbb{P}^1 \to X$$

gives every smooth fibre over $t_0 \in \mathbb{P}^1$ the structure of an elliptic curve (X_{t_0}, O_{t_0}) defined over k, where $O_{t_0} = \sigma_0(t_0)$. This distinguished section is called the *zero section.*

With a chosen zero section σ_0, the generic fibre becomes an elliptic curve (X_η, O) over $k(t)$, where $O = \sigma_0(\mathbb{P}^1) \cap X_\eta$ is the corresponding $k(t)$-rational point of X_η. In this sense, we can view the surface X as a family of elliptic curves

$$\{(X_{t_0}, O_{t_0})\}_{t_0 \in \mathbb{P}^1}$$

or equivalently, as an elliptic curve (X_η, O) defined over $k(t)$.

It is important to note that not every elliptic fibration admits a section. Denote the set of sections by

$$\mathrm{MW}_{\mathrm{f}}(X) := \{\sigma \mid \ \sigma \text{ sections of the fibration } f\colon X \to \mathbb{P}^1\}$$

The presence of a section is useful as it establishes a natural correspondence between sections of the fibration and $k(t)$-rational points of the generic fibre.

Proposition 3.1.14. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration and let X_η denote the generic fibre defined over $k(t)$. Then there is a one-to-one correspondence

$$\mathrm{MW}_{\mathrm{f}}(X) \overset{1:1}{\longleftrightarrow} X_\eta(k(t)).$$

Proof. Let $\sigma : \mathbb{P}^1 \to X$ be a section of the fibration. Then $P := \sigma(\mathbb{P}^1) \cap X_\eta$ is a $k(t)$-rational point of X_η. Conversely, let $P \in X_\eta(k(t))$ be a $k(t)$-rational point of the generic fibre. A priori this only defines a point on the smooth closed fibres, hence does not immediately define a section of f. Let $\bar{P} \subset X$ denote the Zariski closure of P in X. Then $\bar{P}$ is an irreducible curve satisfying $\bar{P} \cap X_\eta = P$. Restricting the fibration f to $\bar{P}$, we obtain a morphism

$$f|_{\bar{P}} : \bar{P} \longrightarrow \mathbb{P}^1.$$

Since P is a rational point of the generic fibre, the morphism $f|_{\bar{P}}$ is birational and since $\mathbb{P}^1$ is smooth, Lemma 1.2.9 implies that $f|_{\bar{P}}$ is an isomorphism. The inverse morphism

$$\sigma_P := (f|_{\bar{P}})^{-1} : \mathbb{P}^1 \longrightarrow X$$

is therefore a section of f associated to the point P. □

Therefore, fixing a section σ_0 gives $\mathrm{MW}_{\mathrm{f}}(X)$ the structure of an abelian group, with neutral element σ_0 and operation induced by the group law on (X_η, O) over $k(t)$.
We will often abuse notation and denote sections by their corresponding points in $X_\eta(k(t))$. In particular, O denotes both the origin of $X_\eta(k(t))$ and the zero section σ_0.

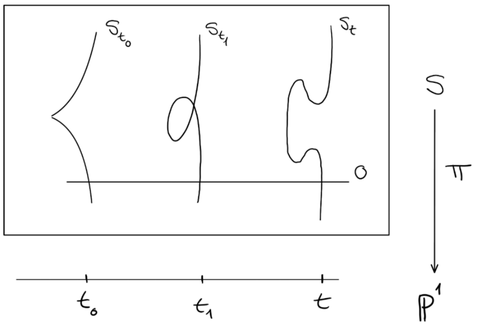


Figure 3.1: Elliptic fibration with section

3.2 Singular fibres of an elliptic surface

Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with a section O. The generic fibre X_η is a smooth curve of genus one, and the presence of a section provides it with the structure of an elliptic curve (X_η, O) over the function field $k(t)$. Not every fibre of X is smooth: at finitely many points of $\mathbb{P}^1$ the fibres X_t are singular. The possible configurations of such singular fibres have been classified by Kodaira and Néron. In this section, we recall their classification following [SS19, Chapter 5] and [Huy16, Section 11.1], restricting to the case $\mathrm{char}(k) = 0$.

3.2.1 Classification of singular fibres

Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface and let τ be a closed point of $\mathbb{P}^1$. As a divisor on X, we see in [Huy16, Corollary 11.1.7] that X_τ decomposes as

$$X_\tau = \sum_{i=0}^{m_{\tau-1}} \mu_{\tau,i}\, \Theta_{\tau,i},$$

where

- m_τ is the number of distinct irreducible components,
- $\Theta_{\tau,i}$ are the irreducible components,
- $\mu_{\tau,i}$ are their multiplicities.

Recall that we fixed a section $O : \mathbb{P}^1 \to X$. Since $f \circ O = \mathrm{id}$, a section meets every fibre only once, and therefore the image of O intersects a singular fibre X_τ in exactly one smooth point of an irreducible component of X_τ. We let this component be $\Theta_{\tau,0}$ and we always have $\mu_{\tau,0} = 1$. Now, we distinguish two cases as in [Huy16, Corollary 11.1.7]:

- If $m_\tau = 1$, then X_τ is irreducible. The fibre is either smooth, a nodal rational curve, or a cuspidal rational curve.
- If $\tau > 1$, then X_τ is reducible. In this case, $\Theta_{\tau,i}$ is a smooth rational curve with self-intersection number -2 for all i.

Kodaira and Néron showed that every singular fibre of a smooth minimal elliptic surface belongs to one of the following types:

- **Multiplicative fibres:** I_n $(n \geq 0)$, consisting of n rational curves arranged in a cycle. By convention, I_0 refers to a smooth fibre.
- **Additive fibres:** II, III, IV, I_n^*, II^*, III^*, and IV^*, each consisting of configurations of smooth rational curves intersecting as listed in [Mir89, Tables I.4.1 and I.4.2].

3.3 The Weierstrass model and its desingularization

An elliptic surface $f : X \to \mathbb{P}^1$ may not be given by a Weierstrass equation in $\mathbb{P}^2_{k(t)}$. However, one can always find a (possibly singular) elliptic surface given by a Weierstrass equation, which has the same generic fibre as $X \to \mathbb{P}^1$. In fact, we will show that there is a canonical such surface, which we will call the Weierstrass model $\bar{X}$ of $X \to \mathbb{P}^1$. We will also discuss how $\bar{X}$ can be desingularized to recover a smooth elliptic surface birational to X.

3.3.1 Valuations and minimality

Let $f : X \to \mathbb{P}^1$ be an elliptic surface equipped with a section O. Since $\operatorname{char}(k) \neq 2, 3$, the generic fibre X_η of f can be written in short Weierstrass form:

$$y^2 = x^3 + a_4 x + a_6 \qquad a_4, a_6 \in k(t).$$

Its discriminant

$$\Delta = -16(4a_4^3 + 27a_6^2) \in k(t),$$

detects the singular fibres of the fibration: for each $t_0 \in \mathbb{P}^1$, the fibre X_{t_0} is smooth if and only if $\Delta(t_0) \neq 0$.

To study singular fibres, we work locally at points $\tau \in \mathbb{P}^1$. We choose t as the inhomogeneous coordinate of $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1,\tau}$ the local ring at τ.

- If $\tau \neq \infty$, take $t - \tau$ as a uniformizer. Any $f \in k(t)$ can be written uniquely as

  $$f = (t - \tau)^m \, u, \qquad u \in \mathcal{O}^*_{\mathbb{P}^1,\tau},$$

  and we define $v_\tau(f) := m$.

- If $\tau = \infty$, set $s = 1/t$ as a uniformizer. Then $f \in k(t)$ can be written as

  $$f = s^m \, u(s), \qquad u(s) \in \mathcal{O}^*_{\mathbb{P}^1,\infty}$$

  and define $v_\infty(f) := m$.

The coefficients a_4, a_6 of the short Weierstrass equation of X_η might have some poles at some $\tau \in \mathbb{P}^1$. For each such point $\tau \in \mathbb{P}^1$, our goal is to make $v_\tau(a_4)$ and $v_\tau(a_6)$ non-negative and as small as possible by some admissible change of coordinates.

Definition 3.3.1. A short Weierstrass equation

$$y^2 = x^3 + a_4 x + a_6 \qquad \text{with } a_4, a_6 \in k(t)$$

is *integral* at $\tau \in \mathbb{P}^1$ if $v_\tau(a_4), v_\tau(a_6) \geq 0$.

Definition 3.3.2. A short Weierstrass equation is *minimal* at τ if it is integral at τ and at least one of

$$v_\tau(a_4) < 4 \qquad \text{or} \qquad v_\tau(a_6) < 6.$$

holds.

Proposition 3.3.3. Over $\mathbb{P}^1$, there exists a Weierstrass equation which is minimal at all $\tau \in \mathbb{P}^1$. We call this the *minimal* Weierstrass form.

Proof. For the finite case $\tau \neq \infty$, after applying the change of coordinates

$$(x, y) \mapsto \left(\frac{1}{(t-\tau)^2} x, \frac{1}{(t-\tau)^3} y \right),$$

to a short Weierstrass equation with coefficients a_4, a_6, we get a new one with coefficients a'_4 and a'_6 given by

$$a'_4 = (t-\tau)^4 a_4 \qquad a'_6 = (t-\tau)^6 a_6.$$

Hence, integrality at any τ can be achieved by rescaling as many times as necessary to eliminate the poles at τ without introducing new ones. Minimality can also be achieved by rescaling by

$$(x, y) \mapsto ((t-\tau)^2 x, (t-\tau)^3 y)$$

since in the non-minimal case at τ, we have $v_\tau(a_4) \geq 4$ and $v_\tau(a_6) \geq 6$ and after rescaling the new Weierstrass equation has coefficients

$$a'_4 = \frac{a_4}{(t-\tau)^4} \qquad a'_6 = \frac{a_6}{(t-\tau)^6}.$$

so $v_\tau(a'_i) = v_\tau(a_i) - i \geq 0$ for $i = 4, 6$. The Weierstrass equation remains integral at τ and repeating this process as many times as necessary we reach minimality at τ. At $\tau = \infty$, use the coordinate $s = 1/t$ to reduce to the finite case. □

Definition 3.3.4. For an elliptic fibration $f : X \to \mathbb{P}^1$, we define the *Weierstrass model* associated to f, denoted by $\bar{X}$, to be the surface in $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the minimal Weierstrass equation of the generic fibre X_η.

3.3.2 Desingularization of the Weierstrass model

The minimal Weierstrass model $\bar{X}$ can be seen as a surface in $\mathbb{A}^2 \times \mathbb{P}^1$ over k, and it comes with the natural projection

$$\bar{f} \colon \bar{X} \to \mathbb{P}^1,$$

onto the t-coordinate. The singular fibres of $\bar{f}$ are exactly over the points where the discriminant of the minimal Weierstrass equation of X_η is zero. Since all but finitely many, fibres are smooth, $\bar{f}$ is an elliptic fibration. In particular, since the fibres are given by Weierstrass equations, the singular fibres of $\bar{f}$ are either nodal or cuspidal curves.

By construction, the generic fibres X_η and $\bar{X}_\eta$ coincide. The surface $\bar{X}$ is in general not smooth, but it is birational to X, since both surfaces have the same function field and agree over the generic point of $\mathbb{P}^1$. On the other hand, the closed fibres of f and $\bar{f}$ need not agree, as the process of passing to the Weierstrass model may contract certain components of reducible fibres.

Let $\Sigma \subset \mathbb{P}^1$ be the finite set of zeros of the discriminant of the minimal Weierstrass equation of X_η. Then the singularities of $\bar{X}$ are contained in the fibres over points of Σ, and are at worst nodes or cusps. In other words, any singular point of $\bar{X}$ lies on some fibre above a point in Σ.

The original surface X can be recovered as the minimal desingularization of $\bar{X}$. More precisely, resolving the singular points of $\bar{X}$ by blowing up, we obtain a smooth surface

$$\pi : X' \to \bar{X},$$

in which each singular point is replaced by a configuration of smooth rational curves corresponding to the exceptional divisors of the blow-up. The resulting smooth surface X' has the induced elliptic fibration

$$f' = \bar{f} \circ \pi : X' \to \mathbb{P}^1$$

that has the same generic fibre as $\bar{f}$, since the blow-ups are performed only over closed points. Blowing down any (-1)-curves that may appear in the resolution, we may assume that X' is minimal. Then, X and X' are smooth, minimal surfaces with the same generic fibre and therefore, Theorem 1.2.8 gives

$$X' \cong X.$$

For more details, see [SS19, Sections 5.6 and 3.7] and [Huy16, Section 11.2].

3.3.3 Singular fibres via Weierstrass equation

Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with generic fibre X_η. Let $\bar{X}$ be its Weierstrass model given by

$$y^2 = x^3 + a_4(t)x + a_6(t).$$

To classify a singular fibre X_τ, we consider the corresponding fibre $\bar{X}_\tau$. It is important to notice that if the singularity of $\bar{X}_\tau$ is not a singularity of $\bar{X}$, then $\bar{X}_\tau \cong X_\tau$. This is because the resolution of singularities only affects the fibres containing the singular points, leaving all other fibres unchanged. If the singularity of $\bar{X}_\tau$ is also a singular point of $\bar{X}$, then $\bar{X}_\tau$ is different from X_τ, but it still determines the singularities of X_τ.Whether the singularity occurs only in the fibre or also in the total surface can be determined by the valuations $v_\tau(\Delta)$ and $v_\tau(a_4), v_\tau(a_6)$. The details of the following classification can be found in [SS19, Section 5.8].

- **Multiplicative fibres.**
  - If $v_\tau(\Delta) = 1$, then $\bar{X}_\tau$ is a nodal cubic curve and the singularity lies only on the fibre; then X_τ has Kodaira fibre type I_1.
  - If $v_\tau(\Delta) = n > 1$ and $v_\tau(a_4) = v_\tau(a_6) = 0$, then $\bar{X}_\tau$ has a node which is also a singularity on the surface; resolving it introduces n irreducible components arranged in a cycle giving fibre type I_n.
- **Additive fibres.**
  - If $v_\tau(\Delta) = 2$ and $v_\tau(a_4) \geq 1$ and $v_\tau(a_6) = 1$, then $\bar{X}_\tau$ has a cusp but the cusp is not a singularity of the surface; then X_τ has fibre type II.
  - For $v_\tau(\Delta) > 2$, $\bar{X}_\tau$ is cuspidal but the cusp is a surface singularity and resolving it gives the remaining fibre types $III, IV, I_n^*, II^*, III^*, IV^*$ depending on $v_\tau(a_4)$, $v_\tau(a_6)$ and $v_\tau(\Delta)$ as shown in the table below.

The following table summarizes the possibilities of Kodaira's classification of the fibres X_τ depending on the orders of vanishing of a_4, a_6, and Δ at τ from the fibres $\bar{X}_\tau$ of the minimal Weierstrass model. It also shows the Euler characteristic of the corresponding type of the fibre.

Fibre type	$\mathrm{ord}_\tau(a_4)$	$\mathrm{ord}_\tau(a_6)$	$\mathrm{ord}_\tau(\Delta)$	$e(X_\tau)$
I_0 (smooth)	≥ 0	≥ 0	0	0
I_n $(n \geq 1)$	0	0	n	n
II	≥ 1	1	2	2
III	1	≥ 2	3	3
IV	≥ 2	2	4	4
I_0^*	≥ 2	3	6	6
	3	≥ 2	6	6
I_n^* $(n \geq 1)$	2	3	$n+6$	$n+6$
IV^*	≥ 3	4	8	8
III^*	3	≥ 5	9	9
II^*	≥ 4	5	10	10

Table 3.1: Classification of the singular fibers

The Euler characteristic of an elliptic surface, is given as the sum of the Euler characteristics of the fibres:

Theorem 3.3.5. ([SS19, Theorem 5.47]) For an elliptic surface X over $\mathbb{P}^1$ we have

$$e(X) = \sum_{\tau \in \mathbb{P}^1} e(X_\tau).$$

In particular, if X is an elliptic K3 surface, then we know $e(X) = 24$ so

$$\sum_{\tau \in \mathbb{P}^1} e(X_\tau) = 24.$$

3.4 Example resolving a singularity

In this section, we present an explicit example of the resolution of a surface singularity by computing the blow-up of a point, following [Moo15, Chapter 6, §3] and [SS19, Example 5.24].

Suppose we have an elliptic fibration $f : X \to \mathbb{P}^1$, whose Weierstrass model $\bar{X}$ is given by the minimal equation

$$\bar{X} : y^2 = x^3 + t^2 x + t^2.$$

The discriminant is

$$\Delta = -16(4t^6 + 27t^4) = -16t^4(4t^2 + 27),$$

so the fibre over $t = 0$ is singular and one can check that the singular point $P = (0, 0, 0)$ is also a singularity of the surface $\bar{X}$ since all partial derivatives vanish at P. We resolve the singularity at $P = (0,0,0)$ by blowing up at P to get X'. This takes us to $\mathbb{A}^3 \times \mathbb{P}^2$ with homogeneous coordinates $(u : v : w)$ on $\mathbb{P}^2$ and extra relations

$$xv = yu, \qquad xw = tu, \qquad yw = tv.$$

We compute the strict transform in each of the three affine charts.
First chart: $U = \{u \neq 0\}$.
Here we have

$$y = xv, \qquad t = xw.$$

Substituting this into the Weierstrass equation gives

$$x^2v^2 = x^3 + w^2x^3 + w^2x^2.$$

The points with $x = 0$ lie over the singular point P. Dividing by the exceptional divisor x^2, we find that X' intersects this chart in the strict transform (in $\mathbb{A}^3$ with coordinates x, v, w) given by

$$X' \cap U : \quad v^2 = x + w^2x + w^2.$$

The fibre over $t = 0$ is defined by $xw = 0$.

- If $x = 0$, then we find $v^2 = w^2$ so

$$\Theta_{0,1} = V(x, v + w), \qquad \Theta_{0,2} = V(x, v - w).$$

- If $w = 0$, then we find $v^2 = x$ so

$$\Theta_{0,3} = V(w, v^2 - x).$$

The components intersect in this chart at the unique point $(x, v, w) = (0, 0, 0)$.
Second chart: $V = \{v \neq 0\}$.
Here we have

$$x = yu, \quad t = yw$$

Substituting gives:

$$y^2 = y^3u^3 + y^3w^2u + y^2w^2$$

The points with $y = 0$ lie over the singular point P. The desingularization X' intersects this chart in the strict transform (in $\mathbb{A}^3$ with coordinates (y, u, w)) given by

$$X' \cap V : \quad 1 = u^3y + uw^2y + w^2.$$

The fibre over $t = 0$ is defined by $yw = 0$.

- If $y = 0$, then we have $1 = w^2$ so

$$\Theta_{0,1} = V(y, w + 1), \qquad \Theta_{0,2} = V(y, w - 1).$$

- If $w = 0$, then we have $1 = u^3y$ so

$$\Theta_{0,3} = V(w, 1 - u^3y).$$

The components do not intersect in this chart.
Third chart $W = \{w \neq 0\}$.
Set $w = 1$, so that

$$x = tu, \quad y = tv.$$

Substituting into the Weierstrass equation gives:

$$t^2v^2 = t^3u^3 + t^3u + t^2$$

The strict transform in this chart is

$$X' \cap W : \quad v^2 = tu^3 + ut + 1$$

The fibre over $t = 0$ is defined by $v^2 = 1$, so we have components

$$\Theta_{0,1} = V(t, v + 1), \qquad \Theta_{0,2} = V(t, v - 1)$$

The components of the fibre over 0 do not intersect in this chart.

Gluing the three affine charts, the fibre over $t = 0$ in the resolved surface consists of three smooth rational components and form the configuration of a Kodaira fibre of type IV. This agrees with the Kodaira classification, since $\operatorname{ord}_0(a_4) = 2$, $\operatorname{ord}_0(a_6) = 2$, and $\operatorname{ord}_0(\Delta) = 4$.

Although we have not considered the singular fibres over the points $t_0, t_1 \in \mathbb{P}^1$ satisfying $4t^2 + 27 = 0$, where the discriminant Δ vanishes, we can check that X is not a K3 surface. From Table 3.1 we find that

$$e(X) = e(X_0) + e(X_{t_0}) + e(X_{t_1}) = 4 + 1 + 1 = 6$$

which differs from the Euler characteristic 24 of a $K3$ surface.

3.5 The Jacobian fibration

Let $f\colon X \to \mathbb{P}^1$ be an elliptic fibration with generic fibre X_η. In the classification of singular fibres, we have assumed that f admits a section. If f does not admit a section, then the generic fibre X_η is a smooth projective curve of genus one over $k(t)$ without a $k(t)$-rational point, and hence has no natural group structure. Recall from Chapter 2, that to any smooth projective genus one curve C over a field K one can associate its Jacobian $\operatorname{Jac}(C)$, which is an elliptic curve over K equipped with a distinguished K-rational point O. Applying this construction to the generic fibre X_η over $k(t)$, we obtain its Jacobian $(\operatorname{Jac}(X_\eta), O)$, which is an elliptic curve over $k(t)$. In this section, we will show that one can associate to the fibration $f\colon X \to \mathbb{P}^1$ a new elliptic fibration

$$f_J\colon \operatorname{Jac}(X) \to \mathbb{P}^1,$$

called the *Jacobian fibration* of f, whose generic fibre is $\operatorname{Jac}(X_\eta)$. By construction, f_J admits a section, given by the identity of the Jacobian.

3.5.1 Jacobian fibration

If an elliptic surface $f\colon X \to \mathbb{P}^1$ does not admit a section, one can associate to it a new elliptic surface that does. This is done via the *Jacobian fibration*, which replaces each fibre by its Jacobian.

Proposition 3.5.1. ([CD89, Proposition 5.2.5],[Huy16, Section 11.4.1]) Let $f : X \to \mathbb{P}^1$ be an elliptic fibration. Then there exists a unique elliptic surface $f_J : \mathrm{Jac}(X) \to \mathbb{P}^1$ such that

$$\mathrm{Jac}(X)_\eta \cong \mathrm{Jac}(X_\eta)$$

We call $f_J : \mathrm{Jac}(X) \to \mathbb{P}^1$ the *Jacobian fibration* of $f : X \to \mathbb{P}^1$.

By construction, $\mathrm{Jac}(X)$ admits a section given by the identity of the Jacobian. Moreover, for each closed $\tau \in \mathbb{P}^1$, the fibre of $\mathrm{Jac}(X)$ over τ is isomorphic to the Jacobian of the fibre of X over τ [CD89, Theorem 5.3.1], that is

$$\mathrm{Jac}(X)_\tau \cong \mathrm{Jac}(X_\tau).$$

We use the Jacobian fibration because it allows us to pass from an elliptic fibration without a section to one with a section, while preserving the relevant fibre data of the original surface. In particular, if X is a K3 surface, the Jacobian fibration $\mathrm{Jac}(X)$ has the same configuration of singular fibres and has the same Picard number as X [CD89, Theorem 5.3.1], so the classification of singular fibres applies to $\mathrm{Jac}(X)$. For more details see also [Huy16, Section 11.4].

3.5.2 Multisection index

For an elliptic fibration $f : X \to \mathbb{P}^1$, the image of a section is a curve on X meeting each fibre once. A *multisection* of degree n is a curve that meets each fibre in n points, counted with multiplicity. The smallest such integer n measures how far the fibration is from having a section. That minimal n is called the *multisection index.*

Definition 3.5.2. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration with generic fibre X_η. The *multisection index* of f is defined as

$$l := \min\{(D.X_\eta) \mid D \in \mathrm{Pic}(X)\}.$$

A divisor D with $(D.X_\eta) = l$ corresponds to a multisection of degree l.

We can see that X has a section if and only if $l = 1$. The multisection index also gives a relation between the discriminants of the Néron-Severi groups of X and $\mathrm{Jac}(X)$.

Theorem 3.5.3. [Keu00, Lemma 2.1] Let $\mathrm{NS}(X)$ and $\mathrm{NS}(\mathrm{Jac}(X))$ denote the Néron–Severi groups of X and $\mathrm{Jac}(X)$, respectively. Their discriminants satisfy the relation

$$\mathrm{disc}(\mathrm{NS}(X)) = l^2 \,\mathrm{disc}(\mathrm{NS}(\mathrm{Jac}(X))),$$

where l is the multisection index of X.

Chapter 4

Elliptic fibrations on the Fermat quartic

In this chapter we describe two explicit elliptic fibrations on the Fermat quartic surface. The first arises from projection from a line and admits a section. The second is obtained by the intersection of two quadrics and does not admit a section. For the latter, we compute its Jacobian fibration using results on pencil of curves arising from intersections of two quadrics as in [Rei72]. In both cases we compute the associated Weierstrass form of the generic fibre and classify the singular fibres.

4.1 Fermat quartic

Consider the *Fermat quartic* $X \subset \mathbb{P}^3$ over $k = \mathbb{C}$ defined by the equation

$$X : \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

The surface X has no singular points, since all partial derivatives

$$\frac{\partial}{\partial x_i}(x_0^4 + x_1^4 + x_2^4 + x_3^4) = 4x_i^3$$

for $i = 0, 1, 2, 3$, vanish simultaenously only when $x_i = 0$ for all i. By Example 3.1.3, every smooth quartic surface in $\mathbb{P}^3$ is a $K3$ surface, therefore the Fermat quartic X is a $K3$ surface.

We can now describe the lines contained in the Fermat quartic. A smooth quartic in $\mathbb{P}^3$ can contain at most 64 lines [RS18, Theorem 1.2]. In the case of the Fermat quartic X, there are exactly 48 lines.

Theorem 4.1.1. Let X be the Fermat quartic surface like above. Then X contains exactly 48 lines.

Proof. Let $L \subset \mathbb{P}^3$ be a line. After a suitable permutation of coordinates, we may assume that L is not contained in the hyperplanes $\{x_2 = 0\}$ and $\{x_3 = 0\}$. Solving the defining linear equations of L for x_0 and x_1, we may write

$$L = V_+\big(x_0 - (a_2x_2 + a_3x_3),\ x_1 - (b_2x_2 + b_3x_3)\big),$$

for some $a_2, a_3, b_2, b_3 \in \mathbb{C}$.

To determine when a line L lies on X, we substitute the defining equations of the line into the quartic and require the resulting polynomial of degree 4 in the remaining coordinates to vanish. We obtain the following system of equations:

$$a_2^4 + b_2^4 + 1 = 0, \tag{4.1}$$
$$a_3^4 + b_3^4 + 1 = 0, \tag{4.2}$$
$$4a_2^3a_3 + 4b_2^3b_3 = 0, \tag{4.3}$$
$$4a_2a_3^3 + 4b_2b_3^3 = 0, \tag{4.4}$$
$$6a_2^2a_3^2 + 6b_2^2b_3^2 = 0. \tag{4.5}$$

The system has finitely many solutions. From equation (4.5) we obtain

$$a_2a_3 = \pm ib_2b_3$$

Equations (4.3) and (4.4) can be written as

$$a_3a_2^3 = -b_3b_2^3, \qquad a_2a_3^3 = -b_2b_3^3.$$

If all solutions were non-zero then we can divide equation (4.3) by (4.4) to obtain

$$\frac{a_2}{a_3} = \pm\frac{b_2}{b_3}$$

Multiplying this now by (4.5) gives

$$a_2^2 = \pm ib_2^2$$

which if we substitute in equation (4.1) gives a contradiction. Hence, there is no solution with all four variables nonzero. Taking $a_2 = 0$, gives $b_2^4 = -1$, so $b_2 = \zeta_8^{2i+1}$ for some $i \in \{0,1,2,3\}$ and ζ_8 primitive 8-th root of unity. The remaining equations then force $b_3 = 0$ and $a_3 = \zeta_8^{2j+1}$ for some $j \in \{0,1,2,3\}$. Thus, we get a family of lines of the form

$$V_+(x_0 - \zeta_8^{2i+1}x_3,\ x_1 - \zeta_8^{2j+1}x_2) \quad i,j = 0,1,2,3.$$

Similarly, setting $b_2 = 0$ produces another family

$$V_+(x_0 - \zeta_8^{2i+1}x_2,\ x_1 - \zeta_8^{2j+1}x_3), \quad i,j = 0,1,2,3.$$

The other cases $a_3 = 0$ and $b_3 = 0$ do not give new families. By symmetry of the Fermat quartic under permutation of coordinates, the choice of x_2 and x_3 as the "free coordinates" is arbitrary. Choosing other pair of coodinates, produces in total three distinct families of lines:

$$\begin{aligned} &V_+\big(x_0 - \zeta_8^{2i+1}x_3,\ x_1 - \zeta_8^{2j+1}x_2\big), \\ &V_+\big(x_0 - \zeta_8^{2i+1}x_2,\ x_1 - \zeta_8^{2j+1}x_3\big), \qquad i,j = 0,1,2,3. \\ &V_+\big(x_0 - \zeta_8^{2i+1}x_1,\ x_2 - \zeta_8^{2j+1}x_3\big), \end{aligned}$$

Each family contains $4 \times 4 = 16$ lines, giving a total of $3 \times 16 = 48$ lines on X. $\square$

We will use the following notation for the 48 lines in Fermat quartic:

$$l_{4a+b} = (\zeta_8^{2a+1}s : s : t : \zeta^{2b+1}t), \quad l_{4a+b+16} = (s : \zeta_8^{2a+1}t : t : \zeta_8^{2b+1}s)$$

$$l_{4a+b+32} = (s : t : \zeta_8^{2a+1}s : \zeta_8^{2b+1}t),$$

where $a, b = 0, 1, 2, 3$ and ζ_8 is a primitive 8-th root of unity.

The Néron-Severi group $\mathrm{NS}(X)$ has rank 20. Among the 48 lines on the Fermat quartic, one can select 20 of them which form a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. In particular,

$$\{\, l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_9, l_{10}, l_{11}, l_{17}, l_{18}, l_{19}, l_{21}, l_{22}, l_{23}, l_{33}, l_{34}, l_{35}, l_{37} \,\}$$

is a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ as shown in [BG14, Section 4.1].

4.2 Line fibrations

Let $X \subset \mathbb{P}^3$ be the Fermat quartic and let $L \subset X$ be a line. We construct an elliptic fibration on X by projecting from L as follows.

All planes containing L form a pencil $\{\pi_t\}_{t\in\mathbb{P}^1}$. For every point $x \in X \setminus L$, the point x together with the line L determines a unique plane $\pi_x \subset \mathbb{P}^3$. This gives a rational map

$$f\colon X \dashrightarrow \mathbb{P}^1, \qquad x \mapsto t_x,$$

defined on $X \setminus L$, where t_x is the parameter corresponding to the plane π_x in the pencil.

At points $p \in L$, the plane containing both p and L is not unique, so the map is initially undefined. However, we can assign a value to $f(p)$ by considering a sequence of points $x \in X \setminus L$ approaching p: the corresponding planes π_x in the pencil converge to a unique plane containing L, giving a natural value for $f(p)$. In this way, the rational map extends to all points of L so we have a morphism

$$f\colon X \to \mathbb{P}^1.$$

The generic fibre of f, denoted X_η, can be described as follows.
Let $\pi \subset \mathbb{P}^3 \times \mathbb{P}^1$ be the family of planes containing L, as above, and let

$$\pi_\eta = \pi \times_{\mathbb{P}^1} \mathrm{Spec}(k(\eta))$$

be the *generic plane* in the pencil, defined over $k(t)$. Then the intersection $\pi_\eta \cap X \subset \pi_\eta$ is a plane quartic curve over $k(t)$ containing L. Removing L gives the residual curve

$$X_\eta = (\pi_\eta \cap X) \setminus L,$$

which has degree 3. Since X is smooth, the residual curve X_η is a smooth plane cubic, and therefore has genus 1 over $k(t)$ (see Example 1.2.7). Hence, f is an elliptic fibration.

Now, let $L' \subset X$ be another line skew to L. For each plane π_t in the pencil, the intersection $\pi_t \cap L'$ consists of a single point, which lies on $X \cap \pi_t$ but not on L. Hence, each fibre X_t contains exactly one point of L', giving a section

$$\sigma_{L'}\colon \mathbb{P}^1 \to X, \qquad t \mapsto \pi_t \cap L'.$$

Therefore, the fibration $f : X \to \mathbb{P}^1$ has a section, and the generic fibre X_η is an elliptic curve over $k(t)$, which can be written in Weierstrass form.

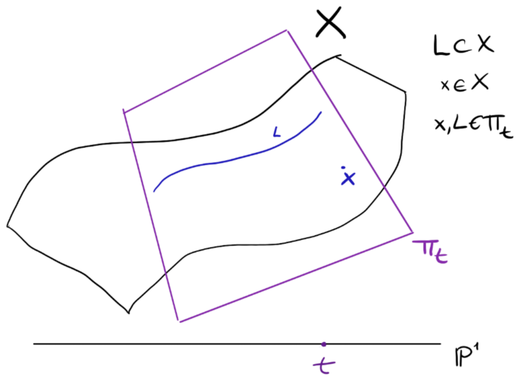


Figure 4.1: Projection from the line $L \subset X$

4.2.1 Line fibration on Fermat quartic

We now choose an explicit line $L \subset X$ and compute the corresponding elliptic fibration and its generic fibre.

Among the 48 lines on X, we fix

$$L : x_0 = \zeta x_3 \qquad x_1 = \zeta x_2, \qquad \text{for } \zeta = e^{\pi i/4}$$

Proposition 4.2.1. The generic fibre of the elliptic fibration on the Fermat quartic X obtained by projecting from the line L from above

$$f \colon X \to \mathbb{P}^1$$

has Weierstrass form

$$y^2 = x^3 - 3x + t^4 + \frac{1}{t^4}.$$

Proof. The pencil of planes containing L has generic plane

$$\pi_\eta : t(x_0 - \zeta x_3) + (x_1 - \zeta x_2) = 0.$$

Let $u = t(x_0 - \zeta x_3)$. We substitute $x_0 = \frac{u}{t} + \zeta x_3$ and $x_1 = -u + \zeta x_2$ on the the Fermat quartic X to get

$$X \cap \pi_\eta \colon \quad (\frac{u}{t} + \zeta x_3)^4 + (\zeta x_2 - u)^4 + x_2^4 + x_3^4 = 0.$$

Expanding gives

$$u\left(4\zeta^3(\frac{x_3^3}{t} - x_2^3) + 6\zeta^2 u(\frac{x_3^2}{t^2} + x_2^2) + 4\zeta u^2(\frac{x_3}{t^3} - x_2) + u^3(1 + \frac{1}{t^4})\right) = 0.$$

where the linear factor $u = 0$ corresponds to the line $L \subset X$.
Removing L and substituting back $u = t(x_0 - \zeta x_3)$, the residual cubic in the plane π_η is

therefore given by the equation

$$\begin{aligned} X_\eta : \quad & \frac{\zeta^3 x_3^3 + \zeta^2 x_3^2 x_0 + \zeta x_2 x_0^2 + x_0^3}{t} - \zeta^3 t^3 x_3^3 - 4\zeta^3 t^2 x_3^2 x_2 \\ & - 6\zeta^3 t x_3 x_2^2 - 4\zeta^3 x_2^3 + 3\zeta^2 t^3 x_3^2 x_0 + 8\zeta^2 t^2 x_3 x_0 x_2 + 6\zeta^2 t x_0 x_2^2 \\ & - 3\zeta t^3 x_3 x_0^2 - 4\zeta t^2 x_0^2 x_2 + t^3 x_0^3 = 0. \end{aligned} \tag{4.6}$$

Now, consider the line $L' \subset X$ defined by

$$L' : x_0 = -\zeta x_3 \qquad x_1 = -\zeta x_2, \qquad \text{for } \zeta = e^{\pi i/4}.$$

Clearly, L and L' are skew and the intersection $\pi_\eta \cap L'$ is the point

$$P_\eta = (-\zeta : \zeta t : -t : 1).$$

This construction gives a section $\sigma_{L'} : t \mapsto P_t$. The generic fibre X_η is therefore a genus one curve over $\mathbb{C}(t)$ with the $\mathbb{C}(t)$-rational point P_η, thus an elliptic curve. Using Magma, the genus one curve defined by the cubic equation 4.6 together with the point P_η can be transformed into Weierstrass form (the Magma code is given in Appendix A.2). The Weierstrass equation returned by Magma, after scaling of the coordinates, is

$$y^2 = x^3 - 3x + t^4 + \frac{1}{t^4}. \tag{4.7}$$

□

Now that we know the Weierstrass equation of the generic fibre of the Fermat quartic obtained by projecting from the line L, we can classify the singular fibres of the fibration as in Section 3.3.3. The Weierstrass equation (4.7) is not minimal, since it has poles at $t = 0$. To remove the pole at $t = 0$, we apply the change of coordinates $(x, y) \mapsto (\frac{1}{t^2}x, \frac{1}{t^3}y)$ which gives the Weierstrass model of f

$$\bar{X} : y^2 = x^3 - 3t^4 x + t^2(t^8 + 1)$$

with discriminant

$$\Delta(\bar{X}) = -16t^4(t^8 - 1)^2.$$

From the vanishing of the discriminant, we see that the singular fibres occur at $t = 0$, $t = \infty$, and at the eight 8-th roots of unity. Using the classification of singular fibres (see Table 3.1), we find that the fibres at $t = 0$ and $t = \infty$ are of type IV, while the fibres over the 8-th roots of unity are of type I_2. From Table 3.1, a fibre of type I_2 has Euler characteristic $e(X_t) = 2$, and for type IV fibres we have $e(X_0) = e(X_\infty) = 4$. Hence, the total Euler characteristic is

$$8 \cdot 2 + 2 \cdot 4 = 24,$$

which agrees with the expected Euler characteristic of the Fermat quartic.

4.3 Elliptic fibration without a section

We now give an explicit example of an elliptic fibration on Fermat quartic which does not admit a section. For this, we give a different description of the Fermat quartic as the intersection of two hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$.

4.3.1 Intersection of two $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$

Let $\mathbb{P}^1 \times \mathbb{P}^3$ have homogenous coordinates $[t_0 : t_1]$ on the first factor and $[x_0 : x_1 : x_2 : x_3]$ on the second. Consider two different bidegree $(1,2)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ given by

$$\begin{aligned} Q_1\colon &\quad f_1t_0 + f_2t_1 = 0 && (4.8)\\ Q_2\colon &\quad g_1t_0 + g_2t_1 = 0 && (4.9)\end{aligned}$$

where f_i and g_i for $i = 1, 2$ are polynomials on $\mathbb{P}^3$ of degree 2. Let $X = Q_1 \cap Q_2$ be the common zero locus of these two equations. From Example 3.1.5, X is a K3 surface.

Proposition 4.3.1. With the above notation we have the following:

(i) The projection $q_2\colon X \to \mathbb{P}^3$ is an isomorphism onto its image.

(ii) The projection $q_1\colon X \to \mathbb{P}^1$ defines an elliptic fibration.

$$\begin{tikzcd} X \arrow[r, "q_1"] \arrow[d, "q_2"'] & \mathbb{P}^1 \\ \mathbb{P}^3 & \end{tikzcd}$$

Proof. For (i), the image $q_2(X)$ is given by the vanishing of the determinant

$$\det \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix} = 0$$

which is a quartic equation in $\mathbb{P}^3$ and by Example 3.1.3 is a K3 surface. For a point $x \in q_2(X)$, the fibre $q_2^{-1}(x)$ consists of a single point. Therefore, q_2 is a bijective morphism between K3 surfaces, and thus an isomorphism.

For (ii), a fibre of q_1 over a closed point $t = t_1/t_0 \in \mathbb{P}^1$ is the curve

$$X_t = (Q_1 \cap Q_2)_t = Q_{1,t} \cap Q_{2,t} \subset \mathbb{P}^3,$$

given as the intersection of the two quadrics $Q_{1,t}$ and $Q_{2,t}$ in $\mathbb{P}^3$, where $Q_{1,t}$ and $Q_{2,t}$ are the quadrics in $\mathbb{P}^3$ for a fixed t. By Example 3.1.4 we have

$$K_{X_t} = \left(K_{\mathbb{P}^3} + \mathcal{O}_{\mathbb{P}^3}(2) + \mathcal{O}_{\mathbb{P}^3}(2)\right)\big|_{X_t} = \mathcal{O}_{\mathbb{P}^3}(-4+2+2)\big|_{X_t} = \mathcal{O}_{\mathbb{P}^3}\big|_{X_t}.$$

Let $g(X_t)$ be the genus of X_t, then the above gives

$$2g(X_t) - 2 = 0$$

so X_t is a genus one curve over $\mathbb{C}$.

Similarly, the generic fibre $X_\eta = (Q_1 \cap Q_2)_\eta = Q_{1,\eta} \cap Q_{2,\eta}$ is the intersection of two quadrics in $\mathbb{P}^3$ over $\mathbb{C}(t)$, where $Q_{1,\eta}$ and $Q_{2,\eta}$ are defined by the same equations as Q_1 and Q_2 seeing them over $\mathbb{C}(t)$. □

We now work with the concrete complete intersection in $\mathbb{P}^1 \times \mathbb{P}^3$ as considered in [Kim18, Section 2.2] given by the following equations

$$\begin{aligned} Q_1\colon &\quad x_0^2 + x_2^2 + 2tx_1x_3 = 0 \\ Q_2\colon &\quad x_1^2 + x_3^2 + 2tx_0x_2 = 0 \end{aligned} \tag{4.10}$$

Let their intersection be $X = Q_1 \cap Q_2 \subseteq \mathbb{P}^1 \times \mathbb{P}^3$. By Proposition 4.3.1, the projection

$$q_2 : X \to \mathbb{P}^3$$

maps X isomorphically onto its image given by

$$(x_0^2 + x_2^2)x_0x_2 - (x_1^2 + x_3^2)x_1x_3 = 0. \tag{4.11}$$

Equivalently, (4.11) can be written as

$$(x_0 + x_2)^4 - (x_0 - x_2)^4 - (x_1 + x_3)^4 + (x_1 - x_3)^4 = 0.$$

The linear change of variables

$$x_0 + x_2 \mapsto x \qquad x_0 - x_2 \mapsto e^{2\pi i/8}y \qquad x_1 + x_3 \mapsto e^{2\pi i/8}z \qquad x_1 - x_3 \mapsto w$$

transforms the defining equation of $q_2(X)$ into

$$x^4 + y^4 + z^4 + w^4 = 0. \tag{4.12}$$

Hence, the intersection X of the two $(1,2)$-hypersurfaces given by (4.10) is isomorphic to the Fermat quartic (4.12).

4.3.2 Pencil of quadrics in $\mathbb{P}^3$ and the associated double cover

Let $Q \subset \mathbb{P}^3$ be a quadric defined over a field K. To any quadric Q, we can associate a symmetric matrix M such that Q is given by $x^TMx = 0$ where $x = (x_0\, x_1\, x_2\, x_3)$ represents the homogenous coordinates on $\mathbb{P}^3$.

Proposition 4.3.2. Let $Q \subset \mathbb{P}^3$ be a quadric defined by a homogeneous quadratic polynomial with associated symmetric matrix M. Then:

1. If $\det(M) \neq 0$, or equivalently, if $\mathrm{rk}(M) = 4$ then Q is smooth. A smooth quadric Q contains two disjoint families $\mathcal{L}_1$ and $\mathcal{L}_2$ of lines (two *rulings*) such that
    - Distinct lines in the same ruling are disjoint:

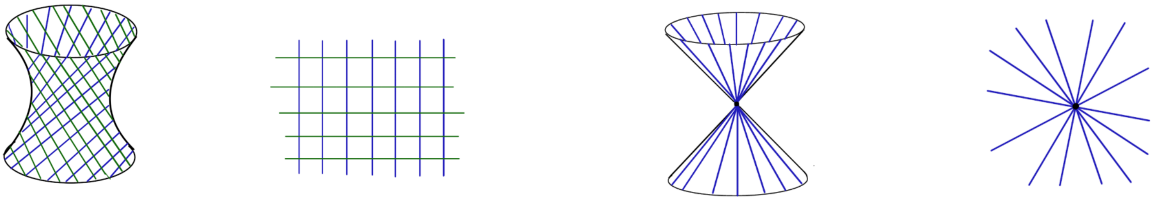
      $$l_1 \cap l_2 = \varnothing \text{ for any } l_1, l_2 \in \mathcal{L}_1 \text{ or } l_1, l_2 \in \mathcal{L}_2 \text{ with } l_1 \neq l_2.$$

    - Two lines from each ruling intersect in exactly one point:

      $$l_1 \cap l_2 = \{\mathit{point}\} \text{ for } l_1 \in \mathcal{L}_1,\ l_2 \in \mathcal{L}_2.$$

    - Each ruling covers the entire quadric:

      $$\bigcup_{l_1 \in \mathcal{L}_1} l_1 = Q = \bigcup_{l_2 \in \mathcal{L}_2} l_2.$$



(a) Smooth quadric with two rulings.

(b) Cone quadric with one ruling.

Figure 4.2: Comparison of two quadrics in $\mathbb{P}^3$.

2. If $\mathrm{rk}(M) = 3$, then Q is a *cone.* A cone Q contains exactly one ruling of lines, and all such lines pass through the unique singular point of Q, the *vertex.*

Let $X = Q_1 \cap Q_2$ be the intersection of two quadrics in $\mathbb{P}^3$. We consider the pencil of quadrics through X

$$\Phi = \{Q_\lambda = Q_1 - \lambda Q_2 \mid \lambda \in \mathbb{P}^1\}.$$

Each quadric Q_λ in the pencil is defined by a symmetric 4×4 matrix.

Definition 4.3.3. A pencil Φ of quadrics is called *non-singular* if there is at least one smooth quadric in the pencil.

Let Φ be a non-singular pencil of quadrics in $\mathbb{P}^3$ through $X = Q_1 \cap Q_2$. For $\lambda \in \mathbb{P}^1$, let Q_λ denote the quadric in the pencil corresponding to λ, and let M_λ be the associated symmetric matrix. Then $\det(M_\lambda)$ is a homogeneous polynomial of degree 4 in λ, and its zeros correspond precisely to the values of λ for which Q_λ is singular.

For a general non-singular pencil, the polynomial $\det(M_\lambda)$ has four distinct roots. These four points determine a smooth projective curve C, defined as the double cover of $\mathbb{P}^1$ branched at the singular members of the pencil. Concretely, the curve C locally in $\mathbb{A}^2_{\lambda,\mu}$ is given by

$$C: \quad \mu^2 = \det(M_\lambda),$$

and the projection

$$(\lambda, \mu) \longmapsto \lambda$$

realizes C as a $2{:}1$ cover of $\mathbb{P}^1$, branched exactly over the four singular values of Φ.

We now introduce the parameter space of lines contained in the quadrics of the pencil. Define

$$\mathrm{Gen}(\Phi) = \big\{(\lambda, L) \in \mathbb{P}^1 \times \mathrm{Gr}(1, \mathbb{P}^3) \bigm| L \subset Q_\lambda\big\},$$

which parametrizes pairs consisting of a quadric in the pencil and a line lying on it.

For a fixed value of λ, a smooth quadric $Q_\lambda \subset \mathbb{P}^3$ admits two rulings by lines, forming a “grid”. Thus, a point of $\mathrm{Gen}(\Phi)$ may be viewed as the choice of a quadric together with a line belonging to one of its two rulings.

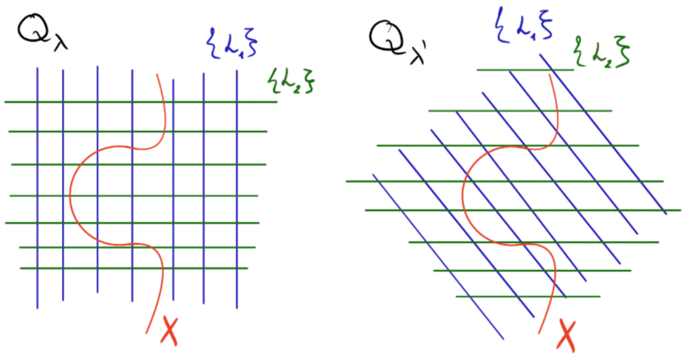


Figure 4.3: Each diagram represents a grid determined by a smooth quadric and the lines contained in its two rulings.

By Reid's description of pencils of quadrics [Rei72, Theorem 1.10], the space $\mathrm{Gen}(\Phi)$ fits into a factorization

$$\begin{tikzcd}
\mathrm{Gen}(\Phi) \arrow[dd, "f"'] \arrow[dr, "p"] & \\
& C \arrow[dl, "q"] \\
\mathbb{P}^1 &
\end{tikzcd} \tag{$**$}$$

where f is projection to $\mathbb{P}^1$, the morphism p records the choice of ruling and the morphism q is the double cover of $\mathbb{P}^1$ branched at the four singular values of the pencil.

4.3.3 Jacobian of the intersection of two quadrics

Let $X = Q_1 \cap Q_2 \subset \mathbb{P}^3$ be the smooth complete intersection of two quadrics over a field K, and let Φ be the pencil of quadrics through X. Let C be the smooth projective curve obtained as the double cover of $\mathbb{P}^1$ branched at the four singular quadrics in Φ. Our goal is to show that the Jacobian of the genus one curve X is isomorphic to the double cover of $\mathbb{P}^1$ associated to the pencil of quadrics through X. To do that, we first show that the curve C isomorphic to the genus one curve X over $\bar{K}$.

Proposition 4.3.4. With the above notation, there exists an isomorphism of curves over $\bar{K}$

$$\phi : X \longrightarrow C.$$

Proof. Consider the incidence variety

$$\mathcal{I} := \{(\lambda, L, x) \in \mathbb{P}^1 \times \mathrm{Gr}(1, \mathbb{P}^3) \times X \mid L \subseteq Q_\lambda,\ x \in L \cap X\},$$

together with the projections $\pi : \mathcal{I} \to \mathrm{Gen}(\Phi)$ and $\psi : \mathcal{I} \to X$.

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "\psi"] \arrow[d, "\pi"'] & X \\
\mathrm{Gen}(\Phi) &
\end{tikzcd}$$

Fixing λ amounts to choosing one quadric in the pencil. Selecting a line $L \subset Q_\lambda$ then specifies one of the lines in that grid. Such a line meets X in exactly two points, and hence each pair (λ, L) in $\mathrm{Gen}(\Phi)$ lifts to two points in $\mathcal{I}$. Therefore, π is a 2:1 map.

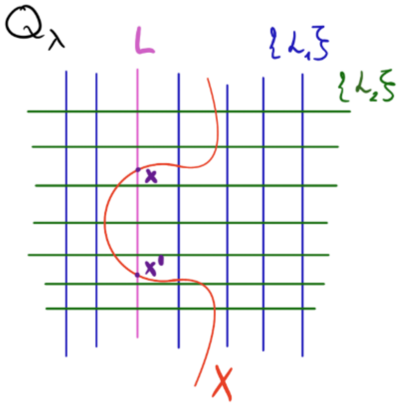


Figure 4.4: For a fixed grid λ and a line $L \subset Q_\lambda$ there are two elements (λ, L, x) and (λ, L, x') in $\mathcal{I}$.

For a fixed $\bar{K}$-rational point $s \in X(\bar{K})$, consider the subset

$$\widetilde{C}_s = \{(\lambda, L) \mid L \subseteq Q_\lambda, s \in X \cap L\} \subseteq \mathrm{Gen}(\Phi).$$

Since X is contained in every member of the pencil, the point s lies on each quadric Q_λ. Thus, the points of $\widetilde{C}_s$ are described by choosing a grid λ and then selecting the lines in that grid that pass through s. For a smooth quadric Q_λ, there are exactly two such lines (one in each ruling), whereas for a singular quadric there is only one.

Observe that

$$\pi^{-1}(\widetilde{C}_s) = \{(\lambda, L, x) \in \mathcal{I} \mid L \subset Q_\lambda,\ x \in L \cap X,\ s \in L \cap X\}.$$

For a fixed pair (λ, L), the intersection $L \cap X$ consists of the two points s and the other point x, so $\pi^{-1}(\widetilde{C}_s)$ contains exactly the two triples (λ, L, s) and (λ, L, x).

Inside $\pi^{-1}(\widetilde{C}_s)$ we have the subvariety

$$D_1 := \{(\lambda, L, s) \mid L \subset Q_\lambda,\ s \in L\} = \psi^{-1}(s),$$

Equivalently, for each grid λ, D_1 consists of choosing a line in that grid passing through s and recording the triple (λ, L, s). The projection π simply forgets the point s, so

$$\pi|_{D_1} : D_1 \to \widetilde{C}_s$$

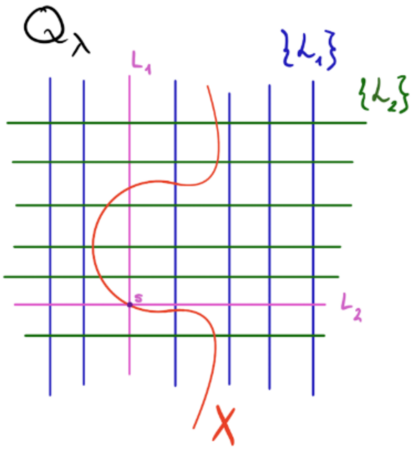


Figure 4.5

is an isomorphism. Write

$$\pi^{-1}(\widetilde{C}_s) = D_1 \cup D_2,$$

where

$$D_2 := \overline{\pi^{-1}(\widetilde{C}_s) \setminus D_1}.$$

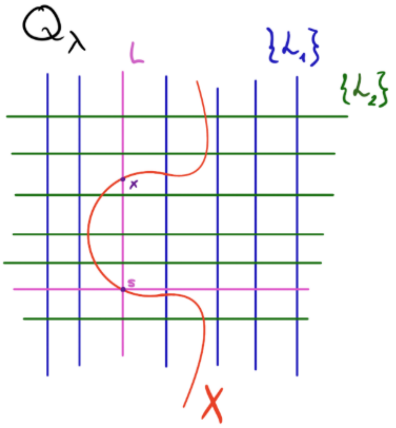


Figure 4.6: In $\pi^{-1}(\widetilde{C}_s)$, the triple $(\lambda, L, s) \in D_1$ and the triple $(\lambda, L, x) \in D_2$.

For each point $x \in X \setminus \{s\}$, there is a unique line through x and s, and this line lies in a unique grid. Indeed, if L were contained in two quadrics of the pencil, it would lie in all quadrics, and hence on X, which is impossible. Therefore, there is a unique triple $(\lambda, L, x) \in \mathcal{I}$ mapping to x, showing that

$$\psi|_{D_2} : D_2 \to X$$

is bijective. Since D_2 is projective, the restriction $\psi|_{D_2}$ is proper. The map $\psi|_{D_2}$ is bijective, and therefore birational. The generic fibre X is smooth, hence normal. So $\psi|_{D_2}$

is a proper birational morphism onto a normal variety thus, by 1.2.9, an isomorphism. Similarly, the restriction

$$\pi|_{D_2} : D_2 \longrightarrow \widetilde{C}_s$$

is bijective by construction. By the same argument, $\pi|_{D_2}$ is an isomorphism.

Composing the inverse of $\psi|_{D_2}$ with $\pi|_{D_2}$ gives the isomorphism

$$\Psi = \pi|_{D_2} \circ (\psi|_{D_2})^{-1} : X \longrightarrow \widetilde{C}_s.$$

Moreover, by [Rei72, Theorem 4.1], the curve $\widetilde{C}_s$ is isomorphic to the double cover C of $\mathbb{P}^1$ under the map $p : \mathrm{Gen}(\Phi) \to C$ from (4.3). Combining this with the previous result gives an isomorphism

$$\phi := p \circ \Psi : X \longrightarrow C. \qquad \square$$

For a fixed $\bar{K}$-rational point $s \in X(\bar{K})$, the isomorphism ϕ sends a point $x \in X$ to the unique point in C determined by the pair $(\lambda(x), L_{xs}) \in \widetilde{C}_s$, where L_{xs} is the unique line through x and s, and $\lambda(x)$ is the unique grid containing $L_{xs} \subset Q_{\lambda(x)}$. This construction is analogous to the map defined in Proposition 2.2.9.

Proposition 4.3.5. There is a natural identification $\mathrm{Jac}(X) \cong C$ over $\bar{K}$.

Proof. Fix a point $s \in X(\bar{K})$. By 4.3.4, there is an isomorphism

$$\phi\colon X \longrightarrow C$$

sending a point $x \in X$ to the unique point of C determined by the pair

$$(\lambda(x), L_{xs}) \in \widetilde{C}_s,$$

where L_{xs} is the unique line through x and s, and $\lambda(x)$ is the unique parameter such that $L_{xs} \subset Q_{\lambda(x)}$. On the other hand, by Proposition 2.2.9, the choice of s also determines an isomorphism

$$\phi_s\colon X \longrightarrow \mathrm{Jac}(X), \qquad x \longmapsto [x - s].$$

Under this map, a point x is sent to the degree-zero divisor class represented by the divisor $x - s$, which corresponds to the line L_{xs}. Therefore, both ϕ and ϕ_s identify a point $x \in X(\bar{K})$ with the line joining x and s. Thus, we have a canonical identification of C with $\mathrm{Jac}(X)$. $\square$

Remark 4.3.6. Over K, the curve C is a torsor under $J = \mathrm{Jac}(X)$, and its class satisfies

$$[C] = 2[X], \qquad 2[C] = 0 \quad \text{in } H^1(K, J).$$

If X has a quadratic point, then $\mathrm{index}(X)$ divides 2, hence $2[X] = 0$. It follows that $[C] = 0$, so $C \cong J$ over K, and in particular $C(K) \neq \varnothing$.

4.3.4 Application to the Fermat quartic

We now apply the general theory of pencils of quadrics to the Fermat quartic surface. Recall that the Fermat quartic can be realized as the intersection of the two bidegree $(1,2)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$:

$$\begin{aligned} Q_1\colon &\quad x_0^2 + x_2^2 + 2tx_1x_3 = 0 \\ Q_2\colon &\quad x_1^2 + x_3^2 + 2tx_0x_2 = 0 \end{aligned} \tag{4.13}$$

Let $X = Q_1 \cap Q_2$. The projection to $\mathbb{P}^1$ gives an elliptic fibration on X as in Proposition 4.3.1.

$$q_1\colon X \to \mathbb{P}^1.$$

The generic fibre $X_\eta = (Q_1 \cap Q_2)_\eta = Q_{1,\eta} \cap Q_{2,\eta}$ is the intersection of two quadrics in $\mathbb{P}^3$ over $\mathbb{C}(t)$. In particular, we apply the theory of Section 4.3.3 to the genus one curve X_η letting $K = \mathbb{C}(t)$. We can consider the pencil of quadrics through X_η given by

$$\Phi = \{Q_\lambda = Q_{1,\eta} - \lambda Q_{2,\eta} \mid \lambda \in \mathbb{P}^1\}.$$

For each $\lambda \in \mathbb{P}^1$, the symmetric matrix associated to Q_λ is

$$M_\lambda = \begin{pmatrix} 1 & 0 & -t\lambda & 0 \\ 0 & -\lambda & 0 & t \\ -t\lambda & 0 & 1 & 0 \\ 0 & t & 0 & -\lambda \end{pmatrix}$$

with determinant

$$\det(M_\lambda) = -t^2\lambda^4 + (t^4+1)\lambda^2 - t^2 = -t^2(\lambda^2 - t^2)(\lambda^2 - t^{-2}),$$

Solving $\det(M_\lambda) = 0$ for λ gives four distinct values corresponding to the singular quadrics in the pencil:

$$\lambda_1 = t, \qquad \lambda_2 = -t, \qquad \lambda_3 = t^{-1}, \qquad \lambda_4 = -t^{-1}.$$

These four points define a smooth projective curve C as the double cover of $\mathbb{P}^1$ branched at the singular members of the pencil locally given by:

$$\begin{aligned} C : \tau^2 &= -t^2(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \\ &= -t^2\lambda^4 + (t^4+1)\lambda^2 - t^2. \end{aligned}$$

The intersection $X_\eta = Q_{1,\eta} \cap Q_{2,\eta}$ has the quadratic point

$$[x_0 : x_1 : x_2 : x_3] = [0 : 1 : \sqrt{-2ti} : i] \text{ in } \mathbb{C}(\sqrt{t}),$$

thus, C is isomorphic to $\mathrm{Jac}(X_\eta)$ over $\mathbb{C}(t)$. We can then compute the Jacobian fibration

$$q_{1J} : \mathrm{Jac}(X) \to \mathbb{P}^1,$$

where the generic fibre is C. Computing the Weierstrass form of C allows us to classify the singular fibres of the fibration q_1 via its Jacobian fibration.

Proposition 4.3.7. The Weierstrass form of C is

$$y^2 = x^3 - \frac{1}{3}\,(t^8 + 14t^4 + 1)\,x + \frac{1}{54}\,(t^{12} - 33t^8 - 33t^4 + 1).$$

Proof. We follow the method in [Cas91, Chapter 8], to transform C into short Weierstrass form.

First, let $G(\lambda) := it\lambda^2 + \frac{t^4+1}{2it}$ and $H(\lambda) := -t^2 + \frac{(t^4+1)^2}{4t^2}$. We can write the right hand side as

$$-t^2\lambda^4 + (t^4+1)\lambda^2 - t^2 = G(\lambda)^2 + H(\lambda)$$

The equation of the curve is now

$$(\tau + G(\lambda))(\tau - G(\lambda)) = H(\lambda)$$

Put

$$\tau + G(\lambda) = T \quad \text{and} \quad \tau - G(\lambda) = \frac{H(\lambda)}{T}$$

so then

$$2G(\lambda) = T - \frac{H(\lambda)}{T}.$$

Multiply both sides by T^2 and set $T\lambda = S$

$$S^2 = \frac{T^3}{2it} + \frac{T^2}{2t^2}(t^4+1) - \frac{T}{8it^3}(t^4-1)^2.$$

After applying the map $T \mapsto \frac{x}{2it}$ and $S \mapsto \frac{y}{2t^2}$ we get

$$y^2 = \frac{1}{4}x^3 - \frac{1}{2}(t^4+1)x^2 + \frac{1}{4}(t^4-1)^2 x.$$

which in short Weierstrass form is

$$y^2 = x^3 - \frac{1}{3}(t^8 + 14t^4 + 1)x + \frac{1}{54}(t^{12} - 33t^8 - 33t^4 + 1). \tag{4.14}$$

□

Now that we know the Weierstrass form we can study the singular fibres of the fibration. The discriminant of the Weierstrass form of C (4.14) is

$$\Delta = 256(t-1)^4 t^4 (t+1)^4 (t^2+1)^4.$$

The discriminant vanishes at $t = 0, \infty, \pm 1, \pm i$. Thus, by studying the orders of the zeros of the discriminant and coefficients of the short Weierstrass form at these singular values, as in Table 3.1, we find six I_4 fibres at $t = 0, \infty, \pm 1, \pm i$. Each I_4 fibre is reducible, consisting of four rational curves arranged in a cycle. Finally, the Euler characteristic is consistent with that of a K3 surface: each fibre of type I_4 has Euler number $e(I_4) = 4$, and therefore

$$\sum e(X_t) = 6 \cdot 4 = 24,$$

as expected for an elliptic K3 surface.

4.3.5 Elliptic fibration without a section

Recall that for a K3 surface X, the second cohomology $H^2(X,\mathbb{Z})$ with the cup product is an even unimodular lattice of signature $(3,19)$. The Néron–Severi lattice $\mathrm{NS}(X) \subset H^2(X,\mathbb{Z})$ and the transcendental lattice $T_X = \mathrm{NS}(X)^{\perp} \subset H^2(X,\mathbb{Z})$ are primitive sublattices. Up to sign, the determinants of these lattices satisfy

$$|\det(\mathrm{NS}(X))| = |\det(T_X)|,$$

see [Huy16, Proposition 14.0.2] for details. Recall from Theorem 3.5.3 that for an elliptic K3 surface $f : X \to \mathbb{P}^1$ of multisection index l, the discriminants of $\mathrm{NS}(X)$ and $\mathrm{NS}(\mathrm{Jac}(X))$ satisfy the relation

$$\det(\mathrm{NS}(X)) = l^2 \det(\mathrm{NS}(\mathrm{Jac}(X))).$$

Let X be Fermat quartic as the intersection of the two bidegree $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ with the elliptic fibration

$$q_1 : X \to \mathbb{P}^1$$

with six I_4 singular fibres. In [SZ01, Table 2], elliptic K3 surfaces of maximal Picard number admitting a section are classified, and their transcendental lattices are computed. Among these, the unique surface with reducible fibre configuration A_3^6 (i.e. six singular fibres of type I_4) has transcendental lattice

$$T_{\mathrm{Jac}(X)} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

On the other hand, the Fermat quartic surface has transcendental lattice [Huy16, Section 3.2.6]

$$T_X = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}.$$

Since $|\det(\mathrm{NS}(X))| = |\det(T_X)|$ and $|\det(\mathrm{NS}(\mathrm{Jac}(X)))| = |\det(T_{\mathrm{Jac}(X)})|$, applying Proposition 3.5.3 we find that the multisection index of q_1 is $l = 2$. Therefore, the elliptic fibration

$$q_1 : X \to \mathbb{P}^1$$

on the Fermat quartic has multisection index 2, and hence does not admit a section.

Finally, we note a result of Keum [Keu00, Corollary 2.2 and Example] which states that any elliptic fibration on the Fermat quartic surface has multisection index either 1 or 2. In this thesis, we have explicitly exhibited examples realizing both possibilities: a fibration with a section obtained by projecting from a line and a fibration with a bisection but no section obtained from the intersection of two quadrics. In both cases, we classified the singular fibres by computing the Weierstrass equation of the generic fibre.

Appendix A

Code computations

A.1 Period-index computations code (Section 2.5)

The following code defines classes for an elliptic curve in Weierstrass form, points on the curve, and the point at infinity in Python. The implementation is taken from publicly available code [Kun23], with minor modifications and omissions.[1]

```

from sympy import simplify, rootof, Symbol


class EllipticCurve(object):
    def _init_(self, a, b):
        # assume we're already in the Weierstrass form
        self.a = a
        self.b = b
        self.discriminant = simplify(-16 * (4 * a*3 + 27 * b*2))

    def _str_(self):
        return "y^2 = x^3 + %sx + %s" % (self.a, self.b)

    def _repr_(self):
        return str(self)


class Point(object):
    def _init_(self, curve, x, y):
        self.curve = curve
        self.x = x
        self.y = y

    def _str_(self):
        return "(%r, %r)" % (self.x, self.y)

    def _repr_(self):
        return str(self)

```

---

[1] https://github.com/j2kun/elliptic-curves-rationals

```
    def _neg_(self):
        return Point(self.curve, self.x, -self.y)

    def _add_(self, Q):
        if self.curve != Q.curve:
            raise Exception("Can’t add points on different curves!")
        if isinstance(Q, O):
            return self

        x_1, y_1, x_2, y_2 = self.x, self.y, Q.x, Q.y

        if (x_1, y_1) == (x_2, y_2):
            if y_1 == 0:
                return O(self.curve)

            # slope of the tangent line
            m = (3 * x_1 * x_1 + self.curve.a) / (2 * y_1)
        else:
            if x_1 == x_2:
                return O(self.curve)

            # slope of the secant line
            m = (y_2 - y_1) / (x_2 - x_1)

        x_3 = m * m - x_2 - x_1
        y_3 = m * (x_3 - x_1) + y_1

        return Point(self.curve, x_3, -y_3)

    def _sub_(self, Q):
        return self + -Q

    def _mul_(self, n):
        if not isinstance(n, int):
            raise Exception("Scaling with not an int")

        if n < 0:
            return -self * -n

        if n == 0:
            return O(self.curve)

        Q = self
        R = self if n & 1 == 1 else O(self.curve)

        i = 2
        while i <= n:
            Q += Q
            if n & i == i:
                R += Q
```

```
            i = i << 1
        return R

    def _rmul_(self, n):
        return self * n

class O(Point):
    def _init_(self, curve):
        self.curve = curve
        self.x = 0
        self.y = 1

    def _neg_(self):
        return self

    def _str_(self):
        return "O"

    def _add_(self, Q):
        return Q

    def _mul_(self, n):
        if not isinstance(n, int):
            raise Exception("Scaling with not an int")
        else:
            return self
```

In the next code, we give the corresponding computations needed for Section 2.5 to verify that, for each σ, the point P_σ acts as a translation on the elliptic curve E. The following Python code checks the relation $\varphi_\sigma(P) = P + P_\sigma$ for a generic point P.

```
# %%
from elliptic import EllipticCurve, Point, O
from IPython.display import display, Latex
import sympy as sp

# %%
A = sp.symbols('A')
a, b = sp.symbols('a b')

# %%
# Define the elliptic curve y^2 = x^3 - 432*A^2
E = EllipticCurve(0, -432*A**2)

# %%
# Define rho as a primitive 3rd root of unity
x = sp.symbols('x')
rho = sp.rootof(x**3 - 1, 1)

# %%
# Define phi_sigma
def phi(P, i=0, j=0):
```

```
    # x coordinate
    xnew = rho*(3-i) * rho*(3-j) * P.x
    # y coordinate
    ynew = (18 * A * (rho*j - rho*i) + (rho*j + rho*i) * P.y / 2)
    # z coordinate
    znew = ((rho*i + rho*j)/2 + (rho*j - rho*i)*P.y/(72*A))
    # return affine coordinates
    Q = Point(E, xnew/znew, ynew/znew)
    return Q

# %%
# Define the point P_sigma in affine coordinates
def P_sigma(i,j):
    if i != j:
        return Point(E, 0, 36*A*(rho*i + rho*j)/(rho*j - rho*i))
    if i == j:
        return O(E)

# %%
# Define a general point Q on E
X = sp.symbols('X')
Y = sp.sqrt(X**3 - 432*A**2)
Q = Point(E, X, Y)

# %%
# Check for all i,j that phi_sigma(Q) == Q + P_sigma
for i in range(0, 3):
    for j in range(0, 3):
        agree_on_x = False
        agree_on_y = False

        transl = Q + P_sigma(i,j)

        if sp.simplify(phi(Q,i,j).x - transl.x) == 0:
            agree_on_x = True
        if sp.simplify(phi(Q,i,j).y - transl.y) == 0:
            agree_on_y = True

        if agree_on_x and agree_on_y:
            print(f'For (i,j)=({i},{j})', 'True')
```

Output.

```
For (i,j)=(0,0) True
For (i,j)=(0,1) True
For (i,j)=(0,2) True
For (i,j)=(1,0) True
For (i,j)=(1,1) True
For (i,j)=(1,2) True
For (i,j)=(2,0) True
For (i,j)=(2,1) True
For (i,j)=(2,2) True
```

The code verifies that the x-coordinate and y-coordinate of P and $P + P_\sigma$ agree for every σ, where σ is determined by the pair (i, j).

The following code calculates the multiples of P_σ to find the order of C.

```

# %%
# For prettier output
rho = sp.Symbol(r'\rho')

def P_sigma(i,j):
    if i != j:
        return Point(E, 0, 36*A*(rho*i + rho*j)/(rho*j - rho*i))
    if i == j:
        return O(E)

# %%
for i in range(0,3):
    for j in range(0,3):
        print(f'For (i,j)=({i},{j}):')
        if P_sigma(i,j) == O(E):
            print('P_sigma=', P_sigma(i,j))
        else:
            print('1*P_sigma= ', 1*P_sigma(i,j))
            print('2*P_sigma= ', 2*P_sigma(i,j))
            print('3*P_sigma= ', 3*P_sigma(i,j))

        print()
```

Output. The points P_σ and their multiples P_σ, $2P_\sigma$, $3P_\sigma$ for all pairs (i, j) are:

$(i, j) = (0, 0)$: $P_\sigma = O$

$(i, j) = (0, 1)$: $P_\sigma = (0, 36A(\rho + 1)/(\rho - 1))$, $2P_\sigma = (0, -36A(\rho + 1)/(\rho - 1))$, $3P_\sigma = O$

$(i, j) = (0, 2)$: $P_\sigma = (0, 36A(\rho^2 + 1)/(\rho^2 - 1))$, $2P_\sigma = (0, -36A(\rho^2 + 1)/(\rho^2 - 1))$, $3P_\sigma = O$

$(i, j) = (1, 0)$: $P_\sigma = (0, 36A(\rho + 1)/(1 - \rho))$, $2P_\sigma = (0, -36A(\rho + 1)/(1 - \rho))$, $3P_\sigma = O$

$(i, j) = (1, 1)$: $P_\sigma = O$

$(i, j) = (1, 2)$: $P_\sigma = (0, 36A(\rho^2 + \rho)/(\rho^2 - \rho))$, $2P_\sigma = (0, -36A(\rho^2 + \rho)/(\rho^2 - \rho))$, $3P_\sigma = O$

$(i, j) = (2, 0)$: $P_\sigma = (0, 36A(\rho^2 + 1)/(1 - \rho^2))$, $2P_\sigma = (0, -36A(\rho^2 + 1)/(1 - \rho^2))$, $3P_\sigma = O$

$(i, j) = (2, 1)$: $P_\sigma = (0, 36A(\rho^2 + \rho)/(-\rho^2 + \rho))$, $2P_\sigma = (0, -36A(\rho^2 + \rho)/(-\rho^2 + \rho))$, $3P_\sigma = O$

$(i, j) = (2, 2)$: $P_\sigma = O$

A.2 Line fibration (Section 4.2.1)

The Magma code below contains all computations necessary to determine the Weierstrass form of the generic fibre of the line fibration in Section 4.2.1. Given a genus one curve X_η and a distinguished point P_η, the code constructs the elliptic curve defined by this curve and outputs its Weierstrass form.

```
-- Step 1: Define Number Field L = Q(z) where z^4 + 1 = 0
```

```
PR<xx> := PolynomialRing(Rationals());
L<z> := NumberField(xx^4 + 1);

-- Step 2: Define Function Field K = L(t)
K<t> := FunctionField(L);

-- Step 3: Polynomial ring in projective variables over K
R<x_0,x_3,x_2> := PolynomialRing(K, 3);

-- Step 4: Define the cubic polynomial X_eta (homogeneous degree 3)
numerator := z^3 * x_3^3 + z^2 * x_3^2 * x_0 + z * x_3 * x_0^2 + x_0^3;
denominator := t;

terms :=
    - z^3 * t^3 * x_3^3
    - 4 * z^3 * t^2 * x_3^2 * x_2
    - 6 * z^3 * t * x_3 * x_2^2
    - 4 * z^3 * x_2^3
    + 3 * z^2 * t^3 * x_3^2 * x_0
    + 8 * z^2 * t^2 * x_3 * x_0 * x_2
    + 6 * z^2 * t * x_0 * x_2^2
    - 3 * z * t^3 * x_3 * x_0^2
    - 4 * z * t^2 * x_0^2 * x_2
    + t^3 * x_0^3;

X_eta := numerator/denominator + terms;

-- Step 5: Define projective plane over K
P2<x_0,x_3,x_2> := ProjectiveSpace(K, 2);

-- Step 6: Define curve C by the homogeneous polynomial X_eta
C := Curve(P2, X_eta);

-- Step 7: Define point P = (-z : 1 : -t)
P_eta := P2![-z, 1, -t];

-- Step 8: Define elliptic curve and compute Weierstrass model
E, phi := EllipticCurve(C, P_eta);
J := Jacobian(C);

Eshort := WeierstrassModel(J);
Eshort;
```

Output. Elliptic Curve defined by

$$y^2 = x^3 - 62208x + (2985984t^8 + 2985984)/t^4$$

over Univariate rational function field over L.

After the change of variables $x = 2^4 3^2\, x$, $y = 2^6 3^3\, y$ and division by $2^{12} 3^6$, this yields the equivalent Weierstrass equation

$$y^2 = x^3 - 3x + t^4 + \frac{1}{t^4}.$$

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