Quasi-periodic orbits in the AB-system

Hagedoorn, Niels

Quasi-periodic orbits in the AB-system

Bachelor thesis
November 3, 2025

Niels Hagedoorn
supervisors: Dock Staal & Arjen Doelman

Universiteit Leiden
The Netherlands

Leiden University
Mathematical Institute

Abstract

We will show the existence of a special type of orbits, that are non-periodic and are contained in a torus, called quasi-periodic, in the AB-system, which approximates the behaviour between a Turing bifurcation, where perturbations suddenly stabilise, and a saddle-node bifurcation. The standard way to approximate the behaviour around a Turing bifurcation, is the Ginzburg-Landau approximation, which is already known to contain these orbits [1]. Hence, our goal arises since the AB-system is derived from the Ginzburg-Landau approximation.

1 Introduction

We begin our study with the derivation of the AB-system, presented in Section 2. The derivation starts from the one-dimensional partial differential equation

$$\partial_t U = \mu U + 2U^2 - U^3 + \nu \partial_x^2 U + 2\partial_x^4 U + \partial_x^6 U + \eta(\partial_x^2 U)^2,$$

which is the simplest form of a system containing both a Turing bifurcation and a fold bifurcation. At a Turing bifurcation, small perturbations around a homogenous solutions become stable, rather than decaying back to the homogeneous solution. This gives rise to Turing pattern formation, the emergence of spatially periodic patterns from an initially homogeneous state in a reaction-diffusion system [4]. Assuming that solutions are spatially homogeneous, we find that, for $\mu < -1$, the only stable fixed point is $U = 0$. Interpreting this system as an ecological setting, this would correspond to a situation where, for $\mu < -1$, all vegetation dies out. However, numerical simulations find sable non-zero solutions for $\mu < -1$ as we let go of the assumption of spatially homogeneity. This means in ecological context, that vegetation can still survive. This is illustrated in Figures 1 and 2.

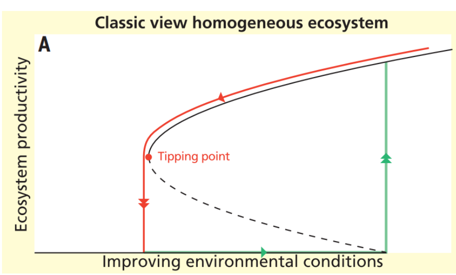


Figure 1: Sketch of the behaviour in the time homogeneous system.

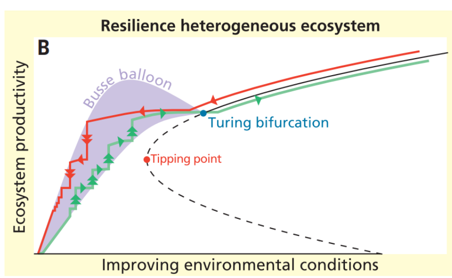


Figure 2: Sketch of the behaviour in the original system, from the Turing bifurcation.

This observation raises the question, why do these non-zero states persist? This motivates investigation into the dynamics between the Turing and fold bifurcations. The standard method for studying the behaviour around a Turing bifurcation, is the Ginzburg-Landau approximation. However, this method is not sufficient to capture the dynamics as we approach the fold bifurcation. Therefore, the AB-system is derived as an adaptation of the Ginzburg-Landau approximation. The existence of periodic and quasi-periodic orbits in the Ginzburg-Landau system has been covered (see [1]), and we will revisit this in Section 3. We shift our focus to determining if these types of solutions occur in the AB-system as well. Periodic orbits in the AB-system have been discussed in [3, §4]. We will revisit this briefly in Section 4. Lastly, we will demonstrate the existence of quasi-periodic orbits in the AB-system in Section 5, using fast-slow analysis. During the course of this analysis, one possible approach was by looking at Hamiltonian systems under certain conditions. The property, the existence of a Hamiltonian function, is discussed in the Appendix.

2 Derivation of the AB-system

Consider the following system,

$$\partial_t U = \mu U + 2U^2 - U^3 + \nu \partial_x^2 U + 2\partial_x^4 U + \partial_x^6 U + \eta(\partial_x^2 U)^2, \tag{2.1}$$

where $\mu, \eta \in \mathbb{R}$ and $\nu > 0$. This partial differential equation is the simplest form of a system that contains both a Turing bifurcation [4], and a fold bifurcation. Our goal is to uncover certain dynamics between these bifurcations.

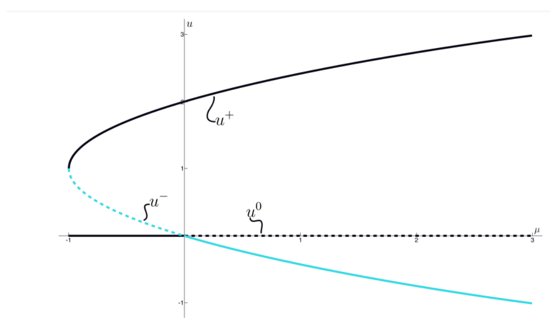


Figure 3: The fold bifurcation at $\mu = -1$, for the ODE case.

Before we proceed, we discuss the fold bifurcation. Suppose the system is spatially homogeneous, meaning $\partial_x U = 0$. This implies that $\nu \partial_x^2 U + 2\partial_x^4 U + \partial_x^6 U + \eta(\partial_x^2 U)^2 = 0$. What remains is the following equation

$$\partial_t U = \mu U + 2U^2 - U^3.$$

The stationary solutions of this equation are $u^0 = 0$, and $u^\pm(\mu) = 1 \pm \sqrt{1+\mu}$. We can conclude that the solutions $u^\pm$ emerge from u^0 through a saddle-node bifurcation as μ crosses through -1. To examine the stability of u^+ with respect to a small perturbation, we consider

$$U = u^+ + \left(\bar{u}e^{ikx+\omega(k;\mu,\nu)t} + c.c.\right).$$

where $\bar{u}$ is the constant factor for the amplitude of the perturbation and k is the wavenumber, the number of waves over a certain unit of length. Furthermore, ω, as a function over k, μ, and ν, determines the amplification factor as time passes. By substituting this perturbated solution into 2.1, and linearising, we can determine ω,

$$\omega(k;\mu,\nu) = \mu + 4u^+ - 3(u^+)^2 - \nu k^2 + 2k^4 - k^6.$$

If we know that $\omega < 0$ for all $k \in \mathbb{R}$ for given μ, ν, then the perturbation waves decrease and converges to u^+, so u^+ is stable. If a Turing bifurcation happens at μ^t, then there is a certain wave number such that u^+ destabilises, which is the critical wave number k^c. For this, we require that $\omega(k^c;\mu^t,\nu) = 0$, and $\frac{\partial}{\partial k}\omega(k^c;\mu^t,\nu) = 0$, which gives us that k^c is the first wave number where the perturbations are stable. This is demonstrated in Figure 4.

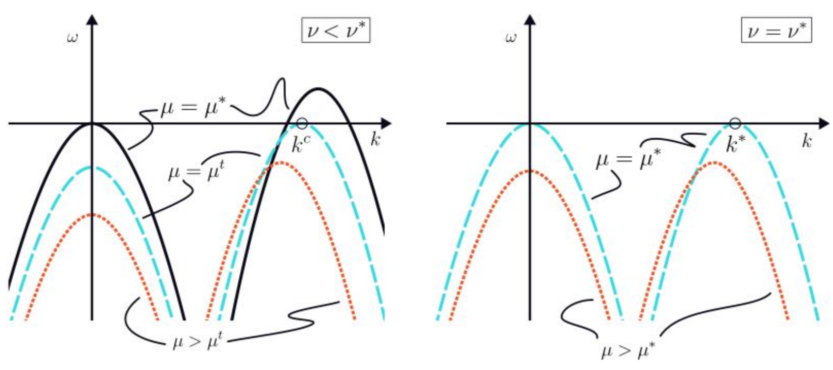


Figure 4: Graph of $\omega(k;,\mu,\nu)$, where (μ^t,ν^t) is the Turing bifurcation point, and (μ^*,ν^*) is the point where the Turing and fold bifurcation coincide. In that case, the critical wave number is denoted as k^*.

We want to examine the dynamics as the Turing bifurcation approaches the fold bifurcation. First, we introduce u^t, the value of u^+ for μ^t, and for $\mu = \mu^t - r\varepsilon^2$, $r \in \mathbb{R}$, $\nu = 1-\delta$, $0 < \varepsilon \ll \delta \ll 1$, we impose the standard Ginzburg-Landau approximation

$$U_{GL}(x,t) = u^t + \left(e^{ik^c x}A(\varepsilon x, \varepsilon^2 t) + c.c.\right), \tag{2.2}$$

where A is the unknown amplitude that varies slowly over $\xi = \varepsilon x$ and $\tau = \varepsilon^2 t$. From substitution of this approximation into 2.1, irrelevant terms emerge.

We utilise the standard Ginzburg-Landau ansatz and determine the unknown functions X_{ij}, to set these terms to zero. Therefore, the nonlinear evolution can be captured by the standard Ginzburg-Landau ansatz:

$$\begin{aligned} & E^0[\varepsilon^2 X_{02} + \varepsilon^3 X_{03} + \mathcal{O}(\varepsilon^4)] \\ U_{GL}(x,t) = u^t + & E[\varepsilon A + \varepsilon^2 X_{12} + \varepsilon^3 X_{13} + \mathcal{O}(\varepsilon^4)] + c.c. \\ & E^2[\varepsilon^2 X_{22} + \varepsilon^3 X_{23} + \mathcal{O}(\varepsilon^4)] + c.c. \\ & E^3[\varepsilon^3 X_{33} + \mathcal{O}(\varepsilon^4)] + c.c., \end{aligned}$$

where $E = e^{ik^c x}$, $X_{0j}(\xi,\tau) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $X_{ij}(\xi,\tau) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$, $i \geq 1$, are the unknown functions we need to determine. We can recognise $u^t + E\varepsilon A$ from 2.2. We observe the terms for each of the $\varepsilon_j E^i$ levels. By separating the terms per these levels, we can express these functions in terms of A. Substituting the uncovered expressions for the functions enables us to determine the unknown functions at the higher levels. In the end, we are left with the Ginzburg-Landau equation

$$A_\tau = -\frac{1}{2}\omega^t_{kk} A_{\xi\xi} - r\omega^t_\mu A + L|A|^2 A, \tag{2.3}$$

where $\omega^t_{kk} = -8 + \mathcal{O}(\delta)$, $\omega^t_\mu = \frac{-2+\mathcal{O}(\delta)}{\delta}$, and $L = \frac{4(1-\eta)}{\delta} + \mathcal{O}(1)$, which yields

$$A_\tau = (4 + \mathcal{O}(\delta))A_{\xi\xi} + \frac{r}{\delta}(2 + \mathcal{O}(\delta))A + \frac{1}{\delta}(4(1-\eta) + \mathcal{O}(\delta))|A|.$$

In Section 3, we will discuss the existence of periodic and quasi-periodic orbits in the general form of the Ginzburg-Landau equation.
This approximation method, however, is not suitable for our goal, to study the behaviour between the Turing bifurcation, and the fold bifurcation, since some small terms increase to the same order as the leading order terms as we approach the fold bifurcation, invalidating this method of approximation. Hence, the range of this approximation method is not great enough to accurately approach the behaviour between both bifurcation points [3]. To solve this problem, an adaptation to the approximation must be made, see [3]. Keeping $\nu = 1 - \delta$, and setting $\mu = \mu_t - r\delta^2$, $\xi = \sqrt{\delta}x$, $\tau = \delta t$, and introducing a second amplitude $B \in \mathbb{R}$ to absorb the effects of the fold bifurcation. We introduce the following adapted ansatz

$$\begin{aligned} U_{AB}(x,t) = 1 + & E^0[\delta B + \delta^{\frac{3}{2}} X_{02} + \delta^2 X_{03} + \delta^{\frac{5}{2}} X_{04} + \mathcal{O}(\delta^3)] \\ & +E[\delta A + \delta^{\frac{3}{2}} X_{12} + \delta^2 X_{13} + \delta^{\frac{5}{2}} X_{14} + \mathcal{O}(\delta^3)] + c.c. \\ & +E^2[\delta^2 X_{23} + \delta^{\frac{5}{2}} X_{24} + \mathcal{O}(\delta^3)] + c.c. \\ & + h.o.t.. \end{aligned}$$

By substitution, repeating the process of determining the unknown functions $X_{0j}(\xi,\tau) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $X_{ij}(\xi,\tau) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$, $i \geq 1$, in terms of A and B, by collecting the terms for each $\varepsilon^i E^j$ level, we find a better approximation.

After introducing some natural scalings, as is done in [3, §2.2.3], we find the canonical form of the AB-system

$$\begin{cases} A_\tau = A_{\xi\xi} + A - AB, \\ \alpha^{-1} B_\tau = d B_{\xi\xi} + 1 - R - B^2 + \beta |A|^2. \end{cases} \tag{2.4}$$

The range of this adapted approximation is indeed great enough to include both bifurcation points. This is demonstrated in Figure 5.

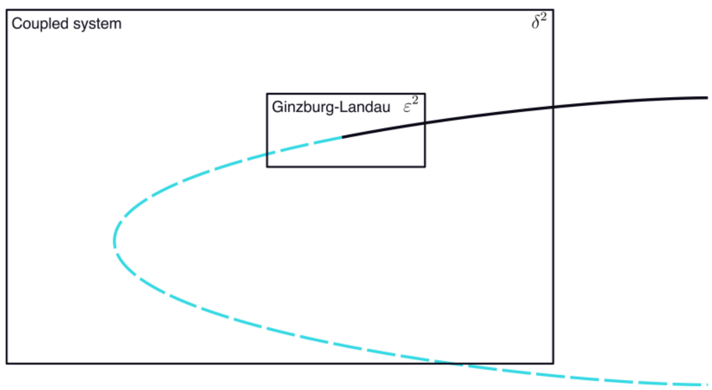


Figure 5: Demonstration of the range of the Ginzburg-Landau system and the AB-system around the Turing bifurcation.

3 Periodic and quasi-periodic solutions in the Ginzburg-Landau system

We now examine the existence of periodic and quasi-periodic solutions in the Ginzburg–Landau system. The Ginzburg–Landau equation serves as the modulation equation describing the nonlinear behaviour of a partial differential equation system near a Turing bifurcation. Later, we will rely on the existence of quasi-periodic orbits within this system to demonstrate the presence of similar orbits in the AB-system. Our analysis follows the approach outlined in [1]. We write down the most general form of the stationary Ginzburg-Landau system for $A(\xi, \tau) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$ as

$$A_\tau = \left(1 - |A|^2\right) A + A_{\xi\xi}. \tag{3.1}$$

This general form can be deduced from Equation 2.3, by setting scalings such that $-\frac{1}{2}\omega^t_{kk} = 1$, $-r\omega^t_\mu = 1$ and $L = -1$. To find periodic and quasi-periodic solutions, we substitute the polar decomposition

$$A(\xi, \tau) = \rho(\xi) e^{i\theta(\xi)},$$

into 3.1. This yields

$$0 = \left(\left(1-\rho^2\right)\rho + \rho_{\xi\xi} + 2i\rho_\xi\theta_\xi + i\rho\theta_{\xi\xi} - \rho\theta_\xi^2\right)e^{i\theta}.$$

Since $e^{i\theta(\xi)} \neq 0$ for all $\theta(\xi) \in \mathbb{R}$, it follows that

$$\begin{aligned} 0 &= \left(1-\rho^2\right)\rho + \rho_{\xi\xi} + 2i\rho_\xi\theta_\xi + i\rho\theta_{\xi\xi} - \rho\theta_\xi^2 = \\ &= \left(\left(1-\rho^2\right)\rho + \rho_{\xi\xi} - \rho\theta_\xi^2\right) + i\left(2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi}\right). \end{aligned}$$

By separating the real and imaginary parts of the equation, we get

$$\begin{cases} 0 = \rho - \rho^3 + \rho_{\xi\xi} - \rho\theta_\xi^2, \\ 0 = 2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi}. \end{cases} \tag{3.2}$$

Integrating the second equation gives us $\rho^2\theta_\xi = \Omega$, where Ω is constant. By splitting the first equation into two first-order differential equations, we obtain the system

$$\begin{cases} \dot{\rho} = V, \\ \dot{V} = \rho^3 - \rho + \frac{\Omega^2}{\rho^3}, \\ \dot{\Omega} = 2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi} = 0. \end{cases} \tag{3.3}$$

Since Ω is a constant, we consider

$$\begin{cases} \dot{\rho} = V, \\ \dot{V} = \rho^3 - \rho + \frac{\Omega^2}{\rho^3}. \end{cases}$$

This is a Hamiltonian system with Hamiltonian

$$H(\rho, V) = \frac{1}{2}V^2 - \frac{1}{4}\rho^4 + \frac{1}{2}\rho^2 + \frac{1}{2}\Omega^2\rho^{-2}.$$

We can verify that

$$\frac{d}{d\xi}H(\rho, V) = \frac{\partial}{\partial\rho}H \cdot \dot{\rho} + \frac{\partial}{\partial V}H \cdot \dot{V} = 0.$$

The function $P : \mathbb{R} \to \mathbb{R}$, defined as

$$P(\rho) = -\frac{1}{4}\rho^4 + \frac{1}{2}\rho^2 + \frac{1}{2}\Omega^2\rho^{-2},$$

describes the potential of the system. The periodic orbits in de Ginzburg-Landau system correspond to the solutions where ρ is constant, which is the case for $P_\rho(\rho^*) = 0$. For solutions periodic in ρ to exist, it is necessary that $P_\rho(\rho^*) = 0$ and $P_{\rho\rho}(\rho^*) < 0$ for a certain $\rho^* \in \mathbb{R}$. These ρ periodic solutions correspond to quasi-periodic solutions in the full Ginzburg-Landau system.
From these conditions, we deduce that

$$-(\rho^*)^6 + (\rho^*)^4 - \Omega^2 = 0, \text{ and } -3(\rho^*)^6 + (\rho^*)^4 + 3\Omega^2 < 0,$$

which gives us that

$$-3(\rho^*)^6 + (\rho^*)^4 + 3\Omega^2 < -(\rho^*)^6 + (\rho^*)^4 - \Omega^2,$$

which simplifies to $\Omega^2 < \frac{1}{2}(\rho^*)^6$. From the condition that $P(\rho^*) = 0$, and the previous inequality, we further derive that $(\rho^*)^2 < \frac{2}{3}$, which gives us that $\Omega^2 < \frac{1}{2}(\rho^*)^6 = \frac{4}{27}$. Thus, ρ periodic orbits, and consequently quasi-periodic orbits, exist for $|\Omega| < \sqrt{\frac{4}{27}}$. For $0 < |\Omega| < \sqrt{\frac{4}{27}}$, the corresponding phase portrait is shown in Figure 6.

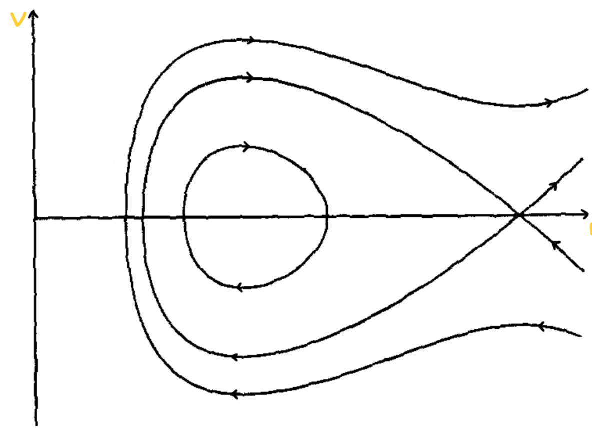


Figure 6: Phase plane of system 3.3 for $0 < |\Omega| < \sqrt{\frac{4}{27}}$.

The solutions in the $\Omega = 0$ plane correspond to periodic solutions of 3.1 where $\rho - \rho^3 + \rho_{\xi\xi} = 0$, and θ is constant. Figure 7 shows the phase plane of system 3.3 for $\Omega = 0$, and for $0 < |\Omega| \ll 1$.

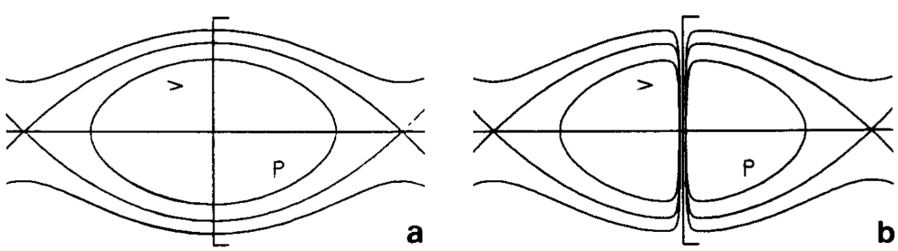


Figure 7: Phase portraits of system 3.3. (a) The case $\Omega_0 = 0$. (b) The case $0 < |\Omega_0| \ll 1$.

4 Periodic orbits in the AB-system

In Section 2, we covered the derivation of the AB-system:

$$\begin{cases} A_\tau = A_{\xi\xi} + A - AB, \\ \alpha^{-1} B_\tau = dB_{\xi\xi} + 1 - R - B^2 + \beta |A|^2, \end{cases} \tag{4.1}$$

where $\alpha, d > 0$, $\beta \in \mathbb{R}$ and $R \in \mathbb{R}$ is the bifurcation parameter. Now we proceed with the analysis of this system, beginning with the existence and stability of homogeneous, and periodic solutions. A numerically observed (seemingly) periodic solution is shown in Figure 11. The solutions we seek are of the form

$$(A(\xi,\tau), B(\xi,\tau)) = (\bar{A}e^{iK\xi}, \bar{B}), \tag{4.2}$$

where $\bar{A}(K,R), \bar{B}(K,R) \in \mathbb{R}$, and $\bar{A} \geq 0$. Here, the bar above A and B denote that this is a constant, instead of the complex conjugate of A and B. Substituting 4.2 into 4.1 yields

$$\begin{cases} 0 = \bar{A}\left(-K^2 + 1 - \bar{B}\right), \\ 0 = 1 - R - \bar{B}^2 + \beta |\bar{A}|^2. \end{cases}$$

This has two homogeneous solutions, where $\bar{A} = 0$ and $\bar{B} = \pm\sqrt{1-R}$. We will denote these as

$$(A_s(\xi,\tau), B_s^{\pm}(\xi,\tau)) = (\bar{A}_s(K,R)e^{iK\xi}, \bar{B}_s^{\pm}(K,R)),$$

where $\bar{A}_s(K,R) = 0$, and $\bar{B}_s^{\pm}(K,R) = \pm\sqrt{1-R}$. These two solutions are homogenous, since they are already constant over time, and since $\bar{A} = 0$, they are constant over the space as well.
We also find a periodic state

$$(A_p(\xi,\tau), B_p(\xi,\tau)) = (\bar{A}_p(K,R)e^{iK\xi}, \bar{B}_p(K,R)),$$

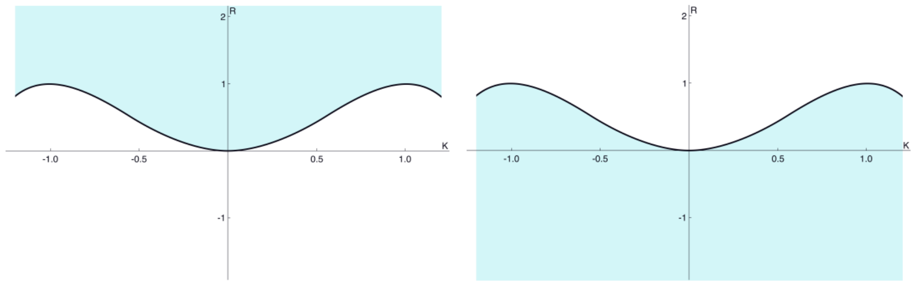
where

$$\bar{B}_p(K,R) = 1 - K^2, \text{ and } \bar{A}_p(K,R) = \sqrt{\frac{1}{\beta}\left((1-K^2)^2 + R - 1\right)}.$$

For this solution to exist, $\bar{A}_p$ must be real (and non-zero to ensure periodicity). Hence, we have the condition for the existence of periodic orbits, that

$$R \geq 1 - (1-K^2)^2 \text{ if } \beta > 0, \text{ and } R \leq 1 - (1-K^2)^2 \text{ if } \beta < 0.$$

The corresponding parameter regions satisfying this condition are shown in Figure 8.



(a) $\beta > 0$ (b) $\beta < 0$

Figure 8: Existence regions for periodic solutions.

4.1 Stability

We now introduce a small perturbation

$$A(\xi, \tau) = (\bar{A} + a(\xi, \tau))e^{iK\xi}, \text{ and } B(\xi, \tau) = \bar{B} + b(\xi, \tau),$$

and we analyse the behaviour of $a(\xi, \tau)$ and $b(\xi, \tau)$ in order to analyse the stability of the found solutions. Substituting these perturbations into 4.1 yields

$$\begin{cases} a_\tau e^{iK\xi} = e^{iK\xi} \left(-K^2(\bar{A} + a)iKa_\xi + a_{\xi\xi} + iKa_\xi + \bar{A} + a - (\bar{A} + a)(\bar{B} + b)\right), \\ \alpha^{-1} b_\tau = db_{\xi\xi} + 1 - R - (\bar{B} + b)^2 + \beta|\bar{A} + a|^2. \end{cases} \tag{4.3}$$

Dividing both sides of the first equation by $e^{iK\xi}$, and decomposing the squared expressions, reduces 4.3 to

$$\begin{cases} a_\tau = a_{\xi\xi} + a(1 - K^2) + 2iKa_\xi - (\bar{B}a + \bar{A}b) + (-K^2 + 1 - \bar{B})\bar{A} - ab, \\ b_\tau = d\alpha b_{\xi\xi} + \alpha(-2\bar{B}b + \beta\bar{A}(a + a^*)) + \alpha(1 - R - \bar{B}^2 + \beta\bar{A}^2 + 2aa^* - b^2). \end{cases}$$

Here, a^* denotes the complex conjugate. Recall that

$$0 = \bar{A}\left(-K^2 + 1 - \bar{B}\right), \text{ and } 0 = 1 - R - \bar{B}^2 + \beta|\bar{A}|^2.$$

This reduces the equations to

$$\begin{cases} a_\tau = a_{\xi\xi} + a(1 - K^2) + 2iKa_\xi - (\bar{B}a + \bar{A}b) - ab, \\ b_\tau = d\alpha b_{\xi\xi} + \alpha(-2\bar{B}b + \beta\bar{A}(a + a^*)) + \alpha(2aa^* - b^2). \end{cases}$$

Linearising gives us

$$\begin{cases} a_\tau = a_{\xi\xi} + a(1 - K^2) + 2iKa_\xi - (\bar{B}a + \bar{A}b), \\ b_\tau = d\alpha b_{\xi\xi} + \alpha(-2\bar{B}b + \beta\bar{A}(a + a^*)). \end{cases}$$

We now decompose $a = U + iV$, and write $b = W$, to derive the system for U, V. and W,

$$\begin{cases} U_\tau = U_{\xi\xi} + U(1-K^2) - 2KV_\xi - \bar{B}U - \bar{A}W, \\ V_\tau = V_{\xi\xi} + V(1-K^2) + 2KU_\xi - \bar{B}V, \\ W_\tau = d\alpha W_{\xi\xi} + 2\alpha(-\bar{B}W + \beta\bar{A}U). \end{cases}$$

Now set

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = e^{ik\xi+\lambda\tau} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which puts the system in the form

$$\begin{cases} \lambda u e^{ik\xi+\lambda\tau} = e^{ik\xi+\lambda\tau}\left(-k^2u + (1-K^2-\bar{B})u - 2iKkv - \bar{A}w\right), \\ \lambda v e^{ik\xi+\lambda\tau} = e^{ik\xi+\lambda\tau}\left(-k^2v + (1-K^2-\bar{B})v + 2iKku\right), \\ \lambda w e^{ik\xi+\lambda\tau} = e^{ik\xi+\lambda\tau}\left(-d\alpha k^2 w + 2\alpha(\beta\bar{A}u - \bar{B}w)\right). \end{cases} \tag{4.4}$$

Dividing $e^{ik\xi+\lambda\tau}$ from both sides of each equation, reduces 4.4 to the following system in matrix form

$$\lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1-K^2-\bar{B}-k^2 & -2ikK & -\bar{A} \\ 2iKk & 1-K^2-\bar{B}-k^2 & 0 \\ 2\alpha\beta\bar{A} & 0 & -2\alpha\bar{B}-d\alpha k^2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Now we can analyse the stability of the homogenic and periodic states.

4.1.1 Homogeneous solutions $(A_s, B_s^\pm)$

For $(\bar{A}_s(K,R), \bar{B}_s^\pm(K,R)) = (0, \pm\sqrt{1-R})$, we set $K = 0$, since $\bar{A} = 0$ implies that there is no rotation. The problem reduces to

$$\lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \mp \sqrt{1-R} - k^2 & 0 & 0 \\ 0 & 1 \mp \sqrt{1-R} - k^2 & 0 \\ 0 & 0 & \mp 2\alpha\sqrt{1-R} - d\alpha k^2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Thus, we observe the eigenvalues

$$\begin{aligned} \lambda_1^\pm(k) &= \lambda_2^\pm(k) = 1 \mp \sqrt{1-R} - k^2, \\ \lambda_3^\pm(k) &= \mp 2\alpha\sqrt{1-R} - d\alpha k^2. \end{aligned}$$

Note that $\operatorname{Re}\lambda_1^+(k) = \operatorname{Re}\lambda_2^+(k) < 0$ for all $k \in \mathbb{R}$ if $R < 0$, and $\operatorname{Re}\lambda_3^+(k) < 0$ for all $k \in \mathbb{R}$ and $R \in \mathbb{R}$. Thus, (A_s, B_s^+) is stable for $R < 0$. In contrast, for (A_s, B_s^-), there exist $k \in \mathbb{R}$ such that $\operatorname{Re}\lambda_1^-(k) = \operatorname{Re}\lambda_2^-(k) > 0$ for all $R \in \mathbb{R}$. Hence, (A_s, B_s^-) is unstable for all $R \in \mathbb{R}$.

4.1.2 Periodic solution (A_p, B_p)

For $(\bar{A}_p(K,R), \bar{B}_p(K,R)) = \left(\sqrt{\frac{1}{\beta}\left((1-K^2)^2 + R - 1\right)}, 1 - K^2\right)$, the eigenvalue problem becomes

$$\lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -k^2 & -2ikK & -\bar{A}_p \\ 2iKk & -k^2 & 0 \\ 2\alpha\beta\bar{A}_p & 0 & -2\alpha\bar{B}_p - d\alpha k^2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

and we denote the square matrix as $M(k)$. The corresponding characteristic polynomial $Q(k,\lambda)$ of this system is

$$\begin{aligned} Q(k,\lambda) = \det(\lambda \cdot I - M(k)) &= \begin{vmatrix} \lambda + k^2 & 2ikK & \bar{A}_p \\ -2iKk & \lambda + k^2 & 0 \\ -2\alpha\beta\bar{A}_p & 0 & \lambda + 2\alpha\bar{B}_p + d\alpha k^2 \end{vmatrix} = \\ &= \lambda^3 + \lambda^2\left(2k^2 + 2\alpha\bar{B}_p + d\alpha k\right) \\ &+ \lambda\left(k^4 + 2k^2(2\alpha\bar{B}_p + d\alpha k^2) - 4k^2K^2 + 2\alpha\beta\bar{A}_p^2\right) \\ &+ k^4(2\alpha\bar{B}_p + d\alpha k^2) - 4k^2K^2(2\alpha\bar{B}_p + d\alpha k^2) + 2\alpha\beta k^2\bar{A}_p^2. \end{aligned} \tag{4.5}$$

For $k = 0$, this yields

$$Q(0,\lambda) = \lambda^3 + 2\alpha\bar{B}_p\lambda^2 + 2\alpha\beta\bar{A}_p^2\lambda,$$

so that

$$\lambda_1(0) = 0 \text{ and } \lambda_\pm(0) = \alpha\left(-(1-K^2) \pm \sqrt{(1-K^2)^2 - 2\frac{\beta}{\alpha}\bar{A}_p^2}\right).$$

Since $\alpha > 0$ we can deduce that stability requires that $1 - K^2 > 0$. If this is the case, meaning that $|K| < 1$, we get

$$\operatorname{Re}\lambda_-(0) = \operatorname{Re}\alpha\left(-(1-K^2) - \sqrt{(1-K^2)^2 - 2\frac{\beta}{\alpha}\bar{A}_p^2}\right) < 0.$$

If $\beta > 0$, we observe that

$$\operatorname{Re}\sqrt{(1-K^2)^2 - 2\frac{\beta}{\alpha}\bar{A}_p^2} < \operatorname{Re}\sqrt{(1-K^2)^2},$$

which gives us that

$$\begin{aligned} \operatorname{Re}\lambda_+(0) = \operatorname{Re}\alpha\left(-(1-K^2) + \sqrt{(1-K^2)^2 - 2\frac{\beta}{\alpha}\bar{A}_p^2}\right) < \\ < \operatorname{Re}\alpha\left(-(1-K^2) + \sqrt{(1-K^2)^2}\right) = \operatorname{Re}\alpha\left(-(1-K^2) + (1-K^2)\right) = 0. \end{aligned}$$

Thus, we have conditions $\beta > 0$ and $|K| < 1$. Now assume that these conditions are satisfied. Setting $k^2 = \varepsilon$, where $0 < \varepsilon \ll 1$ such that $\lambda_\pm(k) = \lambda_\pm(\sqrt{\varepsilon}) < 0$. We know that $\lambda_1(0) = 0$, so the sideband stability depends on the sign of $\operatorname{Re}\lambda_1(k)$. We also set $\lambda_1 = \varepsilon\tilde{\lambda}$. This gives us

$$Q(\sqrt{\varepsilon}, \varepsilon\tilde{\lambda}) = \varepsilon\left(2\alpha\beta\bar{A}_p^2(\tilde{\lambda}+1) - 8K^2\alpha\bar{B}_p\right) + \mathcal{O}(\varepsilon^2).$$

If $Q(\sqrt{\varepsilon}, \varepsilon\tilde{\lambda}) = 0$, we observe that

$$\begin{aligned}\tilde{\lambda} &= \frac{8K^2\alpha\bar{B}_p}{2\alpha\beta\bar{A}_p^2} - 1 = \\ &= \frac{4K^2\left(1-K^2\right) - \beta\left(\frac{1}{\beta}((1-K^2)^2+R-1)\right)}{\beta\bar{A}_p^2} = \frac{6K^2-5K^4-R}{\beta\bar{A}_p^2}.\end{aligned}$$

We assumed that $\beta > 0$ and $|K| < 1$. Thus, we have that $\beta\bar{A}_p^2 > 0$. If we impose the condition that $R > 6K^2 - 5K^4$, then we have that $6K^2 - 5K^4 - R < 0$, and thus, $\tilde{\lambda} < 0$. Hence, (A_p, B_p) is sideband stable if $\beta > 0$, $|K| < 1$ and $R > 6K^2 - 5K^4$. The periodic solution (A_p, B_p) destabilises via a Turing bifurcation if $\lambda(k^c) = \lambda'(k^c) = 0$, where $\lambda(k)$ is a solution of $Q(k, \lambda(k)) = 0$ for all $k \in \mathbb{R}$, and with $k^c > 0$. Since for all $k \in \mathbb{R}$, we defined $\lambda(k)$ such that $Q(k, \lambda(k)) = 0$, we know that

$$\frac{d}{dk}Q(k,\lambda(k)) = \frac{\partial}{\partial k}Q(k,\lambda(k)) + \frac{\partial}{\partial\lambda}Q(k,\lambda(k))\lambda'(k) \equiv 0.$$

If a Turing bifurcation takes place, we know that $Q(k^c, 0) = 0$ and

$$\frac{d}{dk}Q(k^c,0) = \frac{\partial}{\partial k}Q(k^c,0) + \frac{\partial}{\partial\lambda}Q(k^c,0)\lambda'(k^c) = \frac{\partial}{\partial k}Q(k^c,0) = 0.$$

From 4.5, it follows that

$$\begin{aligned}&Q(k^c,0) = \det(M(k^c)) = \\ &\quad = -\alpha(k^c)^2\left((k^c)^2(2\bar{B}_p + d(k^c)^2) - 4K^2(2\bar{B}_p + d(k^c)^2) + 2\beta\bar{A}_p^2\right) = 0,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\frac{\partial}{\partial k}Q(k^c,0) &= \frac{\partial}{\partial k}\det(M(k))|_{k=k^c} = \\ &= -2\alpha k^c\left((k^c)^2(2\bar{B}_p + d(k^c)^2) - 4K^2(2\bar{B}_p + d(k^c)^2) + 2\beta\bar{A}_p^2\right) \\ &\quad - \alpha(k^c)^2\left(2k^c(2\bar{B}_p + d(k^c)^2) + 2d(k^c)^3 - 8dK^2k^c\right) = 0.\end{aligned}$$

So, we observe that

$$\begin{aligned}&2Q(k^c,0) - k^c\frac{\partial}{\partial k}Q(k^c,0) = \\ &\qquad \alpha(k^c)^3\left(2k^c(2\bar{B}_p + d(k^c)^2) + 2d(k^c)^3 - 8dK^2k^c\right) = \\ &\qquad\qquad 4\alpha(k^c)^4\left(\bar{B}_p + d((k^c)^2 - 2K^2)\right) = 0.\end{aligned}$$

Since $k^c \neq 0$, we find that

$$k^c = -\frac{\bar{B}_p}{d} + 2K^2 = \frac{-(1-K^2)+2dK^2}{d} = \frac{(2d+1)K^2-1}{d}. \tag{4.7}$$

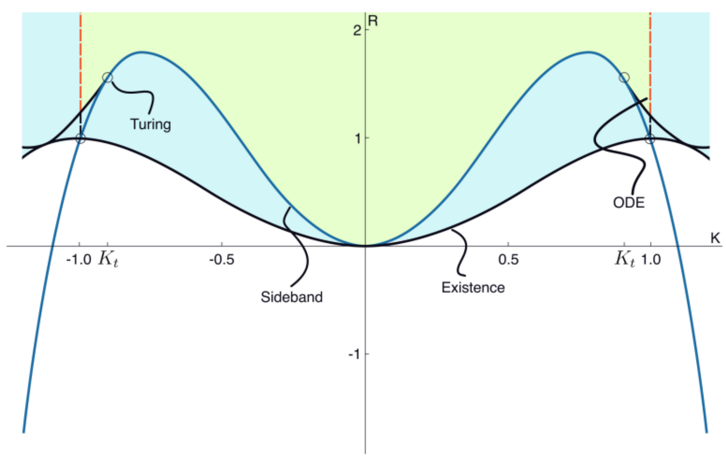
Since $d > 0$, we observe the condition $1 > K^2 > \frac{1}{2d+1} > 0$. Substituting 4.7 into 4.6, yields

$$(k^c)^2(2\bar{B}_p + d(k^c)^2) - 4K^2(2\bar{B}_p + d(k^c)^2) + 2\beta \bar{A}_p^2 =$$
$$-\frac{1}{d}\left(1+(2d-1)K^2\right)^2 + 2(1-K^2)^2 + 2(R-1) = 0.$$

Thus, a Turing bifurcation occurs at

$$R = 1 + \frac{1}{2d}\left(1+(2d-1)K^2\right)^2 + (1-K^2)^2.$$

Figure 9 shows the parameter regions for which (A_p, B_p) exists in blue, and stable in green, for a set value of d. The curves of R for which the solutions exist, are sideband stable, and at which a Turing bifurcation occurs are shown as well.



(a) $d = 0.1$

Figure 9: Region for stability (green) of periodic solutions for a set value of d.

5 Existence of quasi-periodic orbits in the AB-system

We now turn to the existence of quasi-periodic orbits in the AB-system 4.1. Figure 10 shows a numerically observed seemingly quasi-periodic solution.

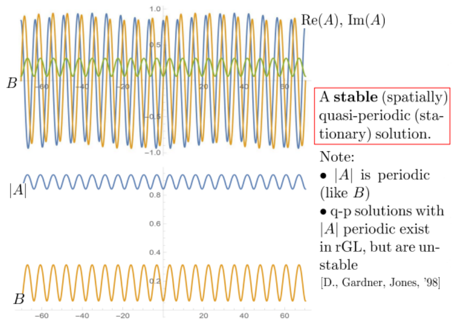


Figure 10: Quasi-periodic solution obtained from simulations.

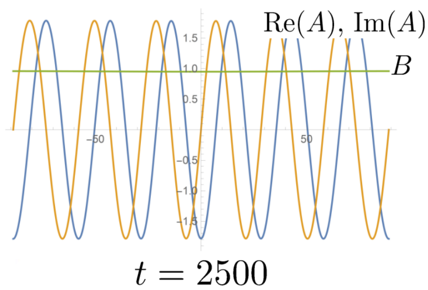


Figure 11: Periodic solution obtained from simulations.

The quasi-periodic solutions take the form $(A(\xi,\tau), B(\xi,\tau)) = (\rho(\xi)e^{i\theta(\xi)}, b(\xi))$, where both ρ and θ_ξ are periodic with the same period. Since A and B are independent of τ, we have $A_\tau = B_\tau = 0$. For the spatial derivatives, we have

$$A_\xi = \rho_\xi e^{i\theta} + i\rho\theta_\xi e^{i\theta}, B_\xi = b_\xi,$$
$$A_{\xi\xi} = \rho_{\xi\xi} e^{i\theta} + i\rho_\xi\theta_\xi e^{i\theta} + i\rho_\xi\theta_\xi e^{i\theta} + i\rho\theta_{\xi\xi} e^{i\theta} - \rho\theta_\xi^2 e^{i\theta}, B_{\xi\xi} = b_{\xi\xi}.$$

Furthermore, we note that $|A| = A\bar{A} = \rho e^{i\theta} \cdot \rho e^{-i\theta} = \rho^2$. Substituting the solution form into 4.1 yields

$$\begin{cases} 0 = \left(\rho_{\xi\xi} + 2i\rho_\xi\theta_\xi + i\rho\theta_{\xi\xi} - \rho\theta_\xi^2 + \rho - b\rho\right) e^{i\theta}, \\ 0 = db_{\xi\xi} + 1 - R - b^2 + \beta\rho^2. \end{cases}$$

Since $e^{i\theta(\xi)} \neq 0$ for all $\xi \in \mathbb{R}$, the equation reduces to

$$\begin{cases} 0 = \rho_{\xi\xi} + 2i\rho_\xi\theta_\xi + i\rho\theta_{\xi\xi} - \rho\theta_\xi^2 + \rho - b\rho, \\ 0 = db_{\xi\xi} + 1 - R - b^2 + \beta\rho^2. \end{cases}$$

We rewrite this as

$$\begin{cases} 0 = \left(\rho_{\xi\xi} - \rho\theta_\xi^2 + \rho - b\rho\right) + i\left(2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi}\right), \\ 0 = db_{\xi\xi} + 1 - R - b^2 + \beta\rho^2. \end{cases}$$

For the top equation to hold, we need that both the real and imaginary parts are equal to zero. Thus, by separating the real and imaginary components, we obtain the following system

$$\begin{cases} 0 = \rho_{\xi\xi} - \rho\theta_\xi^2 + \rho - b\rho, \\ 0 = 2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi}, \\ 0 = db_{\xi\xi} + 1 - R - b^2 + \beta\rho^2. \end{cases}$$

We recognise this second equation from 3.2 in Section 3, which has integral $\rho^2\theta_\xi = \Omega$. Consequently, the second equation reduces to $\dot{\Omega} = 0$. In the first equation, the only term containing θ is the term $\rho\theta_\xi^2$. Using that $\rho\theta_\xi^2 = \frac{\Omega^2}{\rho^3}$, we obtain

$$\begin{cases} \rho_{\xi\xi} = \rho\theta_\xi^2 - \rho + b\rho = \frac{\Omega^2}{\rho^3} - \rho + b\rho, \\ \Omega_\xi = 2\rho_\xi\theta_\xi + \rho\theta_{\xi\xi} = 0, \\ b_{\xi\xi} = \frac{1}{d}\left(R - 1 + b^2 - \beta\rho^2\right). \end{cases}$$

For a quasi-periodic orbit, we seek a solution where ρ and θ_ξ are both periodic with the same period. Since $\Omega = \rho^2\theta_\xi$ is a constant, ρ and θ_ξ automatically share the same period, if ρ is periodic. Hence, by eliminating θ, we only need to find a ρ periodic solution, to obtain a quasi-periodic solution. Keeping $\dot{\Omega} = 0$ in mind, we are left with a two-dimensional system of second-order equations. To express this as a system of four first-order equations, we introduce $\rho_\xi = V$ and $b_\xi = W$, where $V, W : \mathbb{R} \to \mathbb{R}$. This yields

$$\begin{cases} \rho_\xi = V, \\ V_\xi = \frac{\Omega^2}{\rho^3} - \rho + b\rho, \\ b_\xi = W, \\ W_\xi = \frac{1}{d}\left(R - 1 + b^2 - \beta\rho^2\right). \end{cases} \tag{5.1}$$

5.1 Rescaling to the Ginzburg-Landau system

In Section 3, we covered the existence of periodic orbits in the Ginzburg-Landau system. If we reverse these scalings, on 5.1, we can employ fast-slow analysis to show that those ρ periodic orbits persist in the AB-system by retention of the system's reversibility property. We begin by rescaling from 4.1 and apply the rescaling on ρ and b. For $0 < \epsilon \ll 1$, (note that this epsilon is different from the one used in Section 2),

$$B = 1 + \epsilon^2\tilde{B}, A = \epsilon\tilde{A}, \tilde{\xi} = \epsilon\xi, \tilde{\tau} = \epsilon^2\tau, \text{ and } R = \epsilon^2\tilde{R}.$$

Applying these rescalings to 4.1 gives

$$\begin{aligned} \tilde{A}_{\tilde{\tau}} &= \tilde{A}_{\tilde{\xi}\tilde{\xi}} - \tilde{A}\tilde{B}, \\ \alpha^{-1}\epsilon^4\tilde{B}_{\tilde{\tau}} &= \alpha^{-1}B_\tau = dB_{\xi\xi} + 1 - R - B^2 + \beta|A|^2 = \\ &= d\epsilon^4\tilde{B}_{\tilde{\xi}\tilde{\xi}} - \epsilon^2\tilde{R} - 2\epsilon^2\tilde{B} - \epsilon^4\tilde{B}^2 + \beta\epsilon^2|\tilde{A}|^2. \end{aligned}$$

Thus, we obtain

$$\begin{cases} \tilde{A}_{\tilde{\tau}} = \tilde{A}_{\tilde{\xi}\tilde{\xi}} - \tilde{A}\tilde{B}, \\ \alpha^{-1}\epsilon^2\tilde{B}_{\tilde{\tau}} = d\epsilon^2\tilde{B}_{\tilde{\xi}\tilde{\xi}} - \tilde{R} - 2\tilde{B} - \epsilon^2\tilde{B}^2 + \beta|\tilde{A}|^2. \end{cases} \tag{5.2}$$

For these scalings, we have

$$\rho(\xi)e^{i\theta(\xi)} = A = \epsilon\tilde{A} = \epsilon\tilde{\rho}(\tilde{\xi})e^{i\theta(\tilde{\xi})},$$
$$b(\xi) = B = 1 + \epsilon^2\tilde{B} = 1 + \epsilon^2\tilde{b}.$$

Substituting $(\tilde{A}(\tilde{\xi},\tilde{\tau}), \tilde{B}(\tilde{\xi},\tilde{\tau})) = (\tilde{\rho}(\tilde{\xi})e^{i\theta(\tilde{\xi})}, \tilde{b}(\tilde{\xi}))$ into 5.2 yields

$$\begin{cases} 0 = \tilde{\rho}_{\tilde{\xi}\tilde{\xi}} - \tilde{\rho}\theta_{\tilde{\xi}}^2 - \tilde{\rho}\tilde{b} = \tilde{\rho}_{\tilde{\xi}\tilde{\xi}} - \frac{\Omega^2}{\tilde{\rho}^3} - \tilde{\rho}\tilde{b}, \\ 0 = 2i\tilde{\rho}_{\tilde{\xi}}\theta_{\tilde{\xi}} + i\tilde{\rho}\theta_{\tilde{\xi}\tilde{\xi}} = \dot{\Omega}, \\ 0 = (-\tilde{R} - 2\tilde{b} + \beta\tilde{\rho}^2) + \delta(d\tilde{b}_{\tilde{\xi}\tilde{\xi}} - \tilde{b}^2), \end{cases} \tag{5.3}$$

where $\Omega = \tilde{\rho}^2\theta_{\tilde{\xi}}$ and $d\delta = \epsilon^2$.

5.1.1 Slow-fast analysis

For simplicity, we omit the tildes from the variables in the remainder of this section. This is done solely for the purpose of simplification, and does not mean that we are returned to the system before the rescaling.
Again setting $\dot{\Omega} = 0$ to the background and defining $\tilde{\epsilon}^2 = d\epsilon^2$, we obtain

$$\begin{cases} \rho_{\xi\xi} = \frac{\Omega^2}{\rho^3} - \rho b, \\ \tilde{\epsilon}^2 b_{\xi\xi} = R + 2b - \beta\rho^2 - \frac{1}{d}\tilde{\epsilon}^2 b^2. \end{cases} \tag{5.4}$$

Omitting the tilde on the new epsilon as well, and introducing $q = \rho_\xi$ and $c = \epsilon b_\xi$, the system can be written as four first-order equations:

$$\begin{cases} \rho_\xi = q, \\ q_\xi = \frac{\Omega^2}{\rho^3} - \rho b, \\ \epsilon b_\xi = c, \\ \epsilon c_\xi = R + 2b - \beta\rho^2 - \frac{1}{d}\epsilon^2 b^2. \end{cases} \tag{5.5}$$

This is the *slow system*. Taking $\epsilon \to 0$, we find

$$\begin{cases} \rho_\xi = q, \\ q_\xi = \frac{\Omega^2}{\rho^3} - \rho b, \\ 0 = c \\ 0 = R + 2b - \beta\rho^2. \end{cases} \tag{5.6}$$

From the last equation, we find $b = -\frac{1}{2}R + \frac{1}{2}\beta\rho^2$. Substituting this expression into the equation for $\rho_{\xi\xi}$ yields

$$\rho_{\xi\xi} = \frac{\Omega^2}{\rho^3} - \left(-\frac{1}{2}R + \frac{1}{2}\beta\rho^2\right)\rho,$$

which we recognise as the Ginzburg-Landau equation. Hence, in slow time, we recover the Ginzburg-Landau system on the manifold

$$\mathcal{M}_0 = \left\{ c = 0, b = -\frac{1}{2}R + \frac{1}{2}\beta\rho^2 \right\}.$$

This is illustrated in Figure 12.

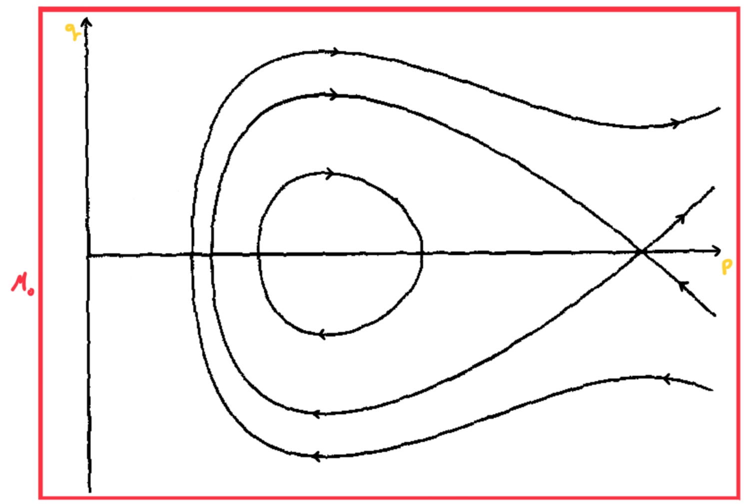


Figure 12: The slow solutions on $\mathcal{M}_0$ (from 5.6) resemble the solutions of the Ginzburg-Landau system for $0 < |\Omega| < \sqrt{\frac{4}{27}}$.

To retrieve the *fast system*, we introduce the rescaling on the spatial variable $\xi = \epsilon x$. This yields the system

$$\begin{cases} \rho_x = \epsilon q, \\ q_x = \epsilon \left(\frac{\Omega^2}{\rho^3} - \rho b \right), \\ b_x = c, \\ c_x = R + 2b - \beta\rho^2 + \frac{1}{d}\epsilon^2 b^2. \end{cases} \tag{5.7}$$

Again taking $\epsilon \to 0$, we find

$$\begin{cases} \rho_x = 0, \\ q_x = 0, \\ b_x = c, \\ c_x = R + 2b - \beta\rho^2. \end{cases} \tag{5.8}$$

It follows that ρ and q are constant in 5.8. Hence, setting $(\rho, q) = (\rho_0, q_0)$ for arbitrary $\rho_0, q_0 \in \mathbb{R}$ yields

$$\begin{cases} b_x = c, \\ c_x = R + 2b - \beta\rho_0^2, \end{cases} \tag{5.9}$$

which can be expressed in matrix form as

$$\begin{pmatrix} b_x \\ c_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ c \end{pmatrix} + \begin{pmatrix} 0 \\ R - \beta\rho_0^2 \end{pmatrix}.$$

For all $\rho_0, q_0 \in \mathbb{R}$ we find an equilibrium

$$\left(\rho_0, q_0, \frac{1}{2}\beta\rho_0^2 - \frac{1}{2}R, 0\right).$$

The characteristic polynomial of the linearised system is $Q(\lambda) = \lambda^2 - 2$, meaning that these equilibria have eigenvalues $\lambda_\pm = \pm\sqrt{2}$. Hence, all these equilibria are saddle-nodes. Therefore, the manifold $\mathcal{M}_0$, containing the solutions of the slow system, is contained in the set of equilibria of the fast system

$$\{b = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R, c = 0\}.$$

Figure 13 shows the phase portrait of the fast system in the in the (ρ, b, c)-plane. Note that the variable q is omitted from the graph, in order to visualise this.

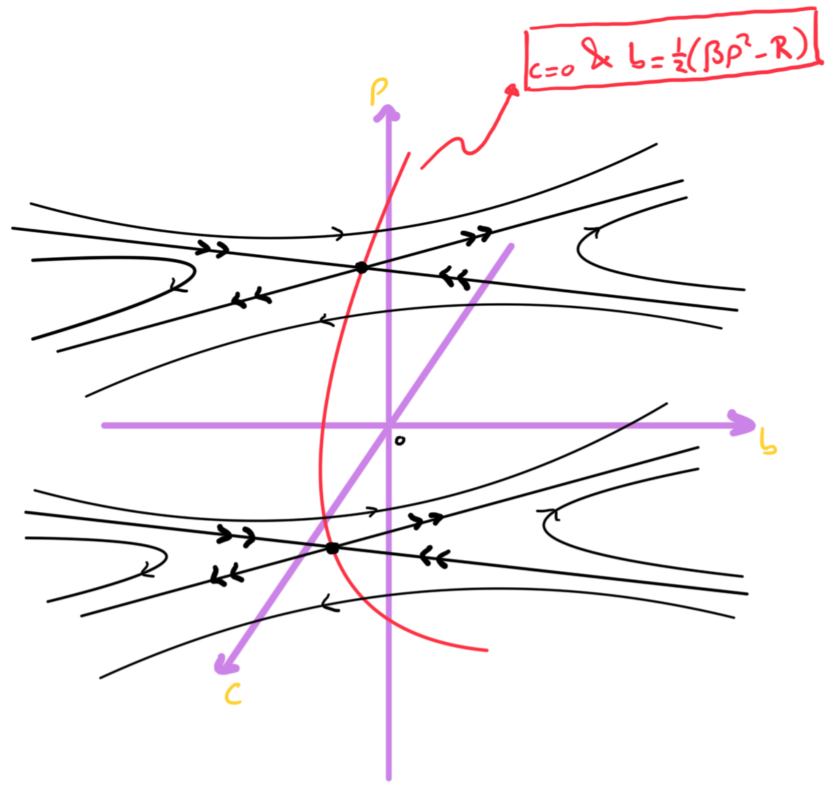


Figure 13: phase portrait of the fast system 5.8 in the (ρ, b, c)-plane. The red curve represents the manifold $\mathcal{M}_0$, which consists of equilibria in the fast system. Each equilibrium $(\rho_0, q_0, \frac{1}{2}\beta\rho_0^2 - \frac{1}{2}R, 0)$ is a saddle node. On $\mathcal{M}_0$, the dynamics in slow time correspond to Figure 12.

We will make use of Fenichel's first theorem, to conclude that for $\epsilon > 0$, there exists a manifold $\mathcal{M}_\epsilon$ that is diffeomorphic to $\mathcal{M}_0$ and locally invariant under the flow of the full problem 5.4, [2, §2, Thm. 2].

$$\begin{cases} \dot{u} = f(u, v, \epsilon), \\ \dot{v} = \epsilon g(u, v, \epsilon), \end{cases} \quad \text{(fast system)} \qquad \begin{cases} \epsilon u' = f(u, v, \epsilon), \\ v' = g(u, v, \epsilon). \end{cases} \quad \text{(slow system)}$$

Theorem 5.1 (Fenichel's first theorem)**.** Suppose $\mathcal{M}_0 \subset \{f(u, v, 0) = 0\}$ is compact, possibly with boundary, and normally hyperbolic, that is, the eigenvalues λ of the Jacobian $\frac{\partial f}{\partial u}(u, v, 0)|_{\mathcal{M}_0}$ all satisfy $\mathrm{Re}(\lambda) \neq 0$. Suppose f and g are smooth. Then for $\epsilon > 0$ and sufficiently small, there exists a manifold $\mathcal{M}_\epsilon$, $\mathcal{O}(\epsilon)$ close and diffeomorphic to $\mathcal{M}_0$, that is locally invariant under the flow of the full problem.

We see that indeed, all the conditions of Fenichel's first theorem are satisfied. Therefore, we can assume that locally, the manifold $\mathcal{M}_\epsilon$ can be written as

$$\mathcal{M}_\epsilon = \{b = b_\epsilon(\rho, q), c = c_\epsilon(\rho, q)\}, \tag{5.10}$$

where

$$b_\epsilon(\rho, q) = b_0(\rho, q) + \epsilon b_1(\rho, q) + \epsilon^2 b_2(\rho, q) + ... = \sum_{i=0}^{\infty} \epsilon^i b_i(\rho, q),$$
$$c_\epsilon(\rho, q) = c_0(\rho, q) + \epsilon c_1(\rho, q) + \epsilon^2 c_2(\rho, q) + ... = \sum_{i=0}^{\infty} \epsilon^i c_i(\rho, q),$$

are unknown functions. We already know that $b_0(\rho, q) = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R$ and $c_0(\rho, q) = 0$, since $\mathcal{M}_\epsilon$ is $\mathcal{O}(\epsilon)$ close to $\mathcal{M}_0$. Equation 5.10 implies is that locally, $\mathcal{M}_\epsilon$ can be written as a graph over $\mathcal{M}_0$. We can now determine the functions b_ϵ and c_ϵ and consequently, the dynamics on $\mathcal{M}_\epsilon$ to any order of ϵ. Our goal is to show that the flow of the system on $\mathcal{M}_\epsilon$ remains reversible, meaning that the system is invariant under the reversal of time. In our case, reversibility means that the equations remain unchanged, if we substitute $\xi \to -\xi$, $q \to -q$ and $c \to -c$. After establishing that the flow on $\mathcal{M}_\epsilon$ remains reversible, we will discuss why this implies the persistence of ρ periodic orbits.
From Fenichel's first theorem, we know that $\mathcal{M}_\epsilon$ is locally invariant under the flow. Furthermore, we have

$$b_x = \frac{\partial b_\epsilon}{\partial \rho}(\rho, q)\rho_x + \frac{\partial b_\epsilon}{\partial q}(\rho, q)q_x, \text{ and} \tag{5.11}$$
$$c_x = \frac{\partial c_\epsilon}{\partial \rho}(\rho, q)\rho_x + \frac{\partial c_\epsilon}{\partial q}(\rho, q)q_x. \tag{5.12}$$

Using 5.7, the Equations 5.11 and 5.12 become

$$b_x = c = c_\epsilon = \frac{\partial b_\epsilon}{\partial \rho}\epsilon q + \frac{\partial b_\epsilon}{\partial q}\epsilon\left(\frac{\Omega^2}{\rho^3} - \rho b\right) = \frac{\partial b_\epsilon}{\partial \rho}\epsilon q + \frac{\partial b_\epsilon}{\partial q}\epsilon\left(\frac{\Omega^2}{\rho^3} - \rho b_\epsilon\right),$$

and

$$c_x = R + 2b - \beta\rho^2 + \frac{1}{d}\epsilon^2 b^2 = R + 2b_\epsilon - \beta\rho^2 + \frac{1}{d}\epsilon^2 b_\epsilon^2 = \frac{\partial c_\epsilon}{\partial \rho}\epsilon q + \frac{\partial c_\epsilon}{\partial q}\epsilon\left(\frac{\Omega^2}{\rho^3} - \rho b_\epsilon\right).$$

With this, we obtain the equations we need to determine the order corrections of $\mathcal{M}_\epsilon$:

$$c_\epsilon = \frac{\partial b_\epsilon}{\partial \rho}\epsilon q + \frac{\partial b_\epsilon}{\partial q}\epsilon\left(\frac{\Omega^2}{\rho^3} - \rho b_\epsilon\right), \tag{5.13}$$
$$R + 2b_\epsilon - \beta\rho^2 + \frac{1}{d}\epsilon^2 b_\epsilon^2 = \frac{\partial c_\epsilon}{\partial \rho}\epsilon q + \frac{\partial c_\epsilon}{\partial q}\epsilon\left(\frac{\Omega^2}{\rho^3} - \rho b_\epsilon\right). \tag{5.14}$$

To determine the first-order corrections, we consider

$$b_\epsilon(\rho, q) = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R + \epsilon b_1(\rho, q) + \mathcal{O}(\epsilon^2), \tag{5.15}$$

$$c_\epsilon(\rho, q) = \epsilon c_1(\rho, q) + \mathcal{O}(\epsilon^2), \tag{5.16}$$

thus, retaining the terms up to $\mathcal{O}(\epsilon^2)$. Substituting 5.15 and 5.16 into 5.13 and 5.14, and collecting all terms up to the first-order of ϵ, neglecting higher-order terms, we obtain

$$\begin{aligned} \mathcal{O}(\epsilon^2) + \epsilon c_1 = \epsilon\beta\rho q, &\quad \text{so } c_1(\rho, q) = \beta\rho q, \text{ and} \\ \mathcal{O}(\epsilon^2) + 2\epsilon b_1 = 0 &\quad \text{so } b_1(\rho, q) = 0. \end{aligned}$$

Thus, the first-order corrections of $\mathcal{M}_\epsilon$ are

$$\mathcal{M}_\epsilon = \{b = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R + \mathcal{O}(\epsilon^2), c = \epsilon\beta\rho q + \mathcal{O}(\epsilon^2)\}. \tag{5.17}$$

By substituting the expressions of b and c from 5.17 into 5.6, we find the first-order correction of the flow on $\mathcal{M}_\epsilon$:

$$\begin{cases} \rho_\xi = q, \\ q_\xi = \frac{\Omega^2}{\rho^3} - \frac{1}{2}\beta\rho^3 + \frac{1}{2}R\rho + \mathcal{O}(\epsilon^2). \end{cases}$$

We observe that this is identical to the Ginzburg-Landau equation, up to higher-order terms. Note that the first-order correction is reversible. To find the second-order correction, we consider

$$b_\epsilon(\rho, q) = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R + \epsilon^2 b_2(\rho, q) + \mathcal{O}(\epsilon^3), \tag{5.18}$$

$$c_\epsilon(\rho, q) = \epsilon\beta\rho q + \epsilon^2 c_2(\rho, q) + \mathcal{O}(\epsilon^3), \tag{5.19}$$

and repeat the process. Substituting 5.18 and 5.19 into 5.13 and 5.14, and collecting all terms up to the second-order of ϵ, neglecting the higher-order terms, we obtain

$$\epsilon\beta\rho q + \epsilon^2 c_2 = \epsilon\beta\rho q \quad \text{so } c_2(\rho, q) = 0, \text{ and from}$$

$$2\epsilon^2 b_2 + \frac{1}{d}\epsilon^2\left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R\right)^2 = \epsilon^2\beta q^2 + \epsilon^2\beta\rho\left(\frac{\Omega^2}{\rho^3} - \rho\left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R\right)\right),$$

we observe

$$b_2 = \frac{1}{2}\left(\beta q^2 + \beta\rho\left(\frac{\Omega^2}{\rho^3} - \rho\left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R\right)\right) - \frac{1}{d}\left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R\right)^2\right).$$

Thus, we obtain the second-order correction of $\mathcal{M}_\epsilon$,

$$\mathcal{M}_\epsilon = \Bigg\{ b = \frac{1}{2}\beta\rho^2 - \frac{1}{2}R +$$
$$+ \frac{1}{2}\epsilon^3 \left(\beta q^2 + \beta\rho \left(\frac{\Omega^2}{\rho^3} - \rho \left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R \right) \right) - \frac{1}{d} \left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R \right)^2 \right) + \mathcal{O}(\epsilon^3),$$
$$c = \epsilon\beta\rho q + \mathcal{O}(\epsilon^3) \Bigg\}. \quad (5.20)$$

Substituting the expressions of b and c from 5.20, into 5.6 yields the second-order correction of the flow on $\mathcal{M}_\epsilon$:

$$\begin{cases} \rho_\xi = q, \\ q_\xi = \frac{\Omega^2}{\rho^3} - \rho \left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R + \frac{1}{2}\epsilon^2 \left(\beta q^2 + \beta\rho \left(\frac{\Omega^2}{\rho^3} - \rho \left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R \right) \right) \right) \right) + \\ + \frac{1}{2}\frac{1}{d}\epsilon^2\rho \left(\frac{1}{2}\beta\rho^2 - \frac{1}{2}R \right)^2 + \mathcal{O}(\epsilon^3). \end{cases} \quad (5.21)$$

If we substitute $\xi \to -\xi$, and $q \to -q$, we have $-\rho_\xi = -q$, which means that the two negative signs cancel out. In the second equation we have $q_\xi \to -(-q_\xi) = q_\xi$, hence, the left-hand side remains unchanged. On the right-hand side, q only appears once as q^2, meaning that the negative sign also cancels out there. Therefore, 5.21 remains reversible. We can repeat the process up to any order. However, the expressions become increasingly complicated. If possible, we want to avoid employing the principle of induction to show that the flow on $\mathcal{M}_\epsilon$ remains reversible for any order. By inspection of 5.5, we find that this is indeed unnecessary, since this system is inherently reversible. Substituting $\xi \to -\xi$, $q \to -q$ and $c \to -c$, we obtain

$$\begin{cases} -\rho_\xi = -q, \\ -(-q_\xi) = \frac{\Omega^2}{\rho^3} - \rho b, \\ -\epsilon b_\xi = -c, \\ -(-\epsilon c_\xi) = R + 2b - \beta\rho^2 - \frac{1}{d}\epsilon^2 b^2, \end{cases}$$

which is identical to 5.5. Hence, the flow on $\mathcal{M}_\epsilon$ remains reversible as well.
The final argument required to establish the existence of quasi-periodic orbits is that ρ periodic orbits persist if the system remains reversible.
Consider a general two-dimensional system $\dot{x} = F(x, \epsilon)$, where $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is smooth, and $0 < \epsilon \ll 1$ introduces a small perturbation. Let $\varphi_{t,\epsilon}$ denote the flow of the system for a given ϵ. Assume that for $\epsilon = 0$, the system contains a periodic orbit Γ_p, and that the system is reversible for all $\epsilon \geq 0$.
Let $T_p > 0$ denote the period of Γ_p, so that $\varphi_{t,0}(x) = \varphi_{t+T_p,0}(x)$. Reversible systems are symmetric in one of the coordinates, since mapping t to $-t$ and x_i to $-x_i$ yields the same system. Without loss of generality, we assume that

the system is symmetric in the second coordinate. Then, for $x = (x_1, x_2) \in \Gamma_p$ there exists $\tilde{T} \in \mathbb{R}$ such that $\varphi_{\tilde{T},0}(x) = \tilde{x} = (x_1, -x_2) \in \Gamma_p$. It follows that $\varphi_{T_p-\tilde{T},0}(\tilde{x}) = x$. For symmetry, we require that for any $t \in \mathbb{R}$, where $\varphi_{t,\epsilon}(x) = (x_1', x_2')$, it holds that $\varphi_{-t,\epsilon}(\tilde{x}) = (x_1', -x_2')$. Consequently, Γ_x and $\Gamma_{\tilde{x}}$ intersect on $x_2 = 0$, which implies that $\Gamma_x = \Gamma_{\tilde{x}}$.
Now suppose that the periodic orbit Γ_p does not persist for $\epsilon > 0$. If Γ_p does not persist, we know that either $\varphi_{\tilde{T},\epsilon}(x) \neq \tilde{x}$ or $\varphi_{T_p-\tilde{T},\epsilon}(\tilde{x}) \neq x$ must hold, since $\varphi_{T_p,\epsilon}(x) \neq x$. In that case however, we have $\tilde{x} \notin \Gamma_x = \{\varphi_{t,\epsilon}(x) : t \in \mathbb{R}\}$, which implies that $\Gamma_x \cap \Gamma_{\tilde{x}} = \emptyset$. This, however, contradicts with the reversibility of the system. Therefore, we can conclude that periodic orbits persist for $\epsilon > 0$. From this conclusion, it follows that quasi-periodic orbits exist in the AB-system. Finally, we note that these quasi-periodic orbits are unstable. Since $\mathcal{M}_\epsilon$ is contained in the set of equilibria of the fast system, and each of these equilibria is a saddle node, hence unstable, we can conclude that the ρ periodic orbits that persist in $\mathcal{M}_\epsilon$ are unstable. Consequently, the corresponding quasi-periodic orbits in the AB-system are unstable as well.

6 Conclusion & Discussion

We began with the derivation of the AB-system, and discussed the existence of periodic and quasi-periodic orbits in the Ginzburg-Landau system, in and the AB-system. The most significant result of this thesis, however, is the existence of quasi-periodic orbits. Although numerical simulations already suggested the existence of these solutions, we provided a theoretical foundation for their existence. It is important to note that the quasi-periodic solutions found through simulations differ from those whose existence we observed. As concluded in 5, the quasi-periodic solutions we found are unstable. It is therefore unlikely to find these unstable quasi-periodic solutions through numerical methods, since this would require the initial conditions to coincide exactly with the solution. Therefore, we suspect that the numerically observed solutions are stable, which means we can hypothesize that the AB-system contains stable quasi-periodic solutions as well, but not in the vicinity of the Ginzburg-Landau system. This presents a promising direction for future research. A possible approach to explore this is to set $d = \epsilon^2$, and repeat the fast-slow analysis. Doing this implies that δ, the parameter used to rescale the system back to the Ginzburg-Landau system, in 5.3, is equal to one. Since for rescaling we set $0 < \delta \ll 1$, the case $\delta = 1$ corresponds to a setting far away from the Ginzburg-Landau system. The existence of other types of solutions warrants further investigation. For example, it is known that the Ginzburg-Landau system contains a homoclinic solution [1]. This raises the question of whether the AB-system contains homoclinic solutions as well.

References

[1] Arjen Doelman. "Slow time-periodic solutions of the Ginzburg-Landau equation". In: *Phys. D* 40.2 (1989), pp. 156–172. ISSN: 0167-2789,1872-8022. DOI: `10.1016/0167-2789(89)90060-2`. URL: `https://doi.org/10.1016/0167-2789(89)90060-2`.

[2] Geertje Hek. "Geometric singular perturbation theory in biological practice". In: *Journal of mathematical biology* 60.3 (2010), pp. 347–386.

[3] Dock Staal and Arjen Doelman. *The evasion of tipping: pattern formation near a Turing-fold bifurcation*. 2025. arXiv: `2506.22251 [math.DS]`. URL: `https://arxiv.org/abs/2506.22251`.

[4] A. M. Turing. "The chemical basis of morphogenesis". In: *Philos. Trans. Roy. Soc. London Ser. B* 237.641 (1952), pp. 37–72. ISSN: 0080-4622.

Appendix

6.1 Hamiltonian conditions

The first direction this study went to show the existence of quasi-periodic orbits in the AB-system, was to see if 5.1 is Hamiltonian, and use properties of those kinds of systems to find periodic orbits. Although we did find conditions for this system to be Hamiltonian, we will not go further into this. However, A system being Hamiltonian is still an interesting property, which is why this section was kept in the appendix. For a Hamiltonian system, we need that there exists a Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$ where

$$\frac{\partial H}{\partial \rho} = -V_\xi, \frac{\partial H}{\partial V} = \rho_\xi, \frac{\partial H}{\partial b} = -W_\xi, \frac{\partial H}{\partial W} = b_\xi, \tag{6.1}$$

such that H is conserved. This means that

$$\frac{dH}{d\xi} = \frac{\partial H}{\partial \rho}\cdot\rho_\xi + \frac{\partial H}{\partial V}\cdot V_\xi + \frac{\partial H}{\partial b}\cdot b_\xi + \frac{\partial H}{\partial W}\cdot W_\xi = -V_\xi\cdot\rho_\xi + \rho_\xi\cdot V_\xi - W_\xi\cdot b_\xi + b_\xi\cdot W_\xi = 0.$$

If we integrate both sides of each of the equations in 6.1 over ρ, V, b and W respectively, we see that

$$H(\rho, V, b, W) = -\int V_\xi d\rho + h_1(V, b, W) = \\ = -\left(-\frac{1}{2}\frac{\Omega^2}{\rho^3} + \frac{1}{2}(b-1)\rho^2\right) + h_1(V, b, W), \tag{6.2}$$

$$H(\rho, V, b, W) = \int \rho_\xi dV + h_2(\rho, b, W) = \frac{1}{2}V^2 + h_2(\rho, b, W), \tag{6.3}$$

$$H(\rho, V, b, W) = -\int W_\xi db + h_3(\rho, V, W) =$$
$$= -\frac{1}{d}\left((R-1)b + \frac{1}{3}b^3\right) + \frac{\beta}{d}\rho^2 b + h_3(\rho, V, W), \quad (6.4)$$

$$H(\rho, V, b, W) = \int b_\xi dW + h_4(\rho, V, b) = \frac{1}{2}W^2 + h_4(\rho, V, b), \quad (6.5)$$

for some unknown functions $h_1, h_2, h_3, h_4 : \mathbb{R}^3 \to \mathbb{R}$. The terms $-\frac{1}{2}b\rho^2$ and $\frac{\beta}{d}\rho^2 b$ are the only terms that do not coincide in 6.2 and 6.4, since h_1 is not a function over ρ and h_3 is not a function over b. This means that the functions h_1, h_2, h_3, h_4 can only exist if $-\frac{1}{2}b\rho^2 = \frac{\beta}{d}\rho^2 b$. In this case we need that $d = -2\beta$, to find a Hamiltonian function. Suppose we indeed have $d = -2\beta$. We find the Hamiltonian:

$$H(\rho, V, b, W) = \frac{1}{2}V^2 + \frac{1}{2}W^2 + \frac{1}{2}\frac{\Omega^2}{\rho^2} + \frac{1}{2}\rho^2 + \frac{1}{2\beta}\left((R-1)b + \frac{1}{3}b^3\right) - \frac{1}{2}b\rho^2. \quad (6.6)$$

Thus, we find that 5.1 is a Hamiltonian system for $d = -2\beta$.