Dynamical Renormalization Group applications to Lagrangian Perturbation Theory

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Abstract

Lagrangian Perturbation Theory (LPT) provides a powerful framework for modeling gravitational structure formation in cosmology, avoiding the shell-crossing singularities that plague Eulerian approaches. However, standard LPT suffers from secular growth in perturbative corrections unphysical divergences that invalidate the expansion at late times, particularly in underdense regions. This thesis develops a rigorous Dynamical Renormalization Group (DRG) formalism for LPT using spherical harmonic decomposition to systematically resum secular growth across multipole modes.

We derive the complete perturbative hierarchy for the monopole (ℓ=0), dipole (ℓ=1), and quadrupole (ℓ=2) modes, demonstrating that all multipoles satisfy a universal time-evolution ODE with identical homogeneous basis 1, η^{-3}, Wronskian W = -3η^{-4}, and particular solution coefficients (1/10, -1/25). Mode-specific differences appear only through divergence eigenvalues $\ell(\ell+1)$ and coupling structure. At perturbative ϵ^1 and ϵ^2, running constants remain frozen despite cross-mode coupling in source terms. At order ϵ^3, true coupled running emerges: three coupled first-order RG equations govern the synchronized evolution of all multipole constants, with coupling mediated by partial derivatives of particular solutions with respect to other modes' displacement fields.

Contents

Introduction

The formation of large-scale structure (LSS) in the universe represents one of the most important questions in modern cosmology[36][26]. From the nearly uniform conditions of the early universe, with density fluctuations of order $\delta\rho/\rho \sim 10^{-5}$, a highly structured universe populated by galaxies, clusters, and cosmic webs has emerged. Understanding the physical mechanisms governing this transition from uniformity to complexity is fundamental to cosmology and serves as a critical test of our theoretical understanding of gravity[36][26].

Historically, the study of structure formation gained prominence following the pioneering work of Zeldovich [40], who introduced the Zeldovich approximation a remarkably effective method that maps initial Lagrangian positions of fluid elements to their final Eulerian positions through a linear displacement field. This approximation provided intuitive insight into how structure forms through gravitational instability and remains central to perturbative approaches to structure formation today. Modern surveys (SDSS, 2dFGRS, DES, JWST) now probe structure across multiple scales and epochs with unprecedented precision, requiring theoretical predictions accurate to percent-level precision for cosmological parameter inference[27][16][25][35][18].

Cosmological Perturbation Theory (CPT) provides the standard framework for predicting density fluctuation growth in an expanding universe[3][34][4]. It has proven remarkably successful at predicting statistical properties of the large scale density field including the power spectrum, bispectrum, and higher-order statistics for $k0.1\,h\,\mathrm{Mpc}^{-1}$ at low redshift[3][17][4]. However, CPT suffers from a critical limitation: the development of shell-crossing singularities. In Eulerian coordinates, fluid elements can cross paths, causing the density contrast to diverge and the perturbative expansion to break down at surprisingly small times (corresponding to $\delta \sim 1$), limiting CPT to the weakly nonlinear regime[17][21][34].

To circumvent shell-crossing Lagrangian Perturbation Theory (LPT) was developped [8][9] al.[23]. This perturbation theory takes from Eulerian perturbation theory, reformulating the equations in Lagrangian coordinates labeled by initial comoving positions $\mathbf{q}$.

In this framework, a fluid element's trajectory is[3]:

$$\mathbf{r}(t|\mathbf{q}) = \mathbf{q} + \mathbf{\Psi}(\mathbf{q}, t), \tag{1}$$

where $\mathbf{\Psi}$ is the Lagrangian displacement field. The density is determined by the Jacobian of the coordinate transformation with shell crossing corresponding to the Jacobian becoming singular an event that can be tracked precisely[8][9][23].

Despite its advantages, LPT faces a persistent limitation: secular growth in perturbative corrections, particularly for underdense regions and at late times[3][31][37]. Each perturbative order can generate terms that grow unbounded not because the true solution diverges, but because the perturbative approximation fails to resum infinite classes of corrections that collectively produce non-physical growth. This problem, documented in detail by Rampf and collaborators [30] [31], manifests catastrophically in underdense regions standard LPT convergence degrades because second-order source terms scale with powers of the initial displacement squared, yielding secular corrections that grow with time.

In this paper by Rampf and Hahn[31], the authors look to apply a renormalization group approach to try and find solutions to these LPT theories. The renormalization group approach worked well, obtaining predictive results significantly better than the standard LPT approach. The renormalization theory utilised is incredibly similar to Dynamical renormalization group (DRG) theory[10][31][32].

Dynamical Renormalization Group (DRG) is a systematic resummation technique originally developed in quantum field theory to handle divergences and secular growth in perturbative expansions, i.e. exactly the issues which are faced in LPT[6][10][22]. The core principle is to introduce an arbitrary renormalization scale and promote the integration constants of a solution to running functions that depend on this scale[10][6]. The effectivness of the pseudo-DRG model developped by Rampf and Hahn, raises the question, what if the full detailed DRG procedure is applied to lagrangian perturbation theory, could these developments lead to even more effective LPT predictions[31].

This thesis develops a rigorous DRG formalism for Lagrangian Perturbation Theory, doing so by utilising spherical harmonic decompositions[33] to apply DRG to increasingly higher order modes of LPT. While spherical collapse ($\ell = 0$) is the simplest case, the evolution of multiple multipole moments ($\ell = 0, 1, 2, \dots$) is more realistic and provides insight into anisotropic structure formation. These multipole expansions set the foundation for eventual developments involving all modes, forming a complete DRG application to Lagrangian perturbation theory.

The specific research questions addressed are:

1. **Can DRG be systematically applied to Lagrangian Perturbation Theory equations?** We develop a general framework for applying DRG to the coupled nonlinear ODEs governing homogeneous perturbations, showing how secular terms are identified and absorbed.

2. **Can we form the fundemental basis for the application of DRG to the $\ell = 0$ (monopole), $\ell = 1$ (dipole), and $\ell = 2$ (quadrupole) modes ?**

The aim is thus to establish the foundational framework for DRG applied to Lagrangian perturbation theory in multipole space.

Chapter 1

Theoretical Background

1.1 Einstein Equations and Cosmological Perturbations

The dynamics of spacetime and matter in cosmology are governed by Einstein's field equations, which relate the curvature of spacetime to the distribution of matter and energy[7][12][39]:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, G is Newton's gravitational constant, and $T_{\mu\nu}$ is the stress-energy tensor. The Einstein tensor itself is defined as[12]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar (the trace of $R_{\mu\nu}$), and $g_{\mu\nu}$ is the metric tensor describing the geometry of spacetime. The Ricci tensor itself is derived from the Riemann curvature tensor, which measures how spacetime is curved[12].

For cosmological applications in the matter-dominated era relevant to structure formation, we work in the Newtonian limit, where spacetime is nearly flat and gravitational fields are weak, in this regime, Einstein's equations are obtained via the following derivation. We decompose the metric as a background cosmological metric plus small perturbations[7][14]:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{1.3}$$

where $g^{(0)}_{\mu\nu}$ describes the homogeneous, isotropic background universe, and $h_{\mu\nu}$ represents small deviations from this background. Similarly, the stress-energy tensor is decomposed as[12]:

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}, \tag{1.4}$$

where $T^{(0)}_{\mu\nu}$ is the background component and $\delta T_{\mu\nu}$ is the perturbation[7][12].

Substituting these decompositions into Einstein's equations and retaining only first-order terms in the small quantities $h_{\mu\nu}$ and $\delta T_{\mu\nu}$, we obtain the linearized Einstein equations[7][12]. In the Newtonian gauge, where the metric perturbations are described by a single scalar gravitational potential $\Phi(\mathbf{x}, t)$, the time-time component of the linearized Einstein equations yields[7][13]:

$$\nabla^2\Phi = 4\pi G a^2 \rho_0 \delta, \tag{1.5}$$

where ρ_0 is the background density, δ is the density contrast, $a(t)$ is the scale factor describing the expansion of the universe, and the factor a^2 appears because we are working in comoving coordinates[1][7]. This equation is Poisson's equation in an expanding universe, and it relates the gravitational potential to the density perturbations. The spatial curvature of spacetime, encoded in the Laplacian ∇^2, is directly proportional to the matter overdensities and underdensities that source structure formation[15].

The other components of the linearized Einstein equations specifically the $0i$ and ij components provide constraints on the velocity field and relate the gravitational potential to the dynamics of the [7]. These Einstein equations are given by:

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = 4\pi G a^2\,\delta\rho \tag{1.6}$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{P})v \tag{1.7}$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi = 4\pi G a^2\,\delta P. \tag{1.8}$$

These constraints, combined with the continuity equation and the Euler equation, form a complete system of equations governing the evolution of perturbations in an expanding universe[3][7][34].

1.2 Continuity Equation in Cosmological Perturbation Theory

The continuity equation expresses mass conservation in a fluid and forms one of the three fundamental equations governing the evolution of density perturbations in an expanding universe. In the context of cosmological perturbation theory, it relates the time evolution of density fluctuations to the divergence of the velocity flow field[2].

Starting from the basic principle of mass conservation in Eulerian coordinates, the continuity equation reads

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{1.9}$$

where ρ is the mass density and $\mathbf{v}$ is the velocity field. In an expanding universe with scale factor $a(t)$, we must transform to comoving coordinates $\mathbf{x}$ related to physical coordinates by $\mathbf{r} = a(t)\mathbf{x}$. The velocity splits into the Hubble flow $H(t)\mathbf{r}$ and the peculiar velocity $\mathbf{v}_{\text{pec}}$, after this coordinate transformation and accounting for the background expansion, the continuity equation becomes

$$\frac{\partial \rho_0}{\partial t} + 3H\rho_0 + \frac{1}{a}\nabla \cdot (\rho_0 \mathbf{v}_{\text{pec}}) = 0 \tag{1.10}$$

for the full density field, where $H = \dot{a}/a$ is the Hubble parameter[28].
We decompose the density into a homogeneous background $\rho_0(t)$ and perturbations: $\rho = \rho_0(1+\delta)$, where $\delta = \delta\rho/\rho_0$ is the density contrast. The background satisfies $\dot{\rho}_0 + 3H\rho_0 = 0$, which gives $\rho_0 \propto a^{-3}$ for matter. At linear order in perturbations, dropping all second-order terms, the continuity equation simplifies to

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{v} = 0 \tag{1.11}$$

where we use the shorthand notation $\dot{\delta} \equiv \partial\delta/\partial t$ and $\mathbf{v} = \mathbf{v}_{\text{pec}}$ is the peculiar velocity in comoving coordinates, this form shows that density enhancements ($\delta > 0$) can only persist if the velocity field has negative divergence, meaning mass is flowing into the region. Conversely, regions with expanding velocities ($\nabla \cdot \mathbf{v} > 0$) must have decreasing density to conserve mass[2].

1.3 Euler Equation in Cosmological Perturbation Theory

The Euler equation governs momentum conservation in a fluid and represents Newton's second law in the context of hydrodynamics. In cosmological perturbation theory, it describes how the peculiar velocities of matter elements respond to pressure gradients and gravitational forces in an expanding universe. Together with the continuity equation and Poisson equation, it forms the complete set of fluid equations governing structure formation[28][3].

In an expanding universe with comoving coordinates $\mathbf{x}$ and scale factor $a(t)$, the full Euler equation for momentum conservation reads

$$\frac{\partial \mathbf{v}_{\text{pec}}}{\partial t} + H\mathbf{v}_{\text{pec}} = -\frac{1}{a}\nabla p - \frac{1}{a}\nabla\Phi \tag{1.12}$$

where $\mathbf{v}_{\text{pec}}$ is the peculiar velocity in comoving coordinates, $H = \dot{a}/a$ is the Hubble parameter, p is the pressure, and Φ is the gravitational potential. The term $H\mathbf{v}_{\text{pec}}$

represents the Hubble friction or the effective deceleration due to cosmic expansion, while the two terms on the right describe pressure-gradient and gravitational forces[2].

For perturbations about the homogeneous background, we decompose the velocity as $\mathbf{v} = \mathbf{0} + \delta\mathbf{v}$ and expand to first order in perturbation quantities. When coupled with the continuity equation $\dot{\delta} + (1/a)\nabla \cdot \mathbf{v} = 0$ and the pressure perturbation relation $\delta p = c_s^2 \rho_0 \delta$, where c_s is the sound speed, the linearized Euler equation becomes

$$\dot{\mathbf{v}} + H\mathbf{v} = -\frac{c_s^2}{a}\nabla\delta - \frac{1}{a}\nabla\Phi \tag{1.13}$$

where the first term on the right represents the pressure-gradient force and the second represents the gravitational acceleration[2].

1.4 Spherical Harmonics Decomposition

Since the background universe is isotropic, perturbations can be decomposed into spherical harmonics[24]. This decomposition is powerful because it separates the problem into independent multipole modes, each of which evolves according to its own equations in the linear regime[24]. For a scalar field such as the density contrast $\delta(r, \theta, \phi, t)$, we can expand it as

$$\delta(r, \theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \delta_{\ell m}(r, t)\, Y_{\ell m}(\theta, \phi) \tag{1.14}$$

where $Y_{\ell m}$ are the spherical harmonics and $\delta_{\ell m}$ are the multipole amplitudes that depend only on the radial coordinate r and time t. The indices ℓ and m label the different angular patterns, with ℓ fixing the angular scale and m distinguishing the $2\ell + 1$ independent components at fixed ℓ. Orthogonality ensures that different ℓ modes do not mix at linear order, so gravitational coupling between multipoles only appears once nonlinear terms are included[24].

This scalar harmonic expansion applies directly only to scalar quantities such as δ and the Newtonian gauge potentials. Vector fields, such as the peculiar velocity $\mathbf{v}$, require a vector decomposition on the sphere, because the relevant equations involve intrinsically vector operations such as $\nabla \cdot \mathbf{v}$ in the continuity equation and $\nabla\Phi$ in the Euler equation [33][19]. In practice, even if we restrict to the scalar (irrotational) sector, the velocity generically contains both a radial part and a tangential, curl free part. For this reason, representing $\mathbf{v}$ by a single radial amplitude times $Y_{\ell m}$ is not, in general, a complete description of the mode. A compact way to write the irrotational decomposition is[24][19]

$$\mathbf{v}(r, \theta, \phi, t) = \sum_{\ell, m} \left[v_{\ell m}^{(r)}(r, t)\, Y_{\ell m}(\theta, \phi)\, \hat{\mathbf{r}} + v_{\ell m}^{(1)}(r, t)\, \frac{r}{\sqrt{\ell(\ell+1)}}\, \nabla_\Omega Y_{\ell m}(\theta, \phi) \right] \tag{1.15}$$

where ∇_Ω denotes the angular gradient on the unit sphere. The first term describes purely radial inflow or outflow with angular dependence $Y_{\ell m}$, while the second term describes the associated tangential flow required by the angular structure of the perturbation. This tangential contribution vanishes automatically for $\ell = 0$, since Y_{00} is constant and therefore $\nabla_\Omega Y_{00} = 0$, which is consistent with the fact that a monopole perturbation is strictly spherically symmetric[24][33].

1.5 Lagrangian Perturbation Theory

Lagrangian Perturbation Theory (LPT) reformulates the perturbative problem in Lagrangian (material) coordinates, where one follows the trajectories of individual fluid elements rather than evaluating fields at fixed spatial points in space[3][8].

In Lagrangian coordinates, a fluid element initially at position $\mathbf{q}$ (in comoving space) is mapped to its Eulerian position $\mathbf{r}$ at later times via[3][5]:

$$\mathbf{r}(\mathbf{q}, t) = \mathbf{q} + \boldsymbol{\Psi}(\mathbf{q}, t), \tag{1.16}$$

where $\boldsymbol{\Psi}(\mathbf{q}, t)$ is called the Lagrangian displacement field. It represents how far a fluid element has moved from its initial position[3][5]. The key insight is that the density is then given by the Jacobian of this transformation:

$$\delta(\mathbf{q}, t) = \frac{1}{\det[\mathbf{I} + \nabla_{\mathbf{q}}\boldsymbol{\Psi}]} - 1, \tag{1.17}$$

where $\mathbf{I}$ is the identity matrix and $\nabla_{\mathbf{q}}$ is the gradient with respect to Lagrangian coordinates. This formula is exact, it follows directly from the requirement that mass be conserved as fluid elements move.

Crucially, unlike in the Eulerian picture, the singularity can be computed exactly within perturbation theory. We do not have to worry about multivaluedness of the density field because each Lagrangian point maps to exactly one Eulerian point, one fluid element occupies one position. This is a fundamental advantage of the Lagrangian approach[8].

The evolution equations for $\boldsymbol{\Psi}$ are derived from the Euler equation in Lagrangian form[3][8]:

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} + 2H\frac{\partial \boldsymbol{\Psi}}{\partial t} = -\nabla_{\mathbf{r}}\Phi(\mathbf{r}), \tag{1.18}$$

where Φ is the gravitational potential evaluated at the physical (Eulerian) position $\mathbf{r} = \mathbf{q} + \boldsymbol{\Psi}$. This equation governs how the displacement field evolves. The left side is the acceleration of a fluid element plus Hubble friction. The right side is the gravitational force.

1.6 Expansion and Solution Scheme in LPT

The key to LPT is to expand the displacement field $\mathbf{\Psi}$ as a power series[30]:

$$\mathbf{\Psi}(\mathbf{q},t) = \mathbf{\Psi}^{(1)}(\mathbf{q},t) + \mathbf{\Psi}^{(2)}(\mathbf{q},t) + \mathbf{\Psi}^{(3)}(\mathbf{q},t) + \cdots , \tag{1.19}$$

where each order is computed self consistently by solving a hierarchy of equations. This is similar to EPT in spirit, but carried out in Lagrangian coordinates with some important differences[8][1].

At first order, neglecting the nonlinear terms in the Euler equation, we obtain:

$$\frac{\partial^2 \mathbf{\Psi}^{(1)}}{\partial t^2} + 2H \frac{\partial \mathbf{\Psi}^{(1)}}{\partial t} = 0. \tag{1.20}$$

This is a homogeneous equation with no sources, simply harmonic oscillations damped by Hubble friction. It has two families of solutions: $\mathbf{\Psi}^{(1)} \propto a$ and $\mathbf{\Psi}^{(1)} \propto a^{-1}$. For structure formation, we keep the growing mode, which means the displacement grows linearly with the scale factor[8][3].

At second order, the displacement satisfies an inhomogeneous equation where the source depends on products of first order terms and the potential sourced by the first order density contrast. The general structure is:

$$\frac{\partial^2 \mathbf{\Psi}^{(2)}}{\partial t^2} + 2H \frac{\partial \mathbf{\Psi}^{(2)}}{\partial t} = f(\mathbf{\Psi}^{(1)}), \tag{1.21}$$

where f is a nonlinear functional of the first order solution. The source arises from the nonlinear coupling in the gravitational potential and the curvature of spacetime[7][8].

The advantage of this hierarchical structure is that each order can be solved in turn, making LPT computationally efficient. Moreover, analytical solutions are often possible, especially for simplified, approximate collapse. This is in contrast to EPT, where computing higher orders becomes increasingly complicated due to the spatial derivatives[3][8].

1.7 Dynamical Renormalization Group

Dynamical renormalization group (DRG) is a resummation technique originally developed in quantum field theory to handle secular growth in perturbative expansions[10][38][6]. While traditionally employed to renormalize ultraviolet divergences and running couplings in QFT, the same mathematical framework can be adapted to cosmological perturbation theory, removing uncontrolled secular growth that arise in perturbative solutions[3][30].

When solving the equations of motion for cosmological perturbations perturbatively, it is standard to expand the solution in powers of a small parameter ϵ that measures the strength of the nonlinearity or initial curvature[34][8]. In many cases, the first-order correction produces terms that grow faster than the zeroth-order solution as time evolves[31][8]. For example, a correction proportional to ϵa^2 grows much faster than the linear term $\sim a$, and as the scale factor increases, this correction becomes comparable to or larger than the leading-order contribution, invalidating the perturbative ordering. Such terms are called secular terms, and they signal that naive perturbation theory breaks down before the physically interesting regime of strong nonlinearity is reached[3][30]

The dynamical renormalization group provides a systematic way to resum secular growth by promoting parameters in the solution to running, scale-dependent functions[10][38]. The key is that the growth of the secular term can be absorbed into a redefinition of the integration constants in the zeroth-order solution[10]. By deriving a flow equation for these running parameters, we can construct a resummed solution that remains uniformly valid over a large range of scales, capturing the correct asymptotic behavior all the way up to the point of collapse or singularity formation[10].

As a concrete example of the dynamical renormalization procedure, consider a perturbative solution to some differential equation:

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y(x) + \mathcal{O}(\epsilon^3) \tag{1.22}$$

For example if the first order y_1 term is a secular term, then we can utilise the following procedure to renormalize. First we introduce an arbitrary variable ν, then we rewrite the expression for $y(x)$ as[10]:

$$y(x) = y_0(x) + \epsilon[y_1(x) - y_1(\nu) + y_1(\nu)] \tag{1.23}$$

Then the last term ($y_1(\nu)$) is absorbed into the integration constant c_1 of the first term, so c_1 passes from being a constant to being an equation of ν absorbing $y_1(\nu)$. This then makes $y(x)$ equal to[10]:

$$y(x) = y_0(c_1(\nu), x) + \epsilon[y_1(x) - y_1(\nu)] \tag{1.24}$$

Then, since the function $y(x)$ is by definiton independent of this arbitrary factor, then $\frac{dy}{d\nu} = 0$, hence:

$$\frac{dy}{d\nu} = (\frac{\partial y_0}{\partial c_1})\frac{dc_1}{d\nu} - \epsilon\frac{\partial y_1}{\partial \nu} = 0 \tag{1.25}$$

This is a differential equation in terms of ν which allows to solve for $c_1(\nu)$ giving a solution of the form $\tilde{c}_1(\nu)$[10]. Then substituting this back into the original equation we get:

$$y(x) = y_0(\tilde{c}_1(\nu), x) + \epsilon[y_1(x) - y_1(\nu)] \tag{1.26}$$

then choosing the arbitrary variable ν such that $\nu = x$, we get the final renormalized form:

$$y(x) = y_0(\tilde{c}_1(x), x) \tag{1.27}$$

This is then the final result of the application of DRG, the method returns a function $y(x)$ which is now renormalized, and has no secular terms.

1.8 Rampf and Hahn

The RG approach employed by Rampf and Hahn [31] begins by recasting the spherical-collapse equation into the form

$$r'^2 = \frac{a}{r} - \epsilon a^2, \tag{1.28}$$

where r is the physical radius, a is the scale factor, a prime denotes differentiation with respect to a, and ϵ is a dimensionless bookkeeping parameter related to the initial curvature[31]. The solution is then expanded perturbatively as

$$r(a) = r_0(a) + \epsilon r_1(a) + \epsilon^2 r_2(a) + \ldots, \tag{1.29}$$

with the zeroth-order solution given by

$$r_0(a) = a\left(1 + \frac{3c_1}{2a^{3/2}}\right)^{2/3}, \tag{1.30}$$

where c_1 is an integration constant. At successive orders in ϵ, this perturbative approach generates secular terms that grow faster than the leading-order contribution, signaling the breakdown of naive perturbation theory at late times[31].

To remedy this, Rampf and Hahn promote the integration constant c_1 to a running parameter $c_1(a)$ that evolves with the scale factor[31]. By demanding that the physical solution remain independent of an arbitrary reference scale, they derive a renormalization-group flow equation. The first-order RG-improved solution is then[31]

$$r_{1\mathrm{RG}}(a) = a\left(1 - \frac{3\epsilon a}{20}\right)^{2/3}, \tag{1.31}$$

which correctly predicts the critical exponent $2/3$ characterizing the power-law divergence of the velocity at shell-crossing. Higher-order corrections can be incorporated systematically; the second-order result is

$$r_{2\mathrm{RG}}(a) = a\left(1 - \frac{3\epsilon a}{20} - \frac{3\epsilon^2 a^2}{1120}\right)^{2/3}. \tag{1.32}$$

1.8.1 Relation to the Dynamical Renormalization Group

While Rampf and Hahn do not explicitly label their approach as a "dynamical renormalization group" (DRG) technique, the mathematical structure of their RG method aligns closely with the DRG framework outlined in Section 1.7[10][31].This is evident by the way they promote integration constants in the zeroth order solution to functions that run with scale, and impose a renormalization condition to determine the running. They are using the running constants in exactly the same way as classical DRG uses them, i.e. using them to absorb secular growth terms. This highlights the similarity between the two methods

Chapter 2

Derivations

First it is essential to verify that applying the DRG method yields coherent results. This can be done by following the Rampf and Hahn method up until the point where they apply their renormalization technique.
Following the steps section IV-A of the by Rampf and Hahn [31] we have the perturbative expansion for r given by:

$$r = r_0 - \epsilon \frac{1}{10} r_0^2 + \dots \tag{2.1}$$

To first order in ϵ, we apply the dynamical RG method discussed in the theoretical background section [10] to cancel out the secular term (r_0^2). To do so we rewrite the expansion using an arbitrary time scale τ as:

$$r = r_0(c_1, a) - \epsilon \frac{1}{10}\big(r_0^2(a) - r_0^2(\tau) + r_0^2(\tau)\big) \tag{2.2}$$

Subsequently, we redefine the variable c_1 as a function of this arbitrary time scale τ, so that it includes the second $r_0^2(\tau)$ term, giving:

$$r = r_0(c_1(\tau), a) - \epsilon \frac{1}{10}\big(r_0^2(a) - r_0^2(\tau)\big) \tag{2.3}$$

r itself shouldn't depend on the arbitrary time scale τ, hence $\frac{dr}{d\tau} = 0$. This gives the following expression:

$$\frac{dr}{d\tau} = \frac{\partial r_0}{\partial c_1}\frac{dc_1}{d\tau} - \epsilon \frac{1}{10}\frac{\partial r_0^2}{\partial \tau}. \tag{2.4}$$

The expression for r_0 is obtained by solving the differential equation:

$$(r_0')^2 = \frac{a}{r_0}, \tag{2.5}$$

which gives:

$$r_0 = a\Big(1 + \frac{3c_1}{2a^{3/2}}\Big)^{2/3}. \tag{2.6}$$

We can rewrite this expression as a binomial expansion, since we are interested in late-time regions where the secular term becomes important, and in this late time region

$a \gg 1$ and $c_1/a^{3/2} \ll 1$. Hence, doing a binomial expansion of eq (2.6) we obtain:

$$r_0 = a + \frac{c_1}{a^{1/2}} + \dots \tag{2.7}$$

Hence r_0^2 is given by:

$$r_0^2 = a^2 + \frac{c_1^2}{a} + 2c_1 a^{1/2} + \dots \tag{2.8}$$

We can then use these expressions to find the derivatives needed in eq (2.4), namely:

$$\frac{\partial r_0}{\partial c_1} = a^{-1/2} \tag{2.9}$$

and

$$\frac{\partial r_0^2}{\partial \tau} = 2\tau + c_1\tau^{-1/2}. \tag{2.10}$$

where in these expressions we have ignored all decaying modes. Since τ is an arbitrary variable, it is possible to choose the variables such that $\tau = a$, hence, using this substitution these expressions then combine to give the expanded eq (2.4):

$$\tau^{-1/2}\,\frac{dc_1}{d\tau} = \epsilon\,\frac{1}{10}\Big[2\tau + c_1\tau^{1/2}\Big]. \tag{2.11}$$

This is a linear ODE which we can solve by utilising an integrating factor $I(\tau)$, this then gives:

$$\Big(c_1 e^{-\frac{\epsilon}{10}c_1\tau} + c\Big) = \int \frac{\epsilon}{10}\,\tau^{3/2}\,\mathrm{d}\tau. \tag{2.12}$$

We then expand the exponential utilising the Taylor expansion definition of the exponential:

$$c_1 = \Big(-\frac{2\epsilon}{25}\tau^{5/2}\Big)\Big(1 + \frac{\epsilon}{10}c_1\tau\Big) + c_1. \tag{2.13}$$

Then, eliminating all higher order contributions, we get the final expression for c_1:

$$c_1 = -\frac{2\epsilon}{25}\tau^{5/2} + c_1. \tag{2.14}$$

Substituting this into the original expression for r, the secular term cancels out as we can choose the arbitrary time scale such that $\tau = a$, hence canceling out the r_0^2 terms, leaving only the first term, expressed in terms of a and the new redefined constant $c_1(\tau)$:

$$r_{\text{DRG}} = a\Big(1 + \frac{3}{2a^{3/2}}\Big(c_1 - \frac{2\epsilon a^{5/2}}{25}\Big)\Big)^{2/3}. \tag{2.15}$$

Eliminating decaying modes, this yields the following expression for r_{DRG}:

$$r_{\text{DRG}} = a\Big(1 - \frac{3}{25}\epsilon a\Big)^{2/3}. \tag{2.16}$$

Hence, it is shown, that it is possible to effectivley apply DRG to LPT. Furthermore, this result for the first order term is comparable to the method Rampf and Hahn utilised, the only difference being a coefficient of $\frac{3}{20}$ instead of one of $\frac{3}{25}$, highlighting the similarity between the two methods.

2.1 Reformulating the problem

To effectively set up a generalized formulation for the application of DRG to LPT, it is necessary to reformulate the method utilized in the Rampf and Hahn paper. More specifically, the aim is to use spherical harmonic decompositions so it is possible to initially look at lower order modes, and then slowly increase the order of the multipole, hence slowly increasing the order of the complexity of the approximation. Ultimately, this formulation will allow us to generalize the DRG implementation to l-th order, i.e. including all possible modes, making it a generalized solution to DRG.

2.2 Einstein Equations for $\ell = 0$ Mode

For the $\ell = 0$ monopole mode, the gravitational potential depends only on time and radial coordinate, $\Phi_0(\eta, r)$.

In matter-dominated cosmology with Hubble parameter $H = 2/\eta$, the time-time component of Einstein's equations for the Newtonian potential yields:

$$\Phi_0'' + \frac{6}{\eta}\Phi_0' = 0. \tag{2.17}$$

This second-order linear ODE has the general solution:

$$\Phi_0(\eta, r) = A(r) + B(r)\eta^{-5}, \tag{2.18}$$

where $A(r)$ is the constant mode and $B(r)\eta^{-5}$ is the decaying mode.

2.2.1 Constraint Equations

The Hamiltonian constraint from spatial components of Einstein's equations relates the potential to density perturbations:

$$\nabla^2\Phi_0 - 3H(\Phi_0' + H\Phi_0) = 4\pi G a^2 \bar{\rho}\delta_0, \tag{2.19}$$

where $\delta_0 = \delta\rho/\bar{\rho}$ is the density contrast.

For spherical symmetry:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_0}{dr}\right) - \frac{6}{\eta}\left(\Phi_0' + \frac{2}{\eta}\Phi_0\right) = \frac{3}{2\eta^2}\delta_0. \tag{2.20}$$

The momentum constraint from spatial-temporal components of Einstein's equations:

$$\frac{d}{dr}\left(\Phi_0' + \frac{2}{\eta}\Phi_0\right) = \frac{3}{\eta^2}v_0, \tag{2.21}$$

where $v_0(r)$ is the radial velocity.

The continuity equation from mass conservation:

$$\delta_0' + \frac{dv_0}{dr} + \frac{2}{r}v_0 - 3\Phi_0' = 0. \tag{2.22}$$

The Euler equation from fluid dynamics:

$$v_0' + \frac{2}{\eta}v_0 + \frac{d\Phi_0}{dr} = 0. \tag{2.23}$$

Together, these equations form the complete system of Eulerian perturbation equations.

2.2.2 Lagrangian Formulation

In the Lagrangian description, we track individual fluid elements labeled by their initial comoving position q. The trajectory is[8]:

$$r(\eta, q) = q + \Xi_0(\eta, q), \tag{2.24}$$

where $\Xi_0(\eta, q)$ is the radial displacement field.

The velocity along the trajectory is:

$$v_0(\eta, r) = \left.\frac{\partial \Xi_0}{\partial \eta}\right|_q = \Xi_0'(\eta, q). \tag{2.25}$$

Starting from the Euler equation, we obtain the equations of motion:

$$\frac{d^2 r}{d\eta^2} + 2H\frac{dr}{d\eta} + \left.\frac{d\Phi_0}{dr}\right|_{r=r(\eta,q)} = 0, \tag{2.26}$$

we substitute $r = q + \Xi_0$. For small displacements, the potential gradient linearizes:

$$\left.\frac{d\Phi_0}{dr}\right|_{r=q+\Xi_0} \approx \frac{d\Phi_0}{dq}. \tag{2.27}$$

The equation of motion becomes:

$$\frac{d^2\Xi_0}{d\eta^2} + 2H\frac{d\Xi_0}{d\eta} + \frac{d\Phi_0}{dq} = 0. \tag{2.28}$$

With $H = 2/\eta$:

$$\Xi_0'' + \frac{4}{\eta}\Xi_0' + \frac{d\Phi_0}{dq} = 0. \tag{2.29}$$

Solving for the general solution, we get:

With $\Phi_0(\eta, q) = A(q) + B(q)\eta^{-5}$:

$$\frac{d\Phi_0}{dq} = A'(q) + B'(q)\eta^{-5}. \tag{2.30}$$

The Lagrangian ODE becomes:

$$\Xi_0'' + \frac{4}{\eta}\Xi_0' = -A'(q) - B'(q)\eta^{-5}. \tag{2.31}$$

The homogeneous equation $u'' + (4/\eta)u' = 0$ has solutions $u_1(\eta) = 1$ and $u_2(\eta) = \eta^{-3}$. The Wronskian is:

$$W(\eta) = -3\eta^{-4}. \tag{2.32}$$

Using variation of parameters:

$$u_p = \frac{1}{3}\int \eta S(\eta) d\eta - \frac{\eta^{-3}}{3}\int \eta^4 S(\eta) d\eta, \tag{2.33}$$

where $S(\eta)$ is the "source" term.
For the $A'(q)$ component:

$$u_{p,A'} = \frac{A'(q)\eta^2}{10}. \tag{2.34}$$

For the $B'(q)\eta^{-5}$ component:

$$u_{p,B'} = -\frac{B'(q)}{25}\eta^{-3}. \tag{2.35}$$

The general solution is:

$$\Xi_0(\eta, q) = C_0(q) + \frac{1}{10}A'(q)\eta^2 - \frac{1}{25}B'(q)\eta^{-3}, \tag{2.36}$$

where $C_0(q)$ is the integration constant determined by initial conditions.
At late times:

$$\Xi_0(\eta, q) \approx C_0(q) + \frac{1}{10}A'(q)\eta^2. \tag{2.37}$$

2.2.3 Coupled Multipole System

Before continuing, it is necessary to mention that in order to simplify the approximations in this section, the ℓ values considered will be only those up until ℓ=2. The procedure utilised to derive these equations is applicable to higher order, but, including these higher order solutions complicates the system significantly.

To begin, we consider the ℓ=0 case, it is necessary to carry out a perturbative expansion to solve these equations, since ϕ_{ell} and Ξ_{ell} depend on each other and it is necessary to isolate them, to do so, we introduce bookkeeping parameter $\epsilon \ll 1$ for each multipole[29]:

$$\Phi_\ell(\eta, q) = \epsilon\phi_\ell(\eta, q), \quad \ell = 0, 1, 2, \tag{2.38}$$

with expansions:

$$\Xi_\ell(\eta, q; \tau) = \sum_{n=1}^{\infty} \epsilon^n \xi_\ell^{(n)}(\eta, q; \tau), \quad \ell = 0, 1, 2. \tag{2.39}$$

The renormalization scale τ enables Dynamical RG with running constants $c_1^{(n),\ell}(\tau)$ and $c_2^{(n),\ell}(\tau)$ for each mode.

At each order n and multipole ℓ:

$$\xi_\ell^{(n)}(\eta, q; \tau) = c_1^{(n),\ell}(\tau) + c_2^{(n),\ell}(\tau)\eta^{-3} + P_\ell^{(n)}(\eta, q; \tau). \tag{2.40}$$

Here, $\xi_\ell^{(n)}$ is the displacement at order n for multipole ℓ, depending on conformal time η, Lagrangian position q, and renormalization scale τ. The three terms represent: (1) $c_1^{(n),\ell}(\tau)$, the growing-mode amplitude (running with scale); (2) $c_2^{(n),\ell}(\tau)\eta^{-3}$, the decaying transient; (3) $P_\ell^{(n)}(\eta, q; \tau)$, the forced response to nonlinear coupling, exhibiting secular growth η^{2n}, this equation is just a generalization of the equation (2.39).

Subsequently, expanding $\phi_0(\eta, r)$ about $r = q$ using the total , i.e. the trajectory including all ℓ values being considered[31]:

$$r = q + \Xi_0 + \Xi_1 + \Xi_2, \tag{2.41}$$

where we sum only over the three active modes, i.e. we are summing and considering only the .

$$\phi_0(\eta, r) = \phi_0(\eta, q) + (r-q)\frac{d\phi_0}{dq} + \frac{(r-q)^2}{2}\frac{d^2\phi_0}{dq^2} + \cdots \tag{2.42}$$

$$= \phi_0(\eta, q) + \epsilon\left[\xi_0^{(1)} + \xi_1^{(1)} + \xi_2^{(1)}\right](q;\tau)\frac{d\phi_0}{dq} \tag{2.43}$$

$$+ \epsilon^2\left[\xi_0^{(2)} + \xi_1^{(2)} + \xi_2^{(2)}\right](q;\tau)\frac{d\phi_0}{dq} \tag{2.44}$$

$$+ \frac{\epsilon^2}{2}\left[\xi_0^{(1)} + \xi_1^{(1)} + \xi_2^{(1)}\right]^2(q;\tau)\frac{d^2\phi_0}{dq^2} + O(\epsilon^3). \tag{2.45}$$

The monopole source now includes contributions from all three modes. To consider the solutions, it is necessary to compare the orders of ϵ in the equaitons:

Order ϵ^1

$$\xi_0^{(1)''} + \frac{4}{\eta}\xi_0^{(1)'} + A'(q) + B'(q)\eta^{-5} = 0. \tag{2.46}$$

At first order, only the monopole self-source appears. No coupling from ℓ=1 or ℓ=2 at this order. The general solution is then of the form:

$$\xi_0^{(1)}(\eta, q; \tau) = c_1^{(1),0} + c_2^{(1),0}\eta^{-3} + \frac{1}{10}A'(q)\eta^2 - \frac{1}{25}B'(q)\eta^{-3}. \tag{2.47}$$

Order ϵ^2

$$\xi_0^{(2)''} + \frac{4}{\eta}\xi_0^{(2)'} + S_0^{\text{self}(2)} + S_0^{\text{cross}(2)} = 0, \tag{2.48}$$

where:

Self-source (monopole self-coupling):

$$S_0^{\text{self}(2)} = \xi_0^{(1)}(q)\phi_0'(\eta, q) = \left[c_1^{(1),0}\eta^2 + \cdots\right][A'(q) + B'(q)\eta^{-5}]. \tag{2.49}$$

Cross-mode source (coupling to dipole and quadrupole):

$$S_0^{\text{cross}(2)} = \left[\xi_1^{(1)}(q;\tau) + \xi_2^{(1)}(q;\tau)\right]\phi_0'(\eta, q). \tag{2.50}$$

Explicitly:

$$S_0^{\text{cross}(2)} = \left[c_1^{(1),1}\eta^2 + c_1^{(1),2}\eta^2 + \cdots\right][A'(q) + B'(q)\eta^{-5}]. \tag{2.51}$$

Since $c_1^{(1),1}$ and $c_1^{(1),2}$ are frozen, the cross-mode source is τ-independent.

General solution:

$$\xi_0^{(2)}(\eta, q) = c_1^{(2),0} + c_2^{(2),0}\eta^{-3} + P_0^{(2)}(\eta, q), \tag{2.52}$$

where the particular solution contains coupling terms:

$$P_0^{(2)}(\eta, q) = \frac{1}{10}\left[c_1^{(1),0} + c_1^{(1),1} + c_1^{(1),2}\right]A'(q)\eta^2 + (\text{higher-order terms}). \tag{2.53}$$

The second-order running constants freeze (sources depend only on frozen first-order constants).

Order ϵ^3

$$\xi_0^{(3)''} + \frac{4}{\eta}\xi_0^{(3)'} + S_0^{\text{self}(3)} + S_0^{\text{cross}(3)} = 0. \tag{2.54}$$

Self-source:

$$S_0^{\text{self}(3)} = \xi_0^{(2)}(q;\tau)\phi_0'(\eta, q) + \frac{1}{2}[\xi_0^{(1)}(q)]^2\phi_0''(\eta, q). \tag{2.55}$$

Since $c_1^{(2),0}$ is frozen, this contribution is τ-independent.

Cross-mode source:

$$S_0^{\text{cross}(3)} = \left[\xi_1^{(2)}(q;\tau) + \xi_2^{(2)}(q;\tau)\right]\phi_0'(\eta, q) + \frac{1}{2}[\xi_0^{(1)} + \xi_1^{(1)} + \xi_2^{(1)}]^2(q)\phi_0''(\eta, q). \tag{2.56}$$

Explicitly, the crucial terms are:

$$\left[c_1^{(2),1}(\tau)\eta^2 + c_1^{(2),2}(\tau)\eta^2 + \cdots\right][A'(q) + B'(q)\eta^{-5}]. \tag{2.57}$$

Now $c_1^{(2),1}(\tau)$ and $c_1^{(2),2}(\tau)$ are running constants that can (and must) evolve at order ϵ^3. This is where true coupling emerges.

General solution:

$$\xi_0^{(3)}(\eta, q;\tau) = c_1^{(3),0}(\tau) + c_2^{(3),0}(\tau)\eta^{-3} + P_0^{(3)}(\eta, q;\tau), \tag{2.58}$$

where the particular solution now has explicit τ-dependence through $c_1^{(2),1}(\tau)$ and $c_1^{(2),2}(\tau)$.

2.2.4 RG Equations for $\ell = 0$ with $\ell_{\max} = 2$

Orders ϵ^1 and ϵ^2:

$$\frac{dc_1^{(n),0}}{d\tau} = 0, \quad \frac{dc_2^{(n),0}}{d\tau} = 0, \quad n = 1, 2. \tag{2.59}$$

Order ϵ^3:

$$\frac{dc_1^{(3),0}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),0}}{d\tau} = -\frac{1}{2}\frac{\partial P_0^{(3)}}{\partial\tau}\bigg|_\eta - \frac{\partial P_0^{(3)}}{\partial c_1^{(2),1}}\frac{dc_1^{(2),1}}{d\tau} - \frac{\partial P_0^{(3)}}{\partial c_1^{(2),2}}\frac{dc_1^{(2),2}}{d\tau}. \tag{2.60}$$

The monopole running at third order depends on, its own particular solution derivatives via the self coupling, and the running of dipole and quadrupole second-order constants: $\frac{dc_1^{(2),1}}{d\tau}$ and $\frac{dc_1^{(2),2}}{d\tau}$, i.e. coupling to other modes.

This equation is coupled to the RG equations for ℓ=1 and ℓ=2 at third order, all three must be solved simultaneously.

2.3 Spherical Harmonic Decomposition and Einstein Equations for $\ell = 1$

Having established the monopole framework, we now extend to the dipole mode, which introduces angular structure beyond the purely radial monopole. Since the dipole involves vector fields (velocities, for instance), we must employ vector spherical harmonics, which provide a complete orthogonal basis for decomposing vector fields on the sphere[19].

To simplify, the approximations, we assume an irrotational vector field[11]. Any irrotational vector field on a sphere can be decomposed into a gradient (polar) component and a curl (axial) component:

$$V = \nabla_\Omega V^{(S)} + \hat{r} \times \nabla_\Omega V^{(A)}, \tag{2.61}$$

where ∇_Ω is the covariant derivative on the unit sphere, $V^{(S)}$ is a scalar potential, and $V^{(A)}$ is an axial potential. The vector spherical harmonics are defined as:

$$Y_{\ell m}^{(E)} \equiv \nabla_\Omega Y_{\ell m} \quad \text{(polar, from gradients)}, \tag{2.62}$$
$$Y_{\ell m}^{(B)} \equiv \hat{r} \times \nabla_\Omega Y_{\ell m} \quad \text{(axial, from curls)}. \tag{2.63}$$

These basis functions satisfy crucial orthogonality and divergence properties:

$$\nabla_\Omega \cdot Y_{\ell m}^{(E)} = -\ell(\ell+1)Y_{\ell m}, \tag{2.64}$$

$$\nabla_\Omega \cdot Y_{\ell m}^{(B)} = 0. \tag{2.65}$$

The first relation shows that the divergence of a polar vector spherical harmonic yields a scalar spherical harmonic with the mode-dependent eigenvalue factor $-\ell(\ell+1)$. This factor will appear in all divergence formulas for modes with $\ell > 0$, making it a key distinguishing feature between the monopole ($\ell = 0$, where the factor is zero) and higher multipoles.

For the dipole mode, the Lagrangian displacement field is written in terms of a scalar displacement potential $\Xi_1(\eta, q)$ multiplying the spherical harmonic $Y_{10}(\theta, \phi)$:

$$\Psi_{\ell=1}(\eta, q) = \nabla_q[\Xi_1(\eta, q)Y_{10}(\theta, \phi)], \tag{2.66}$$

where $Y_{10} = \sqrt{3/(4\pi)}\cos\theta$ and $\Xi_1(\eta, q)$ is the radial displacement potential. This represents a pure gradient (polar) mode with no axial component, reflecting the assumption of irrotational flows, by writing $\Psi = \nabla[\Xi_1 Y_{10}]$, we ensure that the displacement automatically generates only divergent flows without rotation.

In spherical coordinates, the three components of the displacement field are:

$$\Psi_r = \frac{\partial \Xi_1}{\partial q}\cos\theta, \tag{2.67}$$

$$\Psi_\theta = -\frac{\Xi_1}{q}\sin\theta, \tag{2.68}$$

$$\Psi_\phi = 0. \tag{2.69}$$

The radial component Ψ_r describes outward (or inward) motion that is stronger at the poles ($\theta = 0, \pi$) and weaker at the equator ($\theta = \pi/2$), consistent with the $\cos\theta$ angular dependence. The tangential component Ψ_θ couples the radial and angular motion: regions moving outward radially must also develop tangential flows to maintain continuity. The azimuthal component vanishes, reflecting axial symmetry around the polar axis.

For any multipole displacement field of the form $\Psi = \nabla_q[\Xi_\ell(q)Y_{\ell m}(\theta, \phi)]$, the divergence in spherical coordinates is:

$$\nabla_q \cdot \Psi = \left[\frac{1}{q^2}\frac{d}{dq}\left(q^2\frac{d\Xi_\ell}{dq}\right) - \frac{\ell(\ell+1)}{q^2}\Xi_\ell\right]Y_{\ell m}. \tag{2.70}$$

For the dipole mode with $\ell = 1$, the eigenvalue factor is $\ell(\ell+1) = 2$, yielding:

$$\nabla_q \cdot \Psi_{\ell=1} = \left[\frac{d^2\Xi_1}{dq^2} + \frac{2}{q}\frac{d\Xi_1}{dq} - \frac{2\Xi_1}{q^2}\right]\cos\theta. \tag{2.71}$$

The three terms in the brackets have distinct physical meanings, the first term $\frac{d^2\Xi_1}{dq^2}$ represents the radial curvature of the displacement, the second term $\frac{2}{q}\frac{d\Xi_1}{dq}$ arises from geometric divergence (the q^2 volume element), and the third term $-\frac{2\Xi_1}{q^2}$ is the direct consequence of the angular eigenvalue it represents how the angular structure feedsback into the radial dynamics. For comparison, the monopole divergence is $\nabla \cdot \Psi_0 = [\frac{d^2\Xi_0}{dq^2} + \frac{2}{q}\frac{d\Xi_0}{dq}]$ (with no $\ell(\ell+1)$ term), showing that the dipole dynamics are qualitatively different from spherically symmetric monopole evolution.

2.3.1 Einstein Equations and Constraints for $\ell = 1$

The temporal evolution of the dipole gravitational potential is governed by the time-time (00) component of Einstein's equations, for the $\ell = 1$ scalar mode in matter domination, the potential exhibits the same angular structure $\Phi_1(\eta, r)\cos\theta$ as the displacement field, and satisfies an evolution equation identical in form to the monopole:

$$\Phi_1'' + \frac{6}{\eta}\Phi_1' = 0. \tag{2.72}$$

This is identical to the ℓ=0 case, hence the general solution is also identical:

$$\Phi_1(\eta, r) = A(r) + B(r)\eta^{-5}, \tag{2.73}$$

where $A(r)$ is the growing mode and $B(r)\eta^{-5}$ is the decaying mode. The only difference from the monopole is that the coefficients $A(r)$ and $B(r)$ are now determined by the dipole structure.

The momentum constraint, from the spatial-temporal (0i) components of Einstein's equations, relates the potential to the velocity field:

$$\nabla(\Phi_1' + H\Phi_1) = 4\pi G a^2 \bar{\rho} v. \tag{2.74}$$

The continuity equation, expressing mass conservation, is:

$$\delta_1' + \left[\frac{d^2\Xi_1'}{dq^2} + \frac{2}{q}\frac{d\Xi_1'}{dq} - \frac{2\Xi_1'}{q^2}\right] - 3\Phi_1' = 0, \tag{2.75}$$

where the bracketed terms follow the divergence formula above. The factor $-\frac{2\Xi_1'}{q^2}$ again reflects the dipole-specific angular eigenvalue.

The structure of the Einstein equations for the dipole is fundamentally similar to the monopole, but the eigenvalue factor $\ell(\ell + 1) = 2$ modifies the effective coupling strength between displacement and density/velocity fields.

2.4 Lagrangian Equation of Motion and Solution for $\ell = 1$

From the Euler equation in Lagrangian form:

$$\frac{d^2\mathbf{\Psi}}{d\eta^2} + 2H\frac{d\mathbf{\Psi}}{d\eta} + \nabla\Phi(\eta, \mathbf{q}) = 0, \tag{2.76}$$

we decompose the dipole displacement as $\mathbf{\Psi}_{\ell=1} = \nabla_q[\Xi_1(\eta, q)Y_1^0(\theta, \phi)]$, where Ξ_1 is the radial amplitude and Y_1^0 is the dipole harmonic. Since both $\nabla\mathbf{\Psi}_{\ell=1}$ and $\nabla\Phi_{\ell=1}$ are gradients of $\ell = 1$ harmonics, they transform identically under the decomposition. Factoring out the angular dependence yields a scalar equation for the amplitude:

$$\Xi_1'' + 2H\Xi_1' + \Phi_1 = 0. \tag{2.77}$$

With $H = 2/\eta$ in an Einstein-de Sitter universe:

$$\Xi_1'' + \frac{4}{\eta}\Xi_1' + \Phi_1 = 0. \tag{2.78}$$

This is identical in form to the monopole ODE. With $\Phi_1 = A(q) + B(q)\eta^{-5}$, we solve this ODE using variation of parameters. The homogeneous equation $u'' + (4/\eta)u' = 0$ has solutions $u_1(\eta) = 1$ and $u_2(\eta) = \eta^{-3}$ with Wronskian $W(\eta) = -3\eta^{-4}$ (identical to the monopole case). Applying variation of parameters yields particular solutions with the same universal coefficients: $u_{p,A} = \frac{A\eta^2}{10}$ and $u_{p,B} = -\frac{B}{25}\eta^{-3}$.

The general solution is:

$$\Xi_1(\eta, q) = C_1(q) + \frac{1}{10}A(q)\eta^2 - \frac{1}{25}B(q)\eta^{-3}, \tag{2.79}$$

where $C_1(q)$ is an integration constant determined by initial conditions. At late times:

$$\Xi_1(\eta, q) \approx C_1(q) + \frac{1}{10}A(q)\eta^2. \tag{2.80}$$

It is interesting to note that despite the additional angular structure and the eigenvalue factor $\ell(\ell + 1) = 2$ appearing in the divergence formula, the ODE for Ξ_1 is structurally identical to the monopole ODE. This universality is the foundation enabling the multipole decomposition: all multipoles obey the same time-evolution ODE, with their differences manifesting only in the divergence operators that couple displacement to observables.

2.5 Perturbative Hierarchy and Coupled Dynamics with $\ell_{\max} = 2$

We now continue the derivation, this time considering the dipole case, i.e. the case for ℓ=1.

We introduce a bookkeeping parameter $\epsilon \ll 1$,as was done in the monopole case, and expand:

$$\Phi_1(\eta, q) = \epsilon\phi_1(\eta, q), \quad \Xi_1(\eta, q; \tau) = \sum_{n=1}^{\infty} \epsilon^n \xi_1^{(n)}(\eta, q; \tau). \tag{2.81}$$

At each order n, we promote integration constants to running functions of the renormalization scale τ:

$$\xi_1^{(n)}(\eta, q; \tau) = c_1^{(n)}(\tau) + c_2^{(n)}(\tau)\eta^{-3} + P_1^{(n)}(\eta, q; \tau), \tag{2.82}$$

where $P_1^{(n)}$ is the particular solution. In the $\ell_{\max} = 2$ truncation, the full displacement field is:

$$\Psi(\eta, q) = \Psi_0(\eta, q) + \Psi_1(\eta, q) + \Psi_2(\eta, q), \tag{2.83}$$

with each component expanded perturbatively: $\Xi_\ell(\eta, q; \tau) = \sum_{n=1}^{\infty} \epsilon^n \xi_\ell^{(n)}(\eta, q; \tau)$ for $\ell = 0, 1, 2$.

The interaction between modes enters through Taylor expansion of the dipole potential about the initial Lagrangian position:

$$r = q + \Xi_0 + \Xi_1 + \Xi_2, \tag{2.84}$$

leading to:

$$\phi_1(\eta, r) = \phi_1(\eta, q) + \epsilon \left[\xi_0^{(1)} + \xi_1^{(1)} + \xi_2^{(1)}\right](q; \tau)\frac{d\phi_1}{dq} \tag{2.85}$$

$$+ \epsilon^2 \left[\xi_0^{(2)} + \xi_1^{(2)} + \xi_2^{(2)}\right](q; \tau)\frac{d\phi_1}{dq} \tag{2.86}$$

$$+ \frac{\epsilon^2}{2} \left[\xi_0^{(1)} + \xi_1^{(1)} + \xi_2^{(1)}\right]^2 (q; \tau)\frac{d^2\phi_1}{dq^2} + O(\epsilon^3). \tag{2.87}$$

The dipole source now includes self-coupling and cross-mode coupling from monopole and quadrupole at all perturbative orders.

Order ϵ^1: At first order, only the dipole's own source appears:

$$\xi_1^{(1)''} + \frac{4}{\eta}\xi_1^{(1)'} + A(q) + B(q)\eta^{-5} = 0, \tag{2.88}$$

with solution:

$$\xi_1^{(1)}(\eta, q; \tau) = c_1^{(1)} + c_2^{(1)}\eta^{-3} + \frac{1}{10}A(q)\eta^2 - \frac{1}{25}B(q)\eta^{-3}. \tag{2.89}$$

No coupling from other multipoles occurs at this order, and RG analysis shows the constants are frozen: $\frac{dc_1^{(1)}}{d\tau} = 0$.

Order ϵ^2: The equation now contains both self-source and cross-mode source:

$$\xi_1^{(2)''} + \frac{4}{\eta}\xi_1^{(2)'} + S_1^{\text{self}(2)} + S_1^{\text{cross}(2)} = 0. \tag{2.90}$$

The self-source is:

$$S_1^{\text{self}(2)} = \xi_1^{(1)}(q)\frac{d\phi_1}{dq} = \left[c_1^{(1)}\eta^2 + c_2^{(1)}\eta^{-3} + \cdots\right][A'(q) + B'(q)\eta^{-5}], \tag{2.91}$$

while the cross-mode source from monopole and quadrupole is:

$$S_1^{\text{cross}(2)} = \left[c_1^{(1),0}\eta^2 + c_1^{(1),2}\eta^2 + \cdots\right][A'(q) + B'(q)\eta^{-5}]. \tag{2.92}$$

Since first-order constants from all modes are frozen, the total source is τ-independent, and the second-order constants remain frozen. The general solution is:

$$\xi_1^{(2)}(\eta, q) = c_1^{(2)} + c_2^{(2)}\eta^{-3} + P_1^{(2)}(\eta, q), \tag{2.93}$$

with particular solution:

$$P_1^{(2)}(\eta, q) = \frac{1}{10}\left[c_1^{(1)} + c_1^{(1),0} + c_1^{(1),2}\right]A'(q)\eta^2 + (\text{higher-order terms}). \tag{2.94}$$

The cross-mode amplitude enhancement is clear, the second-order secular growth is proportional to the sum of first-order constants from all three multipoles.

Order ϵ^3: This is where genuine running emerges, as can be seen by the following equation:

$$\xi_1^{(3)''} + \frac{4}{\eta}\xi_1^{(3)'} + S_1^{\text{self}(3)} + S_1^{\text{cross}(3)} = 0. \tag{2.95}$$

The self-source remains τ-independent:

$$S_1^{\text{self}(3)} = \xi_1^{(2)}(q)\frac{d\phi_1}{dq} + \frac{1}{2}[\xi_1^{(1)}(q)]^2\frac{d^2\phi_1}{dq^2}. \tag{2.96}$$

However, the cross-mode source now involves second-order displacements:

$$S_1^{\mathrm{cross}(3)} = \left[\xi_0^{(2)}(q;\tau) + \xi_2^{(2)}(q;\tau)\right] \frac{d\phi_1}{dq} + (\text{quadratic products}). \tag{2.97}$$

The crucial difference: $\xi_0^{(2)}(q;\tau)$ and $\xi_2^{(2)}(q;\tau)$ contain running constants $c_1^{(2),0}(\tau)$ and $c_1^{(2),2}(\tau)$ that can evolve. The cross-mode source now contains:

$$\left[c_1^{(2),0}(\tau)\eta^2 + c_1^{(2),2}(\tau)\eta^2 + \cdots\right][A'(q) + \cdots], \tag{2.98}$$

making it explicitly τ dependent. This forces the dipole third order running to depend on how the monopole and quadrupole second-order constants are evolving true coupling emerges at this order.

The secular growth pattern across orders reveals the nonlinearity, order ϵ^1 produces η^2 growth, order ϵ^2 produces resonant η^2 enhancement from coupling, and order ϵ^3 generates faster η^4 secular growth. Without DRG resummation, these secular divergences invalidate perturbation theory.

2.6 Dynamical RG: Flow Equations and Coupled System for $\ell = 1$

Physical observables must be independent of the arbitrary renormalization scale τ. This invariance principle provides the foundation for deriving RG flow equations. For each multipole $\ell \in \{0, 1, 2\}$:

$$\left.\frac{d\Xi_\ell}{d\tau}\right|_{\eta,q} = 0. \tag{2.99}$$

For the dipole, differentiating the perturbative solution with respect to τ at fixed η, q:

$$\left.\frac{d\Xi_1}{d\tau}\right|_{\eta,q} = \sum_{n=1}^{\infty} \epsilon^n \left[\left.\frac{\partial \xi_1^{(n)}}{\partial \tau}\right|_\eta + \sum_{m=1}^{n}\left(\frac{\partial \xi_1^{(n)}}{\partial c_1^{(m),1}}\frac{dc_1^{(m),1}}{d\tau} + \frac{\partial \xi_1^{(n)}}{\partial c_2^{(m),1}}\frac{dc_2^{(m),1}}{d\tau}\right) + \sum_{\ell' \in \{0,2\}}\sum_{m=1}^{n-1}\frac{\partial \xi_1^{(n)}}{\partial \xi_{\ell'}^{(m)}}\frac{d\xi_{\ell'}^{(m)}}{d\tau}\right] = 0. \tag{2.100}$$

At each order n, this yields a constraint equation. The particular solution $P_1^{(n)}(\eta, q; \tau)$ depends on τ through self-sources from $\xi_1^{(m)}$ $(m < n)$ and cross-mode sources from $\xi_0^{(m)}$ and $\xi_2^{(m)}$ $(m < n)$.

Orders ϵ^1 and ϵ^2: Analysis of the constraint equations shows that the solution is to freeze all running constants:

$$\frac{dc_1^{(n),1}}{d\tau} = 0, \quad \frac{dc_2^{(n),1}}{d\tau} = 0, \quad n = 1, 2. \tag{2.101}$$

This occurs because at these orders, the sources depend only on frozen first-order constants from all modes, making the particular solutions τ-independent.

Order ϵ^3 and higher: True running begins when sources contain second-order displacements with running constants. The flow equation for the dipole is:

$$\frac{dc_1^{(3)}}{d\tau} + \eta^{-3}\frac{dc_2^{(3)}}{d\tau} = -\frac{1}{2}\frac{\partial P_1^{(3)}}{\partial \tau}\bigg|_\eta - \frac{\partial P_1^{(3)}}{\partial c_1^{(2),0}}\frac{dc_1^{(2),0}}{d\tau} - \frac{\partial P_1^{(3)}}{\partial c_1^{(2),2}}\frac{dc_1^{(2),2}}{d\tau}. \tag{2.102}$$

This equation is coupled to the corresponding equations for monopole and quadrupole: the running of dipole constants depends on the running rates of second-order constants from the other multipoles. For the $\ell_{\max} = 2$ system, there are exactly 3 coupled first-order ODEs at order ϵ^3:

$$\frac{dc_1^{(3),0}}{d\tau} = f_0^{(3)}\left(\tau, \eta, q, c_1^{(2),\ell}(\tau), \frac{dc_1^{(2),1}}{d\tau}, \frac{dc_1^{(2),2}}{d\tau}\right), \tag{2.103}$$

$$\frac{dc_1^{(3),1}}{d\tau} = f_1^{(3)}\left(\tau, \eta, q, c_1^{(2),\ell}(\tau), \frac{dc_1^{(2),0}}{d\tau}, \frac{dc_1^{(2),2}}{d\tau}\right), \tag{2.104}$$

$$\frac{dc_1^{(3),2}}{d\tau} = f_2^{(3)}\left(\tau, \eta, q, c_1^{(2),\ell}(\tau), \frac{dc_1^{(2),0}}{d\tau}, \frac{dc_1^{(2),1}}{d\tau}\right), \tag{2.105}$$

(with analogous equations for $c_2^{(3),\ell}(\tau)$).

2.7 The $\ell = 2$ Quadrupole Mode and Universal Coupling Behavior

The $\ell = 2$ quadrupole mode represents anisotropic deformation: tidal stretching and compression with angular structure $Y_{20}(\theta, \phi) = \sqrt{5/(16\pi)}(3\cos^2\theta - 1)$. In the coupled system with $\ell_{\max} = 2$, the quadrupole is both sourced by and sources the monopole and dipole modes.

The Lagrangian displacement for $\ell = 2$ is:

$$\Psi_{\ell=2}(q, \eta) = \nabla_q[\Xi_2(q, \eta)Y_{20}(\theta, \phi)], \tag{2.106}$$

with divergence:

$$\nabla_q \cdot \Psi_{\ell=2} = \left[\frac{d^2\Xi_2}{dq^2} + \frac{2}{q}\frac{d\Xi_2}{dq} - \frac{6\Xi_2}{q^2}\right]Y_{20}. \tag{2.107}$$

The eigenvalue factor $\ell(\ell+1) = 6$ appears in the third term, distinguishing the quadrupole from lower multipoles ($\ell(\ell+1) = 0$ for monopole, $\ell(\ell+1) = 2$ for dipole). This factor modulates how the angular structure feedback into radial dynamics and determines the coupling strength of the quadrupole to other modes through the Poisson equation.

Following the structure established in previous sections, the $\ell = 2$ displacement potential satisfies the universal ODE:

$$\Xi_2'' + \frac{4}{\eta}\Xi_2' + A_2(q) + B_2(q)\eta^{-5} = 0, \tag{2.108}$$

with general solution:

$$\Xi_2(\eta, q) = c_1^{(0),2} + c_2^{(0),2}\eta^{-3} + \frac{1}{10}A_2(q)\eta^2 - \frac{1}{25}B_2(q)\eta^{-3}. \tag{2.109}$$

At late times:

$$\Xi_2(\eta, q) \approx c_1^{(0),2} + \frac{1}{10}A_2(q)\eta^2. \tag{2.110}$$

The integration constants $c_1^{(0),2}$ and $c_2^{(0),2}$ are determined by initial conditions on the quadrupole displacement and velocity, just as for $\ell = 0$ and $\ell = 1$. The fundemental ODE is identical for all three modes.

The quadrupole expansion follows the same perturbative structure as monopole and dipole. At **order** ϵ^1, no sources exist and the quadrupole evolves independently with frozen constants. At **order** ϵ^2, sources appear that split into self-coupling and cross-mode coupling:

Self-source:

$$S_2^{\text{self}(2)} = \xi_2^{(1)}(q)\frac{d\phi_2}{dq} \propto \left[c_1^{(1),2}\eta^2 + \cdots\right][A_2'(q) + B_2'(q)\eta^{-5}]. \tag{2.111}$$

Cross-mode source (from monopole and dipole):

$$S_2^{\text{cross}(2)} = \left[\xi_0^{(1)}(q) + \xi_1^{(1)}(q)\right]\frac{d\phi_2}{dq} \propto \left[c_1^{(1),0}\eta^2 + c_1^{(1),1}\eta^2 + \cdots\right][A_2'(q) + B_2'(q)\eta^{-5}]. \tag{2.112}$$

Both sources depend only on frozen first-order constants, so the second-order constants remain frozen. The particular solution exhibits secular growth:

$$P_2^{(2)}(\eta, q) = \frac{1}{10}\left[c_1^{(1),2} + c_1^{(1),0} + c_1^{(1),1}\right]A_2'(q)\eta^2 + \mathcal{O}(\eta^3) \tag{2.113}$$

At **order** ϵ^3, the self-source remains τ-independent, but the cross-mode source becomes crucial:

$$S_2^{\mathrm{cross}(3)} = \left[\xi_0^{(2)}(q;\tau) + \xi_1^{(2)}(q;\tau)\right]\frac{d\phi_2}{dq} + (\text{quadratic products}). \tag{2.114}$$

Now $\xi_0^{(2)}(q;\tau)$ and $\xi_1^{(2)}(q;\tau)$ contain running constants $c_1^{(2),0}(\tau)$ and $c_1^{(2),1}(\tau)$ from monopole and dipole, making the quadrupole third-order source explicitly τ-dependent. The quadrupole's running is thus forced to depend on how monopole and dipole are running completing the three mode coupling.

2.8 Universal Coupling Behavior and Running Hierarchy

2.8.1 Complete Running Pattern for $\ell_{\max} = 2$

The complete picture emerges when examining all three multipoles simultaneously:

Order ϵ^1 (All Modes Decouple):

All running constants freeze with no coupling:

$$\frac{dc_1^{(1),\ell}}{d\tau} = 0, \quad \frac{dc_2^{(1),\ell}}{d\tau} = 0, \quad \ell = 0, 1, 2. \tag{2.115}$$

Each multipole evolves with only its own source.

Order ϵ^2 (Coupling Present, But Constants Frozen):

Cross-mode sources appear for all three modes. The monopole is sourced by dipole and quadrupole, the dipole by monopole and quadrupole, and the quadrupole by monopole and dipole. However, since all sources depend only on frozen first-order constants, they are τ-independent:

$$\frac{dc_1^{(2),\ell}}{d\tau} = 0, \quad \frac{dc_2^{(2),\ell}}{d\tau} = 0, \quad \ell = 0, 1, 2. \tag{2.116}$$

Cross-mode coupling is present in source terms and particular solutions (their amplitudes are enhanced by contributions from all modes), but the RG evolution of constants remains frozen. **Order ϵ^3 (True Coupled Running):**

At this order, cross-mode sources contain second-order displacements from other multipoles, which now carry running constants. The RG system becomes fully coupled. For each multipole ℓ:

$$\frac{dc_1^{(3),\ell}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),\ell}}{d\tau} = -\frac{1}{2}\frac{\partial P_\ell^{(3)}}{\partial \tau}\bigg|_\eta - \sum_{\ell' \neq \ell} \frac{\partial P_\ell^{(3)}}{\partial c_1^{(2),\ell'}}\frac{dc_1^{(2),\ell'}}{d\tau} \quad . \tag{2.117}$$

This is a system of three coupled ODEs monopole running depends on dipole and quadrupole running, dipole depends on monopole and quadrupole running, and quadrupole depends on monopole and dipole running. The three modes evolve synchronously, with their RG flows intertwined through the partial derivatives that dictate how each mode's particular solution responds to changes in other modes' integration constants.

The physical picture is now complete, at early perturbative orders, modes couple in their source terms but evolve their constants independently. At ϵ^3, the coupling becomes dynamical the running of each mode is forced to adjust to preserve physical invariance under renormalization scale changes, creating a unified system where all three multipoles must run together.

2.9 Coupled RG Equations, Multipole Comparison, and Complete System Implementation

At order ϵ^3, three coupled RG constraint equations emerge from the physical invariance requirement $\frac{d\Xi_\ell}{d\tau}|_{\eta,q} = 0$ for each multipole:

For ℓ=0:

$$\frac{dc_1^{(3),0}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),0}}{d\tau} = -\frac{1}{2}\frac{\partial P_0^{(3)}}{\partial \tau}\bigg|_\eta - \frac{\partial P_0^{(3)}}{\partial \xi_1^{(2)}}\frac{d\xi_1^{(2)}}{d\tau} - \frac{\partial P_0^{(3)}}{\partial \xi_2^{(2)}}\frac{d\xi_2^{(2)}}{d\tau}. \tag{2.118}$$

For ℓ=1:

$$\frac{dc_1^{(3),1}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),1}}{d\tau} = -\frac{1}{2}\frac{\partial P_1^{(3)}}{\partial \tau}\bigg|_\eta - \frac{\partial P_1^{(3)}}{\partial \xi_0^{(2)}}\frac{d\xi_0^{(2)}}{d\tau} - \frac{\partial P_1^{(3)}}{\partial \xi_2^{(2)}}\frac{d\xi_2^{(2)}}{d\tau}. \tag{2.119}$$

For ℓ=2:

$$\frac{dc_1^{(3),2}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),2}}{d\tau} = -\frac{1}{2}\frac{\partial P_2^{(3)}}{\partial \tau}\bigg|_\eta - \frac{\partial P_2^{(3)}}{\partial \xi_0^{(2)}}\frac{d\xi_0^{(2)}}{d\tau} - \frac{\partial P_2^{(3)}}{\partial \xi_1^{(2)}}\frac{d\xi_1^{(2)}}{d\tau}. \tag{2.120}$$

The partial derivatives $\frac{\partial P_\ell^{(3)}}{\partial \xi_{\ell'}^{(2)}}$ dictate how each multipole's particular solution responds to changes in other modes' displacement fields. These coupling coefficients, weighted by the τ evolution rates of other modes' displacements, force all three multipoles to evolve their constants in a coordinated, mutually-dependent fashion. The system is closed: three equations for three sets of running constants (growing and decaying modes for each multipole), fully determining the coupled evolution.

2.10 Complete Framework for Implementation

The complete coupled multipole system with $\ell_{\max} = 2$ is:

$$\Xi_\ell(\eta, q; \tau) = \sum_{n=1}^{\infty} \epsilon^n \xi_\ell^{(n)}(\eta, q; \tau), \quad \ell = 0, 1, 2, \tag{2.121}$$

where each order has the universal structure:

$$\xi_\ell^{(n)}(\eta, q; \tau) = c_1^{(n),\ell}(\tau) + c_2^{(n),\ell}(\tau)\eta^{-3} + P_\ell^{(n)}(\eta, q; \tau). \tag{2.122}$$

Physical invariance under renormalization scale changes requires:

$$\left.\frac{d\Xi_\ell}{d\tau}\right|_{\eta,q} = 0, \quad \ell = 0, 1, 2. \tag{2.123}$$

These coupled invariance conditions determine the RG running of all constants.

2.10.1 Running Hierarchy and Constraint Equations

Orders ϵ^1 and ϵ^2 (Frozen for All Modes):

All running constants freeze despite cross-mode sources:

$$c_1^{(n),\ell}(\tau) = c_1^{(n),\ell} = \text{const}, \quad c_2^{(n),\ell}(\tau) = c_2^{(n),\ell} = \text{const} \tag{2.124}$$

for: n = 1, 2 and ℓ = 0, 1, 2

Order ϵ^3 and Higher (Coupled Running):

Running constants evolve according to:

$$\frac{dc_1^{(n),\ell}}{d\tau} + \eta^{-3}\frac{dc_2^{(n),\ell}}{d\tau} = -\frac{1}{2}\left.\frac{\partial P_\ell^{(n)}}{\partial\tau}\right|_\eta - \sum_{\ell' \neq \ell} \frac{\partial P_\ell^{(n)}}{\partial \xi_{\ell'}^{(n-1)}} \frac{d\xi_{\ell'}^{(n-1)}}{d\tau}, \quad n \geq 3, \quad \ell = 0, 1, 2. \tag{2.125}$$

For $n = 3$, this is a system of 3 coupled first-order ODEs. Higher orders follow the same structure with increasing complexity in source terms.

2.10.2 Initial Conditions

At the reference scale $\tau_0 = \eta_0$, determine all running constants by matching to initial perturbations. For each multipole ℓ and order n:

$$\xi_\ell^{(n)}(\eta_0, q; \eta_0) = c_1^{(n),\ell} + c_2^{(n),\ell}\eta_0^{-3} + P_\ell^{(n)}(\eta_0, q; \eta_0) = \xi_{\ell,0}^{(n)}(q), \tag{2.126}$$

$$\left.\frac{\partial \xi_\ell^{(n)}}{\partial \eta}\right|_{\eta_0} = -3c_2^{(n),\ell}\eta_0^{-4} + P_\ell^{(n)\prime}(\eta_0, q; \eta_0) = v_{\ell,0}^{(n)}(q), \tag{2.127}$$

where $\xi_{\ell,0}^{(n)}$ and $v_{\ell,0}^{(n)}$ are the initial displacement and velocity profiles for order n and multipole ℓ. These two equations per mode and order determine both $c_1^{(n),\ell}$ and $c_2^{(n),\ell}$ at initialization.

2.10.3 Particular Solution Computation and Coupled Integration

For each order n, multipole ℓ, and renormalization scale τ, the particular solution is:

$$P_\ell^{(n)}(\eta, q; \tau) = \frac{1}{3} \int_{\eta_*}^{\eta} \eta' S_\ell^{(n)}(\eta', q; \tau) d\eta' - \frac{\eta^{-3}}{3} \int_{\eta_*}^{\eta} \eta'^4 S_\ell^{(n)}(\eta', q; \tau) d\eta', \tag{2.128}$$

where the source $S_\ell^{(n)}(\eta, q; \tau)$ contains self-coupling (interactions of multipole ℓ with itself at lower orders) and cross-mode coupling (interactions with other multipoles). For $n \geq 3$, the source is explicitly τ-dependent through running constants from all modes.

Implementation proceeds as follows: (1) Compute initial constants at $\tau_0 = \eta_0$ from matching. (2) Integrate the coupled RG ODEs from τ_0 to the target scale τ, computing running constants $c_1^{(n),\ell}(\tau)$ and $c_2^{(n),\ell}(\tau)$ at each τ. (3) For each τ value, compute the updated particular solutions using the current running constants. (4) Evaluate the resummed displacement at the target time:

$$\Xi_\ell^{\mathrm{DRG}}(\eta, q) = \sum_{n=1}^{N_{\max}} \epsilon^n \left[c_1^{(n),\ell}(\eta) + c_2^{(n),\ell}(\eta)\eta^{-3} + P_\ell^{(n)}(\eta, q; \eta) \right], \quad \ell = 0, 1, 2. \tag{2.129}$$

Here, $N_{\max}$ is the maximum perturbative order kept, and the choice $\tau = \eta$. The resummed solution is uniformly valid over large ranges of time, overcoming the secular divergences of naive perturbation theory.

Chapter 3

Discussion

The equations obtained in the derivation section of this paper are simply the basis of the development of DRG as a solution to LPT divergence issues. This method has been shown to be a functional and effective way to solve the major issues plaguing LPT, as is shown by Rampf and Hahn's paper. Throughout this paper, it has been shown that the approximate dynamical renormalization group theory which Rampf and Hahn utilise performs incredibly well in making up for the issues of the classic LPT. We improved upon this method by applying an actual DRG procedure, demonstrating the utility of introducing an arbitrary renormalization scale and promoting integration constants to running parameters to absorb and resum secular growth. This development places DRG on a more rigorous theoretical footing, with explicit demonstration of how secular terms can be systematically eliminated from the perturbative solution.

This thesis has, furthermore, established the complete theoretical and mathematical framework for applying Dynamical Renormalization Group methods to Lagrangian Perturbation Theory across multipole modes.

We have shown that all multipole modes $\ell = 0, 1, 2$ satisfy an identical linearized Lagrangian ODE in conformal time.

$$\Xi_\ell'' + \frac{4}{\eta}\Xi_\ell' + \Phi_\ell = 0, \tag{3.1}$$

with universal homogeneous basis $\{1, \eta^{-3}\}$, Wronskian $W = -3\eta^{-4}$, and particular solution coefficients $(1/10, -1/25)$. This universality is the foundation enabling systematic multipole analysis: the time-evolution structure is decoupled from angular complexity, allowing clean separation of multipole-specific divergence operators and coupling mechanisms.

Furthermore, we have derived the complete perturbative expansions for all three active multipoles

$$\xi_\ell^{(n)}(\eta, q; \tau) = c_1^{(n),\ell}(\tau) + c_2^{(n),\ell}(\tau)\eta^{-3} + P_\ell^{(n)}(\eta, q; \tau), \quad \ell = 0, 1, 2, \quad n \geq 1. \tag{3.2}$$

For each order n, secular terms grow as η^{2n} (or equivalently a^{2n} in scale-factor form), with amplitudes enhanced by cross-mode coupling. At orders ϵ^1 and ϵ^2, sources depend only on frozen first-order constants, keeping running constants frozen. At order ϵ^3 and higher, sources contain running constants from other multipoles, forcing synchronized running across all three modes.

Finally, we have obtained the complete coupled DRG equations at order ϵ^3. The physical invariance requirement $\frac{d\Xi_\ell}{d\tau}|_{\eta,q} = 0$ yields three coupled first-order ODEs:

$$\frac{dc_1^{(3),\ell}}{d\tau} + \eta^{-3}\frac{dc_2^{(3),\ell}}{d\tau} = -\frac{1}{2}\frac{\partial P_\ell^{(3)}}{\partial \tau}\bigg|_\eta - \sum_{\ell' \neq \ell} \frac{\partial P_\ell^{(3)}}{\partial \xi_{\ell'}^{(2)}} \frac{d\xi_{\ell'}^{(2)}}{d\tau}, \quad \ell = 0, 1, 2. \tag{3.3}$$

The partial derivatives $\frac{\partial P_\ell^{(3)}}{\partial \xi_{\ell'}^{(2)}}$ form a 3×3 coupling matrix encoding mode-to-mode feedback.

The derivations also provide several consistency checks and demonstrations of completeness. All three multipoles exhibit identical running behavior, frozen at orders ϵ^1-ϵ^2 despite coupling in source terms, coupled running at order ϵ^3 and beyond. This pattern is independent of the divergence eigenvalue $\ell(\ell+1)$, indicating it is a fundamental feature of the EdS dynamics in Lagrangian coordinates. For $\ell = 0$, the framework recovers the spherically symmetric case, reproducing results consistent with Rampf-Hahn renormalization. The critical exponent $2/3$ for collapse time divergence emerges naturally from the RG resummation.

3.1 Research proposal

The complete theoretical framework now enables a clear, step-by-step program for numerical implementation, validation, and generalization:

3.1.1 Step 1: Initial Conditions and RG Scale Setup

At the reference scale $\tau_0 = \eta_0$ (early times before significant nonlinearity), match all perturbative orders and all multipoles to initial conditions:

$$\xi_\ell^{(n)}(\eta_0, q; \eta_0) = \xi_{\ell,0}^{(n)}(q), \tag{3.4}$$

$$\left.\frac{\partial \xi_\ell^{(n)}}{\partial \eta}\right|_{\eta_0} = v_{\ell,0}^{(n)}(q), \tag{3.5}$$

for $\ell = 0, 1, 2$ and $n = 1, 2, \ldots, N_{\max}$ (where $N_{\max}$ is the maximum perturbative order kept, typically 2-4). These $2 \times 3 \times N_{\max}$ equations determine all running constants at initialization:

$$c_1^{(n),\ell}(\eta_0), \quad c_2^{(n),\ell}(\eta_0), \quad \ell = 0, 1, 2, \quad n = 1, \ldots, N_{\max}. \tag{3.6}$$

3.1.2 Step 2: Solve Coupled RG System from τ_0 to Target Time

Orders ϵ^1 and ϵ^2 have frozen constants throughout. Order ϵ^3 (and potentially higher orders) requires integrating the coupled RG ODEs. Starting from $\tau_0 = \eta_0$, integrate to the target time $\tau = \eta$ using standard numerical methods.

$$\frac{d\mathbf{c}^{(3),\ell}(\tau)}{d\tau} = \mathbf{f}^{(3),\ell}(\tau, \eta, q, \{\mathbf{c}^{(2),\ell'}\}, \{\dot{\mathbf{c}}^{(2),\ell'}\}), \quad \ell = 0, 1, 2. \tag{3.7}$$

The coupling matrix $\mathbf{f}^{(3),\ell}$ involves derivatives of the particular solutions and running rates of other modes. Higher orders follow the same pattern with more complex source structures.

3.1.3 Step 3: Compute Resummed Displacements and Observables

At each evaluation time η, the running constants have been evolved from τ_0 to $\tau = \eta$. Compute the DRG-improved displacements for all multipoles:

$$\Xi_\ell^{\mathrm{DRG}}(\eta, q) = \sum_{n=1}^{N_{\max}} \epsilon^n \left[c_1^{(n),\ell}(\eta) + c_2^{(n),\ell}(\eta)\eta^{-3} + P_\ell^{(n)}(\eta, q; \eta)\right] \tag{3.8}$$

From these displacements, compute observables: density contrasts using the multipole divergence formulas:

$$\delta_\ell(\eta, q) \approx -3\left[\frac{d^2\Xi_\ell}{dq^2} + \frac{2}{q}\frac{d\Xi_\ell}{dq} - \frac{\ell(\ell+1)}{q^2}\Xi_\ell\right] Y_{\ell m}, \tag{3.9}$$

$$v_\ell(\eta, q) = \frac{\partial \Xi_\ell}{\partial \eta}, \tag{3.10}$$

$$\rho(\eta, q) \propto [\det(\delta_{ij} + \frac{\partial \Xi_i}{\partial q_j})]^{-1}. \tag{3.11}$$

The collapse time $a_c(\ell)$ for each multipole is determined by the point where the Jacobian becomes singular (first shell-crossing).

3.1.4 Step 4: Validation Against Exact Solutions

Systematic comparison with exact numerical integration of the coupled ODE system:

$$\frac{\Delta a_c(\ell)}{a_c(\ell)} = \frac{|a_c^{\mathrm{DRG}}(\ell) - a_c^{\mathrm{exact}}(\ell)|}{a_c^{\mathrm{exact}}(\ell)}, \quad \ell = 0, 1, 2. \tag{3.12}$$

3.1.5 Step 5: Generalization to Arbitrary $\ell_{\max}$

Once the three-mode system is validated, extension to higher multipoles becomes straightforward. The universal pattern, universal ODE, frozen running at ϵ^1-ϵ^2, coupled running at ϵ^3 scales directly to $\ell_{\max} = 3, 4, \ldots$. The steps are:

1. Identify the universal eigenvalue structure $\lambda_\ell = \ell(\ell+1)$ and its role in divergence operators and source coupling
2. Verify that secular growth rates η^{2n} at order ϵ^n are independent of ℓ
3. Formulate convergence criteria: determine $\ell_{\max}$ needed to achieve specified accuracy (e.g., 0.1% in collapse time)
4. Establish a general computational prescription valid for any multipole truncation

3.1.6 Step 6: Extension to Realistic Initial Conditions

The homogeneous-collapse calculations provide the theoretical template. Extensions to realistic structure formation require:

1. Truncation criteria: determine when higher-ℓ multipoles become negligible
2. Connection to random-field initial conditions[20]

3.2 Conclusion: Complete Framework Ready for Implementation

This thesis has delivered the complete theoretical and mathematical foundation for applying Dynamical Renormalization Group methods to multipole Lagrangian Perturbation Theory. The explicit universal ODE structure, systematic perturbative expansions with identified secular terms, and coupled RG equations at order ϵ^3 provide all necessary input for numerical and analytical continuation. The natural path forward is clear: solve the

RG equations, validate the solutions, and generalize to arbitrary multipole order. The result will be a new level of accuracy and control in predictions of gravitational collapse and the large-scale structure formation in the universe.

Chapter 4

Conclusion

This work has provided the complete set of equations and perturbative structures needed to begin the DRG programme for homogeneous LPT in multipole space.

Starting from the Rampf and Hahn derivation, and it was shown that the their derivations could be modfied utilsing a complete DRG method. The results of this derivation were similar to those already established by Rampf and Hahn, showing the consistency of this derivation method. Subsequently, the basis for applying this same DRG method to more complex cases was derived. To do this, spherical decompositions were utilised, and increasingly complex modes utilised, resulting in the setting up the final equations for DRG to be applied to the ℓ=0,1,2 cases, showing that it is possible to effectively apply DRG to these cases, and setting up the equations for future works. This paper, hence introduced a new renormalization method for LPT, deriving the basis equations for this renormalization method

The next steps following on from these basis equations were detailed in the research proposal section, showing how this project could be continued to effectively derive the DRG solutions for these modes (i.e. ℓ=0,1,2). Furthermore, larger scopes of this project would be to derive the case for some general ℓ, hence allowing for more and more modes to be included in these calculations of LPT. Managing to include more complex modes in the DRG calculation of LPT, could prove incredibly important in theoretical predicitons concerning the formation of large-scale structure, and this new renormalization method could prove to be a major advancement in the field of large scale structure formation.

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