A Journey Through Elliptic Units for Complex Cubic Fields

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To my grandma

Abstract

This master thesis is written with the aim of understanding the 2023 work of Nicolas Bergeron, Pierre Charollois, and Luis E. García on elliptic units for complex cubic fields. On the one hand, we supply numerical and computational details that are only briefly indicated or omitted in their original paper. On the other hand, since their results are to a large extent conjectural at the current stage, we collect and review the relevant classical analogues in the setting of imaginary quadratic fields, in order to place their constructions in context and to gain a clearer conceptual understanding of their work.

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"Denkend aan Holland
zie ik brede rivieren
traag door oneindig
laagland gaan,
rijen ondenkbaar
ijle populieren
als hoge pluimen
aan den einder staan;
en in de geweldige
ruimte verzonken
de boerderijen
verspreid door het land,
boomgroepen, dorpen,
geknotte torens,
kerken en olmen
in een groots verband."

Hendrik Marsman,
Herinnering aan Holland

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(Since my family does not read English, this paragraph is written in Chinese—even though our kitten Yunibao is a British Shorthair.)

Contents

Chapter 1

Introduction

„Wir müssen wissen, wir werden wissen.“

David Hilbert,
Address to the Society of German Scientists and Physicians

Let $G_{\mathbf{Q}}$ denote the absolute Galois group of $\mathbf{Q}$. It is one of the ultimate objects of interest in algebraic number theory. If one considers only its abelianization $G_{\mathbf{Q}}^{\mathrm{ab}}$, then its structure encodes all abelian extensions of $\mathbf{Q}$. This is precisely the content of class field theory, a collection of results developed through the work of Hilbert, Furtwängler, Takagi, Hasse, Artin, and others in the early twentieth century.

As is often the case in post-Hilbert mathematics, one can prove the existence of certain abstract objects while finding it difficult to construct them explicitly. Hilbert's twelfth problem asks for explicit constructions of the class fields of a given number field. At present, this problem is understood only in the cases of $\mathbf{Q}$ and imaginary quadratic fields. More precisely, for $\mathbf{Q}$, one has the celebrated Kronecker–Weber theorem, while for imaginary quadratic fields, the theory of complex multiplication provides the desired explicit class field theory. For more general number fields, however, the existence of infinitely many units creates substantial difficulties, and an analogous explicit theory remains largely out of reach.

Important insight into this problem was provided by Stark in the 1970s, in the form of a series of conjectures relating special values of Artin L-functions to arithmetic invariants. More precisely, these conjectures concern the leading terms in the Taylor expansion at $s = 0$ of the Artin L-functions attached to Galois extensions F/K of number fields. In the case where F/K is abelian and the corresponding L-function vanishes to order one at $s = 0$, one obtains the rank one abelian Stark conjecture. In this setting, Stark predicts that the first derivative at $s = 0$ is given by the logarithm of a distinguished algebraic unit, now called a Stark unit, which may be used as an explicit generator of a class field.

It is not surprising that the cases in which the rank one abelian Stark conjecture is best understood coincide with the cases in which explicit class field theory is already available, namely $\mathbf{Q}$ and imaginary quadratic fields. In the imaginary quadratic case, the theory of complex multiplication provides an explicit construction of elliptic units. By further applying the Kronecker limit formula, one obtains an elliptic unit that is a Stark unit. Although Stark's conjectures remain open in general, they have been supported by extensive numerical evidence and have also served as a practical guide for explicit constructions of class fields.

A striking recent development was made in 2023 by Bergeron, Charollois, and Garcia. Using the elliptic gamma function, together with arithmetic data intrinsic to complex cubic fields, they constructed a family of units for such fields and conjectured that these units satisfy a Shimura reciprocity law. They further established a new Kronecker limit formula, which connects these units to Stark units in the setting of complex cubic fields. Their work thus provides new evidence toward both Hilbert's twelfth problem and the rank one abelian Stark conjecture in this context.

The analogy with the imaginary quadratic case is particularly compelling: the units introduced by Bergeron, Charollois, and Garcia share several features with the classical elliptic units arising from complex multiplication. For this reason, they call them *elliptic units for complex cubic fields*. This suggests that, for complex cubic fields, one may hope for a theory analogous to complex multiplication, and more broadly for a new chapter in explicit class field theory.

This thesis is devoted to the study of their construction and of the classical theories that motivate and surround it.

Table 1.1: Elliptic units: quadratic imaginary vs. complex cubic (conjectural)

Aspect	**Quadratic imaginary fields**	**Complex cubic fields**
Construction	Definition 2.1.1, Proposition 2.2.4	Definition 4.3.15
Distribution relations	Theorem 2.1.7	Proposition 4.3.10
Norm-compatibility / Euler system	Corollary 2.1.8	Conjecture 5.4.1
Galois action / reciprocity	Proposition 2.1.3	Conjecture 5.2.1
Links to L-values / partial zeta	Theorem 2.2.13, Corollary 3.3.3	Theorem 5.3.2
Links to Stark units	(3.4)	(5.1)

The thesis is organized as follows.

- **Chapter 2.** We review the classical theory of elliptic units.
- **Chapter 3.** We gather some basic facts related to the rank one abelian Stark conjecture, and we provide a sketch of proof in the setting of imaginary quadratic fields.
- **Chapter 4.** We give a detailed account of how to construct the conjectural elliptic units and how to compute them numerically in order to test our conjecture.

- **Chapter 5.** We carry out numerical verifications for a range of examples, discuss the conjecture and the new Kronecker limit formula due to Bergeron, Charollois, and Garcia, and formulate a new conjecture.
- **Chapter 6.** We conclude by offering some tentative ideas.

Chapter 2

Classical theory of elliptic units

> "There was never going to be a one-line proof. Nor do proofs come just because one has been born with mathematical perfect pitch. There is no such thing. One has to spend years mastering the problem so that it becomes second nature. Then, and only then, after years of preparation is one's intuition so strong that the answer can come in a flash."
>
> Andrew Wiles,
> *Abel Prize Acceptance Speech*

In this chapter, our main goal is to follow [Rub1999] and provide a somewhat self-contained account of the theory of elliptic units. At several points, however, we shall need to appeal to results from [Sil1994] on the theory of complex multiplication. Much of what is done here amounts to expanding some of the overly concise proofs in [Rub1999] and supplementing them with relevant background from other sources. Since the theory of elliptic units for complex cubic fields is still largely conjectural, we hope that such an exposition will be useful for understanding the work of [BCG2023], and perhaps even for suggesting possible approaches to the conjectures formulated there. Any errors are the author's own.

2.1 | The algebraic side

Throughout this chapter, we fix an imaginary quadratic field K with ring of integers $\mathcal{O}_K$, an elliptic curve $E/\mathbf{C}$ with complex multiplication by $\mathcal{O}_K$, and a nontrivial ideal $\mathfrak{a} \subset \mathcal{O}_K$ that is coprime to 6. Following Rubin, we also assume that K has class number 1 to simplify the exposition. For the general case, see [dS1987].

Definition 2.1.1. Choose a Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_1, a_2, a_3, a_4, a_6 \in \mathbf{C}$$

for E. Define a rational function on E by

$$\Theta_{E,\mathfrak{a}} := \alpha^{-12}\Delta(E)^{\mathrm{N}(\mathfrak{a})-1} \prod_{P \in E[\mathfrak{a}]-O} (x - x(P))^{-6},$$

where α denotes a generator of $\mathfrak{a}$ and $\Delta(E)$ is the discriminant of the chosen Weierstrass model of E. Although α is not unique, α^{12} is (since $\#\mathcal{O}_K^\times \mid 12$), so $\Theta_{E,\mathfrak{a}}$ is well defined.

The following lemma is [Rub1999, Lemma 7.2], except that in (ii) we add the assumption $F \supset K$ to ensure that every endomorphism of E is defined over F and $\alpha \in F$.

Lemma 2.1.2.

(i) *$\Theta_{E,\mathfrak{a}}$ is independent of the choice of Weierstrass model. Equivalently, if $\phi : E' \xrightarrow{\sim} E$ is an isomorphism of elliptic curves, then $\Theta_{E',\mathfrak{a}} = \Theta_{E,\mathfrak{a}} \circ \phi$.*

(ii) *If E is defined over F and $F \supset K$, then the rational function $\Theta_{E,\mathfrak{a}}$ is defined over F.*

Proof. (i) Any other Weierstrass model has coordinate function x', y' given by

$$x = u^2x' + r \quad \text{and} \quad y = u^3y' + u^2sx' + t,$$

where $u, r, s, t \in \mathbf{C}$ and $u \neq 0$. Then

$$u^{12}\Delta' = \Delta$$

and hence that

$$\begin{aligned}\Delta(E)^{\mathrm{N}(\mathfrak{a})-1} \prod_{P \in E[\mathfrak{a}]-O} (x - x(P))^{-6} &= u^{12(\mathrm{N}(\mathfrak{a})-1)}\Delta'(E)^{\mathrm{N}(\mathfrak{a})-1} \prod_{P \in E[\mathfrak{a}]-O} (u^2x' + r - u^2x'(P) - r)^{-6} \\ &= \Delta'(E)^{\mathrm{N}(\mathfrak{a})-1} \prod_{P \in E[\mathfrak{a}]-O} (x' - x'(P))^{-6},\end{aligned}$$

where the last identity follows by [Sil1994, Proposition 1.4(b)]

(ii) It suffices to observe that $\alpha \in F, \Delta(E) \in F$, and $G_F = \mathrm{Gal}(\bar{F}/F)$ permutes the set

$$\{x(P) : P \in E[\mathfrak{a}] - O\}$$

by [Sil1994, Theorem 2.3].

□

By Lemma 2.1.2(i) and [Sil1994, Theorem 4.3(b)], we may assume that E is defined over K. Pick a $Q \in E_{\mathrm{tors}}$. By [Sil1994, Theorem 2.3], we have $\Theta_{E,\mathfrak{a}}(Q) \in K^{\mathrm{ab}}$. The following proposition provides a more refined description, which is [Rub1999, Theorem 7.4(i), (ii)], with the only modification that we do not assume $(\mathfrak{b}, \mathfrak{a}) = 1$.

Proposition 2.1.3. *Let $\mathfrak{b}$ be a nontrivial integral ideal of K, and choose $Q \in E[\mathfrak{b}]$ to be an $\mathcal{O}_K$-generator of the $\mathcal{O}_K$-module $E[\mathfrak{b}]$.*

(i) $\Theta_{E,\mathfrak{a}}(Q) \in K(\mathfrak{b})$.

(ii) If $\mathfrak{c} = c\mathcal{O}_K$ is an integral ideal of K prime to $\mathfrak{b}$ and $\sigma_{\mathfrak{c}} = (\mathfrak{c}, K(\mathfrak{b})/K)$, then

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_{\mathfrak{c}}} = \Theta_{E,\mathfrak{a}}(cQ).$$

Proof.

(i) Fix a complex analytic isomorphism

$$f\colon \mathbf{C}/\mathfrak{n} \to E(\mathbf{C})$$

for some fractional ideal $\mathfrak{n}$ of K. Pick an idele $x \in U_{\mathfrak{b}}$ and let $\sigma_x = [x, K]$. By [Sil1994, Theorem 9.1],there exists a unique $\alpha \in K^{\times}$ such that $\alpha\mathcal{O}_K = \iota(x) = \mathcal{O}_K$, i.e. $\alpha \in \mathcal{O}_K^{\times}$, and the following diagram commutes:

$$\begin{array}{ccc} K/\mathfrak{n} & \xrightarrow{\alpha x^{-1}} & K/\mathfrak{n} \\ \downarrow f & & \downarrow f \\ E(K^{\mathrm{ab}}) & \xrightarrow{\sigma_x} & E(K^{\mathrm{ab}}). \end{array}$$

By [Sil1994, Lemma 8.1], the module $\mathfrak{b}^{-1}\mathfrak{n}/\mathfrak{n}$ admits a decomposition into its $\mathfrak{p}$-primary parts:

$$\mathfrak{b}^{-1}\mathfrak{n}/\mathfrak{n} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}} (\mathfrak{b}^{-1}\mathfrak{n}/\mathfrak{n})_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{p}} \mathfrak{b}_{\mathfrak{p}}^{-1}\mathfrak{n}_{\mathfrak{p}}/\mathfrak{n}_{\mathfrak{p}} = \bigoplus_{\mathfrak{p}|\mathfrak{b}} \mathfrak{b}_{\mathfrak{p}}^{-1}\mathfrak{n}_{\mathfrak{p}}/\mathfrak{n}_{\mathfrak{p}}.$$

The final equality uses that $\mathfrak{b}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ whenever $\mathfrak{p} \nmid \mathfrak{b}$. Since $x \in U_{\mathfrak{b}}$, for all $\mathfrak{p} \mid \mathfrak{b}$, $x_{\mathfrak{p}} \in 1 + \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{b})}\mathcal{O}_{\mathfrak{p}}$ and hence that $x_{\mathfrak{p}}^{-1} \in 1 + \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{b})}\mathcal{O}_{\mathfrak{p}}$ as well. So for all $\mathfrak{p} \mid \mathfrak{b}$ and for all $t_{\mathfrak{p}} \in \mathfrak{b}_{\mathfrak{p}}^{-1}/\mathfrak{n}_{\mathfrak{p}}$, we have $x_{\mathfrak{p}}^{-1}t_{\mathfrak{p}} = t_{\mathfrak{p}}$. In other words, for all $P \in E[\mathfrak{b}]$, we have $P^{\sigma_x} = \alpha Q$. It follows that

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_x} = \Theta_{E,\mathfrak{a}}(Q^{\sigma_x}) = \Theta_{E,\mathfrak{a}}(\alpha Q) = \Theta_{E,\mathfrak{a}}(Q)$$

by Lemma 2.1.2.

Invoking [Sil1994, Theorem 3.6], we know that these σ_x generate $\mathrm{Gal}(K^{\mathrm{ab}}/K(\mathfrak{b}))$. Hence $\Theta_{E,\mathfrak{a}}(Q)$ is fixed by the entire group, so $\Theta_{E,\mathfrak{a}}(Q) \in K(\mathfrak{b})$.

(ii) Pick an idele $x \in \mathbf{A}_K^\times$ such that $\iota(x) = \mathfrak{c}$ and $x_\mathfrak{p} = 1$ for each $\mathfrak{p} \mid \mathfrak{b}$. Then another application of [Sil1994, Theorem 9.1]and Lemma 2.1.2 yields

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_\mathfrak{c}} = \Theta_{E,\mathfrak{a}}(Q)^{\sigma_x} = \Theta_{E,\mathfrak{a}}(cuQ) = \Theta_{E,\mathfrak{a}}(cQ)$$

for some $u \in \mathcal{O}_K^\times$. Here the first identity follows by [Sil1994, Theorem 3.5].

□

To obtain a more precise description of $\Theta_{E,\mathfrak{a}}(Q)$, we need the following lemma, which is [Rub1999, Lemma 7.3], except that our (iii) differs from the statement in the original source. A careful reading of the original proof shows that the assumption $\mathfrak{p} \nmid \mathfrak{bc}$ in the original source is stronger than necessary.

Lemma 2.1.4. *Suppose E is defined over K and $\mathfrak{p}$ is a prime of K where E has good reduction. Fix a Weierstrass model for E which is minimal at $\mathfrak{p}$. Let $\mathfrak{b}$ and $\mathfrak{c}$ be nontrivial relatively prime ideals of $\mathcal{O}_K$. Let $P \in E[\mathfrak{b}]$ be an $\mathcal{O}_K$-generator of $E[\mathfrak{b}]$ and $Q \in E[\mathfrak{c}]$ be an $\mathcal{O}_K$-generator of $E[\mathfrak{c}]$, respectively. Fix an extension of the $\mathfrak{p}$-adic order $\mathrm{ord}_\mathfrak{p}$ to $\bar{K}$, normalized so $\mathrm{ord}_\mathfrak{p}(\mathfrak{p}) = 1$.*

(i) If $n > 0$ and $\mathfrak{b} = \mathfrak{p}^n$ then

$$\mathrm{ord}_\mathfrak{p}(x(P)) = -\frac{2}{\mathrm{N}(\mathfrak{p})^n - \mathrm{N}(\mathfrak{p})^{n-1}}.$$

(ii) If $\mathfrak{b}$ is not a power of $\mathfrak{p}$ then $\mathrm{ord}_\mathfrak{p}(x(P)) \geq 0$.

(iii) If both $\mathfrak{b}$ and $\mathfrak{c}$ are not power of $\mathfrak{p}$ then $\mathrm{ord}_\mathfrak{p}(x(P) - x(Q)) = 0$.

Proof. See [Rub1999, Lemma 7.3]. □

Theorem 2.1.5 ([Rub1999, Theorem 7.4(iii)]). *Let $\mathfrak{b}$ be a nontrivial ideal of $\mathcal{O}_K$ relatively prime to $\mathfrak{a}$ and let $Q \in E[\mathfrak{b}]$ be an $\mathcal{O}_K$-generator of $E[\mathfrak{b}]$. If $\mathfrak{b}$ is not a prime power, then $\Theta_{E,\mathfrak{a}}(Q)$ is a global unit. If $\mathfrak{b} = \mathfrak{p}^n$ for some prime $\mathfrak{p}$ and $n > 0$, then $\Theta_{E,\mathfrak{a}}(Q)$ is a unit at primes outside $\mathfrak{p}$.*

Proof. Let $\mathfrak{p}$ be a prime of K such that $\mathfrak{b}$ is not a power of $\mathfrak{p}$. By [Rub1999, Corollary 5.22], we may assume that our Weierstrass equation for E has good reduction at $\mathfrak{p}$, so that $\Delta(E)$ is prime to $\mathfrak{p}$. Let $n = \mathrm{ord}_\mathfrak{p}(\mathfrak{a})$. If $n = 0$, it follows immediately that

$$\mathrm{ord}_\mathfrak{p}(\Theta_{E,\mathfrak{a}}(Q)) = -6 \sum_{P \in E[\mathfrak{a}]-O} \mathrm{ord}_\mathfrak{p}(x(Q) - x(P)) = 0$$

by Lemma 2.1.4(iii). Otherwise,

$$\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q))/6 = -2n - \sum_{P \in E[\mathfrak{a}]-O} \mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)).$$

- For $P \in E[\mathfrak{a}] - E[\mathfrak{p}^n]$, another application of Lemma 2.1.4(iii) yields $\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) = 0$ (for comparison, [Rub1999, Lemma 7.3] gives merely $\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) \geq 0$, as $\mathfrak{p}$ can divide the exact order of P).

- For $P \in E[\mathfrak{p}^m] - E[\mathfrak{p}^{m-1}]$ with $1 \leq m \leq n$,

$$\mathrm{ord}_{\mathfrak{p}}(x(Q) - x(P)) = -\frac{2}{\mathrm{N}(\mathfrak{p})^m - \mathrm{N}(\mathfrak{p})^{m-1}}$$

  by Lemma 2.1.4(i), (ii).

Thus we can conclude that

$$\begin{aligned}
\mathrm{ord}_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q))/6 &= -2n + \sum_{m=1}^{n} \sum_{P \in E[\mathfrak{p}^m]-E[\mathfrak{p}^{m-1}]} \frac{2}{\mathrm{N}(\mathfrak{p})^m - \mathrm{N}(\mathfrak{p})^{m-1}} \\
&= -2n + 2\sum_{m=1}^{n} \frac{(\mathrm{N}(\mathfrak{p})^m - \mathrm{N}(\mathfrak{p})^{m-1})}{(\mathrm{N}(\mathfrak{p})^m - \mathrm{N}(\mathfrak{p})^{m-1})} \\
&= 0,
\end{aligned}$$

where the second identity follows by [Sil1994, Proposition 1.4(b)]. □

Remark 2.1.6. This agrees with the cyclotomic case: if $F = \mathbf{Q}(\zeta_m)$ with $\zeta_m = e^{2\pi i/m}$, then $\zeta_m - 1 \in \mathcal{O}_F^\times$ if and only if m is divisible by at least two distinct primes.

The function $\Theta_{E,\mathfrak{a}}$ satisfies the distribution relation given in the following theorem.

Theorem 2.1.7. [Rub1999, Theorem 7.6] *Let $\mathfrak{b} = \beta\mathcal{O}_K$ be an ideal relatively prime to $\mathfrak{a}$. Then for every $Q \in E(\bar{K})$,*

$$\prod_{R \in E[\mathfrak{b}]} \Theta_{E,\mathfrak{a}}(Q + R) = \Theta_{E,\mathfrak{a}}(\beta Q).$$

Proof. Let $F := \prod_{R \in E[\mathfrak{b}]} \Theta_{E,\mathfrak{a}} \circ \tau_R$ and let $G := \Theta_{E,\mathfrak{a}} \circ [\beta]$, where τ_R denote translation R. We begin by observing that

$$\mathrm{div}(\Theta_{E,\mathfrak{a}}) = -6 \sum_{P \in E[\mathfrak{a}]-O} ([P] + [-P] - 2[O]) = 12\mathrm{N}(\mathfrak{a})[O] - 12 \sum_{P \in E[\mathfrak{a}]} [P]$$

since the coordinate function x is an even rational function with a double pole at O and no other poles. It follows that

$$\begin{aligned}\operatorname{div}(F) &= \sum_{R\in E[\mathfrak{b}]}\left(12\mathrm{N}(\mathfrak{a})[-R] - 12\sum_{P\in E[\mathfrak{a}]}[P-R]\right)\\ &= 12\mathrm{N}(\mathfrak{a})\sum_{R\in E[\mathfrak{b}]}[R] - 12\sum_{S\in E[\mathfrak{ab}]}[S],\end{aligned}$$

where the second identity holds since $(\mathfrak{a},\mathfrak{b})=1$. Fix $P\in E[\mathfrak{a}]-O$. We view $Q\mapsto x([\beta]Q)-x(P)$ as a function of Q. It has double poles at all point Q with $[\beta]Q=O$, that is, at $Q\in E[\mathfrak{b}]$, and has no other poles. Its zeros occur at the points Q for which $[\beta]Q=\pm P$. Then as P ranges over $E[\mathfrak{a}]-O$, the corresponding Q ranges over $E[\mathfrak{ab}]-E[\mathfrak{b}]$. Hence

$$\operatorname{div}(G) = 12(\mathrm{N}(\mathfrak{a})-1)\sum_{R\in E[\mathfrak{b}]}[R] - 12\sum_{S\in E[\mathfrak{ab}]-E[\mathfrak{b}]}[S] = \operatorname{div}(F).$$

Thus F/G is a constant $\lambda\in K^\times$, and we need to show that $\lambda=1$. Let $e_K=\#\mathcal{O}_K^\times$ and let α be a generator of $\mathfrak{a}$. Evaluating $F(O)/G(O)$ yields

$$\lambda = \frac{\Delta(E)^{(\mathrm{N}(\mathfrak{a})-1)(\mathrm{N}(\mathfrak{b})-1)}}{\alpha^{12(\mathrm{N}(\mathfrak{b})-1)}\beta^{12(\mathrm{N}(\mathfrak{a})-1)}}\prod_{R\in E[\mathfrak{b}]-O}\;\prod_{P\in E[\mathfrak{a}]-O}(x(R)-x(P))^{-6} = \mu^{e_K}$$

with

$$\mu = \frac{\Delta(E)^{(\mathrm{N}(\mathfrak{a})-1)(\mathrm{N}(\mathfrak{b})-1)/e_K}}{\alpha^{12(\mathrm{N}(\mathfrak{b})-1)/e_K}\beta^{12(\mathrm{N}(\mathfrak{a})-1)/e_K}}\prod_{R\in E[\mathfrak{b}]-O}\;\prod_{P\in (E[\mathfrak{a}]-O)/\pm 1}(x(R)-x(P))^{-12/e_K}.$$

Here we use $(2,\mathfrak{a})=1$. Next, we claim that $e_K\mid \mathrm{N}(\mathfrak{a})-1$. Indeed, for any $\mathfrak{p}\mid\mathfrak{a}$, we have $\mathcal{O}_K^\times\hookrightarrow(\mathcal{O}_K/\mathfrak{p})^\times$. More precisely, if $u\in(1+\mathfrak{p})\cap\mathcal{O}_K^\times$, then $\mathfrak{p}\mid 2$ or $\mathfrak{p}\mid 3$ since $e_K=2,4,6$, contradicting our assumption that $(\mathfrak{a},6)=1$. Hence $e_K\mid \mathrm{N}(\mathfrak{p})-1$ and

$$\mathrm{N}(\mathfrak{a})\equiv\prod_{\mathfrak{p}\mid\mathfrak{a}}\mathrm{N}(\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a})}\equiv 1 \pmod{e_K},$$

i.e. all exponents in the definition of μ are integers.

Now, arguing as in the proof of Theorem 2.1.5(iii), we conclude that $\mu\in\mathcal{O}_K^\times$ and $\lambda=1$. □

Corollary 2.1.8. [Rub1999, Corollary 7.7] *Let $\mathfrak{b}$ be an ideal of $\mathcal{O}_K$ prime to $\mathfrak{a}$. Let $\mathfrak{p}=\pi\mathcal{O}_K$ be a prime ideal dividing $\mathfrak{b}$ and write $\mathfrak{b}=\mathfrak{p}\mathfrak{b}'$. Assume that the map $\mathcal{O}_K^\times\to(\mathcal{O}_K/\mathfrak{b}')^\times$ is injective. If*

$Q \in E[\mathfrak{b}]$ *has order exactly* $\mathfrak{b}$*, then*

$$\mathrm{N}_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \begin{cases} \Theta_{E,\mathfrak{a}}(\pi Q) & \text{if } \mathfrak{p} \mid \mathfrak{b}', \\ \Theta_{E,\mathfrak{a}}(\pi Q)^{1-\sigma_{\mathfrak{p}}^{-1}} & \text{if } \mathfrak{p} \nmid \mathfrak{b}', \end{cases}$$

where $\sigma_{\mathfrak{p}} = (\mathfrak{p}, K(\mathfrak{b}')/K)$.

Proof. By [Sil1994, Theorem 3.2],

$$\mathrm{Cl}_K(\mathfrak{b}) = I(\mathfrak{b})/P(\mathfrak{b}) \xrightarrow{\sim} \mathrm{Gal}(K(\mathfrak{b})/K), \quad \mathfrak{c} \mapsto (\mathfrak{c}, K(\mathfrak{b})/K).$$

Furthermore, by [Sut2021, Theorem 21.8],

$$\mathrm{Cl}_K(\mathfrak{b}) \cong (\mathcal{O}_K^\times/\mathfrak{b})^\times/\mathcal{O}_K^\times$$

since we assume $h_K = 1$. The same is true with $\mathfrak{b}$ replaced by $\mathfrak{b}'$. Hence $\mathrm{Gal}(K(\mathfrak{b})/K(\mathfrak{b}'))$ is isomorphic to the kernel of the map

$$(\mathcal{O}_K/\mathfrak{b})^\times/\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\mathfrak{b}')^\times/\mathcal{O}_K^\times. \tag{2.1}$$

Denote by C the multiplicative group $\{(c+\mathfrak{b})\mathcal{O}_K^\times : c \in \mathcal{O}_K, c-1 \in \mathfrak{b}' \text{ and } c+\mathfrak{b} \in (\mathcal{O}_K/\mathfrak{b})^\times\}$. Due to our assumption that $\mathcal{O}_K^\times \hookrightarrow (\mathcal{O}_K/\mathfrak{b}')^\times$, C is isomorphic to the kernel of the map (2.1). Thus, we have an isomorphism

$$C \xrightarrow{\sim} \mathrm{Gal}(K(\mathfrak{b})/K(\mathfrak{b}')), \quad (c+\mathfrak{b})\mathcal{O}_K^\times \mapsto \sigma_c,$$

where $\sigma_c = ((c), K(\mathfrak{b})/K)$ (note that σ_c fixes $K(\mathfrak{b}')$ by construction). By Proposition 2.1.3(ii),

$$\mathrm{N}_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \prod_{(c+\mathfrak{b})\mathcal{O}_K^\times \in C} \Theta_{E,\mathfrak{a}}(Q)^{\sigma_c} = \prod_{(c+\mathfrak{b})\mathcal{O}_K^\times \in C} \Theta_{E,\mathfrak{a}}(cQ).$$

Write $c = 1 + x$ for some $x \in \mathfrak{b}'$. Then

$$\pi c Q = \pi(1+x)Q = \pi Q + \pi x Q = \pi Q.$$

On the other hand, $cQ \notin E[\mathfrak{b}']$ since $c \in \mathrm{Aut}_{\mathcal{O}_K/\mathfrak{b}}(E[\mathfrak{b}])$. This proves that

$$\{cQ : (c+\mathfrak{b})\mathcal{O}_K^\times \in C\} \subset \{P \in E[\mathfrak{b}] : \pi P = \pi Q \text{ and } P \notin E[\mathfrak{b}']\}.$$

In contrast, pick a $P \in E[\mathfrak{b}]$ such that $\pi P = \pi Q$ and $P \notin E[\mathfrak{b}']$. Since $\pi P = \pi Q \neq O$, i.e. $P \notin E[\mathfrak{p}]$, $P \in E[\mathfrak{b}]$ has order exactly $\mathfrak{b}$. So there exists a $c \in \mathcal{O}_K$ such that $c + \mathfrak{b} \in (\mathcal{O}/\mathfrak{b})^\times$

and $P = cQ$. Since $\pi P - \pi Q = \pi(c-1)Q = O$, $c - 1 \in \mathfrak{b}'$. Thus we can conclude that

$$\{cQ : c \in C\} = \{P \in E[\mathfrak{b}] : \pi P = \pi Q \text{and } P \notin E[\mathfrak{b}']\}.$$

If $\mathfrak{p} \mid \mathfrak{b}'$, then $\mathfrak{b}'^2 \subset \mathfrak{b}$, which implies that $1 + x + \mathfrak{b} \in (\mathcal{O}_K/\mathfrak{b})^\times$ for every $x \in \mathfrak{b}'$. So

$$C = \{(c+\mathfrak{b})\mathcal{O}_K^\times : c - 1 \in \mathfrak{b}'\},$$

and, by the same argument as above, we obtain

$$\{cQ : (c+\mathfrak{b})\mathcal{O}_K^\times \in C\} = \{P \in E[\mathfrak{b}] : \pi P = \pi Q\} = \{Q + R : R \in E[\mathfrak{p}]\}.$$

It follows that

$$\mathrm{N}_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \prod_{(c+\mathfrak{b})\mathcal{O}_K^\times \in C} \Theta_{E,\mathfrak{a}}(cQ) = \prod_{R \in E[\mathfrak{p}]} \Theta_{E,\mathfrak{a}}(Q+R) = \Theta_{E,\mathfrak{a}}(\pi Q),$$

where the last identity follows by by Theorem 2.1.7.

If $\mathfrak{p} \nmid \mathfrak{b}'$, then

$$\{cQ : c \in C\} = \{Q + R : R \in E[\mathfrak{p}], R \not\equiv -Q \pmod{E[\mathfrak{b}']}\}.$$

There is a unique $R_0 \in E[\mathfrak{p}]$ satisfying $Q + R_0 \in E[\mathfrak{b}']$. Indeed, if $R_1, R_2 \in E[\mathfrak{p}]$ satisfy $Q + R_i \in E[\mathfrak{b}']$ ($i = 1, 2$), then $R_1 - R_2 \in E[\mathfrak{b}'] \cap E[\mathfrak{p}] = \{O\}$, hence $R_1 = R_2$. Consequently,

$$\Theta_{E,\mathfrak{a}}(Q + R_0)\mathrm{N}_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q) = \Theta_{E,\mathfrak{a}}(\pi Q).$$

By Proposition 2.1.3(ii) (note that $\mathfrak{b}' \neq \mathcal{O}_K$ since we assume $\mathcal{O}_K^\times \hookrightarrow (\mathcal{O}_K/\mathfrak{b}')^\times$),

$$\Theta_{E,\mathfrak{a}}(Q + R_0)^{\sigma_\mathfrak{p}} = \Theta_{E,\mathfrak{a}}(\pi Q + \pi R_0) = \Theta_{E,\mathfrak{a}}(\pi Q).$$

This completes the proof. □

Remark 2.1.9. This agrees with the cyclotomic case. For $m \geq 1$ and a rational prime ℓ,

$$\mathrm{N}_{\mathbf{Q}(\zeta_{ml})/\mathbf{Q}(\zeta_m)}(\zeta_{ml} - 1) = \begin{cases} \zeta_m - 1 & \ell \mid m, \\ (\zeta_m - 1)^{1-\sigma_\ell^{-1}} & \ell \nmid m, \end{cases}$$

where $\zeta_m = e^{2\pi i/m}$ and $\sigma_\ell = ((\ell), \mathbf{Q}(\zeta_m)/\mathbf{Q})$.

2.2 | The analytic side

This section has two purposes. First, using the complex uniformization of elliptic curves, we express the previous results in terms of special functions; this will allow us to carry out numerical computations of elliptic units in the next section. Second, we relate elliptic units to special values of Hecke L-functions, offering only a glimpse of their profound arithmetic significance; a more comprehensive treatment is beyond the scope of the present work.

Henceforth, we assume that the elliptic curve E is defined over K. Let ψ be the Hecke character associated with E via [Sil1994, Theorem 9.2], let $\mathfrak{f}$ denote its conductor, and let $\mathfrak{a}$ be an ideal coprime to both $\mathfrak{f}$ and 6.

Fix an $\mathcal{O}_K$-generator S of $E[\mathfrak{f}]$. We define

$$\Lambda_{E,\mathfrak{a}} = \prod_{\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)} \Theta_{E,\mathfrak{a}} \circ \tau_{S^\sigma},$$

where τ_P denotes translation P. It is independent of the choice of S since by [Sil1994, Theorem 5.6], $S \in E(K(\mathfrak{f}))$.

Proposition 2.2.1. [Rub1999, Proposition 7.8]

(i) $\Lambda_{E,\mathfrak{a}}$ *is a rational function defined over* K.

(ii) If B *is a set of ideals of* $\mathcal{O}_K$*, prime to* $\mathfrak{a}\mathfrak{f}$*, such that the Artin map* $\mathfrak{b} \mapsto (\mathfrak{b}, K(\mathfrak{f})/K)$ *is a bijection from* B *to* $\mathrm{Gal}(K(\mathfrak{f})/K)$*, then*

$$\Lambda_{E,\mathfrak{a}}(P) = \prod_{\mathfrak{b} \in B} \Theta_{E,\mathfrak{a}}(\psi(\mathfrak{b})S + P).$$

(iii) If $\mathfrak{c}$ *is an ideal of* $\mathcal{O}_K$ *and* $Q \in E[\mathfrak{c}]$*,* $Q \notin E[\mathfrak{f}]$*, then* $\Lambda_{E,\mathfrak{a}}(Q)$ *is a global unit in* $K(E[\mathfrak{c}])$*.*

Proof.

(i) It suffices to observe that G_F permutes the set

$$\{S^\sigma : \sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)\}.$$

(ii) Arguing as in the proof of Proposition 2.1.3(i), one shows that

$$Q^{\sigma_\mathfrak{b}} = \psi(\mathfrak{b})Q,$$

where $\sigma_\mathfrak{b} = (\mathfrak{b}, K(\mathfrak{f})/K)$. The assertion follows.

(iii) An easy application of Theorem 2.1.5.

□

As usual, we choose a Weierstrass equation for E/K and denote by $L \subset \mathbf{C}$ the associated period lattice. Then

$$\{\alpha \in \mathbf{C} : \alpha L \subseteq L\} = \mathcal{O}_K,$$

so in particular there exists $\Omega \in \mathbf{C}^\times$ with $L = \Omega\,\mathcal{O}_K$. The uniformization map

$$\xi : \mathbf{C}/L \longrightarrow E(\mathbf{C}), \qquad z \longmapsto \big(\wp(z;L), \wp'(z;L)\big),$$

is a complex-analytic isomorphism, where

$$\wp(z;L) := \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

is the *Weierstrass $\wp$-function (relative to L)*. We define $\Theta_{L,\mathfrak{a}} = \Theta_{E,\mathfrak{a}} \circ \xi$, i.e.

$$\Theta_{L,\mathfrak{a}}(z) = \alpha^{-12} \Delta(L)^{\mathrm{N}(\mathfrak{a})-1} \prod_{w \in \mathfrak{a}^{-1}L/L - 0} (\wp(z;L) - \wp(w;L))^{-6}.$$

Then $\Theta_{L,\mathfrak{a}}$ is an elliptic function with respect to L.

Write $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ so that $\tau = \omega_1/\omega_2 \in \mathcal{H}$.

Definition 2.2.2. Define

$$\begin{aligned}
\sigma(z;L) &= z \prod_{0 \neq \lambda \in L} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2 \right), \\
\Delta(L) &= (2\pi i/\omega_2)^{12} q_\tau \prod_{n=1}^{\infty} (1 - q_\tau^n)^{24}, \quad q_\tau = e^{2\pi i \tau}, \\
A(L) &= (2\pi i)^{-1}(\omega_1 \bar{\omega}_2 - \bar{\omega}_1 \omega_2) = \pi^{-1} \mathrm{Area}(\mathbf{C}/L), \\
s_2(L) &= \lim_{s \to 0^+} \sum_{0 \neq \lambda \in L} \lambda^{-2} |\lambda|^{-2s}, \\
\eta(z;L) &= s_2(L) z + A(L)^{-1} \bar{z}.
\end{aligned}$$

Taking the logarithmic derivative of $\sigma(z;L)$ formally defines the *Weierstrass zeta function*

$$\zeta(z;L) = \frac{\sigma'(z;L)}{\sigma(z;L)} = \frac{1}{z} + \sum_{0 \neq \lambda \in L} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right).$$

Termwise differentiation shows that $\zeta'(z;L) = -\wp(z;L)$. Hence for any $\lambda \in L$,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\zeta(z+\lambda;L) - \zeta(z;L)) = 0,$$

since $\wp$ is L-periodic. Therefore, $\zeta(z+\lambda;L)-\zeta(z;L)$ is constant in z. In fact, we have exactly $\zeta(z+\lambda;L)=\zeta(z;L)+\eta(\lambda;L)$. It follows that $\sigma(z;L)$ satisfies the transformation law

$$\sigma(z+\lambda;L)=\pm\sigma(z;L)\exp\left(\eta(\lambda;L)\left(z+\frac{\lambda}{2}\right)\right),\quad z\in\mathbf{C},\lambda\in L.$$

Here we take $+$ if $\lambda/2\in L$ and $-$ otherwise (see [Lan1987, Chapter 18, Theorem 1]).

Definition 2.2.3. We define the *fundamental theta function* by

$$\theta(z;L)=\Delta(L)e^{-6\eta(z;L)z}\sigma(z;L)^{12}. \tag{2.2}$$

Note that $\theta(z;L)$ is not holomorphic, since $\eta(z;L)$ involves $\bar{z}$. Nevertheless, $\theta(z;L)$ is well suited to arithmetic applications. If $c\neq 0$, then

$$\theta(cz;cL)=\theta(z;L),$$

which explains the presence of $\Delta(L)$ in (2.2). Moreover, the exponential factor is chosen so that $|\theta(z;L)|$ is L-periodic. The function $\theta(z;L)$ also admits an important product expansion. Normalize L so that $\omega_1=\tau$ and $\omega_2=1$, and set $q_z=e^{2\pi iz}$ (so $q_\tau=e^{2\pi i\tau}$). Then ([Wei1976, IV §3(15)])

$$\theta(z;L)=e^{6A(L)^{-1}z(z-\bar{z})}q_\tau(q_z^{1/2}-q_z^{-1/2})^{12}\prod_{n=1}^{\infty}(1-q_\tau^n q_z)^{12}(1-q_\tau^n q_z^{-1})^{12}. \tag{2.3}$$

Proposition 2.2.4. [Rub1999, lemma 7.10] *We have*

$$\Theta_{L,\mathfrak{a}}(z)=\theta(z;L)^{\mathrm{N}(\mathfrak{a})}/\theta(z;\mathfrak{a}^{-1}L).$$

Proof. Let $f(z)=\theta(z;L)^{\mathrm{N}(\mathfrak{a})}/\theta(z;\mathfrak{a}^{-1}L)$. We begin by observing that

$$\begin{aligned} f(z) &= \frac{\Delta(L)^{\mathrm{N}(\mathfrak{a})}\sigma(z;L)^{12\mathrm{N}(\mathfrak{a})}}{\Delta(\mathfrak{a}^{-1}L)\sigma(z;\mathfrak{a}^{-1}L)^{12}}e^{6z(\eta(z;\mathfrak{a}^{-1}L)-\mathrm{N}(\mathfrak{a})\eta(z;L))} \\ &= \frac{\Delta(L)^{\mathrm{N}(\mathfrak{a})}\sigma(z;L)^{12\mathrm{N}(\mathfrak{a})}}{\Delta(\mathfrak{a}^{-1}L)\sigma(z;\mathfrak{a}^{-1}L)^{12}}e^{6z(s_2(\mathfrak{a}^{-1}L)z-\mathrm{N}(\mathfrak{a})s_2(L)z)} \quad \text{since } A(\mathfrak{a}^{-1}L)^{-1}=\mathrm{N}(\mathfrak{a})A(L)^{-1} \end{aligned}$$

is meromorphic. Moreover, f is an elliptic function with respect to L. Indeed, for any $\lambda\in L$,

$$\begin{aligned} f(z+\lambda) &= \frac{\Delta(L)^{\mathrm{N}(\mathfrak{a})}\sigma(z+\lambda;L)^{12\mathrm{N}(\mathfrak{a})}}{\Delta(\mathfrak{a}^{-1}L)\sigma(z+\lambda;\mathfrak{a}^{-1}L)^{12}}e^{6(z+\lambda)(\eta(z+\lambda;\mathfrak{a}^{-1}L)-\mathrm{N}(\mathfrak{a})\eta(z+\lambda;L))} \\ &= \frac{\Delta(L)^{\mathrm{N}(\mathfrak{a})}\sigma(z;L)^{12\mathrm{N}(\mathfrak{a})}}{\Delta(\mathfrak{a}^{-1}L)\sigma(z;\mathfrak{a}^{-1}L)^{12}}e^{12(z+\lambda/2)(\mathrm{N}(\mathfrak{a})\eta(\lambda;L)-\eta(\lambda;\mathfrak{a}^{-1}L))+6(z+\lambda)(\eta(z+\lambda;\mathfrak{a}^{-1}L)-\mathrm{N}(\mathfrak{a})\eta(z+\lambda;L))} \end{aligned}$$

$$
\begin{aligned}
&= f(z)e^{6(\lambda(\eta(z;\mathfrak{a}^{-1}L)-\mathrm{N}(\mathfrak{a})\eta(z;L))+z(\mathrm{N}(\mathfrak{a})\eta(\lambda;L)-\eta(\lambda;\mathfrak{a}^{-1}L)))} \\
&= f(z).
\end{aligned}
$$

Now, by definition of $\sigma(z;L)$, we have

$$
\mathrm{div}(f) = 12\mathrm{N}(\mathfrak{a})[0] - 12 \sum_{v\in\mathfrak{a}^{-1}L/L} [v] = \mathrm{div}(\Theta_{L,\mathfrak{a}}).
$$

Hence these two functions differ by a constant $\lambda \in \mathbf{C}^\times$. At $z=0$, both functions have Laurent series beginning

$$
\alpha^{-12}\Delta(L)^{\mathrm{N}(\mathfrak{a})-1}z^{12(\mathrm{N}(\mathfrak{a})-1)},
$$

so $\lambda = 1$. □

The following proposition is [dS1987, Proposition 2.4] and can be viewed as an analytic version of Proposition 2.1.3 and Theorem 2.1.5 proved above.

Proposition 2.2.5. *Let $\mathfrak{b}$ be a nontrivial ideal of $\mathcal{O}_K$, and v a primitive $\mathfrak{b}$-division point of L (i.e. $v \in \mathfrak{b}^{-1}L$, but $v \notin \mathfrak{b}'L$ for any proper divisor $\mathfrak{b}'$ of $\mathfrak{b}$). Then if $(\mathfrak{b},\mathfrak{a}) = 1$,*

(i) $\Theta_{L,\mathfrak{a}}(v) \in K(\mathfrak{b})$.

(ii) $\Theta_{L,\mathfrak{a}}(v)^{\sigma_\mathfrak{c}} = \Theta_{\mathfrak{c}^{-1}L,\mathfrak{a}}(v) = \Theta_{L,\mathfrak{ac}}(v)\Theta_{L,\mathfrak{c}}(v)^{-\mathrm{N}(\mathfrak{a})}$ *for* $\mathfrak{c} \subset \mathcal{O}_K$ *with* $(\mathfrak{c},\mathfrak{b}) = 1$, *where* $\sigma_\mathfrak{c} = (\mathfrak{c}, K(\mathfrak{f})/K)$.

(iii) $\Theta_{L,\mathfrak{a}}(v) \in \mathcal{O}_{K(\mathfrak{b})}^\times$ *if $\mathfrak{b}$ is not a prime power. If $\mathfrak{b} = \mathfrak{p}^n$, it is a unit outside $\mathfrak{p}$.*

Proof. Only (ii) is not immediate. Let c be a generator of $\mathfrak{c}$. By Proposition 2.1.3,

$$
\begin{aligned}
\Theta_{L,\mathfrak{a}}(v)^{\sigma_\mathfrak{c}} &= \Theta_{L,\mathfrak{a}}(cv) \\
&= \frac{\theta(cv;L)^{\mathrm{N}(\mathfrak{a})}}{\theta(cv;\mathfrak{a}^{-1}L)} \quad \text{by Proposition 2.2.4,} \\
&= \frac{\theta(v;c^{-1}L)^{\mathrm{N}(\mathfrak{a})}}{\theta(v;\mathfrak{a}^{-1}c^{-1}L)} \\
&= \Theta_{\mathfrak{c}^{-1}L,\mathfrak{a}}(v).
\end{aligned}
$$

Another application of Proposition 2.2.4 yields

$$
\begin{aligned}
\Theta_{L,\mathfrak{ac}}(v)\Theta_{L,\mathfrak{c}}(v)^{-\mathrm{N}(\mathfrak{a})} &= \frac{\theta(v;L)^{\mathrm{N}(\mathfrak{a})\mathrm{N}(\mathfrak{c})}}{\theta(v;\mathfrak{a}^{-1}\mathfrak{c}^{-1}L)}\left(\frac{\theta(v;L)^{\mathrm{N}(\mathfrak{c})}}{\theta(v;\mathfrak{c}^{-1}L)}\right)^{-\mathrm{N}(\mathfrak{a})} \\
&= \Theta_{\mathfrak{c}^{-1}L,\mathfrak{a}}(v).
\end{aligned}
$$

□

The following proposition can be found in [dS1987, p. 50], which may be regarded as the analytic formulation of Theorem 2.1.7 proved above.

Proposition 2.2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of K, relatively prime to each other. Then*

$$\prod_{v\in\mathfrak{b}^{-1}L/L}\Theta_{L,\mathfrak{a}}(z+v)=\Theta_{\mathfrak{b}^{-1}L,\mathfrak{a}}(z).$$

The bridge between special values of Hecke L-functions and elliptic units is provided by Eisenstein series, which we define below.

Definition 2.2.7. For $k\geq 1$, define the *Eisenstein series*

$$\begin{aligned}E_k(z;L)&:=\lim_{s\to 0}\sum_{\lambda\in L}\frac{1}{(z-\lambda)^k|z-\lambda|^{2s}}\\&=\sum_{\lambda\in L}\frac{1}{(z+\lambda)^k}\quad\text{if } k\geq 3,\end{aligned}$$

where the limit means evaluation of the analytic continuation at $s=0$.

Proposition 2.2.8. *We have*

$$\begin{aligned}&E_1(z;L)=\zeta(z;L)-\eta(z;L),\\&E_2(z;L)=-E_1(z;L)'=\wp(z;L)+s_2(L),\\&E_k(z;L)=\frac{-E_{k-1}(z;L)'}{k-1}=\frac{(-1)^k}{(k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k-2}\wp(z;L)\quad\textit{if } k\geq 3.\end{aligned}$$

Proof. See [Rub1999, Proposition 7.12]. □

Proposition 2.2.9. [Rub1999, Theorem 7.13] *For every $k\geq 1$,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k\log\Theta_{L,\mathfrak{a}}(z)=12(-1)^{k-1}(k-1)!(\mathrm{N}(\mathfrak{a})E_k(z;L)-E_k(z;\mathfrak{a}^{-1}L)).$$

Proof. By Proposition 2.2.4,

$$\log\Theta_{L,\mathfrak{a}}(z)=\mathrm{N}(\mathfrak{a})\log\theta(z;L)-\log\theta(z;\mathfrak{a}^{-1}L).$$

Recall that $\theta(z;L)=\Delta(L)e^{-6\eta(z;L)z}\sigma(z;L)^{12}$. So

$$\frac{\mathrm{d}}{\mathrm{dz}}\log\theta(z;L)=-6s_2(L)z-6\eta(z;L)+12\zeta(z;L)=12E_1(z;L)+6A(L)^{-1}\bar{z}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{dz}}\log\Theta_{L,\mathfrak{a}}(z) = 12(\mathrm{N}(\mathfrak{a})E_1(z;L) - E_1(z;\mathfrak{a}^{-1}L)) + 6(\mathrm{N}(\mathfrak{a})A(L)^{-1} - A(\mathfrak{a}^{-1}L)^{-1})\bar{z}$$
$$= 12(\mathrm{N}(\mathfrak{a})E_1(z;L) - E_1(z;\mathfrak{a}^{-1}L)) \quad \text{since } A(\mathfrak{a}^{-1}L) = \mathrm{N}(\mathfrak{a})^{-1}A(L).$$

This proves the assertion for $k = 1$. For $k \geq 2$, it follows from Proposition 2.2.8. □

Definition 2.2.10. Let $k \geq 1$. The Hecke L-function associated to $\bar{\psi}^k$ is defined as the analytic continuation of the Dirichlet series

$$L(\bar{\psi}^k, s) = \sum_{\mathfrak{b}} \frac{\bar{\psi}^k(\mathfrak{b})}{\mathrm{N}(\mathfrak{b})^s},$$

where the sum is taken over ideals of $\mathcal{O}_K$ prime to the conductor of $\bar{\psi}^k$. Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$ divisible by $\mathfrak{f}$ and let $\mathfrak{c}$ be an ideal of K prime to $\mathfrak{m}$. The partial L-function $L_{\mathfrak{m}}(\bar{\psi}^k, s, \mathfrak{c})$ is defined by the same formula, but with the sum restricted to ideals of $\mathcal{O}_K$ prime to $\mathfrak{m}$ such that $(\mathfrak{b}, K(\mathfrak{m})/K) = (\mathfrak{c}, K(\mathfrak{m})/K)$.

Proposition 2.2.11. [Rub1999, Proposition 7.15] *Let $v \in KL/L$ be a point of order $\mathfrak{m}$, where $\mathfrak{m}$ is divisible by $\mathfrak{f}$. Recall that $L = \Omega\mathcal{O}_K$ and let $\mathfrak{c} = \Omega^{-1}v\mathfrak{m}$. Then for all $k \geq 1$, we have*

$$E_k(v;L) = v^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\psi}^k, k, \mathfrak{c}).$$

Proof. Let μ be a generator of $\mathfrak{m}$. Then $c := \Omega^{-1}v\mu \in \mathcal{O}_K$ and $(c, \mathfrak{m}) = 1$ since v has order exactly $\mathfrak{m}$ and $L = \Omega\mathcal{O}_K$. Using

$$E_k(z;L) = \lim_{s\to k}\sum_{\lambda\in L}\frac{1}{(z-\lambda)^k|z-\lambda|^{2s-2k}} = \lim_{s\to k}\sum_{\lambda\in L}\frac{(\bar{z}-\bar{\lambda})^k}{|z-\lambda|^{2s}},$$

we have

$$\begin{aligned} E_k(v;L) &= \lim_{s\to k}\sum_{\lambda\in L}\frac{(\bar{v}-\bar{\lambda})^k}{|v-\lambda|^{2s}} \\ &= \lim_{s\to k}\sum_{\lambda_1\equiv v \mod L}\frac{\bar{\lambda}_1^k}{|\lambda_1|^{2s}} \\ &= \lim_{s\to k}\frac{\bar{\Omega}^k}{|\Omega|^{2s}}\sum_{\lambda_2\equiv v/\Omega \mod \mathcal{O}_K}\frac{\bar{\lambda}_2^k}{|\lambda_2|^{2s}} \\ &= \lim_{s\to k}\frac{\mathrm{N}(\mathfrak{m})^s}{\bar{\mu}^k}\frac{\bar{\Omega}^k}{|\Omega|^{2s}}\sum_{\substack{\lambda_3\in\mathcal{O}_K \\ \lambda_3\equiv c \mod \mathfrak{m}}}\frac{\bar{\lambda}_3^k}{|\lambda_3|^{2s}}. \end{aligned}$$

By [Sil1994, Theorem 9.1],

$$\epsilon(\alpha) := \psi(\alpha\mathcal{O}_K)/\beta$$

defines a multiplicative map from $\{\alpha \in \mathcal{O}_K : \beta \text{ is prime to } \mathfrak{f}\}$ to $\mathcal{O}_K^\times$. Furthermore, by definition of the conductor, ϵ factors through $(\mathcal{O}_K/\mathfrak{f})^\times$. Thus, if $\alpha \equiv c \pmod{\mathfrak{m}}$ then $\epsilon(\alpha) = \epsilon(c)$, and hence that

$$\bar{\alpha} = \bar{\psi}(\alpha\mathcal{O}_K)\overline{\left(\frac{c}{\psi(\mathfrak{c})}\right)} = \bar{\psi}(\alpha\mathcal{O}_K)\frac{\psi(\mathfrak{c})}{c}.$$

Here the second equality holds since

$$(\psi(\mathfrak{c})/c)^{\#\mathcal{O}_K^\times} = 1.$$

It follows that

$$\begin{aligned} E_k(v;L) &= \lim_{s\to k} \frac{\mathrm{N}(\mathfrak{m})^s}{\bar{\mu}^k}\frac{\bar{\Omega}^k}{|\Omega|^{2s}}\frac{\psi(\mathfrak{c})^k}{c^k} \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_K \\ (\mathfrak{b},K(\mathfrak{m})/K)=(\mathfrak{c},K(\mathfrak{m})/K)}} \frac{\bar{\psi}(\mathfrak{b})^k}{\mathrm{N}(\mathfrak{b})^s} \\ &= \lim_{s\to k}\left(\frac{\mathrm{N}(\mathfrak{m})}{\mathrm{N}((\Omega))}\right)^{s-k} v^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\psi}^k, s, \mathfrak{c}) \quad \text{since } c = \Omega^{-1}v\mu, \\ &= v^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\psi}^k, s, \mathfrak{c}). \end{aligned}$$

□

Definition 2.2.12. Let f generate $\mathfrak{f}$. Fix a set B of ideals of $\mathcal{O}_K$ prime to $\mathfrak{af}$ which forms a system of representatives for $\mathrm{Gal}(K(\mathfrak{f})/K)$ via the Artin map $\mathfrak{b} \mapsto (\mathfrak{b}, K(\mathfrak{f})/K)$. Put $u = \Omega/f \in \mathfrak{f}^{-1}L$. We then define

$$\Lambda_{L,\mathfrak{a}}(z) = \Lambda_{E,\mathfrak{a}}(\xi(z)) = \prod_{\mathfrak{b}\in B}\Theta_{L,\mathfrak{a}}(\psi(\mathfrak{b})u + z).$$

Theorem 2.2.13. [Rub1999, Theorem 7.17] *For all* $k \geq 1$, *we have*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \log\Lambda_{L,\mathfrak{a}}(z)\bigg|_{z=0} = 12(-1)^{k-1}(k-1)!f^k(\mathrm{N}(\mathfrak{a}) - \psi(\mathfrak{a})^k)\Omega^{-k}L(\bar{\psi}^k, k).$$

Proof. By Proposition 2.2.9, we have

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \log\Lambda_{L,\mathfrak{a}}(z)\bigg|_{z=0} &= \sum_{\mathfrak{b}\in B}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \log\Theta_{L,\mathfrak{a}}(z)\bigg|_{z=\psi(\mathfrak{b})u} \\ &= 12(-1)^{k-1}(k-1)!\sum_{\mathfrak{b}\in B}\left(\mathrm{N}(\mathfrak{a})E_k(\psi(\mathfrak{b})u;L) - E_k(\psi(\mathfrak{b})u;\mathfrak{a}^{-1}L)\right). \end{aligned}$$

Since $u \in KL/L$ has order $\mathfrak{f}$ and $(\mathfrak{b},\mathfrak{f}) = 1$, $\psi(\mathfrak{b})u$ also has order $\mathfrak{f}$. Since $u = \Omega/f$, $\Omega^{-1}\psi(\mathfrak{b})u\mathfrak{f} =$

$\mathfrak{b}$. By Proposition 2.2.11,

$$E_k(\psi(\mathfrak{b})u; L) = (\psi(\mathfrak{b})u)^{-k}\psi(\mathfrak{b})^k L_{\mathfrak{f}}(\bar{\psi}^k, k, \mathfrak{b}) = u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k, \mathfrak{b})$$

and

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})u; L) = u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k).$$

Similarly, since

$$E_k(z; \mathfrak{a}^{-1}L) = \lim_{s\to 0} \sum_{\lambda\in L} \frac{1}{(z-\psi(\mathfrak{a}^{-1})\lambda)^k |z-\psi(\mathfrak{a}^{-1})\lambda|^{2s}} = \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a})z; L),$$

we have

$$E_k(\psi(\mathfrak{b})u; \mathfrak{a}^{-1}L) = \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a}\mathfrak{b})u; L) = \psi(\mathfrak{a})^k u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k, \mathfrak{a}\mathfrak{b})$$

and

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})u; \mathfrak{a}^{-1}L) = \psi(\mathfrak{a})^k u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k).$$

Since $u = \Omega/f$, we are done. □

Corollary 2.2.14 (Damerell's theorem)**.** *For every* $k \geq 1$*,*

$$\Omega^{-k} L(\bar{\psi}^k, k) \in K.$$

Proof. By Proposition 2.2.1(i), $\Lambda_{L,\mathfrak{a}}(z) \in K(\wp(z;L), \wp'(z;L))$. Recall that the Weierstrass $\wp$-function and its derivative satisfy the relation

$$\wp'(z;L)^2 = 4\wp(z;L)^3 - 60G_4(L)\wp(z;L) - 140G_6(L),$$

where

$$G_{2n}(L) = \sum_{0\neq\lambda\in L} \lambda^{-2n}$$

is the *Eisenstein series of weight* $2n$ *for* L. So all derivatives $\wp^{(k)}(z;L)$ belongs to $K(\wp(z;L), \wp'(z;L))$ as well. Then the assertion follows from Theorem 2.2.13. □

Remark 2.2.15. By a theorem of Deuring ([Sil1994, Theorem 10.5]), we have $L(E/K, s) = L(\psi, s)L(\bar{\psi}, s)$. In particular, $L(E/K, 1) \neq 0$ implies that $L(\bar{\psi}, 1) \neq 0$. Invoking Theorem 2.2.13, this nonvanishing can be translated into information about elliptic units, which in turn yields arithmetic information on $E(K)$. In this way, Coates–Wiles [CW1977] proved a special case of the Birch–Swinnerton-Dyer conjecture (namely, the rank-zero case for CM elliptic curves).

2.3 | The computational side

The goal of this section is to perform, for imaginary quadratic fields, numerical computations parallel to those in [BCG2023]. The basic facts about the function θ_0 are taken from [FV2000], and those about Siegel functions are taken from [Lan1987].

We introduce the following theta function θ_0. For our numerical computations we will use it, rather than the fundamental theta function from the previous section. This choice is motivated by [BCG2023], where the conjectural construction of elliptic units for complex cubic fields employs the elliptic gamma function, which may be viewed as a higher-dimensional analogue of the theta function defined below.

Definition 2.3.1. We define

$$\theta_0(z,\tau) \coloneqq (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}), \quad z\in\mathbf{C}, \operatorname{Im}\tau>0.$$

One checks (by the Jacobi triple product identity) that

$$\theta_0(z,\tau) = -i\frac{e^{\pi i(z-\tau/6)}}{\eta(\tau)}\theta(z,\tau),$$

where

$$\eta(\tau) \coloneqq e^{\pi i\tau/12}\prod_{n\geq 1}\left(1-e^{2n\pi i\tau}\right), \quad \operatorname{Im}\tau>0$$

is the *Dedekind eta function*, and

$$\theta(z,\tau) \coloneqq -\sum_{j\in\mathbf{Z}} e^{i\pi\tau(j+1/2)^2+2\pi i(j+1/2)(z+1/2)}, \quad z\in\mathbf{C}, \operatorname{Im}\tau>0$$

is the *Jacobi's first theta function*. Its modular transformation laws are inherited from those of $\theta(z,\tau)$ and $\eta(\tau)$. In particular,

$$\theta_0(z,\tau+1) = \theta_0(z,\tau),$$

and under $\tau\mapsto -1/\tau$ one has

$$\exp\bigl(\pi i(\tau/6-z)\bigr)\,\theta_0(z,\tau) = i\,\exp\left(-\frac{\pi i}{\tau}\left(z^2+z+\frac{1}{6}\right)\right)\,\theta_0\left(\frac{z}{\tau},-\frac{1}{\tau}\right).$$

We extend θ_0 to a meromorphic function on $\mathbf{C}\times\{\operatorname{Im}\tau\neq 0\}$ by setting $\theta_0(z,-\tau)=\theta_0(z+\tau,\tau)^{-1}$.

This function obeys ([FV2000, § 2.3, (6)])

$$\begin{aligned}\theta_0(z+1,\tau) &= \theta_0(z,\tau),\\ \theta_0(z+\tau,\tau) &= -e^{-2\pi i z}\theta_0(z,\tau),\\ \theta_0(\tau-z) &= \theta_0(z,\tau).\end{aligned} \tag{2.4}$$

The final property of θ_0 that we shall need is the following expansion, which is particularly useful for numerical computations ([FV2000, § 2.3, (8)]):

$$\theta_0(z,\tau) = \exp\left(-i\sum_{j=1}^{\infty}\frac{\cos\pi j(2z-\tau)}{j\sin\pi j\tau}\right), \quad 0 < \operatorname{Im} z < \operatorname{Im}\tau. \tag{2.5}$$

Although this will not be used in what follows, we note that a geometric interpretation of θ_0 is given in [FHRZ2008].

To relate the function θ_0 introduced above to the fundamental theta function from the previous section, we will need the Siegel functions defined below. They will also play a key role in the proof of the rank one abelian Stark conjecture for imaginary quadratic fields in the next chapter.

Definition 2.3.2. Let $\tau \in \mathcal{H}$ and let $a_1, a_2 \in \mathbf{R}$. The *Siegel function* is defined by

$$g_{a_1,a_2}(\tau) = -q_\tau^{B_2(a_1)/2}e^{\pi i a_2(a_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}), \tag{2.6}$$

where $z = a_1\tau + a_2$ and $B_2(x) = x^2 - x + 1/6$ is the second Bernouli polynomial.

Recall that $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with $\omega_1/\omega_2 = \tau \in \mathcal{H}$. By definition, for $a_1, a_2 \in \mathbf{R}$,

$$g_{a_1,a_2}(\tau)^{12} = \theta(a_1\tau + a_2; \omega_2^{-1}L) = \theta(a_1\omega_1 + a_2\omega_2; L). \tag{2.7}$$

On the other hand,

$$g_{a_1,a_2}(\tau) = -q_\tau^{B_2(a_1)/2}e^{\pi i a_2(a_1-1)}\theta_0(a_1\tau + a_2, \tau).$$

Consequently, the elliptic units constructed in the previous section can be expressed purely in terms of θ_0 only for those inputs for which the extra prefactors cancel in the relevant quotient. More precisely, if $\big(\omega_1, \omega_2/\mathrm{N}(\mathfrak{a})\big)$ is also a $\mathbf{Z}$-basis of $\mathfrak{a}^{-1}L$, then

$$\begin{aligned}\Theta_{L,\mathfrak{a}}(a_1\omega_1 + a_2\omega_2) &= \frac{\theta(a_1\omega_1 + a_2\omega_2; L)^{\mathrm{N}(\mathfrak{a})}}{\theta(a_1\omega_1 + a_2\omega_2; \mathfrak{a}^{-1}L)} && \text{by Proposition 2.2.4}\\ &= \frac{g_{a_1,a_2}(\tau)^{12\mathrm{N}(\mathfrak{a})}}{g_{a_1,\mathrm{N}(\mathfrak{a})a_2}(\mathrm{N}(\mathfrak{a})\tau)^{12}} && \text{by (2.7)}\end{aligned}$$

$$= \frac{\theta_0(a_1\tau + a_2, \tau)^{12\mathrm{N}(\mathfrak{a})}}{\theta_0(\mathrm{N}(\mathfrak{a})(a_1\tau + a_2), \mathrm{N}(\mathfrak{a})\tau)^{12}}.$$

In what follows, we review the motivating example from the introduction of [BCG2023].

Example 2.3.3. Let $K = \mathbf{Q}(i)$. Set $L = \frac{(2+i)\mathcal{O}_K}{5} = \mathbf{Z}\frac{2+i}{5} + \mathbf{Z}$ and $\mathfrak{a} = (2+i)\mathcal{O}_K$. Then $\mathfrak{a}^{-1}L = \mathbf{Z}\frac{2+i}{5} + \mathbf{Z}\frac{1}{5}$. Let $v = 1/3$. Then v is a primitive (3)-division point of L. Set $\tau = (2+i)/5$. By Proposition 2.2.5, we have

$$\Theta_{L,\mathfrak{a}}(v) = \frac{\theta_0(v,\tau)^{60}}{\theta_0(5v, 5\tau)^{12}} \in \mathcal{O}_{K(3)}.$$

We compute

$$\frac{\theta_0(v,\tau)^{60}}{\theta_0(5v, 5\tau)^{12}} = -1969757.7301998150\ldots,$$

a complex number that PARI/GP identifies as root of

$$P(x) = x^2 + 1969758x + 531441$$

to 1000 decimal digits. More precisely,

$$\frac{\theta_0(v,\tau)^{60}}{\theta_0(5v, 5\tau)^{12}} \approx -3(405 + 234\sqrt{3})^2,$$

where $\approx$ represents an error less than 10^{-1000}. On the other hand, we know $K(3) = \mathbf{Q}(i, \sqrt{3})$ and $\mathcal{O}_{K(3)} = \mathbf{Z}[\zeta_{12}]$, where $\zeta_{12} = e^{2\pi i/12}$. Hence $-3(405 + 234\sqrt{3})^2 \in \mathcal{O}_{K(3)}$. Moreover,

$$\mathrm{N}_{K(3)/\mathbf{Q}}\left(-3(405 + 234\sqrt{3})^2\right) = 3^{24},$$

so $-3(405 + 234\sqrt{3})^2$ is a unit outside 3. Our numerical results agree very well with the theory.

Remark 2.3.4. In [BCG2023], the authors in fact claim that

$$\frac{\theta_0(v,\tau)^{5}}{\theta_0(5v, 5\tau)} \in \mathcal{O}_{K(3)},$$

which is stronger than the conclusion obtained above. In view of Proposition 2.3.5 below and the fact that $\zeta_{12} \in K(3)$, their assertion is indeed correct. However, [dS1987] does not provide a proof or an explicit reference for this statement, so in the example above we retain the weaker conclusion. Moreover, the sentence in [BCG2023, p. 18]—“Complex Multiplication theory implies ..., see [18, Prop. 2.4, p. 51]”—suggests that the authors of [BCG2023] may have conflated θ_0 with the fundamental theta function.

Proposition 2.3.5. [dS1987, p. 56] *Let $\mathfrak{b} \neq (1)$ be a non-trivial ideal of $\mathcal{O}_K$, and v be a primitive*

$\mathfrak{b}$*-division point of* L*. Then for* $(\mathfrak{c}, 6\mathfrak{b}) = 1$*,* $\Theta_{L,\mathfrak{c}}(v)$ *is a* $12w_\mathfrak{b}$ *power in* $K(\mathfrak{b})^\times$*, where* $w_\mathfrak{b}$ *is the number of roots of unity congruent to* 1 *modulo* $\mathfrak{b}$*.*

Chapter 3

Interlude: Stark's conjectures

„Viele Umwege werde ich noch gehen, viele
Erfüllungen noch werden mich enttäuschen.
Alles wird seinen Sinn einst zeigen. Dort,
wo die Gegensätze erlöschen, ist Nirwana.
Mir brennen sie noch hell, geliebte Sterne
der Sehnsucht.“

Hermann Hesse,
Rotes Haus

We give a brief introduction to the Stark conjecture, following [Das1999, DG2011].

3.1 | Motivation

3.1.1 | Dirichlet class number formula

One of the most compelling threads in number theory is the mutual influence of its analytic and algebraic sides. A striking manifestation of this phenomenon is the class number formula.

Definition 3.1.1. Let K be a number field. The *Dedekind zeta function of* K is defined by the series

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{N}(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ runs over the nonzero ideals of $\mathcal{O}_K$.

As in the case of the Riemann zeta function, the Dedekind zeta function continues meromorphically to $\mathbf{C}$, with its only singularity a simple pole at $s = 1$.

Theorem 3.1.2 (Class number formula)**.** [BK2016, Theorem 1.5] *Let K be a number field. Let Δ_K denote the discriminant of K, h_K the class number, R_K the regulator, and e_K the number of roots* *of unity in K, and let r_1 and r_2 denote the number of real and complex places of K, respectively. Then $\zeta_K(s)$ has a simple pole at $s=1$ with residue*

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{|\Delta_K|^{1/2}e_K}.$$

Completing $\zeta_K(s)$ to

$$Z_K(s) = |\Delta_K|^{s/2}\left(\pi^{-s/2}\Gamma(s/2)\right)^{r_1}\left((2\pi)^{1-s}\Gamma(s)\right)^{r_2}\zeta_K(s),$$

and using the functional equation for $Z_K(s)$, namely

$$Z_K(s) = Z_K(1-s),$$

we obtain

$$\lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{e_K}.$$

This can be generalized to the so-called S*-class number formula*. More precisely, let S be a finite set of places of K containing all archimedean places. We define the S*-imprimitive Dedekind zeta function* by

$$\zeta_{K,S}(s) \coloneqq \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K\\(\mathfrak{a},S)=1}}\frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

where $(\mathfrak{a},S)=1$ means that $\mathfrak{a}$ is relatively prime to all finite primes in S. Similarly, it admits a meromorphic continuation to $\mathbf{C}$, with a simple pole at $s=1$.

Theorem 3.1.3 (S-class number formula at $s=0$)**.** [Das1999, Corollary 3.1.4] *Let K be a number field. Let S be a finite set of places of K containing all archimedean place. Let $h_{K,S}$ denote the S-class number, $R_{K,S}$ the S-regulator, and e_K the number of roots of unity in K. We have*

$$\lim_{s\to 0}\frac{\zeta_{K,S}(s)}{s^{|S|-1}} = -\frac{h_{K,S}R_{K,S}}{e_K}.$$

We further consider a finite abelian extension of number fields F/K with $G\coloneqq\mathrm{Gal}(F/K)$ (in fact, one may work more generally with a finite Galois extension, but in this thesis we focus on the abelian case). Let S be a finite set of places of K containing all archimedean places and those that ramify in F. For each $\sigma\in G$, we define the S*-imprimitive partial zeta function*

$$\zeta_{F/K,S}(\sigma,s) \coloneqq \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K\\(\mathfrak{a},S)=1\\\sigma_\mathfrak{a}=\sigma}}\frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

where $\sigma_\mathfrak{a} = \left(\mathfrak{a}, F/K\right)$ denotes the Artin symbol. It admits a meromorphic continuation to $\mathbf{C}$, which is holomorphic on $\mathbf{C}\setminus\{1\}$ and has a simple pole at $s=1$. For each character $\chi\colon G\to\mathbf{C}^\times$, we define the *$S$-imprimitive Artin L-function*

$$L_S(\chi,s) \coloneqq \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_K\\(\mathfrak{a},S)=1}} \frac{\chi(\sigma_{\mathfrak{a}})}{\mathrm{N}(\mathfrak{a})^s}.$$

It admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s=0$. The partial zeta functions $\zeta_{F/K,S}(\sigma,s)$ and the Artin L-functions $L_S(\chi,s)$ are related by Fourier inversion on G. More precisely, for each character $\chi\in\widehat{G}$ one has

$$L_S(\chi,s) = \sum_{\sigma\in G}\chi(\sigma)\,\zeta_{F/K,S}(\sigma,s),$$

and conversely, for each $\sigma\in G$,

$$\zeta_{F/K,S}(\sigma,s) = \frac{1}{|G|}\sum_{\chi\in\widehat{G}}\overline{\chi}(\sigma)\,L_S(\chi,s),$$

where $\widehat{G}$ is the *character group of* G.

Denote by $r_S(\chi)$ the order of $L_S(\chi,s)$ at $s=0$.

Lemma 3.1.4. [Das1999, Proposition 3.2.4] *We have*

$$r_S(\chi) = \begin{cases} |S|-1 & \text{if } \chi=\mathbf{1}_G,\\ |v\in S:\chi(G_v)=1| & \text{otherwise,}\end{cases}$$

where G_v is the decomposition group over v.

Let S_F be the set of places of F lying above the places of S. Then

$$\zeta_{F,S_F}(s) = \prod_{\chi\in\widehat{G}} L_S(\chi,s). \tag{3.1}$$

Indeed,

$$\begin{aligned}\prod_{\chi\in\widehat{G}} L_S(\chi,s) &= \prod_{\mathfrak{p}\notin S}\prod_{\chi\in\widehat{G}}(1-\chi(\sigma_{\mathfrak{p}})\mathrm{N}(\mathfrak{p})^{-s})^{-1}\\ &= \prod_{\mathfrak{p}\notin S}(1-\mathrm{N}(\mathfrak{p})^{-sf_{\mathfrak{p}}})^{-[F:K]/f_{\mathfrak{p}}}\\ &= \prod_{\mathfrak{P}\notin S_F}(1-\mathrm{N}(\mathfrak{P})^{-s})^{-1}\end{aligned}$$

$$= \zeta_{F,S_F}(s).$$

Broadly speaking, Stark's conjectures aim to extend Dirichlet class number formula from Dedekind zeta functions to Artin L-functions. More concretely, it predicts that the first nonvanishing Taylor coefficient of $L_S(\chi, s)$ at $s = 0$ can be described as the product of an algebraic number and the determinant of an $r_S(\chi) \times r_S(\chi)$ matrix built from linear forms in the logarithms of archimedean absolute values of units of K. As noted in [Das1999], this viewpoint is not only a conceptual passage from Dedekind zeta functions to more general Artin L-functions, but also sheds light on how the leading coefficients $-h_{F,S_F} R_{F,S_F}/e_F$ in (3.1) factors into contributions coming from the L-functions $L_S(\chi, s)$.

3.1.2 | A toy model

The following example is taken from [DG2011]. Let f be a positive integer, and let a be an integer coprime to f. Set $K = \mathbf{Q}$ and let $F = \mathbf{Q}(\zeta_f)^+$ be the maximal real subfield of $\mathbf{Q}(\zeta_f)$, where $\zeta_f = e^{2\pi i/f}$. Let $S = \{\infty\} \cup \{\, p : \ p \mid f \,\}$. Then

$$G \coloneqq \mathrm{Gal}(F/K) \cong (\mathbf{Z}/f\mathbf{Z})^\times/\{\pm 1\},$$

and the class of $a \in (\mathbf{Z}/f\mathbf{Z})^\times$ corresponds to the automorphism $\sigma_a \in G$ determined by

$$\sigma_a : \ \zeta_f + \zeta_f^{-1} \longmapsto \zeta_f^a + \zeta_f^{-a}.$$

Define the *Hurwitz zeta function* by

$$\zeta_H(x, s) \coloneqq \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}, \qquad \mathrm{Re}\, x > 0, \mathrm{Re}\, s > 1.$$

Choosing a representative a with $0 < a < f$, we obtain

$$\zeta_{F/K,S}(\sigma_a, s) = \sum_{\substack{n \geq 1 \\ n \equiv \pm a \mod f}} \frac{1}{n^s} = f^{-s} \left(\zeta_H(a/f, s) + \zeta_H((f-a)/f, s) \right).$$

Since $\zeta_H(a/f, 0) = 1/2 - a/f$, it follows that

$$\zeta_{F/K,S}(\sigma_a, 0) = \frac{1}{2} - \frac{a}{f} + \frac{1}{2} - \frac{f-a}{f} = 0.$$

Moreover, using the classical identities

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \zeta_H(x, s) = \log \Gamma(x) - \frac{1}{2}\log(2\pi)$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \zeta_{F/K,S}(\sigma_a, s) &= \log \frac{\Gamma(a/f)\Gamma(1-a/f)}{2\pi} \\ &= -\log(2\sin(\pi a/f)) \\ &= -\frac{1}{2}\log(2 - 2\cos(2\pi a/f)) \\ &= -\frac{1}{2}\log u(a, f),\end{aligned}$$

where

$$u(a, f) = (\zeta_f^a - 1)(\zeta_f^{-a} - 1)$$

is an f-unit in F. If f is divisible by at least two distinct primes, then $u(a, f)$ is in fact a global unit of F.

3.2 | The rank one abelian Stark conjecture

3.2.1 | The conjecture

Fix the following data:

- an abelian extension of number fields F/K;
- a finite set of places S of K containing all archimedean places and all places that ramify in F, and satisfying $|S| \geq 2$ as well as the existence of a place $v \in S$ that splits completely in F;
- the Galois group $G \coloneqq \mathrm{Gal}(F/K)$;
- a place w of F lying above v.

Let $U_{v,S}$ denote the set of elements $u \in F^\times$ satisfying the following conditions.

- If $|S| \geq 3$, then $|u|_{w'} = 1$ for every place w' of F with $w' \nmid v$.

- If $S = \{v, v'\}$, then $|u|_{\sigma w'} = |u|_{w'}$ for all $\sigma \in G$ and all $w' \mid v'$, and $|u|_{w''} = 1$ for every place w'' of F with $w'' \notin S_F$.

The assumption that S contains a place that splits completely in F and that $|S| \geq 2$ ensure that $\zeta_{F/K,S}(\sigma, 0) = 0$ for all $\sigma \in G$. We may now state the rank one abelian Stark conjecture.

Conjecture 3.2.1. There exists a $u \in U_{v,S}$ such that

$$\zeta'_{F/K,S}(\sigma, 0) = -\frac{1}{e_F} \log |\sigma(u)|_w \quad \text{for all } \sigma \in G \tag{3.2}$$

and such that $F(u^{1/e_F})/K$ is an abelian extension.

Remark 3.2.2. (3.2) is equivalent to

$$L'_S(\chi, 0) = -\frac{1}{e_F} \sum_{\sigma \in G} \chi(\sigma) \log |\sigma(u)|_w \quad \text{for all } \chi \in \widehat{G}.$$

Remark 3.2.3. It is clear that the validity of the conjecture is independent of the choice of the place w above v. The S_F-unit u appearing in the conjecture is called a *Stark unit*, and we denote it by $\mathbf{u}_{\text{Stark}}$. The defining conditions determine the absolute values of u at every place of F; consequently, u is uniquely determined up to multiplication by a root of unity in F. This is immediate when $|S| \geq 3$, and in the case $|S| = 2$ it follows from the product formula.

Example 3.2.4. In Section 3.1.2, we have

$$u = u(1, f) = (\zeta_f - 1)(\zeta_f^{-1} - 1) = 2 - 2\cos(2\pi/f).$$

Indeed,

$$u = 2 - \zeta_f - \zeta_f^{-1} = (i \cdot (\zeta_{2f} + \zeta_{2f}^{-1}))^2 \in \mathbf{Q}(\zeta_{4f}).$$

Hence in particular, the extension $\mathbf{Q}(u^{1/2}, \zeta_f + \zeta_f^{-1})/\mathbf{Q}$ is abelian.

We collect below some basic facts concerning Conjecture 3.2.1.

Proposition 3.2.5. *Conjecture 3.2.1 is true if S contains at least two places which split completely.*

Proof. See [Das1999, Proposition 4.3.4]. □

In view of the Proposition 3.2.5, the validity of the conjecture is independent of the choice of v. Accordingly, we simply write $\mathrm{St}(F/K, S)$ for Conjecture 3.2.1.

Proposition 3.2.6. *St$(F/K, S)$ implies St$(F/K, S')$ for $S \subset S'$.*

Proof. See [Das1999, Proposition 4.3.7]. □

Proposition 3.2.7. *If $k \subset K \subset F$, then St$(F/k, S)$ implies St$(K/k, S)$.*

Proof. See [Das1999, Proposition 4.3.8]. □

Remark 3.2.8. By Proposition 3.2.5, it suffices to treat the situation in which S contains exactly one place v that splits completely in F. Since every complex place splits completely in any extension, we are reduced to the following three cases ([DG2011, § 1.4]):

- **Case** TR_∞**.** The field K is totally real and v is a real (archimedean) place. The places of F above v are real, while all other archimedean places of F are complex.
- **Case** ATR**.** The field K is almost totally real, i.e. it has exactly one complex place v and all other archimedean places are real. The field F is totally complex.
- **Case** TR_p**.** The field K is totally real and v is a finite place. The field F is totally complex.

Theorem 3.2.9. *St$(F/K, S)$ is true if $K = \mathbf{Q}$ or if K is an imaginary quadratic field.*

We do not give a complete proof of the above theorem here; this was established by Stark himself in [Sta1980]. Very roughly speaking, in the case TR_∞ with $K = \mathbf{Q}$, the Stark units are given by cyclotomic units, which we have essentially established in Section 3.1.2. In the case TR_p with $K = \mathbf{Q}$, the Stark units are given by Gauss sums. In the ATR case, when K is an imaginary quadratic field, the Stark units are given by elliptic units (which differ slightly from the elliptic units defined in the previous chapter), as we will explain in the next section.

3.3 | The conjecture for quadratic imaginary fields

3.3.1 | The Kronecker second limit formula

Recall that we define the Siegel function in (2.6). Suppose a_1, a_2 have denominator N. Define

$$f_{a_1,a_2}(\tau) = g_{a_1,a_2}(\tau)^{12N}.$$

By the transformation law of Klein forms, f_{a_1,a_2} depends only on the class of (a_1, a_2) in $(\mathbf{Q}/\mathbf{Z})^2$ (cf. [Lan1987, p. 262]).

Let K be an imaginary quadratic field, and let $\mathfrak{f}$ be a nonzero proper ideal of $\mathcal{O}_K$. Denote by f the smallest positive integer contained in $\mathfrak{f} \cap \mathbf{Z}$. Let $G := \mathrm{Gal}(K(\mathfrak{f})/K)$, and let $\sigma \in G$. Choose an ideal $\mathfrak{a} \subset \mathcal{O}_K$, relatively prime to $\mathfrak{f}$, such that $\sigma_\mathfrak{a} = \sigma$. Finally, set $S = \{\infty\} \cup \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{f}\}$. Suppose that

$$\mathfrak{a}^{-1}\mathfrak{f} = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2 = \omega_2(\mathbf{Z}\tau + \mathbf{Z}), \quad \tau := \omega_1/\omega_2 \in \mathcal{H}.$$

Then $1 \in (f\mathfrak{a})^{-1}\mathfrak{f}$, so there exists a pair $a = (a_1, a_2) \in \left(\frac{1}{f}\mathbf{Z}\right)^2$ such that

$$1 = a_1\omega_1 + a_2\omega_2.$$

We define the *Siegel-Ramachandra invariant*

$$u(\sigma, \mathfrak{f}) = f_{a_1,a_2}(\tau) = g_{a_1,a_2}(\tau)^{12f}.$$

Using (2.7),

$$u(\sigma, \mathfrak{f}) = \theta(1; \mathfrak{f}\mathfrak{a}^{-1})^f.$$

Therefore $u(\sigma, \mathfrak{f})$ does not depend on the choices of τ and (a_1, a_2).

The following theorem is a slight modification of [DG2011, Theorem 3.27].

Theorem 3.3.1. *The Siegel-Ramachandra invariants have the following properties:*

(i) The quantity $u(\sigma, \mathfrak{f})$ depends only on σ and not on our choice of $\mathfrak{a}$.

(ii) If $\mathfrak{f}$ is divisible by two distinct primes then $u(\sigma, \mathfrak{f})$ is a unit in $\mathcal{O}_{K(\mathfrak{f})}$. If $\mathfrak{f} = \mathfrak{p}^n$, it is a $\mathfrak{p}$-unit. Moreover, $u(\sigma, \mathfrak{f})/u(\tau, \mathfrak{f})$ is a unit for all $\sigma, \tau \in G$.

(iii) The Shimura reciprocity law holds:

$$u(\sigma, \mathfrak{f})^\tau = u(\sigma\tau, \mathfrak{f}), \quad \sigma, \tau \in G.$$

Proof. Below we provide only the proof of (i). The proofs of (ii) and (iii) depend on the theory of complex multiplication and are similar to the argument in Proposition 2.1.3 and Theorem 2.1.5. Unfortunately, we have not been able to locate a reference containing a complete proof. For complete proofs of weaker versions, see [Lan1987, Chapter 19, § 3, Theorem 3].

Suppose $[\mathfrak{b}] = [\mathfrak{a}] \in \mathrm{Cl}_K(\mathfrak{f})$. There exists $\alpha \in K^\times$ such that $\mathfrak{b} = \alpha\mathfrak{a}$ and $\alpha \equiv 1 \pmod{\mathfrak{f}}$. It follows that $\alpha - 1 \in \mathfrak{f}\mathfrak{a}^{-1}$. Then there exists a pair $(b_1, b_2) \in \left(\frac{1}{f}\mathbf{Z}\right)^2$ such that

$$\alpha = b_1\omega_1 + b_2\omega_2 \quad \text{and} \quad (b_1, b_2) \in (a_1, a_2) + \mathbf{Z}^2.$$

Hence

$$\theta(1; \mathfrak{f}\mathfrak{a}^{-1})^f = f_{a_1,a_2}(\tau) = f_{b_1,b_2}(\tau) = \theta(\alpha; \mathfrak{f}\mathfrak{a}^{-1})^f = \theta(1; \mathfrak{f}\mathfrak{b}^{-1})^f.$$

□

Note that we can extend $u(\cdot, \mathfrak{f})$ linearly to $\mathbf{Z}[G]$. Then the relation between $\Theta_{\mathfrak{f},\mathfrak{a}}(1)$ and $u(\sigma, \mathfrak{f})$ is given by

$$u(\mathrm{N}(\mathfrak{a}) - \sigma, \mathfrak{f}) = \frac{\theta(1; \mathfrak{f})^{f\mathrm{N}(\mathfrak{a})}}{\theta(1; \mathfrak{f}\mathfrak{a}^{-1})^f} = \Theta_{\mathfrak{f},\mathfrak{a}}(1)^f. \tag{3.3}$$

The moral is that to get well defined units from $\theta(z; L)$ evaluated at a primitive $\mathfrak{f}$-division point, we may either raise to power f, or twist by $\mathrm{N}(\mathfrak{a}) - \sigma$, and the two operations are related by (3.3).

The last ingredient we need is the following Kronecker second limit formula. The version stated here is taken from [DG2011, Theorem 3.28]; unfortunately, no proof is provided there. Another version with a complete proof can be found in [Lan1987, Chapter 20, § 5]. We believe that these two versions are equivalent, although the equivalence does not seem to be completely immediate.

Theorem 3.3.2. *Suppose $\omega = (\omega_1, \omega_2)$ with $\tau = \omega_1/\omega_2 \in \mathcal{H}$. We define*

$$Z(z, \omega, s) := \sum_{m,n \in \mathbf{Z}} |z + m\omega_1 + n\omega_2|^{-2s}, \quad z \notin \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2.$$

Assume that a_1, a_2 are not both integral. Then $Z(a_1\omega_1 + a_2\omega_2, \omega, s)$ vanishes at $s = 0$ and

$$Z'(a_1\omega_1 + a_2\omega_2, \omega, 0) = -\log |g_a(\tau)|^2.$$

Let $w_{\mathfrak{f}}$ be the number of roots of unity congruent to 1 modulo $\mathfrak{f}$. Since $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2 + 1 = 1 + \mathfrak{a}^{-1}\mathfrak{f}$, we have

$$w_{\mathfrak{f}}\zeta_{K(\mathfrak{f})/K,S}(\sigma, s) = w_{\mathfrak{f}} \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_K, (\mathfrak{b},\mathfrak{f})=1 \\ \sigma_{\mathfrak{b}} = \sigma}} \frac{1}{\mathrm{N}(\mathfrak{b})^s} = \mathrm{N}(\mathfrak{a})^{-s} \sum_{\alpha \in 1+\mathfrak{a}^{-1}\mathfrak{f}} \mathrm{N}(\alpha)^{-s} = \mathrm{N}(\mathfrak{a})^{-s} Z(1, \omega, s).$$

Corollary 3.3.3. *We have*

$$\zeta'_{K(\mathfrak{f})/K,S}(\sigma, 0) = -\frac{1}{12 f w_{\mathfrak{f}}} \log |f_a(\tau)|^2 = -\frac{1}{12 f w_{\mathfrak{f}}} \log |u(\sigma, \mathfrak{f})|^2.$$

3.3.2 | Sketch of the proof

We are now in the position to sketch the proof of rank one abelian Stark conjecture for quadratic imaginary fields. Let F/K be a finite abelian extension, and S a set of places of K satisfying the assumption of Conjecture 3.2.1.

We may take an integral ideal $\mathfrak{f}$ of K such that

- $\mathfrak{p} \mid \mathfrak{f}$ if and only if $\mathfrak{p} \in S - \{\infty\}$,
- $w_{\mathfrak{f}} = 1$,
- $F \subseteq K(\mathfrak{f})$.

Indeed, it suffices to take $\mathfrak{f} = \left(\prod_{\mathfrak{p} \in S - \{\infty\}} \mathfrak{p}\right)^n$ for n large enough. Note that the set S also satisfies the assumption of Conjecture 3.2.1 for the extension $K(\mathfrak{f})/K$. So we can reduce to the case $\mathrm{St}(K(\mathfrak{f})/K, S)$ by Proposition 3.2.7. We take v the unique infinite place ∞ of K and w a place of $K(\mathfrak{f})$ lying above ∞.

By Theorem 3.3.1(ii), if $|S| \geq 3$ then it follows immediately that $|u(\sigma, \mathfrak{f})|_{w'} = 1$ for all $w' \nmid \infty$, and if $S = \{\infty, \mathfrak{p}\}$ then $|u(\sigma, \mathfrak{f})|_{\tau w''} = |u(\tau^{-1}\sigma, \mathfrak{f})|_{w''} = |u(\sigma, \mathfrak{f})|_{w''}$ for every $\tau \in \mathrm{Gal}(K(\mathfrak{f})/K)$ and any w'' lying above $\mathfrak{p}$. Hence $u(\sigma, \mathfrak{f}) \in U_{\infty,S}$. Stark established in [Sta1980] that the number $e_{\mathfrak{f}}$ of roots of unity contained in $K(\mathfrak{f})$ is a divisor of $12f$. He also showed that, for every $\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)$, one may write

$$u(\sigma, \mathfrak{f}) = u(\sigma)^{12f/e_{\mathfrak{f}}}$$

for a suitable element $u(\sigma) \in K(\mathfrak{f})$, and that these elements are compatible under Galois action in the sense that

$$u(\sigma) = \sigma(u(1)).$$

In addition, $u(1)^{1/e_{\mathfrak{f}}}$ generates an abelian extension of K.

Now, by Corollary 3.3.3,

$$\zeta'_{K(\mathfrak{f})/K,S}(\sigma, 0) = -\frac{1}{e_{\mathfrak{f}}} \log |u(1)^{\sigma}|_w.$$

So we can conclude that $\mathrm{St}(F/K, S)$ is true.

Although the stark unit $\mathbf{u}_{\mathrm{Stark}}$ is defined only up to multiplication by a root of unity, the quotient $\mathbf{u}_{\mathrm{Stark}}^{\mathrm{N}(\mathfrak{a})}/\sigma_{\mathfrak{a}}(\mathbf{u}_{\mathrm{Stark}})$ is is uniquely determined whenever $(\mathfrak{a}, 6\mathfrak{f}) = 1$ (see [Tat1984, Lemma IV.1.1]). We also note that Stark predicted that $\mathbf{u}_{\mathrm{Stark}}^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}}$ should be an $e_{\mathfrak{f}}$-power in $K(\mathfrak{f})$ (cf. [Sta1980, Lemma 6]). Suppose $\mathbf{u}_{\mathrm{Stark}}^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}} = \alpha^{e_{\mathfrak{f}}}$ for some $\alpha \in K(\mathfrak{f})$. Using (3.3), we have

$$\left(\mathbf{u}_{\mathrm{Stark}}^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}}\right)^{12f/e_{\mathfrak{f}}} = \left(u(1)^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}}\right)^{12f/e_{\mathfrak{f}}} = u(\mathrm{N}(\mathfrak{a}) - \sigma_{\mathfrak{a}}, \mathfrak{f}) = \Theta_{\mathfrak{f},\mathfrak{a}}(1)^f.$$

On the other hand, by Proposition 2.3.5, there exists $\beta \in K(\mathfrak{f})$ such that $\Theta_{\mathfrak{f},\mathfrak{a}}(1) = \beta^{12}$. Hence $\alpha^{12f} = \beta^{12f}$. Since $\alpha, \beta \in K(\mathfrak{f})$, there exists ζ such that $\zeta^{e_{\mathfrak{f}}} = 1$ and $\alpha = \beta \cdot \zeta$. It follows that

$$\mathbf{u}_{\mathrm{Stark}}^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}} = \beta^{e_{\mathfrak{f}}}. \tag{3.4}$$

Chapter 4

Elliptic units for complex cubic fields

> "This frozen daiquiri, so well beaten as it is, looks like the sea where the wave falls away from the bow of a ship when she is doing thirty knots."
>
> Ernest Hemingway,
> *Islands in the Stream*

The main goal of this chapter is to introduce the elliptic units for complex cubic fields constructed in [BCG2023] and to work out detailed computations in explicit examples.

4.1 | Elliptic gamma functions

In this section we briefly recall some basic facts about the elliptic gamma function, following [FV2000].

Definition 4.1.1. Let $z \in \mathbf{C}$ and $\operatorname{Im}\tau, \operatorname{Im}\sigma > 0$. We define

$$\Gamma(z,\tau,\sigma) := \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1 - e^{2\pi i(j\tau+k\sigma+z)}}.$$

We extend Γ to a meromorphic function on $\mathbf{C} \times (\mathbf{C} - \mathbf{R}) \times (\mathbf{C} - \mathbf{R})$ by setting

$$\Gamma(z,-\tau,\sigma) = \frac{1}{\Gamma(z+\tau,\tau,\sigma)}, \quad \Gamma(z,\tau,-\sigma) = \frac{1}{\Gamma(z-\sigma,\tau,-\sigma)}.$$

The zeros and poles of this function can be read off directly from its infinite product expansion, and it satisfies the following interesting identity involving θ_0 ([FV2000, Theorem 3.1]):

$$\begin{aligned} \Gamma(z,\tau,\sigma) &= \Gamma(z,\sigma,\tau), \\ \Gamma(z+1,\tau,\sigma) &= \Gamma(z,\tau,\sigma), \\ \Gamma(z+\sigma,\tau,\sigma) &= \theta_0(z,\tau)\Gamma(z,\tau,\sigma), \\ \Gamma(z+\tau,\tau,\sigma) &= \theta_0(z,\sigma)\Gamma(z,\tau,\sigma). \end{aligned} \tag{4.1}$$

Just as for θ_0, the function Γ admits a summation formula that is useful for numerical computations. Define

$$S(z,\tau,\sigma) := -\frac{i}{2}\sum_{j=1}^{\infty}\frac{\sin(\pi j(2z-\tau-\sigma))}{j\sin(\pi j\tau)\sin(\pi j\sigma)}. \tag{4.2}$$

We have ([FV2000, § 3.3, (15)]):

$$\Gamma(z,\tau,\sigma) = \exp\left(S(z,\tau,\sigma)\right), \quad |\mathrm{Im}(2z-\tau-\sigma)| < |\mathrm{Im}\,\tau| + |\mathrm{Im}\,\sigma|. \tag{4.3}$$

$$|\mathrm{Im}(2z-\tau-\sigma)| < |\mathrm{Im}\,\tau| + |\mathrm{Im}\,\sigma|.$$

Moreover, in analogy with the transformation properties of θ_0 under $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$, one is naturally led to expect that the elliptic gamma function admits similar transformation properties under $\mathrm{SL}_3(\mathbf{Z}) \ltimes \mathbf{Z}^3$. This was shown to be the case by Felder and Varchenko.

Theorem 4.1.2. *Suppose that* $\tau, \sigma, \sigma/\tau, \tau+\sigma \in \mathbf{C} - \mathbf{R}$. *Let*

$$\begin{aligned} P_3(z,\tau,\sigma) = \frac{z^3}{3\tau\sigma} - \frac{\tau+\sigma-1}{2\tau\sigma}z^2 + \frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{6\tau\sigma}z \\ + \frac{1}{12}(\tau+\sigma-1)(\tau^{-1}+\sigma^{-1}-1). \end{aligned}$$

Then

$$\begin{aligned} \Gamma(z,\tau+1,\sigma) = \Gamma(z,\tau,\sigma+1) &= \Gamma(z,\tau,\sigma), \\ \Gamma(z,\tau+\sigma,\sigma) &= \frac{\Gamma(z,\tau,\sigma)}{\Gamma(z+\tau,\tau,\sigma+\tau)}, \\ \Gamma(z/\sigma,\tau/\sigma,-1/\sigma) &= e^{i\pi P_3(z,\tau,\sigma)}\Gamma((z-\sigma)/\tau,-1/\tau,-\sigma/\tau)\Gamma(z,\tau,\sigma), \\ \Gamma(z/\tau,-1/\tau,\sigma/\tau) &= e^{i\pi P_3(z,\tau,\sigma)}\Gamma((z-\tau)/\sigma,-\tau/\sigma,-1/\sigma)\Gamma(z,\tau,\sigma). \end{aligned}$$

Proof. See [FV2000, Theorem 4.1]. □

4.2 | Elliptic gamma functions associated with wedges

In this section, we briefly review the construction of elliptic gamma functions associated with wedges introduced by Felder, Henriques, Rossi, and Zhu in [FHRZ2008]. The material in Remark 4.2.7 is motivated by observations from explicit computations.

Let L be a free $\mathbf{Z}$-module of rank 3. Fix a $\mathbf{Z}$-basis of L, and hence an orientation. This choice identifies L with $\mathbf{Z}^3$, and therefore determines an action of $\mathrm{SL}_3(\mathbf{Z})$ on L. This choice also gives rise to a map $\det\colon \wedge^3_{\mathbf{Z}} L \longrightarrow \mathbf{Z}$. Denote by $L^* = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ the dual of L. For $F = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, set $L_F = L \otimes_{\mathbf{Z}} F$, and similarly define $L^*_F = L^* \otimes_{\mathbf{Z}} F$.

An element l of L is called *primitive* if the condition $l \in \lambda L$ for some $\lambda \in \mathbf{Z}$ implies that $\lambda = \pm 1$. In particular, 0 is not primitive. Given a primitive element $a \in L^*$, we associate to it the oriented plane

$$H(a) \coloneqq \ker a \otimes_{\mathbf{Z}} \mathbf{Q}.$$

An ordered basis (λ, μ) of $H(a)$ is said to be oriented if $\det(\lambda, \mu, \delta) > 0$ for every $\delta \in L_{\mathbf{Q}}$ such that $a(\delta) > 0$. We then define

$$U_a \coloneqq \left\{ z \in L^*_{\mathbf{C}} \,\middle|\, \mathrm{Im}\left(z(\lambda)\overline{z(\mu)}\right) > 0 \right\},$$

where (λ, μ) is any oriented basis of $H(a)$. The following lemma shows that this definition is independent of the choice of oriented basis.

Lemma 4.2.1. *Let $z \in L^*_{\mathbf{C}}$. The following are equivalent:*

(i) $\mathrm{Im}\left(z(\lambda)\overline{z(\mu)}\right) > 0$ *for some oriented basis* (λ, μ) *of* $H(a)$.

(ii) $\mathrm{Im}\left(z(\lambda)\overline{z(\mu)}\right) > 0$ *for all oriented bases* (λ, μ) *of* $H(a)$.

(iii) $z(\mu) \neq 0$ *and* $\mathrm{Im}\left(z(\lambda)/z(\mu)\right) > 0$ *for some oriented basis* (λ, μ) *of* $H(a)$.

(iv) $z(\lambda) \neq 0$ *and* $\mathrm{Im}\left(z(\mu)/z(\lambda)\right) > 0$ *for some oriented basis* (λ, μ) *of* $H(a)$.

Proof. See [FHRZ2008, Lemma 3.2]. □

Next we consider pairs of primitive elements of L^*.

Definition 4.2.2. A *wedge* is an ordered pair of oriented planes through 0 in $L_{\mathbf{Q}}$, or equivalently, an ordered pair $(a, b) \in (L^*_{\mathrm{prim}})^2$. We say that a wedge is *in general position* if a and b are $\mathbf{Z}$-linearly independent, or equivalently, if the corresponding planes intersect in a line.

Fix a wedge (a, b) in general position. Then there exist a unique positive integer $s(a, b)$ and a unique primitive element $\gamma(a, b) \in L$ such that

$$\det(a, b, \cdot) = s(a, b)\gamma(a, b). \tag{4.4}$$

When no confusion is likely to arise, we simply write s and γ. We also associate to the wedge (a,b) the cone

$$C(a,b) := \{\, l \in L \mid a(l) > 0,\ b(l) \le 0 \,\}.$$

Definition 4.2.3. Let (a,b) be a wedge in general position. We define

$$\Gamma_{a,b}(w,z;L) := \frac{\prod_{l \in C(a,b)/\mathbf{Z}\gamma} \left(1 - e^{-2\pi i (z(l)-w)/z(\gamma)}\right)}{\prod_{l \in C(b,a)/\mathbf{Z}\gamma} \left(1 - e^{2\pi i (z(l)-w)/z(\gamma)}\right)}, \quad w \in \mathbf{C}, z \in U_a \cap U_b, \tag{4.5}$$

where the group $\mathbf{Z}\gamma$ acts on $C(a,b)$ and $C(b,a)$ by translation.

Proposition 4.2.4. [FHRZ2008, Proposition 3.3] *The product* (4.5) *converges to a meromorphic function on* $\mathbf{C} \times (U_a \cap U_b)$*. It has simple zeros at* $w = z(l) + nz(\gamma)$*, with* $l \in C(a,b), n \in \mathbf{Z}$*, and simple poles at* $w = z(l) + nz(\gamma)$*, with* $l \in C(b,a), n \in \mathbf{Z}$*.*

Proof. Let $\alpha, \beta \in L$ such that $b(\alpha) = a(\beta) = 0$ and $a(\alpha), b(\beta) > 0$. By construction, (β,γ) is an oriented basis of $H(a)$ and (γ,α) is an oriented basis of $H(b)$. By Proposition 4.2.6, $\Gamma_{a,b}(w,z;L)$ can be expressed as a finite product, with each factor being meromorphic for (w,z) with $\mathrm{Im}\left(-\frac{z(\alpha)}{z(\gamma)}\right) > 0$ and $\mathrm{Im}\left(\frac{z(\beta)}{z(\gamma)}\right) > 0$, i.e. for $(w,z) \in \mathbf{C} \times (U_a \cap U_b)$. In this domain, the divisor of $\Gamma_{a,b}(w,z;L)$ can be determined directly from the zeros of the factors in (4.5). □

Remark 4.2.5. The above proof was suggested to the author by Pierre Charollois in response to an inquiry. The original proof in [FHRZ2008, Proposition 3.3] does not appear to be entirely rigorous, and making it precise was not straightforward for the present author. Pierre Charollois also pointed out that it is in fact more convenient to take (4.6) as the definition. Of course, if this definition is adopted, one must verify that it is independent of the choice of α and β.

The following proposition shows that the elliptic gamma function associated with a wedge can be expressed in various ways as a finite product of ordinary elliptic gamma functions.

Proposition 4.2.6. [FHRZ2008, Proposition 3.5] *Let* $\alpha, \beta \in L$ *such that* $b(\alpha) = a(\beta) = 0$ *and* $a(\alpha), b(\beta) > 0$*. Set*

$$F(\alpha,\beta) := \{l \in L : 0 \le a(l) < a(\alpha), 0 \le b(l) < b(\beta)\}.$$

Then

$$\Gamma_{a,b}(w,z;L) = \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \Gamma\left(\frac{w+z(l)}{z(\gamma)}, \frac{z(\alpha)}{z(\gamma)}, \frac{z(\beta)}{z(\gamma)}\right). \tag{4.6}$$

Proof. Note that, modulo $\mathbf{Z}\gamma$, each element $l \in C(a,b)$ admits a unique expression of the form $l = -l' + (j+1)\alpha - k\beta$, where $l' \in F(\alpha,\beta)$ and $j,k \in \mathbf{Z}_{\ge 0}$. Similarly, any $l \in C(b,a)$ can be uniquely written modulo $\mathbf{Z}\gamma$ as $l = -l' - j\alpha + (k+1)\beta$, with $l' \in F(\alpha, \beta)$ and $j, k \in \mathbf{Z}_{\geq 0}$. It follows that

$$\prod_{l \in C(a,b)/\mathbf{Z}\gamma} \left(1 - e^{-2\pi i (z(l)-w)/z(\gamma)}\right) = \prod_{j,k=0}^{\infty} \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \left(1 - e^{-2\pi i(-z(l)+(j+1)z(\alpha)-kz(\beta)-w)/z(\gamma)}\right),$$

and similarly,

$$\prod_{\delta \in C(b,a)/\mathbf{Z}\gamma} \left(1 - e^{2\pi i (z(l)-w)/z(\gamma)}\right) = \prod_{j,k=0}^{\infty} \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \left(1 - e^{2\pi i(-z(l)-jz(\alpha)+(k+1)z(\beta)-w)/z(\gamma)}\right).$$

Putting these together, we obtain

$$\begin{aligned}
\Gamma_{a,b}(w, z; L) &= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i(-(j+1)z(\alpha)/z(\gamma)+kz(\beta)/z(\gamma)+(z(l)+w)/z(\gamma))}}{1 - e^{2\pi i(-jz(\alpha)/z(\gamma)+(k+1)z(\beta)/z(\gamma)-(z(l)+w)/z(\gamma))}} \\
&= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i(-jz(\alpha)/z(\gamma)+kz(\beta)/z(\gamma)+(z(l)+w)/z(\gamma)-z(\alpha)/z(\gamma))}}{1 - e^{2\pi i(-(j+1)z(\alpha)/z(\gamma)+(k+1)z(\beta)/z(\gamma)-((z(l)+w)/z(\gamma)-z(\alpha)/z(\gamma)))}} \\
&= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \Gamma\left(\frac{z(l)+w}{z(\gamma)} - \frac{z(\alpha)}{z(\gamma)}, -\frac{z(\alpha)}{z(\gamma)}, \frac{z(\beta)}{z(\gamma)}\right)^{-1} \\
&= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \Gamma\left(\frac{w+z(l)}{z(\gamma)}, \frac{z(\alpha)}{z(\gamma)}, \frac{z(\beta)}{z(\gamma)}\right),
\end{aligned}$$

where the last equality is merely a change of notation. □

Remark 4.2.7. There is a standard (though not canonical) choice of α, β such that $|F(\alpha, \beta)/\mathbf{Z}\gamma| = s$. We proceed to show how to find this standard choice of α and β. Let (l_1, l_2, l_3) be an oriented basis of L. Let (l_1^*, l_2^*, l_3^*) be the dual basis, i.e. $l_i^*(l_j) = \delta_{ij}$. Then there are unique $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{Z}$ such that $a = a_1 l_1^* + a_2 l_2^* + a_3 l_3^*$, $b = b_1 l_1^* + b_2 l_2^* + b_3 l_3^*$, and $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = 1$. We compute

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{pmatrix} = a_2b_3 - a_3b_2, \quad \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 1 & 0 \end{pmatrix} = a_3b_1 - a_1b_3, \quad \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{pmatrix} = a_1b_2 - a_2b_1.$$

Hence

$$\det(a, b, \cdot) = (a_2b_3 - a_3b_2)l_1 + (a_3b_1 - a_1b_3)l_2 + (a_1b_2 - a_2b_1)l_3,$$

$$s = \gcd(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1),$$

and

$$\gamma = (a_2b_3 - a_3b_2)l_1/s + (a_3b_1 - a_1b_3)l_2/s + (a_1b_2 - a_2b_1)l_3/s.$$

Write $\gamma = c_1l_1 + c_2l_2 + c_3l_3$ for some $(c_1, c_2, c_3) \in \mathbf{Z}^3$. There exist $(d_1, d_2, d_3) \in \mathbf{Z}$ such that

$$(c_1, c_2, c_3) \cdot (d_1, d_2, d_3) = 1$$

Let

$$\alpha_0 = \det(\cdot, b, d_1l_1^* + d_2l_2^* + d_3l_3^*) = (b_1, b_2, b_3) \times (d_1, d_2, d_3)(l_1, l_2, l_3)^t.$$

Then $b(\alpha_0) = 0$ and

$$a(\alpha_0) = s(c_1, c_2, c_3) \cdot (d_1, d_2, d_3) = s > 0.$$

Similarly, let

$$\beta_0 = \det(a, \cdot, d_1l_1^* + d_2l_2^* + d_3l_3^*) = (d_1, d_2, d_3) \times (a_1, a_2, a_3)(l_1, l_2, l_3)^t.$$

Then $a(\beta_0) = 0$ and $b(\beta_0) = s > 0$. Let

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3).$$

Consider the homomorphism

$$T\colon \mathbf{Z}^3 \to \mathbf{Z}^2, \quad T(x) = (A \cdot x, B \cdot x).$$

We have $\ker T = \mathbf{Z}(c_1, c_2, c_3)$. Therefore

$$\#\left(F(\alpha_0, \beta_0)/\mathbf{Z}\gamma\right) = \#(\operatorname{im} T \cap [0, s-1]^2).$$

Observe that $\operatorname{im} T \subset \mathbf{Z}^2$ is a rank 2 sublattice and by Smith normal form, $[\mathbf{Z}^2 : \operatorname{im} T] = s$. It follows that the reduction map $\mathbf{Z}^2 \to (\mathbf{Z}/s\mathbf{Z})^2$ injects $\mathbf{Z}^2/\operatorname{im} T$ into $(\mathbf{Z}/s\mathbf{Z})^2$. Further, we have

$$[(\mathbf{Z}/s\mathbf{Z})^2 : \operatorname{im} T/s\mathbf{Z}^2] = [\mathbf{Z}^2 : \operatorname{im} T] = s.$$

But $|(\mathbf{Z}/s\mathbf{Z})^2| = s^2$. Hence

$$|\operatorname{im} T/s\mathbf{Z}^2| = s.$$

So modulo s, exactly s residue classes are hit by $\operatorname{im} T$. Picking the unique representatives of these classes in the box $[0, s-1]^2$, we get exactly s integer pairs in that square that lie in $\operatorname{im} T$.

We claim that the above choice of α, β minimizes $|F(\alpha, \beta)/\mathbf{Z}\gamma|$. Let $N_1 = a(\alpha)$ and $N_2 = b(\beta)$. Consider the reduction map $\pi : \mathbf{Z}^2 \to \mathbf{Z}/N_1\mathbf{Z} \times \mathbf{Z}/N_2\mathbf{Z}$. We have

$$|F(\alpha, \beta)/\mathbf{Z}\gamma| = |\pi(\operatorname{im} T)|$$

$$
\begin{aligned}
&= [\operatorname{im} T : N_1\mathbf{Z} \times N_2\mathbf{Z}] \\
&= \frac{[\mathbf{Z}^2 : N_1\mathbf{Z} \times N_2\mathbf{Z}]}{[\mathbf{Z}^2 : \operatorname{im} T]} \\
&= \frac{N_1 N_2}{s}.
\end{aligned}
$$

On the other hand, observe that

$$\ker b = \mathbf{Z}(b_2, -b_1, 0)(l_1, l_2, l_3)^t + \mathbf{Z}(b_3, 0, -b_1)(l_1, l_2, l_3)^t + \mathbf{Z}(0, b_3, -b_2)(l_1, l_2, l_3)^t.$$

It follows that

$$N_1 \in \gcd(a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2)\mathbf{Z} = s\mathbf{Z}.$$

This also proves that $\ker b = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\gamma$. Similarly, one can show that $s \mid N_2$. Thus we can conclude that

$$|F(\alpha, \beta)/\mathbf{Z}\gamma| \geq s.$$

We conclude this section with a lemma describing the behavior of the elliptic gamma function under the action of $\mathrm{SL}(L)$. Here and throughout, we use the symbol "$\cdot$" to denote the induced action on L^*. More explicitly, for $g \in \mathrm{SL}(L)$, $a \in L^*$, and $l \in L$, we define

$$(g \cdot a)(l) = a(g^{-1}l).$$

Lemma 4.2.8. [BCG2023, (2.8)] *Let (a, b) be a wedge in general position, and let $g \in \mathrm{SL}(L)$. Then $g \cdot a$ is primitive, $U_{g\cdot a} = g \cdot U_a$, and*

$$\Gamma_{g\cdot a, g\cdot b}(w, g \cdot z; L) = \Gamma_{a,b}(w, z; L) \tag{4.7}$$

for all $w \in \mathbf{C}$ and $z \in U_a \cap U_b$.

Proof. The first assertion is immediate. For the second assertion, let (μ, δ) be an oriented basis of $H(a)$. Since $g \in \mathrm{SL}(L)$, we have $(g\mu, g\delta)$ is an oriented basis of $H(g \cdot a)$. On the other hand, for every $z \in L^*_{\mathbf{C}}$, we have

$$\operatorname{Im}\left(z(g\mu)\overline{z(g\delta)}\right) = \operatorname{Im}\left(g^{-1} \cdot z(\mu)\overline{g^{-1} \cdot z(\delta)}\right).$$

This implies that the map $z \mapsto g^{-1} \cdot z$ gives a bijection between $U_{g\cdot a}$ and U_a. It follows that $U_{g\cdot a} = g \cdot U_a$.

Now, for all $c \in L^*$, we have

$$\det(g \cdot a, g \cdot b, c) = \det(a, b, g^{-1} \cdot c) = g^{-1} \cdot c(s\gamma(a, b)) = c(s(g\gamma(a, b))).$$

Moreover, $g\gamma(a,b)$ is primitive. This implies that $\gamma(g\cdot a, g\cdot b) = g\gamma(a,b)$. Let $\gamma = \gamma(a,b)$. Observe that the map $l \mapsto gl$ gives a bijection between $C(a,b)$ (resp. $C(b,a)$) and $C(g\cdot a, g\cdot b)$ (resp. $C(g\cdot b, g\cdot a)$), which also restricts to a bijection between $\mathbf{Z}\gamma$ and $\mathbf{Z}g\gamma$. Thus, for $z \in U_a \cap U_b$ and $w \in \mathbf{C}$, we have

$$\begin{aligned}\Gamma_{g\cdot a, g\cdot b}(w, g\cdot z; L) &= \frac{\prod_{l\in C(g\cdot a, g\cdot b)/\mathbf{Z}g\gamma}\left(1 - e^{-2\pi i((g\cdot z(l)-w)/g\cdot z(g\gamma))}\right)}{\prod_{l\in C(g\cdot b, g\cdot a)/\mathbf{Z}g\gamma}\left(1 - e^{2\pi i((g\cdot z(l)-w)/g\cdot z(g\gamma))}\right)} \\ &= \frac{\prod_{l'\in C(a,b)/\mathbf{Z}\gamma}\left(1 - e^{-2\pi i((g\cdot z(gl')-w)/z(\gamma))}\right)}{\prod_{l'\in C(b,a)/\mathbf{Z}\gamma}\left(1 - e^{2\pi i((g\cdot z(gl')-w)/z(\gamma))}\right)} \\ &= \Gamma_{a,b}(w, z; L).\end{aligned}$$

This proves the third assertion. □

4.3 | Elliptic gamma functions associated to complex cubic fields and their values

In this section, we follow [BCG2023] in constructing elliptic units for complex cubic fields. Our setup differs slightly from theirs in that it allows for greater flexibility in the choice of input data. This extension is motivated by our study of Section 2.3 and by extensive computations in examples, where such additional freedom proved useful. Although it does not lead to new units, it does simplify the calculations.

4.3.1 | Admissible elements

Let K be a complex cubic field, and let $\sigma_{\mathbf{R}}\colon K \to \mathbf{R}$ be its unique real embedding. Fix also, once and for all, a complex embedding $\sigma_{\mathbf{C}}\colon K \to \mathbf{C}$. The map

$$x \longmapsto (\sigma_{\mathbf{R}}(x), \sigma_{\mathbf{C}}(x))$$

induces an isomorphism

$$K \otimes \mathbf{R} \cong \mathbf{R} \times \mathbf{C}.$$

We therefore endow K with the orientation obtained by pulling back the standard orientation on $\mathbf{R} \times \mathbf{C}$.

Let L be an ideal of K. Let $\mathfrak{f}$ be a nontrivial ideal of $\mathcal{O}_K$, and let f be the smallest positive integer in $\mathfrak{f} \cap \mathbf{Z}$. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ such that $\mathrm{N}(\mathfrak{a})$ is a rational prime and $(\mathrm{N}(\mathfrak{a}), f) = 1$. While this hypothesis on $\mathfrak{a}$ may seem somewhat ad hoc at first, we shall show later that it ensures the existence of a $\mathbf{Z}$-basis $(\omega_1, \omega_2, \omega_3)$ of L with the property that $(\omega_1, \omega_2, \omega_3/\mathrm{N}(\mathfrak{a}))$ is a

$\mathbf{Z}$-basis of $\mathfrak{a}^{-1}L$ enjoying certain additional properties. We call such an ideal $\mathfrak{a}$ a *smoothing ideal for* $\mathfrak{f}$.

Example 4.3.1. Let $K = \mathbf{Q}(\rho)$ with $\rho^3 - 3 = 0$. Let $\mathfrak{f} = (\rho)$ be the unique ideal of norm 3. Then $\mathfrak{a} = (\rho - 2)$, the unique ideal of norm 5, is a smoothing ideal for $\mathfrak{f}$.

Definition 4.3.2. Let v be a primitive $\mathfrak{f}$-division point of L. An element $\lambda \in L$ is *v-admissible for the data* $L, \mathfrak{f}, \mathfrak{a}$ if $\lambda/f \equiv v \pmod{L}$ and $\lambda/\mathrm{N}(\mathfrak{a}) \in \mathfrak{a}^{-1}L - L$.

Lemma 4.3.3.

(i) Admissible elements for the data $L, \mathfrak{f}, \mathfrak{a}$ *exist.*

(ii) Let $\mathfrak{c}$ *be an integral ideal of* K *with* $(\mathfrak{c}, \mathfrak{f}) = 1$*, and let* λ *be* v*-admissible for the data* $\mathfrak{c}^{-1}L, \mathfrak{f}, \mathfrak{a}$*. Then, for every* $\alpha \in 1 + \mathfrak{f}\mathfrak{c}^{-1}$*, the element* $\alpha^{-1}\lambda$ *is* v*-admissible for the data* $\alpha^{-1}\mathfrak{c}^{-1}L, \mathfrak{f}, \mathfrak{a}$*. In particular, if* $\mathfrak{c} = \mathcal{O}_K$ *and* $\alpha \in 1 + \mathfrak{f}$*, then* $\alpha^{-1}\lambda$ *is* v*-admissible for the data* $\alpha^{-1}L, \mathfrak{f}, \mathfrak{a}$*. If moreover* $\alpha \in \mathcal{O}_K^\times \cap (1 + \mathfrak{f})$*, then* $\alpha^{-1}\lambda$ *is* v*-admissible for* $L, \mathfrak{f}, \mathfrak{a}$*.*

Proof. (i) It is immediate that

$$\lambda \text{ is } v\text{-admissible} \iff \begin{cases} \lambda \in \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L, \\ \lambda \equiv fv \mod fL, \\ \lambda \not\equiv 0 \mod \mathrm{N}(\mathfrak{a})L. \end{cases}$$

By the Chinese remainder theorem, we have

$$L/(f\mathrm{N}(\mathfrak{a}))L \cong L/fL \times L/\mathrm{N}(\mathfrak{a})L,$$

which restricts to

$$\begin{aligned} \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L/(f\mathrm{N}(\mathfrak{a}))L &\cong \left(\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L + fL\right)/fL \times \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L/\mathrm{N}(\mathfrak{a})L \\ &\cong L/fL \times \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L/\mathrm{N}(\mathfrak{a})L \quad \text{by } (\mathrm{N}(\mathfrak{a}), f) = 1. \end{aligned}$$

This completes the proof since $fv \in L$.

(ii) It suffices to show that $\alpha^{-1}\lambda \equiv fv \mod f\alpha^{-1}\mathfrak{c}^{-1}L$. Indeed,

$$\alpha^{-1}\lambda - fv = \alpha^{-1}(\lambda - fv) + fv\alpha^{-1}(1 - \alpha) \in f\alpha^{-1}\mathfrak{c}^{-1}L.$$

□

Remark 4.3.4. The proof of the first part of Lemma 4.3.3 is very useful for finding the initial admissible elements, while the second part allows one to construct further admissible elements from those already obtained.

Example 4.3.5. Recall from Example 4.3.1 that $K = \mathbf{Q}(\rho)$ with $\rho^3 = 3$, $\mathfrak{f} = (\rho)$, and $\mathfrak{a} = (\rho - 2)$. We next characterize the 1-admissible elements λ for the data $\mathfrak{f}, \mathfrak{f}, \mathfrak{a}$. It is clear that $f = 3$. Guided by the proof of Lemma 4.3.3.(i), we construct admissible elements as follows.

1. We find $t_1, t_2 \in \mathbf{Z}$ such that $\mathrm{N}(\mathfrak{a})t_1 + ft_2 = 1$. In our case, we may take $t_1 = -1$ and $t_2 = 2$.

2. We pick any element $\lambda_0 \in \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L - \mathrm{N}(\mathfrak{a})L$. We have $L = 3\mathbf{Z} + \mathbf{Z}\rho + \mathbf{Z}\rho^2$ and $\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L = 15\mathbf{Z} + 5\mathbf{Z}\rho + \mathbf{Z}(\rho^2 + 2\rho + 9)$. Moreover, since $\mathfrak{a}$ is a smoothing ideal, we also have $\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L/\mathrm{N}(\mathfrak{a})L \cong \mathbf{Z}/5\mathbf{Z}$. It follows that

$$\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L - \mathrm{N}(\mathfrak{a})L = \{(3n+5m_{11}, 2n+5m_{12}, n+5m_{13})(3, \rho, \rho^2)^t : m_{1j} \in \mathbf{Z}, 1 \le n \le 4\}. \tag{4.8}$$

3. Let $\lambda_1 = \mathrm{N}(\mathfrak{a})t_1 f + ft_2\lambda_0 = 6\lambda_0 - 15$. Then $\lambda_1 \equiv q \mod fL$ and $\lambda_1 \equiv \lambda_0 \mod \mathrm{N}(\mathfrak{a})L$. On the other hand, it is clear that $\lambda \in \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L$ by construction. Hence λ_1 is admissible. Moreover, all admissible λ with $\lambda \equiv \lambda_0 \mod \mathrm{N}(\mathfrak{a})L$ are of the form $\lambda_1 + l$ for some $l \in f\mathrm{N}(\mathfrak{a})L$. Then by (4.8), we have

$$\begin{aligned}\{\lambda : \lambda \text{ is 1-admissible}\} &= \{(18n + 30m_{11} + 15m_{21} - 5, 12n + 30m_{12} + 15m_{22},\\ &\quad 6n + 30m_{13} + 15m_{23})(3, \rho, \rho^2)^t : m_{ij} \in \mathbf{Z}, 1 \le n \le 4\}\\ &= \{(18n + 15m_1 - 5, 12n + 15m_2, 6n + 15m_3)(3, \rho, \rho^2)^t :\\ &\quad m_i \in \mathbf{Z}, 1 \le n \le 4\}.\end{aligned} \tag{4.9}$$

The general case proceeds similarly (with the obvious modifications). We may take

$$\lambda = (4, 6, 3)(3, \rho, \rho^2)^t = 3\rho^2 + 6\rho + 12.$$

The reason for choosing this particular λ will become clear in Remark 4.3.17.

4.3.2 | Wedges associated to admissible elements and smoothed elliptic gamma functions

Consider the group of units

$$\mathcal{O}_{\mathfrak{f}}^{+,\times} := \{u \in \mathcal{O}_K^\times \mid u - 1 \in \mathfrak{f} \text{ and } \sigma_{\mathbf{R}}(u) > 0\}.$$

By Dirichlet's unit theorem, it is of rank one. Fix a generator $\epsilon \in \mathcal{O}_{\mathfrak{f}}^{+,\times}$ such that

$$\epsilon_{\mathbf{R}} := \sigma_{\mathbf{R}}(\epsilon) \in (0, 1)$$

and set

$$\epsilon_{\mathbf{C}} := \sigma_{\mathbf{C}}(\epsilon).$$

Multiplication by ϵ preserves both L and its orientation, since

$$\mathrm{N}_{K/\mathbf{Q}}(\epsilon) = \epsilon_{\mathbf{R}}|\epsilon_{\mathbf{C}}|^2 = 1 > 0.$$

Hence ϵ defines an element of $\mathrm{SL}(L)$.

We assume throughout that

$$\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_K^{\times} \cap (1+\mathfrak{f}).$$

Equivalently, there is no unit in $\mathcal{O}_K^{\times} \cap (1+\mathfrak{f})$ of negative norm. In particular, this implies that $-1 \notin 1+\mathfrak{f}$, and hence that $f \geq 3$. Although this assumption may seem somewhat ad hoc at first, it ensures that the real place of K ramifies in the narrow ray class field $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$, thereby excluding the trivial situation. Namely, if the real place of K splits completely in $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$, then both Conjecture 5.2.1 and $\mathrm{St}(K(\mathfrak{f}\cdot\infty_{\mathbf{R}})/K, \{\infty_{\mathbf{R}}, \infty_{\mathbf{C}}\} \cup \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{f}\})$ would be trivial.

Fix a v-admissible element λ and set $h := \lambda/f$. Let $\mathbf{x} \in L_{\mathbf{C}}^*$ be the element induced by the fixed complex embedding $\sigma_{\mathbf{C}}$.

Lemma 4.3.6. [BCG2023, Lemma 2.1] *There exists a unique primitive element $a_\lambda \in L^*$ such that:*

(i) $\det(a_\lambda, \epsilon \cdot a_\lambda, \cdot) \in (L^*)^* = L$ *is a nonzero multiple of* λ*;*

(ii) $\mathbf{x} \in U_{a_\lambda}$.

Proof. Since $\epsilon \notin \mathbf{Q}$, the elements λ and $\epsilon^{-1}\lambda$ are linearly independent in $L_{\mathbf{Q}}$. We claim that for $a \in L^*_{\mathrm{prim}}$, $\det(a, \epsilon\cdot a, \cdot) = n\lambda \in L$ for some $n \in \mathbf{Q}^\times$ if and only if $a \in \ker\lambda \cap \ker(\epsilon^{-1}\lambda)$. To see this, suppose that $\det(a, \epsilon\cdot a, \cdot) = n\lambda$. Then $a(\lambda) = \det(a, \epsilon\cdot a, a)/n = 0$. Similarly, $a(\epsilon^{-1}\lambda) = \det(a, \epsilon\cdot a, \epsilon\cdot a)/n = 0$. Conversely, assume that $a \in \ker\lambda \cap \ker(\epsilon^{-1}\lambda)$. Then $a(\lambda) = \epsilon\cdot a(\lambda) = 0$. This implies that $\lambda = r\gamma(a, \epsilon\cdot a)$ for some $r \in \mathbf{Z}_{\neq 0}$. It follows that $\det(a, \epsilon\cdot a, \cdot) = s\lambda/r$.

Now, there are exactly two primitive elements $\pm a \in \ker\lambda \cap \ker(\epsilon^{-1}\lambda)$. They correspond to the two orientations on the same plane $H(a) \subset K$, and only one of them will satisfy (ii). □

Example 4.3.7. Recall from Example 4.3.5 that $K = \mathbf{Q}(\rho)$, $\rho^3 = 3$, with $L = \mathfrak{f} = (\rho)$, $\mathfrak{a} = (\rho - 2)$ and $\lambda = 3\rho^2 + 6\rho + 12$. Fix the complex embedding of K sending ρ to $3^{1/3}e^{2\pi i/3}$. We continue with this example, illustrating how to explicitly compute the element a_λ constructed in Lemma 4.3.6.

1. We first determine ϵ. Since $\mathcal{O}_K^\times = \{\pm 1\} \times \langle \rho^2 - 2\rangle$, $\rho^2 - 2 \in 1+\mathfrak{f}$, and $0 < \sigma_{\mathbf{R}}(\rho^2 - 2) < 1$, we have $\epsilon = \rho^2 - 2$.

2. We find an oriented basis of L. On one hand, $L = 3\mathbf{Z} + \mathbf{Z}\rho + \mathbf{Z}\rho^2$. On the other hand,

$$\det \begin{pmatrix} \sigma_{\mathbf{R}}(3) & \mathrm{Re}(\sigma_{\mathbf{C}}(3)) & \mathrm{Im}(\sigma_{\mathbf{C}}(3)) \\ \sigma_{\mathbf{R}}(\rho) & \mathrm{Re}(\sigma_{\mathbf{C}}(\rho)) & \mathrm{Im}(\sigma_{\mathbf{C}}(\rho)) \\ \sigma_{\mathbf{R}}(\rho^2) & \mathrm{Re}(\sigma_{\mathbf{C}}(\rho^2)) & \mathrm{Im}(\sigma_{\mathbf{C}}(\rho^2)) \end{pmatrix} = \det \begin{pmatrix} 3 & 3 & 0 \\ 3^{1/3} & -3^{1/3}/2 & 3^{5/6}/2 \\ 3^{2/3} & -3^{2/3}/2 & -3^{7/6}/2 \end{pmatrix} > 0.$$

Thus we can conclude that $(3, \rho, \rho^2)$ is an oriented basis of L.

3. We use condition (i) of Lemma 4.3.6 to determine $\pm a_\lambda$. With respect to the basis $(3, \rho, \rho^2)$, we have $\lambda = (4, 6, 3)(3, \rho, \rho^2)^t$ and $\epsilon^{-1}\lambda = (37, 78, 54)(3, \rho, \rho^2)^t$. Write

$$a_\lambda = (a_1, a_2, a_3)(3^*, \rho^*, (\rho^2)^*)^t \quad \text{with } a_1, a_2, a_3 \in \mathbf{Z},$$

where $(3^*, \rho^*, (\rho^2)^*)$ is the dual basis of $(3, \rho, \rho^2)$. Since $a_\lambda \in \ker(\lambda) \cap \ker(\epsilon^{-1}\lambda)$ and

$$(4, 6, 3) \times (37, 78, 54) = 15(6, -7, 6),$$

we have

$$(a_1 : a_2 : a_3) = (6 : -7 : 6).$$

Moreover, a_λ is primitive, so $a_\lambda = \pm(6, -7, 6)(3^*, \rho^*, (\rho^2)^*)^t$.

4. We use condition (ii) of Lemma 4.3.6 to determine a_λ. Let $a = (6, -7, 6)(3^*, \rho^*, (\rho^2)^*)^t$. Since $a(3) > 0$ and

$$\det(\lambda, \epsilon^{-1}\lambda, 3) = \det \begin{pmatrix} 4 & 6 & 3 \\ 37 & 78 & 54 \\ 1 & 0 & 0 \end{pmatrix} > 0,$$

we have $(\lambda, \epsilon^{-1}\lambda)$ is an oriented basis of $H(a)$. But

$$\mathrm{Im}\Big(\sigma_{\mathbf{C}}(\lambda)\overline{\sigma_{\mathbf{C}}(\epsilon^{-1}\lambda)}\Big) = -|\sigma_{\mathbf{C}}(\lambda)|^2 \mathrm{Im}\left(\epsilon_{\mathbf{C}}^{-1}\right) < 0,$$

so we conclude that

$$a_\lambda = -a = (-6, 7, -6)(3^*, \rho^*, (\rho^2)^*)^t.$$

Let a_λ be the primitive element of L^* constructed in Lemma 4.3.6, and set $b_\lambda := \epsilon \cdot a_\lambda$. When no confusion is likely to arise, we write simply a, b. Recall that $\gamma = \gamma(a, b)$ denotes the unique primitive element of L such that $\det(a, b, \cdot) = s\gamma(a, b)$ for some $s \in \mathbf{Z}_{\geq 1}$ (see (4.4)).

Lemma 4.3.8. *We have* $\mathfrak{a}^{-1}L = L + \mathbf{Z}\gamma/\mathrm{N}(\mathfrak{a})$

Proof. It is clear that $\mathfrak{a}^{-1}L = L + \mathbf{Z}\lambda/\mathrm{N}(\mathfrak{a}) \subset L + \mathbf{Z}\gamma/\mathrm{N}(\mathfrak{a})$. So it suffices to show that $\gamma/\mathrm{N}(\mathfrak{a}) \in \mathfrak{a}^{-1}L$. By construction, there is a unique $n \in \mathbf{Z}_{\neq 0}$ such that $\lambda = n\gamma$. Since $\lambda/\mathrm{N}(\mathfrak{a}) \notin L$ and $\mathrm{N}(\mathfrak{a})$ is a rational prime, we have $(n, \mathrm{N}(\mathfrak{a})) = 1$. Otherwise, $\lambda = \frac{n}{\mathrm{N}(\mathfrak{a})}\gamma \in L$, a contradiction. Hence there are $t_1, t_2 \in \mathbf{Z}$ such that $t_1 n + t_2 \mathrm{N}(\mathfrak{a}) = 1$. It follows that

$$t_1\lambda/\mathrm{N}(\mathfrak{a}) = nt_1\gamma/\mathrm{N}(\mathfrak{a}) = \gamma/\mathrm{N}(\mathfrak{a}) - t_2\gamma.$$

In other words,

$$\gamma/\mathrm{N}(\mathfrak{a}) \in \mathfrak{a}^{-1}L.$$

□

By condition (i) of Lemma 4.3.6, both a and b vanish on λ. Since $\mathfrak{a}^{-1}L = L + \mathbf{Z}\lambda/\mathrm{N}(\mathfrak{a})$, they may therefore also be viewed as elements of $(\mathfrak{a}^{-1}L)^*$. Moreover, both a and b are primitive in $\mathfrak{a}^{-1}L^*$. It follows that the elliptic gamma function $\Gamma_{a,b}(w, z; \mathfrak{a}^{-1}L)$ is well defined and admits the following expression.

Lemma 4.3.9. *We have*

$$\Gamma_{a,b}(w, z; \mathfrak{a}^{-1}L) = \frac{\prod_{l \in C(a,b)/\mathbf{Z}\gamma} \left(1 - e^{-2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)}{\prod_{l \in C(b,a)/\mathbf{Z}\gamma} \left(1 - e^{2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)}.$$

Proof. We extend γ to a basis $\{l_1, l_2, \gamma\}$ of L. By Lemma 4.3.8, $\{l_1, l_2, \gamma/\mathrm{N}(\mathfrak{a})\}$ is a basis of $\mathfrak{a}^{-1}L$. On one hand, by construction, we have $\gamma(a, b) = \gamma/\mathrm{N}(\mathfrak{a})$ when viewing a, b as elements in $\mathfrak{a}^{-1}L_{\mathrm{prim}}$. On the other hand, the map

$$k_1 l_1 + k_2 l_2 + k_3\gamma \mapsto k_1 l_1 + k_2 l_2 + k_3\gamma/\mathrm{N}(\mathfrak{a}), \quad k_1, k_2, k_3 \in \mathbf{Z}$$

restricts to a bijection between $C(a, b) = \{l \in L \mid a(l) > 0, b(l) \leq 0\}$ (resp. $C(b, a)$) and $C'(a, b) = \{l' \in \mathfrak{a}^{-1}L \mid a(l') > 0, b(l') \leq 0\}$ (resp. $C'(b, a)$). Thus we can conclude that

$$\begin{aligned}
\Gamma_{a,b}(w, z; \mathfrak{a}^{-1}L) &= \frac{\prod_{l \in C'(a,b)/\mathbf{Z}\gamma/\mathrm{N}(\mathfrak{a})} \left(1 - e^{-2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)}{\prod_{l \in C'(b,a)/\mathbf{Z}\gamma/\mathrm{N}(\mathfrak{a})} \left(1 - e^{2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)} \\
&= \frac{\prod_{l \in C(a,b)/\mathbf{Z}\gamma} \left(1 - e^{-2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)}{\prod_{l \in C(b,a)/\mathbf{Z}\gamma} \left(1 - e^{2\pi i (z(l) - w)/z(\gamma/\mathrm{N}(\mathfrak{a}))}\right)}.
\end{aligned}$$

□

The relation between the two elliptic gamma functions associated with the same wedge (a, b), but with respect to the lattices L and $\mathfrak{a}^{-1}L$, is given by the following proposition.

Proposition 4.3.10. [BCG2023, Lemma 2.2] *We have*

$$\Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L) = \prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \Gamma_{a,b}\left(w + k\frac{z(\gamma)}{\mathrm{N}(\mathfrak{a})}, z; L\right).$$

Proof. We have

$$\begin{aligned}
\prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \Gamma_{a,b}\left(w + k\frac{z(\gamma)}{\mathrm{N}(\mathfrak{a})}, z; L\right) &= \prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \frac{\prod_{l\in C(a,b)/\mathbf{Z}\gamma}\left(1 - e^{-2\pi i((z(l)-w)/z(\gamma)-k/\mathrm{N}(\mathfrak{a}))}\right)}{\prod_{l\in C(b,a)/\mathbf{Z}\gamma}\left(1 - e^{2\pi i((z(l)-w)/z(\gamma)-k/\mathrm{N}(\mathfrak{a}))}\right)} \\
&= \frac{\prod_{l\in C(a,b)/\mathbf{Z}\gamma}\prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} e^{-2\pi i(z(l)-w)/z(\gamma)}\left(e^{2\pi i(z(l)-w)/z(\gamma)} - e^{2\pi ik/\mathrm{N}(\mathfrak{a})}\right)}{\prod_{l\in C(b,a)/\mathbf{Z}\gamma}\prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} e^{2\pi i(z(l)-w)/z(\gamma)}\left(e^{-2\pi i(z(l)-w)/z(\gamma)} - e^{-2\pi ik/\mathrm{N}(\mathfrak{a})}\right)} \\
&= \frac{\prod_{l\in C(a,b)/\mathbf{Z}\gamma} e^{-2\pi i\mathrm{N}(\mathfrak{a})(z(l)-w)/z(\gamma)}\left(e^{2\pi i\mathrm{N}(\mathfrak{a})(z(l)-w)/z(\gamma)} - 1\right)}{\prod_{l\in C(b,a)/\mathbf{Z}\gamma} e^{2\pi i\mathrm{N}(\mathfrak{a})(z(l)-w)/z(\gamma)}\left(e^{-2\pi i\mathrm{N}(\mathfrak{a})(z(l)-w)/z(\gamma)} - 1\right)} \\
&= \Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L) \quad \text{by Lemma 4.3.9},
\end{aligned}$$

where the third equality holds since

$$\prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1}\left(X - \zeta_{\mathrm{N}(\mathfrak{a})}^{k}\right) = X^{\mathrm{N}(\mathfrak{a})} - 1.$$

□

Remark 4.3.11. This is reminiscent of Proposition 2.2.6.

Definition 4.3.12. We define the smoothed elliptic gamma function by

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(w,z) := \frac{\Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L)}{\Gamma_{a,b}(w,z;L)^{\mathrm{N}(\mathfrak{a})}}.$$

It follows from Propositions 4.2.4 and 4.3.10 that this defines a meromorphic function on $\mathbf{C}\times(U_a\cap U_b)$. Using $\Gamma_{b,a} = \Gamma_{a,b}^{-1}$ for the ordinary elliptic gamma function (see (4.5)), we can rewrite

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(w,z) = \frac{\Gamma_{b,\epsilon^{-1}\cdot b}(w,z;L)^{\mathrm{N}(\mathfrak{a})}}{\Gamma_{b,\epsilon^{-1}\cdot b}(w,z;\mathfrak{a}^{-1}L)},$$

which more closely matches the form of Proposition 2.2.4.

Recall that, by construction, $\mathbf{x}\in U_a$. We claim that $\mathbf{x}\in U_b$ as well. Indeed, let (μ,δ) be an oriented basis of $H(b)$. Since $b = \epsilon\cdot a$, it follows that $(\epsilon^{-1}\mu,\epsilon^{-1}\delta)$ is an oriented basis of $H(a)$. Then our claim follows from

$$\mathrm{Im}\left(\mathbf{x}(\epsilon^{-1}\mu)\overline{\mathbf{x}(\epsilon^{-1}\delta)}\right) = \mathrm{Im}\left(\epsilon\cdot\mathbf{x}(\mu)\overline{\epsilon\cdot\mathbf{x}(\delta)}\right)$$

$$
\begin{aligned}
&= |\epsilon_{\mathbf{C}}|^{-2}\mathrm{Im}\left(\mathbf{x}(\mu)\overline{\mathbf{x}(\delta)}\right) \\
&> 0.
\end{aligned}
$$

Hence $\mathbf{x} \in U_b$, so $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(w, \mathbf{x})$ is well defined.

We conclude this subsection by proving that $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(\mathbf{x}(h), \mathbf{x})$ is a well defined complex number.

Lemma 4.3.13. *We have $h \notin \mathfrak{a}^{-1}L$.*

Proof. If $h \in \mathfrak{a}^{-1}L$, then $\lambda \in f\mathfrak{a}^{-1}L$. By the Chinese remainder theorem, we further have $\lambda \in f\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L \subset fL$, contradicting our assumption that λ is admissible. □

Lemma 4.3.14. *The complex number $\mathbf{x}(h)$ is neither a zero nor a pole of $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(w, \mathbf{x})$.*

Proof. We first show that $\mathbf{x}(h)$ avoids the zeroes and poles of $\Gamma_{a,b}(w, z; L)$. By Proposition 4.2.4, if $\mathbf{x}(h)$ is a zero (resp. pole) of $\Gamma_{a,b}(w, \mathbf{x}; L)$, then

$$
\mathbf{x}(h) = \mathbf{x}(l) + n\mathbf{x}(\gamma),
$$

or equivalently

$$
h = l + n\gamma
$$

for some $l \in C(a, b)$ (resp. $C(b, a)$) and $n \in \mathbf{Z}$. In either case, we have $l \in L$. This implies that

$$
h = l + n\gamma \in L + \mathbf{Z}\gamma \subset \mathfrak{a}^{-1}L,
$$

contradicting Lemma 4.3.13. Similarly, if $\mathbf{x}(h)$ is a zero or pole of $\Gamma_{a,b}(w, \mathbf{x}; \mathfrak{a}^{-1}L)$, then by Proposition 4.3.10, we have

$$
\mathbf{x}(h) = \mathbf{x}(l) + n\mathbf{x}(\gamma) - k\mathbf{x}(\gamma)/\mathrm{N}(\mathfrak{a})
$$

for some $0 \leq k \leq \mathrm{N}(\mathfrak{a}) - 1$, $n \in \mathbf{Z}$, and $l \in L$, which implies that

$$
h \in L + \mathbf{Z}\gamma/\mathrm{N}(\mathfrak{a}) = \mathfrak{a}^{-1}L,
$$

again a contradiction. □

4.3.3 | Values of elliptic gamma functions

We are now ready to give the main definition of this chapter.

Definition 4.3.15. We define

$$
\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda) := \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(\mathbf{x}(h), \mathbf{x}) \in \mathbf{C}^{\times}.
$$

Example 4.3.16. Recall from Example 4.3.7 that $K = \mathbf{Q}(\rho)$, $\rho^3 = 3$, with $L = \mathfrak{f} = (\rho)$, $\mathfrak{a} = (\rho - 2)$, $\lambda = 3\rho^2 + 6\rho + 12$, $\epsilon = \rho^2 - 2$ and $a = (-6, 7, -6)(3^*, \rho^*, (\rho^2)^*)^t$. We continue with this example, illustrating how to explicitly compute $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(1, \lambda)$.

1. We first determine $b = \epsilon \cdot a$. Since
$$\epsilon^{-1} \begin{pmatrix} 3 \\ \rho \\ \rho^2 \end{pmatrix} = \begin{pmatrix} 4 & 9 & 6 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ \rho \\ \rho^2 \end{pmatrix}, \tag{4.10}$$
we have
$$b = \epsilon \cdot a = \begin{pmatrix} 3^* & \rho^* & (\rho^2)^* \end{pmatrix} \begin{pmatrix} 4 & 9 & 6 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix} \begin{pmatrix} -6 \\ 7 \\ -6 \end{pmatrix} = (3, -2, 0)(3^*, \rho^*, (\rho^2)^*)^t.$$

2. We then determine γ. Write $\gamma = 3x + y\rho + z\rho^2$ with $x, y, z \in \mathbf{Z}$. Since $s\gamma = \det(a, b, \cdot)$ and
$$(-6, 7, -6) \times (3, -2, 0) = 3 \cdot (-4, -6, -3),$$
we have
$$\gamma = (-4, -6, -3)(3, \rho, \rho^2)^t = -\lambda \quad \text{and} \quad s = 3.$$

3. We apply Proposition 4.2.6 to express $\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; L)$ and $\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; \mathfrak{a}^{-1}L)$ in terms of the ordinary elliptic gamma function. We first find a $c \in L^*$ such that $c(\gamma) = 1$. Since
$$(2, 0, -3) \cdot (-4, -6, -3) = 1,$$
we can take $c = (2, 0, -3)(3^*, \rho^*, (\rho^2)^*)^t$. Then we set
$$\alpha = \det(\cdot, b, c) = (3, -2, 0) \times (2, 0, -3)(3, \rho, \rho^2)^t = (6, 9, 4)(3, \rho, \rho^2)^t$$
and
$$\beta = \det(a, \cdot, c) = (2, 0, -3) \times (-6, 7, -6)(3, \rho, \rho^2)^t = (21, 30, 14)(3, \rho, \rho^2)^t.$$
As in Remark 4.2.7, we have $a(\beta) = b(\alpha) = 0$, $a(\alpha) = b(\beta) = 3$, and $\#(F(\alpha, \beta)/\mathbf{Z}\gamma) = 3$. Since
$$(1, 1, 0) \cdot (-6, 7, -6) = 1 \quad \text{and} \quad (1, 1, 0) \cdot (3, -2, 0) = 1,$$
we have $\delta := (1, 1, 0)(3, \rho, \rho^2)^t \in F(\alpha, \beta)$. Moreover, it is clear that $\delta + \mathbf{Z}\gamma \neq 2\delta + \mathbf{Z}\gamma$, so
$$F(\alpha, \beta)/\mathbf{Z}\gamma = \{\mathbf{Z}\gamma, \delta + \mathbf{Z}\gamma, 2\delta + \mathbf{Z}\gamma\}.$$

By construction, $\{\gamma, \epsilon\gamma\}$ spans $H(b)$. Since $b(\alpha) = 0$, there exist $m, n \in \mathbf{Q}$ such that $\alpha = m\epsilon\gamma + n\gamma$. Solving the equation

$$(6, 9, 4) = m(2, 3, -6) + n(-4, -6, -3)$$

gives $m = 1/15, n = -22/15$. Hence

$$\frac{\alpha}{\gamma} = \frac{\epsilon - 22}{15}.$$

Similarly, we have

$$\frac{\beta}{\gamma} = \frac{\epsilon^{-1} - 88}{15}.$$

On the other hand, we also have $\{\gamma, \epsilon\gamma, \epsilon^{-1}\gamma\}$ is a basis of $L_{\mathbf{Q}}$, so there exist $m, n, k \in \mathbf{Q}$ such that $\delta = m\gamma + n\epsilon\gamma + k\epsilon^{-1}\gamma$. Solving the equation

$$(1, 1, 0) = m(2, 3, -6) + n(-37, -78, -54) + k(-4, -6, -3)$$

gives $m = 1/45, n = 1/45, k = -4/9$. Hence

$$\frac{\delta}{\gamma} = \frac{\epsilon + \epsilon^{-1} - 20}{45}.$$

It follows that

$$\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; L) = \Gamma\left(-\frac{1}{3}, \frac{\epsilon_{\mathbf{C}} - 22}{15}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{15}\right) \Gamma\left(-\frac{1}{3} + \frac{\epsilon_{\mathbf{C}} + \epsilon_{\mathbf{C}}^{-1} - 20}{45}, \frac{\epsilon_{\mathbf{C}} - 22}{15}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{15}\right) \Gamma\left(-\frac{1}{3} + \frac{2\epsilon_{\mathbf{C}} + 2\epsilon_{\mathbf{C}}^{-1} - 40}{45}, \frac{\epsilon_{\mathbf{C}} - 22}{15}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{15}\right)$$

by Proposition 4.2.6. By Lemma 4.3.9, we also have

$$\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; \mathfrak{a}^{-1}L) = \Gamma\left(-\frac{5}{3}, \frac{\epsilon_{\mathbf{C}} - 22}{3}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{3}\right) \Gamma\left(-\frac{5}{3} + \frac{\epsilon_{\mathbf{C}} + \epsilon_{\mathbf{C}}^{-1} - 20}{9}, \frac{\epsilon_{\mathbf{C}} - 22}{3}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{3}\right) \Gamma\left(-\frac{5}{3} + \frac{2\epsilon_{\mathbf{C}} + 2\epsilon_{\mathbf{C}}^{-1} - 40}{9}, \frac{\epsilon_{\mathbf{C}} - 22}{3}, \frac{\epsilon_{\mathbf{C}}^{-1} - 88}{3}\right).$$

4. We use (4.3) to compute $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(1, \lambda)$. Since

$$\operatorname{Im}\left(\frac{22 - \epsilon_{\mathbf{C}}}{15}\right), \operatorname{Im}\left(\frac{\epsilon_{\mathbf{C}}^{-1} - 88}{15}\right) > 0$$

and

$$\left|\mathrm{Im}\left(\frac{22-\epsilon_{\mathbf{C}}}{15}-\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)\right|<\mathrm{Im}\left(\frac{22-\epsilon_{\mathbf{C}}}{15}\right)+\mathrm{Im}\left(\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right),$$

we can apply (4.3) to obtain

$$\Gamma\left(-\frac{1}{3},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)=\exp\left(S\left(-\frac{1}{3},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)\right).$$

Similarly, we have

$$\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-35}{45},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)=\exp\left(S\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-35}{45},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)\right)$$

and

$$\Gamma\left(\frac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}-55}{45},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)=\exp\left(S\left(\frac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}-55}{45},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)\right).$$

It follows that

$$\begin{aligned}\Gamma_{a,b}(\mathbf{x}(h),\mathbf{x};L)=\exp\Bigg(&S\left(-\frac{1}{3},\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\right)+S\Bigg(\frac{-35+\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}}{45},\\ &\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\Bigg)+S\Bigg(\frac{-55+2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}}{45},\\ &\frac{\epsilon_{\mathbf{C}}-22}{15},\frac{\epsilon_{\mathbf{C}}^{-1}-88}{15}\Bigg)\Bigg).\end{aligned}$$

We also have an analogous result for $\Gamma_{a,b}(\mathbf{x}(h),\mathbf{x};\mathfrak{a}^{-1}L)$. Putting these together, we obtain

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v,\lambda)\approx 0.1810828736\ldots+i\cdot 0.2174691148\ldots,$$

a complex number that PARI/GP recognizes, to 1000 decimal digits, as a root of

$$P(x)=x^6-3x^5+6x^4+17x^3+6x^2-3x+1.$$

Over K this polynomial factors as

$$(x^2+(2\rho-1)x+1)(x^2+(-\rho^2-\rho-1)x+2\rho^2+3\rho+4)(x^2+(\rho^2-\rho-1)x+\rho^2-2).$$

Recall that we fix the complex embedding $\sigma_{\mathbf{C}}:K\hookrightarrow\mathbf{C}$ sending ρ to $3^{1/3}e^{2\pi i/3}$. So $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v,\lambda)$ can be identified as a root of

$$x^2+(2\sigma_{\mathbf{C}}(\rho)-1)x+1$$

to 1000 digits. Let $M = K(\nu)$ where $\nu^2 + (2\rho - 1)\nu + 1 = 0$. Using the command **rnfconductor** in PARI/GP (or by hand), one can show that the conductor of the extension M/K is precisely $\mathfrak{f} \cdot \infty_{\mathbf{R}}$. On the other hand, we also have $|\mathrm{Cl}_K(\mathfrak{f} \cdot \infty_{\mathbf{R}})| = 2$. Thus we can conclude that $M = K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$. Let $\widetilde{\sigma}_{\mathbf{C}}$ be a complex embedding $K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) \hookrightarrow \mathbf{C}$ extending $\sigma_{\mathbf{C}}$. Then there is a unit $\mathbf{u} \in \mathcal{O}^{\times}_{K(\mathfrak{f} \cdot \infty_{\mathbf{R}})}$ such that

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda) \approx \widetilde{\sigma}_{\mathbf{C}}(\mathbf{u}),$$

where $\approx$ represents an error less than 10^{-1000}.

Remark 4.3.17. As the computation in Example 4.3.16 shows, even for $s = 3$ the calculation is already quite involved. As we shall see later, the value of $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)$ appears to be independent of the choice of λ. It is therefore natural to ask whether one can choose a particularly convenient admissible element to simplify matters. Unfortunately, by (4.9) one can show that $3 \mid s$ for every 1-admissible λ. To see this, we fixed the oriented basis $(3, \rho, \rho^2)$ of L and the corresponding basis in L^*. Henceforth all elements will be written in coordinates relative to these bases. For every 1-admissible λ, we have

$$\lambda \equiv (1, 0, 0)^t \pmod{3}.$$

Let

$$M = \begin{pmatrix} 4 & 2 & 3 \\ 9 & 4 & 6 \\ 6 & 3 & 4 \end{pmatrix}.$$

Then by (4.10), we have

$$\epsilon^{-1}\lambda = M(1, 0, 0)^t \equiv (1, 0, 0)^t \pmod{3}.$$

Write $a = (a_x, a_y, a_z)^t$. Then $a_x \equiv 0 \pmod{3}$ since $a \in \ker \lambda \cap \ker \epsilon^{-1}\lambda$. It follows that

$$b = M^t (a_x, a_y, a_z)^t \equiv (0, a_y, a_z)^t \pmod{3},$$

which implies that

$$a \times b \equiv (0, 0, 0)^t \pmod{3}.$$

In other words, $s \mid 3$. As a consequence, our chosen 1-admissible λ is already the most convenient for the computation in view of Remark 4.2.7.

We collect several consequences of the preceding results concerning values of $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)$, which will be useful when we formulate Conjecture 5.2.1.

Proposition 4.3.18.

(i) For $u \in \mathcal{O}_{\mathfrak{f}}^{+,\times}$,

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, u\lambda) = \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda).$$

(ii) For $\alpha \in 1 + \mathfrak{f}$ *with* $\sigma_{\mathbf{R}}(\alpha) > 0$,

$$\mathbf{\Gamma}_{\alpha^{-1}L,\mathfrak{f},\mathfrak{a}}(v, \alpha^{-1}\lambda) = \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda). \tag{4.11}$$

(iii) For $\alpha \in -1 + \mathfrak{f}$ *with* $\sigma_{\mathbf{R}}(\alpha) > 0$,

$$\mathbf{\Gamma}_{\alpha^{-1}L,\mathfrak{f},\mathfrak{a}}(v, -\alpha^{-1}\lambda) = \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(-\mathbf{x}(h), \mathbf{x}). \tag{4.12}$$

(iv) Write temporarily $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda, \sigma_{\mathbf{C}})$ *to emphasize the role of the fixed complex embedding* $\sigma_{\mathbf{C}}$*. We have*

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda, \overline{\sigma}_{\mathbf{C}}) = \overline{\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda, \sigma_{\mathbf{C}})}.$$

Proof. (i) We first show that $a_{u\lambda} = u \cdot a_\lambda$. On one hand, $u \cdot a_\lambda \in \ker u\lambda \cap \ker \epsilon^{-1}u\lambda$ since

$$u \cdot a_\lambda(u\lambda) = a_\lambda(\lambda) = 0 = a_\lambda(\epsilon^{-1}\lambda) = u \cdot a_\lambda(\epsilon^{-1}u\lambda).$$

On the other hand, $u \in \mathcal{O}_{\mathfrak{f}}^{+,\times}$, we have $\mathrm{N}_{K/\mathbf{Q}}(u) = 1$ and hence that $u \in \mathrm{SL}(L)$. Then by Lemma 4.2.8, $u \cdot a_\lambda$ is primitive. Moreover,

$$\mathbf{x} \in U_{u \cdot a_\lambda} = u \cdot U_{a_\lambda}$$

since $\mathbf{x}$ is an eigenvector of ϵ, and hence also of u.

Next, going through the proof of Lemma 4.2.8, we find that $\gamma(u \cdot a_\lambda, u \cdot b_\lambda) = u\gamma(a_\lambda, b_\lambda)$. By (4.7), we obtain

$$\begin{aligned}
\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; L) &= \Gamma_{u\cdot a,u\cdot b}(\mathbf{x}(h), u \cdot \mathbf{x}; L) \\
&= \frac{\prod_{l \in C(u\cdot a,u\cdot b)/\mathbf{Z}u\gamma} \left(1 - e^{-2\pi i (u\cdot\mathbf{x}(l) - \mathbf{x}(h))/u\cdot\mathbf{x}(u\gamma)}\right)}{\prod_{l \in C(u\cdot b,u\cdot a)/\mathbf{Z}u\gamma} \left(1 - e^{2\pi i (u\cdot\mathbf{x}(l) - \mathbf{x}(h))/u\cdot\mathbf{x}(u\gamma)}\right)} \\
&= \frac{\prod_{l \in C(u\cdot a,u\cdot b)/\mathbf{Z}u\gamma} \left(1 - e^{-2\pi i (\mathbf{x}(l) - \mathbf{x}(uh))/\mathbf{x}(u\gamma)}\right)}{\prod_{l \in C(u\cdot b,u\cdot a)/\mathbf{Z}u\gamma} \left(1 - e^{2\pi i (\mathbf{x}(l) - \mathbf{x}(uh))/\mathbf{x}(u\gamma)}\right)} \\
&= \Gamma_{u\cdot a,u\cdot b,}(\mathbf{x}(uh), \mathbf{x}; L).
\end{aligned}$$

Similarly, applying Proposition 4.3.10 and by the same argument above, one shows that

$$\Gamma_{u\cdot a,u\cdot b}(\mathbf{x}(uh), \mathbf{x}; \mathfrak{a}^{-1}L) = \prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \Gamma_{u\cdot a,u\cdot b}\left(\mathbf{x}(uh) + k\frac{\mathbf{x}(u\gamma)}{\mathrm{N}(\mathfrak{a})}, \mathbf{x}; L\right)$$

$$= \prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \Gamma_{u\cdot a, u\cdot b}\left(\mathbf{x}(h) + k\frac{\mathbf{x}(\gamma)}{\mathrm{N}(\mathfrak{a})}, u\cdot \mathbf{x}; L\right)$$

$$= \prod_{k=0}^{\mathrm{N}(\mathfrak{a})-1} \Gamma_{a,b}\left(\mathbf{x}(h) + k\frac{\mathbf{x}(\gamma)}{\mathrm{N}(\mathfrak{a})}, \mathbf{x}; L\right) \quad \text{by (4.7)}$$

$$= \Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; \mathfrak{a}^{-1}L) \quad \text{again by Proposition 4.3.10.}$$

Thus we can conclude that

$$\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, u\lambda) = \frac{\Gamma_{u\cdot a,u\cdot b}(\mathbf{x}(uh), \mathbf{x}; \mathfrak{a}^{-1}L)}{\Gamma_{u\cdot a,u\cdot b}(\mathbf{x}(uh), \mathbf{x}; L)^{\mathrm{N}(\mathfrak{a})}} = \frac{\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; \mathfrak{a}^{-1}L)}{\Gamma_{a,b}(\mathbf{x}(h), \mathbf{x}; L)^{\mathrm{N}(\mathfrak{a})}} = \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda).$$

(ii) We first show that when viewing $a_{\alpha^{-1}\lambda}, a_\lambda$ as elements in $L^*_{\mathbf{Q}}$, we have $a_{\alpha^{-1}\lambda} = \alpha^{-1}\cdot a_\lambda$. Indeed, $\alpha^{-1}\cdot a_\lambda \in \ker \alpha^{-1}\lambda \cap \ker \alpha^{-1}\epsilon^{-1}\lambda$. Moreover, $\alpha^{-1}\cdot a_\lambda \in (\alpha^{-1}L)^*_{\mathrm{prim}}$ since $a_\lambda \in L^*_{\mathrm{prim}}$. On the other hand, since $\mathrm{N}_{K/\mathbf{Q}}(\alpha) > 0$, we also have $\mathbf{x} \in U_{\alpha^{-1}\cdot a_\lambda} = \alpha^{-1}\cdot U_{a_\lambda}$. More precisely, for every oriented basis (μ, δ) of $H(a_\lambda)$, $(\alpha^{-1}\mu, \alpha^{-1}\delta)$ is an oriented basis of $H(\alpha^{-1}\cdot a_\lambda)$ and

$$\mathrm{Im}\left(\mathbf{x}(\alpha^{-1}\mu)\overline{\mathbf{x}(\alpha^{-1}\delta)}\right) = |\sigma_{\mathbf{C}}(\alpha)|^{-2}\mathrm{Im}\left(\mathbf{x}(\mu)\overline{\mathbf{x}(\delta)}\right) > 0.$$

Next we show that $\gamma(\alpha^{-1}\cdot a_\lambda, \alpha^{-1}\cdot b_\lambda) = \alpha^{-1}\gamma(a_\lambda, b_\lambda)$. It is clear that

$$\alpha^{-1}\cdot a_\lambda(\alpha^{-1}\gamma(a_\lambda, b_\lambda)) = \alpha^{-1}\cdot b_\lambda(\alpha^{-1}\gamma(a_\lambda, b_\lambda)) = 0$$

and $\alpha^{-1}\gamma(a_\lambda, b_\lambda) \in (\alpha^{-1}L)_{\mathrm{prim}}$. Hence $\gamma(\alpha^{-1}\cdot a_\lambda, \alpha^{-1}\cdot b_\lambda) = \pm\alpha^{-1}\gamma(a_\lambda, b_\lambda)$. Pick a $c \in L^*$ such that $\det(a_\lambda, b_\lambda, c) > 0$, or equivalently $c(\gamma(a_\lambda, b_\lambda)) > 0$. Then

$$\det(\alpha^{-1}\cdot a_\lambda, \alpha^{-1}\cdot b_\lambda, \alpha^{-1}\cdot c) = \mathrm{N}_{K/\mathbf{Q}}(\alpha)^{-1}\det(a_\lambda, b_\lambda, c) > 0,$$

or equivalently $c(\alpha\gamma(\alpha^{-1}\cdot a_\lambda, \alpha^{-1}\cdot b_\lambda)) > 0$. This proves that $\gamma(\alpha^{-1}\cdot a_\lambda, \alpha^{-1}\cdot b_\lambda) = \alpha^{-1}\gamma(a_\lambda, b_\lambda)$.

The rest of the proof proceeds in exactly the same way as for (i).

(iii) The proof proceeds in exactly the same way as for (ii).

(iv) Write temporarily $K_{\sigma_{\mathbf{C}}}, \mathbf{x}_{\sigma_{\mathbf{C}}}, a_{\lambda,\sigma_{\mathbf{C}}}, \gamma(a, b, \sigma_{\mathbf{C}})$. On one hand, it is clear that $a_{\lambda,\overline{\sigma}_{\mathbf{C}}} = \pm a_{\lambda,\sigma_{\mathbf{C}}}$ by construction in Lemma 4.3.6. On the other hand, replacing $\sigma_{\mathbf{C}}$ by $\overline{\sigma}_{\mathbf{C}}$ changes the orientation of K. Namely, if (μ, δ) be an oriented basis of $H(a_{\lambda,\sigma_{\mathbf{C}}}) \subset K_{\sigma_{\mathbf{C}}}$, then (δ, μ)

is an oriented basis of $H(a_{\lambda,\sigma_{\mathbf{C}}}) \subset K_{\overline{\sigma}_{\mathbf{C}}}$. Since

$$\operatorname{Im}\left(\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\delta)\overline{\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\mu)}\right) = \operatorname{Im}\left(\mathbf{x}_{\sigma_{\mathbf{C}}}(\mu)\overline{\mathbf{x}_{\sigma_{\mathbf{C}}}(\delta)}\right) > 0,$$

we have $a_{\lambda,\sigma_{\mathbf{C}}} = a_{\lambda,\overline{\sigma}_{\mathbf{C}}}$. It follows immediately that $\gamma(a, b, \overline{\sigma}_{\mathbf{C}}) = -\gamma(a, b, \sigma_{\mathbf{C}})$.

Let $\alpha, \beta \in L$ such that $b(\alpha) = a(\beta) = 0$ and $a(\alpha), b(\beta) > 0$. We compute

$$\begin{aligned}
\Gamma_{a,b}(\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(h), \mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}; L) &= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}(-\gamma)} \Gamma\left(\frac{\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(h) + \mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(l)}{-\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\gamma)}, \frac{\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\alpha)}{-\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\gamma)}, \frac{\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\beta)}{-\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(\gamma)}\right) \quad \text{by (4.6)} \\
&= \prod_{l \in F(\alpha,\beta)/\mathbf{Z}\gamma} \overline{\Gamma\left(\frac{\mathbf{x}_{\sigma_{\mathbf{C}}}(h) + \mathbf{x}_{\sigma_{\mathbf{C}}}(l)}{\mathbf{x}_{\sigma_{\mathbf{C}}}(\gamma)}, \frac{\mathbf{x}_{\sigma_{\mathbf{C}}}(\alpha)}{\mathbf{x}_{\sigma_{\mathbf{C}}}(\gamma)}, \frac{\mathbf{x}_{\sigma_{\mathbf{C}}}(\beta)}{\mathbf{x}_{\sigma_{\mathbf{C}}}(\gamma)}\right)} \\
&= \overline{\Gamma_{a,b}(\mathbf{x}_{\sigma_{\mathbf{C}}}(h), \mathbf{x}_{\sigma_{\mathbf{C}}}; L)} \quad \text{again by (4.6)},
\end{aligned}$$

where the second equality holds since

$$\Gamma(-\overline{z}, -\overline{\tau}, -\overline{\sigma}) = \overline{\Gamma(z, \tau, \sigma)}.$$

Similarly, applying Proposition 4.3.10 and by the same argument above, we obatin

$$\Gamma_{a,b}(\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(h), \mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}; \mathfrak{a}^{-1}L) = \overline{\Gamma_{a,b}(\mathbf{x}_{\sigma_{\mathbf{C}}}(h), \mathbf{x}_{\sigma_{\mathbf{C}}}; \mathfrak{a}^{-1}L)}.$$

Thus we can conclude that

$$\boldsymbol{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda, \overline{\sigma}_{\mathbf{C}}) = \frac{\Gamma_{a,b}(\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(h), \mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}; \mathfrak{a}^{-1}L)}{\Gamma_{a,b}(\mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}(h), \mathbf{x}_{\overline{\sigma}_{\mathbf{C}}}; L)^{\mathrm{N}(\mathfrak{a})}} = \frac{\overline{\Gamma_{a,b}(\mathbf{x}_{\sigma_{\mathbf{C}}}(h), \mathbf{x}_{\sigma_{\mathbf{C}}}; \mathfrak{a}^{-1}L)}}{\overline{\Gamma_{a,b}(\mathbf{x}_{\sigma_{\mathbf{C}}}(h), \mathbf{x}_{\sigma_{\mathbf{C}}}; L)}^{\mathrm{N}(\mathfrak{a})}} = \overline{\boldsymbol{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda, \sigma_{\mathbf{C}})}.$$

□

Remark 4.3.19. In fact, one has

$$\boldsymbol{\Gamma}_{L,\mathfrak{f},\mathfrak{a},v,\lambda}(-\mathbf{x}(h), \mathbf{x}) = \boldsymbol{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)^{-1}$$

up to a root of unity, which is trivial when $(\mathfrak{a}, 6) = 1$ (see [BCG2023, p. 36, (4.32)]).

Remark 4.3.20. Let $\mathfrak{c}$ be an integral ideal of K with $(\mathfrak{c}, \mathfrak{f}) = 1$, and let λ be v-admissible for the data $\mathfrak{c}^{-1}L, \mathfrak{f}, \mathfrak{a}$. One shows that

$$\boldsymbol{\Gamma}_{\alpha^{-1}\mathfrak{c}^{-1}L,\mathfrak{f},\mathfrak{a}}(v, \alpha^{-1}\lambda) = \boldsymbol{\Gamma}_{\mathfrak{c}^{-1}L,\mathfrak{f},\mathfrak{a}}(v, \lambda) \tag{4.13}$$

for every $\alpha \in 1 + \mathfrak{f}\mathfrak{c}^{-1}$ with $\sigma_{\mathbf{R}}(\alpha) > 0$ in the same way as for (ii).

Chapter 5

The conjecture

> “Et toute science, quand nous l’entendons non comme un instrument de pouvoir et de domination, mais comme aventure de connaissance de notre espèce à travers les âges, n’est autre chose que cette harmonie, plus ou moins vaste et plus ou plus riche d’une époque à l’autre, qui se déploie au cours des générations et des siècles, par le délicat contrepoint de tous les thèmes apparus tour à tour, comme appelés du néant, pour se joindre en elle et s’y entrelacer.”
>
> Alexander Grothendieck,
> *Récoltes et Semailles*

In this chapter, we carry out numerical computations for a range of examples, discuss the conjecture formulated in [BCG2023], and elucidate its connection with the rank one abelian Stark conjecture in the setting of complex cubic fields. Although all of the examples presented here are drawn from [BCG2023], our perspective differs slightly from theirs. Moreover, the fact that we appear to have chosen different admissible elements provides further evidence for the plausibility of their conjecture. We have also carried out numerical computations for many examples not treated in [BCG2023], and in all cases the results agree very well with the conjecture. Since these additional examples exhibit features similar to those discussed in this chapter, we do not present them here in detail.

5.1 | Numerical experiments

We would like to see that the conjectural units constructed in the previous chapter satisfy properties similar to those of elliptic units. To this end, we perform further numerical computations for several choices of parameters.

Example 5.1.1. We now continue the study of Example 4.3.5.

Table 5.1: Numerical values of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$ for 1-admissible elements λ

	$\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$	numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$
$6\rho^2-3\rho-6$	$\dfrac{\Gamma\left(-\frac{5}{3},-\frac{\epsilon_{\mathbf{C}}+23}{3},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)\Gamma\left(-\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-5}{9},-\frac{\epsilon_{\mathbf{C}}+23}{3},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)\Gamma\left(-\frac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}-25}{9},-\frac{\epsilon_{\mathbf{C}}+23}{3},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)}{\left(\Gamma\left(-\frac{1}{3},-\frac{\epsilon_{\mathbf{C}}+23}{15},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\Gamma\left(-\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-5}{45},-\frac{\epsilon_{\mathbf{C}}+23}{15},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\Gamma\left(-\frac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}-25}{45},-\frac{\epsilon_{\mathbf{C}}+23}{15},-\frac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\right)^5}$	$0.1810828736\ldots+i\cdot 0.2174691148\ldots$
$-3\rho^2+9\rho+3$	$\dfrac{\Gamma\left(-\frac{5}{3},\frac{\epsilon_{\mathbf{C}}-7}{6},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{6}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-50}{18},\frac{\epsilon_{\mathbf{C}}-7}{6},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{6}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-35}{9},\frac{\epsilon_{\mathbf{C}}-7}{6},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{6}\right)}{\left(\Gamma\left(-\frac{1}{3},\frac{\epsilon_{\mathbf{C}}-7}{30},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{30}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-50}{90},\frac{\epsilon_{\mathbf{C}}-7}{30},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{30}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-35}{45},\frac{\epsilon_{\mathbf{C}}-7}{30},\frac{\epsilon_{\mathbf{C}}^{-1}-13}{30}\right)\right)^5}$	$0.1810828736\ldots+i\cdot 0.2174691148\ldots$
$9\rho^2+3\rho+21$	$\dfrac{\Gamma\left(-\frac{5}{3},\frac{\epsilon_{\mathbf{C}}+53}{12},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{12}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-50}{18},\frac{\epsilon_{\mathbf{C}}+53}{12},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{12}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}+10}{36},\frac{\epsilon_{\mathbf{C}}+53}{12},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{12}\right)}{\left(\Gamma\left(-\frac{1}{3},\frac{\epsilon_{\mathbf{C}}+53}{60},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{60}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}-50}{90},\frac{\epsilon_{\mathbf{C}}+53}{60},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{60}\right)\Gamma\left(\frac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}+10}{180},\frac{\epsilon_{\mathbf{C}}+53}{60},\frac{\epsilon_{\mathbf{C}}^{-1}+197}{60}\right)\right)^5}$	$0.1810828736\ldots+i\cdot 0.2174691148\ldots$

As shown in Table 5.1, varying λ does not change the numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$ to 1000 decimal places. We also tested additional admissible elements with substantially larger associated values of s[1] and obtained the same numerical value to 1000 decimal places. These computations suggest that $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v,\lambda)$ is independent of the choice of λ.

Suppose that $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v,\lambda)$ is independent of the choice of λ. Write $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v)=\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v,\lambda)$. We fix the complex embedding $\widetilde{\sigma}_{\mathbf{C}}\colon K(\mathfrak{f}\cdot\infty_{\mathbf{R}})\to\mathbf{C}$ with $\widetilde{\sigma}_{\mathbf{C}}(\rho)=3^{1/3}e^{2\pi i/3}$ and

$$\widetilde{\sigma}_{\mathbf{C}}(\nu)=0.1810828736\ldots+i\cdot 0.2174691148\ldots.$$

Via this embedding we regard $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$ as a subfield of $\mathbf{C}$. Assume further that $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v)\in K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$. Motivated by Proposition 2.2.5.(ii), we conjecture that

$$\sigma_{\mathfrak{c}}(\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v))=\mathbf{\Gamma}_{\mathfrak{c}^{-1}L,\mathfrak{f},\mathfrak{a}}(v)\quad\text{for }\mathfrak{c}\text{ integral ideal with }(\mathfrak{f},\mathfrak{c})=1.$$

We take $\lambda=-6\rho^2+3\rho+6$. Then λ is 2-admissible for the data $\mathfrak{f},\mathfrak{f},\mathfrak{a}$ and we compute

$$\begin{aligned}\mathbf{\Gamma}_{\mathfrak{f}/2,\mathfrak{f},\mathfrak{a}}(1)=\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(2)&=\Bigg(\Gamma\left(\tfrac{5}{3},\tfrac{\epsilon_{\mathbf{C}}-22}{3},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)\Gamma\left(\tfrac{5}{3}+\tfrac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}+25}{9},\tfrac{\epsilon_{\mathbf{C}}-22}{3},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)\\&\quad\Gamma\left(\tfrac{5}{3}+\cdot\tfrac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}+50}{9},\tfrac{\epsilon_{\mathbf{C}}-22}{3},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{3}\right)\Bigg)\Bigg(\Gamma\left(\tfrac{1}{3},\tfrac{\epsilon_{\mathbf{C}}-22}{15},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\\&\quad\Gamma\left(\tfrac{1}{3}+\tfrac{\epsilon_{\mathbf{C}}+\epsilon_{\mathbf{C}}^{-1}+25}{45},\tfrac{\epsilon_{\mathbf{C}}-22}{15},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\Gamma\left(\tfrac{1}{3}+\tfrac{2\epsilon_{\mathbf{C}}+2\epsilon_{\mathbf{C}}^{-1}+50}{45},\tfrac{\epsilon_{\mathbf{C}}-22}{15},\tfrac{\epsilon_{\mathbf{C}}^{-1}+2}{15}\right)\Bigg)^{-3}\\&=2.2611666966\ldots-i\cdot 2.7155186478\ldots\end{aligned}$$

[1]We omit the explicit expressions, which are too lengthy to display.

$$\approx \mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1),$$

where $\approx$ represents an error less than 10^{-1000}.

Table 5.2: Numerical values of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)$ and conjectured minimal polynomials for smoothing ideals $\mathfrak{a}$

	numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)$	conjectured minimal polynomial
$(\rho+2,11)$	$-0.0754589478\ldots + i \cdot 0.0268209974\ldots$	$x^6+24x^5+168x^4+98x^3+168x^2+24x+1$
$(\rho-7,17)$	$-0.0002568055\ldots + i \cdot 0.0226615343\ldots$	$x^6+3x^5+1950x^4+3895x^3+1950x^2+3x+1$
$(\rho-12,23)$	$-0.0059928137\ldots - i \cdot 0.0022843209\ldots$	$x^6+291x^5+24198x^4-9305x^3+24198x^2+291x+1$
$(\rho+11,29)$	$-0.0005884266\ldots - i \cdot 0.0017169033\ldots$	$x^6+357x^5+303486x^4-82063x^3+303486x^2+357x+1$

In Table 5.2 we report, for each smoothing ideal $\mathfrak{a}$, the numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)$ together with its conjectured minimal polynomial. For every $\mathfrak{a}$ we computed more than 10 admissible elements, and—as noted earlier—the values of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)$ obtained from different admissible λ agree to at least 1000 decimal places. We therefore omit the admissible elements from the table. As in Example 4.3.16, factoring the polynomials over K, we single out the irreducible factor with a root numerically approximating $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)$; its splitting field turns out to be exactly $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$.

Motivated by Proposition 2.3.5, we are led to expect that, when $(\mathfrak{a},6)\neq 1$, it is necessary to pass to a suitable power. For $\mathfrak{a}=(\rho-1,2)$, we performed extensive numerical computations, whose results are summarized in Table 5.3. They strongly suggest that $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$ is algebraic and that $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)^4$ belongs to $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$. Furthermore, up to sign, the quantity $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)^4$ appears to be independent of the choice of λ, to at least 1000 decimal places. This agrees with the results of [BCG2023, § 5.1].

Table 5.3: Numerical values of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$, $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)^4$, and conjectured minimal polynomials for $\mathfrak{a}=(\rho-1,2)$ and admissible elements λ

	numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)$	conjectured minimal polynomial
$3\rho^2+3\rho+3$	$0.3427120088\ldots - i\cdot 0.4068652507\ldots$	$x^{24}+24x^{20}+168x^{16}+98x^{12}+168x^8+24x^4+1$
$3\rho^2-3\rho+3$	$-0.5300311633\ldots + i\cdot 0.0453631924\ldots$	$x^{12}-6x^{10}+6x^8+10x^6+6x^4-6x^2+1$
$-3\rho^2+3\rho+3$	$0.3427120088\ldots - i\cdot 0.4068652507\ldots$	$x^{24}+24x^{20}+168x^{16}+98x^{12}+168x^8+24x^4+1$
$3\rho^2+9\rho+3$	$-0.0453631924\ldots - i\cdot 0.5300311633\ldots$	$x^{12}+6x^{10}+6x^8-10x^6+6x^4+6x^2+1$
$27\rho^2+9\rho+3$	$-0.0453631924\ldots - i\cdot 0.5300311633\ldots$	$x^{12}+6x^{10}+6x^8-10x^6+6x^4+6x^2+1$
$9\rho^2+15\rho+3$	$0.4068652507\ldots + i\cdot 0.3427120088\ldots$	$x^{24}+24x^{20}+168x^{16}+98x^{12}+168x^8+24x^4+1$
$87\rho^2+69\rho+3$	$0.5300311633\ldots - i\cdot 0.0453631924\ldots$	$x^{12}-6x^{10}+6x^8+10x^6+6x^4-6x^2+1$
$9\rho^2+3\rho+21$	$0.5300311633\ldots - i\cdot 0.0453631924\ldots$	$x^{12}-6x^{10}+6x^8+10x^6+6x^4-6x^2+1$
$-9\rho^2-3\rho+39$	$0.0453631924\ldots + i\cdot 0.5300311633\ldots$	$x^{12}+6x^{10}+6x^8-10x^6+6x^4+6x^2+1$
$15\rho^2+-3\rho+57$	$-0.4068652507\ldots - i\cdot 0.3427120088\ldots$	$x^{24}+24x^{20}+168x^{16}+98x^{12}+168x^8+24x^4+1$

(continued)

	numerical value of $\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda)^4$	conjectured minimal polynomial
$3\rho^2+3\rho+3$	$-0.0754589478\ldots + i\cdot 0.0268209974\ldots$	$x^6+24x^5+168x^4+98x^3+168x^2+24x+1$
$\rho^2-3\rho+3$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$-3\rho^2+3\rho+3$	$-0.0754589478\ldots + i\cdot 0.0268209974\ldots$	$x^6+24x^5+168x^4+98x^3+168x^2+24x+1$
$3\rho^2+9\rho+3$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$27\rho^2+9\rho+3$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$9\rho^2+15\rho+3$	$-0.0754589478\ldots + i\cdot 0.0268209974\ldots$	$x^6+24x^5+168x^4+98x^3+168x^2+24x+1$
$87\rho^2+69\rho+3$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$9\rho^2+3\rho+21$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$-9\rho^2-3\rho+39$	$0.0754589478\ldots - i\cdot 0.0268209974\ldots$	$x^6-24x^5+168x^4-98x^3+168x^2-24x+1$
$15\rho^2-3\rho+57$	$-0.0754589478\ldots + i\cdot 0.0268209974\ldots$	$x^6+24x^5+168x^4+98x^3+168x^2+24x+1$

Remark 5.1.2. Set $\mathfrak{f}=(\rho-2,5)$ and $\mathfrak{a}=(\rho,3)$. Note that $\mathcal{O}_{\mathfrak{f}}^{+,\times}\subset\mathcal{O}_K^\times\cap(1+\mathfrak{f})$. We nevertheless keep this choice, because the numerical computations for other nontrivial $\mathfrak{f}$ coprime to $(\rho,3)$ would take more than a day on a typical personal computer. For this choice, the narrow ray class field $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$ coincide with K. Let $\lambda=(-239,360,115)(5,\rho+3,\rho^2+1)^t$ be the admissible element. Our computation shows that

$$\begin{aligned}\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1,\lambda) &= \left(\Gamma\left(\tfrac{3}{5},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{682915},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{682915}\right)\Gamma\left(\tfrac{3}{5}+\tfrac{-\epsilon_{\mathbf{C}}^4-\epsilon_{\mathbf{C}}^{-4}+2965367}{2048745},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{682915},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{682915}\right)\right.\\ &\quad\left.\Gamma\left(\tfrac{3}{5}+2\cdot\tfrac{-\epsilon_{\mathbf{C}}^4-\epsilon_{\mathbf{C}}^{-4}+2965367}{2048745},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{682915},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{682915}\right)\right)\left(\Gamma\left(\tfrac{1}{5},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{2048745},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{2048745}\right)\right.\\ &\quad\Gamma\left(\tfrac{1}{5}+\tfrac{-\epsilon_{\mathbf{C}}^4-\epsilon_{\mathbf{C}}^{-4}+2965367}{6146235},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{2048745},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{2048745}\right)\\ &\quad\left.\Gamma\left(\tfrac{1}{5}+2\cdot\tfrac{-\epsilon_{\mathbf{C}}^4-\epsilon_{\mathbf{C}}^{-4}+2965367}{6146235},\tfrac{-\epsilon_{\mathbf{C}}^4+677146}{2048745},\tfrac{-\epsilon_{\mathbf{C}}^{-4}-108344009}{2048745}\right)\right)^{-3}\\ &= -0.5000000000\ldots - i\cdot 0.8660254037\ldots\\ &\approx e^{4\pi i/3},\end{aligned}$$

where $\approx$ represents an error less than 10^{-1000}. Here we only test one admissible element, because the convergence of (4.2) deteriorates markedly when $|\mathrm{Im}\,\tau|$ and $|\mathrm{Im}\,\sigma|$ are small. As observed in [BCG2023], this constitutes the main bottleneck for computations involving larger input data.

5.2 | The main conjecture

We are now ready to state the conjecture proposed in [BCG2023]. We caution, however, that our notation and setup differ slightly from those of [BCG2023]. The original formulation of their conjecture is recovered by taking $L = \mathfrak{f}\mathfrak{b}^{-1}$ for an integral ideal $\mathfrak{b}$ satisfying $(\mathfrak{f}, \mathfrak{b}) = 1$, and $v = 1$.

Conjecture 5.2.1. Fix the following quantities:

- K a complex cubic field,
- $\mathfrak{f}$ an ideal of $\mathcal{O}_K$ such that the ray class field $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ of K modulo $\mathfrak{f} \cdot \infty_{\mathbf{R}}$ is totally complex (or equivalently, $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_K^\times \cap (1 + \mathfrak{f})$),
- $\sigma_{\mathbf{C}} \colon K \hookrightarrow \mathbf{C}$ a complex embedding,
- $\tilde{\sigma}_{\mathbf{C}}$ a complex embedding $K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) \hookrightarrow \mathbf{C}$ extending $\sigma_{\mathbf{C}}$ and ν its corresponding archimedean place,
- L a fractional ideal of K,
- $\mathfrak{a}$ a smoothing ideal for $\mathfrak{f}$ and relatively prime to 6,
- v a primitive $\mathfrak{f}$-division point of L,
- f the smallest positive integer in $\mathfrak{f}$,
- λ a v-admissible element, namely $\begin{cases} \lambda \in \mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}L, \\ \lambda \equiv fv \mod fL, \\ \lambda \not\equiv 0 \mod \mathrm{N}(\mathfrak{a})L. \end{cases}$

Then

(i) The value $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)$ is independent of the choice of λ, and arises as the image in $\mathbf{C}$ of a unit $\mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v} \in \mathcal{O}_{K(\mathfrak{f}\cdot\infty_{\mathbf{R}})}^\times$.

(ii) For every archimedean place ν' of $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ lying above the real archimedean place of K, one has

$$|\mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}|_{\nu'} = 1.$$

(iii) For every integral ideal $\mathfrak{c}$ of K relatively prime to $\mathfrak{f}$, one has

$$\sigma_{\mathfrak{c}}(\mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}) = \mathbf{u}_{\mathfrak{c}^{-1}L,\mathfrak{f},\mathfrak{a},v}.$$

Remark 5.2.2.

1. Conjecture 5.2.1 is consistent with the result established in Proposition 4.3.18.
2. In numerical examples, one often finds that the minimal polynomial of $\mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}$ over $\mathbf{Q}$ is palindromic and that $\mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}$ generates the full narrow ray class field.
3. Part (ii) of the conjecture reveals its connection with Stark's conjecture, as will become apparent in (5.1).

5.3 | A Kronecker limit formula

As anticipated, our conjectural elliptic units are closely connected with Stark units, and the link between the two is furnished by the Kronecker limit formula established by Bergeron, Charollois, and García in [BCG2023].

Set $S = \{\infty_{\mathbf{R}}, \infty_{\mathbf{C}}\} \cup \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{f}\}$. Write

$$\zeta_{\mathfrak{f}}(\mathfrak{c}, s) = \zeta_{K(\mathfrak{f}\cdot\infty_{\mathbf{R}})/K,S}(\sigma_{\mathfrak{c}}, s).$$

Definition 5.3.1. For every integral ideal $\mathfrak{b}$ of K relatively prime to $\mathfrak{f}$, we define the *smoothed partial zeta function*

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b}, s) := \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b}, s) - \mathrm{N}(\mathfrak{a})\zeta_{\mathfrak{f}}(\mathfrak{b}, s).$$

The following theorem closely resembles the limit formulas that express the derivative at $s = 0$ of Hecke L-functions for imaginary quadratic fields in terms of logarithms of absolute values of elliptic units.

Theorem 5.3.2. *The absolute value of $\mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)$ is independent of the choice of λ, and one has*

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b}, 0) = \log|\mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-1},\mathfrak{f},\mathfrak{a}}(1)|^2.$$

Proof. See [BCG2023, Theorem 3.2]. □

Assuming Conjecture 5.2.1, the above theorem can be reformulated as

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b}, 0) = \log|\mathbf{u}_{\mathfrak{f}\mathfrak{b}^{-1},\mathfrak{f},\mathfrak{a},1}|_{\nu}.$$

Let $e_{\mathfrak{f}}$ be the number of roots of unity in $K(\mathfrak{f}\cdot\infty_{\mathbf{R}})$. Set $G = \mathrm{Gal}(K(\mathfrak{f}\cdot\infty_{\mathbf{R}})/K)$. Assuming $\mathrm{St}(K(\mathfrak{f}\cdot\infty_{\mathbf{R}})/K, S)$, there exists a unit $\mathbf{u}_{\mathrm{Stark}} \in K(\mathfrak{f}\cdot\infty_{\mathbf{R}})^{\times}$ such that

- If ν' is a place of $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ not lying above the complex archimedean place of K, then $|\mathbf{u}_{\text{Stark}}|_{\nu'} = 1$;
- For all $\sigma \in G$, we have
$$\zeta'_{\mathfrak{f}}(\sigma, 0) = -\frac{1}{e_{\mathfrak{f}}} \log |\mathbf{u}_{\text{Stark}}^{\sigma}|_{\nu};$$
- $K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) \left(\mathbf{u}_{\text{Stark}}^{1/e_{\mathfrak{f}}}\right) / K$ is an abelian extension.

It follows that
$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b}, 0) = -\frac{1}{e_{\mathfrak{f}}} \log \left| \mathbf{u}_{\text{Stark}}^{(\sigma_{\mathfrak{a}} - \mathrm{N}(\mathfrak{a}))\sigma_{\mathfrak{b}}} \right|_{\nu}.$$
As in Section 3.3.2, $\mathbf{u}_{\text{Stark}}^{\mathrm{N}(\mathfrak{a}) - \sigma_{\mathfrak{a}}}$ is an $e_{\mathfrak{f}}$-power in $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$. Suppose that $\mathbf{u}_{\text{Stark}}^{\mathrm{N}(\mathfrak{a}) - \sigma_{\mathfrak{a}}} = \alpha^{e_{\mathfrak{f}}}$ for some $\alpha \in K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$. By Theorem 5.3.2, we have
$$\alpha^{e_{\mathfrak{f}}} = \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1}^{e_{\mathfrak{f}}}.$$
It follows that there exists a root of unity ζ with $\zeta^{e_{\mathfrak{f}}} = 1$ such that $\alpha = \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1}^{e_{\mathfrak{f}}} \cdot \zeta$. Assuming Conjecture 5.2.1 and Conjecture 3.2.1, we therefore obtain
$$\mathbf{u}_{\text{Stark}}^{\mathrm{N}(\mathfrak{a}) - \sigma_{\mathfrak{a}}} = \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1}^{e_{\mathfrak{f}}}. \tag{5.1}$$
This is reminiscent of (3.4).

5.4 | More examples

Throughout this section, unless explicitly stated otherwise, for each conjectural unit we compute at more than 10 admissible elements to ensure that the numerical results are independent of the particular admissible choices; accordingly, we silently omit the specific admissible elements used.

5.4.1 | An example due to Dasgupta

This example was first investigated by Dasgupta in [Das1999], and later studied at length in [BCG2023]. We use it as a running example to illustrate, step by step, our numerical verification of Conjecture 5.2.1. While the computations involved are substantial, the basic strategy is entirely inspired by [BCG2023]; the only real difference from their computations is the extra flexibility afforded by the parameter v.

Let $K = \mathbf{Q}(\rho)$, where $\rho \in \mathbf{C}$ is a root of $x^3 - x^2 + 5x + 1$ with positive imaginary part. The discriminant of K over $\mathbf{Q}$ is $-588 = -2^2 \cdot 3 \cdot 7^2$. Therefore (3) ramifies in K and we have

$$(3) = \mathfrak{p}_1\mathfrak{p}_2^2 \quad \text{where} \quad \mathfrak{p}_1 = (3, \rho + 1) \quad \text{and} \quad \mathfrak{p}_2 = (3, \rho - 1).$$

The Galois closure of K over $\mathbf{Q}$ is $K(\sqrt{-3})$. The Hilbert class field of K is $H = K(\theta)$, where $\theta = \zeta_7 + \zeta_7^{-1}$ satisfies the equation $\theta^3 + \theta^2 - 2\theta - 1 = 0$. The extension $H(\sqrt{-3})/K$ is abelian, since $H(\sqrt{-3})$ is the compositum of $K(\sqrt{-3})$ and H. The Galois group $G := \mathrm{Gal}(H(\sqrt{-3})/K)$ is cyclic of order 6. In addition, $H(\sqrt{-3})$ has 6 roots of unity. One can show that $H(\sqrt{-3}) = K(\mathfrak{p}_1 \cdot \infty_{\mathbf{R}})$ by hand (cf. [Das1999, Proposition 5.1.1]) or by PARI/GP. We take $\mathfrak{f} = \mathfrak{p}_1$ and $S = \{\infty_{\mathbf{R}}, \infty_{\mathbf{C}}, \mathfrak{p}_1\}$.

Following Bergeron, Charollois, and García, let $\mathfrak{a}$ be the unique ideal of norm 5, which we take as our smoothing ideal, and let $\mathfrak{b} = (2, \rho - 1)$. Then $\epsilon = -\rho$, and the class $[\mathfrak{a}] = [\mathfrak{b}]$ is a generator of the narrow ray class group

$$\mathrm{Cl}_K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) \cong \mathbf{Z}/6\mathbf{Z}.$$

Set $L_k = \mathfrak{f}\mathfrak{b}^{-k}$ for $0 \leq k \leq 5$. There are 2 primitive $\mathfrak{f}$-division points of L_k. We compute (to 1000 decimal digits):

$$\begin{aligned}
\mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)^{-1}, \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-3},\mathfrak{f},\mathfrak{a}}(2)^{-1} &\approx -1.3795863226\ldots + i \cdot 2.0250077123\ldots \approx \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-3},\mathfrak{f},\mathfrak{a}}(1), \mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(2)\\
\mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-1},\mathfrak{f},\mathfrak{a}}(1)^{-1}, \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-4},\mathfrak{f},\mathfrak{a}}(2)^{-1} &\approx 1.3269203008\ldots - i \cdot 1.2639106201\ldots \approx \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-4},\mathfrak{f},\mathfrak{a}}(1), \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-1},\mathfrak{f},\mathfrak{a}}(2)\\
\mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-2},\mathfrak{f},\mathfrak{a}}(1)^{-1}, \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-5},\mathfrak{f},\mathfrak{a}}(2)^{-1} &\approx -4.8390562074\ldots - i \cdot 7.5167566542\ldots \approx \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-5},\mathfrak{f},\mathfrak{a}}(1), \mathbf{\Gamma}_{\mathfrak{f}\mathfrak{b}^{-2},\mathfrak{f},\mathfrak{a}}(2).
\end{aligned} \tag{5.2}$$

Our numerical results agree with those reported in [BCG2023] to at least ten decimal places. Furthermore, as we shall see below, they give rise to the same minimal polynomial. We therefore consider our computations to be fully consistent with those of [BCG2023], despite the fact that the admissible elements involved were very likely quite different. This agreement may be viewed as further numerical evidence in support of Conjecture 5.2.1.

We use the command **algdep** in PARI/GP to identify them as roots of

$$\begin{aligned}
P(x) = {} & x^{18} + 12x^{17} + 106x^{16} + 219x^{15} + 284x^{14} + 16x^{13} + 1208x^{12} + 3414x^{11} + 2314x^{10} + 477x^9\\
& + 2314x^8 + 3414x^7 + 1208x^6 + 16x^5 + 284x^4 + 219x^3 + 106x^2 + 12x + 1.
\end{aligned}$$

Over K this polynomial factors as

$$\begin{aligned}
&(x^6 + (3\rho + 3)x^5 + (-\rho^2 + \rho + 2)x^4 + (7\rho^2 + 3\rho + 1)x^3 + (-\rho^2 + \rho + 2)x^2 + (3\rho + 3)x + 1)\\
&(x^6 + (-2\rho^2 - 2)x^5 + (-5\rho^2 + 4\rho + 3)x^4 + (-2\rho^2 + 3\rho + 2)x^3 + (-9\rho - 1)x^2 + (-3\rho^2 - \rho)x\\
&- \rho^2 + 5\rho + 1)
\end{aligned}$$

$$(x^6 + (2\rho^2 - 3\rho + 11)x^5 + (19\rho^2 - 23\rho + 98)x^4 + (35\rho^2 - 42\rho + 184)x^3 + (53\rho^2 - 64\rho + 279)x^2 + (-54\rho^2 + 64\rho - 282)x + 26\rho^2 - 31\rho + 136).$$

Our alleged units agree, to 1000 decimal places, with the roots of

$$Q(x) = x^6 + (3+3\rho)x^5 + (2+\rho-\rho^2)x^4 + (1+3\rho+7\rho^2)x^3 + (2+\rho-\rho^2)x^2 + (3+3\rho)x + 1.$$

Pick a root $w \in \mathbf{C}$ of Q and consider the field extension $K(w)/K$. Using the command **rnfconductor** in PARI/GP, we find that this extension is abelian and the conductor of this extension is precisely $\mathfrak{f} \cdot \infty_{\mathbf{R}}$. Since $\deg Q = 6 = \#\mathrm{Cl}_K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$, $K(w) = K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$. This agrees with part (i) of Conjecture 5.2.1.

Note that Q is palindromic. Hence, if z is a root of Q, then so is z^{-1}. It follows that the unique element $\tau \in G$ of order 2 acts on the roots by

$$\tau(z) = z^{-1}.$$

Consequently, for every place ν' of $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ lying above the real archimedean place of K, we have

$$|z|_{\nu'} = 1.$$

This agrees with part (ii) of Conjecture 5.2.1.

Let $\mathfrak{p}$ be a prime of K such that $(\mathfrak{p}, \mathfrak{f}) = 1$, and let $\mathfrak{P}$ be a prime of $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ lying over $\mathfrak{p}$. Recall that $\sigma_{\mathfrak{p}}$ is uniquely determined by the condition

$$\sigma_{\mathfrak{p}}(x) \equiv x^{\mathrm{N}(\mathfrak{p})} \mod \mathfrak{P} \quad \text{for all } x \in \mathcal{O}_{K(\mathfrak{f} \cdot \infty_{\mathbf{R}})}.$$

Using (5.2), we can numerically verify the reciprocity law. The results agree with the third part of Conjecture 5.2.1.

In [Das1999, § 5.5], Dasgupta showed that, if the Stark unit $\mathbf{u}_{\mathrm{Stark}}$ exists, then after possibly rescaling by a root of unity in $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$, it must be a root of

$$x^6 - (\rho^2 - 4\rho + 1)x^5 + (-7\rho^2 + 10\rho + 5)x^4 - (22\rho^2 + 17\rho + 6)x^3 + (-7\rho^2 + 10\rho + 5)x^2 - (\rho^2 - 4\rho + 1)x + 1.$$

We choose the complex embedding of $K(\mathbf{u}_{\mathrm{Stark}})$ determined by the condition that $\mathbf{u}_{\mathrm{Stark}}$ be sent to the root whose initial decimal digits are

$$-0.256\ldots - i \cdot 0.077\ldots.$$

With respect to this embedding, we find numerically that

$$\mathbf{u}_{\text{Stark}}^{\mathrm{N}(\mathfrak{a})-\sigma_{\mathfrak{a}}} \approx \mathbf{\Gamma}_{\mathfrak{f},\mathfrak{f},\mathfrak{a}}(1)^6,$$

with an error of less than 10^{-1000}. This provides numerical confirmation of (5.1).

Finally, in [Das1999, § 5.2.7], Dasgupta computed $\zeta_{\mathfrak{f}}'(\mathfrak{b}^k, 0)$ numerically to 25 decimal places. Using these values, we numerically verified Theorem 5.3.2, and found excellent agreement with the theoretical prediction.

5.4.2 | An example due to Ren and Sczech

Table 5.4: Input data for Section 5.4.2

Symbol	Specification
K	$\mathbf{Q}(\rho)$ with $\rho \in \mathbf{C}$ a root of $x^3 - x + 1$ such that $\operatorname{Im} \rho < 0$.
L	$(5) \subset \mathcal{O}_K$.
$\mathfrak{f}$	$(5) \subset \mathcal{O}_K$.
$\mathfrak{a}$	$(\rho - 2, 7) \subset \mathcal{O}_K$.

This example first appeared in [RS2009], and was later briefly revisited in the appendix of [BCG2023]. From our perspective, its main interest lies in the fact that it is one of the few examples currently known to the author for which $\mathfrak{f}$ is nonprime and yet the associated computations remain reasonably tractable.

With these choices, K is the complex cubic field of smallest discriminant, with

$$\Delta_K = -23.$$

Its class number is 1, $-\rho$ is a fundamental unit, and $\epsilon = \rho^{24}$. Moreover, $\operatorname{Gal}(K(\mathfrak{f} \cdot \infty_{\mathbf{R}})/K)$ is cyclic of order 4, $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ contains 10 roots of unity, and $[\mathfrak{a}]$ is a generator of

$$\mathrm{Cl}_K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) \cong \mathbf{Z}/4\mathbf{Z}.$$

Using their refinement of Stark's conjecture over complex cubic fields, Ren and Sczech are able to compute numerical values of the conjectural Stark unit, so—just as in Section 5.4.1—we can numerically test our Conjecture Conjecture 5.2.1 and (5.1). An attentive reader will notice that here we take $\mathfrak{f} = (5)$, rather than a prime ideal of norm 3 as in nearly all examples in this thesis. In fact, as mentioned in Remark 5.1.2, due to the convergence rate in (4.2), examples that can be run quickly on a personal computer almost invariably require $\mathfrak{f}$ of very small norm. The present case is one of the few examples we currently know with comparatively large $\mathrm{N}(\mathfrak{f})$ that can still be carried out within a reasonable time (though not effortlessly: for example, for all 1-admissible elements we have $35 \mid s$).

A larger norm of $\mathfrak{f}$ affords more flexibility in the choice of v. In the present example,

$$\left|(\mathcal{O}_K/\mathfrak{f})^\times\right| = 96,$$

so there are 96 possible choices of v. Owing to time limitations, we have not yet tested our conjectures for every v; so far we have computed the numerical values of only the following units:[2]

$$\begin{aligned}
\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1} &= 86.3507600524\ldots + i \cdot 58.4192940826\ldots, \\
\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},2} &= -0.1098351352\ldots - i \cdot 0.3630246796\ldots, \\
\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},3} &= -0.7635357653\ldots + i \cdot 2.5236216634\ldots, \\
\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},4} &= 0.0079444873\ldots - i \cdot 0.0053747221\ldots.
\end{aligned}$$

[2]For each $v \in \{1, 2, 3, 4\}$ we computed 3 admissible elements

5.4.3 | An evocative example

Table 5.5: Input data for Section 5.4.3

Symbol	Specification
K	$\mathbf{Q}(\rho)$ with $\rho \in \mathbf{C}$ a root of $x^3 - 7$ such that $\operatorname{Im} \rho > 0$.
L	$(\rho - 1,\, 3) \subset \mathcal{O}_K$.
$\mathfrak{f}$	$(\rho - 1,\, 3) \subset \mathcal{O}_K$.
$\mathfrak{a}$	$(\rho + 2,\, 5) \subset \mathcal{O}_K$.
$\mathfrak{p}$	$(\rho - 1,\, 2) \subset \mathcal{O}_K$.

This example is briefly discussed in the appendix of [BCG2023]. Its striking simplicity makes the relevant computations especially manageable.

As in Section 5.4.1, we can numerically verify Conjecture 5.2.1. In this example one can find many admissible elements with $s = 1$, which makes the numerics straightforward; for this reason we recommend that readers who wish to try their own computations begin with this example. However, this is not why we end the chapter with it. What is interesting here is that the narrow ray class field $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ happens to coincide with $K(\mathfrak{f}\mathfrak{p} \cdot \infty_{\mathbf{R}})$. More precisely,

$$K(\mathfrak{f}\mathfrak{p} \cdot \infty_{\mathbf{R}}) = K(\mathfrak{f} \cdot \infty_{\mathbf{R}}) = K[t]/P(t),$$

where

$$\begin{aligned} P(t) = t^6 &+ (6\rho^2 - 14\rho + 2)t^5 + (4\rho^2 - 6\rho + 2)t^4 + (106\rho^2 - 152\rho - 103)t^3 \\ &+ (4\rho^2 - 6\rho + 2)t^2 + (6\rho^2 - 14\rho + 2)t + 1. \end{aligned}$$

Motivated by Corollary 2.1.8, it is natural to conjecture that

$$\mathbf{u}_{\mathfrak{p}L,\mathfrak{f}\mathfrak{p},\mathfrak{a},v} = \mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}^{1-\sigma_{\mathfrak{p}}^{-1}},$$

and our computations support this. Concretely, taking $t = -0.0626162938\ldots + i \cdot 0.2669850710\ldots$ (so that we may view $K(\mathfrak{f} \cdot \infty_{\mathbf{R}})$ as a subfield of $\mathbf{C}$), we obtain

$$\begin{aligned} \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1} &= -0.0626162938\ldots + i \cdot 0.2669850710\ldots, \\ \sigma_{\mathfrak{p}}^{-1}(\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1}) &= -4.0240295453\ldots + i \cdot 41.8559517671\ldots, \\ \mathbf{u}_{\mathfrak{f}\mathfrak{p},\mathfrak{f}\mathfrak{p},\mathfrak{a},1} &= 0.0064627547\ldots + i \cdot 0.0008746659\ldots = \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},1}^{1-\sigma_{\mathfrak{p}}^{-1}}, \\ \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},5} &= -0.8326432366\ldots - i \cdot 3.5502470704\ldots, \\ \sigma_{\mathfrak{p}}^{-1}(\mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},5}) &= -0.0022758886\ldots - i \cdot 0.0236726609\ldots, \\ \mathbf{u}_{\mathfrak{f}\mathfrak{p},\mathfrak{f}\mathfrak{p},\mathfrak{a},5} &= 151.9495511279\ldots - i \cdot 20.5647760487\ldots = \mathbf{u}_{\mathfrak{f},\mathfrak{f},\mathfrak{a},5}^{1-\sigma_{\mathfrak{p}}^{-1}}. \end{aligned}$$

We close this chapter with the following admittedly very tentative conjecture.

Conjecture 5.4.1. Let $\mathfrak{p}$ be a prime ideal of K. Withe notation as in Conjecture 5.2.1, we assume further that $(\mathfrak{p}, \mathfrak{a}) = 1$.Assuming Conjecture 5.2.1, and if necessary, further hypotheses, we have

$$\mathrm{N}_{K(\mathfrak{f}\mathfrak{p}\cdot\infty_{\mathbf{R}})/K(\mathfrak{f}\cdot\infty_{\mathbf{R}})}\mathbf{u}_{\mathfrak{p}L,\mathfrak{f}\mathfrak{p},\mathfrak{a},v} = \begin{cases} \mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}^{1-\sigma_{\mathfrak{p}}^{-1}}, & \text{if } \mathfrak{p} \nmid \mathfrak{f}, \\ \mathbf{u}_{L,\mathfrak{f},\mathfrak{a},v}, & \text{if } \mathfrak{p} \mid \mathfrak{f}. \end{cases}$$

Chapter 6

Concluding thoughts

"孤帆远影碧空尽，唯见长江天际流。"

李白，《黄鹤楼送孟浩然之广陵》

Naively, I feel that there should exist an algebraic variety B whose analytification B^{an} is determined by the lattice L. The primitive $\mathfrak{f}$-division point v of L should then be regarded as the torsion points of B. Moreover, there should exist a function $\mathbf{\Gamma}_{B,\mathfrak{a}}$ on B, together with a positive integer n, such that

$$\mathbf{\Gamma}_{B,\mathfrak{a}}(v) = \mathbf{\Gamma}_{L,\mathfrak{f},\mathfrak{a}}(v, \lambda)^n$$

for every $\mathfrak{f}$ such that $(\mathfrak{a}, \mathfrak{f}) = 1$, every primitive $\mathfrak{f}$-division point v of L and every admissible element λ associated with the data $L, \mathfrak{f}, \mathfrak{a}$.

The introduction of the seemingly redundant admissible elements λ may be understood in the same way as in Section 2.3: they are needed to cancel certain extraneous factors. Then, by finding an analogue of [Sil1994, Theorem 8.2], one should be able to translate the algebraic Galois action on B_{tors} into an analytic action, and in this way the conjecture in [BCG2023] could be resolved. More ambitiously, a deeper study of B and of its moduli space may lead to a theory of complex multiplication for complex cubic fields.

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